Seminar 7 — Vector spaces. Subspaces

Let $(K,+,\cdot)$ be a field.

A K-vector space is an Abelian group $(V,+)$ with an external operation

$$\cdot : K\times V \to V$$

which fulfills the following conditions:

(L_1) $k(v_1+v_2) = kv_1 + kv_2$

(L_2) $(k_1+k_2)\cdot v = k_1 v + k_2 v$ $\quad\forall k,k_1,k_2\in K,\ \forall v, v_1, v_2 \in V.$

(L_3) $(k_1\cdot k_2)\cdot v = k_1(k_2\cdot v)$

(L_4) $1\cdot v = v$

Let ${}_KV$, $S\subseteq V$

$$S\leq_K V \iff \begin{cases} 0\in S \\ \forall x,y\in S,\ x+y\in S \\ \forall \alpha\in K,\ \forall x\in S,\ \alpha x\in S\end{cases} \iff \begin{cases} 0\in S \\ \forall k_1,k_2\in K, \forall x,y\in S,\ k_1x+k_2y\in S\end{cases}$$

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1) Show that $(\overset{\text{Abelian group}}{\mathbb{R}_+^*,\cdot})$ is a $\underline{\mathbb{R}}$-v.s. with $\qquad (\mathbb{R},+,\cdot)$

$$\alpha * x = x^\alpha,\quad \alpha\in\mathbb{R},\quad x\in\mathbb{R}_+^*.$$

Solution: We must show that $*$ satisfies $(L_1)-(L_4)$.

Let $\alpha,\beta\in\mathbb{R}$, $x,y\in\mathbb{R}_+^*$

(L_1) $\alpha*(x\cdot y) \overset{?}{=} (\alpha*x)\cdot(\alpha*y)$

$$\alpha*(x\cdot y) = (xy)^\alpha = x^\alpha\cdot y^\alpha = (\alpha*x)\cdot(\alpha*y)$$

(L_2) $(\alpha+\beta)*x \overset{?}{=} (\alpha*x)\cdot(\beta*x)$

$$(\alpha+\beta)*x = x^{\alpha+\beta} = x^\alpha\cdot x^\beta = (\alpha*x)\cdot(\beta*x)$$

(L_3) $(\alpha\cdot\beta)*x \overset{?}{=} \alpha*(\beta*x)$

$$(\alpha\cdot\beta)*x = x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha = (\beta*x)^\alpha = \alpha*(\beta*x)$$

(L_4) $1*x = x^1 = x.$

2) Let $M\neq\emptyset$ be a set, ${}_KV$, $V^M = \{f \mid f: M\to V\}$

$f,g: M\to V$, $f+g: M\to V$, $(f+g)(x) = f(x)+g(x)$, $\forall x\in M$ $\qquad (1)$

$\alpha\in K$, $\alpha f: M\to V$, $(\alpha f)(x) = \alpha\cdot f(x)$, $\forall x\in M$ $\qquad (2)$

Show that V^M is a K-v.s. (?)

Solution: $(V^M,+)$ Abelian group (?)

$+$ assoc.: $\forall f, g, h : M \to V,\ (f+g)+h \overset{?}{=} f+(g+h)$ $\leftarrow$

$\forall x \in M,\ ((f+g)+h)(x) \overset{?}{=} (f+(g+h))(x)$ $\leftarrow$ in V

$$((f+g)+h)(x) \overset{(1)}{=} (f+g)(x)+h(x) \overset{(1)}{=} (f(x)+g(x)) + h(x) \underset{(V,+)}{\overset{\text{in}}{=}}$$

$$= f(x)+(g(x)+h(x)) \overset{(1)}{=} f(x)+(g+h)(x) \overset{(1)}{=} (f+(g+h))(x)$$

$+$ comm (homework)

$\exists^{?}\ \theta : M \to V,\ \forall f : M \to V,\ f+\theta \overset{?}{=} f\ (= \cancel{\theta + f})$

$\theta(x) = 0,\ \forall x \in M$ $\qquad (f+\theta)(x) \overset{?}{=} f(x),\ \forall x \in M$

Let $x \in M$, $(f+\theta)(x) \overset{(1)}{=} f(x)+\theta(x) = f(x)+0 = f(x)$

$\forall f : M \to V,\ \exists^{?} -f : M \to V,\ f+(-f) \overset{?}{=} \theta$

$(-f)(x) = -f(x),\ \forall x \in M$

$\forall x \in M,\ (f+(-f))(x) \overset{(1)}{=} f(x)+(-f)(x) = f(x)-f(x) = 0 = \theta(x)$

Let $\alpha, \beta \in K$, $f, g : M \to V$.

(L_1) $\alpha\cdot(f+g) \overset{?}{=} \alpha f + \alpha g$

$\forall x \in M,\ (\alpha\cdot(f+g))(x) \overset{?}{=} (\alpha f + \alpha g)(x)$ $\leftarrow$ in ${}_K V$

$$(\alpha\cdot(f+g))(x) \overset{(2)}{=} \alpha\cdot(f+g)(x) \overset{(1)}{=} \alpha\cdot(\underset{\in V}{f(x)}+\underset{\in V}{g(x)}) \underset{{}_K V}{\overset{\text{in}}{=}}$$

$$= \alpha\cdot f(x) + \alpha\cdot g(x) \overset{(2)}{=} (\alpha f)(x)+(\alpha g)(x) \underset{(1)}{=} (\alpha f + \alpha g)(x).$$

(L_2) $(\alpha+\beta)\cdot f = \alpha f + \beta f$
(L_3) $(\alpha\beta) f = \alpha(\beta\cdot f)$ $\Big\}$ homework
(L_4) $1\cdot f = f$

Particular case: $K = \mathbb{R}$, $V = \mathbb{R}$, $M = I$ interval $\Longrightarrow V^M = \mathbb{R}^I$

$\Longrightarrow \mathbb{R}^I$ $\mathbb{R}$-v.s.

In particular, $I = \mathbb{R} \Longrightarrow \mathbb{R}^{\mathbb{R}}$ $\mathbb{R}$-v.s. $\left({}_{\mathbb{R}}\mathbb{R}^{\mathbb{R}} \right)$

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3) Let $M \neq \emptyset$ finite and K be an infinite field. ${}_K M$?

I) $|M| = 1$, $M = \{0\}$

$\exists!\ + : M \times M \to M,\ 0+0 = 0$
$\exists!\ \cdot : K \times M \to M,\ \alpha\cdot 0 = 0,\ \forall \alpha \in K$ $\Big\} \Longrightarrow M$ is the zero K-v.s.

The answer is YES!

II) $|M| \geq 2$

Assume by contradiction that there exists a K-v.s. structure on M.

Let $x \in M$, $x \neq 0$, $t'_x : K \to M$, $t'_x(k) = kx$

Let $k_1, k_2 \in K$, $t'_x(k_1) = t'_x(k_2) \Longleftrightarrow k_1 x = k_2 x \Longleftrightarrow$

$\Longleftrightarrow k_1 x - k_2 x = 0 \Longleftrightarrow (k_1 - k_2) \cdot \underset{\neq 0}{x} = 0 \Longrightarrow k_1 - k_2 = 0 \Longleftrightarrow k_1 = k_2$

Thus $\underset{\text{infinite}}{t'_x : K} \longrightarrow \underset{\text{finite}}{M}$ is injective. contrad.

The answer in this case is NO!

4) $p \in \mathbb{N}$ prime $(p \geq 2 \ldots)$

Can we organize $\underbrace{(\mathbb{Z}, +)}_{\text{Abelian group}}$ as a $\underbrace{(\mathbb{Z}_p, +, \cdot)}_{\text{field.}}$-v.s. ?

Solution:

Assume by contradiction that $\exists * : \mathbb{Z}_p \times \mathbb{Z} \to \mathbb{Z}$ which satisfies $(L_1)-(L_4)$.

Let $x \in \mathbb{Z}$, $x \neq 0$,

$p \cdot x = \underbrace{x + \ldots + x}_{p \text{ terms}} = \underbrace{(\hat{1} * x) + \ldots + (\hat{1} * x)}_{p \text{ terms}} = (\underbrace{\hat{1} + \ldots + \hat{1}}_{p \text{ terms}}) * x = \hat{p} * x =$

$= \hat{0} * x = 0.$ contradiction.

Thus the answer is NO!

5) a) $a, b \in \mathbb{R}$ given, $A = \{(x,y) \in \mathbb{R}^2 \mid ax + by = 0\} \overset{?}{\leq}_{\mathbb{R}} \mathbb{R}^2$

$(0,0) \in A$ $(a \cdot 0 + b \cdot 0 = 0)$

Let $\alpha, \beta \in \mathbb{R}$, $(x,y), (x',y') \in A$, $\alpha(x,y) + \beta(x',y') \overset{?}{\in} A$

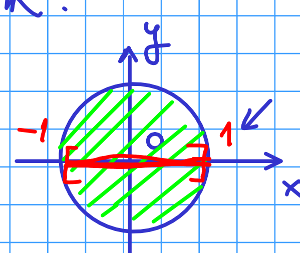
$\alpha \cdot (x,y) + \beta \cdot (x',y') = (\alpha x, \alpha y) + (\beta x', \beta y') = (\alpha x + \beta x', \alpha y + \beta y') \overset{?}{\in} A$

$a(\alpha x + \beta x') + b(\alpha y + \beta y') = a\alpha x + a\beta x' + b\alpha y + b\beta y' =$

$= \alpha \underbrace{(ax + by)}_{=0} + \beta \underbrace{(ax' + by')}_{=0} = 0$ Thus $A \leq_{\mathbb{R}} \mathbb{R}^2$.

b) $D = [-1, 1] \not\leq_{\mathbb{R}} \mathbb{R}$ since

$1 \in D$, $1 + 1 = 2 \notin D$



b') $D' = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\} \not\leq_{\mathbb{R}} \mathbb{R}^2$ since

$(1,0) \in D'$, $(1,0)+(1,0) = (2,0) \notin D'$

b'') $D'' = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_1^2+\dots+x_n^2 \leq 1\} \not\leq_{\mathbb{R}} \mathbb{R}^n$ since

$(1,\underbrace{0,\dots,0}_{n-1}) \in D''$, $(1,0,\dots,0)+(1,0,\dots,0) = (2,0,\dots,0) \notin D''$

c) $n \in \mathbb{N}$ given, $\cancel{P_n(\mathbb{R})} = \{f \in \mathbb{R}[X] \mid \deg f \leq n\} = \mathbb{R}_n[X] \overset{?}{\leq}_{\mathbb{R}} \mathbb{R}[X]$

$f \in \mathbb{R}_n[X] \iff \exists\, a_0, a_1, \dots, a_n \in \mathbb{R} : f = a_0 + a_1X + \dots + a_nX^n \quad (*)$

$0 \in \mathbb{R}_n[X]$ $(\deg 0 = -\infty \leq n)$

Let $f, g \in \mathbb{R}_n[X]$, $f = a_0 + a_1X + \dots + a_nX^n$, $g = b_0 + b_1X + \dots + b_nX^n$

$f+g = (a_0+b_0) + (a_1+b_1)X + \dots + (a_n+b_n)X^n \in \mathbb{R}_n[X]$

$\alpha \in \mathbb{R}$, $\alpha f = (\alpha a_0) + (\alpha a_1)X + \dots + (\alpha a_n)X^n \overset{(*)}{\in} \mathbb{R}_n[X]$

d) $n \in \mathbb{N}$ given, $B = \{f \in \mathbb{R}[X] \mid \deg f = n\} \overset{?}{\leq}_{\mathbb{R}} \mathbb{R}[X]$

$f = X^n$, $g = -X^n$, $f, g \in B$

$f+g = 0 \notin B$ $(\deg 0 = -\infty \neq n) \implies B \not\leq_{\mathbb{R}} \mathbb{R}[X]$

6) Let ${}_KV$, $A \leq_K V$, $C_VA = V \setminus A$

a) $C_VA \not\leq_K V \Longleftarrow 0 \notin C_VA = V \setminus A \Longleftarrow 0 \in A$

b) $C_VA \cup \{0\}$ is not necessarily a subspace of ${}_KV$.

<u>Ex</u>: $K = \mathbb{R}$, $V = \mathbb{R}[X]$, $A = \mathbb{R}_n[X]$ ($n \in \mathbb{N}$ given)

$C_VA = \{f \in \mathbb{R}[X] \mid \deg f \geq n+1\}$

$f = -X^{n+1}$, $g = X^{n+1}+1$, $f, g \in C_VA \cup \{0\}$

$f+g = 1 \notin C_VA \cup \{0\}$

which has the degree $0 \neq n+1$

$\implies C_VA \cup \{0\} \not\leq_{\mathbb{R}} \mathbb{R}[X]$