COURSE 4

Determinants

Let $(K,+,\cdot)$ be a field, $n \in \mathbb{N}^*$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \in M_n(K).$$

Definition 1. **The determinant of (the squ**a**re matrix)** A is

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} (\in K).$$

The map $M_n(K) \to K$, $A \mapsto \det A$ is also called **determinant**.

Remark 2. None of the products from the above definition contains 2 elements from the same row or the same column.

We also denote the determinant of A by $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$.

Examples 3. a) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \in K$.

b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{31} + a_{12}a_{23}a_{31}$

$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$ $\quad - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \in K.$

Lemma 4. The determinant of A and the determinant of the transpose matrix ${}^tA$ are equal.

Proof. $\det A = (a_{ij}) \in M_n(K)$, ${}^tA = (a_{ji}) \in M_n(K)$ □

$$\det {}^tA = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} \cdots a_{n\sigma^{-1}(n)}.$$

$\sigma(i) = j \iff i = \sigma^{-1}(j)$

The map $S_n \to S_n$, $\sigma \mapsto \sigma^{-1}$ is a bijective.

$\varepsilon : S_n \to \{-1, 1\} = U_2$, $\varepsilon(\sigma^{-1}) \cdot \varepsilon(\sigma) = \varepsilon(\sigma^{-1} \cdot \sigma) = \varepsilon(e) = 1 \Rightarrow \varepsilon(\sigma) = \varepsilon(\sigma^{-1})$

$$\Rightarrow \det {}^tA = \sum_{\sigma^{-1} \in S_n} \varepsilon(\sigma^{-1}) a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} \cdots a_{n\sigma^{-1}(n)} = \sum_{\tau \in S_n} \varepsilon(\tau) a_{1\tau(1)} a_{2\tau(2)} \cdots a_{n\tau(n)} = \det A.$$

Remark 5. Any property which refers to the rows of the determinant of a certain matrix A can also be written for the columns of A and any property valid for the columns of $\det A$ is also valid for its rows.

Proposition 6. If $n \in \mathbb{N}^*$ and $i \in \{1, \dots, n\}$ then

$$\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array}\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \dots & a_{i-1,n} \\ a_{i1}+a'_{i1} & a_{i2}+a'_{i2} & \dots & a_{in}+a'_{in} \\ a_{i+1,1} & a_{i+1,2} & \dots & a_{i+1,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}_{\text{the determinant of } A} + \underbrace{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \dots & a_{i-1,n} \\ a'_{i1} & a'_{i2} & \dots & a'_{in} \\ a_{i+1,1} & a_{i+1,2} & \dots & a_{i+1,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}_{\text{the determinant of } A'} . \quad i\text{'th row.}$$

This property can be generalized and restated for columns (homework).

Proof.

$$\sum_{\sigma\in S_n} \varepsilon(\sigma)\, a_{1\sigma(1)} \cdots a_{i-1\,\sigma(i-1)} (a_{i\sigma(i)} + a'_{i\sigma(i)})\, a_{i+1\,\sigma(i+1)} \cdots a_{n\sigma(n)} =$$

the distributivity of $\cdot$ w.r.t $+$ in K

$$= \sum_{\sigma\in S_n} \varepsilon(\sigma)\, a_{1\sigma(1)} \cdots a_{i-1\,\sigma(i-1)}\, a_{i\sigma(i)}\, a_{i+1\,\sigma(i+1)} \cdots a_{n\sigma(n)} +$$

$$+ \sum_{\sigma\in S_n} \varepsilon(\sigma)\, a_{1\sigma(1)} \cdots a_{i-1\,\sigma(i-1)}\, a'_{i\sigma(i)}\, a_{i+1\,\sigma(i+1)} \cdots a_{n\sigma(n)} =$$

$$= \det A + \det A'.$$

□

For the next part of the section, we consider a field $(K, +, \cdot)$, $n \in \mathbb{N}$, $n \geq 2$ and $A = (a_{ij}) \in M_n(K)$.

Proposition 7. If the matrix B results from A by multiplying each element of a row (column) of A by $\alpha \in K$ then $\det B = \alpha \det A$.

Proof.

$$\det B = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ \alpha a_{i1} & \alpha a_{i2} & \dots & \alpha a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} =$$

$$= \sum_{\sigma\in S_n} \varepsilon(\sigma)\, a_{1\sigma(1)} \cdots a_{i-1\,\sigma(i-1)} (\alpha a_{i\sigma(i)})\, a_{i+1\,\sigma(i+1)} \cdots a_{n\sigma(n)} =$$

$$= \alpha \sum_{\sigma\in S_n} \varepsilon(\sigma)\, a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)} = \alpha \cdot \det A.$$

□

Proposition 8. If all the elements of a row (column) of A are 0, then $\det A = 0$.

Proof. Let us consider that that the i'th row of A consists only of 0.

$\Longrightarrow$ $\det A$ is a sum of products such that $\underbrace{\text{each product}}_{=0}$ contains exactly one elem. from the i'th row

$\Longrightarrow$ each product of the sum $\det A$ is 0 $\Longrightarrow$ $\det A = 0$. ☐

Proposition 9. If B results from A by switching two rows (columns) of A then $\det B = -\det A$.

Proof. Let us take $1 \le i < j \le n$ and

$$B = \begin{array}{cc} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} & \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \end{array}.$$

Let us consider $\underline{\tau = (ij)}$

$$\det B = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} \cdots a_{i\sigma(j)} \cdots a_{j\sigma(i)} \cdots a_{n\sigma(n)} =$$

$$\sigma(1) = \sigma(\tau(1)),\quad \sigma(j) = \sigma(\tau(i)),\quad \sigma(i) = \sigma(\tau(j)),\quad \sigma(n) = \sigma(\tau(n))$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1(\sigma\circ\tau)(1)} \cdots a_{i(\sigma\circ\tau)(i)} \cdots a_{j(\sigma\circ\tau)(j)} \cdots a_{n(\sigma\circ\tau)(n)}$$

The map $S_n \to S_n$, $\sigma \mapsto \sigma \circ \underline{\tau}$ is bijective. (home-work).

$$-\varepsilon(\sigma) = \varepsilon(\sigma) \cdot \underbrace{\varepsilon(\tau)}_{-1} = \varepsilon(\sigma \circ \tau).$$

$$\det B = -\sum_{\sigma\circ\tau \in S_n} \varepsilon(\sigma\circ\tau) a_{1(\sigma\circ\tau)(1)} \cdots a_{i(\sigma\circ\tau)(i)} \cdots a_{j(\sigma\circ\tau)(j)} \cdots a_{n(\sigma\circ\tau)(n)} = -\det A$$

☐

$\longrightarrow$ **Proposition 10.** If A has two equal rows (columns) then $\det A = 0$.

Proof.

$$A = \begin{array}{cc} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ a_{n1} & a_{2n} & \dots & a_{nn} \end{pmatrix} & \begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} \end{array}$$

$1 \le i < j \le n$

I) If $1+1 \ne 0$ in K.

Switching r_i with r_j we get A, hence $\det A = -\det A$.

$\Longrightarrow$ $\det A + \det A = 0$ $\Longrightarrow$ $\det A \cdot \underbrace{(1+1)}_{\ne 0} = 0$ $\Longrightarrow$ $\det A = 0$

→ II) If $1+1=0$ in K. Then:

$\forall a \in K$, $a+a = a\underbrace{(1+1)}_{=0} = a \cdot 0 = 0 \implies a = -a$.

$$\implies \det A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

$$A = \begin{array}{c} \\ \left(\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ \cdots & & a_{ik} & \cdots & a_{il} & \cdots & \\ & & \shortparallel & & \shortparallel & & \\ \cdots & & a_{jk} & \cdots & a_{jl} & \cdots & \\ & & \vdots & & \vdots & & \end{array}\right) \\ \end{array} \begin{array}{c} i \\ \\ j \end{array}$$

$$\begin{array}{ccc} & k & l \end{array}$$

$1 \leq k < l \leq n$

$r_i = r_j$

$$\det A = \sum_{1 \leq k < l \leq n} \underbrace{\left(a_{ik} a_{jl} + \overset{a_{jl}}{\overset{\shortparallel}{a_{il}}}\, \overset{a_{jl}}{\overset{\shortparallel}{a_{jk}}}\right)}_{=0} \cdot S_{ijkl} = 0$$

↖ the sum of all the products with $n-2$ factors taken from all the other rows and columns except for rows i, j and columns k, l such that there are no 2 element from the same row or the same column.

□

Let us denote by $r_1, r_2, \cdots, r_n$ the rows and by $c_1, c_2, \cdots, c_n$ the columns of A. We say that **the rows (columns)** i **and** j ($i, j \in \{1, \ldots, n\}$, $i \neq j$) **are proportional** if there exists $\alpha \in K$ such that all the elements of a row (column) are the elements of the other one multiplied by α. We write, correspondingly, $\cancel{l_i = \alpha l_j}$ or $\cancel{l_j = \alpha l_i}$ or $c_i = \alpha c_j$ or $c_j = \alpha c_i$.

$r_i = \alpha r_j$ $\quad r_j = \alpha r_i$

Corollary 11. If A has two proportional rows (columns) then $\det A = 0$.

$r_j = \alpha r_i \; (i < j)$

$$\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & - & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \overset{P7}{=} \alpha \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & - & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \overset{P10}{=} 0.$$

→ **Definition 12.** We say that the i'th row of the matrix A **is a linear combination of (all) the other rows** ($i \in \{1, \ldots, n\}$) if there exists $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in K$ such that

$$a_{ij} = \alpha_1 a_{1j} + \cdots + \alpha_{i-1} a_{i-1,j} + \alpha_{i+1} a_{i+1,j} + \cdots + \alpha_n a_{nj}, \; \forall j \in \{1, \ldots, n\}. \leftarrow$$

We write $\quad r_i = \alpha_1 r_1 + \cdots + \alpha_{i-1} r_{i-1} + \alpha_{i+1} r_{i+1} + \cdots + \alpha_n r_n$

$$\cancel{l_i = \alpha_1 l_1 + \cdots + \alpha_{i-1} l_{i-1} + \alpha_{i+1} l_{i+1} + \cdots + \alpha_n l_n}.$$

An analogous definition can be given for columns (homework). ←

The property from the previous corollary can be generalized as follows:

→ **Corollary 13.** If a row (column) of A is a linear combination of the other rows (columns) then $\det A = 0$.

This results, straightforward from Proposition 6 and Corollary 11. (home-work)

Corollary 14. If the matrix B results from A by adding the i'th row (column) multiplied by $\alpha \in K$ to the j'th one ($i \neq j$) then $\det B = \det A$.

$$\det B = \begin{array}{c|c} \begin{vmatrix} \cdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \cdots & & & \\ a_{j1}+\alpha a_{i1} & a_{j2}+a_{i2} & \dots & a_{jn}+\alpha a_{in} \\ \cdots & & & \end{vmatrix} & \begin{matrix} i \\ \vdots \\ j \end{matrix} \end{array} =$$

$$= \underbrace{\begin{vmatrix} \cdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \cdots & & & \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \cdots & & & \end{vmatrix}}_{= \det A} + \underbrace{\begin{vmatrix} \cdots & & & \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \cdots & & & \\ \alpha a_{i1} & \alpha a_{i2} & \dots & \alpha a_{in} \\ \cdots & & & \end{vmatrix}}_{=0} \begin{matrix} i \\ j \end{matrix} = \det A.$$

Definition 15. Let $A = (a_{ij}) \in M_n(K)$, $n \geq 2$ and $i, j \in \{1, \dots, n\}$. Let $A_{ij} \in M_{n-1}(K)$ be the matrix resulted from A by eliminating the i'th row and the j'th column (i.e. the row and the column of a_{ij}). The determinant

$$d_{ij} = \det A_{ij}$$

is called **the minor of** a_{ij} and

$$\alpha_{ij} = (-1)^{i+j} d_{ij}$$

is called **the cofactor of** a_{ij}.

Then:

Theorem 16. (the cofactor expansion of $\det A$ along the i'th row)

$$\det(A) = a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in}, \ \forall i \in \{1, \dots, n\}.$$

Proof. Let us denote
(optional)

$$S_i = a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in}. \qquad (*)$$

(A) For $i = 1$, we have $S_1 = a_{11}\alpha_{11} + a_{12}\alpha_{12} + \cdots + a_{1n}\alpha_{1n}$. Let us consider the term $a_{11}\alpha_{11} = a_{11}d_{11}$. We notice that d_{11} is the sum of all the products of the form

$$a_{2k_2}a_{3k_3}\cdots a_{nk_n} \ \overset{\text{with}}{\cancel{\text{cu}}} \ \{k_2, \dots, k_n\} = \{2, \dots, n\},$$

and each term has the sign $(-1)^{Inv\,\tau}$ where $\tau = \begin{pmatrix} 2 & 3 & \dots & n \\ k_2 & k_3 & \dots & k_n \end{pmatrix}$. Each term of S_1 which contains a_{11} comes from $a_{11}\alpha_{11}$. Therefore, these terms are the products

$$(-1)^{Inv\,\tau} a_{11}a_{2k_2}a_{3k_3}\cdots a_{nk_n}.$$

On the other side, the terms of $\det A$ which contain a_{11} are (all) the products

$$a_{11}a_{2k_2}a_{3k_3}\cdots a_{nk_n} \ \overset{\text{with}}{\cancel{\text{cu}}} \ \{k_2, \dots, k_n\} = \{2, \dots, n\},$$

and the sign of each such term is $(-1)^{Inv\,\sigma}$ with $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & k_2 & k_3 & \dots & k_n \end{pmatrix}$.

Since $1 < k_2$, …, $1 < k_n$, we have $Inv\,\sigma = Inv\,\tau$ thus the terms which contain a_{11} are the same in S_1 and $\det A$ (and when we say the same we refer, of course, to the fact that they have the same signs in the both sums).

(B) Let us consider the general case. Let $i, j \in \{1, \dots, n\}$, and let us take the term

$$a_{ij}\alpha_{ij} = (-1)^{i+j} a_{ij} d_{ij}$$

from $(*)$. This term provides us with all the (products which are) terms of S_i which contain a_{ij}. On the other side, let us rewrite $\det A$ in the following way: by successively permuting adjacent rows, we bringa_{ij} on the first row, then, by permuting adjacent columns, we bring it in the position $(1, 1)$. Let us denote by D the resulted determinant. Since we applied i row switches and j column switches this way, we have

$$\det A = (-1)^{i+j} D.$$

Based on this equality, all the terms of $\det A$ containing a_{ij} result from D as in (A). As we already saw the element which lays in the $(1, 1)$ position of D is a_{ij}; from the way D occured, one deduces that its minor d_{ij}, and its cofactor is $(-1)^{1+1} d_{ij} = d_{ij}$. Therefore, the terms which contain a_{ij} are the same as in

$$(-1)^{i+j} a_{ij} d_{ij} = a_{ij}\alpha_{ij},$$

hence they are exactly the terms of S_i which contain a_{ij}.

We also notice that S_i has n terms and each such term is a sum of $(n-1)!$ products of elements of A (each one considered with the corresponding sign), thus S_i has $(n-1)!n = n!$ terms which are exactly the terms of $\det A$. This remark completes proof. □

We also have:

Teorema 16'. (the cofactor expansion of $\det(A)$ along the j'th column)

$$\det A = a_{1j}\alpha_{1j} + a_{2j}\alpha_{2j} + \dots + a_{nj}\alpha_{nj}, \ \forall j \in \{1, \dots, n\}.$$

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Corollary 17. If $i, k \in \{1, \dots, n\}$, $i \neq k$, then

$$a_{i1}\alpha_{k1} + a_{i2}\alpha_{k2} + \dots + a_{in}\alpha_{kn} = 0.$$

Also, if $j, k \in \{1, \dots, n\}$, $j \neq k$ then

$$a_{1j}\alpha_{1k} + a_{2j}\alpha_{2k} + \dots + a_{nj}\alpha_{nk} = 0.$$

Corollary 18. If $d = \det A \neq 0$ then A is a unit of the ring $M_n(K)$ and

$$A^{-1} = d^{-1} \cdot A^*,$$

where A^* is the matrix

$$A^* = {}^t(\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{pmatrix}$$

(called **the adjugate of** A).

Remark 19. We will see later that the converse of the previous statement is also valid, i.e. *if* A *is invertible then* $\det A \neq 0$.

Corollary 20. **(Cramer)** Let us consider the following system with n equations with n unknowns

$$(S) \qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}, \ a_{ij}, b_i \in K \ (i, j \in \{1, \dots, n\}).$$

Denote by d the determinant $d = \det A$ of $A = (a_{ij}) \in M_n(K)$ and by d_j the determinant of the matrix resulted from A by replacing the j'th column by

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If $d \neq 0$ then (S) has a unique solution. This solution is given by the equalities

$$x_i = d_i \cdot d^{-1}, \ i = 1, \dots, n.$$