Homework 5

E1:

a) $\sum_{n\geq 3} \frac{7}{9^n} = 7\cdot \sum_{n\geq 3}\left(\frac{1}{9}\right)^n \to$ the geometric series of ratio $q=\frac{1}{9}$

$|q|<1 \Rightarrow$ the series is convergent

$$\sum_{n=3}^{\infty}\frac{7}{9^n} = 7\cdot\sum_{n=3}^{\infty}\left(\frac{1}{9}\right)^n = 7\left(\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n - {}^{9)}\frac{1}{9} - \frac{1}{9^2}\right) =$$

$$= 7\left(\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n - \frac{8}{81}\right) = 7\left(\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^{n-1}\cdot\frac{1}{9} - \frac{8}{81}\right) =$$

$$= 7\left(\frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^{n-1} - \frac{8}{81}\right) = 7\left(\frac{1}{9}\cdot\frac{1}{{}^{9)}1-\frac{1}{9}} - \frac{8}{81}\right) =$$

$$= 7\left(\frac{1}{9}\cdot\frac{1}{\frac{8}{9}} - \frac{8}{81}\right) = 7\left(\frac{1}{\cancel{9}}\cdot\frac{\cancel{9}}{8} - \frac{8}{81}\right) =$$

$$= 7\left({}^{81/}\frac{1}{8} - {}^{8/}\frac{8}{81}\right) = 7\left(\frac{81-64}{648}\right) = 7\cdot\frac{17}{648} = \frac{119}{648}$$

b) $$\sum_{n\geq 4}\frac{3^{n-3}+(-4)^{n+3}}{5^n} = \sum_{n\geq 4}\left(\left(\frac{3}{5}\right)^{n-3}\cdot\frac{1}{5^3} + \left(\frac{-4}{5}\right)^n\cdot(-4)^3\right) =$$

$$= \underbrace{\sum_{n\geq 4}\left(\frac{3}{5}\right)^{n-3}\cdot\frac{1}{5^3}}_{\text{the geometric series of ratio } q_1=\frac{3}{5}} + \underbrace{\sum_{n\geq 4}\left(-\frac{4}{5}\right)^n\cdot(-4)^3}_{\text{the geometric series of ratio } q_2=-\frac{4}{5}}$$

$\Rightarrow$ convergent because $|q_1|<1$

$\Rightarrow$ convergent because $|q_2|<1$

$$\sum_{n=4}^{\infty}\left(\frac{3}{5}\right)^{n-3}\cdot\frac{1}{125}=\frac{1}{125}\sum_{n=4}^{\infty}\left(\frac{3}{5}\right)^{n-3}=\frac{1}{125}\left(\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^{n-3}-\left(\frac{5}{3}\right)^{2}-\frac{5}{3}-1\right)=$$

$$=\frac{1}{5^3}\left(\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^{n-1}\cdot\left(\frac{3}{5}\right)^{-2}-\left(\frac{5}{3}\right)^{2}-\frac{5}{3}-1\right)=$$

$$=\frac{1}{\cancel{5^3}}\cdot\frac{\cancel{5^2}}{3^2}\cdot\left(\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^{n-1}\right)-\frac{1}{5^3}\left(\frac{5^2}{3^2}-\frac{5}{3}-1\right)=$$

$$=\frac{1}{45}\left(\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^{n-1}\right)-\frac{1}{\cancel{5^3}}\cdot\frac{\cancel{5^2}}{3^2}-\frac{1}{\cancel{5^3}}\cdot\frac{\cancel{5}}{3}-\frac{1}{5^3}\cdot 1=$$

$$=\frac{1}{45}\cdot\frac{1}{1-\frac{3}{5}}-\frac{1}{45}-\frac{1}{75}-\frac{1}{125}=$$

$$=\frac{1}{\cancel{45}_{9}}\cdot\frac{\cancel{5}}{2}-\frac{1}{45}-\frac{1}{75}-\frac{1}{125}=\frac{{}^{5/}1}{18}-\frac{{}^{2/}1}{45}-\frac{1}{75}-\frac{1}{125}=$$

$$=\frac{3}{90}-\frac{{}^{5/}1}{75}-\frac{{}^{3/}1}{125}=\cancel{\frac{3}{90}}-\frac{2}{375}=\frac{{}^{25/}1}{30}-\frac{{}^{2/}2}{375}=$$

$$=\frac{25-4}{750}=\cancel{\frac{21}{750}}=\frac{7}{250}\quad(1)$$

$$\sum_{n=4}^{\infty}\left(-\frac{4}{5}\right)^{n}\cdot(-4)^{3}=\sum_{n=4}^{\infty}\left(-\frac{4}{5}\right)^{n}\cdot-64=(-64)\left(\sum_{n=4}^{\infty}\left(-\frac{4}{5}\right)^{n}\right)=$$

$$=-64\cdot\left(\sum_{n=1}^{\infty}\left(-\frac{4}{5}\right)^{n}-\left(\frac{-4}{5}\right)-\left(\frac{-4}{5}\right)^{2}-\left(\frac{-4}{5}\right)^{3}\right)=$$

$$=-64\cdot\left(\sum_{n=1}^{\infty}\left(-\frac{4}{5}\right)^{n-1}\cdot\left(\frac{-4}{5}\right)+\frac{4}{5}-\frac{16}{25}+\frac{64}{125}\right)=$$

$$=-64\left(-\frac{4}{5}\cdot\frac{1}{1+\frac{4}{5}}+\frac{4}{5}-\frac{16}{25}+\frac{64}{125}\right)=$$

$$=\frac{64\cdot4}{\cancel{5}}\cdot\frac{\cancel{5}}{9}-\frac{64\cdot4}{5}+\frac{64\cdot16}{25}-\frac{64\cdot64}{125}=$$

$$=\frac{64\cdot4}{9}-\frac{64\cdot4}{5}+\frac{64\cdot16}{25}-\frac{64\cdot64}{125}=\frac{{}^{5/}4^4}{3^2}-\frac{{}^{9/}4^4}{5}+\frac{{}^{5/}4^6}{5^2}-\frac{4^8}{5^3}=$$

$$=\frac{5\cdot4^4-9\cdot4^4}{45}+\frac{5\cdot4^6-4^8}{5^3}=\frac{4^4(5-9)}{45}+\frac{4^6(5-4^2)}{5^3}=$$

$$= \frac{25/4^4 \cdot (-4)}{45} + \frac{9/4^6(-11)}{125} =$$

$$= \frac{-4^5 \cdot 25 + 4^6(-99)}{1125} = \frac{4^5(4\cdot(-99) - 25)}{1125} = \frac{4^5(-421)}{1125} = \frac{(-421)4^5}{1125} =$$

$$= \frac{(-421)\cdot 1024}{1125} = -\frac{421\cdot 4^5}{1125} \quad (2)$$

(1) & (2) $\Rightarrow$ The final solution is:

$$\frac{7}{250} - \frac{421\cdot 4^5}{1125} =$$

c) $\sum\limits_{n\geq 5} e^n \Rightarrow \sum\limits_{n=1}^{\infty} e^n - e - e^2 - e^3 - e^4 = \sum\limits_{n=1}^{\infty} e^{n-1} \cdot e - \frac{e^5 - e}{e-1} =$

the geometric series of ratio $q = e$

$\Rightarrow$ divergent because $|q| > 1$

$$= e \cdot \sum_{n=1}^{\infty} e^{n-1} - \frac{e^5 - e}{e-1} =$$

$$= e \cdot (+\infty) - \frac{e^5 \cdot e}{e-1} =$$

$$= +\infty$$

d) $\sum\limits_{n\geq 2} \left(-\frac{1}{\pi}\right)^n \Rightarrow \sum\limits_{n=2}^{\infty} \left(-\frac{1}{\pi}\right)^n = \sum\limits_{n=1}^{\infty} \left(-\frac{1}{\pi}\right)^n - \frac{-1}{\pi} =$

the geometric series of ratio $q = -\frac{1}{\pi}$

$\Rightarrow$ convergent because $|q| < 1$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{\pi}\right)^{n-1} \cdot \frac{-1}{\pi} + \frac{1}{\pi} =$$

$$= \frac{-1}{\pi} \sum_{n=1}^{\infty} \left(-\frac{1}{\pi}\right)^{n-1} + \frac{1}{\pi} =$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1 + \frac{1}{\pi}} + \frac{1}{\pi} =$$

$$= -\frac{1}{\pi} \cdot \frac{\pi}{\pi + 1} + \frac{1}{\pi} =$$

$$= \frac{1}{\pi} - \frac{1}{\pi + 1} = \frac{\cancel{\pi} + 1 - \cancel{\pi}}{\pi(\pi+1)} = \frac{1}{\pi(\pi+1)}$$

c) $\underbrace{\sum_{n\geq 3} (-4)^n}$ $\Rightarrow$

the geometric series
of ratio $q=-4$
$\Rightarrow$ divergent because $|q| \geq 1$

$$\nexists \lim_{n\to\infty} (-4)^n \Rightarrow \nexists \sum_{n=3}^{\infty} (-4)^n$$

E2:

a) $\sum_{n\geq 1} \frac{1}{4n^2-1}$

$$u_n = \frac{1}{(2n-1)(2n+1)} = \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\cdot\frac{1}{2}$$

$$S_n = \frac{1}{2}\left(\frac{1}{2-1} - \frac{1}{2+1}\right) + \frac{1}{2}\left(\frac{1}{2\cdot 2-1} - \frac{1}{2\cdot 2+1}\right) + \dots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right) =$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{2}(1-0) = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

b) $\sum_{n\geq 1} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

$$u_n = \frac{\cancel{\frac{\sqrt{n}-\sqrt{n+1}}{1}}}{\sqrt{n}+\sqrt{n+1}} = \frac{\sqrt{n}-\sqrt{n+1}}{n-(n+1)} = \frac{\sqrt{n}-\sqrt{n+1}}{\cancel{n}-\cancel{n}-1} =$$

$$= \sqrt{n+1} - \sqrt{n}$$

$$S_n = u_1 + \dots + u_n = \sqrt{2} - \sqrt{1} + \cancel{\sqrt{3}} - \cancel{\sqrt{2}} + \dots + \sqrt{n+1} - \cancel{\sqrt{n}} = \sqrt{n+1} - 1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\sqrt{n+1}-1) = +\infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} = +\infty$$

c) $\sum_{n\geq 5} \frac{1}{n(n+1)(n+2)}$

$$u_n = \frac{1}{n(n+1)(n+2)} = \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\cdot\frac{1}{2}$$

$$S_n = u_5 + u_6 + \dots + u_n = \frac{1}{2}\left(\frac{1}{5\cdot 6} - \frac{1}{6\cdot 7}\right) + \frac{1}{2}\left(\frac{1}{6\cdot 7} - \frac{1}{7\cdot 8} + \dots +\right.$$

$$+ \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{1\cdot 2} - \cancel{\frac{1}{2\cdot 3}} + \cancel{\frac{1}{2\cdot 3}} - \cancel{\frac{1}{3\cdot 4}} + \dots + \cancel{\frac{1}{n(n+1)}} - \frac{1}{(n+1)(n+2)}\right) - \frac{1}{2}\left(\frac{1}{1\cdot 2} - \cancel{\frac{1}{2\cdot 3}} + \cancel{\frac{1}{2\cdot 3}} - \cancel{\frac{1}{3\cdot 4}} + \cancel{\frac{1}{4\cdot 5}} - \frac{1}{5\cdot 6}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{30}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) - \frac{1}{2}\cdot\frac{14}{30}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \frac{1}{4} - \frac{14}{60} = \frac{1}{60} \Rightarrow \sum_{n=1}^{n} \frac{1}{n(n+1)(n+2)} = \frac{1}{60}$$

d) $\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$

$$u_n = \ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$$

$$S_n = u_1 + \dots + u_n = \cancel{\ln 2} - \ln 1 + \ln 3 - \cancel{\ln 2} + \dots + \ln(n+1) - \cancel{\ln n} =$$

$$= \ln(n+1) - 0 = \ln(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = +\infty \Rightarrow \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = +\infty$$

e) $\displaystyle\sum_{n\geq 2} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)}$

$$u_n = \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+2)}\right)} = \frac{\ln\frac{n+1}{n}}{\ln(n+1)\cdot\ln n} = \frac{\ln(n+1)-\ln n}{\ln(n+1)\cdot\ln n} =$$

$$= \frac{\cancel{\ln(n+1)}}{\cancel{\ln(n+1)}\cdot\ln n} - \frac{\cancel{\ln n}}{\ln(n+1)\cdot\cancel{\ln n}} = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$$

$$S_n = u_2 + u_3 + \ldots + u_n = \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \ldots +$$

$$+ \frac{1}{\ln n} - \frac{1}{\ln(n+1)} =$$

$$= \frac{1}{\ln 2} - \frac{1}{\ln(n+1)}$$

$$\lim_{n\to\infty} S_n = \frac{1}{\ln 2} \Rightarrow \sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)} = \frac{1}{\ln 2}$$