ANALYTIC GEOMETRY, PROBLEM SET 4

Projections. Dot product. Cross product.

1. Find the orthogonal projection $pr_{\overline{u}}(\overline{v})$, where $\overline{v} = 10\overline{a} + 2\overline{b}$, $\overline{u} = 5\overline{a} - 12\overline{b}$, if $\overline{a} \perp \overline{b}$ and $||\overline{a}|| = ||\overline{b}|| \neq 0$.

2. Using the dot product, prove the **Cauchy-Buniakowski-Schwarz** inequality, i.e. show that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

3. For a tetrahedron $ABCD$, show that $\cos(\widehat{\overline{AB}, \overline{CD}}) = \frac{AD^2+BC^2-AC^2-BD^2}{2AB\cdot CD}$. (the 3D version of the **cosine theorem**)

4. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds: $9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$.

5. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ and $A'B'$, respectively, are concurrent. Then the perpendicular lines through A', B', C' on BC, CA and AB, respectively are also concurrent.
(Steiner's theorem on **orthologic triangles**)

6. Find the area of the plane triangle having the vertices $A(1, 0, 1)$, $B(0, 2, 3)$, $C(2, 1, 0)$.

7. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \overline{a}$, $\overline{CA} = \overline{b}$ and $\overline{AB} = \overline{c}$ if and only if $\overline{a} \times \overline{b} = \overline{b} \times \overline{c} = \overline{c} \times \overline{a}$.

8. Find a vector orthogonal on both $\overline{u}$ and $\overline{v}$, if:
a) $\overline{u} = -7\overline{i} + 3\overline{j} + \overline{k}$ and $\overline{v} = 2\overline{i} + 4\overline{k}$ b) $\overline{u} = (-1, -1, -1)$ and $\overline{v} = (2, 0, 2)$.

9. Let a, b, and c denote the lengths of the sides of $\triangle ABC$. We write O for its circumcenter, R for the length of its circumradius, H for its orthocenter and G for the centroid. Show that
a) $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$; b) $OG^2 = R^2 - 1/9(a^2 + b^2 + c^2)$.

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