CHAPTER 8

Bilinear and quadratic forms

Contents

8.1 Bilinear forms

Definition 8.1. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$. A map

$$b : \mathbf{V} \times \mathbf{V} \to \mathbf{K}$$

is a *bilinear form on* $\mathbf{V}$ if it is linear in each of its arguments. That is, if it satisfies

(BF1) $b(\mathbf{v}+\mathbf{v}',\mathbf{w}) = b(\mathbf{v},\mathbf{w}) + b(\mathbf{v}',\mathbf{w})$

(BF2) $b(\mathbf{v},,\mathbf{w}+\mathbf{w}') = b(\mathbf{v},\mathbf{w}) + b(\mathbf{v},\mathbf{w}')$

(BF3) $b(c\mathbf{v},\mathbf{w}) = b(\mathbf{v},c\mathbf{w}) = cb(\mathbf{v},\mathbf{w})$

for every $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{w}$, $\mathbf{w}' \in \mathbf{V}$ and $c \in \mathbf{K}$. The set of all bilinear forms on $\mathbf{V}$ is denoted by $\mathrm{Bil}(\mathbf{V})$. The bilinear form is called *symmetric* if

$$b(\mathbf{v},\mathbf{w}) = b(\mathbf{w},\mathbf{v}) \quad \text{for every} \quad \mathbf{v},\mathbf{w} \in \mathbf{V}.$$

Example 8.2. Let $A = (a_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbf{K})$. One obtains a bilinear form on $\mathbf{K}^n$ by putting

$$b(\mathbf{x},\mathbf{y}) = \mathbf{x}^t A\mathbf{y} = \sum_{i,j=1}^{n} a_{ij}x_i y_j$$

for every $\mathbf{x} = (x_1,\dots,x_n)^t$ and $\mathbf{y} = (y_1,\dots,y_n)^t$. So, to a matrix we can associate a bilinear form. The converse is also true.

Definition 8.3. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space of dimension n, let $\boldsymbol{e} = (\mathbf{e}_1,\dots,\mathbf{e}_n)$ be a basis of $\mathbf{V}$ and let $b : \mathbf{V}\times\mathbf{V} \to \mathbf{K}$ be a bilinear form on $\mathbf{V}$. The *matrix of the bilinear form b with respect to the basis $\boldsymbol{e}$* is the matrix $A = (a_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbf{K})$ defined by

$$a_{ij} = b(\mathbf{e}_i,\mathbf{e}_j), \quad 1 \le i,j \le n.$$

Example 8.4. If $\boldsymbol{e} = (\mathbf{e}_1,\dots,\mathbf{e}_n)$ is the canonical basis of $\mathbf{K}^n$ and b is the bilinear form defined in the previous example with the matrix $A = (a_{ij})$, then, for each i and j

$$b(\mathbf{e}_i,\mathbf{e}_j) = a_{ij}.$$

If A is the identity matrix Id_n then one obtains the *standard symmetric form on* $\mathbf{K}^n$:

$$b(\mathbf{x},\mathbf{y}) = \mathbf{x}^t \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Proposition 8.5. Let $\mathbf{V}$ be a finite dimensional vector space with basis $\boldsymbol{e} = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. Associating to each bilinear form its matrix with respect to $\boldsymbol{e}$ gives rise to a bijection between the set $\mathrm{Bil}(\mathbf{V})$ of bilinear forms on $\mathbf{V}$ and the set $\mathrm{Mat}_{n\times n}(\mathbf{K})$. This bijection induces a bijection of the set of symmetric bilinear forms with the set of symmetric matrices.

- The bijection in Proposition 8.5 depends on the choice of basis.

Definition 8.6. Two matrices $A,B \in \mathrm{Mat}_{n\times n}(\mathbf{K})$ are *congruent* if there exists $M \in \mathrm{GL}_n(\mathbf{K})$ satisfying

$$B = M^t AM.$$

- The congruency of matrices is an equivalence relation in $\mathrm{Mat}_{n\times n}(\mathbf{K})$.

Proposition 8.7. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space of dimension n. Two matrices represent the same bilinear form b on $\mathbf{V}$ with respect to two bases if and only if they are congruent.

- It follows that two congruent matrices have the same rank.

Definition 8.8. By Proposition 8.7, the rank r of a matrix which represents a bilinear form b with respect to some basis does not depend on the basis, but only on b. We call r the *rank of the bilinear form b*.

If b has rank $r = \dim(\mathbf{V})$, we say that the bilinear form b is *non-degenerate*. Otherwise it is a *degenerate* bilinear form.

Proposition 8.9. Let $\mathbf{V}$ be a finite dimensional $\mathbf{K}$-vector space and let $b : \mathbf{V}\times\mathbf{V} \to \mathbf{K}$ be a bilinear form. The following properties are equivalent:

1. b is non-degenerate.
2. For every $\mathbf{v} \neq 0$ in $\mathbf{V}$ there is a $\mathbf{w} \in \mathbf{V}$ for which $b(\mathbf{v},\mathbf{w}) \neq 0$.
3. For every $\mathbf{w} \neq 0$ in $\mathbf{V}$ there is a $\mathbf{v} \in \mathbf{V}$ for which $b(\mathbf{v},\mathbf{w}) \neq 0$.

8.2 Symmetric bilinear forms

Definition 8.10. Let b be a symmetric form on a vector space $\mathbf{V}$ and let $\mathbf{v} \in \mathbf{V}$. A vector $\mathbf{w}$ is said to be *orthogonal to* $\mathbf{v}$ *(with respect to the bilinear form b)* if $b(\mathbf{v},\mathbf{w}) = 0$. In this case, one also says that *the two vectors are orthogonal (with respect to the bilinear form b)* and we write $\mathbf{v} \perp_b \mathbf{w}$ or just $\mathbf{v} \perp \mathbf{w}$.

Let S be a non-empty subset of $\mathbf{V}$. The set of vectors orthogonal to every vector in S is denoted by $S^\perp$:

$$S^\perp = \{\mathbf{w} \in \mathbf{V} : b(\mathbf{v},\mathbf{w}) = 0 \text{ for every } \mathbf{v} \in S\}.$$

The set $S^\perp$ is a vector subspace of $\mathbf{V}$ called the *subspace orthogonal to* S. Two subspaces $\mathbf{U}$ and $\mathbf{W}$ of $\mathbf{V}$ are said to be *orthogonal* if $\mathbf{U} \subseteq \mathbf{W}^\perp$, which is equivalent to $\mathbf{W} \subseteq \mathbf{U}^\perp$.

If $\mathbf{V}$ is finite dimensional and $\boldsymbol{e} = (\mathbf{e}_1,\ldots,\mathbf{e}_n)$ is a basis whose vectors are pairwise orthogonal, that is, if $b(\mathbf{e}_i,\mathbf{e}_j) = 0$ whenever $i \neq j$, then $\boldsymbol{e}$ is called a *diagonalizing basis for b* or *orthogonal basis for b*.

- If $\boldsymbol{e}$ is an orthogonal basis for b, then the matrix $A = (a_{ij})$ of b with respect to $\boldsymbol{e}$ is a diagonal matrix. In such a basis, the bilinear form can therefore be written in the following manner:

$$b(\mathbf{x},\mathbf{y}) = a_{11}x_1y_1 + a_{22}x_2y_2 + \cdots + a_{nn}x_ny_n.$$

- Note that if a diagonalizing basis exists, it is not unique. For example one can rescale the vectors of such a basis.

8.3 Quadratic forms

Definition 8.11. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space and let b be a symmetric bilinear form. The *quadratic form associated to b* is the map $q : \mathbf{V} \to \mathbf{K}$, defined by

$$q(\mathbf{v}) = b(\mathbf{v},\mathbf{v}) \quad \text{for every} \quad \mathbf{v} \in \mathbf{V}.$$

Example 8.12. The quadratic form associated to the standard symmetric bilinear form on $\mathbf{K}^n$ is

$$q(\mathbf{x}) = x_1^2 + x_2^2 + \cdots + x_n^2,$$

and is called *standard quadratic form on* $\mathbf{K}^n$.

Proposition 8.13. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$ with symmetric bilinear form $b : \mathbf{V} \times \mathbf{V} \to \mathbf{K}$. The quadratc form q associated to b satisfies the following two condition:

$$\begin{aligned} q(c\mathbf{v}) &= c^2 q(\mathbf{v}) \\ 2b(\mathbf{v},\mathbf{w}) &= q(\mathbf{v}+\mathbf{w}) - q(\mathbf{v}) - q(\mathbf{w}) \end{aligned}$$

for every $c \in \mathbf{K}$ and every $\mathbf{v},\mathbf{w} \in \mathbf{V}$.

Definition 8.14. From Proposition 8.13 it follows that the quadratic form q determines the symmetric bilinear form b to which it is associated, since b can be expressed in terms of q. We call b the *polar bilinear form* of the quadratic form q. Consequently, assigning a symmetric bilinear form to a vector space $\mathbf{V}$ is equivalent to assigning the associated quadratic form.

Example 8.15. If $\boldsymbol{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a basis of $\mathbf{V}$ and if $A = (a_{ij})$ is the symmetric matrix which represents the bilinear form b, then for the associated quadratic form, we have for every $\mathbf{x}(x_1, \ldots, x_n) \in \mathbf{V}$,

$$q(\mathbf{x}) = \mathbf{x}^t A\mathbf{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

Thus, we may view q as a homogeneous polynomial of degree 2 in the unknowns $x_1, \ldots, x_n$. Conversely, if a homogeneous polynomial

$$Q(\mathbf{x}) = \sum_{1 \leq i \leq j \leq n} q_{ij} x_i x_j$$

is given, then Q determines a quadratic form corresponding to the symmetric bilinear form with matrix A where

$$\begin{array}{ll} a_{ii} = q_{ii}, & i = 1, \ldots, n \\ a_{ij} = a_{ji} = \frac{1}{2} q_{ij} & i \leq j. \end{array}$$

Thus, we will make no distinction between quadratic forms and homogeneous polynomials of degree two, due to the above identification.

- The rank of a quadratic form is by definition the rank of the corresponding polar bilinear form.
- The matrix of a quadratic form relative to a basis $\boldsymbol{e}$ is the matrix of the corresponding polar bilinear form relative to $\boldsymbol{e}$.
- Two matrices A and B represent the same quadratic form relative to two different bases if the symmetric matrices A and B are congruent.
- Suppose that there is a symmetric bilinear form $b : \mathbf{V} \times \mathbf{V} \to \mathbf{K}$ on a vector space $\mathbf{V}$, and let q be the associated quadratic form. If $\mathbf{W}$ is a vector subspace of $\mathbf{V}$ then b induces a map

  $$b' : \mathbf{W} \times \mathbf{W} \to \mathbf{K}$$

  which obviously satisfies the conditions given in the definition of a symmetric bilinear form. Therefore b' is a symmetric bilinear form on $\mathbf{W}$. In the same manner, one has a quadratic form

  $$q' : \mathbf{W} \to \mathbf{K}$$

  obtained by restricting q to $\mathbf{W}$. It is the quadratic form associated to b'.

8.4 Diagonalizing quadratic forms

Theorem 8.16. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space and let b be a symmetric bilinear form on $\mathbf{V}$. Then there exists a diagonalizing basis for b. Equivalently, any symmetric matrix is congruent to a diagonal matrix.

Theorem 8.17 (Sylvester)**.** Let $\mathbf{V}$ be a real vector space of dimension n, and let $b : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ be a symmetric bilinear form on $\mathbf{V}$ of rank $r \leq n$. Then there is an integer p with $0 \leq p \leq r$ depending only on b, and a basis $\boldsymbol{e} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbf{V}$ with respect to which the matrix associated to b has the form

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_{r-p} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.1}$$

where 0 denotes zero matrices of appropriate sizes.

Equivalently, every symmetric matrix $A \in \mathrm{Mat}_{n\times n}(R)$ is congruent to a diagonal matrix of the form (8.1) in which $r = \mathrm{rank}(A)$ and p depends only on A.

- With respect to the basis $\boldsymbol{e}$ in the above theorem, the quadratic form q associated to b is

$$q(\mathbf{x}) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_r^2$$

  where $\mathbf{x} = (x_1, \dots, x_n)$. This is called the *normal form* of the quadratic form q.

- The pair $(p, r-p)$ is called the *signature* of b and of q.

- We have the following notions:
  - if $q(\mathbf{v}) > 0$ for all $\mathbf{v} \neq 0$ then q is *positive definite*, in this case the signature is $(n, 0)$ and the normal form is
  $$x_1^2 + \dots + x_n^2.$$
  - If $q(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbf{V}$ then q is *positive semi-definite*, in this case the signature is $(r, 0)$ and the normal form is
  $$x_1^2 + \dots + x_r^2.$$
  - If $q(\mathbf{v}) < 0$ for all $\mathbf{v} \neq 0$ then q is *negative definite*, in this case the signature is $(0, n)$ and the normal form is
  $$-x_1^2 - \dots - x_n^2.$$
  - If $q(\mathbf{v}) \leq 0$ for all $\mathbf{v} \in \mathbf{V}$ then q is *negative semi-definite*, in this case the signature is $(0, r)$ and the normal form is
  $$-x_1^2 - \dots - x_r^2.$$
  - In all other cases we say that q is *indefinite*, in this case the signature is $(p, r-p)$ and the normal form is
  $$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_r^2.$$

- For a symmetric matrix we have the analogue terminology depending on the type of the quadratic form associated to such a matrix.

Corollary 8.18. A symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is positive definite if and only if there is a matrix $M \in \mathrm{GL}_n(\mathbb{R})$ for which $A = M^T M$.

- The scalar product in $\mathbf{E}^3$ is a positive definite symmetric bilinear form. If q is the associated quadratic form then
$$d(A,B) = \sqrt{q(\overrightarrow{AB})}.$$
- In $\mathbb{R}^4$ the quadratic form
$$q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2$$
is called the *Minkowski form*. It is non-degenerate and indefinite of signature $(3,1)$. The pair $(\mathbb{R}^4, q)$ is called *Minkowski space* and is important in special relativity.
- Covariance matrices are symmetric positive semi-definite.