Seminar 5 + 6

1) Solve the following systems with Gauss elimination:

a) $$\begin{cases} x_1 + x_2 + 2x_3 = -1 \\ 2x_1 - x_2 + 2x_3 = -4 \\ 4x_1 + x_2 + 4x_3 = -2 \end{cases} \quad (\text{in } \mathbb{R}^3)$$

The augmented matrix of our system is:

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & -1 \\ 2 & -1 & 2 & -4 \\ 4 & 1 & 4 & -2 \end{array}\right) \underset{\substack{r_2-2r_1 \\ r_3-4r_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & \textcircled{-3} & -2 & -2 \\ 0 & \underline{-3} & -4 & 2 \end{array}\right) \underset{r_3-r_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -2 & -2 \\ 0 & 0 & -2 & 4 \end{array}\right)$$

The system is consistent, equivalent to:

$$\begin{cases} x_1 + x_2 + 2x_3 = -1 \\ \quad -3x_2 - 2x_3 = -2 \\ \qquad\quad -2x_3 = 4 \end{cases} \Longleftrightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = -2 \end{cases}$$

The solution of our system $(1, 2, -2)$.

b) $$\begin{cases} 3x_1 + 4x_2 + x_3 + 2x_4 = 3 \\ 6x_1 + 8x_2 + 2x_3 + 5x_4 = 7 \\ 9x_1 + 12x_2 + 3x_3 + 10x_4 = 13 \end{cases} \quad (\text{in } \mathbb{R}^4)$$

The augmented matrix of our system is:

$$\left(\begin{array}{cccc|c} \textcircled{3} & 4 & 1 & 2 & 3 \\ 6 & 8 & 2 & 5 & 7 \\ 9 & 12 & 3 & 10 & 13 \end{array}\right) \underset{\substack{r_2-2r_1 \\ r_3-3r_1}}{\sim} \left(\begin{array}{cccc|c} 3 & 4 & 1 & 2 & 3 \\ 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right) \underset{r_3-4r_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 3 & 4 & 1 & 2 & 3 \\ 0 & \underline{0} & 0 & \underline{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \underset{c_2 \leftrightarrow c_4}{\sim} \left(\begin{array}{cc|cc|c} x_1 & x_4 & x_3 & x_2 & \\ 3 & 2 & 1 & 4 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The system is consistent, equivalent to:

$$\begin{cases} 3x_1 + 2x_4 + x_3 + 4x_2 = 3 \\ \qquad x_4 \qquad\qquad = 1 \end{cases} \Longleftrightarrow \begin{cases} 3x_1 + 2x_4 = 3 - x_3 - 4x_2 \\ \qquad\quad x_4 = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 = \frac{1}{3}(1 - x_3 - 4x_2) \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \\ x_4 = 1 \end{cases}$$

The solution set is $S = \{(\frac{1}{3}(1-x_3-4x_2), x_2, x_3, 1) \mid x_2, x_3 \in \mathbb{R}\}$.

c) $$\begin{cases} x_1 + x_2 - 3x_3 = -1 \\ 2x_1 + x_2 - 2x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \\ x_1 + 2x_2 - 3x_3 = +1 \end{cases} \quad (\text{in } \mathbb{R}^3).$$

The augmented matrix of our system is:

$$\left(\begin{array}{ccc|c} \enclose{circle}{1} & 1 & -3 & -1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & -3 & 1 \end{array}\right) \underset{\substack{r_2-2r_1 \\ r_3-r_1 \\ r_4-r_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & \enclose{circle}{-1} & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & \underline{1} & 0 & 2 \end{array}\right) \underset{r_4+r_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & \enclose{circle}{4} & 4 \\ 0 & 0 & 4 & 5 \end{array}\right) \underset{r_4-r_3}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$\Longrightarrow$ the system is inconsistent $(S = \emptyset)$

2) Discuss on $\alpha \in \mathbb{R}$ the consistency of the following systems, then solve them:

a) $$\begin{cases} 5x_1 - 3x_2 + 2x_3 + 4x_4 = 3 \\ 4x_1 - 2x_2 + 3x_3 + 7x_4 = 1 \\ 8x_1 - 6x_2 - x_3 - 5x_4 = 9 \\ 7x_1 - 3x_2 + 7x_3 + 17x_4 = \alpha \end{cases} \quad (\text{in } \mathbb{R}^4)$$

The augmented matrix of our system is

$$\bar{A} = \left(\begin{array}{cccc|c} 5 & -3 & 2 & 4 & 3 \\ 4 & -2 & 3 & 7 & 1 \\ 8 & -6 & -1 & -5 & 9 \\ 7 & -3 & 7 & 17 & \underline{\underline{\alpha}} \end{array}\right)$$

Solution 1 (the algorithm based on Rouché Thm.)

$$\underline{\underline{d}} = \begin{vmatrix} 5 & -3 \\ 4 & -2 \end{vmatrix} = 2 \neq 0$$

There are 4 ways to complete d to a 3-size minor of the system matrix:

$$\begin{vmatrix} 5 & -3 & 2 \\ 4 & -2 & 3 \\ 8 & -6 & \enclose{circle}{-1} \end{vmatrix} \overset{r_1+2r_3}{\underset{r_2+3r_3}{=\!=}} \begin{vmatrix} 21 & -15 & 0 \\ 28 & -20 & 0 \\ 8 & -6 & -1 \end{vmatrix} = 0 \quad \left(\tfrac{1}{3} r_1 = \tfrac{1}{4} r_2\right.$$

$$\begin{vmatrix} 5 & -3 & 4 \\ 4 & -2 & 7 \\ 8 & -6 & -5 \end{vmatrix} = 0 \; ; \; \begin{vmatrix} 5 & -3 & 2 \\ 4 & -2 & 3 \\ 7 & -3 & 7 \end{vmatrix} = 0 \; ; \; \begin{vmatrix} 5 & -3 & 4 \\ 4 & -2 & 7 \\ 7 & -3 & 17 \end{vmatrix} = 0 .$$

There d is a 2-size non zero minor which gives us the rank(2) of the system's matrix; there are 2 ways to complete d to a a 3-size minor by adding constant terms columns:

$$\begin{vmatrix} 5 & -3 & 3 \\ 4 & -2 & \textcircled{1} \\ 8 & -6 & 9 \end{vmatrix} \underset{\substack{c_1-4c_3 \\ c_2+2c_3}}{=\!=} \begin{vmatrix} -7 & 3 & 3 \\ 0 & 0 & 1 \\ -28 & 12 & 9 \end{vmatrix} = 0$$

$$\begin{vmatrix} 5 & -3 & 3 \\ 4 & -2 & 1 \\ 7 & -3 & \alpha \end{vmatrix} = \begin{vmatrix} -7 & 3 & 3 \\ 0 & 0 & \textcircled{1} \\ 7-4\alpha & -3+2\alpha & \alpha \end{vmatrix} = -(21-14\alpha-21+12\alpha) = 2\alpha$$

1) If $\alpha \in \mathbb{R}\setminus\{0\}$, the system is inconsistent.

2) If $\alpha = 0$ the system is consistent, equivalent to

$$\begin{cases} 5x_1 - 3x_2 = 3 - 2x_3 - 4x_4 \\ 4x_1 - 2x_2 = 1 - 3x_3 - 7x_4 \end{cases} \Longleftrightarrow \begin{cases} x_1 = \dots \\ x_2 = \dots \\ x_3 \in \mathbb{R} \\ x_4 \in \mathbb{R} \end{cases}$$ homework

The solution is

$$S = \{(\quad , \quad , x_3, x_4) / x_3, x_4 \in \mathbb{R}\}.$$

Solution 2 (Gauss elimination):

$$\bar{A} \underset{r_1-r_2}{\sim} \left(\begin{array}{cccc|c} \textcircled{1} & -1 & -1 & -3 & 2 \\ 4 & -2 & 3 & 7 & 1 \\ 8 & -6 & -1 & -5 & 9 \\ 7 & -3 & 7 & 17 & \alpha \end{array}\right) \underset{\substack{r_2-4r_1 \\ r_3-8r_1 \\ r_4-7r_1}}{\sim} \left(\begin{array}{cccc|c} 1 & -1 & -1 & -3 & 2 \\ 0 & \textcircled{2} & 7 & 19 & -7 \\ 0 & 2 & 7 & 19 & -7 \\ 0 & 4 & 14 & 38 & \alpha-14 \end{array}\right)$$

$$\underset{\substack{r_3-r_2 \\ r_4-2r_2}}{\sim} \left(\begin{array}{cccc|c} 1 & -1 & -1 & -3 & 2 \\ 0 & 2 & 7 & 19 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{array}\right) \underset{r_3 \leftrightarrow r_4}{\sim} \left(\begin{array}{cccc|c} 1 & -1 & -1 & -3 & 2 \\ 0 & 2 & 7 & 19 & -7 \\ 0 & 0 & 0 & 0 & \alpha \swarrow \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

1) If $\alpha \in \mathbb{R}\setminus\{0\}$, the system is inconsistent.

2) If $\alpha = 0$ the system is consistent, equivalent to:

$$\begin{cases} x_1 - x_2 - x_3 - 3x_4 = 2 \\ 2x_2 + 7x_3 + 19x_4 = -7 \end{cases} \Longleftrightarrow \begin{cases} x_1 - x_2 = 2 + x_3 + 3x_4 \\ 2x_2 = -7 - 7x_3 - 19x_4 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 = \dots \\ x_2 = \frac{1}{2}(-7 - 7x_3 - 19x_4) \\ x_3 \in \mathbb{R} \\ x_4 \in \mathbb{R} \end{cases}$$ homework

The solution set is $S=\{(\qquad , \frac{1}{2}(-7-7x_3-19x_4), x_3, x_4) / x_3, x_4 \in \mathbb{R}\}$

b) $\begin{cases} 2x_1 - x_2 + 3x_3 + 4x_4 = 5 \\ 4x_1 - 2x_2 + 5x_3 + 6x_4 = 7 \\ 6x_1 - 3x_2 + 7x_3 + 8x_4 = 9 \\ \alpha x_1 - 4x_2 + 9x_3 + 10x_4 = 11 \end{cases}$ (in $\mathbb{R}^4$)

The augmented matrix of our system is

$$\bar{A} = \left(\begin{array}{cccc:c} 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ 6 & -3 & 7 & 8 & 9 \\ \alpha & -4 & 9 & 10 & 11 \end{array}\right) \begin{array}{c} \leftarrow \\ \\ \\ \leftarrow \end{array}$$
$$\qquad\qquad\qquad\qquad\uparrow$$

<u>Solution 1</u>:

$$d_1 = \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = -2 \neq 0$$

There are 4 ways to complete d_1 to a 3-size minor of the system's matrix:

$$\begin{vmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{vmatrix} \overset{r_1+r_3=2r_2}{=\!=} 0 = \begin{vmatrix} -1 & 3 & 4 \\ -2 & 5 & 6 \\ -3 & 7 & 8 \end{vmatrix} = \begin{vmatrix} -2 & 5 & 6 \\ -3 & 7 & 8 \\ -4 & 9 & 10 \end{vmatrix}$$

$$d_2 = \begin{vmatrix} 4 & 5 & 6 \\ 6 & 7 & 8 \\ \alpha & 9 & 10 \end{vmatrix} \underset{r_2-\frac{1}{2}(r_1+r_3)}{=\!=} \begin{vmatrix} 4 & 5 & 6 \\ 6-\frac{1}{2}(\alpha+4) & 0 & 0 \\ \alpha & 9 & 10 \end{vmatrix} = -\left(\frac{1}{2}\alpha - 4\right)\cdot 4 =$$
$$= -2\alpha + 16 = -2(\alpha-8)$$

1) If $\alpha = 8$, $d_2 = 0$ and d_1 is a 2-size non-zero minor which gives us the system matrix rank (2); there are 2 ways to complete d_1 to a 3-size minor, by adding constant terms columns of $\bar{A}$:

$$\begin{vmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{vmatrix} \underset{(r_1+r_3=2r_2)}{=\!=} 0 = \begin{vmatrix} 5 & 6 & 7 \\ 7 & 8 & 9 \\ 9 & 10 & 11 \end{vmatrix}$$

$\Longrightarrow$ the system is consistent, equivalent to:

$$\begin{cases} 5x_3 + 6x_4 = 7 - 4x_1 - 2x_2 \\ 7x_3 + 8x_4 = 9 - 6x_1 - 3x_2 \end{cases} \Longleftrightarrow \begin{cases} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \\ x_3 = \dots \\ x_4 = \dots \end{cases}$$

The solution set is

$$S = \{(x_1, x_2, \qquad , \qquad) \mid x_1, x_2 \in \mathbb{R}\}$$

2) If $\alpha \in \mathbb{R} \setminus \{8\}$, then $d_2 \neq 0$, the system's matrix has the rank 3 since the the system's matrix determinant is zero $(r_1+r_3=2r_2)$

$\Longrightarrow$ d_2 is a 3-size non-zero minor which gives us the rank of the system's matrix; there is only one way to complete d_2 to a 4-size minor by adding constant terms

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \\ \alpha & 9 & 10 & 11 \end{vmatrix} \underset{r_1+r_3=2r_2}{=\!=\!=} 0$$

$\Longrightarrow$ the system is consistent, equivalent to:

$$\begin{cases} 4x_1 + 5x_3 + 6x_4 = 7 + 2x_2 \\ 6x_1 + 7x_3 + 8x_4 = 9 + 3x_2 \\ \alpha x_1 + 9x_3 + 10x_4 = 11 + 4x_2 \end{cases}$$

Cramer system with the system's matrix determinant d_2

$$\Longrightarrow \begin{cases} x_1 = \dfrac{\Delta_1}{d_2} = \cdots \\ x_3 = \dfrac{\Delta_2}{d_2} = \cdots \\ x_4 = \dfrac{\Delta_3}{d_2} = \cdots \\ x_2 \in \mathbb{R} \end{cases}$$

<u>homework</u>

The solution set

$$S = \{ (\quad , x_2 , \quad , \quad) \mid x_2 \in \mathbb{R} \}$$

<u>Solution 2</u>: The augmented matrix of our system is:

$$\bar{A} = \left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 2 & -1 & 3 & 4 & 5 \\ 4 & -2 & 5 & 6 & 7 \\ 6 & -3 & 7 & 8 & 9 \\ \alpha & -4 & 9 & 10 & 11 \end{array}\right) \underset{c_1 \leftrightarrow c_2}{\sim} \left(\begin{array}{cccc|c} x_2 & x_1 & x_3 & x_4 & \\ \boxed{-1} & 2 & 3 & 4 & 5 \\ -2 & 4 & 5 & 6 & 7 \\ -3 & 6 & 7 & 8 & 9 \\ -4 & \alpha & 9 & 10 & 11 \end{array}\right) \underset{\substack{r_3-3r_1 \\ r_4-4r_1}}{\overset{r_2-2r_1}{\sim}} \left(\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -2 & -4 & -6 \\ 0 & \underline{\alpha-8} & -3 & -6 & -9 \end{array}\right) \sim$$

$$\underset{c_2 \leftrightarrow c_3}{\sim} \left(\begin{array}{cccc|c} x_2 & x_3 & x_1 & x_4 & \\ -1 & 3 & 2 & 4 & 5 \\ 0 & \boxed{-1} & 0 & -2 & -3 \\ 0 & -2 & 0 & -4 & -6 \\ 0 & -3 & \alpha-8 & -6 & -9 \end{array}\right) \underset{r_4-3r_2}{\overset{r_3-2r_2}{\sim}} \left(\begin{array}{cccc|c} -1 & 3 & 2 & 4 & 5 \\ 0 & -1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha-8 & 0 & 0 \end{array}\right) \overset{r_3 \leftrightarrow r_4}{\sim} \left(\begin{array}{cccc|c} -1 & 3 & 2 & 4 & 5 \\ 0 & -1 & 0 & -2 & -3 \\ 0 & 0 & \alpha-8 & 0 & \boxed{0} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) = B$$

this does not lead us to an inconsistency case

1) If $\alpha = 8$, then B is a trapezoidal form with 2 non-zero rows, the system is

consistent, equivalent to:

$$\begin{cases} -x_2 + 3x_3 + 2x_1 + 4x_4 = 5 \\ \quad - x_3 \qquad\qquad - 2x_4 = -3 \end{cases} \Longleftrightarrow \begin{cases} x_1 \in \mathbb{R} \\ x_2 = 4 - 2x_4 + 2x_1 \\ x_3 = 3 - 2x_4 \\ x_4 \in \mathbb{R} \end{cases}$$

The solution set is

$$S = \{(x_1, 4-2x_4+2x_1, 3-2x_4, x_4) \mid x_1, x_4 \in \mathbb{R}\}.$$

2) If $\alpha \in \mathbb{R} \setminus \{8\}$, then B is a trapezoidal form with 3 non-zero rows, the system is consistent, equivalent to:

$$\begin{cases} -x_2 + 3x_3 + 2x_1 = -4x_4 + 5 \\ \quad - x_3 \qquad = 2x_4 - 3 \\ \quad \underbrace{(\alpha-8)}_{\neq 0} x_1 = 0 \end{cases} \Longleftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 4 - 2x_4 \\ x_3 = 3 - 2x_4 \\ x_4 \in \mathbb{R} \end{cases}$$

The solution set is $S = \{(0, 4-2x_4, 3-2x_4, x_4) \mid x_4 \in \mathbb{R}\}$.

c) $\begin{cases} \alpha x_1 + x_2 + x_3 = 1 \\ x_1 + \alpha x_2 + x_3 = 1 \\ x_1 + x_2 + \alpha x_3 = 1 \end{cases}$ (in $\mathbb{R}^3$)

<u>Solution 1</u>: The system matrix $A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}$.

$$d = \det A \underset{r_1+r_2+r_3}{=} (\alpha+2) \underline{\underline{\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix}}} \underset{c_2-c_1}{=} (\alpha+2) \begin{vmatrix} 1 & 0 & 1 \\ 1 & \boxed{\alpha-1} & 1 \\ 1 & 0 & \alpha \end{vmatrix} = (\alpha+2)(\alpha-1)^2.$$

1) If $\alpha \in \mathbb{R} \setminus \{-2, 1\}$ the system is consistent, with a unique solution given by Cramer formulas:

$$x_1 = \frac{\Delta_1}{d}, \text{ where } \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = (\alpha-1)^2 \Longrightarrow x_1 = \frac{1}{\alpha+2}$$

$$x_2 = \frac{\Delta_2}{d}, \text{ where } \Delta_2 = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \alpha \end{vmatrix} = (\alpha-1)^2 \Longrightarrow x_2 = \frac{1}{\alpha+2}$$

$$x_3 = \frac{\Delta_3}{d}, \text{ where } \Delta_3 = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & 1 \end{vmatrix} = (\alpha-1)^2 \Longrightarrow x_3 = \frac{1}{\alpha+2}$$

The solution of our system is $\left(\frac{1}{\alpha+2}, \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)$.

2) If $\alpha = -2$, the augmented matrix of our system is

$$\bar{A} = \left(\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{array}\right), \quad d = 0, \quad d' = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 \neq 0$$

$\Longrightarrow$ d' is a non-zero 2-size minor of A (whose rank is 2); there is only one way to add constant terms to d':

$$\begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (-2-1)^2 = 9 \neq 0 \Rightarrow$$ the system is inconsistent.

3) If $\alpha = 1$, the system is consistent, equivalent to

$$x_1 + x_2 + x_3 = 1 \iff \begin{cases} x_1 = 1 - x_2 - x_3 \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \end{cases}$$

The solution set is

$$S = \{(1 - x_2 - x_3, x_2, x_3) / x_2, x_3 \in \mathbb{R}\}.$$

<u>**Solution 2**</u>: The augmented matrix of our system is:

$$\bar{A} = \left(\begin{array}{ccc|c} \underline{\alpha} & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \end{array}\right) \underset{r_1 \leftrightarrow r_3}{\sim} \left(\begin{array}{ccc|c} \boxed{1} & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 \\ \alpha & 1 & 1 & 1 \end{array}\right) \underset{r_3 - \underline{\alpha} r_1}{\overset{r_2 - r_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \boxed{\alpha - 1} & 1-\alpha & 0 \\ 0 & 1-\alpha & \underset{\substack{\shortparallel \\ (1-\alpha)(1+\alpha)}}{1-\alpha^2} & 1-\alpha \end{array}\right)$$

$$\underset{r_3 + r_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & \alpha & 1 \\ 0 & \alpha - 1 & 1-\alpha & \underline{0} \\ 0 & 0 & \underset{= 0}{(1-\alpha)(2+\alpha)} & \underset{\neq 0}{1-\alpha} \end{array}\right) = B$$

1) If $\alpha = -2$ then $B = \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right) \Rightarrow$ the system is inconsistent.

2) If $\alpha = 1$ then

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$ the system is consistent,

equivalent to

$$x_1 + x_2 + x_3 = 1 \iff \begin{cases} x_1 = 1 - x_2 - x_3 \\ x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \end{cases}$$

The solution set is $S = \{(1 - x_2 - x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$.

3) If $\alpha \in \mathbb{R} \setminus \{-2, 1\}$ the system is consistent, equivalent to:

$$\begin{cases} x_1 + x_2 + \alpha x_3 = 1 \\ (\alpha - 1) x_2 + (1-\alpha) x_3 = 0 \\ (1-\alpha)(2+\alpha) x_3 = 1 - \alpha \end{cases} \iff \begin{cases} x_1 = \frac{1}{\alpha+2} \\ x_2 = \frac{1}{\alpha+2} \\ x_3 = \frac{1}{\alpha+2} \end{cases}$$

The solution is $\left(\frac{1}{\alpha+2}, \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)$.

3) <u>Homework</u> (according to my suggestions) $\Leftarrow$ finding the inverse of a matrix

<u>**Remark**</u>: Concerning elementary operations:

Application 1 – determinant computing – we can perform both row and column operations

Application 2 – rank computing – —"— "— "— "— "— "—

Application 3 – linear systems solving (with Gauss elimination)

– we can perform only row operations and, if necessary, we can switch columns except for the last one

Application 4 – finding the inverse of a matrix (if possible) –

– we can perform only row operations.

a) Find the inverse of each of the following matrices (if possible):

$$A=\begin{pmatrix}1&1&2\\2&-1&1\\4&1&4\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc}\enclose{circle}{1}&1&2&1&0&0\\2&-1&1&0&1&0\\4&1&4&0&0&1\end{array}\right)\underset{r_3-4r_1}{\overset{r_2-2r_1}{\sim}}\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&\enclose{circle}{-3}&-3&-2&1&0\\0&-3&-4&-4&0&1\end{array}\right)\underset{r_3-r_2}{\overset{r_1+\frac{1}{3}r_2}{\sim}}$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&1&\frac{1}{3}&\frac{1}{3}&0\\0&\underline{-3}&-3&-2&1&0\\0&0&\underline{-1}&-2&-1&1\end{array}\right)\underset{(-1)r_3}{\overset{\left(-\frac{1}{3}\right)r_2}{\sim}}\left(\begin{array}{ccc|ccc}1&0&1&\frac{1}{3}&\frac{1}{3}&0\\0&1&1&\frac{2}{3}&-\frac{1}{3}&0\\0&0&\enclose{circle}{1}&2&1&-1\end{array}\right)\underset{r_2-r_3}{\overset{r_1-r_3}{\sim}}$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&0&-\frac{5}{3}&-\frac{2}{3}&1\\0&1&0&-\frac{4}{3}&-\frac{4}{3}&1\\0&0&1&2&1&-1\end{array}\right)$$

with the left block $\underline{}$ $=I_3$ and the right block $\underbrace{}_{A^{-1}}$. Thus $A^{-1}=\begin{pmatrix}-\frac{5}{3}&-\frac{2}{3}&1\\-\frac{4}{3}&-\frac{4}{3}&1\\2&1&-1\end{pmatrix}$.

b) $A=\begin{pmatrix}3&4&2\\6&8&5\\9&12&10\end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc}\enclose{circle}{3}&4&2&1&0&0\\6&8&5&0&1&0\\9&12&10&0&0&1\end{array}\right)\underset{r_3-3r_1}{\overset{r_2-2r_1}{\sim}}\left(\begin{array}{ccc|ccc}3&4&2&1&0&0\\0&0&1&-2&1&0\\0&0&4&-3&0&1\end{array}\right)$$

$\Longrightarrow \det A=0 \Longrightarrow \nexists\, A^{-1}$.

c) $$A=\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc}\enclose{circle}{1}&1&1&1&1&0&0&0\\1&1&-1&-1&0&1&0&0\\1&-1&1&-1&0&0&1&0\\1&-1&-1&1&0&0&0&1\end{array}\right)\underset{\substack{r_3-r_1\\r_4-r_1}}{\overset{r_2-r_1}{\sim}}\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&\underline{0}&-2&-2&-1&1&0&0\\0&-2&0&-2&-1&0&1&0\\0&-2&-2&0&-1&0&0&1\end{array}\right)\sim$$

$$\overset{r_2\leftrightarrow r_3}{\sim}\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&\underline{-2}&0&-2&-1&0&1&0\\0&0&\underline{-2}&-2&-1&1&0&0\\0&-2&-2&0&-1&0&0&1\end{array}\right)\underset{\left(-\frac{1}{2}\right)r_3}{\overset{\left(-\frac{1}{2}\right)r_2}{\sim}}\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&\enclose{circle}{1}&0&1&\frac{1}{2}&0&-\frac{1}{2}&0\\0&0&1&1&\frac{1}{2}&-\frac{1}{2}&0&0\\0&-2&-2&0&-1&0&0&1\end{array}\right)\underset{r_4+2r_2}{\overset{r_1-r_2}{\sim}}$$

$$\sim \left(\begin{array}{cccc:cccc} 1 & 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \textcircled{1} & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & -1 & 1 \end{array}\right) \overset{r_1-r_3}{\underset{r_4+2r_3}{\sim}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \underline{4} & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\overset{\frac{1}{4}r_4}{\sim} \left(\begin{array}{cccc:cccc} 1 & 0 & 0 & -1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right) \underset{\substack{r_2-r_4 \\ r_3-r_4}}{\overset{r_1+r_4}{\sim}} \left(\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{array}\right)$$

$$\underbrace{}_{=I_4} \qquad \underbrace{}_{=A^{-1}}$$

Thus $A^{-1} = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \frac{1}{4} \cdot A.$

d)

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \overset{r_1-r_2}{\sim} \left(\begin{array}{cccc|c|c|c|c} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \overset{r_2-r_3}{\sim}$$

$$\sim \left(\begin{array}{cccc|c|c|c|c} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 0 & 0 & 1 \end{array}\right) \underset{r_3-r_4}{\sim} \left(\begin{array}{cccc|c|c|c|c} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\underline{}$$
$$A^{-1}$$