# Substitutions in Integrals

## 1 Euler's substitions

Sometime when in the formulation of the function to be integrated we encounter

$$\sqrt{ax^2+bx+c},$$

where $a, b, c \in \mathbb{R}$, we consider a new variable $t$, in one of the following cases:

$$\sqrt{ax^2+bx+c} = \begin{cases} \pm\sqrt{a}x \pm t & \text{if } a > 0 \\ \pm x \cdot t \pm \sqrt{c} & \text{if } c > 0 \\ t(x-x_0) & \text{if } x_0 \text{ is a solution of the equation } ax^2+bx+c=0. \end{cases}$$

## 2 Weirstras' (trigonometric) substitutions

For functions in whose formulations are involved trigonometric functions, there is a usual substitution, namely:

$$tg\frac{x}{2} = t.$$

If we denote by $R(\sin x, \cos x)$ the expression to be integrated, sometimes we may consider other substitutions, which might lead us faster to the expected solution. Hence:P

- If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, then choose $\cos x = t$.
- If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, then choose $\sin x = t$.
- If $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$, then choose $tg\ x = t$.

Recall the following trigonometric identities:

$$\cos^2 x = \frac{1}{1+tg^2 x} \qquad \sin^2 x = \frac{tg^2 x}{1+th^2 x}.$$

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} \qquad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

# 3 Other tigonometric substitution

Sometimes, when the integrating function contains square roots of second degree polynomials ( alternatively to using Euler's substitutions) we may pass to trigonometric functions, in the following situations:

- When$\int R(x, \sqrt{r^2 - x^2}dx$ choose $x = r\sin$ or $x = r\cos t$.
- When $\int R(x, \sqrt{r^2 + x^2}dx$ choose $x = rtgt$ or $x = rctgt$.
- When $\int R(x, \sqrt{x^2 - r^2}dx$ choose $x = \frac{r}{\cos x}$ or $x = \frac{r}{\sin x}$.

## Exercise 1:

a) $\int \frac{1}{1+\frac{1}{sinx}}dx, \quad x \in (\pi, \pi);$

b) $\int \frac{1}{3\sin x+4\cos x}dx \quad x \in (\pi, \pi);$

c) $\int \frac{\sqrt{9-x^2}}{x^2}dx, \quad x \in (-3, 3);$

d) $\int \frac{1}{\sqrt{(x^2+1)^3}}dx, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);$

e) $\int \frac{1}{\sqrt{(x^2-8)^3}}dx \quad x \in (-\sqrt{8}, \sqrt{8});$

f) $\int \sqrt{2x - x^2}dx \quad x \in (0, 2);$

g) $\int \sqrt{4 - x^2}dx \quad x \in (-2, 2);$

h) $\int x\sqrt{1 + x^2}dx.$

### Exercise 2:

Determine

$$a) \int \frac{2x-1}{x^2-3x+2}\mathrm{d}x, \;\; x \in ]2, +\infty[;$$

*b)* $\displaystyle\int \frac{4}{(x-1)(x+1)^2}\mathrm{d}x, \ \ x > 1;$

*c)* $\displaystyle\int \frac{1}{x^3 - x^4}\mathrm{d}x, \ \ x > 1;$

*d)* $\displaystyle\int \frac{2x+5}{x^2+5x+10}, \ x \in \mathbb{R};$

*e)* $\displaystyle\int \frac{1}{x^2+x+1}, \ x \in \mathbb{R}.$

## Exercise 3:

Determine:

*a)* $\displaystyle I = \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\mathrm{d}x, \ \ x \in ]0, +\infty[;$

*b)* $\displaystyle I = \int \frac{1}{x+\sqrt{x-1}}\mathrm{d}x, \ \ x \in ]1, +\infty[.$

## Exercise 4:

Determine:

*a)* $\displaystyle I = \int \frac{1}{1+\sqrt{x^2+2x-2}}\mathrm{d}x, \quad x \in ]\sqrt{3}-1, +\infty[;$

*b)* $\displaystyle I = \int \frac{1}{(x+1)\sqrt{-4x^2-x+1}}\mathrm{d}x, \quad x \in ]\frac{-1-\sqrt{17}}{8}, \frac{\sqrt{17}-1}{8}[.$

# Exercise 5:

Determine:

*a)* $\displaystyle\int_1^2 \frac{1}{x^3+x^2+x+1}\mathrm{d}x;$ *b)* $\displaystyle\int_1^3 \frac{1}{x(x^2+9)}\mathrm{d}x;$

*c)* $\displaystyle\int_{-1}^1 \frac{x^2+1}{x^4+1}\mathrm{d}x;$ *d)* $\displaystyle\int_{-1}^1 \frac{x}{x^2+x+1}\mathrm{d}x.$

# Exercise 6:

Determine:

$a)\ \int_{-3}^{-2} \frac{x}{(x+1)(x^2+3)}\mathrm{d}x;\qquad b)\ \int_{0}^{1} \frac{x+1}{(x^2+4x+5)^2}\mathrm{d}x;$

$c)\ \int_{1}^{2} \frac{1}{x^3+x}\mathrm{d}x;\qquad d)\ \int_{0}^{2} \frac{x^3+2x^2+x+4}{(x+1)^2}\mathrm{d}x. e)\ \int_{0}^{1} \frac{1}{(x+1)(x^2+4)}\mathrm{d}x;$

$f)\ \int_{2}^{3} \frac{2x^3+x^2+2x-1}{x^4-1}\mathrm{d}x;\qquad g)\ \int_{0}^{1} \frac{x^3+2}{(x+1)^3}\mathrm{d}x.$

# Exercise 7:

Determine:

$a)\ \int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\mathrm{d}x;\qquad b)\ \int_{0}^{1} \frac{1}{\sqrt{x^2+x+1}}\mathrm{d}x;$

$c)\ \int_{-1}^{1} \frac{1}{\sqrt{4x^2+x+1}}\mathrm{d}x;\qquad d)\ \int_{2}^{3} \frac{x^2}{(x^2-1)\sqrt{x^2-1}}\mathrm{d}x.$

# Exercise 8:

Determine

$a)\ \int_{2}^{3} \sqrt{x^2+2x-7}\mathrm{d}x;\qquad b)\ \int_{0}^{1} \sqrt{6+4x-2x^2}\mathrm{d}x;$

$c)\ \int_{0}^{3/4} \frac{1}{(x+1)\sqrt{x^2+1}}\mathrm{d}x;\qquad d)\ \int_{2}^{3} \frac{1}{x\sqrt{x^2-1}}\mathrm{d}x.$

# Exercise9:

Determine:

$a)\ 2\sqrt{2} < \int_{-1}^{1} \sqrt{x^2+4x+5}\mathrm{d}x < 2\sqrt{10};$

$b)\ \mathrm{e}^2(\mathrm{e}-1) < \int_{\mathrm{e}}^{\mathrm{e}^2} \frac{x}{\ln x}\mathrm{d}x < \frac{\mathrm{e}^3}{2}(\mathrm{e}-1).$

## Exercise 1:

a) $\int \frac{1}{1+\frac{1}{sinx}}dx, \quad x \in (\pi, \pi);$

b) $\int \frac{1}{3\sin x + 4\cos x}dx \quad x \in (\pi, \pi);$

a) $$J = \int \frac{1}{1+\frac{1}{\sin x}}\,dx = \int \frac{1}{\frac{tg^2(\frac{x}{2})+1}{2\,tg(\frac{x}{2})}+1}\,dx$$

Apply Weierstrass: $u = tg\left(\frac{x}{2}\right)$, $dx = \frac{2}{u^2+1}\,du$

$$\Rightarrow J_u = 4\int \frac{u}{(u^2+1)(u^2+2u+1)}\,du = 4\int \frac{u}{(u+1)^2(u^2+1)}\,du =$$

$$= 4\int \left(\frac{1}{2(u^2+1)} - \frac{1}{2(u+1)^2}\right) du =$$

$$= 4\cdot\left(\frac{1}{2}\int \frac{1}{(u^2+1)}\,du - \frac{1}{2}\int \frac{1}{(u+1)^2}\,du\right) =$$

$$= 2\int \frac{1}{(u^2+1)}\,du - 2\int \frac{1}{(u+1)^2}\,du =$$

$$= 2\arctan(u) - 2\cdot\left(-\frac{1}{u+1}\right) =$$

$$= 2\arctan(u) + \frac{2}{u+1}$$

$$J = \frac{2}{\tan(\frac{x}{2})+1} + 2\arctan\left(\tan\left(\frac{x}{2}\right)\right) + \mathcal{C}$$

b) $$J = \int \frac{1}{3\sin x + 4\cos x}\,dx$$

Apply Weierstrass: $u = \tan\left(\frac{x}{2}\right)$, $dx = \frac{2}{u^2+1}\,du$

partial fraction decomposition

$$\Rightarrow J' = -\int \frac{1}{2u^2-3u-2}\,du = -\int \frac{1}{(u-2)(2u+1)}\,du =$$

$$= -\int \left(\frac{1}{5(u-2)} - \frac{2}{5(2u+1)}\right) du =$$

$$= -\frac{1}{5}\int \frac{1}{(u-2)}\,du + \frac{2}{5}\int \frac{1}{(2u+1)}\,du =$$

$$= -\frac{1}{5}\cdot\ln(u-2) + \frac{2}{5}\cdot\frac{1}{2}\cdot\ln(2u+1) =$$

$$= -\frac{1}{5}\ln(u-2) + \frac{1}{5}\ln(2u+1) =$$

$$= \frac{\ln(2u+1)}{5} - \frac{\ln(u-2)}{5}$$

$$J = \frac{\ln\left(\left|2\tan\left(\frac{x}{2}\right)+1\right|\right)}{5} - \frac{\ln\left(\left|\tan\left(\frac{x}{2}\right)-2\right|\right)}{5} + \mathcal{C}$$

**Exercise 2:**

Determine

a) $\int \frac{2x-1}{x^2-3x+2}\mathrm{d}x,\ \ x \in ]2, +\infty[;$

a) $$J = \int \frac{2x-1}{x^2-3x+2}\,dx = \int\left(\frac{2x-3}{x^2-3x+2} + \frac{2}{x^2-3x+2}\right)dx = \int \frac{2x-3}{x^2-3x+2}\,dx + 2\int \frac{1}{x^2-3x+2}\,dx =$$

$u = x^2-3x+2$

$dx = \frac{1}{2x-3}\,du$

partial fraction decomposition

$$= \int \frac{1}{u}\,du + 2\int \frac{1}{(x-2)(x-1)}\,dx =$$

$$= \ln u + 2\int\left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx =$$

$$= \ln(x^2-3x+2) + 2\left(\ln(x-2) - \ln(x-1)\right) =$$

$$= \ln(x^2-3x+2) - 2\ln(x-1) + 2\ln(x-2)$$

$$J = \ln(|x-2||x-1|) - 2\ln(|x-1|) + 2\ln(|x-2|) + \mathcal{C}$$

b) $\int \frac{4}{(x-1)\,(x+1)^2}\mathrm{d}x,\ \ x > 1;$

b) $$J = 4\int \frac{1}{(x-1)(x+1)^2}\,dx = 4\int\left(-\frac{1}{4(x+1)} - \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)}\right)dx =$$

$$= 4\cdot\left[\left(-\frac{1}{4}\right)\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx =\right.$$

$$= -\ln(|x+1|) + \frac{2}{x+1} + \ln(|x-1|) + \mathcal{C}$$

c) $\int \frac{1}{x^3-x^4}\mathrm{d}x,\ \ x > 1;$

$$J = -\int \frac{1}{x^4-x^3}\,dx = -\int \frac{1}{(x-1)x^3}\,dx = -\int\left(-\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x-1}\right)dx =$$

$$= \int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx - \int \frac{1}{x^3}\,dx - \int \frac{1}{x-1}\,dx =$$

$$= \ln(|x|) - \frac{1}{x} - \frac{1}{2x^2} - \ln(|x-1|) + \mathcal{C}$$

**Exercise 3:**

Determine:

$a)\ \ I = \displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\mathrm{d}x, \ \ x \in ]0, +\infty[;$

$$a)\ \ J = \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\, dx = \int \left(\sqrt{x+1}-\sqrt{x}\right) dx = \int \sqrt{x+1}\, dx - \int \sqrt{x}\, dx = \frac{2(x+1)^{\frac{3}{2}}}{3} - \frac{2x^{\frac{3}{2}}}{3} + \mathcal{C} =$$

$$= \frac{2\left((x+1)^{\frac{3}{2}} - (x)^{\frac{3}{2}}\right)}{3} + \mathcal{C}$$

$$\int \sqrt{x+1}\, dx = \int \sqrt{u}\, du = \frac{2u^{\frac{3}{2}}}{3} = \frac{2(x+1)^{\frac{3}{2}}}{3}$$

$u = x+1$

$dx = du$

$$\int \sqrt{x}\, dx = \frac{2x^{\frac{3}{2}}}{3}$$

$b)\ \ I = \displaystyle\int \frac{1}{x+\sqrt{x-1}}\mathrm{d}x, \ \ x \in ]1, +\infty[.$

$$b)\ \ J = \int \frac{1}{x+\sqrt{x-1}}\, dx =$$

$u = \sqrt{x-1}$

$dx = 2\sqrt{x-1}\, du$

$x = u^2+1$

$$\Rightarrow J' = 2\int \frac{u}{u^2+u+1}\, du = 2\int \left(\frac{2u+1}{2(u^2+u+1)} - \frac{1}{2(u^2+u+1)}\right) du =$$

$$= \cancel{2}\,\frac{1}{\cancel{2}} \int \frac{2u+1}{u^2+u+1}\, du - \cancel{2}\,\frac{1}{\cancel{2}} \int \frac{1}{u^2+u+1}\, du =$$

$$= \ln(u^2+u+1) - \frac{2\,\mathrm{arctg}\left(\frac{2u+1}{\sqrt{3}}\right)}{\sqrt{3}} =$$

$$= \frac{\ln(x+\sqrt{x-1})}{2} - \frac{\mathrm{arctg}\left(\frac{2\sqrt{x-1}+1}{\sqrt{3}}\right)}{\sqrt{3}}\Bigg) + \mathcal{C}$$

$$\int \frac{1}{u^2+u+1}\, du = \int \frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}\, du$$

$v = \dfrac{2u+1}{\sqrt{3}}$

$du = \dfrac{\sqrt{3}}{2}\, dv$

$$\Rightarrow \int \frac{\sqrt{3}}{2\left(\frac{3v^2}{4}+\frac{3}{4}\right)}\, dv = \frac{2}{\sqrt{3}} \int \frac{1}{v^2+1}\, dv = \frac{2}{\sqrt{3}}\, \mathrm{arctg}(v) =$$

$$= \frac{2\,\mathrm{arctg}\left(\frac{2u+1}{\sqrt{3}}\right)}{\sqrt{3}}$$