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Task 2. Consider events

- H_1: the car is behind door number 1
- H_2: the car is behind door number 2
- H_3: the car is behind door number 3
- A: the host opened door number 2.

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1. **What can you say about probability of H_3 before any door is opened?**

$\texttt{Answer}: \Omega = \{H1, H2, H3\} \implies P(H_3) = \frac{1}{3}$

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2. **Assume that the host opened door number 2. What would you say about probability of $H3$ after you observe that?**

$\texttt{Answer}$: by definition of the game restriction, the host always opens the non-chosen empty door. This gives us the information that under the general setup $1/2$ of the total decoy doors are eliminated by the host. Since we chose door 1, which is empty, we eliminated the possibility of the host opening this door. Likewise, the host won't open the door 3 since there's a car behind it. Using the *law of total probability*, we obtain the following setup for $P(A)$ calculation:

- $P(A \mid H_1) = \frac{1}{2}$, as the participant correctly guessed the car door, the host can open any remaining decoy doors 2 or 3 (assuming the host chooses randomly with equal probabilities)
- $P(A \mid H_2) = 0$, since the car is behind door 2, which cannot be opened by the game restriction.
- $P(A \mid H_3) = 1$, since the participant already chose one of the decoy doors, and the host cannot open the door 3 because it has a car behid it.

$$\implies P(A) = (\tfrac{1}{2} \cdot \tfrac{1}{3}) + (0 \cdot \tfrac{1}{3}) + (1 \cdot \tfrac{1}{3}) = \tfrac{1}{2}$$

$$\implies P(H_3 \mid A) = \frac{P(A \cap H_3) \cdot P(H_3)}{P(A)} = \frac{P(A|H_3) \cdot P(H_3)}{P(A)} = \frac{(1) \cdot (\frac{1}{3})}{\frac{1}{2}} = \frac{2}{3}$$

3. **Are events A and H_1 independent? A and H_2? A and H_3?**

`Answer` : given $P(A) = \frac{1}{2}$, $P(A \mid H_1) = \frac{1}{2}$, $P(A \mid H_2) = 0$, $P(A \mid H_3) = 1$, we obtain

- $P(A \mid H_1) = P(A) \implies$ independent
- $P(A \mid H_2) \neq P(A) \implies$ not independent; moreover, such case violates the game restriction and these events are disjoint
- $P(A \mid H_3) \neq P(A) \implies$ not independent

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4. **Use Bayes' rule to find $P(H_1 \mid A)$ and $P(H_3 \mid A)$ (find all necessary probabilities that are used in Bayes' rule first). Compare with your previous answers.**

`Answer`

$$P(H_1 \mid A) = \frac{P(H_1 \cap A)}{P(A)} = \frac{P(A|H_1) \cdot P(H_1)}{P(A)} = \frac{(\frac{1}{2}) \cdot (\frac{1}{3})}{\frac{1}{2}} = \frac{1}{3}$$

$$P(H_3 \mid A) = \frac{P(H_3 \cap A)}{P(A)} = \frac{P(A|H_3) \cdot P(H_3)}{P(A)} = \frac{(1) \cdot (\frac{1}{3})}{\frac{1}{2}} = \frac{2}{3}$$

$\therefore$ the answers are consistent with what we obtained in *Q2*.

5. **Should the participant change the initial decision to increase probability of winning?**

`Answer` : yes, under giving conditions the participant should always switch to maximize the winning probability. Non-switching strategy results in $P(win) = \frac{1}{3}$, while switching allows to obtain $P(win) = \frac{2}{3}$, which is confirmed by the calculations above. Although being a bit complex, the same conclusion can be drawn if we brute force the solution using the backward induction. The participant wins by switching doors $\iff$ the wrong door is chosen initially, which happens in $\frac{2}{3}$ cases.