Analytic Geometry

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November 15, 2021

P_0 is a point on the line.

We saw the following ways in which one can describe a line in the plane:

- As a vector equation: $\overline{OP} = \overline{OP_0} + t \cdot \overline{v}$, $t \in \mathbb{R}$.
- As two parametric equations:

$$\begin{cases} x = x_0 + t \cdot a \\ y = y_0 + t \cdot b \end{cases}$$, where $P_0(x_0, y_0)$ and $\overline{v}(a, b)$. $t \in \mathbb{R}$.

- Via a symmetric equation: $\frac{x - x_0}{a} = \frac{y - y_0}{b}$ for some
- A general equation: $Ax + By + C = 0$, $A, B, C \in \mathbb{R}$.
- A reduced equation:

$y = mx + n$, where $m, n \in \mathbb{R}$.

Intersection of two lines

Let $d_1 : a_1x + b_1y + c_1 = 0$ and $d_2 : a_2x + b_2y + c_2 = 0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

A

$$\begin{pmatrix} a_1 & b_1 & -c_1 \\ a_2 & b_2 & -c_2 \end{pmatrix}$$

$\overline{A}$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, the system has a unique solution (x_0, y_0) and the lines have a unique intersection point $P_0(x_0, y_0)$. They are *secant*.

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has infinitely many solutions, and the lines coincide. They are *identical*.

not parallel lines

If $d_i : a_i x + b_i y + c_i = 0$, $i = \overline{1,3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{array}{cc} A & \overline{A} \\ \end{array}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{1}$$

If d_1, d_2, d_3 are concurrent, then

$$\begin{cases} a_1 x + b_1 y + c_1 = 0 \\ a_2 x + b_2 y + c_2 = 0 \\ a_3 x + b_2 y + c_3 = 0 \end{cases}$$ has a solution $(x,y) \in \mathbb{R}^2$. $\Leftrightarrow$ $\operatorname{rank}(A) = \operatorname{rank}(\overline{A})$

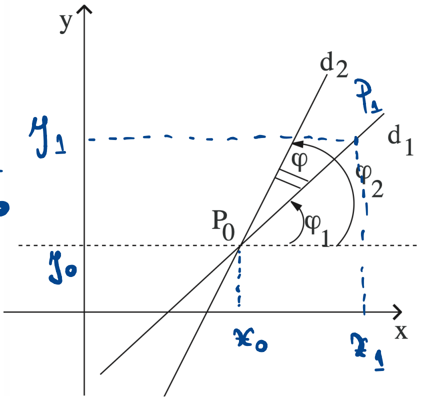
$\therefore \operatorname{rank}(\overline{A}) \leq 2$. Hence $\det(\overline{A}) = 0$.

The angle between two lines

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1 x + n_1 \quad \text{and} \quad d_2 : y = m_2 x + n_2.$$

$$\tan \varphi_1 = \frac{y_1 - y_0}{x_1 - x_0} = m_1.$$



$$\tan \varphi_2 = m_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$.

One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that

$\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.

The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan\varphi = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{2}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan\varphi = 0$, therefore

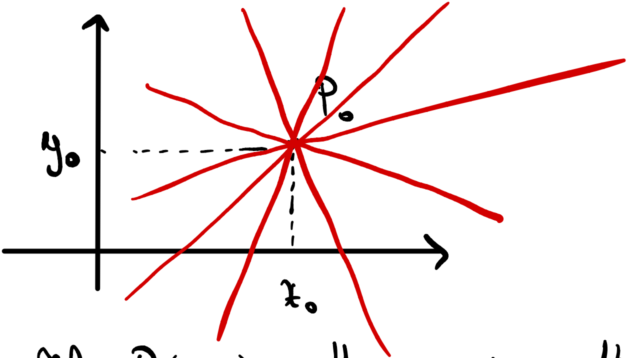
$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{3}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \tag{4}$$

A bundle of lines

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.



Example: If $P(0,0)$ is the origin. How do lines that pass through P look like?

$$\{Ax + By = 0 \mid A, B \in \mathbb{R},\ A^2 + B^2 > 0\}.$$

In general, given $P(x_0, y_0)$.

$$\{A \cdot (x - x_0) + B \cdot (y - y_0) = 0 \mid A, B \in \mathbb{R} \text{ and } A^2 + B^2 > 0\}.$$ consists of all lines that pass through P_0.

Remark: If $B \neq 0$, then

$$y - y_0 = s \cdot (x - x_0), \text{ where } s \in \mathbb{R}.\qquad \left(\frac{A}{B}\right)$$

The bundle is $\{x = x_0\} \cup \{y - y_0 = s(x - x_0) \mid s \in \mathbb{R}\}$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1x + b_1y + c_1 = 0 \\ d_2 : a_2x + b_2y + c_2 = 0 \end{cases},$$

supposed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \quad (5)$$

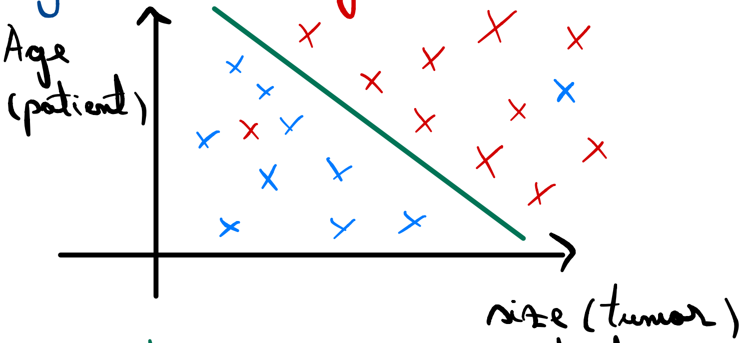
Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (or, respectively except d_2).

$t = \frac{s}{r}$.

$\{a_2x + b_2y + c_2 = 0\} \cup \{a_1x + b_1y + c_1 + t\cdot(a_2x + b_2y + c_2) = 0 \mid t \in \mathbb{R}\}.$

An interlude, if time permits...

Problem: Decide whether a tumor is benign or malignant.



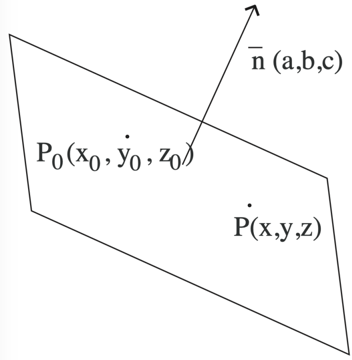
$$ax + by = c$$

New patient:

$$a \cdot x_p + b \cdot y_p \overset{?}{\underset{?}{\gtrless}} c$$

The analytic representation of planes in space

- Recall that if we endow the 3-dimensional Euclidean space $\mathcal{E}_3$ with a rectangular system of coordinates $Oxyz$, a point $P \in \mathcal{E}_3$ is characterized by three real numbers, the coordinates of the point, $P(x, y, z)$



- A plane π in the 3-dimensional space can be uniquely determined by specifying a point $P_0(x_0, y_0, z_0)$ in the plane and a nonzero vector $\overline{n}(a, b, c)$, orthogonal to the plane. $\overline{n}$ is called the *normal vector* to the plane π.
- An arbitrary point $P(x, y, z)$ is contained into the plane π if and only if

$$\overline{n} \perp \overline{P_0P},$$

or

$$\overline{n} \cdot \overline{P_0P} = 0.$$

- But $\overline{P_0P}(x - x_0, y - y_0, z - z_0)$ and one obtains the *normal* equation of the plane π containing the point $P_0(x_0, y_0, z_0)$ and of normal vector $\overline{n}(a, b, c)$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{6}$$

Remark: The equation (6) can be written in the form $ax + by + cz + d = 0$.

Theorem

Given $a, b, c, d \in \mathbb{R}$, with $a^2 + b^2 + c^2 > 0$, the equation

$$\pi: \; ax + by + cz + d = 0 \tag{7}$$

describes a plane in $\mathcal{E}_3$. This plane has $\overline{n}(a, b, c)$ as a normal vector.

Short proof: Let's take $P_0, P_1, P_2 \in \pi$.

$$\overline{P_0P_1} \cdot \overline{n} = \overline{P_0P_2} \cdot \overline{n} = 0$$

$$\left(a(x_1 - x_0) + b \cdot (y_1 - y_0) + c(z_1 - z_0) = 0 \right)$$

- The equation $ax + by + cz + d = 0$ with $(a, b, c) \neq (0, 0, 0)$ is sometimes referred to as the "general equation" of the plane.
- Given a fixed point O in the 3-space, any point P is characterized by its position vector $\overline{r}_P = \overline{OP}$.

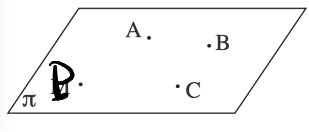
Theorem

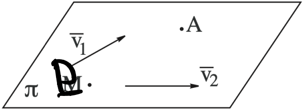
a) The vector equation of the plane π, determined by three noncollinear points A, B and C, is

$$\overline{r}_{P} = (1 - \alpha - \beta)\overline{r}_A + \alpha\overline{r}_B + \beta\overline{r}_C, \qquad \alpha, \beta \in \mathbb{R}. \tag{8}$$

b) The vector equation of the plane π, determined by a point A and two nonparallel directions $\overline{v}_1$ and $\overline{v}_2$ contained into the plane, is

$$\overline{r}_{P} = \overline{r}_A + \alpha\overline{v}_1 + \beta\overline{v}_2, \qquad \alpha, \beta \in \mathbb{R}. \tag{9}$$





a) $P \in \pi$ iff $\overline{AP}$, $\overline{AB}$ and $\overline{AC}$ are linearly dependent. So $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$\overline{AP} = \alpha \cdot \overline{AB} + \beta \cdot \overline{AC} \iff$$

$$\overline{r}_P - \overline{r}_A = \alpha \cdot (\overline{r}_B - \overline{r}_A) + \beta(\overline{r}_C - \overline{r}_A)$$

b) $P \in \pi$ iff. $\overline{AP}, \overline{v}_1, \overline{v}_2$ are linearly dependent. $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$\overline{AP} = \alpha \cdot \overline{v}_1 + \beta \cdot \overline{v}_2$$

$$\overline{r}_P = \overline{r}_A + \alpha \cdot \overline{v}_1 + \beta \cdot \overline{v}_2$$

$\blacksquare$

If the points A, B and C which determine the plane π are of coordinates $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$ and an arbitrary point of π is $M(x, y, z)$, then the equation (8) decomposes into three linear equations:

$$\begin{cases} x = (1-\alpha-\beta)x_A + \alpha x_B + \beta x_C \\ y = (1-\alpha-\beta)y_A + \alpha y_B + \beta y_C \\ z = (1-\alpha-\beta)z_A + \alpha z_B + \beta z_C \end{cases}.$$

This system must have solutions (α, β), so that

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0, \tag{10}$$

which is the analytic equation of the plane determined by three noncollinear points.

The points A, B, C and D are coplanar if and only if:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{11}$$

Replacing now, in (9), the vectors $\overline{v}_1(p_1, q_1, r_1)$ and $\overline{v}_2(p_2, q_2, r_2)$ and the points $A(x_A, y_A, z_A)$ and $M(x, y, z)$, the equation (9) becomes

$$\begin{cases} x = x_A + \alpha p_1 + \beta p_2 \\ y = y_A + \alpha q_1 + \beta q_2 \\ z = z_A + \alpha r_1 + \beta r_2 \end{cases}, \qquad \alpha, \beta \in \mathbb{R}, \tag{12}$$

and these are the parametric equations of the plane. Again, this system must have solutions (α, β), so that

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \tag{13}$$

which is the analytic equation of the plane determined by a point and two nonparallel directions.

The problem set for this week will be posted soon. Ideally you would think about it before the seminar.

Thank you very much for your attention!