Analytic Geometry

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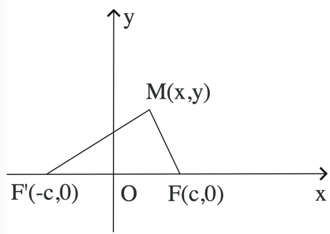
Recap... The ellipse

- An *ellipse* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant sum.

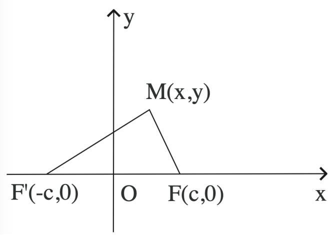


- The two fixed points are called the *foci* of the ellipse and the distance between the foci is the *focal distance*.

- Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$.
- If M is an arbitrary point of the ellipse, it must verify the condition

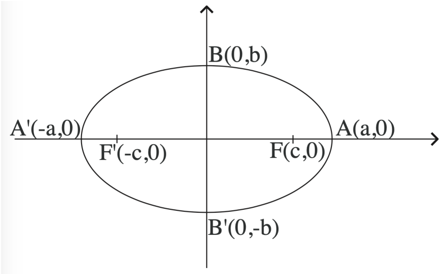
$$|MF| + |MF'| = 2a.$$

- One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c,0)$ and $F'(-c,0)$.



- Remark that, by triangle inequality, we have $|MF| + |MF'| > |FF'|$, hence $2a > 2c$.
- Let us determine the equation of an ellipse. Starting with the definition, $|MF| + |MF'| = 2a$, after a few slides of computations we found that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0. \tag{1}$$



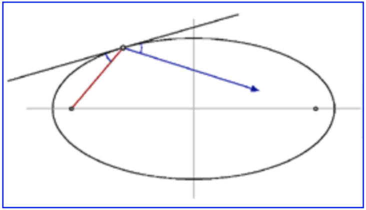
- In order to sketch the graph of the ellipse, remark that is it enough to represent the function

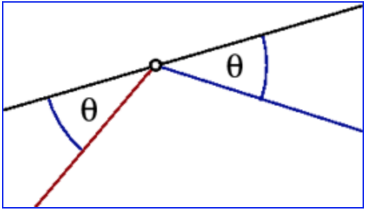
$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to Ox.

Applications

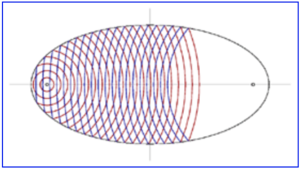
- Ellipses have an interesting reflective property

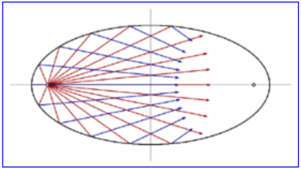




- This property affects both light and sound.

- Sound emanating from a focus in any direction will always travel along the focal lengths. And since the sum of these focal lengths is constant, it will always travels the same total distance. So, regardless of direction, the ray takes the same amount of time to leave one focus, reflect off of the ellipse and pass through the other focus (since the velocity of sound is the same). So when sound is emitted radially the reflections arrive at the other focus at the same time.





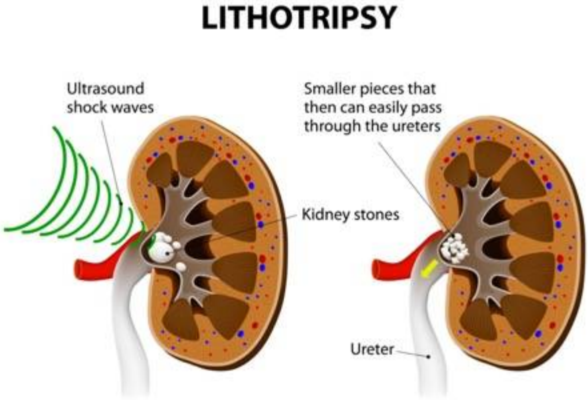
Ellipses in art... Whispering galleries

Figure 1: St Pauls' Cathedral

Figure 2: Whispering Gallery on the dome of St Pauls'

Figure 3: Orecchio Di Dionisio, Sicily

Medical applications

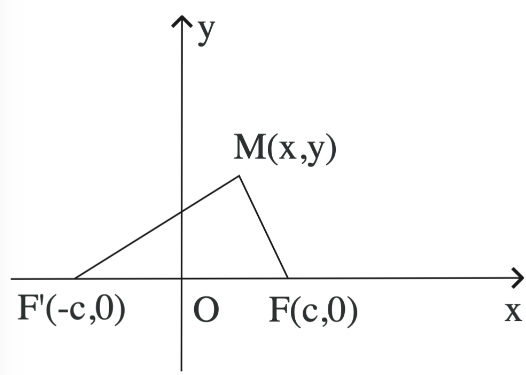


The hyperbola

- A *hyperbola* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant difference.
- The two fixed points are called the *foci* of the hyperbola, and the distance between the foci is the *focal distance*.
- Denote by F and F' the foci of the hyperbola and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

- Choose a Cartesian system of coordinates, having the center at the midpoint of the segment $[FF']$ and such that $F(c, 0)$, $F'(-c, 0)$.



- Remark: In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so $a < c$.

$M(x,y)$
$F(c,0)$, $F'(-c,0)$

- The metric relation $|MF| - |MF'| = \pm 2a$ becomes

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a,$$

or

$$\sqrt{(x-c)^2+y^2} = \pm 2a + \sqrt{(x+c)^2+y^2}.$$

- This is

$$x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2+y^2} + x^2 + 2cx + c^2 + y^2 \iff$$

$$\iff cx + a^2 = \pm a\sqrt{(x+c)^2+y^2} \iff$$

$$\iff c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \iff$$

$$\iff (c^2-a^2)x^2 - a^2y^2 - a^2(c^2-a^2) = 0. \quad \Big| : a^2(c^2-a^2)$$

- Denote $c^2 - a^2 = b^2$ (possible, since $c > a$) and one obtains the equation of the hyperbola

$b^2 = c^2 - a^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \qquad \Rightarrow \boxed{c^2 = a^2 + b^2} \tag{2}$$

- *Remark*: The equation (2) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Then, the coordinate axes are axes of symmetry for the hyperbola. Their intersection point is the *center* of the hyperbola.

- To sketch the graph of the hyperbola, is it enough to represent the function

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

by taking into account that the hyperbola is symmetrical with respect to Ox.

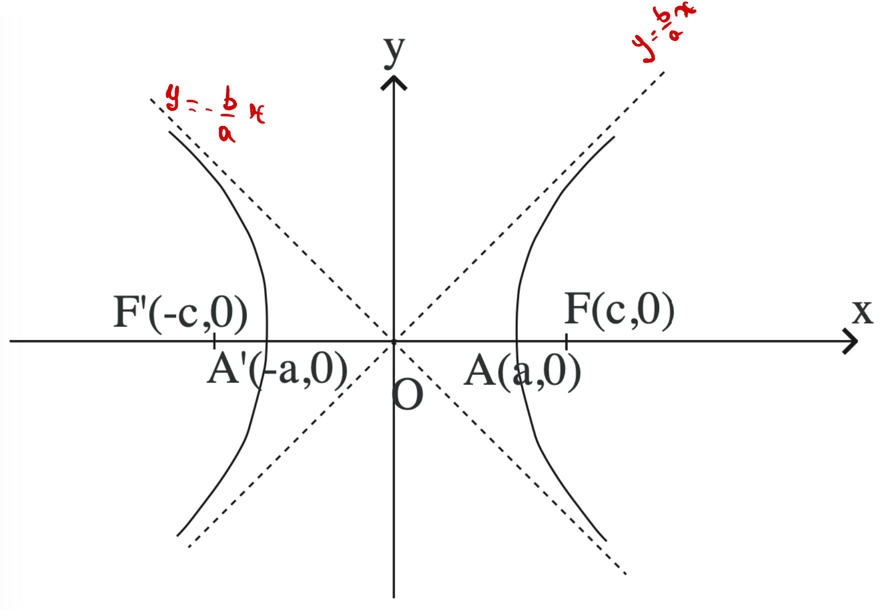
- Since $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \dfrac{b}{a}$ and $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = -\dfrac{b}{a}$, it follows that $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$ are asymptotes of f.

- One has, also,

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$				$-a$				a				∞
$f'(x)$	$-$	$-$	$-$	$-$	$\vert$	/	/	/	$\vert$	$+$	$+$	$+$	$+$
$f(x)$	∞		$\searrow$		$0\vert$	/	/	/	$\vert 0$		$\nearrow$		∞
$f''(x)$	$-$	$-$	$-$	$-$	$\vert$	/	/	/	$\vert$	$-$	$-$	$-$	$-$

The graph of the hyperbola



A few remarks

- If $a = b$, the equation of the hyperbola becomes $x^2 - y^2 = a^2$. In this case, the asymptotes are the bisectors of the system of coordinates and one deals with an *equilateral* hyperbola.
- As in the case of an ellipse, one can consider the hyperbola having the foci on Oy.
- The number $e = \dfrac{c}{a}$ is called the *eccentricity* of the hyperbola. Since $c > a$, then the eccentricity is always greater than 1.
- Moreover,

$$e^2 = \frac{c^2}{a^2} = 1 + \left(\frac{b}{a}\right)^2,$$

hence e gives informations about the shape of the hyperbola. For e closer to 1, the hyperbola has the branches closer to Ox.

Intersection of a Hyperbola and a Line

- Let $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ be a hyperbola and $d : y = mx + n$ be a line in $\mathcal{E}_2$. Their intersection is given by the system of equations

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases}.$$

- By substituting y in the first equation, one obtains

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0. \tag{3}$$

If $a^2m^2 - b^2 = 0$, (or $m = \pm\frac{b}{a}$), then the equation (3) becomes

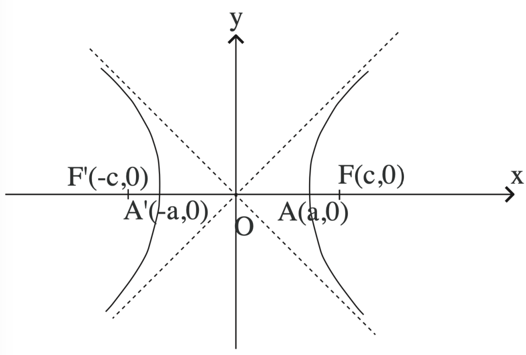
$$\pm 2bnx + a(n^2 + b^2) = 0.$$

- If $n = 0$, there are no solutions (this means, geometrically, that the two asymptotes do not intersect the hyperbola);
- If $n \neq 0$, there exists a unique solution (geometrically, a line d, which is parallel to one of the asymptotes, intersects the hyperbola at exactly one point);

If $a^2m^2 - b^2 \neq 0$, then the discriminant of the equation (3) is

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 - b^2)(n^2 + b^2)].$$

- If $\Delta < 0$, then the line does not intersect the hyperbola;
- If $\Delta = 0$, then the line is *tangent* to the hyperbola (they have a double intersection point);
- If $\Delta > 0$, then the line and the hyperbola have two intersection points.



The tangent to a hyperbola

The line $d : y = mx + n$ is tangent to the hyperbola $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ if the discriminant Δ of the equation

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0$$

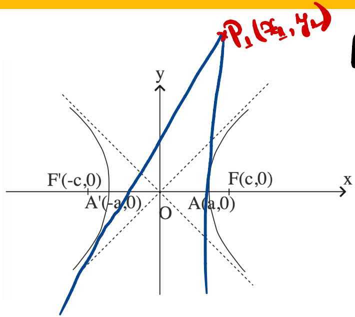
is zero, which is equivalent to $a^2m^2 - n^2 - b^2 = 0$.

- If $a^2m^2 - b^2 \geq 0$, i.e. $m \in \left(-\infty, -\dfrac{b}{a}\right] \cup \left[\dfrac{b}{a}, \infty\right)$, then $n = \pm\sqrt{a^2m^2 - b^2}$. The equations of the tangent lines to $\mathcal{H}$, having the angular coefficient m are

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{4}$$

- If $a^2m^2 - b^2 < 0$, there are no tangent lines to $\mathcal{H}$, of angular coefficient m.

The Tangent at a Point of the Hyperbola



$\ell: \; y - y_1 = m(x - x_1)$

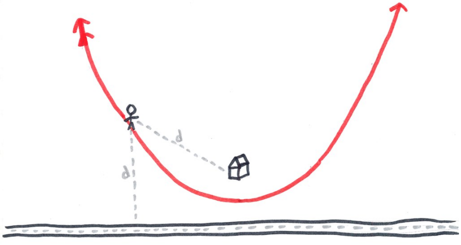
$\ell: \; y = mx \underbrace{- m x_1 + y_1}_{n}$

- One can prove, as in the case of the ellipse that, if $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ is a hyperbola, and $P_0(x_0, y_0)$ is a point of $\mathcal{H}$, then the equation of the tangent to $\mathcal{H}$ at P_0 is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} - 1 = 0. \tag{5}$$

The parabola

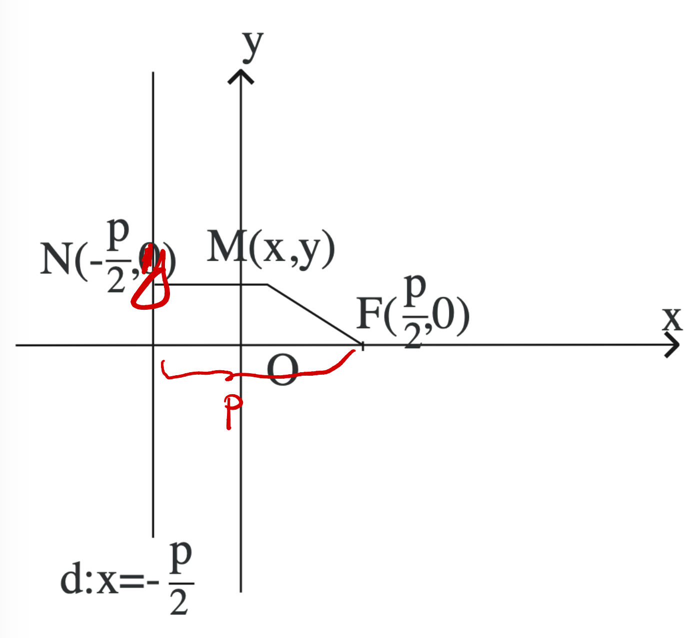
The *parabola* is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.



- The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.
- Consider a Cartesian system of coordinates xOy, in which $F\left(\frac{p}{2}, 0\right)$ and $d : x = -\frac{p}{2}$. If $M(x, y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on d.


y
N(-p/2, y)
M(x,y)
F(p/2,0)
x
O
p
d:x=-p/2

Thus, the coordinates of a point of the parabola verify

$(y-y)^2$

$$\sqrt{\left(x+\frac{p}{2}\right)^2+0}=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2} \Leftrightarrow$$

$$\Leftrightarrow \left(x+\frac{p}{2}\right)^2 = \left(x-\frac{p}{2}\right)^2 = y^2 \Leftrightarrow$$

$$\Leftrightarrow x^2+px+\frac{p^2}{4}=x^2-px+\frac{p^2}{4}+y^2,$$

and the equation of the parabola is

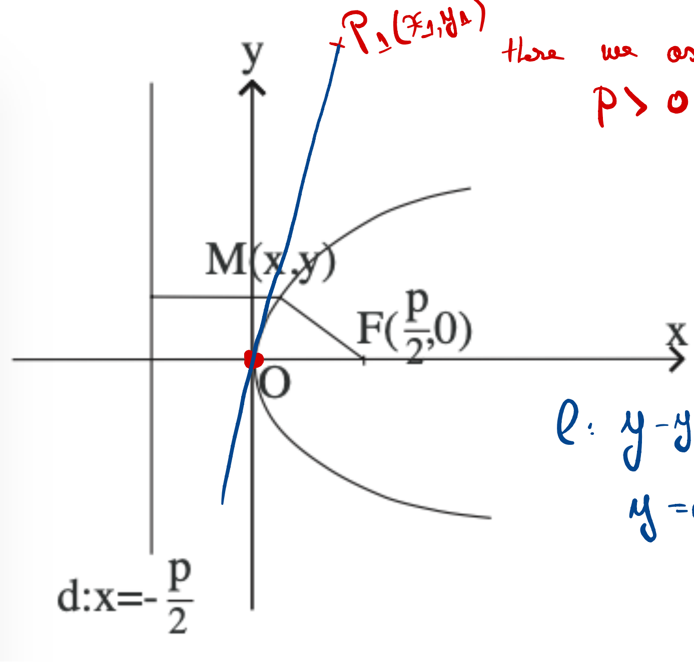
$$y^2 = 2px. \tag{6}$$

Remark: The equation (6) is equivalent to $y=\pm\sqrt{2px}$, so that the parabola is symmetrical with respect to Ox.

Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to Ox, one obtains the graph of the parabola. One has

$$f'(x) = \frac{p}{\sqrt{2px}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0				∞
$f'(x)$	$\vert$	$+$	$+$	$+$	$+$
$f(x)$	0		$\nearrow$		∞
$f''(x)$	$-$	$-$	$-$	$-$	$-$



here we assumed $p > 0$

$\ell: \ y - y_1 = m(x - x_1)$

$y = mx \underbrace{- m x_1 + y_1}_{n}$

Intersection of a Parabola and a Line

Let $\mathcal{P} : y^2 = 2px$ be a parabola, $d : y = mx + n$ $(m \neq 0)$ be a line and

$$\begin{cases} y^2 = 2px \\ y = mx + n \end{cases}$$

be the system determined by their equations.

This leads to a second degree equation

$$m^2x^2 + 2(mn - p)x + n^2 = 0,$$

having the discriminant

$$\Delta = 4p(2mn - p) \quad \Rightarrow \quad n = \frac{p}{2m}. \tag{7}$$

- If $\Delta < 0$, then the line does not intersect the parabola;
- If $\Delta > 0$, then there are two intersection points between the line and the parabola;
- If $\Delta = 0$, then the line is *tangent* to the parabola and they have a unique intersection point.

The tangent to a parabola with a given direction

A line $d : y = mx + n$ (with $m \neq 0$) is tangent to the parabola $\mathcal{P} : y^2 = 2px$ if the discriminant Δ which appears in (7) is zero, i.e. $2mn = p$. Then, the equation of the tangent line to $\mathcal{P}$, having the angular coefficient m, is

$$y = mx + \frac{p}{2m}. \tag{8}$$

The tangent to a parabola with a given point

Let $\mathcal{P} : y^2 = 2px$ be a parabola and $P_0(x_0, y_0)$ be a point of $\mathcal{P}$. Suppose that $y_0 > 0$, so that the point P_0 belongs to the graph of the function $f : [0, \infty) \to [0, \infty)$, $f(x) = \sqrt{2px}$. The angular coefficient of the tangent at P_0 to the curve is

$$f'(x_0) = \frac{p}{\sqrt{2px_0}} = \frac{p}{y_0}.$$

A similar computation leads to the angular coefficient of the tangent for $y_0 < 0$, which is still $\dfrac{p}{y_0}$.

The equation of the tangent at P_0 to $\mathcal{P}$ is

$$y - y_0 = f'(x_0)(x - x_0),$$

or, replacing $f'(x_0)$,

$$y - y_0 = \frac{p}{y_0}(x - x_0) \Leftrightarrow$$

$$\Leftrightarrow yy_0 - y_0^2 = p(x - x_0) \Leftrightarrow$$

$$yy_0 - 2px_0 = p(x - x_0),$$

hence the equation of the tangent is

$$yy_0 = p(x + x_0). \tag{9}$$

The problem set for this week will be posted soon. Try to solve some of the problems before the seminar.

Thank you very much for your attention!