Markov's inequality

Let $X \geq 0$ be a r.v. Then for all $a > 0$:

$$P[X \geq a] \leq \frac{\mathbb{E}X}{a}$$

Proof.

$$\mathbb{I}\{\underbrace{X \geq a}_{\text{event}}\} = \begin{cases} \overbrace{1}, & \overbrace{X \geq a}^{p} \\ \underbrace{0}, & \underbrace{\text{otherwise}} \end{cases}$$

$$\mathbb{E}[\mathbb{I}\{X \geq a\}] = 1 \cdot P[X \geq a] + \cancel{0 \cdot P[\text{otherwise}]} =$$

$$= P[X \geq a].$$

LHS:

$$[P[X \geq a] = \mathbb{E}[\mathbb{I}\{X \geq a\}] \leq \mathbb{E}\left[\frac{X}{a} \cdot \mathbb{I}\{X \geq a\}\right] \leq$$

$$\underbrace{\mathbb{I}\{X \geq a\} \leq \frac{X}{a} \cdot \mathbb{I}\{X \geq a\}}$$

1) $X < a$. Both parts equal to 0 ✓

2) $X \geq a$. LHS $= 1$. RHS $= \frac{X}{a}$

$$1 \leq \frac{X}{a} \iff a \leq X \quad ✓$$

$$\leq \mathbb{E}\left[\frac{X}{a}\right] = \frac{\mathbb{E}X}{a}] \quad \square$$

Exampe X - dice $\{1 \dots 6\}$

$$P[X \geq 5] \leq \frac{\mathbb{E}X}{5} = \frac{3.5}{5} = 0.7$$

$$P[X = 5 \text{ or } 6] = 1/3$$

$$1/3 < \boxed{0.7}$$

Chebyshev's inequality

X - r.v. $\mathrm{Var}\, X < +\infty$ Then $\forall a > 0$

$$P[\underbrace{|X - \mathbb{E}X| \geq a}_{\text{more common}}] \leq \frac{\mathrm{Var}\, X}{a^2}$$

Proof

$$P[|X - \mathbb{E}X| \geq a] = P[(X - \mathbb{E}X)^2 \geq a^2] \leq \{\text{Markov's in.}\}$$

$$\leq \frac{\mathbb{E}(X - \mathbb{E}X)^2}{a^2} = \frac{\mathrm{Var}\, X}{a^2} \quad \square$$

$$P[X \geq a] \leq \frac{\mathbb{E}X}{a}$$

Example X - dice

$$P[|X - \mathbb{E}X| \geq 2] \leq \frac{\mathrm{Var}\, X}{2^2} = \frac{2.9}{4} \approx 0.74$$

$$\mathrm{Var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{6} \cdot (1 + 4 + 9 + 16 + 25 + 36) - 3.5^2 \approx 2.9$$

$$P[|X - \mathbb{E}X| \geq 2] = P[|X - 3.5| \geq 2] = P[X = 1, 6] = \frac{1}{3}$$

$$P[|X-\mathbb{E}X| \geq 1] \leq \frac{Var X}{1^2} \approx 2.5$$

$\parallel$

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Example #2.

$$P[\ldots] \leq \frac{\sigma^2}{a^2}$$

$$X \sim \mathcal{N}(0, \sigma^2)$$

$$\mathbb{P}[|X-\mathbb{E}X| \geq a]$$

$\parallel$

$$\mathbb{P}[|X| \geq a] \leq \frac{Var X}{a^2} = \frac{\sigma^2}{a^2}$$

Doesn't have a radical extension

$\Phi(a)$

polynomials, sin, $\sqrt{\ }$, +, *, a^b, log

Task

X - r.v. $Var X = 1.2$

$$\frac{Var \bar{X}}{1.8^2} =$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{X_1 + \dots + X_n}{n} \quad \bar{X}$$

W/ {Cheb.}

Find min n: $P\left[\left|\frac{X_1 + \dots + X_n}{n} - \mathbb{E}\bar{X}\right| \leq 1.8\right] \geq 0.55$

$$\mathbb{E}\bar{X} = \mathbb{E}X = \frac{1}{n}\sum \mathbb{E}X_i = \frac{1}{n}\sum \mathbb{E}X = \frac{1}{n}\cdot n \cdot \mathbb{E}X = \mathbb{E}X$$

$\mathbb{E}X_i$ $\parallel$ $\mathbb{E}X$

$$\frac{\operatorname{Var}\bar{X}}{1.8^2} = \frac{\operatorname{Var}\left[\frac{1}{n}\sum X_i\right]}{1.8^2} = \frac{\frac{1}{n^2}\cdot\operatorname{Var}\left[\sum X_i\right]}{1.8^2} =$$

$$\operatorname{Var} X = 1.2 \qquad n \cdot 1.2$$

$$= \frac{\frac{1}{n^2}\sum \operatorname{Var}[X_i]}{1.8} = \frac{\frac{1}{n^2}\sum_{i=1}^{n} 1.2}{1.8^2} = \frac{1.2}{n\cdot 1.8^2} =$$

$$= \frac{6/5}{n\cdot 81/25} = \frac{5\cdot 2}{n\cdot 27} = \frac{10}{n\cdot 27}$$

$$P\left[\,|\bar{X} - \mathbb{E}\bar{X}| \leq 1.8\right] \geq 0.95$$

$$\Updownarrow$$

$$1 - P\left[\,|\bar{X} - \mathbb{E}\bar{X}| \geq 1.8\right] \geq 0.95$$

min n s.t. $\Updownarrow$

$$\underline{P\left[\,|\bar{X} - \mathbb{E}\bar{X}| \geq 1.8\right]} \leq 0.05$$

$$\parallel$$

$$\frac{10}{n\cdot 27} \leq 0.05 \iff n \geq \frac{10}{27\cdot 1/20} = \frac{200}{27} \approx 7.4$$

$$n \geq 7.4 \implies n \geq 8 \quad \checkmark$$