Course 11 – Line Integrals

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Definition (functions of bounded variation)

Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}^n$ be a vector valued function. If $P := (a = x_0 < x_1 < \ldots < x_k = b)$ is an arbitrary partition of $[a, b]$, then the real number, defined by $V(f, P) := \sum_{j=1}^{k} \|f(x_j) - f(x_{j-1})\|$ is called *the variation of f relative to P*.

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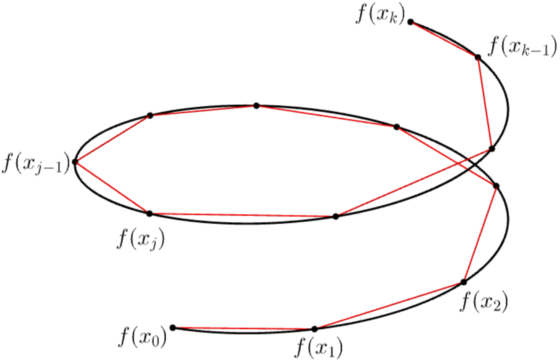

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$$BV([a,b],\mathbb{R}^n) := \left\{f : [a,b] \to \mathbb{R}^n \mid \bigvee_a^b(f) < \infty\right\}.$$

In the special case when $n = 1$, the simplified notation $BV[a,b] := BV([a,b],\mathbb{R})$ is used.

Examples of functions of bounded variation

a) Every monotonic function $f : [a, b] \to \mathbb{R}$ is of bounded variation and $\bigvee_a^b (f) = |f(b) - f(a)|$. Indeed, for every partition P of $[a, b]$ we have $V(f, P) = |f(b) - f(a)|$, hence the preceding assertion holds.

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Consequently, $f \in BV([a, b], \mathbb{R}^n)$ and $\bigvee_a^b(f) \leq \alpha(b - a)$.

Additivity of the total variation with respect to the interval

Given an arbitrary function $f : [a, b] \to \mathbb{R}^n$, for every point $c \in (a, b)$ one has

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The total variation formula

If $f : [a, b] \to \mathbb{R}^n$ is a function of class C^1, then f is of bounded variation on $[a, b]$ and one has

$$\bigvee_a^b (f) = \int_a^b \|f'(x)\| dx.$$

Example of a continuous function that is not of bounded variation

The function $f : [0, 1] \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} x \cos \frac{\pi}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0, \end{cases}$$

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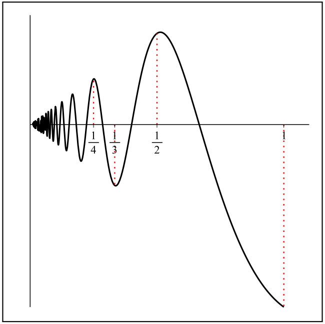

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Letting $n \to \infty$, we deduce that $\bigvee\limits_0^1 (f) = \infty$, hence f is not of bounded variation on $[0, 1]$.

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Definition (parameterized paths)

Let $a, b \in \mathbb{R}$ with $a < b$, and let $\gamma : [a, b] \to \mathbb{R}^n$ be a vector valued function. One says that γ is a *parameterized path* or a *parameterized curve* în $\mathbb{R}^n$ if γ is continuous on $[a, b]$.

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Examples

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The first example on the preceding slide shows that the trace of a parameterized path is far from giving an accurate description of that parameterized path. There is a wide variety of parameterized paths with the same trace.

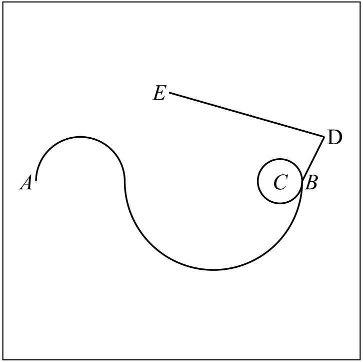

Figure 1: A parameterized path from A to E: it may trace the loop C several times, it may follow the segment from B to D, then return to B, after that continue back to D; then it may stand at D (i.e., it may transform an entire segment of the parameter space into the single point D).

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Then $\ell(\gamma) \in [0, \infty]$ is called the *arclength* (or the *total variation of*) γ.

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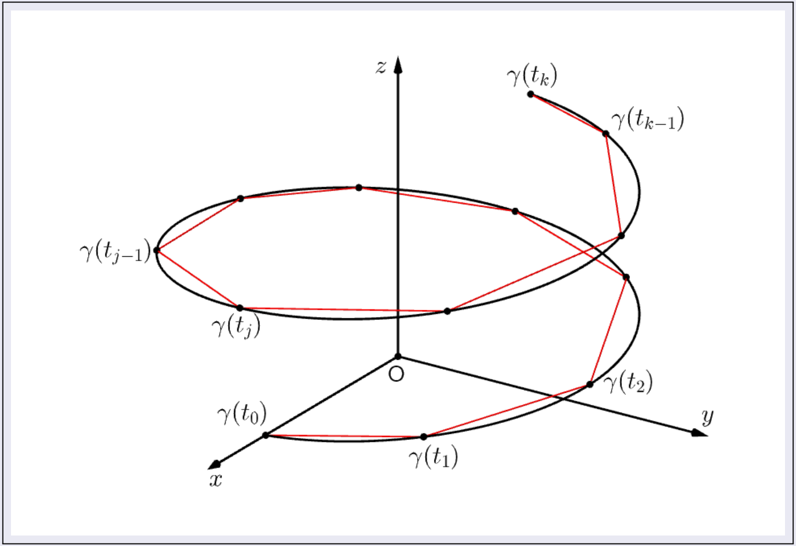

Figure 2: The variation of a parameterized path γ relative to a partition represents to total length of a polygonal line whose vertices lie on the trace of γ.

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f) *smooth*, if γ is of class C^1 and $\|\gamma'(t)\| \neq 0$ for all $t \in [a, b]$.

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If $\gamma(b) = \rho(c)$, then we consider a new function $\gamma \vee \rho : [a, b + d - c] \to \mathbb{R}^n$, defined by

$$(\gamma \vee \rho)(t) := \begin{cases} \gamma(t) & \text{dacă } t \in [a, b] \\ \rho(t - b + c) & \text{dacă } t \in [b, b + d - c]. \end{cases}$$

This new function is continuous,

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If $\gamma(b) = \rho(c)$, then we consider a new function $\gamma \vee \rho : [a, b + d - c] \to \mathbb{R}^n$, defined by

$$(\gamma \vee \rho)(t) := \begin{cases} \gamma(t) & \text{dacă } t \in [a, b] \\ \rho(t - b + c) & \text{dacă } t \in [b, b + d - c]. \end{cases}$$

This new function is continuous, hence it is a parameterized path in $\mathbb{R}^n$, called the *union* of γ and ρ.

Definition (equivalent parameterized paths)

Two parameterized paths $\gamma : [a, b] \to \mathbb{R}^n$ and $\rho : [c, d] \to \mathbb{R}^n$ are called *equivalent* if there exists an homeomorphism $\varphi : [a, b] \to [c, d]$ (i.e., φ is bijective and continuous) such that $\gamma = \rho \circ \varphi$.

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$$\gamma(t) := (\cos t, \sin t) \quad \text{and} \quad \rho(x) := \left(x, \sqrt{1 - x^2}\right), \text{ respectively,}$$

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have both the set C as their traces. In the case of γ the semicircle C is traced counterclockwise, while in the case of ρ it is traced clockwise. We have $\gamma \sim \rho$ via the homeomorphism $\varphi : [0, \pi] \to [-1, 1]$, $\varphi(t) := \cos t$, but $\gamma \not\approx \rho$.

Example (the opposite of a parameterized path)

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parameterized path in $\mathbb{R}^n$. Then the function $\bar{\gamma} : [a, b] \to \mathbb{R}^n$, defined by $\bar{\gamma}(t) := \gamma(a + b - t)$, is a parameterized path, too. It is called the *opposite of the parameterized path* γ.

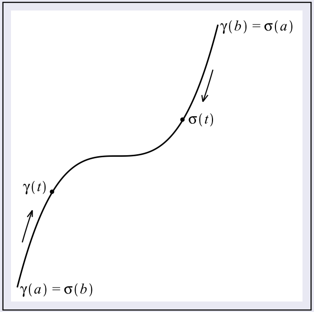

Figure 3: The opposite $\sigma = \bar{\gamma}$ of a parameterized path γ.

Obviously, the parameterized paths γ and $\bar{\gamma}$ are equivalent and they have the same trace. However, $\gamma \not\approx \bar{\gamma}$.

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We denote by $\mathcal{P}^n$ the set of all parameterized paths in $\mathbb{R}^n$. It is easy to prove that the binary relations $\sim$ and $\approx$ are both reflexive, symmetric and transitive, hence they are

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Proposition

Let $\gamma : [a, b] \to \mathbb{R}^n$ *and* $\rho : [c, d] \to \mathbb{R}^n$ *be* *equivalent* *parameterized paths in* $\mathbb{R}^n$*. Then the following assertions are true:*

1° *One has* $I(\gamma) = I(\rho)$*, i.e., two equivalent parameterized paths have the same trace.*

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The subset of $\mathbb{R}^n$ defined by $I(\Gamma) := I(\gamma)$, where $\gamma \in \Gamma$, is called *the trace or the image of the curve* Γ.

Theorem

Every piecewise C^1 parameterized path $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable and its arclength is given by the formula

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.$$

Definition (the integral of a scalar function along a parameterized path)

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a rectifiable parameterized path in $\mathbb{R}^n$, and let $s : [a, b] \to \mathbb{R}$, $s(t) := \bigvee_a^t(\gamma)$ be the *length function of γ*.

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$$\int_\gamma f \, \mathrm{d}s \quad \text{or by} \quad \int_\gamma f(x) \, \mathrm{d}s \quad \text{or by} \quad \int_\gamma f(x_1, \ldots, x_n) \, \mathrm{d}s.$$

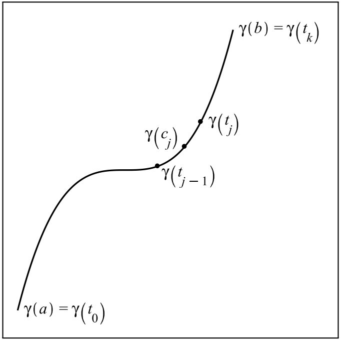

Figure 4: The physical meaning of an integral of the first kind along a parameterized path.

Consider a wire whose shape is the trace $I(\gamma)$ of a rectifiable parameterized path $\gamma : [a, b] \to \mathbb{R}^3$. We assume that the wire is not homogeneous, but that its linear density (mass per unit length) $f(x)$ is known at each point $x \in I(\gamma)$. Let $P := (a = t_0 < t_1 < \ldots < t_k = b)$ be a partition of $[a, b]$.

For every $j \in \{1, \ldots, k\}$ we select a point $c_j \in [t_{j-1}, t_j]$ and we consider that the part of the wire between $\gamma(t_{j-1})$ and $\gamma(t_j)$ has the density $f(\gamma(c_j))$. Since the length of this part of the wire is

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$$m \approx \sum_{j=1}^{k} f(\gamma(c_j))\big[s(t_j) - s(t_{j-1})\big] = \sigma(f \circ \gamma, s, P, \xi),$$

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where $\xi := (c_1, \dots, c_k)$. The exact value of the mass of the wire is

$$m = \int_a^b (f \circ \gamma)(t)\, \mathrm{d}s(t) = \int_\gamma f(x, y, z)\, \mathrm{d}s.$$

Theorem (computation of line integrals of the first kind by means of Riemann integrals)

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^1 parameterized path, and let $f : I(\gamma) \to \mathbb{R}$ be a continuous function. Then f is integrable with respect to the arclength along γ and one has

$$\int_\gamma f\, ds = \int_a^b f(\gamma(t)) \left\| \gamma'(t) \right\| dt.$$

Theorem (independence on parametrization of the line integrals of the first kind along a parameterized path)

Let $\gamma : [a, b] \to \mathbb{R}^n$ and $\rho : [c, d] \to \mathbb{R}^n$ be equivalent rectifiable parameterized paths in $\mathbb{R}^n$. If a function $f : I(\gamma) \to \mathbb{R}$ is integrable with respect to the arclength along γ, then f is integrable with respect to the arclength also along ρ and one has

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In particular, if $f : I(\gamma) \to \mathbb{R}$ is integrable with respect to the arclength along γ, then f is integrable with respect to the arclength also along $\bar{\gamma}$ and one has

$$\int_\gamma f\,ds = \int_{\bar{\gamma}} f\,ds.$$

Definition (the integral of a scalar function along a curve)

The preceding theorem enables us to define the integral of a scalar function (the line integral of the first kind) along a curve. Let Γ be a rectifiable curve in $\mathbb{R}^n$, and let $f : I(\Gamma) \to \mathbb{R}$ be a scalar function. If there exists a parameterized path $\gamma \in \Gamma$ such that f is integrable with respect to the arclength along γ, then one says that f *is integrable with respect to the arclength along the curve* Γ. The real number, defined by $\int_\gamma f \, \mathrm{d}s$ is called *the integral with respect to the arclength of f along Γ* or *the integral of the first kind of f along Γ*. It will be denoted by $\int_\Gamma f \, \mathrm{d}s$.

Definition (vector fields)

Let A be a subset of $\mathbb{R}^n$. A continuous vector function $F = (F_1, \ldots, F_n) : A \to \mathbb{R}^n$ is called a *vector field in A*.

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Given an open set $A \subseteq \mathbb{R}^n$ and a vector field $F : A \to \mathbb{R}^n$, one says that F is a *conservative vector field* if there exists a scalar function $U : A \to \mathbb{R}$, of class C^1 on A and with the property that $F = \nabla U$. In this case the scalar function U is called a *scalar potential* for F.

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A set $A \subseteq \mathbb{R}^n$ is called *connected* if there do not exist open sets $G, H \subseteq \mathbb{R}^n$ such that $A \subseteq G \cup H$, $A \cap G \neq \emptyset$, $A \cap H \neq \emptyset$, and $A \cap G \cap H = \emptyset$.

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Definition (vector fields)

Let A be a subset of $\mathbb{R}^n$. A continuous vector function $F = (F_1, \dots, F_n) : A \to \mathbb{R}^n$ is called a *vector field in A*.

Examples of vector fields

a) Suppose that A is an open subset of $\mathbb{R}^n$ and that $U : A \to \mathbb{R}$ is a function of class C^1 on A. Then $\nabla U : A \to \mathbb{R}^n$ is a vector field in A.

Given an open set $A \subseteq \mathbb{R}^n$ and a vector field $F : A \to \mathbb{R}^n$, one says that F is a *conservative vector field* if there exists a scalar function $U : A \to \mathbb{R}$, of class C^1 on A and with the property that $F = \nabla U$. In this case the scalar function U is called a *scalar potential* for F.

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Examples of vector fields

b) $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) := (x,y) = x \cdot \overrightarrow{i} + y \cdot \overrightarrow{j}$ is a vector field in $\mathbb{R}^2$.

Examples of vector fields

b) $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) := (x, y) = x \cdot \overrightarrow{i} + y \cdot \overrightarrow{j}$ is a vector field in $\mathbb{R}^2$. It is a conservative vector field because $F = \nabla U$, where the potential $U : \mathbb{R}^2 \to \mathbb{R}$ is defined by

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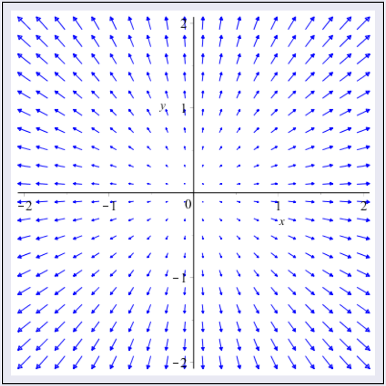

Figure 5: The graph of the vector field $F(x, y) := (x, y) = x \cdot \overrightarrow{i} + y \cdot \overrightarrow{j}$.

Examples of vector fields

c) $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y):=(y,x)=y\cdot\overrightarrow{i}+x\cdot\overrightarrow{j}$ is a vector field in $\mathbb{R}^2$.

Examples of vector fields

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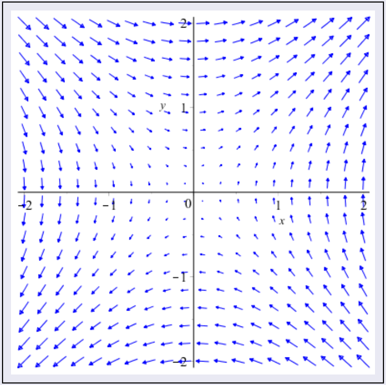

Figure 6: The graph of the vector field $F(x, y) := (x, y) = y \cdot \overrightarrow{i} + x \cdot \overrightarrow{j}$.

Examples of vector fields

d) $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) := (y, x) = y \cdot \overrightarrow{i} - x \cdot \overrightarrow{j}$ is a vector field in $\mathbb{R}^2$.

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d) $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) := (y, x) = y \cdot \overrightarrow{i} - x \cdot \overrightarrow{j}$ is a vector field in $\mathbb{R}^2$. This is not a conservative vector field.

Examples of vector fields

d) $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y):=(y,x)=y\cdot\overrightarrow{i}-x\cdot\overrightarrow{j}$ is a vector field in $\mathbb{R}^2$. This is not a conservative vector field.

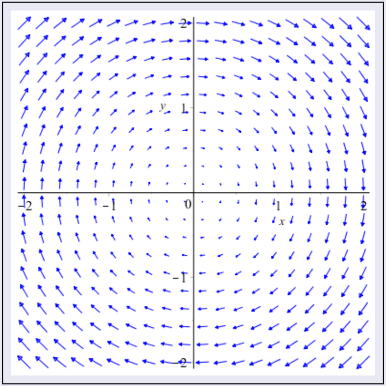

Figure 7: The graph of the vector field $F(x,y):=(x,y)=y\cdot\overrightarrow{i}-x\cdot\overrightarrow{j}$.

Definition (the integral of a vector field along a parameterized path)

Let $A \subseteq \mathbb{R}^n$ be a nonempty set, let $F := (F_1, \ldots, F_n) : A \to \mathbb{R}^n$ be a vector field in A, and let $\gamma := (\gamma_1, \ldots, \gamma_n) : [a, b] \to A$ be a parameterized path whose trace is contained in A.

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$$\int_\gamma F_1(x_1, \dots, x_n)\, \mathrm{d}x_1 + \dots + F_n(x_1, \dots, x_n)\, \mathrm{d}x_n.$$

Theorem

Let A be a nonempty subset of $\mathbb{R}^n$, let $\gamma : [a, b] \to A$ be a rectifiable parameterized path, and let $F := (F_1, \ldots, F_n) : A \to \mathbb{R}^n$ be a vector field in A. Then F is integrable along γ and it holds $\left| \int_\gamma \overrightarrow{F} \cdot d\overrightarrow{r} \right| \leq M\,\ell(\gamma)$, where $M := \max_{x \in I(\gamma)} \|F(x)\|$.

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Theorem (computation of line integrals of vector fields by means of Riemann integrals)

Let A be a nonempty subset of $\mathbb{R}^n$, let $\gamma : [a, b] \to A$ be a parameterized path of class C^1, and let $F := (F_1, \dots, F_n) : A \to \mathbb{R}^n$ be a vector field in A. Then one has

$$\int_\gamma \overrightarrow{F} \cdot d\overrightarrow{r} = \sum_{i=1}^{n} \int_a^b (F_i \circ \gamma)(t)\, \gamma_i'(t)\, dt.$$

Remark

If $\gamma(t) = (x(t), y(t))$, $t \in [a, b]$ is a parameterized path in $\mathbb{R}^2$, then

$$\begin{aligned}\int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= \int_\gamma F_1(x, y)\,\mathrm{d}x + F_2(x, y)\,\mathrm{d}y \\ &= \int_a^b F_1(x(t), y(t))x'(t)\,\mathrm{d}t + \int_a^b F_2(x(t), y(t))y'(t)\,\mathrm{d}t.\end{aligned}$$

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In the case of a parameterized path $\gamma(t) = (x(t), y(t), z(t))$, $t \in [a, b]$ in $\mathbb{R}^3$, we have

$$\begin{aligned}\int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= \int_\gamma F_1(x, y, z)\,\mathrm{d}x + F_2(x, y, z)\,\mathrm{d}y + F_3(x, y, z)\,\mathrm{d}z \\ &= \int_a^b F_1(x(t), y(t), z(t))x'(t)\,\mathrm{d}t \\ &\qquad + \int_a^b F_2(x(t), y(t), z(t))y'(t)\,\mathrm{d}t \\ &\qquad + \int_a^b F_3(x(t), y(t), z(t))z'(t)\,\mathrm{d}t.\end{aligned}$$

Theorem (additivity of the line integral of a vector field with respect to the path)

Let A be a nonempty subset of $\mathbb{R}^n$, let $\gamma : [a, b] \to A$ and $\rho : [c, d] \to A$ be parameterized paths such that $\gamma(b) = \rho(c)$, and let $F := (F_1, \dots, F_n) : A \to \mathbb{R}^n$ be a vector field in A. If F is integrable along $\gamma \vee \rho$, then F is integrable along both γ and ρ and one has

$$\int_{\gamma\vee\rho} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{\gamma} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{\rho} \overrightarrow{F} \cdot d\overrightarrow{r}.$$

Theorem

Let A be a nonempty subset of $\mathbb{R}^n$, let $\gamma : [a,b] \to A$ and $\rho : [c,d] \to A$ be equivalent parameterized paths, and let $F := (F_1, \ldots, F_n) : A \to \mathbb{R}^n$ be a vector field in A, such that F is integrable along γ. Then F is integrable along ρ, too and it holds $\int_\rho \overrightarrow{F} \cdot d\overrightarrow{r} = \varepsilon_{\gamma,\rho} \int_\gamma \overrightarrow{F} \cdot d\overrightarrow{r}$, where

$$\varepsilon_{\gamma,\rho} := \begin{cases} 1 & \text{if } \gamma \approx \rho \\ -1 & \text{otherwise.} \end{cases}$$

In particular, F is integrable along $\bar{\gamma}$ and $\int_{\bar{\gamma}} \overrightarrow{F} \cdot d\overrightarrow{r} = -\int_\gamma \overrightarrow{F} \cdot d\overrightarrow{r}$.

Definition (the integral of a vector field along an oriented curve)

The preceding theorem enables us to define the integral (or work) of a vector field along an oriented curve. Let A be a nonempty subset of $\mathbb{R}^n$, let $F := (F_1, \ldots, F_n) : A \to \mathbb{R}^n$ be a vector field in A, and let Γ be an oriented curve in $\mathbb{R}^n$ with $I(\Gamma) \subseteq A$. If there exists a parameterized path $\gamma \in \Gamma$ such that F is integrable along γ, then one says that F is *integrable along the curve* Γ. The real number, defined by $\int_\gamma f$ is called *the integral or the work of the vector field* F *along the curve* Γ and it will be denoted by $\displaystyle\int_\Gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}$ or by

$$\int_\Gamma F_1(x_1, \ldots, x_n)\,\mathrm{d}x_1 + \cdots + F_n(x_1, \ldots, x_n)\,\mathrm{d}x_n.$$

Remark (the physical meaning of the integral of a vector field along a parameterized path)

Consider a particle moving in $\mathbb{R}^3$ under the action of a vector field $F := (P, Q, R) : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose that the trajectory of the particle is the trace $I(\gamma)$ of a parameterized path $\gamma : [a, b] \to \mathbb{R}^3$. Then the work W, done by the vector field F in moving the particle is

$$W = \int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} = \int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z.$$

The work represents the amount of energy necessary to move the particle along its trajectory.