Seminar 4

We say that $f : (G_1, \circ) \to (G_2, *)$ is a **group homomorphism** if $\forall x, y \in G_1$ we have $f(x \circ y) = f(x) * f(y)$. For f to be an **isomorphism** it has to be bijective, too. If the domain and the codomain are the same group, then f is an **endomorphism**. And if f is an endomorphism and it's bijective, then it is an **automorphism**.

1. (i) Use $\forall z_1, z_2 \in \mathbb{C}^*$ we have $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.
   (ii) Use $\forall x_1, x_2 \in \mathbb{Z}$ we have $\widehat{x_1 + x_2} = \widehat{x_1} + \widehat{x_2}$.

2. (i) Use $det(A \cdot B) = detA \cdot detB$.
   (ii) Take two random matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Compute $det(A + B) = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} + b_{11} \cdot b_{22} - b_{21} \cdot b_{12} + a_{11} \cdot b_{22} + a_{22} \cdot b_{11} - a_{21} \cdot b_{12} - a_{12} \cdot b_{21}$.
And $detA + detB = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} + b_{11} \cdot b_{22} - b_{21} \cdot b_{12}$.
The two results are not equal in general.
An example: for $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have $det\,A + det\,B = 0$, but $det(A + B) = 1$.
So, in conclusion, β is not a group homomorphism.

3. For $f(z) =| z |$ we have:

$$Ker(f) = \{x \in \mathbb{C}^* \mid |z| = 1\}$$

$$Im(f) = \{|z| \mid z \in \mathbb{C}^*\} = \mathbb{R}^*{}_+$$

For $g(x) = \hat{x}$ we have:

$$Ker(g) = \{x \in \mathbb{Z} \mid \hat{x} = \hat{0}\} = n \cdot \mathbb{Z}$$

$$Im(g) = \{\hat{x} \mid x \in \mathbb{Z}\} = \{\hat{0}, \hat{1}, \hat{2}, \ldots, \widehat{(n-1)}\} = \mathbb{Z}_n$$

For $\alpha(A) = detA$ we have:

$$Ker(\alpha) = \{A \in GL_n(\mathbb{R}) \mid detA = 1\} = SL_n(\mathbb{R})$$

$$Im(\alpha) = \{detA \mid A \in GL_n(\mathbb{R})\} = \mathbb{R}^*$$

4. It is easy to prove this: $\forall z_1 = a_1 + b_1 \cdot i, z_2 = a_2 + b_2 \cdot i \in \mathbb{C}^*$, $f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$.

5. Suppose $f \in Hom \Rightarrow \forall z_1, z_2 \in \mathbb{C}^* \Rightarrow f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2) \Rightarrow f(z_1 \cdot z_2) = a \cdot |z_1 \cdot z_2| + b$.

   Also, if $f \in Hom \Rightarrow f(1) = 1 \Rightarrow a + b = 1$.

   And $f(z_1) \cdot f(z_2) = a^2 \cdot |z_1 \cdot z_2| + ab \cdot (|z_1| + |z_2|) + b^2$.

   From the last two equations we get the following relations:

$$a^2 = a, a \neq 0 \Rightarrow a = 1$$

$$b^2 - b + ab \cdot (|z_1| + |z_2|) = 0$$

   Hence $b = 0$ or $b = 1 - |z_1| + |z_2|$, which does not happen, as $a + b = 1$.

   In conclusion, for f to be a homomorphism, $a = 1$ and $b = 0$.

6. We have: $f \in End(G) \iff \forall x, y \in G,\ f(x \cdot y) = f(x) \cdot f(y) \iff \forall x, y \in G, (x \cdot y)^{-1} = x^{-1} \cdot y^{-1} \iff \forall x, y \in G, (x \cdot y)^{-1} = (y \cdot x)^{-1} \iff \forall x, y \in G, x \cdot y = y \cdot x \iff G$ is abelian.

7. We need to find $f : \mathbb{Z}_n \to U_n$ such that f is an isomorphism, i.e. bijective homomorphism. Recall that $\mathbb{Z}_n = \{\hat{k} \mid k \in \{0, \ldots, n-1\}\}$ and $U_n = \{\varepsilon^k \mid k \in \{0, \ldots, n-1\}\}$, where $\varepsilon^k = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$.

   Take $f(\hat{k}) = \varepsilon^k$ for every $k \in \{0, \ldots, n-1\}$. Then f is bijective.

   For every $k_1, k_2 \in \{0, \ldots, n-1\}$, we can easily see that $f(\widehat{k_1} + \widehat{k_2}) = f(\widehat{k_1 + k_2}) \iff \varepsilon^{k_1 + k_2} = \varepsilon^{k_1} \cdot \varepsilon^{k_2}$, which is true by using the above trigonometrical form.

8. Klein's group $K = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where σ_0 is the identity element, has the next two properties:

(a) Each element is it's self-inverse.

(b) Multiplying any two elements, different from the identity element, we get the third element.

And, also $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{z_0, z_1, z_2, z_3\}$ has the same two properties with respect to addition, where $z_0 = (\hat{0}, \hat{0})$ is the identity element and $z_1 = (\hat{0}, \hat{1}), z_2 = (\hat{1}, \hat{0}), z_3 = (\hat{1}, \hat{1})$.

If $f : K \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is a group homomorphism we must have $f(\sigma_0) = z_0$. For the other elements, we may take, for instance, $f(\sigma_i) = z_i$, $\forall i \in \{1, 2, 3\}$. Then f is bijective.

For every $i \in \{1, 2, 3, 4\}$, we show that $f(\sigma_i \cdot \sigma_j) = f(\sigma_i) \cdot f(\sigma_j)$.

If $i = 0$ or $j = 0$, then the equality is clear. Assume that $i, j \neq 0$.

If $i = j$, then $\sigma_i \cdot \sigma_j = \sigma_0$ and the equality becomes $f(\sigma_0) = z_i^2$, that is, $f(\sigma_0) = z_0$, which is true.

If $i \neq j$, then $\sigma_i \cdot \sigma_j = \sigma_k$, where $k \in \{1, 2, 3\} \setminus \{i, j\}$, and the equality becomes $f(\sigma_k) = z_i \cdot z_j$, that is, $f(\sigma_k) = z_k$, which is true. Hence f is a group isomorphism.

9. So, we need to find an isomorphism $f : \mathbb{R} \to \mathbb{R}_+^*$.

   Take $a \in \mathbb{R}, a > 0, a \neq 1$, and define $f(x) = a^x$, $\forall x \in \mathbb{R}$. Then f is bijective, as we know the inverse $log_a(x)$. Also, $\forall x, y \in \mathbb{R}$, we have $f(x + y) = a^{x+y} = a^x \cdot a^y = f(x) \cdot f(y)$.

10. Let $G = \{e, x, y\}$ be a group, where e is the identity element. From the operation table, which can be filled in a unique way, it follows that y is the inverse of x and x is the inverse of y. If $f \in End(G)$, then $f(e) = e$ and $f(y) = f(x^{-1}) = f(x)^{-1}$ is determined by the value of $f(x)$. But $f(x)$ may take 3 values, namely e, x or y. Hence there are 3 possible endomorphisms of G: the identity function, $f(x) = x$, the trivial function $f(x) = e$ and the inverse function $f(x) = x^{-1}$. Two of them are bijections, and so they are automorphisms of G, i.e. the identity function and the inverse function.

11. We know that $U_4 = \{\varepsilon^k \mid k \in \{0, 1, 2, 3\}\}$, where $\varepsilon^k = \cos(\frac{2k\pi}{4}) + i \sin(\frac{2k\pi}{4})$. Hence $U_4 = \{1, i, -1, -i\}$.

    If $f \in Aut(U_4)$, then $f(1) = 1$.

We have $f(-1) \cdot f(-1) = f((-1) \cdot (-1)) = f(1) = 1$. Because f is bijective, we must have $f(-1) = -1$.

It follows that $f(i) \in \{i, -i\}$. If $f(i) = i$, then $f(-i) = -i$. If $f(i) = -i$, then $f(-i) = i$.

Hence there are two possible automorphisms of U_4, the identity function and the inverse function. (as in *Exercise 10*)

12. (i) First, take $n > 0$, then $f(n) = f(1 + (n-1)) = f(1) + f(n-1)$, as f is an endomorphism. Now $f(1) + f(n-1) = f(1) + f(1 + (n-2)) = f(1) + f(1) + f(n-2)$. By induction, we get $f(n) = f(1) + \cdots + f(1) = nf(1)$.

    If $n < 0$, then $n = (-1) \cdot (1 + 1 + \cdots + 1)$ and with the same reasoning, we get again $f(n) = nf(1)$. And if $n = 0$, then $f(0) = f(n-n) = f(n) + f(n) = nf(1) - nf(1) = 0$.

    (ii) $t_a \in End(\mathbb{Z}, +) \iff \forall x, y \in \mathbb{Z} : t_a(x+y) = t_a(x) + t_a(y) \iff a(x+y) = ax + ay$ (True).

    Now, let $t_a(1) = a$ and $n > 0 \Rightarrow t_a(n) = t_a(1) + \cdots + t_a(1) = a + \cdots + a = a \cdot n$.

    From $t_a \in End(\mathbb{Z}, +) \Rightarrow t_a(0) = 0$. And with the same reasoning $t_a(-1) = -a$ and $t_a(-n) = -an$.