Assume that continuous random variable X has PDF that is non-zero only on segment $[a,b]$ and strictly positive on open interval (a,b). Median of random variable X is value $m \in (a,b)$ such that $P(X<m)=P(X>m)=1/2$.

```
1.1 Prove that for symmetrical PDFs median is equal to expected value.
```

To prove that, we need to confirm that the point of symmetry of the PDF is equal to the median, then that the point of symmetry is equal to the $\mathbb{E}(X)$. Let us start by proving that the point of symmetry is equal to the median.

Assume x_0 is the point of symmetry $\implies$ it should follow that $\int_{-\infty}^{X_0} p(x)dx = \int_{x_0}^{\infty} p(x)dx$

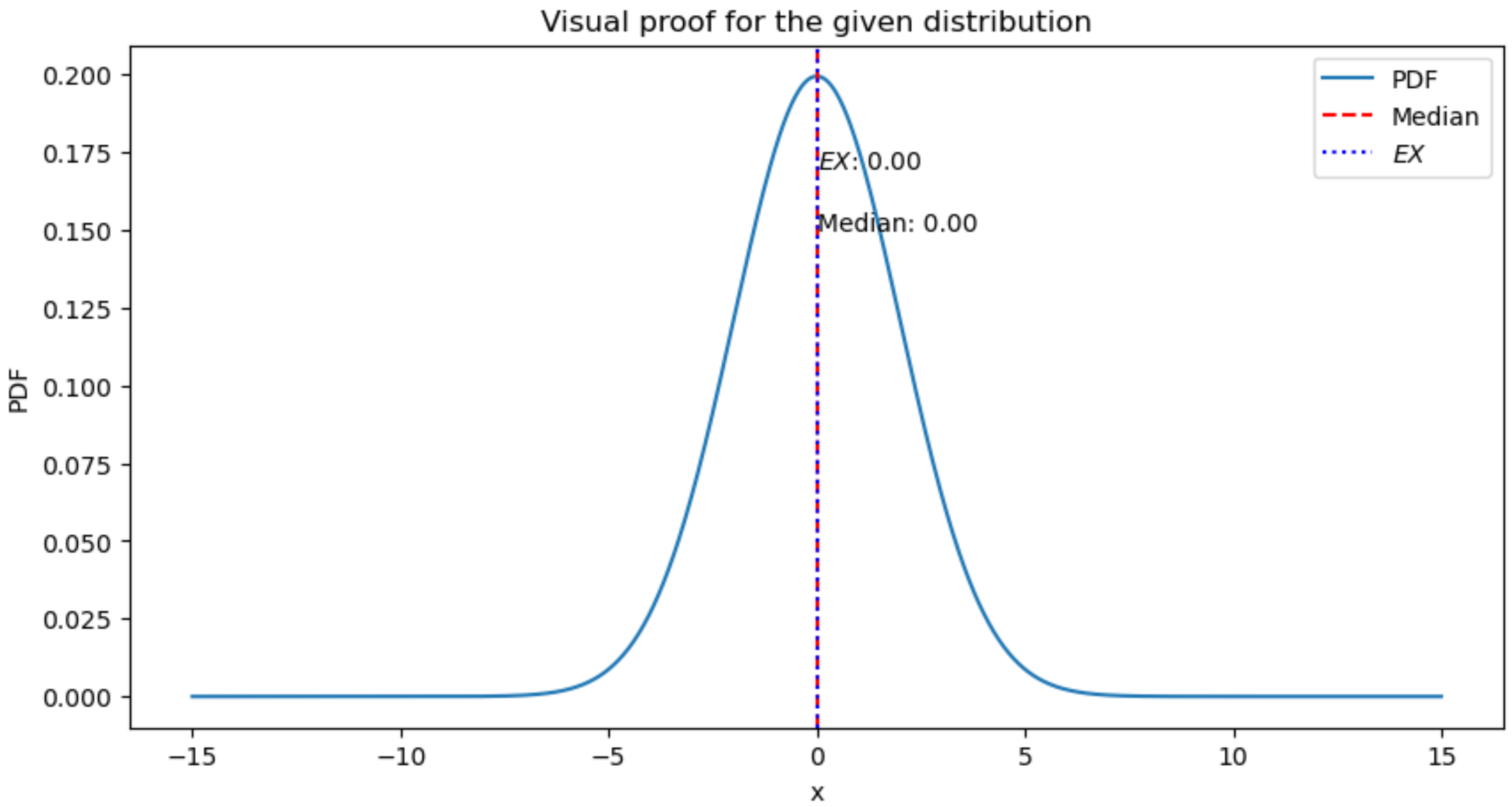
$\forall x_0 - \Delta x$ we obtain $\int_{-\infty}^{X_0} p(x_0 - \Delta x)dx = \int_0^{\infty} p(x_0 - \Delta x)dx$
$\forall x_0 + \Delta x$ we obtain $\int_{x_0}^{\infty} p(x_0 + \Delta x)dx = \int_0^{\infty} p(x_0 - \Delta x)dx$
$\therefore$ the point of symmetry equals to the median value of the pdf.

Now let us prove that $\mathbb{E}(X)$ = point of symmetry. And **by definition**, if $\mathbb{E}(X)$ exists (and it does, which follows from the task condition) and pdf is symmetric with respect to $X = X_0$, then $\mathbb{E}(X)$ = point of symmetry.
$\therefore \mathbb{E}(X) = X_0$

The code below also highlights that the proof is correct.

In [52]:
```python
import matplotlib.pyplot as plt
import scipy.stats
import numpy as np


def solution(x, pdf):
    med = np.median(x)
    exp_val = abs(round(np.sum(x * pdf) * (x[1] - x[0]), 5))
    print(f'Median: {med}\nExpected value: {exp_val}')

    plt.figure(figsize=(10, 5))
    plt.plot(x, pdf, label='PDF')
    plt.axvline(med, ls='--', color='r', label='Median')
    plt.axvline(exp_val, ls=':', color='b', label='$EX$')
    plt.text(med, .15, f'Median: {med:.2f}', color='black')
    plt.text(exp_val, .17, f'$EX$: {exp_val:.2f}', color='black')
    plt.xlabel('x')
    plt.ylabel('PDF')
    plt.title('Visual proof for the given distribution')
    plt.legend()
    plt.show()


# checking  for a normal distribution
x = np.linspace(-15, 15, 1000)
pdf = scipy.stats.norm.pdf(x, loc=0, scale=2)

solution(x, pdf)
```

```
Median: 0.0
Expected value: 0.0
```



`1.2 Provide an example of PDF such that median is larger than the expected value.`

Let us consider some arbitrary the exponensial distribution. The pdf of the exponential distribution is given by $f(x) = \lambda e^{-\lambda x}$, where λ is the rate parameter. The median of the exponential distribution is given by $m = \frac{1}{\lambda}$, where λ is the rate parameter.

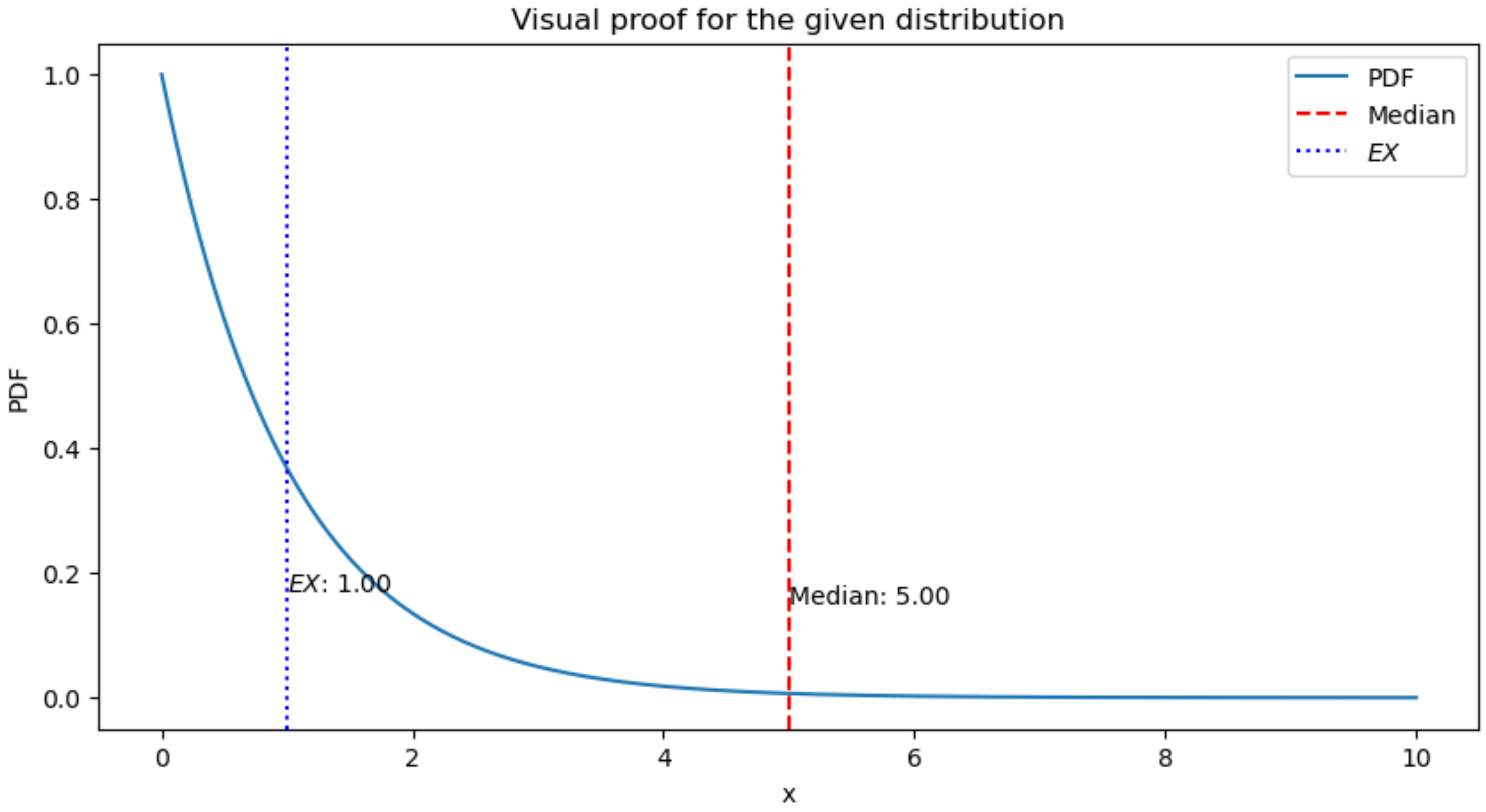
Let us assume that $\lambda = 1$ and $m = 1$.

The expected value of the exponential distribution is given by $\mathbb{E}(X) = \frac{1}{\lambda} = \frac{1}{1} = 1$

The code below also highlights that the proof is correct.

In [53]:
```
x = np.linspace(0, 10, 1000)
pdf = scipy.stats.expon.pdf(x, loc=0, scale=1)

solution(x, pdf)
```

```
Median: 5.0
Expected value: 0.99949
```



1.3 Let X be random variable, f be strictly increasing function and $Y=f(X)$. What can you say about medians of X and Y?

Assuming that X is a random variable, f is a strictly increasing function and $Y = f(X)$, the median of X is m_X, the median of Y is m_Y

$\implies$ it follows that $m_X = f^{-1}(m_Y)$

$\implies \forall x_i < x_{i+1} \implies \forall f(x_i) < f(x_{i+1})$