Exercises- Predicate Logic

Exercise 1

Transform the following sentences from natural language into predicate formulas. Explain the syntactic elements used in the predicate formulas: variables, constants, functions symbols, predicate symbols.

1. In a plane if a line x is perpendicular to a constant line d then all the lines parallel to x are perpendicular to d.
2. In a plane there are lines parallel to a constant line d and there are lines perpendicular to d.
3. If x is a nonzero integer divisible by 10, it can be decomposed in two factors such that one is divisible by 2 and the other one is divisible by 5, and x can be written as a sum of 2 even numbers.
4. Every positive number can be written as a product of two positive numbers and as a product of two negative numbers.
5. If x and y are positive prime numbers, $x \neq y$, $x \neq 2$ and $y \neq 2$, their sum and difference are even numbers and their product is an odd number.
6. For every positive integer x, if x is a square of an integer, then there exists an integer y such that $(y+1)*(y-1)=x-1$
7. For every positive integer x, if x is not a prime, then there exists a prime y such that y divides x and y is less than x.
8. The sum of two even numbers is an even number and their product is divisible by 4.

Exercise 2

Transform the following statements from natural language into predicate formulas choosing the appropriate constants, function symbols and predicate symbols:

1. Every student who makes good grades is brilliant or studies.
2. Some of John's colleagues like to draw and some like to dance.
3. CS students like either algebra or logic, all of them like Java but only Bill likes history.
4. All Mary's relatives live in Cluj-Napoca, only her cousin John lives in Bucharest.
5. Anyone who owns a rabbit hates anything that chases any rabbit.
6. All birds have wings but only penguins do not fly.
7. If Santa has some reindeer with a red nose, then every child loves Santa.
8. Every investor who bought something that falls is not happy.
9. Anyone who has any cats will not have any mice.
10. Caterpillars and snails are much smaller than birds, which are much smaller than foxes, which in turn are much smaller than wolves.
11. Caterpillars and snails like to eat some plants.
12. Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants.

Exercise 3.

Check whether the conclusion C is derivable from the set of hypotheses using the definition of deduction and the appropriate inference rules.

1. **Succession to the British throne**

   Hypotheses:

   H_1. If x is the king and y is his oldest son, then y can become the king.

   H_2. If x is the king and y defeats x, then y will become the king.

   H_3. *RichardIII* is the king.

   H_4. *HenryVII* defeated *RichardIII*.

   H_5. *HenryVIII* is *HenryVII*'s oldest son.

   Conclusion: C. Can *HenryVIII* become the king?

2. **Scrooge, Santa and Rudolph**

   Hypotheses:

   H_1. Every child loves *Santa*.

   H_2. Everyone who loves *Santa* loves any reindeer.

   H_3. *Rudolph* is a reindeer, and *Rudolph* has a red nose.

   H_4. Anything which has a red nose is weird or is a clown.

   H_5. No reindeer is a clown.

   H_6. *Scrooge* does not love anything which is weird.

   Conclusion: C. *Scrooge* is not a child.

Exercise 4

Using the given interpretations evaluate the following formulas:

1. $U = (\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\forall x)(A(x) \vee B(x))$

   Interpretation $I = < D, m >$, where: $D =$ the set of all straight lines of a plane P

   Let $d \in P$, a constant straight line belonging to the interpretation domain

   $m(A): D \rightarrow \{T, F\}, m(A)(x)$: " $x \perp d$ ";

   $m(B): D \rightarrow \{T, F\}, m(B)(x)$: " $x \parallel d$ ";

2. $U = (\exists x)(P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \vee Q(12)$

   Interpretation $I = < D, m >$ where $D = \mathbf{N}$ (the set of natural numbers)

   $m(P): \mathbf{N} \rightarrow \{T, F\}, m(P)(x)$: "$x \vdots 5$";

   $m(Q): \mathbf{N} \rightarrow \{T, F\}, m(Q)(x)$: "$x \vdots 7$";

3. $U = (\forall x)(P(x) \wedge Q(x) \rightarrow P(sq(x)) \wedge Q(prod(x, 5)))$

   Interpretation $I = < D, m >$, where $D = \mathbf{Z}$ (the set of integer numbers),

   $m(P): \mathbf{Z} \rightarrow \{T, F\}, m(P)(x)$: "*x is even*";

   $m(Q): \mathbf{Z} \rightarrow \{T, F\}, m(Q)(x)$: "$x<0$";

   $m(prod): \mathbf{Z}^2 \rightarrow \mathbf{Z},\ m(prod)(x, y) = x * y$

4. $U = (\forall x)(A(x) \rightarrow B(x)) \wedge (\exists x)(B(x) \wedge \neg A(x))$

   Interpretation $I = < D, m >$, where:

   $D = \mathbf{N}$ (the set of natural numbers)

   $m(A): D \rightarrow \{T, F\}, m(A)(x)$: "x is a prime";

   $m(B): D \rightarrow \{T, F\}, m(B)(x)$: "x is an odd number".

5. $U = (\forall x)(\forall y)(A(x) \wedge A(y) \rightarrow \neg A(sum(x, y)) \wedge (\forall z)A(prod(y, z)))$

   Interpretation $I = < D, m >$, where $D = \mathbf{N}$ (the set of natural numbers)

   $m(sum): \mathbf{N}^2 \rightarrow \mathbf{N},\ m(sum)(x, y) = x + y$ and

   $m(prod): \mathbf{N}^2 \rightarrow \mathbf{N},\ m(prod)(x, y) = x * y$

   $m(A): \mathbf{N} \rightarrow \{T, F\}, m(A)(x)$: "x is an odd number",

6. $U = ((\forall x)A(x) \rightarrow (\exists x)B(x)) \rightarrow ((\exists x)A(x) \rightarrow (\forall x)B(x))$

   Interpretation $I = < D, m >$, where:

   $D =$ the set of all persons from Romania

   $m(A): D \rightarrow \{T, F\}, m(A)(x)$: "the person x lives in a city";

   $m(B): D \rightarrow \{T, F\}, m(B)(x)$: "the person x has a job".

7. $U = (\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\forall x)(A(x) \wedge B(x))$.

   Interpretation $I = < D, m >$, where:

   $D =$ N (the set of natural numbers)

   $m(A): D \rightarrow \{T, F\}, m(A)(x)$: "x is a perfect square";

   $m(B): D \rightarrow \{T, F\}, m(B)(x)$: "x is divisible by 10".

Exercise 5

Choose an arbitrary interpretation with a finite domain (2 elements) for the formula $U_j, j \in \{1,2,...,8\}$ and prove that it is a model of $U_j, j \in \{1,2,...,8\}$.

1. $U_1 = (\forall x)(A(x) \leftrightarrow B(x)) \rightarrow ((\forall x)A(x) \leftrightarrow (\forall x)B(x))$;
2. $U_2 = (\forall x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\forall x)B(x))$;
3. $U_3 = (\forall x)(A(x) \leftrightarrow B(x)) \rightarrow ((\exists x)A(x) \leftrightarrow (\exists x)B(x))$;
4. $U_4 = (\exists x)(A(x) \rightarrow B(x)) \leftrightarrow ((\forall x)A(x) \rightarrow (\exists x)B(x))$;

5. $U_5 = ((\exists x)A(x) \to (\forall x)B(x)) \to (\forall x)(A(x) \to B(x))$;
6. $U_6 = (\forall x)(A(x) \vee B(x)) \to ((\forall x)A(x) \vee (\exists x)B(x))$;
7. $U_7 = (\forall x)(A(x) \to B(x)) \to ((\exists x)A(x) \to (\exists x)B(x))$;
8. $U_8 = (\forall x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\exists x)B(x))$.

Exercise 6

Prove that the following formulas are not valid by finding anti-models for them.

1. $U_1 = ((\exists x)P(x) \to (\exists x)Q(x)) \to (\forall x)(P(x) \to Q(x))$
2. $U_2 = (\exists x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$;
3. $U_3 = ((\forall x)P(x) \to (\forall x)Q(x)) \to (\forall x)(P(x) \to Q(x))$;
4. $U_4 = (\exists x)P(x) \wedge (\exists x)Q(x) \to (\exists x)(P(x) \wedge Q(x))$;
5. $U_5 = (\forall x)(P(x) \vee Q(x)) \to (\forall x)P(x) \vee (\forall x)Q(x)$;
6. $U_6 = ((\forall x)P(x) \to (\exists x)Q(x)) \to (\forall x)(P(x) \to Q(x))$;
7. $U_7 = (\exists x)P(x) \wedge (\exists x)Q(x) \to (\forall x)(P(x) \wedge Q(x))$.

Exercise 1

Transform the following sentences from natural language into predicate formulas. Explain the syntactic elements used in the predicate formulas: variables, constants, functions symbols, predicate symbols.

The sum of two even numbers is an even number and their product is divisible by 4.

$D = \mathbb{Z}$

Predicate symbols (Predicate $\to$ T/F)

$even \in P_1$, $even(x)$: "x even", $even : D \to \{T, F\}$

$divisible \in P_2$, $divisible(x, y)$: "$x \vdots y$", $divisible : D \times D \to \{T, F\}$

Function symbols

$sum \in F_2$, $sum(x, y) =$ "$x+y$", $sum : D \times D \to D$

$product \in F_2$, $product(x, y) =$ "$x \cdot y$", $product : D \times D \to D$

$(\forall x)(\forall y)\ (even(x) \wedge even(y) \to even(sum(x, y)) \wedge divisible(product(x, y), 4))$

- PROPERTIES

① divisible
- reflexive : $(\forall x)\ (divisible(x, x))$
- symmetrical : X
- transitivity : $(\forall x)(\forall y)(\forall z)\ (divisible(x, y) \wedge divisible(y, z) \to divisible(x, z))$

② sum
- commutative : $(\forall x)(\forall y)\ (x + y = y + x)$
- associative : $(\forall x)(\forall y)(\forall z)\ ((x+y)+z = x+(y+z))$
- identity element : $(\exists i)(\forall x)\ (x + i = x)$

③ product
- commutative : $(\forall x)(\forall y)\ (x \cdot y = y \cdot x)$
- associative : $(\forall x)(\forall y)(\forall z)\ ((x \cdot y) \cdot z = x \cdot (y \cdot z))$
- identity element : $(\exists i)(\forall x)\ (x \cdot i = x)$

Exercise 3.

Check whether the conclusion C is derivable from the set of hypotheses using the definition of deduction and the appropriate inference rules.

Every investor who bought something that falls is not happy.

D_1 = all people, D_2 = all stock

PREDICATES

↳ $investor \in P_1$, $investor(x)$: "x is investor", $investor : D_1 \to \{T, F\}$

↳ $bought \in P_2$, $bought(x, y)$: "x bought y", $bought : D_1 \times D_2 \to \{T, F\}$

↳ $falls \in P_1$ ...

↳ $happy \in P_1$ ...

$(\forall x)\ (investor(x) \wedge (\exists y)\ bought(x, y) \wedge falls(y)) \to \neg happy(x))$

Exercise 3.
Check whether the conclusion C is derivable from the set of hypotheses using the definition of deduction and the appropriate inference rules.

Scrooge, Santa and Rudolph

Hypotheses:

H_1. Every child loves *Santa*.

H_2. Everyone who loves *Santa* loves any reindeer.

H_3. *Rudolph* is a reindeer, and *Rudolph* has a red nose.

H_4. Anything which has a red nose is weird or is a clown.

H_5. No reindeer is a clown.

H_6. *Scrooge* does not love anything which is weird.

Conclusion: C. *Scrooge* is not a child.

PREDICATES

- child(x): "x is a child"
- loves(x,y): "x loves y"
- rd(x): "x is reindeer"
- rn(x): "x has a red nose"
- w(x): "x is weird"
- cl(x): "x is clown"

$C: \neg child(Sc)$

$f_1: (\forall x)(child(x) \rightarrow loves(x, Sa))$

$f_2: (\forall x)(\forall y)(loves(x, Sa) \wedge rd(y) \rightarrow loves(x, y))$

$f_3: rd(R) \wedge rn(R)$

$f_4: (\forall x)(rn(x) \rightarrow w(x) \vee cl(x))$

$f_5: (\forall x)(rn(x) \rightarrow \neg cl(x))$

$f_6: (\forall x)(w(x) \rightarrow \neg loves(Sc, x))$

$f_7: child(Sc) \rightarrow loves(Sc, Sa)$ $[f_1 \vdash_{univ.\ inst.}]$

$f_8: (\forall y)(loves(Sc, Sa) \wedge rd(y) \rightarrow loves(Sc, y))$ $[f_2 \vdash_{univ.\ inst.}]$

$f_9: loves(Sc, Sa) \wedge rd(R) \rightarrow loves(Sc, R)$ $[f_8 \vdash_{univ.\ inst.}]$

$f_{10}: w(R) \vee cl(R) \equiv \neg cl \rightarrow w(R)$ $[f_4 \vdash_{univ.\ inst.}]$

$f_{11}: rd(R) \rightarrow \neg cl(R)$ $[f_5 \vdash_{univ.\ inst.}]$

$f_{12}: \neg cl(R)$ $[f_3, f_{11} \vdash_{mp}]$

$f_{13}: w(R)$ $[f_{10}, f_{12} \vdash_{mp}]$

$f_{14}: w(R) \rightarrow \neg loves(Sc, R)$ $[f_6 \vdash_{univ.\ inst.}]$

$f_{15}: \neg loves(Sc, R)$ $[f_{13}, f_{14} \vdash_{mp}]$

$f_{16}: \neg loves(Sc, R) \rightarrow \neg loves(Sc, Sa) \vee \neg rd(R)$ $[f_9 \vdash_{int}]$

$f_{17}: \neg loves(Sc, Sa) \vee \neg rd(R) \equiv rd(R) \rightarrow \neg loves(Sc, Sa)$ $[f_{15}, f_{16} \vdash_{mp}]$

$f_{18}: rd(R)$ $[f_3 \vdash_{simpl.}]$

$f_{19}: \neg loves(Sc, Sa)$ $[f_{18,17} \vdash_{mp}]$

$f_{20}: \neg loves(Sc, Sa) \rightarrow \neg child(Sc)$ $[f_7 \vdash_{int}]$

$f_{21}: \neg child(Sc)$ $[f_{19}, f_{20} \vdash_{mp}]$ ✓

Exercise 4

Using the given interpretations evaluate the following formulas:

$U_2 = (\exists x)(P(x) \land Q(x)) \to (\exists x)(P(x) \lor Q(12))$

DISTRIBUTIVITY OF QUANTIFIERS

$(\exists x)(A(x) \lor B(x)) \equiv (\exists x)(A(x)) \lor (\exists x)(B(x))$

$(\forall x)(A(x) \land B(x)) \equiv (\forall x)(A(x)) \land (\forall x)(B(x))$

Interpretation $I = \langle D, m \rangle$, where $D = \mathbb{N}$

$m(P)(x)$: "$x \vdots 5$"

$m(Q)(x)$: "$x \vdots 7$"

$$v^I(U_2) = v^I(\exists x)(P(x) \land Q(x)) \to v^I(\exists x)(P(x) \lor Q(12))$$

$$= \underbrace{(\exists x)_{x \in \mathbb{N}}(x \vdots 5 \land x \vdots 7)}_{T} \to (\exists x)_{x \in \mathbb{N}}(x \vdots 5 \lor 12 \vdots 7)$$

$$T \quad \to \quad T \lor F$$

$$= T \to T = T$$

Exercise 5

Choose an arbitrary interpretation with a finite domain (2 elements) for the formula $U_j, j \in \{1,2,\ldots,8\}$ and prove that it is a model of $U_j, j \in \{1,2,\ldots,8\}$.

$U_2 = (\forall x)(A(x) \to B(x)) \to ((\forall x)\ A(x) \to (\forall x)\ B(x))$

$(\forall x)_{x \in \{a,b\}}\ P(x) \equiv P(a) \land P(b)$

$(\exists x)_{x \in \{a,b\}}\ P(x) \equiv P(a) \lor P(b)$

$I_1 = \langle D, m \rangle$, where $D = \{2,3\}$

$m(A)(x)$ = "$x \vdots 2$"

$m(B)(x)$ = "$x = 3$"

$$v^I(U_2) = v^I(\forall x)(A(x) \to B(x)) \to v^I((\forall x)\ A(x) \to (\forall x)\ B(x))$$

$$= \underbrace{\underbrace{(\underbrace{A(2)}_{T} \to \underbrace{B(2)}_{F})}_{F} \land \underbrace{(\underbrace{A(3)}_{F} \to \underbrace{B(3)}_{T})}_{T}}_{F} \to \underbrace{(\underbrace{\underbrace{A(2)}_{T} \land \underbrace{A(3)}_{F}}_{F} \to \underbrace{\underbrace{B(2)}_{F} \land \underbrace{B(3)}_{T}}_{F})}_{T}$$

$$T$$

Exercise 6

Prove that the following formulas are not valid by finding anti-models for them.

$$U_2 = \underbrace{(\exists x)(P(x) \to Q(x))}_{T} \to \underbrace{(\overset{T}{(\exists x)\ P(x)} \to \overset{F}{(\exists x)\ Q(x)})}_{F}$$

In order to have antimodels, the premise should be true but the conclusion false.

$I = \langle D, m \rangle$, D = all even numbers $\qquad D = \{1\}$

$m(P)(x)$ = "x is odd" $\qquad p$ = odd

$m(Q)(y)$ = "x is even" $\qquad \{2\}$

$$v^I(U_2) = v^I(\exists x)(P(x) \to Q(x)) \to v^I((\exists x)\ P(x) \to (\exists x)\ Q(x))$$

$$= \underbrace{(\exists x)_{x \in D}(\neg \mathrm{odd}(x) \lor \mathrm{even}(x)}_{T} \to (\underbrace{(\exists x)_{x \in D}\ \mathrm{odd}(x)}_{F} \to \underbrace{(\exists x)_{x \in D}\ \mathrm{even}(x)}_{T})$$

$$= \qquad T \qquad F$$

1. **Succession to the British throne**

Hypotheses:

H_1. If x is the king and y is his oldest son, then y can become the king.

H_2. If x is the king and y defeats x, then y will become the king.

H_3. *RichardIII* is the king.

H_4. *HenryVII* defeated *RichardIII*.

H_5. *HenryVIII* is *HenryVII*'s oldest son.

Conclusion: C. Can *HenryVIII* become the king? $c(H8)$

$c(x)$ = "x can become king"

$k(x)$ = "x is the king"

$o(x, y)$ = "x oldest son of y"

$d(x, y)$ = "x defeats y"

$f_1: (\forall x)(\forall y)(k(x) \wedge o(y, x) \rightarrow c(y))$

$f_2: (\forall x)(\forall y)(k(x) \wedge d(y, x) \rightarrow k(y))$

$f_3: k(R)$

$f_4: d(H7, R)$

$f_5: o(H8, H7)$

$f_6: (\forall y)(k(H7) \wedge o(y, H7) \rightarrow c(y))$

$f_7: k(H7) \wedge o(H8, H7) \rightarrow c(H8)$

$f_8: (\forall y)(k(R) \wedge d(y, R) \rightarrow k(y))$

$f_9: k(R) \wedge d(H7, R) \rightarrow k(H7)$

$f_{10}: k(H7)$ (f_3, f_4, f_9 tmp)

$f_{11}: c(H8)$ (f_{10}, f_5, f_6 tmp)