Analytic Geometry

George Țurcaș

Maths & Comp. Sci., UBB Cluj-Napoca

January 3, 2022

Recap... Plane isometries

A map $f : \mathcal{E}_2 \to \mathcal{E}_2$ is said to be an *isometry* of the plane $\mathcal{E}_2$ if f conserves the distances, i.e.

$$|f(A)f(B)| = |AB|, \quad \forall\, A, B \in \mathcal{E}_2.$$

(One denotes $|AB| = d_2(A, B)$).

We briefly list a few properties of isometries. These are all proved in Chapter 4 of our textbook.

1) The image of a segment through an isometry is a segment.

2) The image of a half-line is a half-line;

3) The image of a line is a line;

4) If A, B and C are three noncollinear points on $\mathcal{E}_2$, then so are their images $f(A)$, $f(B)$ and $f(C)$;

5) The image of a triangle ΔABC is triangle $\Delta f(A)f(B)f(C)$, such that

$$\Delta ABC \equiv \Delta f(A)f(B)f(C);$$

6) The image of an angle $\widehat{AOB}$ is an angle $f(A)\widehat{f(O)}f(B)$ having the same measure;

7) Two orthogonal lines are transformed into two orthogonal lines;

8) Two parallel lines are transformed into two parallel lines.

9) Any isometry $f : \mathcal{E}_2 \to \mathcal{E}_2$ is surjective.

Denote the set of isometries of the plane by $\mathsf{Iso}(\mathcal{E}_2)$;

$$\mathsf{Iso}(\mathcal{E}_2) = \{f : \mathcal{E}_2 \to \mathcal{E}_2, f \text{ isometry}\}.$$

Theorem

$(\mathrm{Iso}(\mathcal{E}_2), \circ)$ *is a group, called the group of isometries of the plane.*

Denote the set of isometries of the plane by $\text{Iso}(\mathcal{E}_2)$;

$$\text{Iso}(\mathcal{E}_2) = \{f : \mathcal{E}_2 \to \mathcal{E}_2, f \text{ isometry}\}.$$

Theorem

$(\mathrm{Iso}(\mathcal{E}_2), \circ)$ *is a group, called the group of isometries of the plane.*

- A point $A \in \mathcal{E}_2$ is a *fixed point* for the isometry f if $f(A) = A$;
- A line $d \in \mathcal{E}_2$ is said to be *invariant* with respect to f if $f(d) = d$ (obviously, a line whose points are all fixed is invariant, while the converse is not necessarily true).

Examples. Symmetries (reflections)

Let d be a line in $\mathcal{E}_2$. The map $s_d : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$s_d(P) = P', \quad \text{where } P' \text{ is the symmetrical of } P \text{ with respect to the line } d,$$

is called *axial symmetry*. The line d is the *axis* of the symmetry.

Let be given a point O in the plane. The map $s_O : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$s_O(P) = P'$, where P' is the symmetrical of P with respect to the point P,

is called *central symmetry*. The point O is the *center* of the symmetry.

Another example. Translations

Let $\overline{v}$ be a vector in V_2. The map $t_{\overline{v}} : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$t_{\overline{v}}(M) = M', \qquad \text{where} \quad \overline{MM'} = \overline{v},$$

is called *translation* of vector $\overline{v}$.

Rotations

An angle $\widehat{AOB}$ is said to be *oriented* if the pair of half-lines $\{[OA, [OB\}$ is ordered. The angle $\widehat{AOB}$ is *positively oriented* if $[OA$ gets over $[OB$ counterclockwisely. Otherwise, $\widehat{AOB}$ is *negatively oriented*. If the measure of the *nonoriented* angle $\widehat{AOB}$ is θ, then the measure of the oriented angle $\widehat{AOB}$ is either θ, or $-\theta$, depending on the orientation of $\widehat{AOB}$.

Let $O \in \mathcal{E}_2$ be a point and $\theta \in [-2\pi, 2\pi]$ be a number. The map $r_{O,\theta} : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$r_{O,\theta}(M) = M', \quad \text{where} \quad \begin{cases} |OM| = |OM'| \\ m(\widehat{MOM'}) = \theta \end{cases},$$

is called *rotation* of center O and oriented angle θ.

Analytic form of isometries

Theorem

Let $P(x_0, y_0)$ be the center of the central symmetry s_P. The map s_P can be expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto -I_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2x_0 \\ 2y_0 \end{pmatrix}$$

Proof.

Let $M(x, y)$ be an arbitrary point on $\mathcal{E}_2$ and $M' = s_P(M)$ its symmetrical with respect to P, $M' = (x', y')$.

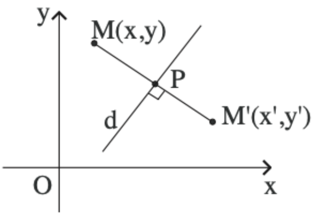
Since P is the midpoint of the segment $[MM']$, then $x_0 = \dfrac{x + x'}{2}$ and $y_0 = \dfrac{y + y'}{2}$, and the conclusion follows. □

Let us now see the analytic form of an axial symmetry.

Theorem

Let $d : ax + by + c = 0$, $a^2 + b^2 > 0$, be a line in $\mathcal{E}_2$. The axial symmetry s_d can be expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & -\frac{b^2-a^2}{a^2+b^2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{2ac}{a^2+b^2} \\ -\frac{2bc}{a^2+b^2} \end{pmatrix}.$$



Proof.

- One may suppose that $b \neq 0$.
- Let $M(x, y)$ be an arbitrary point and $M' = s_d(M)$, $M'(x', y')$.
- The points M and M' are symmetric with respect to d if and only if the line passing through M and M' is orthogonal on d and the midpoint P of the segment $[MM']$ belongs to d.
- The equation of the line determined by M and M' is $\dfrac{X-x}{x'-x} = \dfrac{Y-y}{y'-y}$. The orthogonality condition gives $a(y'-y) = b(x'-x)$.
- The midpoint of $[MM']$ is a point of d if and only if

$$a\left(\frac{x+x'}{2}\right) + b\left(\frac{y+y'}{2}\right) + c = 0.$$

□

Continuation of the proof.

Then, the coordinates (x', y') of M' are the solution of the system of equation

$$\begin{cases} ax' + by' &= -(ax + by + 2c) \\ bx' - ay' &= bx - ay \end{cases}$$

and one obtains

$$\begin{cases} x' = \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y - \dfrac{2ac}{a^2 + b^2} \\ y' = -\dfrac{2ab}{a^2 + b^2}x - \dfrac{b^2 - a^2}{a^2 + b^2}y - \dfrac{2bc}{a^2 + b^2} \end{cases}.$$

In vector form, this can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & -\frac{b^2-a^2}{a^2+b^2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{2ac}{a^2+b^2} \\ -\frac{2bc}{a^2+b^2} \end{pmatrix}.$$

$\square$

A few remarks

- If the line d passes through the origin O, then $c = 0$ and the coordinates of M' become

$$\begin{cases} x' = \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y \\ y' = -\dfrac{2ab}{a^2 + b^2}x - \dfrac{b^2 - a^2}{a^2 + b^2}y \end{cases}. \tag{1}$$

- If the line d is parallel to Ox, then $a = 0$ and the coordinates of M' become

$$\begin{cases} x' = x \\ y' = -y - \dfrac{2c}{b} \end{cases}. \tag{2}$$

- If the line d is parallel to Oy, then $b = 0$ and the coordinates of M' become

$$\begin{cases} x' = -x - \dfrac{2c}{a} \\ y' = y \end{cases}. \tag{3}$$

$P_0(x_0, y_0)$
$(x - x_0)^2 + (y - y_0)^2 = R^2.$

Let $\overline{v}(x_0, y_0)$ be a vector. The translation $t_{\overline{v}}$ of vector $\overline{v}$ can be expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

$\boxed{x^2 + y^2 = R^2.}$

Theorem

If f is an arbitrary isometry of $\mathcal{E}_2$, then its analytic form is given by

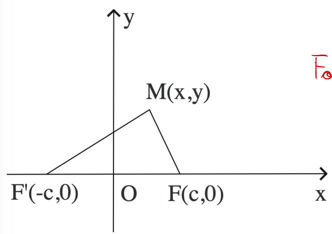
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -\epsilon b \\ b & \epsilon a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

where $a^2 + b^2 = 1$ and $\epsilon = \pm 1$.

Back to conics... The ellipse

Remark: Isometries are not examinable.

- An *ellipse* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant sum.



Focal distance: $2 \cdot c$.

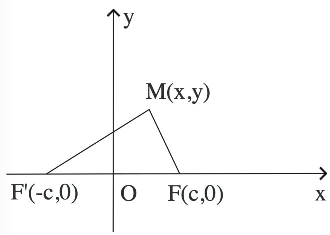
- The two fixed points are called the *foci* of the ellipse and the distance between the foci is the *focal distance*.

- Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$.
- If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

$\leftarrow$ By def. of the ellipse.

- One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.



- Remark that, by triangle inequality, we have $|MF| + |MF'| > |FF'|$, hence $2a > 2c$.
- Let us determine the equation of an ellipse. $M(x, y)$ Starting with the definition, $|MF| + |MF'| = 2a$, or

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

This is equivalent to

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

and by squaring

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2.$$

- One obtains

$$a\sqrt{(x+c)^2+y^2} = cx + a^2,$$

which gives

$$a^2(x^2+2xc+c^2)+a^2y^2 = c^2x^2+2a^2cx+a^2,$$

or

$$(a^2-c^2)x^2+a^2y^2-a^2(a^2-c^2)=0.$$

- Denoting $a^2-c^2=b^2$ (possible, since $a>c$), one has

$$b^2x^2+a^2y^2-a^2b^2=0.$$

- Dividing by a^2b^2, one obtains the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0. \tag{4}$$

The canonical equation of the ellipse.

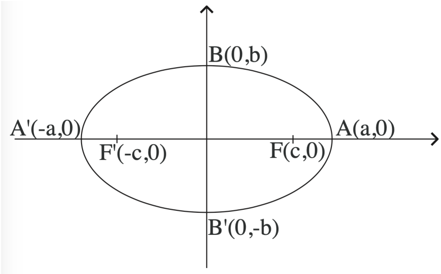
A few remarks

Think of y as $f(x)$

- The equation (4) is equivalent with

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \qquad x = \pm\frac{a}{b}\sqrt{b^2 - y^2},$$

  which means that the ellipse is symmetrical with respect to both Ox and Oy.
- In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its *center*.



- In order to sketch the graph of the ellipse, remark that is it enough to represent the function

$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

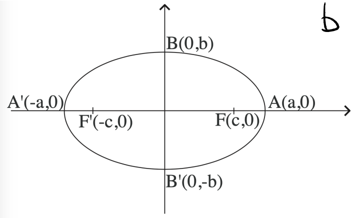
and to complete the ellipse by symmetry with respect to Ox.

- One has

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}, \qquad f''(x) = -\frac{ab}{(a^2-x^2)\sqrt{a^2-x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\vert$	$+\ +\ +$	0	$-\ -\ -$	$\vert$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\vert$	$-\ -\ -$	$-$	$-\ -\ -$	$\vert$

- The graph of the ellipse is



$b^2 = a^2 - c^2$

Some other remarks

$$\frac{x^2}{25} + \frac{y^2}{36} = 1.$$

$F(0, c), F'(0, -c).$

$c^2 = b^2 - a^2.$

- In particular, if $a = b$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, one obtains the equation $x^2 + y^2 - a^2 = 0$ of the circle centered at the origin and of radius a. This happens when $c = 0$, i.e. when the foci coincide, so that the circle may be seen as an ellipse whose foci are identical.
- All the considerations can be done in a similar way, by taking the foci of the ellipse on Oy. One obtains a similar equation for such an ellipse.

The eccentricity

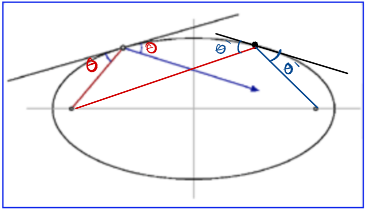
The number $e = \dfrac{c}{a}$ is called the *eccentricity* of the ellipse. Since $a > c$, then $0 \leq e < 1$, hence any ellipse has the eccentricity smaller that 1.

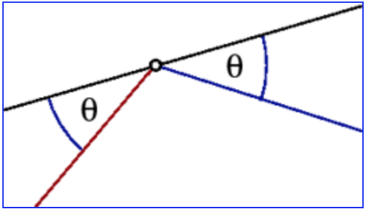
$\boxed{c^2 = a^2 - b^2.}$

On the other hand, $e^2 = \dfrac{c^2}{a^2} = 1 - \left(\dfrac{b}{a}\right)^2$, so that e gives informations about the shape of the ellipse. When e is closer and closer to 0, then the ellipse is "closer and closer" to a circle; and when e is closer to 1, then the ellipse is flattened to Ox.

Applications

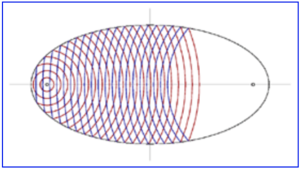
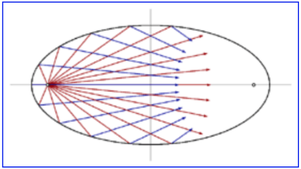
- Ellipses have an interesting reflective property





- This property affects both light and sound.

- Sound emanating from a focus in any direction will always travel along the focal lengths. And since the sum of these focal lengths is constant, it will always travels the same total distance. So, regardless of direction, the ray takes the same amount of time to leave one focus, reflect off of the ellipse and pass through the other focus (since the velocity of sound is the same). So when sound is emitted radially the reflections arrive at the other focus at the same time.

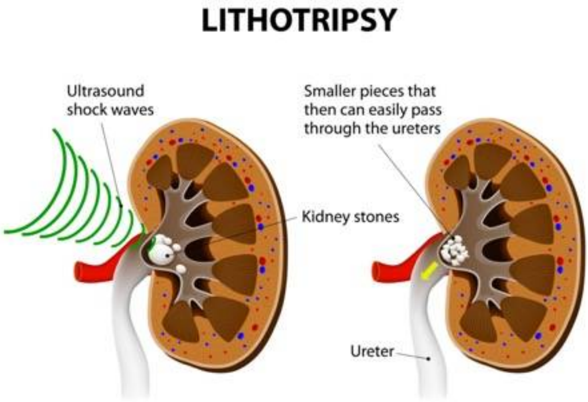
Ellipses in art... Whispering galleries

Figure 1: St Pauls' Cathedral

Figure 2: Whispering Gallery on the dome of St Pauls'

Figure 3: Orecchio Di Dionisio, Sicily

Medical applications



Intersection of a line and an ellipse

- Given an ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ and a line $d : y = mx + n$, their intersection is given by the solutions of the system of equations

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases}.$$

- By replacing y in the equation of the ellipse,

$$(a^2m^2 + b^2)x^2 + 2a^2mnx + a^2(n^2 - b^2) = 0.$$

The discriminant Δ of the last equation is given by

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 + b^2)(n^2 - b^2)].$$

We distinguish the following cases:

- If $\Delta < 0$, then d does not intersect $\mathcal{E}$. The line is *exterior* to the ellipse;
- If $\Delta = 0$, then the line is *tangent* to the ellipse. There is a *tangency* point between d and $\mathcal{E}$;
- If $\Delta > 0$, then there are two intersection points between d and $\mathcal{E}$. The line is *secant* to the ellipse.

The tangent (to an ellipse) with a given direction

If $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ is an ellipse and $m \in \mathbb{R}$ a given real number, there exist exactly two lines, having the angular coefficient m and tangent to $\mathcal{E}$.

Since a line $d : y = mx + n$ is tangent to the ellipse if and only if $a^4m^2n^2 - a^2(a^2m^2 + b^2)(n^2 - b^2) = 0$, then $n = \pm\sqrt{a^2m^2 + b^2}$. The equations of the tangent lines of direction m are

$$y = mx \pm \sqrt{a^2m^2 + b^2}. \tag{5}$$

The tangent (to an ellipse) at a given point

Let $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ be an ellipse and $P_0(x_0, y_0)$ be a point of $\mathcal{E}$. Suppose that $y_0 > 0$, so that P_0 is situated on the graph of the function $f : [-a, a] \to \mathbb{R}$, $f(x) = \frac{b}{a}\sqrt{a^2 - \overset{x}{\cancel{b}}^2}$.

The angular coefficient of the tangent at P_0 to $\mathcal{E}$ is

$$f'(x_0) = -\frac{b}{a}\frac{x_0}{\sqrt{a^2 - \underset{x_0}{\cancel{b}}^2}} = -\frac{b^2 x_0}{a^2 y_0}.$$

If $y_0 < 0$, a similar argument shows that the angular coefficient of the tangent at P_0 is still $-\frac{b^2 x_0}{a^2 y_0}$.

The equation of the tangent to $\mathcal{E}$ at P_0 is

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0),$$

equivalent to

$$b^2 x_0(x - x_0) + a^2 y_0(y - y_0) = 0,$$

or

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right) = 0.$$

Since P_0 belongs to the ellipse, then $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$, and the equation of the tangent to the ellipse at the point P_0 is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - 1 = 0. \tag{6}$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ (eq. of the ellipse)

The problem set for this week will be posted soon. Ideally you would think about it before the seminar on Friday.

Thank you very much for your attention!