EXERCISES RESOLUTION IN PROPOSITIONAL LOGIC

Exercise 1

Using general resolution prove that the following formulas are theorems.

1. $U_1 = (A \to B \wedge C) \to (A \to B) \wedge (A \to C)$;
2. $U_2 = (B \to A) \wedge (C \to A) \to (B \wedge C \to A)$
3. $U_3 = (B \to A) \wedge (C \to A) \to (B \vee C \to A)$
4. $U_4 = (A \to C) \to ((\neg A \to B) \to (\neg B \to C))$
5. $U_5 = A \vee (B \to C) \to (A \vee B) \to (A \vee C)$
6. $U_6 = (A \to B) \to ((C \to A) \to (C \to B))$
7. $U_7 = (A \to B) \to ((\neg A \to C) \to (\neg B \to C))$
8. $U_8 = (\neg B \to \neg A) \to ((\neg B \to A) \to B)$

Exercise 2

Consider the following *hypotheses*:

H_1. Mary will go to London this summer if both her friends Kate and Susan go.

H_2. If Kate passes the English exam in May then she will go to London.

H_3. Kate was in hospital from April until July and she didn't take the English exam.

H_4. This summer Susan will go to London on a business trip.

and the *conclusion*: *C*. Mary will go to London this summer.

Try to simplify the initial set of clauses by applying the transformations based on Davis-Putman procedure.

Using level-saturation strategy and the deletion strategy in propositional resolution check whether the following deduction holds:

$$H_1, H_2, H_3, H_4 \overset{?}{\vdash} C$$

Exercise 3. Party

Hypotheses:

H_1. Mary will go to the party if Lucy will go and George will not go.

H_2. If John will go to the party then Lucy will go too.

H_3. If John is in town he will go to the party.

H_4. George is sick and can't go to the party.

H_5. Yesterday John has returned in town from Paris.

Conclusion: C : Will Mary go to the party?

Try to simplify the initial set of clauses by applying the transformations based on Davis-Putman procedure.

Using general resolution in propositional logic check whether the following deduction holds: $H_1, H_2, H_3, H_4, H_5 \overset{?}{\vdash} C$

Exercise 4

Build a linear refutation from the following set of clauses:

1. $S_1 = \{p \vee q \vee r, \neg q \vee r, \neg r, \neg p \vee r\}$;
2. $S_2 = \{p \vee \neg r, q \vee r, \neg q \vee r, \neg p \vee \neg r\}$;
3. $S_3 = \{q \vee r, \neg p, \neg q \vee r, p \vee \neg r\}$;
4. $S_4 = \{\neg p \vee q, p \vee \neg q \vee r, \neg r, p \vee q \vee r, \neg p \vee \neg q\}$;
5. $S_5 = \{p \vee \neg r, \neg q, \neg p \vee \neg r, q \vee r\}$;
6. $S_6 = \{q \vee \neg r, p \vee r, \neg q \vee \neg r, \neg p \vee r\}$
7. $S_7 = \{p, q \vee r, \neg p \vee q \vee \neg r, \neg p \vee \neg q\}$;
8. $S_6 = \{p \vee q, \neg p \vee q, \neg p \vee \neg q, p \vee \neg q\}$;

Exercise 5

Prove the consistency of the following sets of clauses using linear resolution.

1. $S_1 = \{\neg q \vee \neg r, p \vee r, q \vee \neg p\}$
2. $S_2 = \{p \vee q \vee r, \neg p \vee q, \neg p \vee \neg q\}$
3. $S_3 = \{q \vee r, p \vee q \vee \neg r, \neg p \vee \neg q\}$
4. $S_4 = \{p \vee \neg q, \neg p \vee \neg q \vee \neg r, p \vee q\}$;
5. $S_5 = \{p \vee q, r \vee p \vee \neg q, \neg r \vee \neg p\}$;
6. $S_6 = \{q \vee r \vee \neg p, \neg q \vee r, \neg q \vee \neg r\}$

7. $S_7 = \{\neg q \vee p, q \vee \neg r, r \vee \neg p\}$
8. $S_8 = \{p \vee q \vee r, \neg q \vee r, \neg r, p \vee r, \neg r \vee \neg p\}$.

Exercise 6

Using lock resolution prove the inconsistency of the following sets of clauses.
Choose two different indexings for the literals. For one indexing combine lock resolution with level-saturation strategy.

1. $S_1 = \{p \vee r, \neg p \vee \neg q \vee r, \neg p \vee q \vee r, \neg r\}$;
2. $S_2 = \{q \vee \neg r, \neg q \vee \neg p \vee \neg r, \neg q \vee p \vee \neg r, r\}$;
3. $S_3 = \{p \vee q, p \vee \neg q \vee \neg r, \neg p \vee \neg r, r\}$;
4. $S_4 = \{p \vee q, \neg p \vee q \vee \neg r, \neg p \vee q \vee r, \neg q \vee \neg r, \neg q \vee r\}$;
5. $S_5 = \{r \vee q, r \vee \neg q \vee \neg p, \neg p \vee \neg r, p\}$;
6. $S_6 = \{p \vee q, \neg p \vee q \vee \neg r, \neg p \vee \neg q \vee \neg r, p \vee \neg q, r\}$;
7. $S_7 = \{r \vee p, r \vee \neg p \vee \neg q, \neg q \vee \neg r, q\}$;
8. $S_8 = \{p \vee \neg r, \neg p \vee \neg q \vee \neg r, \neg p \vee q \vee \neg r, r\}$.

Exercise 7

Check the consistency of the following sets of clauses using lock resolution.
Choose two different indexings for the literals:

1. $S_1 = \{p \vee q \vee r, \neg q \vee r, \neg r \vee \neg p\}$;
2. $S_2 = \{p \vee \neg r, q \vee r, \neg p \vee \neg r, \neg q \vee \neg r\}$;
3. $S_3 = \{p \vee r, q \vee \neg r, \neg p \vee \neg q \vee r\}$;
4. $S_4 = \{q \vee r, p \vee \neg r, \neg q \vee r, \neg p \vee r\}$;
5. $S_5 = \{\neg p \vee \neg q \vee \neg r, q \vee \neg r, p \vee r\}$;
6. $S_6 = \{p \vee q, r \vee \neg q, \neg p \vee q, \neg r \vee q\}$;
7. $S_7 = \{p \vee q, \neg q \vee r, \neg p \vee \neg q \vee \neg r\}$;
8. $S_8 = \{p \vee \neg q \vee r, q, \neg p \vee q \vee r, p \vee \neg r\}$.

Exercise 1

Using general resolution prove that the following formulas are theorems.

$U_2 = (B \to A) \wedge (C \to A) \to (B \wedge C \to A)$

$= \overline{(B \to A) \wedge (C \to A)} \vee (B \wedge C \to A)$

$= \overline{(B \to A)} \vee (\overline{C \to A}) \vee (\overline{B \wedge C} \vee A)$

$= (B \wedge \bar{A}) \vee (C \wedge \bar{A}) \vee (\bar{B} \vee \bar{C} \vee A)$

$\boxed{U \text{ theorem if } CNF(\neg U) \vdash_{res} \square}$

$\neg U_2 = (\neg B \vee A) \wedge (\neg C \vee A) \vee B \vee C \vee \neg A = CNF(\neg U_2)$

$S = \{\neg B \vee A, \neg C \vee A, B, C, \neg A\}$

$C_1: \neg B \vee A$

$C_2: \neg C \vee A$

$C_3: B$

$C_4: C$

$C_5: \neg A$

$C_6: A \quad [Res_B(C_1, C_3)]$

$C_7: \square \quad [Res_A(C_5, C_6)]$

Exercise 2

Consider the following *hypotheses*:

H_1. Mary will go to London this summer if both her friends Kate and Susan go.

H_2. If Kate passes the English exam in May then she will go to London.

H_3. Kate was in hospital from April until July and she didn't take the English exam.

H_4. This summer Susan will go to London on a business trip.

and the *conclusion*: C. Mary will go to London this summer.

Try to simplify the initial set of clauses by applying the transformations based on Davis-Putman procedure.

Using level-saturation strategy and the deletion strategy in propositional resolution check whether the following deduction holds:

$H_1, H_2, H_3, H_4 \overset{?}{\vdash} C$

$H_1: KL \wedge SL \to ML \equiv \neg KL \vee \neg SL \vee ML \ (C_1)$

$H_2: KE \to KL \equiv \neg KE \vee KL \ (C_2)$

$H_3: KH \wedge \neg KE \Rightarrow KH \ (C_3), \neg KE \ (C_4)$

$H_4: SL \ (C_5)$

$\neg C: \neg ML \ (C_6)$

$H_1, H_2, H_3, H_4 \vdash C$ if $H_1 \wedge H_2 \wedge H_3 \wedge H_4 \wedge \neg C \vdash_{Res} \square$

$X = \{C_1, \cancel{C_2}, \cancel{C_3}, \cancel{C_4}, C_5, C_6\}$

C_2: subsumed by C_4; C_3: pure lit.; C_4: pure lit.

$L_0 \quad S^0 = S = \{C_1, C_5, C_6\}$

$L_1 \quad S^1 = \{Res(C_i, C_j) \mid C_i \in S^0, C_j \in S^0\}$

$$\left.\begin{array}{l} C_7 = Res(C_1, C_5) = \neg KL \vee ML \\ C_8 = Res(C_1, C_6) = \neg KL \vee \neg SL \end{array}\right\} \Rightarrow S^1 = \{C_7, C_8\}$$

$L_2 \quad S^2 = \{Res(C_i, C_j) \mid C_i \in S^1, C_j \in S^0 \cup S^1\}$

$$\left.\begin{array}{l} C_9 = Res(C_5, C_8) = \neg KL \\ C_{10} = Res(C_6, C_7) = \neg KL \end{array}\right\} \Rightarrow S^2 = \{C_9\}$$

$L_3 \quad S^3 = \{Res(C_i, C_j) \mid C_i \in S^2, C_j \in S^0 \cup S^1 \cup S^2\}$

$S^3 = \emptyset$

Exercise 3. Party

Hypotheses:

H_1. $\overset{M}{\underline{\text{Mary will go to the party}}}$ if $\overset{L}{\underline{\text{Lucy will go}}}$ and $\overset{\neg G}{\underline{\text{George will not go}}}$.

H_2. If $\overset{J}{\underline{\text{John will go to the party}}}$ then $\overset{L}{\underline{\text{Lucy will go too}}}$.

H_3. If $\overset{JT}{\underline{\text{John is in town}}}$ $\overset{J}{\underline{\text{he will go to the party}}}$.

H_4. $\overset{GS}{\underline{\text{George is sick}}}$ and $\overset{\neg G}{\underline{\text{can't go to the party}}}$.

H_5. Yesterday $\overset{JT}{\underline{\text{John has returned in town}}}$ from Paris.

Conclusion: C : Will $\overset{M}{\underline{\text{Mary go to the party}}}$?

Try to simplify the initial set of clauses by applying the transformations based on Davis-Putman procedure.

Using general resolution in propositional logic check whether the following deduction holds: $H_1, H_2, H_3, H_4, H_5 \overset{?}{\vdash} C$

$$\left.\begin{aligned}
&H_1: L \wedge \neg G \to M\\
&H_2: J \to L\\
&H_3: JT \to J\\
&H_4: GS \wedge \neg G\\
&H_5: JT\\
&C: M
\end{aligned}\right\} \Rightarrow$$

$C_1: \neg L \vee G \vee M$

$C_2: \neg J \vee L$

$C_3: \neg JT \vee J$

$C_4: \neg G$

~~C_5: GS (pure lit.)~~

$C_6: JT$

$C_7: \neg M$

$C_8: \neg J \vee G \vee M = Res_L(C_1, C_2)$

$C_9: \neg J \vee M = Res_G(C_3, C_5)$

$C_{10}: J = Res_{JT}(C_3, C_6)$

$C_{11}: M = Res_J(C_{10}, C_9)$

$C_{12}: \square = Res_M(C_7, C_{11})$

Exercise 4

Build a linear refutation from the following set of clauses:

$S = \{ \underset{C_1 \text{ (top)}}{p \vee \neg r},\ \underset{C_2}{q \vee r},\ \underset{C_3}{\neg q \vee r},\ \underset{C_4}{\neg p \vee \neg r} \}$

I.

C_1: $p \vee \neg r$; $\underline{C_2}, C_3, C_4$: $q \vee r$

$\xrightarrow{r}$ C_5: $p \vee q$; $\underline{C_3}, C_4$: $\neg q \vee r$

$\xrightarrow{q}$ C_6: $p \vee r$; $\underset{(p)}{\underline{C_4}}, \underset{(r)}{C_4}, C_1$: $\neg p \vee \neg r$

$\xrightarrow{p}$ C_7: $r \vee \neg r$

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II.

C_6: $p \vee r$; $\underset{(q)}{C_4}, \underset{(r)}{\underline{C_4}}, C_1$: $\neg p \vee \neg r$

$\xrightarrow{r}$ C_7: $p \vee \neg p$

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III.

C_6: $p \vee r$; $C_4, C_4, \underline{C_1}$: $p \vee \neg r$

$\rightarrow$ C_7: p ; C_4: $\neg p \vee \neg r$

$\rightarrow$ C_8: $\neg r$; $C_2, \underline{C_3}, C_6$: $\neg q \vee r$

$\xrightarrow{r}$ C_9: $\neg q$; C_2: $q \vee r$

$\xrightarrow{q}$ C_{10}: r ; C_8: $\neg r$

$\rightarrow$ C_{11}: $\square$

Exercise 5

Prove the consistency of the following sets of clauses using linear resolution.

$S = \{ \underbrace{p \vee q \vee r}_{C_1}, \underbrace{\neg p \vee q}_{C_2}, \underbrace{\neg p \vee \neg q}_{C_3} \}$

I. C_1 ; $\underline{C_2}$, $\overset{(p)}{C_3}$, $\overset{(q)}{C_3}$

$C_1: p \vee q \vee r$, $\quad C_2: \neg p \vee q$

$\downarrow p$

$C_4: q \vee r$, $\quad C_3: \neg p \vee \neg q$

$\downarrow q$

$C_5: \neg p \vee r$, $\quad C_1: p \vee q \vee r$

$\downarrow p$

$C_8 = C_4$ → process blocked

$r \vee q$

II. C_1 ; C_2, $\overset{(p)}{C_3}$, $\overset{(q)}{C_3}$

$C_1: p \vee q \vee r$, $\quad \neg p \vee \neg q$

$\downarrow p$

$C_4: q \vee r \vee \neg q$

TAUTOLOGY → process blocked

III. C_1 ; C_2, $\overset{(p)}{C_3}$, $\overset{(q)}{C_3}$

$C_1: p \vee q \vee r$, $\quad \neg p \vee \neg q$

$\downarrow q$

$p \vee r \vee \neg p$

TAUTOLOGY → process blocked

The empty clause wasn't obtained after a complete linear system ⇒ consistent

Exercise 6

Using lock resolution prove the inconsistency of the following sets of clauses.
Choose two different indexings for the literals. For one indexing combine lock resolution with level-saturation strategy.

$$S = \{\underbrace{\underset{(1)}{q} \vee \underset{(2)}{\neg r}}_{C_1},\ \underbrace{\underset{(3)}{\neg q} \vee \underset{(4)}{\neg p} \vee \underset{(5)}{\neg r}}_{C_2},\ \underbrace{\underset{(6)}{\neg q} \vee \underset{(7)}{p} \vee \underset{(8)}{\neg r}}_{C_3},\ \underbrace{\underset{(9)}{r}}_{C_4}\}$$

L_0 $\quad S^0 = \{C_1, C_2, C_3, C_4\}$

L_1 $\quad C_5 = \underset{(4)}{\neg p} \vee \underset{(5)}{\neg r} \vee \underset{(2)}{\neg r} = \underset{(2)}{\neg r} \vee \underset{(4)}{\neg p} = Res_q^{lock}(C_1, C_2)$

$\quad\quad C_6 = \underset{(2)}{\neg r} \vee \underset{(7)}{p} \vee \underset{(8)}{\neg r} = \underset{(2)}{\neg r} \vee \underset{(7)}{p} = Res_q^{lock}(C_1, C_3)$ $\quad \Rightarrow S_1 = \{C_5, C_6\}$

L_2 $\quad C_7 = \underset{(4)}{\neg p} = Res_r^{lock}(C_5, C_4)$

$\quad\quad C_8 = \underset{(7)}{p} = Res_r^{lock}(C_6, C_4)$ $\quad \Rightarrow S^2 = \{C_7, C_8\}$

L_3 $\quad C_9 = Res_p^{lock}(C_7, C_8) = \square$

Exercise 7

Check the consistency of the following sets of clauses using lock resolution.

$$S = \{\underbrace{\underset{(1)}{p} \vee \underset{(2)}{\neg r}}_{C_1},\ \underbrace{\underset{(3)}{q} \vee \underset{(4)}{r}}_{C_2},\ \underbrace{\underset{(5)}{\neg p} \vee \underset{(6)}{\neg r}}_{C_3},\ \underbrace{\underset{(7)}{\neg q} \vee \underset{(8)}{\neg r}}_{C_4}\}$$

$?\ Res_q^{lock}(C_2, C_4)$

We have to combine lock resolution with level-saturation strategy

L_0 $\quad S^0 = \{C_1, C_2, C_3, C_4\}$

L_1 $\quad C_5 = \underset{(2)}{\neg r} = Res_p^{lock}(C_1, C_3)$

$\quad\quad S^1 = \{C_5\}$

L_2 $\quad C_6 = \underset{(3)}{q} = Res_r^{lock}(C_5, C_2)$

$\quad\quad S_2 = \{C_6\}$

L_3 $\quad C_7 = \underset{(8)}{\neg r} = C_5 = Res_q(C_6, C_4)$

$\quad\quad S_3 = \{\emptyset\} \Rightarrow S$ is consistent