Seminar 9 — Linear maps

Let K be a field, V, V' K-v.s., $f : V \to V'$

f linear map (K-linear map) $\overset{\text{def}}{\Longleftrightarrow} \begin{cases} f(x+y) = f(x)+f(y), \forall x,y\in V \\ f(\alpha x) = \alpha \cdot f(x), \forall \alpha \in K, \forall x \in V \end{cases} \Longleftrightarrow$

$\Longleftrightarrow f(\alpha x+\beta y) = \alpha f(x) + \beta f(y), \forall \alpha,\beta\in K, \forall x,y\in V.$

1) a) $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $f_1(x,y) = (-x,y)$ $\quad$ $\mathbb{R}$-linear map (?) (= endomorphism)

Let $\alpha,\beta\in\mathbb{R}$, $(x,y),(x',y')\in\mathbb{R}^2$, $\quad f_1(\alpha(x,y)+\beta(x',y')) \overset{?}{=} \alpha f_1(x,y)+\beta f_1(x',y')$

$f_1(\alpha(x,y)+\beta(x',y')) = f_1((\alpha x,\alpha y)+(\beta x',\beta y')) = f_1(\alpha x+\beta x', \alpha y+\beta y') =$

$= (-\alpha x-\beta x', \alpha y+\beta y') = (-\alpha x, \alpha y) + (-\beta x', \beta y') = \alpha(-x,y)+\beta(-x',y') =$

$= \alpha\cdot f_1(x,y)+\beta\cdot f_1(x',y')$.

f_1 bijective (?) $\qquad f_1\circ f_1 \overset{?}{=} 1_{\mathbb{R}^2}$ (homework)

$\overset{\parallel}{\text{injective + surjective}}$ (?) : $\forall (a,b)\in\mathbb{R}^2, \exists! (x,y)\in\mathbb{R}^2 : f_1(x,y)=(a,b)$

$f_1(x,y) = (a,b) \Longleftrightarrow (-x,y) = (a,b) \Longleftrightarrow \begin{cases} x=-a \\ y = b\end{cases}$

Thus f_1 is an isomorphism (= automorphism).

b) homework.

c) $\varphi\in\mathbb{R}$ given, $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi+y\cos\varphi)$
$\mathbb{R}$-linear map (?) (= endom.)

Let $(x,y),(x',y')\in\mathbb{R}^2$, $\quad f((x,y)+(x',y')) \overset{?}{=} f(x,y)+f(x',y')$

$f((x,y)+(x',y')) = f(x+x',y+y') =$

$= ((x+x')\cos\varphi - (y+y')\sin\varphi, (x+x')\sin\varphi + (y+y')\cos\varphi) =$

$= (x\cos\varphi + x'\cos\varphi - y\sin\varphi - y'\sin\varphi, x\sin\varphi + x'\sin\varphi + y\cos\varphi + y'\cos\varphi)$.

$f(x,y)+f(x',y') = (x\cos\varphi - y\sin\varphi, x\sin\varphi+y\cos\varphi) +$

$+ (x'\cos\varphi - y'\sin\varphi, x'\sin\varphi + y'\cos\varphi) =$

$= (x\cos\varphi + x'\cos\varphi - y\sin\varphi - y'\sin\varphi, x\sin\varphi + x'\sin\varphi + y\cos\varphi + y'\cos\varphi)$

Let $\alpha\in\mathbb{R}$, $(x,y)\in\mathbb{R}^2$, $\quad f(\alpha(x,y)) \overset{?}{=} \alpha\cdot f(x,y)$

$f(\alpha(x,y)) = f(\alpha x,\alpha y) = (\alpha x\cos\varphi - \alpha y\sin\varphi, \alpha x\sin\varphi + \alpha y\cos\varphi) =$

$= (\alpha(x\cos\varphi - y\sin\varphi), \alpha(x\sin\varphi+y\cos\varphi)) = \alpha(x\cos\varphi - y\sin\varphi, x\sin\varphi+y\cos\varphi) =$

$= \alpha \cdot f(x,y)$.

f bijective (?) : $\forall (a,b) \in \mathbb{R}^2, \exists! (x,y) \in \mathbb{R}^2 : f(x,y) = (a,b)$

$f(x,y) = (a,b) \iff (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi) = (a,b) \iff$

$$\iff \begin{cases} x\cos\varphi - y\sin\varphi = a \\ x\sin\varphi + y\cos\varphi = b \end{cases} \quad (1)$$

f bijective $\iff$ (1) is consistent with a unique solution $\iff$

$\iff$ the determinant of the system (1) matrix is not 0 (?)

$$\begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix} = \cos^2\varphi + \sin^2\varphi = 1 \neq 0$$

Hence f is an isomorphism (= autom.)

d) $f_4 : \mathbb{R}^2 \to \mathbb{R}^3$, $f_4(x,y) = (x+y, 2x-y, 3x+2y)$ $\mathbb{R}$-linear map (?)

~~endom.~~, ~~autom.~~

Let $\alpha, \beta \in \mathbb{R}$, $(x,y), (x',y') \in \mathbb{R}^2$

$f_4(\alpha(x,y) + \beta(x',y')) = f_4(\alpha x + \beta x', \alpha y + \beta y') =$

$= ((\alpha x + \beta x') + (\alpha y + \beta y'), 2(\alpha x + \beta x') - (\alpha y + \beta y'), 3(\alpha x + \beta x') + 2(\alpha y + \beta y')) =$

$= (\alpha x + \beta x' + \alpha y + \beta y', 2\alpha x + 2\beta x' - \alpha y - \beta y', 3\alpha x + 3\beta x' + 2\alpha y + 2\beta y') =$

$= (\alpha x + \alpha y, 2\alpha x - \alpha y, 3\alpha x + 2\alpha y) + (\beta x' + \beta y', 2\beta x' - \beta y', 3\beta \cdot x' + 2\beta y')$

$= \alpha \cdot \underbrace{(x+y, 2x-y, 3x+2y)} + \beta \cdot \underbrace{(x'+y', 2x'-y', 3x'+2y')} =$

$= \alpha \cdot f_4(x,y) + \beta \cdot f_4(x',y')$.

f_4 ~~bijective~~ (?) $\qquad$ f_4 injective (?) $\leftarrow$ homework.

f_4 ~~surjective~~ (?) : Let $(0,0,1) \in \mathbb{R}^3$, $\nexists (x,y) \in \mathbb{R}^2 : f_4(x,y) = (0,0,1)$

since $f_4(x,y) = (0,0,1) \iff \begin{cases} x+y = 0 \\ 2x-y = 0 \\ 3x+2y = 1 \end{cases}$ inconsistent

Thus f_4 is not an isomorphism.

R: e) $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x \cdot y$ is not an $\mathbb{R}$-linear map, since

$4 = 2 \cdot 2 = g(2,2) = g(2(1,1)) \neq 2 \cdot g(1,1) = 2 \cdot 1 = 2$.

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2) NO! $\qquad$ Assume by contradiction that such f exists. Then

$(2,1) = f(-2,0,-6) = f((-2)\cdot(1,0,3)) = (-2)\,f(1,0,3) = (-2)(1,1) = (-2,-2)$ imp.

3) V, V_1, V_2 K-v.s., $f: V \to V_1$, $g: V \to V_2$

$$h: V \to V_1 \times V_2 \ , \ h(x) = (f(x), g(x))$$

h is a linear map $\overset{?}{\iff}$ f and g are linear maps

Solution: h linear map $\iff$

$\iff \forall \alpha,\beta\in K, \forall x,y\in V, \ h(\alpha x+\beta y) = \alpha h(x) + \beta h(y) \iff$

$\iff$ ——"—— ——"—— ——, $(f(\alpha x+\beta y), g(\alpha x+\beta y)) = \alpha\cdot(f(x),g(x)) + \beta\cdot(f(y),g(y))$

$\iff$ —— " ——" ——" ——" —— $= (\alpha f(x), \alpha g(x)) + (\beta f(y), \beta g(y))$

$\iff$ ——" ——, $(f(\alpha x+\beta y), g(\alpha x+\beta y)) = (\alpha f(x)+\beta f(y), \alpha g(x)+\beta g(y))$

$\iff \forall \alpha,\beta\in K, \forall x,y\in X$, $f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$ and $g(\alpha x+\beta y) = \alpha g(x)+\beta g(y)$ $\iff$ f, g linear maps

Generalization:

$V, V_1, \dots, V_n$ K-v.s. $(n\in\mathbb{N}^*)$

$V_1\times\dots\times V_n = \{(x_1,\dots,x_n) \mid x_1\in V_1, \dots, x_n\in V_n\}$ is a K-v.s. with the operations: $\leftarrow$ the direct product

$(x_1,\dots,x_n) + (x_1',\dots,x_n') = (x_1+x_1', \dots, x_n+x_n')$

$\alpha(x_1,\dots,x_n) = (\alpha x_1, \dots, \alpha x_n), \ \alpha\in\mathbb{R}$.

$f_i: V \to V_i$, $i=\overline{1,n}$

$f: V \to V_1\times\dots\times V_n$, $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$

f linear map $\iff$ $f_1, f_2, \dots, f_n$ are linear maps.

4) a) Let $m\in\mathbb{N}^*$ given, $f:\mathbb{R}^m\to\mathbb{R}$

f $\mathbb{R}$-linear map $\overset{?}{\iff}$ $\exists\, a_1,\dots,a_m\in\mathbb{R}$ uniquely determined such that

$$f(x_1,\dots,x_m) = a_1x_1+\dots+a_mx_m, \ \forall (x_1,\dots,x_m)\in\mathbb{R}^m \qquad (1)$$

Solution: „$\Leftarrow$" Let us consider f defined by (1). We show that f is a linear map (over $\mathbb{R}$)

Let $\alpha,\beta\in\mathbb{R}$, $(x_1,\dots,x_m), (y_1,\dots,y_m)\in\mathbb{R}^m$

$f(\alpha(x_1,\dots,x_m)+\beta(y_1,\dots,y_m)) = f(\alpha x_1+\beta y_1, \dots, \alpha x_m+\beta y_m) \overset{(1)}{=}$

$$= a_1(\alpha x_1+\beta y_1)+\dots+a_m(\alpha x_m+\beta y_m) = \underline{a_1\alpha x_1}+a_1\beta y_1+\dots+\underline{a_m\alpha x_m}+a_m\beta y_m =$$

$$= \alpha(\underbrace{a_1x_1+\dots+a_mx_m}) + \beta(\underbrace{a_1y_1+\dots+a_my_m}) \overset{(1)}{=}$$

$$= \alpha\cdot f(x_1,\dots,x_m) + \beta\cdot f(y_1,\dots,y_m)$$

"$\Longrightarrow$" The existence of $a_1,\dots,a_m$:

Let $e_1=(1,0,\dots,0),\ e_2=(0,1,0,\dots,0),\ \dots,\ e_m=(0,\dots,0,1)$ from $\mathbb{R}^m$.

$\forall (x_1,\dots,x_m)\in\mathbb{R}^m$, $(x_1,\dots,x_m)=(x_1,0,\dots,0)+(0,x_2,0,\dots,0)+\dots+(0,\dots,0,x_m)=$

$$= x_1\cdot(1,0,\dots,0)+x_2(0,1,0,\dots,0)+\dots+x_m(0,\dots,0,1)=x_1e_1+x_2e_2+\dots+x_me_m$$

$$\Longrightarrow f(x_1,\dots,x_m)=f(x_1e_1+x_2e_2+\dots+x_me_m) \underset{f \text{ linear map}}{=} x_1\underbrace{f(e_1)}_{=a_1}+x_2\underbrace{f(e_2)}_{=a_2}+\dots+x_m\underbrace{f(e_m)}_{=a_m}$$

Let us take $a_i=f(e_i)\in\mathbb{R}$, $\forall i=\overline{1,m} \Longrightarrow (1)$.

The uniqueness of $a_1,\dots,a_m$: Suppose that $b_1,\dots,b_m\in\mathbb{R}$ such that

$$f(x_1,\dots,x_m)=b_1x_1+\dots+b_mx_m \qquad (2)$$

Then $a_i=f(e_i)=f(0,\dots,0,\underset{i}{1},0,\dots,0)\overset{(2)}{=}\cancel{b_1}\cdot 0+\dots+\cancel{b_{i-1}}\cdot 0+\underline{b_i\cdot 1}+\cancel{b_{i+1}}\cdot 0+\dots+\cancel{b_m}\cdot 0=$

$$= b_i,\ \forall i=\overline{1,m}.$$

6) Determine the $\mathbb{R}$-linear maps $f:\mathbb{R}^m\to\mathbb{R}^n$ $(m,n\in\mathbb{N}^*)$ ← how do these maps look like?

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "f"] \arrow[rr, bend right, "p_j\circ f=f_j"'] & \mathbb{R}^n \arrow[r, "p_j"] & \mathbb{R} \end{tikzcd}$$

$\{$ $p_j(y_1,\dots,y_n)=y_j$, $j=\overline{1,n}$
the canonical projection of $\mathbb{R}^n$

$\forall (x_1,\dots,x_m)\in\mathbb{R}^m$, $f(x_1,\dots,x_m)=(f_1(x_1,\dots,x_m), f_2(x_1,\dots,x_m),\dots,f_n(x_1,\dots,x_m))$

$f:\mathbb{R}^m\to\mathbb{R}^n$ is a linear map (of $\mathbb{R}$-v.s.) $\Longleftrightarrow f_1,\dots,f_n$ $(:\mathbb{R}^m\to\mathbb{R})$ are $\mathbb{R}$-linear maps $\overset{a)}{\Longleftrightarrow}$

$\Longleftrightarrow \exists\, a_1^1,\dots,a_m^1, a_1^2,\dots,a_m^2,\dots,a_1^n,\dots,a_m^n\in\mathbb{R}$ uniquely determined such that

$$f(x_1,\dots,x_m)=(a_1^1x_1+\dots+a_m^1x_m,\ a_1^2x_1+\dots+a_m^2x_m,\ \dots,\ a_1^nx_1+\dots+a_m^nx_m).$$

5) Find $f:\mathbb{R}^2\to\mathbb{R}^2$ $\mathbb{R}$-linear map such that $f(1,1)=(2,5)$ and $f(1,0)=(1,4)$.

Compute $f(2,3)$. Is f isomorphism? ← homework.

Solution: The form that f should have is (based on 4b)):

$$f(x,y)=(ax+by,\ cx+dy) \qquad (a,b,c,d\in\mathbb{R})$$

$$f(1,1)=(2,5) \Longleftrightarrow (a+b,\ c+d)=(2,5) \qquad (1)$$

$f(1,0) = (1,4) \Leftrightarrow (a, c) = (1,4)$ (2)

$$(1) \Leftrightarrow \begin{cases} a+b=2 \\ c+d=5 \\ a=1 \\ c=4 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=1 \\ c=4 \\ d=1 \end{cases} \Longrightarrow f(x,y) = (x+y, 4x+y)$$

$(2) \Leftrightarrow$