I can buy one lottery ticket out of two available.

1. In the first lottery I can win \$100 with probability 0.1, and the price of ticket is \$10.
2. In the second lottery, I can win \$50 with probability 0.1 and \$500 with probability 0.01. The price of ticket is \$20.

To decide which ticket to buy I toss a fair coin once. I chose first ticket in case of head and second otherwise.

Let X be random variable that denotes my net payout (taking into account price of a ticket).

- Find probability mass function of X (hint: use law of total probability).
- Show that expected value of X is an average of expected values of net payouts for each of two lotteries. Explain, why. Will it still hold if lotteries has different payouts or probabilities? Prove it.

Let us formalize the probabilities (P) and net profits (X).

For the first lottery F :

P	**0.1**	**0.9**
X	90	-10

For the second lottery S :

P	**0.1**	**0.01**	**0.89**
X	30	480	-20

Since we use a fair coin to determine which lottery to play, the probability to obtain heads (H) and tails (T) is equal, namely

$\implies P(H) = P(T) = \frac{1}{2} \therefore P(F) = P(S) = \frac{1}{2}$

The expected value of the net profit is the sum of the probabilities of each outcome multiplied by the value of the outcome.

For F, we obtain $\mathbb{E}(F) = (0.1 \cdot 90) + (0.9 \cdot -10) = 0$
For S, we obtain $\mathbb{E}(S) = (0.1 \cdot 30) + (0.01 \cdot 480) + (0.89 \cdot -20) = -10$

Since the probabilities of lotteries $P(F) = P(S) = \frac{1}{2}$
$\implies \mathbb{E}(X) = (\mathbb{E}(F) \cdot P(F)) + (\mathbb{E}(S) \cdot P(S)) = (\frac{1}{2} \cdot 0) + (\frac{1}{2} \cdot -10) = -5$

Now to check that $\mathbb{E}(X) = 5$ is indeed correct and is the average of $\mathbb{E}(F)$ and $\mathbb{E}(S)$, we need to find PMF of X using the low of total probability:

$P(A) = \sum_n P(A \cap B_n) = \sum_n P(B_n) \cdot P(A \mid B_n)$

- $P(X = -10) = (\frac{1}{2} \cdot 0.9) + (\frac{1}{2} \cdot 0) = 0.45$
- $P(X = 90) = (\frac{1}{2} \cdot 0.1) + (\frac{1}{2} \cdot 0) = 0.05$
- $P(X = 30) = (\frac{1}{2} \cdot 0.1) + (\frac{1}{2} \cdot 0) = 0.05$
- $P(X = 480) = (\frac{1}{2} \cdot 0.01) + (\frac{1}{2} \cdot 0) = 0.005$
- $P(X = -20) = (\frac{1}{2} \cdot 0.89) + (\frac{1}{2} \cdot 0) = 0.445$

Sanity check that the sum of probabilities equals to 1:

$\sum_n P(X_n) = 0.45 + 0.05 + 0.05 + 0.005 + 0.445 = 1$

Given the obtained PMF, we can construct the final table of probabilities (P) and outcomes (X):

P	**0.005**	**0.05**	**0.05**	**0.445**	**0.45**
X	480	30	90	-20	-10

$\implies \mathbb{E}(X) = (0.005 \cdot 480) + (0.05 \cdot 30) + (0.05 \cdot 90) + (0.445 \cdot -20) + (0.45 \cdot -10) = -5$

$\implies \mathbb{E}(X) = (\mathbb{E}(F) \cdot P(F)) + (\mathbb{E}(S) \cdot P(S)) = \frac{\mathbb{E}(F)+\mathbb{E}(S)}{2}$

$\therefore$ if the probabilities of choosing arbitrary lottery n are equal, we can formulate the general solution as follows:

$\mathbb{E}(X) = \sum_n \mathbb{E}(X_n) \cdot P(X_n) = \sum_n \mathbb{E}(X_n) \cdot \frac{1}{N}$, where N is the number of lotteries.