APPM 2360 Project 1: Black Holes

Due: February 22, 2018 by 11:59 PM on D2L

---

1 Introduction

Black holes are one of the stranger predictions of Einsteins beautiful theory of general relativity. When matter becomes extremely dense, then the gravitational field around that matter becomes so strong that not even light can escape. This density can be compared to the mass of 20,000 cars packed into the volume of a grain of salt. The strong gravitational field effects not only energy and matter, but time itself.

The behavior of objects near a black hole can be highly unintuitive. We will investigate the physical observation of an object falling in towards the black hole, and approaching the event horizon. The event horizon is the spherical region around the black hole from which nothing can escape; at this distance, the escape velocity is equal to the speed of light. The distance from the center of the black hole to the event horizon is called the Schwartzschild radius. In this lab, we will measure distance in units of half-Schwartschild radii. See Figure 2 for a diagrammatic explanation.

We will focus on a seemingly paradoxical physical phenomenon. Suppose you sit on a space station, orbiting the black hole. You launch your space-buddy out of the space station towards the black hole. From his point of view, he will get pulled into the black hole and eventually pass the event horizon. However, from your point of view, he will move slower and slower as he approaches the event horizon. In fact, from your point of view, he will never cross the event horizon at all! This is due to the distortion of time caused by the immense gravitational field generated by the black hole. We will explore this phenomenon using a simple physical model, formulated as an initial value problem.

We wish to test the hypothesis that, from the vantage point of someone on the space station, the astronaut never reaches the black hole. We will model the astronauts position with the differential equation.

$$\frac{dx}{dt} = \left(\frac{2}{x} - 1\right)\frac{1}{\sqrt{x}}; \quad x(0) = 5, \tag{1}$$

where $x(t)$ represents the radial distance from the center of the black hole to the astronaut, and the event horizon is at $x = 2$. See Figure 1 for an illustration. Time t is measured in seconds, relative to the reference frame in which the space station is at rest, and we set $t = 0$ to be the time when the astronaut leaves the space station. For this problem, we require that $x \geq 2$. That is to say, this initial value problem only models the physics of the situation when we are outside the event horizon.

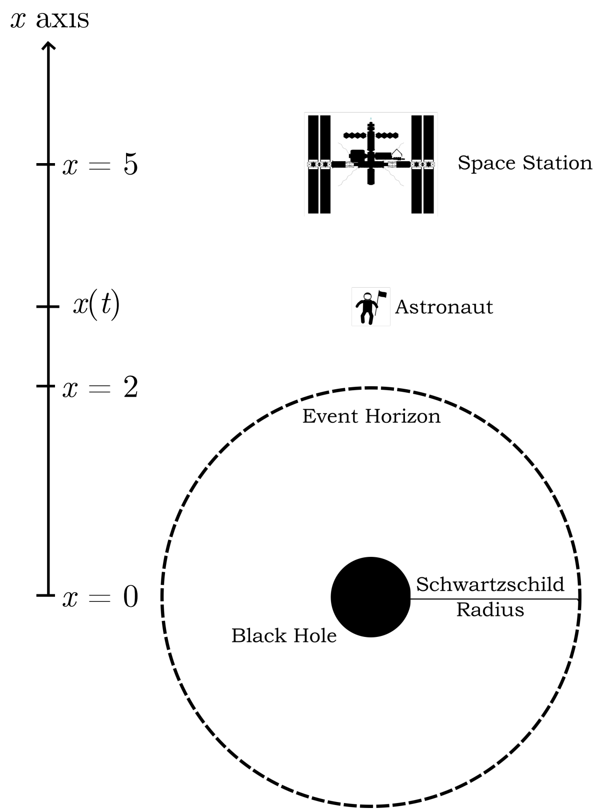

Figure 1: Diagram of the problem. Distance x from the center of the black hole is measured in units of Schwartzschild radii. The space station is fixed at $x = 5$. The position $x(t)$ of the astronaut is what we want to find.

2 Analysis of the model equation

In this section, you will not need to write any original MATLAB code. Most of this work in this section should be done by hand. Any relevant results of calculations should be included in the submitted report. Detailed handwritten calculations may be included in an appendix if necessary.

1. Classify the differential equation. Is it autonomous? Is it linear? Is it homogeneous? What is the degree? Discuss the physical significance of the autonomy and homogeneity of the equation.

2. How can you find the velocity $v(t)$ of the astronaut given their position $x(t)$? What is the initial velocity of the probe? Find this initial velocity without solving equation (1).

3. What do we know about the existence and uniqueness of solutions of (1)? Justify your response.

4. Find all equilibrium solutions to the differential equation (1). For this problem, ignore the initial condition $x(0) = 5$

5. Describe the long-time behavior of the solution to eq. (1). Justify your answer by including a direction field for equation (1). Use the provided script `dfield.m` (found on D2L). You will need slightly modify the script to generate the direction field for the ODE (1). Make sure to adjust the t and x grids used in `dfield.m` so that your directional field is as visually clear as possible. Make sure to use the condition $x \geq 2$ and a time scale so the long term behavior is visible.

3 Numerical methods

In this section we will develop Matlab code to numerically solve equation (1). First we will solve a *test problem*–a similar equation that we know the exact solution to ensure that our code is working properly. We then can modify our test problem code to solve equation (1).

1. Consider the test initial value problem

$$\frac{dx}{dt} = \frac{1}{x}; \quad x(0) = 5. \tag{2}$$

   (a) Solve equation (2) via separation of variables.

   (b) Use Euler's method to solve equation (2) with $h = 0.1$ on the interval $t \in [0, 20]$. On one plot, display the true solution from part (a) along with your numerical solution. The plot of your solutions should include a legend indicating each curve. See the APPM 2460 course page for some information on how to use Matlab.

2. Modify your Euler's method from the previous problem to solve equation (1). On one plot, include solutions for step sizes $h = 2, h = 1, h = 0.1$, and $h = 0.01$ up to $t = 100$. Discuss similarities and differences between the solutions for different timesteps. The plot of your solutions should include a legend indicating which plot corresponds to which timestep.

3. Using a smaller time step takes more computational time. Therefore, we want to find the largest timestep that gives us a reasonably accurate answer. Discuss what you think the optimal time step is for this problem. Explain why you think this time step gives an accurate solution.

4. Using the timestep h = 0.01, plot numerical solutions to the differential equation in (1) with various initial conditions $x(0) = 5$, $x(0) = 4$, $x(0) = 3$, and $x(0) = 2$. These solutions should all appear together on one plot. Your plot should include

a legend. Looking at this plot, does the long-time behavior of this system depend on the initial condition? Does this make sense physically?

5. We have examined the long-time behavior analytically (using the direction field) and numerically (using our Euler solver). Do these results agree? Why or why not?

4 Report Guidelines

Your group will submit your project on D2L, in the appropriate dropbox (you can find these under the “assessments” tab in D2L). Adhere to the following guidelines:

- Do not put off finding a group, do this early. You should have a group set up within one week of the project assignment date.
- Submit your project in pdf format. Contents of .zip files will not necessarily be graded. (Word documents not acceptable because equations are commonly jumbled around by D2L.)
- Submit ALL code used for your project (.nb files for Mathematica, .m files for MatLab, etc).
- Have only ONE group member submit the project. Having multiple people in your group submit the project to D2L will result in multiple grades, and we will take the LOWEST one.
- Include the names of all group members working on the project.

Your report needs to accurately and consistently describe the steps you took in answering the questions asked. This report should have the look and feel of a technical paper. Presentation and clarity are very important. Here are some important items to remember:

- Absolutely make sure your recitation number is on your submitted report.
- Start with an introduction that describes what you will discuss in the body of your document. A brief summary of important concepts used in your discussion could be useful here as well.
- Summarize what you have accomplished in a conclusion. No new information or new results should appear in your conclusion. You should only review the highlights of what you wrote about in the body of the report.
- Always include units in your answers.
- Always label plots and refer to them in the text. The main body of your paper should NOT include lengthy calculations. These should be included in an appendix, and referred to in the main body.

- Labs must be typed. Including the equations in the main body (part of your learning experience is to learn how to use an equation editor). An exception can be made for lengthy calculations in the appendix, which can be hand written (as long as they are neat and clear), and minor labels on plots, arrows in the text and a few subscripts.
- Your report does not have to be long. You need quality, not quantity of work. Of course you cannot omit any important piece of information, but you need not add any extras.
- DO NOT include printouts of computer software screens. This will be considered as garbage. You simply need to state which software you used in each step, and what it did for you.
- You must include any plot that supports your conclusions or gives you insight in your investigations.
- Write your report in an organized and logical fashion. Section headers such as Introduction, Background, Problem Statement, Calculations, Results, Conclusion, Appendix, etc... are not mandatory, but are highly recommended. They not only help you write your report, but help the reader navigate through your paper, besides giving it a clearer look.