Seminar 4

1) a) $A=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 0 & 1 & -1\\ 1 & 3 & 0 & -3\end{pmatrix}\quad (\in M_{3,4}(\mathbb{R}))$

Sol 1: $\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix} = -1 \neq 0 \Longrightarrow \operatorname{rank} A \geq 2$

$$\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 1 & 3 & 0\end{vmatrix} \underset{c_1-c_3}{=} \begin{vmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 1 & 3 & 0\end{vmatrix} = -(-1) = 1 \neq 0 \Longrightarrow \operatorname{rank} A = 3.$$

Sol 2:
$$A=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 0 & 1 & -1\\ 1 & 3 & 0 & -3\end{pmatrix} \underset{\substack{r_2-r_1\\ r_3-r_1}}{\sim} \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & -1 & 0 & -2\\ 0 & 2 & -1 & -4\end{pmatrix} \underset{\substack{c_2-c_1\\ c_3-c_1\\ c_4-c_1}}{\sim} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & -2\\ 0 & 2 & -1 & -4\end{pmatrix}$$

at once

$$\underset{c_4-2c_2}{\sim} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & -8\end{pmatrix} \underset{c_4-8c_3}{\sim} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\end{pmatrix} \Longrightarrow \operatorname{rank} A = 3.$$

c) $A=\begin{pmatrix}3 & 0 & 3 & 0 & 3\\ 0 & 2 & 0 & 2 & 0\\ 3 & 2 & 0 & 3 & 2\\ 0 & 2 & 0 & 2 & 0\end{pmatrix}$

Sol 1: $\begin{vmatrix}0 & 3\\ 2 & 0\end{vmatrix} = -6 \neq 0 \Longrightarrow \operatorname{rank} A \geq 2$

$$d=\begin{vmatrix}3 & 0 & 3\\ 0 & 2 & 0\\ 3 & 2 & 0\end{vmatrix} = 3\cdot(-6) = -18 \neq 0 \Longrightarrow \operatorname{rank} A \geq 3$$

$\exists$ 2 ways of completing d to a 4-size minor of A:

$$\begin{vmatrix}3 & 0 & 3 & 0\\ 0 & 2 & 0 & 2\\ 3 & 2 & 0 & 3\\ 0 & 2 & 0 & 2\end{vmatrix} \underset{r_2=r_4}{=} 0 \underset{r_2=r_4}{=} \begin{vmatrix}3 & 0 & 3 & 3\\ 0 & 2 & 0 & 0\\ 3 & 2 & 0 & 2\\ 0 & 2 & 0 & 0\end{vmatrix}.$$ Thus $\operatorname{rank} A = 3$.

Sol 2:
$$A=\begin{pmatrix}3 & 0 & 3 & 0 & 3\\ 0 & 2 & 0 & 2 & 0\\ 3 & 2 & 0 & 3 & 2\\ 0 & 2 & 0 & 2 & 0\end{pmatrix} \underset{r_3-r_1}{\sim} \begin{pmatrix}3 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & 2 & 0\\ 0 & 2 & -3 & 3 & -1\\ 0 & 2 & 0 & 2 & 0\end{pmatrix} \underset{c_4-c_2}{\sim}$$

$$\sim \begin{pmatrix}3 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0 & 0\\ 0 & 0 & -3 & 1 & -1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \sim \begin{pmatrix}3 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0 & 0\\ 0 & 0 & -3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \Longrightarrow \operatorname{rank} A = 3.$$

d) $A = \begin{pmatrix} 2 & \alpha & -2 & 2 \\ 4 & -1 & 2\alpha & 5 \\ 2 & 10 & -12 & 1 \end{pmatrix} \quad (\alpha \in \mathbb{C})$

Sol 1: $\begin{vmatrix} 4 & -1 \\ 2 & 10 \end{vmatrix} = 42 \neq 0 \Longrightarrow \operatorname{rank} A \geq 2$

$$d = \begin{vmatrix} 2 & \alpha & 2 \\ 4 & -1 & 5 \\ 2 & 10 & 1 \end{vmatrix} \underset{\substack{r_2-2r_1 \\ r_3-r_1}}{=} \begin{vmatrix} 2 & \alpha & 2 \\ 0 & -1-2\alpha & 1 \\ 0 & 10-\alpha & -1 \end{vmatrix} = 2(1+2\alpha-10+\alpha) = 6(\alpha-3)$$

i.) If $\alpha \in \mathbb{C}\setminus\{3\}$ then $d \neq 0 \Longrightarrow \operatorname{rank} A = 3$

ii) If $\alpha = 3$, $d = 0$

$$\begin{vmatrix} 2 & 3 & -2 \\ 4 & -1 & 6 \\ 2 & 10 & -12 \end{vmatrix} \underset{\substack{r_2-2r_1 \\ r_3-r_1}}{=} \begin{vmatrix} 2 & 3 & -2 \\ 0 & -7 & 10 \\ 0 & 7 & -10 \end{vmatrix} = 0 \Longrightarrow \operatorname{rank} A = 2$$

Sol 2:

$$A = \begin{pmatrix} 2 & \alpha & -2 & 2 \\ 4 & -1 & 2\alpha & 5 \\ 2 & 10 & -12 & 1 \end{pmatrix} \underset{\substack{r_2-2r_1 \\ r_3-r_1}}{\sim} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1-2\alpha & 2\alpha+4 & 1 \\ 0 & 10-\alpha & -10 & -1 \end{pmatrix} \underset{c_2 \leftrightarrow c_4}{\sim}$$

$$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 2\alpha+4 & -1-2\alpha \\ 0 & -1 & -10 & 10-\alpha \end{pmatrix} \underset{r_3+r_2}{\sim} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2(\alpha-3) & 3(3-\alpha) \end{pmatrix} \underset{c_4+\frac{3}{2}c_3}{\sim}$$

$$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2(\alpha-3) & 0 \end{pmatrix}$$

i) If $\alpha = 3$, $\operatorname{rank} A = 2$

ii) If $\alpha \in \mathbb{C}\setminus\{3\}$, $\operatorname{rank} A = 3$.

2) a) $\begin{cases} x_1 + x_2 + 2x_3 = -1 \\ 2x_1 - x_2 + 2x_3 = -4 \\ 4x_1 + x_2 + 4x_3 = -2 \end{cases}$ (in $\mathbb{R}^3$).

The system matrix is:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix}$$

$$\det A \underset{c_3-c_1}{=} \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 6 \neq 0 \Longrightarrow$$ the system is consistent, with a unique solution,

$x_1 = \frac{\Delta_1}{6}$, where $\Delta_1 = \begin{vmatrix} -1 & 1 & 2 \\ -4 & -1 & 2 \\ -2 & 1 & 4 \end{vmatrix} = 6 \implies x_1 = 1$

$x_2 = \frac{\Delta_2}{6}$, where $\Delta_2 = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -4 & 2 \\ 4 & -2 & 4 \end{vmatrix} = 12 \implies x_2 = 2$

$x_3 = \frac{\Delta_3}{6}$, where $\Delta_3 = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & -4 \\ 4 & 1 & -2 \end{vmatrix} = -12 \implies x_3 = -2$

The solution of our system is $(1, 2, -2)$.

b) $\begin{cases} 3x_1 + 4x_2 + x_3 + 2x_4 = 3 \\ 6x_1 + 8x_2 + 2x_3 + 5x_4 = 7 \\ 9x_1 + 12x_2 + 3x_3 + 10x_4 = 13 \end{cases}$ (in $\mathbb{R}^4$)

The augmented matrix of our system is:

$$\bar{A} = \left(\begin{array}{cccc:c} 3 & 4 & 1 & 2 & 3 \\ 6 & 8 & 2 & 5 & 7 \\ 9 & 12 & 3 & 10 & 13 \end{array}\right) \begin{array}{c} \leftarrow \\ \leftarrow \\ \end{array}$$

$$d = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 \neq 0$$

$\exists$ 2 ways of completing d to a 3-size minor of the system matrix:

$$\begin{vmatrix} 3 & 1 & 2 \\ 6 & 2 & 5 \\ 9 & 3 & 10 \end{vmatrix} \overset{c_1 = 3c_2}{=\!=} 0 \underset{c_1 = 4c_2}{=\!=} \begin{vmatrix} 4 & 1 & 2 \\ 8 & 2 & 5 \\ 12 & 3 & 10 \end{vmatrix}$$

$\underline{\exists !}$ way to complete d to a 3-size minor of $\bar{A}$ by adding constant terms:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 10 & 13 \end{vmatrix} \overset{c_1 + c_2 = c_3}{=\!=} 0$$

Thus, our system is consistent, equivalent:

$$\rightarrow \begin{cases} x_3 + 2x_4 = 3 - 3x_1 - 4x_2 \\ 2x_3 + 5x_4 = 7 - 6x_1 - 8x_2 \end{cases} \Bigg| (-2) \iff \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \\ x_3 = 1 - 3x_1 - 4x_2 \\ x_4 = 1 \end{cases}$$

The solution set of our system is

$$S = \{(x_1, x_2, 1 - 3x_1 - 4x_2, 1) \mid x_1, x_2 \in \mathbb{R}\}$$

c) $\begin{cases} x_1+x_2-3x_3=-1 \\ 2x_1+x_2-2x_3=1 \\ x_1+x_2+x_3=3 \\ x_1+2x_2-3x_3=1 \end{cases}$ (in $\mathbb{R}^3$)

The augmented matrix of our system is:

$$\bar{A}=\left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & -3 & 1 \end{array}\right).$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0$$

$$d=\begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} \overset{c_1-c_2}{=} \begin{vmatrix} 0 & 1 & -3 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{vmatrix} = -4 \neq 0$$

$\exists!$ way to complete d to a 4-size minor of $\bar{A}$

$$\det \bar{A} \underset{r_1-2r_2,\, r_3-r_1,\, r_4-r_1}{=} \begin{vmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 4 & 3 \\ 0 & 4 & 4 \\ 1 & 0 & 2 \end{vmatrix} \underset{r_3+r_1}{=} \begin{vmatrix} -1 & 4 & 3 \\ 0 & 4 & 4 \\ 0 & 4 & 5 \end{vmatrix}$$

$= -4 \neq 0 \Longrightarrow$ the system is inconsistent.

The solution set of our system is $S=\emptyset$.