COURSE 7

Elementary operations on matrices. Applications

Let K be a field, $m, n \in \mathbb{N}^*$ and $A = (a_{ij}) \in M_{m,n}(K)$.

Definition 1. By an **elementary operation on the rows (columns)** of a matrix we understand one of the following:
(I) the interchange of two rows (columns).
(II) multiplying a row (column) by a non-zero element $\alpha \in K$.
(III) multiplying a row (column) by an element $\alpha \in K$ (also called scalar) and adding the result to another row (column).

Application 1. Computing determinants.

Application 2. Computing the rank of a matrix.

Application 3. Solving systems of linear equations by using **Gauss elimination** algorithm.
Let K be a field and let us consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

over K with the augmented matrix $\overline{A}$. This algorithm is based on the fact that
(i) interchanging of two equations of (1),
(ii) multiplying an equation of (1) by a non-zero element $\alpha \in K$,
(iii) multiplying an equation of (1) by $\alpha \in K$ and adding the resulted equation to another one,
are operations which lead us to systems which are equivalent to (1). Since all these operations act on the coefficients and constant terms of the system, it is quite obvious that these operations can be performed as elementary row operations on the system augmented matrix.

Thus, the purpose of Gauss elimination is to successively use elementary operations on the rows of the augmented matrix $\overline{A}$ of (1) in order to bring it to an echelon form B. If we manage to do this, then B is the augmented matrix of an equivalent system. In some forms of Gauss elimination, and we plan to use this form, the purpose is to bring $\overline{A}$ to a trapezoidal form

$$B = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & \dots & a'_{1k} & a'_{1,k+1} & \dots & a'_{1n} & b'_1 \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2k} & a'_{2,k+1} & \dots & a'_{2n} & b'_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a'_{kk} & a'_{k,k+1} & \dots & a'_{kn} & b'_k \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

has k non-zero rows

Some information on the given system can be easily read from this form. E.g. the rank of $\overline{A}$ is (the rank of B which is) the number of the nonzero elements on the diagonal of B and these nonzero elements on the diagonal of B provide us with the main unknowns.

Remarks 2. a) Finding a trapezoidal form is not always possible by using only row elementary operations. Sometimes, we have to interchange two columns of the first n columns, hence columns corresponding to the matrix of a certain equivalent system. This is, obviously, allowed since this means that we commute the two corresponding terms in each equation of this system and this is possible based on the commutativity of the addition in K.

→ b) If, during this algorithm, one finds a row for which all the elements are 0, except for the last one, which is $a \in K^*$, then (1) is inconsistent since it is equivalent to a system which contains the equality $0 = a$ which is not possible. (the inconsistency case)

Assume that we brought A to the above mentioned trapezoidal form B. This means that (1) is equivalent to a system of the form

→ the side unknowns.

$$\begin{cases} a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1,k-1}x_{k-1} + a'_{1k}x_k + a'_{1,k+1}x_{k+1} + \cdots + a'_{1n}x_n = b'_1 \\ \qquad a'_{22}x_2 + \cdots + a'_{2,k-1}x_{k-1} + a'_{2k}x_k + a'_{2,k+1}x_{k+1} + \cdots + a'_{2n}x_n = b'_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \qquad\qquad a'_{k-1,k-1}x_{k-1} + a'_{k-1,k}x_k + a'_{k-1,k+1}x_{k+1} + \cdots + a'_{k-1,n}x_n = b'_{k-1} \\ \qquad\qquad\qquad a'_{kk}x_k + a'_{k,k+1}x_{k+1} + \cdots + a'_{kn}x_n = b'_k \end{cases}$$

(possibly with the unknowns succeeding in a different way, if we permuted columns) The main unknowns $x_1, \dots, x_k$ can be easily computed starting from the last equation of this system.

Remarks 3. a) A few more steps in Gauss elimination allow us to bring $\overline{A}$ by elementary row operations and, if necessary, by switching columns different from the last one to the following trapezoidal form

$$\begin{array}{c} \begin{array}{ccccccccc} x_{i_1} & x_{i_2} & & & x_{i_k} & & & & \end{array} \\ B' = \begin{pmatrix} a''_{11} & 0 & 0 & \dots & 0 & a''_{1,k+1} & \dots & a''_{1n} & b''_1 \\ 0 & a''_{22} & 0 & \dots & 0 & a''_{2,k+1} & \dots & a''_{2n} & b''_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a''_{kk} & a''_{k,k+1} & \dots & a''_{kn} & b''_k \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}$$

with $a''_{11}, a''_{22}, \dots, a''_{kk}$ non-zero (of course, this is possible only if the system (1) is consistent, otherwise, some non-zero elements may appear in the last column, bellow b''_k). One can easily notice the advantage we have when we form the equivalent system of (1) provided by B. This algorithm is known as **Gauss-Jordan elimination**.

$\exists\ \overbrace{r_1 - a'^{-1}_{22} a_{12} r_2}^{a_{22} \neq 0}$ $\dots$ $\Longrightarrow \begin{cases} a''_{11} x_{i_1} = \dots \\ \vdots \end{cases} \Longrightarrow \begin{cases} x_{i_1} = a''^{-1}_{11}(\dots \end{cases}$

(the way we apply row operations here is similar to the one described in Remark 8 b) (course 6))

I_k

b) Moreover, we can bring the augmented matrix of a consistent system to the following form:

$$
B'' = \begin{array}{c} \begin{array}{ccccc} x_{i_1} & x_{i_2} & & & \end{array} \\ \left(\begin{array}{ccccc|cccc}
1 & 0 & 0 & \dots & 0 & a'''_{1,k+1} & \dots & a'''_{1n} & b'''_1 \\
0 & 1 & 0 & \dots & 0 & a'''_{2,k+1} & \dots & a'''_{2n} & b'''_2 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 1 & a'''_{k,k+1} & \dots & a'''_{kn} & b'''_k \\
\hline
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0
\end{array}\right) \end{array}.
$$

Now, it is straightforward to express the main unknowns as linear combinations of the side unknowns.

$$B' \underset{\substack{a_{11}''^{-1}\cdot r_1 \\ a_{22}''^{-1}\cdot r_2 \\ \vdots}}{\sim} B'' \Longrightarrow \begin{array}{l} x_{i_1} = \dots \\ x_{i_2} = \dots \\ \vdots \end{array}$$

Application 4. Computing the inverse of a matrix: Let K be a field, $n \in \mathbb{N}^*$ and let us consider $A = (a_{ij}) \in M_n(K)$ a matrix with $d = \det A \neq 0$. We ~~rmind~~ remind that the matrix equation

$$
A^{-1}\cdot \Big|\ A \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \qquad \text{Cramer system} \tag{2}
$$

is an equivalent form of a (consistent) Cramer system and that its unique solution is

$$
\begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = A^{-1} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}.
$$

the first column of I_n

Let us take $j = 1$ and $\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then $\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$ is the first column of the matrix A^{-1},

the first column of A^{-1}.

i.e.

$$
\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} d^{-1}\alpha_{11} \\ d^{-1}\alpha_{12} \\ \vdots \\ d^{-1}\alpha_{1n} \end{pmatrix}
$$

(we remind that in our previous courses we denoted by α_{ij} the cofactor of a_{ij}). Of course,

$$
\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}}_{= I_n} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = I_n \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}
$$

Remark 36)

By means of Gauss-Jordan algorithm, one deduces that the augmented matrix of the system (2) can be brought by elementary row operations to the following form

Remark 8c)
(Course 6)

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & d^{-1}\alpha_{11} \\ 0 & 1 & 0 & \dots & 0 & d^{-1}\alpha_{12} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & d^{-1}\alpha_{1n} \end{pmatrix}.$$

the second column of I_n

Taking, successively, $j = 2$ and $\begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then $j = 3$ and $\begin{pmatrix} b_{13} \\ b_{23} \\ \vdots \\ b_{n3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,..., $j = n$ and $\begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, we form the corresponding systems (2) and we use the Gauss-Jordan algorithm to solve them. We perform exactly the same elementary operations as in the case $j = 1$ on the rows of each augmented matrix of a resulted system in order to bring the system matrix to the form I_n. We get:

the second column of A^{-1}

$$\begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{21} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{22} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{2n} \end{pmatrix}, \begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{31} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{32} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{3n} \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{n1} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{n2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{nn} \end{pmatrix},$$

respectively. The constant terms column and, consequently, the solution of each system we solved is the column 2 of A^{-1}, column 3 of A^{-1},..., column n of A^{-1}, respectively.

Since we performed the same row operations on each of the previously mentioned n systems, we can solve all of them using the same algorithm. This way one can finds an algorithm for computing the inverse of the matrix A: we start from the $n \times 2n$ matrix resulted by attaching the matrices A and I_n

$$(A \mid I_n) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array}\right) \in M_{n,2n}(K)$$

and we perform successive elementary row operations (and only row operations) on this matrix and on the matrices successively resulted from this in order to transform the left size block into I_n. Remark 8 c) of the previous course ensures us that this is possible (if and only if A is invertible). The resulted matrix is:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & d^{-1}\alpha_{11} & d^{-1}\alpha_{21} & \dots & d^{-1}\alpha_{n1} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{12} & d^{-1}\alpha_{22} & \dots & d^{-1}\alpha_{n2} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{1n} & d^{-1}\alpha_{2n} & \dots & d^{-1}\alpha_{nn} \end{array}\right) = (I_n \mid A^{-1})$$

Thus, the right side block of the resulted matrix is the exactly the inverse matrix of A.

Definition 4. A square matrix resulted from the identi~~last~~ty matrix after performing only one elementary operation is called **elementary matrix**.

I

Remarks 5. (and examples ...)

a) The elementary matrices resulted by switching rows (columns):

$$I_n \quad \begin{array}{c@{\;}c} \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \cancel{las}t0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ & \begin{array}{cccccccc} & & & i & & j & & \end{array} \end{array} \qquad \underbrace{r_i \leftrightarrow r_j}_{(c_i \leftrightarrow c_j)} \qquad \begin{array}{c} \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{cccccccc} & & & i & & j & & \end{array} \end{array}$$

have the determinant -1.

b) The elementary matrices resulted by multiplying a row (column) with $\alpha \in K^*$:

$$\begin{array}{c@{\;}c} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ & \begin{array}{cccccccc} & & & i & & j & & \end{array} \end{array} \qquad \underbrace{\alpha r_i}_{(\alpha c_i)} \qquad \begin{array}{c} \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & \alpha & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{cccccccc} & & & i & & j & & \end{array} \end{array}$$

have the determinant α.

c) The elementary matrices resulted by multiplying a row (column) by $\alpha \in K$ and adding the result to another row (column):

$$\begin{array}{c@{\;}c} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ & \begin{array}{cccccccc} & & & i & & j & & \end{array} \end{array} \qquad \underbrace{r_i + \alpha r_j}_{(c_j + \alpha c_i)} \qquad \begin{array}{c} \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \dots & \alpha & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{cccccccc} & & & i & & j & & \end{array} \end{array}$$

have the determinant 1.

For the elementary matrices resulted by switching rows (columns), we have:

$$\begin{array}{cc} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ & \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array} \begin{array}{c} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array} = \begin{array}{c} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array}$$

For the elementary matrices resulted by multiplying a row (column) with $\alpha \in K^*$, we have:

$$\begin{array}{cc} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \alpha & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ & \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array} \begin{array}{c} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \alpha^{-1} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array} = \begin{array}{c} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array}$$

For the elementary matrices resulted by multiplying a row (column) by $\alpha \in K$ and adding the result to another row (column), we have:

$$\begin{array}{cc} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & \alpha & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ & \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array} \begin{array}{c} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & -\alpha & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array} = \begin{array}{c} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \\ \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array}$$

Therefore, we can state the following:

Lemma 6. The inverse of an elementary matrix is also an elementary matrix.

Lemma 7. Let $m, n \in \mathbb{N}^*$. Any elementary operation on a matrix $A = (a_{ij}) \in M_{m,n}(K)$ is the result of the multiplication of A with an elementary matrix. More precisely, any elementary operation on the rows (columns) of A results by multiplying A on the left (right) side with the elementary matrix resulted by performing the same elementary operation on I_m (I_n, respectively).

Proof. We check this property for rows. For columns — HOMEWORK.

Let us switch the rows i and j of I_m and let us multiply the resulted elementary matrix with

A. The matrix

$$\begin{array}{ccc}
 & & A \\
\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 0 & \dots & 1 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 1 & \dots & 0 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 0 & \dots & 0 & \dots & 1
\end{pmatrix} &
\begin{pmatrix}
a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn}
\end{pmatrix} \\
 & \begin{array}{ccccccc} & & i & & j & & \end{array} & \begin{array}{ccccccc} & & i & & j & & \end{array}
\end{array}$$

is

$$\begin{array}{cc}
\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} &
\begin{pmatrix}
a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn}
\end{pmatrix}
\end{array}$$

which is exactly the matrix resulted from A by switching the rows i and j.

Let $\alpha \in K^*$, let us multiply the i'th row of I_m by α and let us multiply the resulted elementary matrix with A. The matrix

$$\begin{array}{ccc}
\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & \alpha & \dots & 0 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 0 & \dots & 1 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 0 & \dots & 0 & \dots & 1
\end{pmatrix} &
\begin{pmatrix}
a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn}
\end{pmatrix}
\end{array}$$

is

$$\begin{array}{cc}
\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} &
\begin{pmatrix}
a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\alpha a_{i1} & \dots & \alpha a_{ii} & \dots & \alpha a_{ij} & \dots & \alpha a_{in} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn}
\end{pmatrix}
\end{array}$$

which is exactly the matrix resulted from A by multiplying the i'th row by α.

Let $\alpha \in K$, let us take the elementary matrix that we get from I_m after multiplying the j'th row by α and adding the result to the i'th row, and let us multiply this elementary matrix with

A. The matrix

$$
\begin{array}{c}
\\ \\ i \\ \\ j \\ \\ \\
\end{array}
\begin{pmatrix}
1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 1 & \dots & \alpha & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 0 & \dots & 1 & \dots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \dots & 0 & \dots & 0 & \dots & 1
\end{pmatrix}
\begin{pmatrix}
a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn}
\end{pmatrix}
$$

is

$$
\begin{array}{c}
\\ \\ i \\ \\ j \\ \\ \\
\end{array}
\begin{pmatrix}
a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i1}+\alpha a_{j1} & \dots & a_{ii}+\alpha a_{ji} & \dots & a_{ij}+\alpha a_{jj} & \dots & a_{in}+\alpha a_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1} & \dots & a_{mi} & \dots & a_{mj} & \dots & a_{mn}
\end{pmatrix}
$$

which is exactly the matrix that we get from A after multiplying the j'th row by α and adding the result to the i'th row. □

Corollary 8. Any invertible matrix is a product of elementary matrices. ✓

Proof: Let A be an invertible matrix, $A \in M_n(K)$.

$A \underset{\text{Appl. it}}{\sim} I_n \overset{C7}{\Longrightarrow} \exists\, E_1, E_2, \dots, E_q \in M_n(K)$ elem. matrices

performing row-operations s.t. $E_q(\dots(E_2(E_1A))\dots) = I_n \Rightarrow$

$\Rightarrow (E_q \dots E_2E_1)A = I_n \,\big|\cdot A^{-1} \Rightarrow A^{-1} = E_q \dots E_2 E_1 \Rightarrow$

$\Rightarrow A = (E_q \cdot \dots \cdot E_2 E_1)^{-1} = E_1^{-1} \cdot E_2^{-1} \cdot \dots \cdot E_q^{-1}$

elem. matrices (L6).

Theorem 9. Let $n \in \mathbb{N}^*$. For any matrices $A, B \in M_n(K)$ we have $\det(AB) = \det A \cdot \det B$.

Proof. □

I) A is not invertible $\Rightarrow \det A = 0$ ✓

We intend to prove that AB is not invertible $\Rightarrow \det(AB) = 0 =$

$= 0 \cdot \det B$.

Assume by contradiction that AB is invertible $\Rightarrow$

$\Rightarrow \exists\, C \in M_n(K)$ s.t. $(AB)C = I_n \Rightarrow A\cdot\underbrace{(BC)}_{=D \in M_n(K)} = I_n$

Let $D = BC$ and the system

$$A \cdot \Big/ D \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \underbrace{(AD)}_{=I_n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

The system $D \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is consistent with a unique solution homogeneous system.

$\Rightarrow \operatorname{rank} D = n \Rightarrow \det D \neq 0 \Rightarrow D$ is invertible (D^{-1} its inverse)

$AD = I_n \,/\cdot D^{-1} \Rightarrow A = D^{-1}$ is invertible, which contradicts the fact that A is not invertible.

Thus AB is not invertible.

II) A is invertible $\xRightarrow{CB} \exists\, E_1, E_2, \dots, E_q \in M_n(K)$ $(q \in \mathbb{N}^*)$ elementary matrices s.t.

$A = E_1 E_2 \dots E_q \Rightarrow AB = (E_1 E_2 \dots E_q) B \Leftrightarrow$

$\Leftrightarrow AB = E_1(E_2(\dots(E_q B))\dots)$. $(*)$

Homework: If E is an elementary matrix from $M_n(K)$ and $B \in M_n(K)$ then

$$\det(EB) = \underline{\det E} \cdot \det B = \det(B \cdot E).$$

(Hint: Use L7 and RJ).

$\det(AB) = \det E_1 \cdot \det(E_2(\dots(E_q B)\dots)) = \dots$

$= \underline{\det E_1 \cdot \det E_2 \cdot \dots \cdot \det E_q} \cdot \det B =$

$= \det(\underbrace{E_1 \dots E_q}_{=A}) \cdot \det B = \det A \cdot \det B.$