Seminar 3

Let $A \in M_n(K)$, $n \in \mathbb{N}$, $n \geq 2$, K field.

How do we compute $\det A$? $n = 2, 3$

$\boxed{\text{If } n \geq 4}$ The answer: Step 1 – We apply the properties we know about determinants in order to get $n-1$ zeros in a row (column)

Step 2 – We expand the resulted determinant along this row (column)

$\Rightarrow$ $\det A$ = the determinant of $n-1$ size matrix.

Exercises: 1) Compute:

a) $$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} \underset{r_1 - r_4}{=} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} \underset{c_1 - c_3}{=}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \underbrace{1\cdot(-1)^{1+1}}_{=1} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1.$$

b) $$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} \underset{c_1 - 2c_2}{=} \begin{vmatrix} 0 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 1\cdot(-1)^{1+2} \begin{vmatrix} -3 & 1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \underset{c_3 - 2c_2}{=}$$

$$= - \begin{vmatrix} -3 & 1 & -2 \\ -2 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = - \underbrace{1\cdot(-1)^{3+2}}_{=1} \begin{vmatrix} -3 & -2 \\ -2 & -3 \end{vmatrix} = 5.$$

c) $$\begin{vmatrix} -1 & a & a & \dots & a & a \\ a & -1 & a & \dots & a & a \\ \dots & & & & & \\ a & a & a & \dots & a & -1 \end{vmatrix} \underset{r_1 + \dots + r_n}{=} \begin{vmatrix} -1+(n-1)a & -1+(n-1)a & \dots & -1+(n-1)a \\ a & -1 & \dots & a \\ \vdots & \vdots & & \vdots \\ a & a & \dots & -1 \end{vmatrix} =$$

$$= [-1+(n-1)a] \cdot \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ a & -1 & a & \dots & a & a \\ \dots & & & & & \\ a & a & a & \dots & a & -1 \end{vmatrix} \underset{\substack{c_2 - c_1 \\ c_3 - c_1 \\ \vdots \\ c_n - c_1}}{=} [-1+(n-1)a] \begin{vmatrix} 1 & 0 & \dots & 0 \\ a & -1-a & \dots & 0 \\ \vdots & \vdots & & \vdots \\ a & 0 & \dots & -1-a \end{vmatrix} =$$

$$= [-1+(n-1)a] \cdot 1\cdot(-1-a)^{n-1} = (-1)^{n-1}(a+1)^{n-1}(na - a - 1).$$

d) $x_1, x_2, x_3 \in \mathbb{C}$ are the roots of $X^3 - 2X^2 + 2X + 17 \in \mathbb{Q}[X]$.

$$d = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{vmatrix} = ?$$

$$\begin{cases} \underline{x_1+x_2+x_3} = 2 \\ \underline{x_1x_2+x_2x_3+x_3x_1} = 2 \\ x_1x_2x_3 = -17 \end{cases}$$

$$d \xlongequal[c_1+c_2+c_3]{} (x_1+x_2+x_3)\begin{vmatrix} 1 & x_2 & x_3 \\ 1 & x_3 & x_1 \\ 1 & x_1 & x_2 \end{vmatrix} =$$

$$= (x_1+x_2+x_3)\left[\underline{x_2x_3+x_2x_1+x_1x_3} - x_1^2 - x_2^2 - x_3^2\right] =$$

$$= \underline{(x_1+x_2+x_3)}\left[-\underline{(x_1+x_2+x_3)}^2 + 3\underline{(x_1x_2+x_2x_3+x_3x_1)}\right] = 2(-4+6) = 4$$

e) The determinant is 0. We can approach this exercises as in d) (homework).

2) a) Let us denote $d = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} \xlongequal[r_1+r_2+r_3+r_4]{} (x+3a)\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} \xlongequal[\substack{c_2-c_1 \\ c_3-c_1 \\ c_4-c_1}]{}$

$$= (x+3a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ a & x-a & 0 & 0 \\ a & 0 & x-a & 0 \\ a & 0 & 0 & x-a \end{vmatrix} = (x+3a)(x-a)^3$$

The given equation is equivalent to $(d=0)$

$$(x+3a)(x-a)^3 = 0 \implies x_1 = x_2 = x_3 = a.$$
$$x_4 = -3a.$$

b) an invitation ...

3) $n \in \mathbb{N}$, $n \geq 2$, $a_1, \dots, a_n \in \mathbb{C}$

$$P(n):\ V(a_1,\dots,a_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le} (a_j - a_i)$$

$n=2$, $V(a_1,a_2) = \begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1$ and the formula works.

$\underset{\text{true}}{P(n)} \implies \underset{\text{true}}{P(n+1)}$ Let $a_1, \dots, a_n, a_{n+1} \in \mathbb{C}$.

$$V(a_1,\dots,a_n,a_{n+1}) =$$

$$= \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ a_1 & a_2 & \dots & a_n & a_{n+1} \\ a_1^2 & a_2^2 & \dots & a_n^2 & a_{n+1}^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} & a_{n+1}^{n-1} \\ a_1^n & a_2^n & \dots & a_n^n & a_{n+1}^n \end{vmatrix} \underset{\substack{r_2 - a_{n+1} r_1 \\ r_3 - a_{n+1} r_2 \\ \vdots \\ r_{n+1} - a_{n+1} r_n}}{=\!=\!=} \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ a_1 - a_{n+1} & a_2 - a_{n+1} & \dots & a_n - a_{n+1} & 0 \\ a_1(a_1 - a_{n+1}) & a_2(a_2 - a_{n+1}) & \dots & a_n(a_n - a_{n+1}) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^{n-1}(a_1 - a_{n+1}) & a_2^{n-1}(a_2 - a_{n+1}) & \dots & a_n^{n-1}(a_n - a_{n+1}) & 0 \end{vmatrix}$$

$$= (-1)^n (a_1 - a_{n+1})(a_2 - a_{n+1}) \dots (a_n - a_{n+1}) \cdot V(a_1, a_2, \dots, a_n) =$$

$$= (a_{n+1} - a_1)(a_{n+1} - a_2) \dots (a_{n+1} - a_n) \cdot \prod_{1 \le i < j \le n} (a_j - a_i) = \prod_{1 \le i < j \le n+1} (a_j - a_i).$$

4)

a)

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

$$\det A \underset{\substack{r_2 + r_1 \\ r_3 - r_1}}{=} \begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix} = -(6-9) = 3 \neq 0 \Longrightarrow \exists A^{-1} = \frac{1}{3} \cdot A^*$$

$$A^* = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}$$

$$\alpha_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 4 \end{vmatrix} = -5 \; ; \quad \alpha_{21} = -2 \; ; \quad \alpha_{31} = 3 \; ;$$

$$\alpha_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} = -4 \; ; \quad \alpha_{22} = -4 \; ; \quad \alpha_{32} = 3 \; ;$$

$$\alpha_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \; ; \quad \alpha_{23} = 3 \; ; \quad \alpha_{33} = -3 \, .$$

$$A^{-1} = \frac{1}{3} \begin{pmatrix} -5 & -2 & 3 \\ -4 & -4 & 3 \\ 6 & 3 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & -\frac{2}{3} & 1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

b) For the given matrix $\frac{1}{3} c_1 = \frac{1}{4} c_2 \;\left(\Longleftrightarrow c_2 = \frac{3}{4} c_1\right)$

$\Longrightarrow$ the determinant of this matrix is 0 $\Longrightarrow$ the given matrix is not invertible.

c) Let A be the given matrix., $\det A = 4$

$$A^{-1} = \frac{1}{4} A. \text{ (homework)}$$