1. Let $\mathbf{A}$ be an affine space and S and T two affine subspaces of $\mathbf{A}$. Show that if $S \subseteq T$ then $S \parallel T$.

2. In $\mathbf{A}^4(\mathbb{R})$ consider the affine subspaces

$$\alpha = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} \quad \beta = \langle\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\rangle + \begin{bmatrix}1\\3\\0\\0\end{bmatrix} \quad \gamma = \langle\begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix}\rangle + \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \delta = \langle\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\rangle.$$

Which of the following is true?

1.	$\alpha \in \beta$	5.	$\beta \parallel \delta$
2.	$\alpha \in \gamma$	6.	$\gamma \parallel \delta$
3.	$\alpha \in \delta$	7.	$\beta \subseteq \gamma$
4.	$\beta \parallel \gamma$	8.	$\gamma \subseteq \delta$

3. Consider the following affine subspaces of $\mathbf{A}^4(\mathbb{R})$

$$Y : \begin{cases} x_1 + x_3 - 2 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 &= 0 \end{cases}$$

$$Z : \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 + x_3 - 3x_4 &= -1 \\ x_1 - x_2 + 3x_4 &= 3 \end{cases}$$

1. Determine the dimensions of Y and Z.
2. What are the parametric equations of the two affine subspaces?
3. Is $Y \parallel Z$?

4. In $\mathbf{A}^n(\mathbb{R})$ ($n \geq 2$) consider the line

$$L = P + \langle\begin{bmatrix}v_1\\ \vdots \\ v_n\end{bmatrix}\rangle \quad \text{and the hyperplane} \quad H : \alpha_1 x_1 + \cdots + \alpha_n x_n + \beta = 0.$$

Show that $L \parallel H$ if and only if

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

5. Let $\mathbf{K}$ be a finite field. Determine

1. the number of points in an affine subspace of $\mathbf{A}^n(\mathbf{K})$,
2. the number of lines passing through a given point of $\mathbf{A}^n(\mathbf{K})$ and
3. the number of hyperplanes passing through a given point of $\mathbf{A}^n(\mathbf{K})$.

6. Which of the following are affine subspaces?

1. $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 - 2 = 0\} \subseteq \mathbf{A}^3(\mathbb{R})$.
2. $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 0\} \subseteq \mathbf{A}^3(\mathbb{R})$.
3. $C = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^4 + x_2 - 2x_3 + x_4 = 0\} \subseteq \mathbf{A}^4(\mathbb{R})$.

7. For $n, m \in \mathbb{N}$ let $a_1, \ldots, a_n$, $b_1, \ldots, b_m \in \mathbb{R}$. Fix a function $g : \mathbb{R} \to \mathbb{R} \in \mathcal{C}^\infty(\mathbb{R})$. Which of the following are affine subspaces?

1. $A = \{f \in \mathcal{C}^\infty(\mathbb{R}) : a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + g = 0\} \subseteq \mathcal{C}^\infty(\mathbb{R})_a$.
2. $B = \{f \in \mathcal{C}^\infty(\mathbb{R}) : f^3 - 5f^2 + 6f = 0\} \subseteq \mathcal{C}^\infty(\mathbb{R})_a$.

8. Let H be the affine plane in $\mathbf{A}^3(\mathbb{C})$ with equation $2x + y - 1 = 0$. In each of the following cases calculate the coordinates of $\mathrm{Pr}_{H,\mathbf{u}}(x, y, z)$ where $(x, y, z) \in \mathbf{A}^3$ and $\mathrm{Pr}_{H,\mathbf{u}} : \mathbf{A}^3 \to H$ is the projection in the direction of $\mathbf{u} \in \mathbb{C}^3$, where

1. $\mathbf{u} = (1, 0, 0)$
2. $\mathbf{u} = (i, 0, 0)$
3. $\mathbf{u} = (2i, i, 1)$

9. Show that the hyperplane $H = \mathbf{a} + \langle \mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \rangle$ of $\mathbf{A}^n(\mathbb{R})$ is described by the equation

$$\begin{vmatrix} x_1 - a_1 & \ldots & x_n - a_n \\ v_{1,1} & \ldots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n-1,1} & \ldots & v_{n-1,n} \end{vmatrix} = 0$$

where $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,n})$ and $\mathbf{a} = (a_1, \ldots, a_n)$.

10. In an affine n-dimensional space X let H be a hyperplane and Y a d-dimensional affine subspace. Show that for $d \in \{1, \ldots, n-1\}$ exactly one of the following holds

1. $\dim(H \cap Y) = d - 1$,
2. $H \parallel Y$.

11. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space and let $\mathbf{W}$ be a vector subspace of $\mathbf{V}$. Show that $\mathbf{V}/\mathbf{W}$ has a structure of a vector space. (*Hint.* this is a quotient of vector spaces, in particular it is a quotient of groups.)