## REMARK

There are only 3 <u>clear</u> situations in which we can compute the sum:

- geometric series : $\sum_{n=1}^{\infty} q^{n-1} = \begin{cases} \infty & : q \geq 1 \quad \text{D.} \\ \frac{1}{1-q} & : 0 < |q| < 1 \quad \text{C.} \\ \nexists & : q \leq -1 \quad \text{D.} \end{cases}$

- telescopic series : $x_n = a_n - a_{n+1}$ $\quad \exists (a_n) \subseteq \mathbb{R}$

  If $\lim_{n \to \infty} a_n \in \overline{\mathbb{R}} \Rightarrow \exists \sum_{n=k}^{\infty} x_n = a_k - \lim_{n \to \infty} a_n$

  or

  $x_n = b_{n+1} - b_n$ $\quad \exists (b_n) \in \mathbb{R}$

  If $\lim_{n \to \infty} b_n \in \overline{\mathbb{R}} \Rightarrow \exists \sum_{n=k}^{\infty} x_n = \lim_{n \to \infty} - b_k$

- for SPT (series with positive terms) $\rightarrow$ D (it is $\infty$)

# Geometric series

A geometric series is a series of the type $\sum_{n\geq k} q^{n-1}$, $q \in \mathbb{R}$.

$$\sum_{n=1}^{\infty} q^{n-1} = \begin{cases} 0 & : q = 0 \\ \frac{1}{1-q} & : q \in (-1,1)\setminus\{0\} \\ +\infty & : q \geq 1 \\ \text{doesn't have a sum} & : q \leq -1 \end{cases}$$

Proof:

$q = 0$

In this case $s_n = 0$, $\forall n \geq 1$.
Therefore its limit is 0,
thus the sum of the series is 0.

$q \in (-1,1)\setminus\{0\}$

In this case,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1-0}{1-q} = \frac{1}{1-q}$$

$q \geq 1$

In this case,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1-\infty}{1-q} = \frac{-\infty}{1-q} = \infty \quad \text{because } 1-q < 0.$$

$q \leq -1$

In this case,
we notice that $\lim_{n\to\infty} q^n$ does not exist,
therefore the $\lim_{n\to\infty} s_n$ does not exist as well.

Thus, in this case the geometric series does not have a sum.

# Telescopic series

If the general term of degree $n$ of the series of real numbers $\sum_{n \geq m} x_n$ is defined as the difference of two succesive terms of a sequence of real numbers $(a_n)_{n \geq m}$

i.e. $x_n = a_n - a_{n+1}, \ \forall n \geq m$

then is called a telescopic series.

If the sequence $(a_n)_{n \geq m}$ has the limit

$$l = \lim_{n \to \infty} a_n$$

then the series $\sum_{n \geq m} x_n$ has a sum

$$\sum_{n=m}^{\infty} x_n = a_m - l$$

Proof:

We write the general term of degree $n$ of the sequence of partial sums of the series:

$$s_n = x_m + x_{m+1} + \ldots + x_n = a_m - a_{m+1} + a_{m+1} - a_{m+2} + \ldots + a_{n-1} - a_n + a_n - a_{n+1} =$$
$$= a_m - a_{n+1}$$

In conclusion,

$s_n = a_m - a_{n+1}, \quad \forall n \geq m$

We notice that $a_m$ is a constant, thus, due to the fact that the sequence $(a_n)_{n \geq m}$ has a limit, the sequence $(s_n)_{n \geq m}$ has a limit as well and it is:

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} (a_m - a_{n+1}) = a_m - \lim_{n \to \infty} a_{n+1} = a_m - l$$

Since the limit of the sequence of partial sums is the sum of the series, we reach the conclusion that

$$\sum_{n=m}^{\infty} x_n = a_m - l$$

# Series of real numbers - part 1

## Exercises

**Exercise 1:** Compute the sums for the following geometric series (if they exists):

$$a) \sum_{n\geq 3} \frac{7}{9^n}, \quad b) \sum_{n\geq 4} \frac{3^{n-3} + (-4)^{n+3}}{5^n}, \quad c) \sum_{n\geq 5} e^n, \quad d) \sum_{n\geq 2} \left(-\frac{1}{\pi}\right)^n \quad e) \sum_{n\geq 3} (-4)^n.$$

**Exercise 2:** Compute the sums of the following telescopic series:

$$a) \sum_{n\geq 1} \frac{1}{4n^2 - 1}, \quad b) \sum_{n\geq 1} \frac{1}{\sqrt{n} + \sqrt{n+1}}, \quad c) \sum_{n\geq 5} \frac{1}{n(n+1)(n+2)}$$

$$d) \sum_{n\geq 1} \ln\left(1 + \frac{1}{n}\right), \quad e) \sum_{n\geq 2} \frac{\ln\left(1 + \frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)}.$$