A fair coin is tossed 400 times. Let X be number of heads. Prove that $P(X > 240) \leq \frac{1}{32}$

Given that the fair tossing with such a high sample is a binomial distribution and that $|\mu - \mathbb{E}| \to 0$, we can use Chebyshev's inequality to prove that $P(X > 240) \leq \frac{1}{32}$.

The Chebyshev's inequality is defined as $P(|X - \mathbb{E}X| \geq \alpha) \leq \frac{VarX}{\alpha^2} \mid \alpha > 0$

Now, let us plug in the values for the inequality.

- $\forall x\, \mathbb{E}x = \frac{1}{2} \implies \mu = \mathbb{E}X = 200$
- $Var(X)$, given that this is a binomial distribution, is defined by $\sigma^2 = np(1-p) = 400 \cdot \frac{1}{2} \cdot \frac{1}{2} = 100 \therefore \sigma = 10$

Given that the probability of r.v. X deviating from its μ by more than n standard deviations is at most $\frac{1}{n^2}$

$$\implies P(|X - 200| \geq 40) \leq \frac{1}{n^2} \implies \frac{1}{n^2} \leq P(X - 200 \leq 4 \cdot n) \leq \frac{\mathbb{E}(X-200)^2}{(4n^2)}$$

$$\implies \frac{1}{n^2} \leq P(X - 200 \leq 4 \cdot n) \leq \frac{100}{4n^2}$$

$$\implies n^2 \leq 25, n \leq 5$$

$$\therefore P(X > 240) \leq P(X - 200 \geq 40) \leq \frac{1}{n^2} \leq \frac{1}{25} \leq \frac{1}{32} \quad \blacksquare$$