COURSE 11

Linear maps

Let V and V' be vector spaces over K. The map $f : V \to V'$ is called a **(vector space) homomorphism** or a **linear map** (or a **linear transformation**) if

$$f(x+y) = f(x) + f(y)\,,\ \forall x, y \in V\,, \checkmark$$

$$f(kx) = kf(x),\ \forall k \in K,\ \forall x \in V \checkmark$$

(or, equivalently,

$$\to \quad f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2)\,,\ \forall k_1, k_2 \in K,\ \forall v_1, v_2 \in V). \leftarrow$$

$V'^V = \{f \mid f : V \to V'\}$ K-v.s. with the op. defined by $(*)$ and $(**)$ below.

$Hom_K(V,V') = \{f : V \to V' \mid f \text{ linear map}\} \subseteq V'^V$

$\to Hom_K(V,V') \overset{?}{\leq_K} V'^V$

$\to$ **Theorem 1.** Let V and V' be vector spaces over K. For any $f, g \in Hom_K(V,V')$ and for any $k \in K$, we consider $f+g, k \cdot f \in H\cancel{om_K(V,V')}$: $V \to V'$.

$$\to \quad (f+g)(x) = f(x) + g(x),\ \forall x \in V\,, \qquad (*)$$

$$\to \quad (kf)(x) = kf(x),\ \forall x \in V\,. \qquad (**)$$

These equalities define an addition and a scalar multiplication on $Hom_K(V,V')$ and $Hom_K(V,V')$ is a vector space over K.

Proof. $Hom_K(V,V')$ is closed under $+$ and $\cdot$ defined by $(*)$ and $(**)$, respectively.

- $f+g : V \to V'$ linear map$\overset{?}{.}$ Let $\alpha, \beta \in K$, $x, y \in V$

$(f+g)(\alpha x+\beta y) \overset{(*)}{=} f(\alpha x+\beta y) + g(\alpha x+\beta y) = \alpha f(x) + \beta f(y) + \alpha g(x) + \beta g(y) = (\alpha f(x) + \alpha g(x)) + (\beta f(y) + \beta g(y)) = \alpha(f(x)+g(x)) + \beta(f(y)+g(y)) \overset{(*)}{=} \alpha(f+g)(x) + \beta(f+g)(x)$

- $kf : V \to V'$ linear map$\overset{?}{.}$ Let $\alpha, \beta \in K$, $x, y \in V$. Then

$(kf)(\alpha x+\beta y) \overset{(**)}{=} k \cdot f(\alpha x+\beta y) = k(\alpha f(x) + \beta f(y)) = \alpha(kf(x)) + \beta(kf(y)) \overset{(**)}{=} \alpha (kf)(x) + \beta(kf)(y)$

• $\theta : V \to V'$, $\theta(x) = 0'$, $\forall x \in V$

$\theta \in Hom_K(V, V')$ (it was proved)

Thus $Hom_K(V,V') \leq_K V'^V$ and the theorem is proved.

${}_KV = {}_KV' \Rightarrow Hom_K(V,V') = End_K(V)$ K-v.s. (as previously)

$\Downarrow$

$\square$

Corollary 2. If V is a K-vector space, then $End_K(V)$ is a vector space over K.

Remarks 3. a) Let V be a K-vector space. Then $End_K(V)$ is ~~a subgroupoid of~~ closed in $(V^V, \circ)$ the usual map composition and it follows that $(End_K(V), \circ)$ is a monoid. Moreover, the endomorphism composition $\circ$ is distributive with respect to endomorphism addition $+$, thus $End_K(V)$ also has a unitary ring structure, $(End_K(V), +, \circ)$. homework

$\forall f, g, h \in End_K(V)$, $\left.\begin{array}{l} f\circ(g+h) = f\circ g + f\circ h \\ (g+h)\circ f = g\circ f + h\circ f. \end{array}\right\}$

$(R, +, \cdot)$ unitary ring, $U(R) = \{a \in R \mid \exists a' \in R : a\cdot a' = 1 = a' \cdot a\}$

b) The set $Aut_K(V)$ is the group of the units of $(End_K(V), \circ)$. (hence of $(End_K(V), +, \circ)$).

$U(End_K(V)) = \{f \in End_K(V) \mid \exists g \in End_K(V), f\circ g = 1_V = g \circ f\}$

$= \{f \in End_K(V) \mid f \text{ bijective}\} = Aut_K(V)$.

bij. endom = autom.

Bases. Dimension

Let $(K, +, \cdot)$ be a field and let V be a vector space over K.

Definition 4. We say that the vectors $v_1, \ldots, v_n \in V$ are (or the set of vectors $\{v_1, \ldots, v_n\}$ is):

(1) **linearly independent** in V if for any $k_1, \ldots, k_n \in K$,

$$k_1v_1 + \cdots + k_nv_n = 0 \Rightarrow k_1 = \cdots = k_n = 0\,.$$

(2) **linearly dependent** in V if they are not linearly independent, that is,

$$\exists k_1, \ldots, k_n \in K \text{ not all zero, such that } k_1v_1 + \cdots + k_nv_n = 0\,.$$

More generally, an infinite set of vectors of V is said to be:

(1) **linearly independent** if any finite subset is linearly independent.

(2) **linearly dependent** if there exists a finite subset which is linearly dependent.

$(\underset{\neq 0}{\alpha} \cdot v = 0 \Rightarrow v = 0)$

Remarks 5. (1) A set consisting of a single vector v is linearly dependent if and only if $v = 0$.

(2) As an immediate consequence of the definition, we notice that if V is a vector space over K and $X, Y \subseteq V$ such that $X \subseteq Y$, then:

(i) If Y is linearly independent, then X is linearly independent.

(ii) If X is linearly dependent, then Y is linearly dependent. Thus, every set of vectors containing the zero vector is linearly dependent.

Theorem 6. Let V be a vector space over K. Then the vectors $v_1, \ldots, v_n \in V$ are linearly dependent iff one of the vectors is a linear combination of the others, that is,

$\Longleftrightarrow$

$$\exists j \in \{1, \ldots, n\},\ \exists \alpha_i \in K:\ v_j = \sum_{\substack{i=1\\ i\neq j}}^{n} \alpha_i v_i.$$

Proof. "$\Longrightarrow$" $\exists\, k_1, \ldots, k_n \in K$ not all zero s.t. $k_1 v_1 + \ldots + k_n v_n = 0$. □

Assume that $k_j \neq 0 \Longrightarrow \exists\, k_j^{-1} \in K$.

Then $k_j v_j = -k_1 v_1 - \ldots - k_{j-1} v_{j-1} - k_{j+1} v_{j+1} - \ldots - k_n v_n \Big/ \cdot k_j^{-1} \Longrightarrow$

$$\Longrightarrow v_j = \sum_{\substack{i=1\\ i\neq j}}^{n} (-k_j^{-1} \cdot k_i) v_i.$$ We take $\alpha_i = -k_j^{-1} \cdot k_i$, $i = 1, \ldots, n$, $i \neq j$

$$\Longrightarrow v_j = \sum_{\substack{i=1\\ i\neq j}}^{n} \alpha_i v_i.$$

"$\Longleftarrow$" $v_j = \alpha_1 v_1 + \ldots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \ldots + \alpha_n v_n \Longleftrightarrow$

$\Longleftrightarrow \alpha_1 v_1 + \ldots + \alpha_{j-1} v_{j-1} + \underset{\neq 0}{(-1)} v_j + \alpha_{j+1} v_{j+1} + \ldots + \alpha_n v_n = 0 \Longrightarrow$

$\Longrightarrow v_1, \ldots, v_n$ l. dependent vectors.

Examples 7. (a) $\emptyset$ is linearly independent in any vector space.
(b) Let V_2 be the real vector space of all vectors (in the classical sense) in the plane with a fixed origin O. Recall that the addition is the usual addition of two vectors by the parallelogram rule and the external operation is the usual scalar multiplication of vectors by real scalars. Then:

(i) one vector v is linearly dependent in $V_2 \Leftrightarrow v = 0$;
(ii) two vectors are linearly dependent in $V_2 \Leftrightarrow$ they are collinear;
(iii) three vectors are always linearly dependent in V_2.

$V_2 \simeq \mathbb{R}^2$ as $\mathbb{R}$-v.s.

(c) Let V_3 be the real vector space of all vectors (in the classical sense) in the space with a fixed origin O. Then:

(i) one vector v is linearly dependent in $V_3 \Leftrightarrow v = 0$;
(ii) two vectors are linearly dependent in $V_3 \Leftrightarrow$ they are collinear;
(iii) three vectors are linearly dependent in $V_3 \Leftrightarrow$ they are coplanar;
(iv) four vectors are always linearly dependent in V_3.

$V_3 \simeq \mathbb{R}^3$ as $\mathbb{R}$-v.s.

(d) If K is a field and $n \in \mathbb{N}^*$, then the vectors

$$(1,0,0,\dots,0),(0,1,0,\dots,0),\dots,(0,0,0,\dots,1)$$

homework

from K^n are linearly independent in the K-vector space K^n.
(e) Let K be a field and $n \in \mathbb{N}$. Then the vectors $1, X, X^2, \dots, X^n$ are linearly independent in the vector space $K_n[X] = \{f \in K[X] \mid \deg f \leq n\}$ over K and, more generally, the vectors $1, X, X^2, \dots, X^n, \dots$ are linearly independent in the K-vector space $K[X]$.

We are going to define a key notion concerning vector spaces, namely *basis*, which will perfectly determine a vector space. We will discuss *only the case of finitely generated vector spaces*. This is why, till the end of the chapter, *by a vector space we will understand a finitely generated vector space*. However, many results from the next part hold for arbitrary vector spaces.

finitely generated = generated by a finite set

Definition 8. Let V be a vector space over K. By a **list of vectors** in V we understand an n-tuple $(v_1,\dots,v_n) \in V^n$ for some $n \in \mathbb{N}^*$.

Definition 9. Let V be a vector space over K. An n-tuple $B = (v_1,\dots,v_n) \in V^n$ is called a **basis** of V if:

(1) B is a system of generators for V, that is, $\langle B \rangle = V$;
(2) B is linearly independent in V.

Theorem 10. Let V be a vector space over K. A list $B = (v_1,\dots,v_n)$ of vectors in V is a basis of V if and only if each vector $v \in V$ can be uniquely written as a linear combination of the vectors $v_1,\dots,v_n$, i.e.

$$\forall v \in V,\ \exists k_1,\dots,k_n \in K:\ v = k_1v_1 + \dots + k_nv_n.$$

uniquely determined.

Proof. □

$V = \langle B \rangle \Longleftrightarrow \forall v \in V, \exists k_1,\dots,k_n \in K: v = k_1v_1 + \dots + k_nv_n.$ (1)

B linearly indep. $\overset{?}{\Longleftrightarrow}$ $k_1,\dots,k_n \in K$ from (1) are uniquely determined

"$\Longrightarrow$" Let $k_1',\dots,k_n' \in K$ s.t. $v = k_1v_1 + \dots + k_nv_n = k_1'v_1 + \dots + k_n'v_n$.

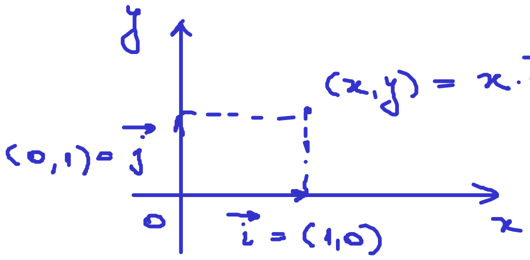
$v_1,\dots,v_n$ l. indep

$\Rightarrow (k_1-k_1')v_1+\dots+(k_n-k_n')v_n=0 \Longrightarrow k_1-k_1'=\dots=k_n-k_n'=0$

$\Rightarrow k_i=k_i', \forall i=\overline{1,n}$.

"$\Leftarrow$" Let $k_1,\dots,k_n\in K$, $k_1v_1+\dots+k_nv_n=0=0\cdot v_1+\dots+0\cdot v_n$.

$\Rightarrow k_1=k_2=\dots=k_n=0$. Thus B is l. indep.



$(x,y)=x\cdot\vec{i}+y\cdot\vec{j}=x(1,0)+y\cdot(0,1)$.

$\mathbb{R}^2$

$\mathbb{R}$

Definition 11. Let V be a vector space over K, $B=(v_1,\dots,v_n)$ a basis of V and $v\in V$. Then the scalars $k_1,\dots,k_n\in K$ from the unique writing of v as a linear combination

$$v=k_1v_1+\dots+k_nv_n$$

of the vectors of B are called the **coordinates of** v **in the basis** B.

l. indep.

Examples 12. (a) $\emptyset$ is basis for the zero vector space. $\{0\}=\langle\emptyset\rangle$

(b) If K is a field and $n\in\mathbb{N}^*$, then the list $E=(e_1,\dots,e_n)$ of vectors of K^n, where

$$e_1=(1,0,0,\dots,0), e_2=(0,1,0,\dots,0),\dots,e_n=(0,0,0,\dots,1)$$

is a basis of the canonical vector space K^n over K, called the **standard basis**. Indeed, we saw that E is linearly independent and each vector $(x_1,\dots,x_n)\in K^n$ can be written as a linear combination of the vectors of E,

$$(x_1,\dots,x_n)=x_1e_1+\dots+x_ne_n.$$

$\Longrightarrow K^n=\langle e_1,\dots,e_n\rangle$

Notice that the coordinates of a vector in the standard basis are just the components of the vector, fact that is not true in general.

In particular, if $n=1$, the set $\{1\}$ is a basis of the canonical vector space K over K. For instance, $\{1\}$ is a basis of the vector space $\mathbb{C}$ over $\mathbb{C}$.

(c) Consider the canonical real vector space $\mathbb{R}^2$. We already know a basis of $\mathbb{R}^2$, namely the standard basis $((1,0),(0,1))$. But it is easy to show that the list $((1,0),(1,1))$ is also a basis of $\mathbb{R}^2$. Therefore, a vector space may have more than one basis.

(d) Let V_3 be the real vector space of all vectors (in the classical sense) in the space with a fixed origin O. Any 3 vectors which are not coplanar form a basis of V_3; e.g. the three pairwise orthogonal *unit vectors* $\vec{i}$, $\vec{j}$, $\vec{k}$.

(e) The sets $S=\{(x,y,z)\in\mathbb{R}^3\mid x+y+z=0\}$ and $T=\{(x,y,z)\in\mathbb{R}^3\mid x=y=z\}$ are subspaces of ${}_{\mathbb{R}}\mathbb{R}^3$. As a matter of fact, $S=\langle(1,0,-1),(0,1,-1)\rangle$ and $T=\langle(1,1,1)\rangle$. Since the two generators of S are linearly independent, they form a basis of S. The only generator of T is clearly linearly independent, hence it forms a basis of T.

(f) Since for any $z \in \mathbb{C}$, there exist the uniquely determined real numbers $x, y \in \mathbb{R}$ such that $z = x \cdot 1 + y \cdot i$, the list $B = (1, i)$ is a basis of the vector space $\mathbb{C}$ over $\mathbb{R}$ (see Theorem 10). The coordinates of a vector $z \in \mathbb{C}$ in the basis B are just its real and its imaginary part.

(g) Let K be a field and $n \in \mathbb{N}$. Then the list $B = (1, X, X^2, \dots, X^n)$ is a basis of the vector space $K_n[X] = \{f \in K[X] \mid \deg f \leq n\}$ over K, because each vector (polynomial) $f \in K_n[X]$ can be uniquely written as a linear combination

$$f = a_0 \cdot 1 + a_1 \cdot X + \cdots + a_n \cdot X^n$$

($a_0, \dots, a_n \in K$) of the vectors of B (see Theorem 10). In this case, the coordinates of a vector $f \in K_n[X]$ in the basis B are just its coefficients as a polynomial.

(h) If V_1 and V_2 are K-vector spaces and $B_1 = (x_1, \dots, x_m)$ and $B_2 = (y_1, \dots, y_n)$ are bases for V_1 and V_2, respectively, then $((x_1, 0), \dots, (x_m, 0), (0, y_1), \dots, (0, y_n))$ is a basis for the direct product $V_1 \times V_2$. ← homework

Theorem 13. Every vector space has a basis.

Proof. Let V be a K-v.s., B a finite list of vectors from V.

$V = \{0\} \iff B = \emptyset$ is a basis for ${}_K V$

Let us consider that $V \neq \{0\}$. Then $B \neq \emptyset$ and we consider $B = (v_1, v_2, \dots, v_n)$, $n \in \mathbb{N}^*$.

If B is l. indep. $\Rightarrow$ B is a basis for ${}_K V$ and the thm. is proved.

If B is l. dep. $\Rightarrow \exists j_1 \in \{1, \dots, n\}$: v_{j_1} is a linear comb. of all the other vectors $\Rightarrow$

$\Rightarrow V = \langle B \rangle \subseteq \langle B \setminus \{v_{j_1}\} \rangle \subseteq V \Rightarrow V = \langle B \setminus \{v_{j_1}\} \rangle$.

If $B \setminus \{v_{j_1}\}$ l. indep. $\Rightarrow B \setminus \{v_{j_1}\}$ is a basis for V.

If $B \setminus \{v_{j_1}\}$ l. dep. $\Rightarrow \exists j_2 \in \{1, \dots, n\} \setminus \{j_1\}$ s.t. $v_{j_2} \in \langle B \setminus \{v_{j_1}, v_{j_2}\} \rangle$

$\Rightarrow V = \langle B \setminus \{v_{j_1}, v_{j_2}\} \rangle$ and so on ...

$V = \langle \underbrace{B \setminus \{v_{j_1}, \dots, v_{j_{n-1}}\}}_{\{v_{j_n}\}} \rangle = \langle v_{j_n} \rangle$. If $v_{j_n} \neq 0 \Rightarrow v_{j_n}$ l. indep.

$\Rightarrow (v_{j_n})$ is a basis for ${}_K V$

If $v_{j_n} = 0 \Rightarrow V = \{0\}$ contrad.

COURSE 12.

Remarks 14. (1) We have proved the existence of a basis of a vector space. As we saw in Example 12 (c) such a basis not necessarily unique.
(2) In the proof of Theorem 13 we saw that if B is an n-elements set which generates V one can successively eliminate elements from B in order to find a basis for V. It follows that any basis of V has at most n vectors. Later we will prove even a stronger result, namely if a vector space has a basis of n elements, then all its bases have n elements.

Theorem 15. i) Let $f : V \to V'$ be a K-linear map and let $B = (v_1, \dots, v_n)$ be a basis of V. Then f is determined by its values on the vectors of the basis B.
ii) Let $f, g : V \to V'$ be K-linear maps and let $B = (v_1, \dots, v_n)$ be a basis of V. If $f(v_i) = g(v_i)$, for any $i \in \{1, \dots, n\}$, then $f = g$.

Proof. $\square$

Remark 16. From the previous theorem one deduces that *given two K-vector spaces V, V', a basis B of V and a function $f' : B \to V'$, there exists a unique linear map $f : V \to V'$ which extends f' (i.e. $f|_B = f'$ or, equivalently, $f(x_i) = f'(x_i)$, $i = 1, \dots, n$)*, result also known as **universal property of vector spaces**.

Theorem 17. Let $f : V \to V'$ be a K-linear map. Then:
(i) f is injective if and only if for any X linearly independent in V, $f(X)$ is linearly independent in V'.
(ii) f is surjective if and only if for any X system of generators for V, $f(X)$ is a system of generators for V'.
(iii) f is bijective if and only if for any X basis of V, $f(X)$ is a basis of V'.

Proof. $\square$

Recall that we consider only finitely generated vector spaces. Let us begin with a very useful lemma, that will be often implicitly used.

Lemma 18. Let V be a K-vector space and let $Y = \langle y_1, \ldots, y_n, z \rangle$. If $z \in \langle y_1, \ldots, y_n \rangle$, then $Y = \langle y_1, \ldots, y_n \rangle$.

Proof. $\square$