Law of Large numbers

$X_1, \dots, X_n$ independent identically distributed r.v.

$\mathrm{Var}\, X_i < +\infty$

Then

$$\frac{X_1 + \dots + X_n}{n} \xrightarrow{P} \mathbb{E} X_1$$

$$\frac{\bar{o}_p(n)}{n} \xrightarrow{P} 0$$

In other words: $X_1 + \dots + X_n = n \cdot \mathbb{E} X_1 + \bar{o}_p(n)$

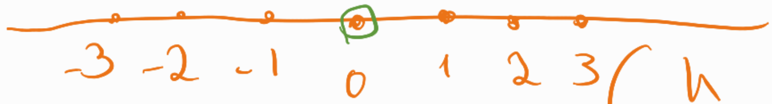
Example

Random walk $P[X_i = 1] = P[X_i = -1] = \frac{1}{2}$

$S_n = X_1 + \dots + X_n$, $S_0 = 0$

$S_i = S_{i-1} + X_i$

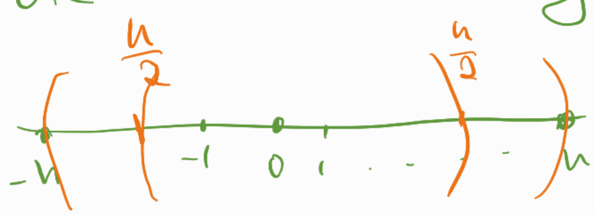
S_0



$\left(-\frac{n}{100}; \frac{n}{100}\right) \ni S_n$

LLN: $S_n = \bar{o}_p(n)$

On step n we are in some range less than n.



Central limit theorem

$X_1 \dots X_n$ iid

$0 < \mathrm{Var}\, X_1 < +\infty$

$$\frac{X_1+\dots+X_n}{\sqrt{n}} - \sqrt{n}\cdot\mathbb{E}X_1 \xrightarrow{d} \mathcal{N}(0, \mathrm{Var}X_1) \;\Big|\, * \sqrt{n}$$

$$X_1+\dots+X_n = n\cdot\mathbb{E}X_1 + \sqrt{n}\cdot\mathcal{N}(0,\mathrm{Var}X_1) + o_p(\sqrt{n})$$

LLN

$$X_1+\dots+X_n = n\cdot\mathbb{E}X_1 + o_p(n)$$

$O_p(n)$

CLT

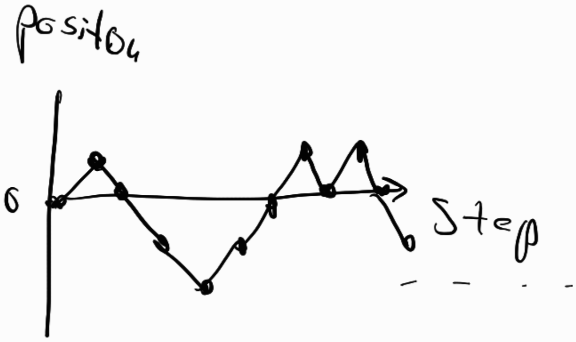
$\mathcal{N}(0,\mathrm{Var}X_1) \sim$

$$X_1+\dots+X_n = n\cdot\mathbb{E}X_1 + \sqrt{n}\cdot Y + o_p(\sqrt{n})$$

Apply to random walk

$X_1 = \pm 1$

$S_n = X_1+\dots+X_n$



LLN:

$S_n = o_p(n)$

$\mathrm{Var}X_1 = 1$

CLT:

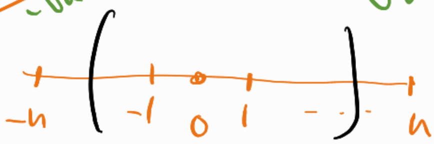
$\mathcal{N}(0,1) \sim$

$n^{2/3} > \sqrt{n}$

$$S_n = \underbrace{\sqrt{n}\cdot Y}_{\text{r.v.}} + o_p(\sqrt{n})$$

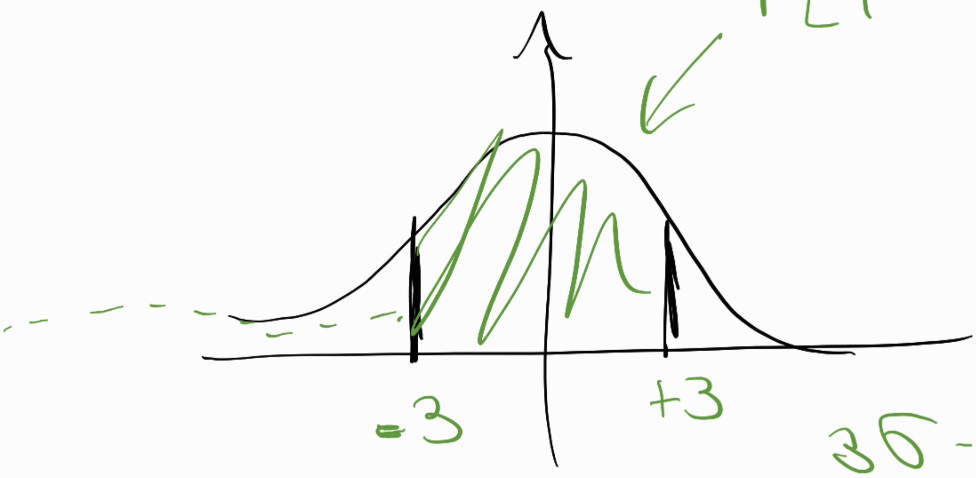
r.v.

$-\sqrt{n}\cdot 3$ $\sqrt{n}\cdot 3$



$$\boxed{S_n \in (-\sqrt{n}\cdot 3, \sqrt{n}\cdot 3)}$$

$P[Y \in \mathcal{M}]$ $\quad \sigma = 1$



$Y \sim N(0,1)$

$P[Y \in \mathcal{M}] = 0.997$

3σ-rule

Multinomial distribution $\quad k=2$. Binomial

(n – number of trials

(k – number of possible outcomes

($\underline{p_1 \ldots p_k}$ – probabilities of these outcomes

$p_1 + \ldots + p_k = 1$

$k=2 \quad n$ trials $\quad$ 2 outcomes

$p, \; 1-p$

Binomial case $P[m$ heads, $n-m$ tails$]$

Example $n = 9$

$k = 6, \quad \underline{p_1 = \ldots = p_6 = \frac{1}{n}}$

fair dice

we are tossing n dice

PMF is the probability of x_1 ones

x_2 twos

....

x_6 sixes

What's the prob. of $X_1 = 3$, $X_2 = 3$, $X_3 = 3$, $X_4 = X_5 = X_6 = 0$

PMF:

$$\frac{n!}{x_1! \cdot \ldots \cdot x_6!} \cdot p_1^{x_1} \cdot \ldots \cdot p_6^{x_6}$$

$$\frac{9!}{3! \cdot 3! \cdot 3! \cdot 0! \cdot 0! \cdot 0!} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{1}{6}\right)^0 \cdot \ldots$$

$$= \frac{9!}{6^3} \cdot \frac{1}{6^9} = \frac{9!}{6^{12}} \approx 0.000167$$

$$\frac{1}{6^3 \cdot 6^3 \cdot 6^3} = \frac{1}{6^9}$$

n exp ~~2~~ k outcomes

P[k of outcome 1, n-k of outcome 2]

11 212 333 3

Good outcome

121233321

Bad

216 54 3641