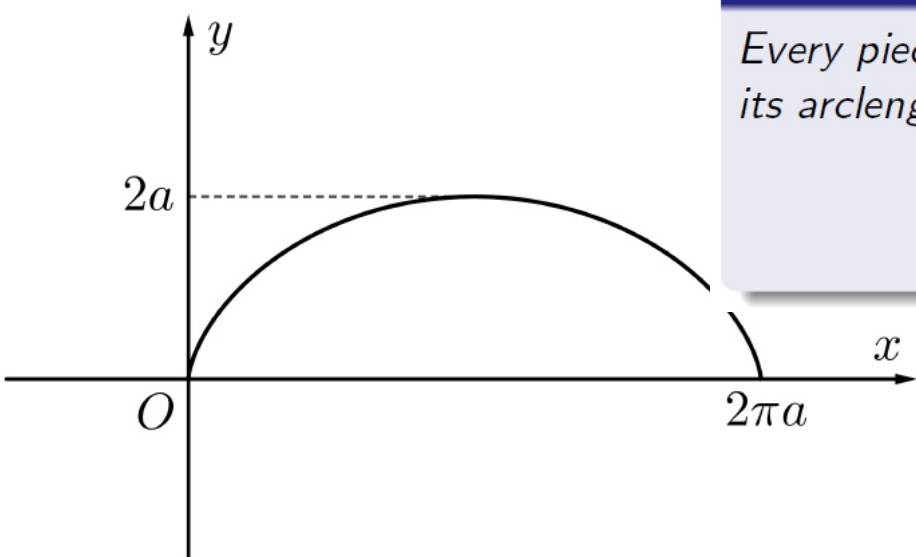
Let $a>0$ and let $\gamma=(x,y):[0,2\pi]\to\mathbb{R}^2$ be the parameterized path defined by
$$\begin{cases} x(t) = a(t-\sin t) \\ y(t) = a(1-\cos t) \end{cases}$$

The trace of γ is a path in $\mathbb{R}^2$, called the cycloid. Determine the arclength of one loop of the cycloid.



Theorem

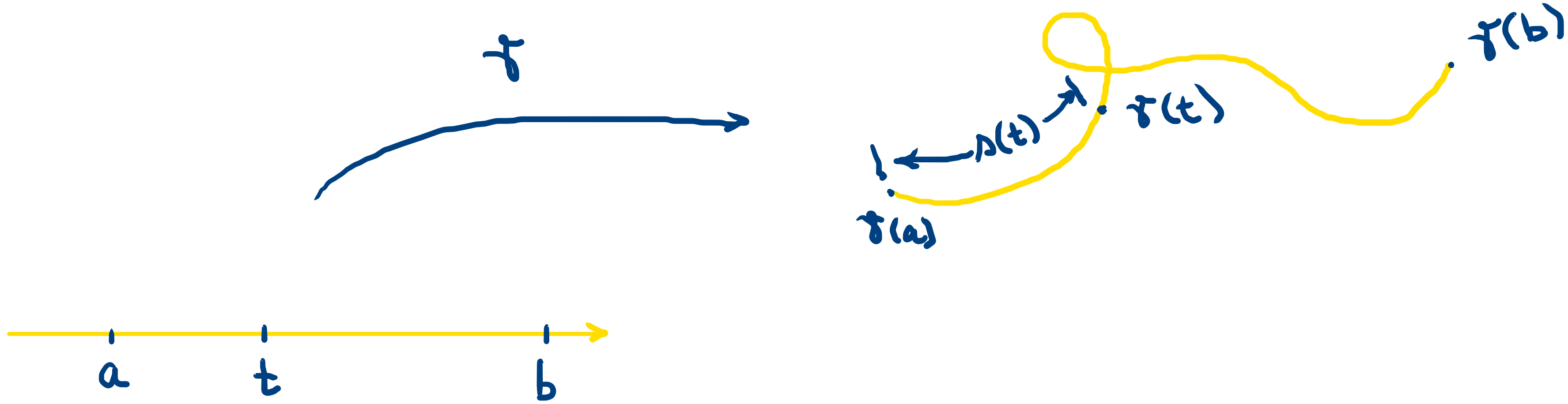
Every piecewise C^1 parameterized path $\gamma:[a,b]\to\mathbb{R}^n$ is rectifiable and its arclength is given by the formula
$$\ell(\gamma)=\int_a^b \|\gamma'(t)\|\,dt.$$

$$\gamma(t) = \big(a(t-\sin t),\ a(1-\cos t)\big)$$
$$\gamma'(t) = \big(a(1-\cos t),\ a\sin t\big)$$

$$\|\gamma'(t)\|^2 = a^2(1-\cos t)^2 + a^2\sin^2 t = a^2(1-2\cos t+\cos^2 t+\sin^2 t) = 2a^2\underbrace{(1-\cos t)}_{2\sin^2\frac{t}{2}}$$

$$\|\gamma'(t)\| = 2a\sin\frac{t}{2}$$

$$\ell(\gamma) = \int_0^{2\pi} 2a\sin\frac{t}{2}\,dt = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = \boxed{8a}$$



Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be the parameterized path defined by

$$\gamma(t) = \left(a(t - \sin t),\ a(1-\cos t) \right), \quad t \in [0, 2\pi], \quad a > 0.$$

Evaluate $I = \int_\gamma y^2 \, ds$.

$f(x,y) = y^2$

Theorem (computation of line integrals of the first kind by means of Riemann integrals)

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a C^1 parameterized path, and let $f : I(\gamma) \to \mathbb{R}$ be a continuous function. Then f is integrable with respect to the arclength along γ and one has

$$\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt.$$

$$I = \int_0^{2\pi} a^2 (\overbrace{1-\cos t}^{2\sin^2\frac{t}{2}})^2 \cdot 2a \sin\frac{t}{2} \, dt$$

$$= 2a^3 \int_0^{2\pi} 4 \sin^5 \frac{t}{2} \, dt \qquad \frac{t}{2} = u$$

$$= 8a^3 \int_0^{\pi} \sin^5 u \cdot 2\,du$$

$$= 16 a^3 \int_0^{\pi} \sin^5 u \, du \qquad \cos u = v$$

$$= \dots$$