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Conference Paper · October 2008

DOI: 10.1109/TIWDC.2008.4649038 · Source: IEEE Xplore

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Proceedings of ESAV'08 - September 3 - 5 - Capri, Italy

An Efficient Multilateration Algorithm

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***Abstract*—Two new closed form localization algorithms for Multilateration Systems are derived and analyzed. The derivation neglects the time difference of arrival in favor of the direct use of the time of arrival (TOA). The algorithms work for arbitrary spatial dimensions and overdetermined systems. A strategy for quick rejection of obviously false time measurements based on coding theory is also proposed.**

I. Introduction

Two localization algorithms that operate on TOA measurements instead of TDOA measurements are proposed. They differ from all the state of the art multilateration systems that use a combination of TOA measurements of each receiving station to compute the time difference of arrival (TDOA). These novel algorithms don't have to resort to expensive hyperbola intersection algorithms; rather then rely on a simple and efficient least-square method and on the solution of a single quadratic equation. The proposed methods set themselves apart from the well-known efficient methods presented by [1], [2], [3], [4], [5], [6], [7] by their derivation that never explicitly uses differences of time measurements TDOA). It turns out that this approach delivers simple formulas that easily generalize to arbitrary spatial dimensions as well as over- and under determined cases.

The derived results from one of the method proposed can also be used as syndrome in an approach based on coding theory that may allow quick detection and rejection of false individual time measurements (for system integrity monitoring and improvement of the total accuracy).

II. The Multilateration Problem Solution

We consider the situation where the target with unknown position $\mathbf{p} \in \mathbb{R}^d$ emits an SSR Mode S reply at the unknown time instant t that is detected, after propagation through a medium with speed c, by a set K of receiving units. Each receiver $k \in K$ is located at a fixed position $\mathbf{p}_k$ and receives the pulse, transmitted at the time t, at the time instant t_k. Under the assumption that the time measurements are indipendent and distributed with $N(t_k, \sigma)$, the following probability density function results for the estimated target position:

$$\prod_k \frac{1}{\sqrt{2\pi}\sigma_r} e^{-\frac{\sigma_r^{-2}}{2}(|\mathbf{x}-\mathbf{p_k}|-|\mathbf{p}_0-\mathbf{p_k}|)^2}$$

where $\mathbf{p}_0$ is the real target position, and $\sigma_r = c\sigma$ is the standard deviation of the range measurements $\rho_k = ct_k$ (that is Gaussian distributed: $N(ct_k, \sigma_r)$).

Examples of this distribution are shown in Fig. 1 for several sensor configurations.

Considering the set of squared distances between each receiving station and the target and the propagation time of the signal, it is possible to write a set of K equations:

$$\begin{aligned} |\ \mathbf{p}_k - \mathbf{p}\ |^2 &= c^2 (t_k - t)^2 \\ |\mathbf{p_k}|^2 + |\mathbf{p}|^2 - 2\mathbf{p_k^T}\mathbf{p} &= c^2 t_k^2 - 2c^2 t_k t + c^2 t^2 \\ |\mathbf{p}_k|^2 - c^2 t_k^2 + |\mathbf{p}|^2 - c^2 t^2 &= 2\left(\mathbf{p}_k . \mathbf{p} - c^2 t_k t\right) \end{aligned}$$

which, using the space-time vectors
$\mathbf{q} = (p_1 \dots p_d\ t)^{\mathrm{T}}$, $\mathbf{q}_k = (p_{k,1} \dots p_{k,d}\ t_k)^{\mathrm{T}}$
and the matrix:

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -c^2 \end{pmatrix},$$

could be arranged into the following equations system:

$$2 \operatorname*{Col}_k \left(\mathbf{q}_k^{\mathrm{T}} \mathbf{S} \mathbf{q}\right) = \operatorname*{Col}_k \left(\mathbf{q}_k^{\mathrm{T}} \mathbf{S} \mathbf{q}_k + \mathbf{q}^{\mathrm{T}} \mathbf{S} \mathbf{q}\right)$$

(we make use of the Row and Col operators which are described in Appendix).

It is possible to introduce the scalar quantity $v = \mathbf{q}^{\mathrm{T}}\mathbf{S}\mathbf{q}$, obtaining:

$$2 \operatorname*{Col}_k \left(\mathbf{q}_k^{\mathrm{T}} \mathbf{S} \mathbf{q}\right) = \operatorname*{Col}_k \left(\mathbf{q}_k^{\mathrm{T}} \mathbf{S} \mathbf{q}_k + v\right) . \tag{1}$$

Two direct approaches to solve this equation system in both the fully determined case, using the matrix inverse, and the overdetermined case, using the pseudo-inverse, will be presented. Both cases will be denoted by the pseudo-inverse [8]. We recall that the pseudo-inverse provides a linear least-square solution in the face of additive white Gaussian noise that disturbs the time measurements t_k, and we omit the notational burden of an error term in the equations below. We further note that, in reality, biases e.g. due to multipath propagation are the most important cause of errors, which invalidates the assumption about the normally distributed, zero mean, error.

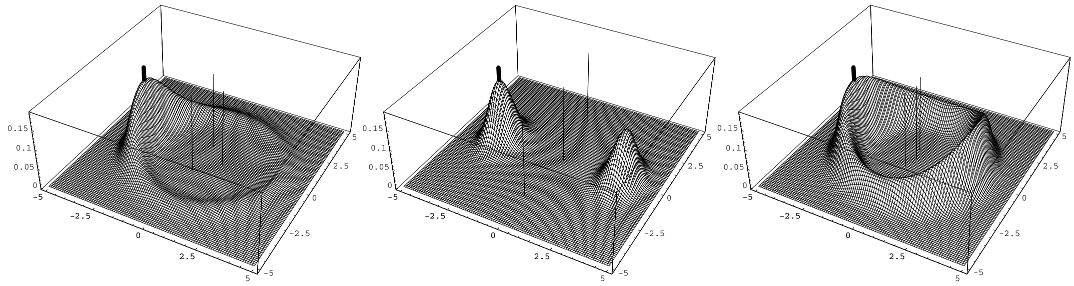

Figure 1. Three ground stations whose positions are indicated by thin vertical lines, and the probability density generated by the target with position marked by the thick vertical line.

Constant-v Approach: The system (1) could be solved for $\mathbf{q}$ (using the pseudoinverse matrix) **:**

$$\begin{aligned}\mathbf{q} &= \frac{1}{2}\left(\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\right)\cdot\mathbf{S}\right)^{\dagger}\cdot\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}_k+v\right)\\ &= \underbrace{\frac{1}{2}\left(\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\right)\cdot\mathbf{S}\right)^{\dagger}\cdot\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}_k\right)+}_{\mathbf{a}}\\ &+\; v\underbrace{\frac{1}{2}\left(\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\right)\cdot\mathbf{S}\right)^{\dagger}\cdot\operatorname*{Col}_k 1}_{\mathbf{b}}=\mathbf{a}+v\mathbf{b}\end{aligned}\tag{2}$$

where we introduce the vectors $\mathbf{a}$ and $\mathbf{b}$ which entirely depend on given values (i.e the positions of the receiving stations and the pertaining reception time instants). Expanding:

$$\begin{aligned}\mathbf{q}^{\mathrm{T}}\mathbf{S}\mathbf{q}=v &= \mathbf{a}^{\mathrm{T}}\mathbf{S}\mathbf{a}+v\mathbf{a}^{\mathrm{T}}\mathbf{S}\mathbf{b}+v\mathbf{b}^{\mathrm{T}}\mathbf{S}\mathbf{a}+v^2\mathbf{b}^{\mathrm{T}}\mathbf{S}\mathbf{b}\\ 0 &= \mathbf{a}^{\mathrm{T}}\mathbf{S}\mathbf{a}+v\left(2\mathbf{a}^{\mathrm{T}}\mathbf{S}\mathbf{b}-1\right)+v^2\mathbf{b}^{\mathrm{T}}\mathbf{S}\mathbf{b}\end{aligned}$$

which is a quadratic equation for v with the solutions:

$$v_{1,2}=\frac{\frac{1}{2}-\mathbf{a}^{\mathrm{T}}\mathbf{S}\mathbf{b}\pm\sqrt{\left(\frac{1}{2}-\mathbf{a}^{\mathrm{T}}\mathbf{S}\mathbf{b}\right)^2-\mathbf{a}^{\mathrm{T}}\mathbf{S}\mathbf{a}\,\mathbf{b}^{\mathrm{T}}\mathbf{S}\mathbf{b}}}{\mathbf{b}^{\mathrm{T}}\mathbf{S}\mathbf{b}}\tag{3}$$

A proper choice of one solution, whose details are not given here for the sake of conciseness, completes the "constant v" method.

Variable-v Approach: The second way to solve Eq. 1 is shown in the following.

It is possible to introduce the vector/matrix:

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}\mathbf{p}\\ t\\ v\end{pmatrix}\\ \mathbf{R} &= \operatorname*{Col}_k\begin{pmatrix}\mathbf{p}_k^{\mathbf{T}} & -c^2t_k & -\frac{1}{2}\end{pmatrix}\end{aligned}$$

which allows us to write:

$$\begin{aligned}2\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}\right) &= \operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}_k+\mathbf{q}^{\mathrm{T}}\mathbf{S}\mathbf{q}\right)\\ &= \operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}_k+v\right)\\ 2\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}-\frac{v}{2}\right)=2\mathbf{R}\mathbf{r} &= \operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}_k\right) \qquad (4)\\ \mathbf{r} &= \frac{1}{2}\mathbf{R}^{\dagger}\cdot\operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}_k\right) \qquad (5)\end{aligned}$$

This estimation method ignores the v component in $\mathbf{r}$, except for the detection of inconsistent measurements (see below). In the overdetermined cases, the method might deliver better results than the constant-v approach, partly because it avoids the cases where, due to disturbed TOA measurements, the radicand in Eq. 3 becomes negative.

For systems where $|K|\leq d+1$, Eq. 4 is underdetermined. In these and in the cases of degenerated layout of the receiver stations, the equation determines a linear subspace $\mathbf{R}$, $\mathbf{r}\in\mathbf{R}$, inside which the solution can be confined (restricted) using $v=\mathbf{q}^{\mathrm{T}}\mathbf{S}\mathbf{q}=\mathbf{p}^2-c^2t^2$.

This method doesn't take care of the relationship between $\mathbf{p}$ and v.

Detecting and Eliminating Inconsistent Solutions

For $|K|\leq d+2$, and non-degenerate $\mathbf{p}_k$, we obtain fully determined solutions for the variable-v approach, and

$$\epsilon=|\mathbf{p}|^2-c^2t^2-v=\begin{pmatrix}\mathbf{p}\\ -c^2t\\ -1\end{pmatrix}^T\cdot\mathbf{r}$$

is the least-square approximation error. When it is zero (or near to zero considering the measurement noise), the solution is consistent (which doesn't necessarily mean that it is correct).

The most frequent cause of multilateration errors is multipath propagation which results in too high values of individual time measurements t_k. When a small subset $E\in\{t_k\}$ is disturbed (for example from multipath or masking), solving Eq. 5 for appropriately chosen subsets and determining ϵ_E can be used to find E and, by omitting the disturbed measurements, arriving at a much more precise position estimate.

The goal is to determine the error $\epsilon=|\mathbf{p}|^2-c^2t^2-v$, which should be zero (or near zero) for time measurements that permit a consistent solution of eq. (5). It is possible to determine how strong ϵ responds to a measurement error Δt_j and how big $|\epsilon|$ has to be such that t_j is to be regarded as erroneous. To this purpose, we form the derivative of Eq. 5 with respect to $t_j$that is:

$$\begin{aligned}
\mathbf{r} &= \frac{1}{2}\mathbf{R}^\dagger \operatorname*{Col}_k\left(\mathbf{q}_k^{\mathrm{T}}\mathbf{S}\mathbf{q}_k\right)\\
2\mathbf{r} &= \mathbf{R}^\dagger \mathrm{Col}\left(|\mathbf{p}_k|^2 - c^2t_k^2\right)\\
2\frac{d}{dt_j}\mathbf{r} &= \frac{d}{dt_j}\mathbf{R}^\dagger \operatorname*{Col}_k\left(|\mathbf{p}_k|^2 - c^2t_k^2\right)\\
&\quad +\mathbf{R}^\dagger \frac{d}{dt_j}\operatorname*{Col}_k\left(|\mathbf{p}_k|^2 - c^2t_k^2\right)\\
2\frac{d}{dt_j}\mathbf{r} &= -\mathbf{R}^\dagger\left(\frac{d}{dt_j}\mathbf{R}\right)\mathbf{R}^\dagger\cdot \operatorname*{Col}_k\left(|\mathbf{p}_k|^2 - c^2t_k^2\right)\\
&\quad -2c^2\mathbf{R}^\dagger\cdot\operatorname*{Col}_k\left([j=k]\,t_k\right)\\
2\frac{d}{dt_j}\mathbf{r} &= -\mathbf{R}^\dagger\left(\frac{d}{dt_j}\mathbf{R}\right)\cdot 2\mathbf{r} - 2c^2\mathbf{R}^\dagger\operatorname*{Col}_k\left([j=k]\,t_k\right)\\
\frac{d}{dt_j}\mathbf{r} &= c^2\mathbf{R}^\dagger\left(\operatorname*{Col}_k(0\ldots0\ [j=k]\ 0)\right)\mathbf{r}\\
&\quad -c^2\mathbf{R}^\dagger\operatorname*{Col}_k\left([j=k]\,t_k\right)\\
&= c^2\mathbf{R}^\dagger\cdot\\
&\quad\left(\operatorname*{Col}_k(0\ldots0\ [j=k]\ 0)\cdot\mathbf{r} - \operatorname*{Col}_k\ \left([j=k]\,t_k\right)\right)\\
&= c^2\mathbf{R}^\dagger\operatorname*{Col}_k\left(t-t_k\right)[j=k]\\
\frac{d}{d\mathbf{t}}\mathbf{r} &= c^2\mathbf{R}^\dagger.\operatorname*{Diag}_j\left(t-t_j\right)
\end{aligned}$$

This allows to express the first-order Taylor series approximation of a multilateration disturbed by an error Δt_j in the measurement of receiver station j:

$$\tilde{\mathbf{r}} \approx \mathbf{r} + \sum_j \Delta t_j\frac{d}{dt_j}\mathbf{r} = \mathbf{r} + \frac{d}{d\mathbf{t}}\mathbf{r}\cdot\operatorname*{Col}_j\Delta t_j$$

The derivative of ϵ with respect to the vector $\mathbf{t}$ of time measurements is:

$$\begin{aligned}
\frac{d}{d\mathbf{t}}\epsilon &= \frac{d}{d\mathbf{t}}\begin{pmatrix}\mathbf{p}\\ -c^2t\\ -1\end{pmatrix}^T\cdot\mathbf{r}\\
&= \begin{pmatrix}\mathbf{p}\\ -c^2t\\ -1\end{pmatrix}^T\cdot\frac{d}{d\mathbf{t}}\mathbf{r}
\end{aligned}$$

and the first-order Taylor series expansion of ϵ around $\epsilon_0 = \mathbf{p}^2 - c^2t^2 - v$ is

$$\begin{aligned}
\epsilon_{\mathbf{t}+\Delta\mathbf{t}} &\approx |\mathbf{p}|^2 - c^2t^2 - v\\
&\quad +c^2\underbrace{\begin{pmatrix}\mathbf{p}\\ -c^2t\\ -1\end{pmatrix}^T\cdot\mathbf{R}^\dagger\cdot\operatorname*{Diag}_j\left(t-t_j\right)}_{\mathbf{n}}\cdot\operatorname*{Col}_j\Delta t_j
\end{aligned}$$

We note that disturbance vectors orthogonal to $\mathbf{n}$ locally do not affect ϵ. This means that for certain configurations of receiver stations, inconsistent measurement vectors $\mathbf{t}$ cannot be detected this way.

Using some simple results from coding theory (e.g. [9]), it is possible to find subsets of the receiver stations that permit a detection of erroneous t_k with maximal reliability and with a minimal number of applications of Eq. (5). In the plane ($d = 2$), 4 time measurements are required to solve Eq.5 . Given the measurements from 7 receiver stations, the control matrix

$$H = \begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1\end{pmatrix}$$

can be used to find out the erroneous measurement with merely 3 solutions of Eq. (5). Each column represents one receiver station, each row represents one solution of Eq. (5), and a 1 means that the time measurement from the respective receiver station enters the solution. Assuming that at most one receiver station has delivered a false measurement, the position of the syndrome vector $\mathbf{z} = ([|\epsilon_0| > \vartheta]\ \ [|\epsilon_1| > \vartheta]\ \ [|\epsilon_2| > \vartheta])^{\mathbf{T}}$ for a suitably chosen $\vartheta \in \mathbb{R}$ in K indicates the odd measurement or, when it is (0 0 0), that there is no detectable error. The matrix H is the control matrix of the Hamming (7,4) code. This code can detect and correct single bit errors (corresponding to disturbed time measurements) due to its minimum Hamming distance of 3.

In three-dimensional space ($d = 3$), a suitable control matrix is:

$$H = \begin{pmatrix} 1&1&0&1&0&0&1&1&0&0&0\\ 1&0&1&0&1&0&1&0&1&0&0\\ 0&1&1&1&0&1&0&0&0&1&0\\ 1&1&0&0&0&1&1&0&0&0&1\end{pmatrix}$$

This is a variant of the Hamming (11,7) code which puts 5 time measurements into each solution of Eq. (5). It doesn't have the nice error location property of the Hamming (7,4) code, where the binary number encoded by the syndrome vector corresponds to the erroneous measurement.

	1	2	3	4	5	6	7	8	9	10
x	280	324	440	802	1490	2234	1418	2018	1762	730
y	3552	3046	2798	1984	920	1460	3084	4104	4628	4878
z	49	42	44	21	52	18	24	3	3	7

Table I
MLAT RECEIVER STATION POSITIONS IN METERS.

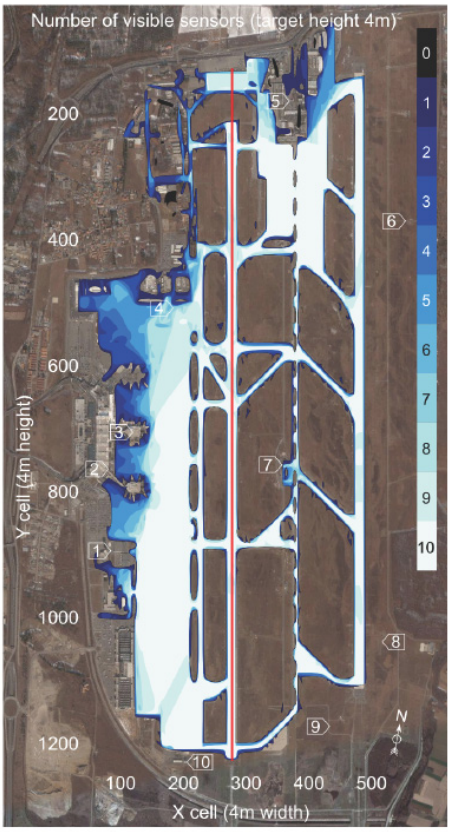

Figure 2. Number of stations that are visible from each point of the airport (the vertical line shows the path used in the simulations) and receiver station positions.

III. SIMULATION RESULTS

The two proposed algorithms, i.e. constant and variable v, are tested in a simulated scenario that reproduces the Malpensa Airport situation, where ten receiver stations are installed. The receiver positions are shown in Table I and in Figure 2.

Not all stations are visible from every point of the airport due to the blockage of line of sight by buildings. The visibility map (number of visible station for each position on the airport) is also reported in Figure 2. The visibility information is taken into account to select which receiver stations must be used in the localization algorithms.

The simulation is referred to the positions running on the runway center line displayed as vertical line in Figure 2. The measurement error (in meters) for each station is imposed independent of the SNR (i.e. independent of the target-receiver distance) normal distributed, zero mean, with a standard deviation of 0.3 meters; the R.M.S. error is calculated on 1000 trials.

The two algorithms are compared with algorithms well know in the literature, i.e. Chan-Ho algorithms [6].

In Figure 3 the calculated RMS horizontal error for the simulated algorithms is plotted. The positions of the target, whose height is 3 meters, are on the straightline segment shown in Figure 2.The performance of the proposed methods is quite similar to the ones of [6] but they are easier to implement and require less computational effort. It is worth noting that the variable-v approach has the same results as the first fix approximation computed by the Chan-Ho algorithm (formula 14b in [6]).

The importance of closed form multilateration algorithms lies in their ability to produce an initial guess for an iterative position estimation process (using Taylor expansion around the guess in order to produce very accurate results if the guess is near the real position of the target). The multilateration position estimation using iterative methods is depending strongly on the initial guess and it is possible, using an erroneous guess, to obtain a divergence of the results. The usefulness of the proposed algorithms for initial guesses is compared to the other methods; in the following some results are reported (Figure 4 shows the RMS 2D position error in the case of iterative algorithms that use the closed form algorithms as initial guess). A Taylor linearisation of the problem in the guess computed by the closed form algorithms is performed. In the case of the Chan-Ho algorithms the guess is the first approximation computed from the algorithms (Formula 14b in [6]). Also in this case we have high performance in relation to the complexity of the algorithms.

In the case of wide-area multilateration (WAM), we have a typical change of conditions: less stations, bigger measurement errors and less favorable geometry.

This case was also analyzed using a possible Malpensa Airport WAM deployment. The receiver station positions are reported in Table II and all stations are assumed to be visible from the target position which simulates an aircraft taking off, flying from 0 to 35 km along the x axis in Fig. 5 at a flight path angle of 3° in the vertical plane. The measurement error (in meters) for each stations is imposed independent of the SNR, normal distributed with zero mean and standard deviation of 0.5 meters. In this case the results are also compared with Schau-Robinson algorithm [1] that performes good results in case of wide area scenario.

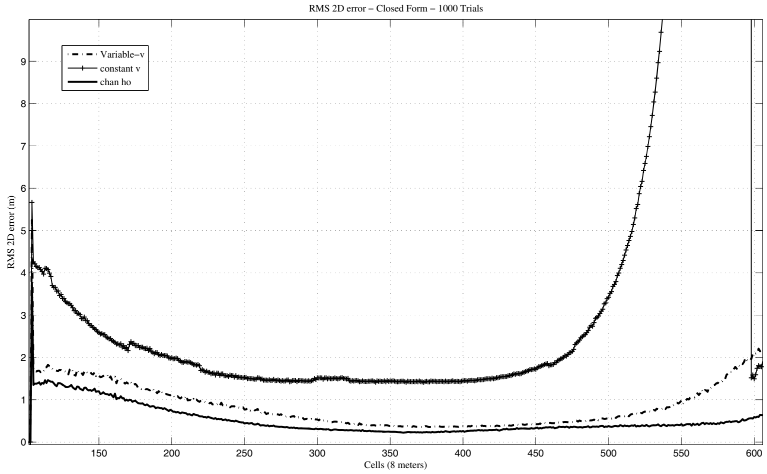

Figure 3. MLAT 2D position error. Only Closed Form Algorithms

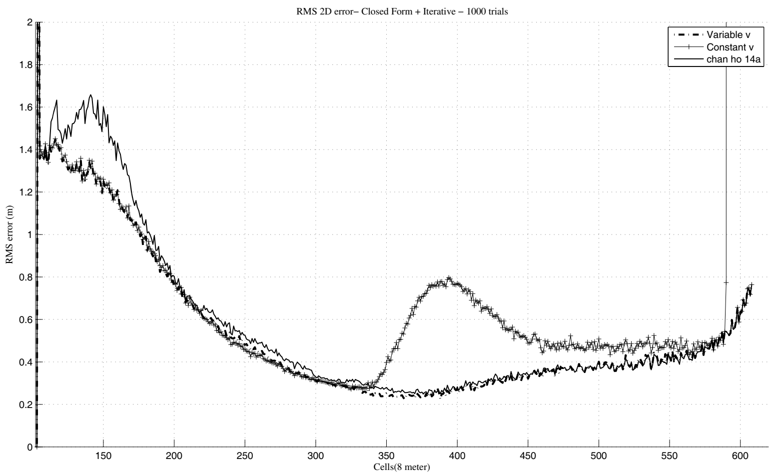

Figure 4. MLAT 2D position error. Closed Form + Iterative Algorithms

Figure 5. WAM sensor layout used in the simulations.

station	1	2	3	4	5
x	-1100	8600	22000	-1800	0
y	7200	-7000	300	-800	0
z	40	-25	-70	10	0

Table II
WAM RECEIVER STATION POSITIONS IN KM FOR THE WIDE-AREA MULTILATERATION SIMULATION.

In Figure 6 the 2D position errors (R.M.S. on 1000 trials) for the various algorithms are reported. The constant-v approach leads to the best results when used to produce initial guesses for the iterative algorithms (discarding two points in which there is some geometric or algebraic instability which requires further investigation).

It is interesting to note that the constant v method leads to good localization results for both MLAT and WAM without any modification, and with a very simple and “clean” algoritmic approach.

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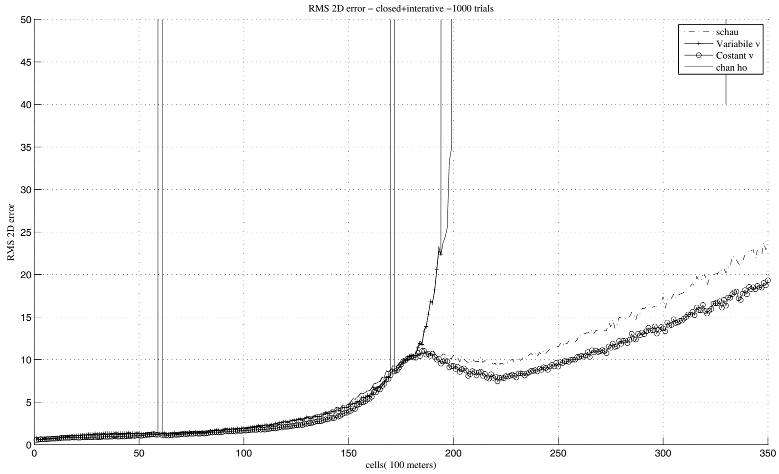

Figure 6. WAM 2D position Error. Closed Form + Iterative algorithms.

IV. Conclusion

Two new multilateration algorithms were introduced and tested in a realistic simulated environment (Malpensa airport installation in Milan). Preliminary results show that these simple to implement and efficient algorithms may produce competitive results, in particular when used to produce an initial guess for an iterative algorithm. This observation is confirmed also the for wide-area multilateration scenario.

More generally, also considering simulation not reported here, the results are comparable with other algorithms and are slightly better or worse depending on the geometrical scenario. It seems that the constant-v approach works better for remote station scenarios, and the variable-v approach works better for target close to the receiver stations. Future work will investigate this point, and will also encompass the evaluation of the coding theory based error detection method, as well as the investigation of the divergence of the constant-v approach in some WAM configurations.

V. Acknowledgement

The authors wish to thank Selex Sistemi Integrati for supplying the MAT and WAM geometries related to the Malpensa projects.

Appendix

Matrix and Vector Notation

In General vector, vector are indicated by boldface small letters and matrixes by boldface capital letters.

We use the operator $\underset{k\in K}{\mathrm{Col}}\left(\mathbf{x}_k\right)$ which stacks up its vector (or matrix) arguments $\mathbf{x_k}$ for all k in the index set K into a vector (or matrix respectively), and the operator $\underset{k\in K}{\mathrm{Diag}} x_k$ which builds a (block) diagonal matrix out of its scalar or square matrix arguments x_k for all k in K. When the index set is obvious from the index variable name, it is omitted.

The Penrose-Moore pseudoinverse of a Matrix $\mathbf{A}$ is denoted $\mathbf{A}^{\dagger} = \left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}$ and it is assumed that $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is invertible.

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