COURSE 12

Dimension

Let $(K, +, \cdot)$ be a field and let V be a vector space over K.

• An n-tuple $B = (v_1, \dots, v_n) \in V^n$ is **a basis** of V if $\langle B \rangle = V$ and B is linearly independent in V.

• Let V be a vector space over K. A list $B = (v_1, \dots, v_n)$ of vectors in V is a basis of V if and only if each vector $v \in V$ can be uniquely written as a linear combination of the vectors $v_1, \dots, v_n$, i.e.

$$\forall v \in V,\ \exists k_1, \dots, k_n \in K:\ v = k_1 v_1 + \dots + k_n v_n.$$

uniquely determined

Theorem 1. Every vector space has a basis.

Remarks 2. (1) We have proved the existence of a basis of a vector space. We saw in a previous example that a space may have more than one basis.

(2) In the proof of Theorem 1 we saw that if B is an n-elements set which generates V one can successively eliminate elements from B in order to find a basis for V. It follows that any basis of V has at most n vectors. Later we will prove even a stronger result, namely if a vector space has a basis of n elements, then all its bases have n elements.

Theorem 3. i) Let $f : V \to V'$ be a K-linear map and let $B = (v_1, \dots, v_n)$ be a basis of V. Then f is determined by its values on the vectors of the basis B.

ii) Let $f, g : V \to V'$ be K-linear maps and let $B = (v_1, \dots, v_n)$ be a basis of V. If $f(v_i) = g(v_i)$, for any $i \in \{1, \dots, n\}$, then $f = g$.

Proof. i) B basis for $V \Rightarrow \forall v \in V$, $\exists\, k_1, \dots, k_n \in K$ uniquely determined s.t. $v = k_1 v_1 + \dots + k_n v_n$ $(*)$

$\Rightarrow f(v) = f(k_1 v_1 + \dots + k_n v_n) = k_1 f(v_1) + \dots + k_n f(v_n)$

ii) $\forall v \in V$ given by $(*)$. Then

$f(v) = k_1 f(v_1) + \dots + k_n f(v_n) = k_1 g(v_1) + \dots + k_n g(v_n) =$

$= g(k_1 v_1 + \dots + k_n v_n) \overset{(*)}{=} g(v)$.

Thus $f = g$.

□

Remark 4. From the previous theorem one deduces that *given two K-vector spaces V, V', a basis B of V and a function $f' : B \to V'$, there exists a unique linear map $f : V \to V'$ which extends f' (i.e. $f|_B = f'$ or, equivalently, $f(x_i) = f'(x_i)$, $i = 1, \ldots, n$)*, result also known as **universal property of vector spaces**.

Theorem 5. Let $f : V \to V'$ be a K-linear map. Then:
(i) f is injective if and only if for any X linearly independent in V, $f(X)$ is linearly independent in V'.
(ii) f is surjective if and only if for any X system of generators for V, $f(X)$ is a system of generators for V'.
(i)+(ii) $\Rightarrow$ (iii) f is bijective (isom.) if and only if for any X basis of V, $f(X)$ is a basis of V'.

Proof.
(i) "$\Longrightarrow$" f injective, $X=(x_1,\ldots,x_n)$ l. indep. (hence mutually different)
$f(X) = (f(x_1),\ldots,f(x_n))$ l. indep. (?) Let $k_1,\ldots,k_n \in K$ arbitrary.
$k_1 f(x_1)+\ldots+k_n f(x_n) = 0 \Longleftrightarrow f(k_1x_1+\ldots+k_nx_n) = 0 = f(0) \xRightarrow{f\text{ inj.}}$
$\Longrightarrow k_1x_1+\ldots+k_nx_n = 0$ (in V) $\underset{\text{l. indep.}}{\Longrightarrow} k_1=\ldots=k_n=0$.

"$\Longleftarrow$" Let $x,y\in V$, $x\neq y \Rightarrow x-y\neq 0 \Longleftrightarrow x-y$ l. indep. vector in $V \Longrightarrow$
$\Longrightarrow f(x-y)$ l. indep. in $V' \Longrightarrow f(x)-f(y) = f(x-y)\neq 0 \Longrightarrow f(x)\neq f(y)$.
Hence f injective.

(ii) $X=(x_1,\ldots,x_n)$ arbitrary list of (mutually different) vectors from V which generates V.
$\langle f(X)\rangle = f(\langle X\rangle) = f(V)$.
f surjective $\Longleftrightarrow f(V)=V' \Longleftrightarrow \langle f(X)\rangle = V'$.

(iii) is obvious.

□

Let us now discuss a key theorem for proving that any two bases of a vector space have the same number of elements. But it is worth mentioning that it has a much broader importance in Linear Algebra.

Theorem 6. (Steinitz, The Exchange Theorem) Let V be a K-vector space, $X = (x_1, \dots, x_m)$ be a linearly independent list of vectors of V and $Y = (y_1, \dots, y_n)$ a system of generators of V ($m, n \in \mathbb{N}^*$). Then $m \leq n$ and m vectors of Y can be replaced by the vectors of X in order to obtain a system of generators for V.

Proof. We prove this result by way of induction on m. Let us take $m = 1$. Then clearly $m \leq n$. Since Y is a system of generators for V, we have $x_1 = \sum_{i=1}^{n} k_i y_i$ for some $k_1, \dots, k_n \in K$. The list $X = \{x_1\}$ is linearly independent, hence $x_1 \neq 0$. It follows that there exists $j \in \{1, \dots, n\}$ such that $k_j \neq 0$. Then

$$y_j = k_j^{-1} x_1 - \sum_{\substack{i=1 \\ i \neq j}}^{n} k_j^{-1} k_i y_i ,$$

that is, y_j is a linear combination of the vectors $y_1, \dots, y_{j-1}, x_1, y_{j+1}, \dots, y_n$. Hence, in any linear combination of $y_1, \dots, y_n$, the vector y_j can be expressed as a linear combination of the other vectors and x_1. Therefore, we have

$$V = \langle y_1, \dots, y_n \rangle = \langle y_1, \dots, y_{j-1}, x_1, y_{j+1}, \dots, y_n \rangle .$$

Thus, we have obtained again a system of n generators for V containing x_1.

Let us assume that the statement holds for a list with $m-1$ linearly independent vectors of V ($m \in \mathbb{N}$, $m \geq 2$) and let us prove it for the linearly independent list $X = (x_1, \dots, x_m)$. Then $(x_1, \dots, x_{m-1})$ is also linearly independent in V. By the induction step hypothesis, we have $m - 1 \leq n$. If necessary, we can reindex the elements of Y and we have

$$V = \langle x_1, \dots, x_{m-1}, y_m, \dots, y_n \rangle .$$

Assume by contradiction that $m - 1 = n$. Then from $V = \langle x_1, \dots, x_{m-1} \rangle$ it follows that $x_m \in \langle x_1, \dots, x_{m-1} \rangle$, which is absurd since X is linearly independent in V. Thus $m - 1 < n$ or, equivalently, $m \leq n$.

We have $x_m \in V = \langle x_1, \dots, x_{m-1}, y_m, \dots, y_n \rangle$, hence

$$x_m = \sum_{i=1}^{m-1} k_i x_i + \sum_{i=m}^{n} k_i y_i$$

for some $k_1, \dots, k_n \in K$. The list X being linearly independent in V, it follows that there exists $m \leq j \leq n$ such that $k_j \neq 0$ (otherwise, $x_m = \sum_{i=1}^{m-1} k_i x_i$ and the list X would be linearly dependent in V). For simplicity of writing, assume that $j = m$. It follows that

$$y_m = k_m^{-1} x_m - \sum_{i=1}^{m-1} k_m^{-1} k_i x_i - \sum_{i=m+1}^{n} k_m^{-1} k_i y_i .$$

Thus, $y_m \in \langle x_1, \dots, x_m, y_{m+1}, \dots, y_n \rangle$. Therefore, we have

$$V = \langle x_1, \dots, x_{m-1}, y_m, \dots, y_n \rangle = \langle x_1, \dots, x_m, y_{m+1}, \dots, y_n \rangle .$$

Thus, we have obtained again a system of generators for V, where m vectors of the list Y have been replaced by the vectors of the list X. This completes the proof. $\square$

Theorem 7. Any two bases of a vector space have the same number of elements.

Proof. Let B_1, B_2 bases for ${}_K V$, $|B_1| = m$, $|B_2| = n$, $m, n \in \mathbb{N}^*$ $\square$

$$\left.\begin{array}{l} B_1 \text{ l. indep. set} \\ \langle B_2 \rangle = V \end{array}\right| \xRightarrow{T6} m \le n$$
$$\left.\begin{array}{l} B_2 \text{ l. indep. set} \\ \langle B_1 \rangle = V \end{array}\right| \xRightarrow{T6} n \le m$$
$$\Rightarrow m = n.$$

R: $m = 0 \Rightarrow V = \{0\} \Rightarrow B_1 = B_2 = \emptyset \Rightarrow m = n = 0$.

Definition 8. Let V be a vector space over K. Then the number of elements of any of its bases is called the **dimension of** V and is denoted by $\dim_K V$ or simply by $\dim V$.

Examples 9. Using the bases given in the previous course examples, one can easily determine the dimension of those vector spaces.

(a) If $V = \{0\}$, V has the basis $\emptyset$ and $\dim V = 0$.

(b) Let K be a field and $n \in \mathbb{N}^*$. Then $\dim_K K^n = n$. In particular, $\dim_{\mathbb{C}} \mathbb{C} = 1$.

(c) $\dim_{\mathbb{R}} \mathbb{C} = 2$. $\forall z \in \mathbb{C}, \exists!\, a, b \in \mathbb{R}$ s.t. $z = a \cdot 1 + b \cdot i \Rightarrow \{1, i\}$ basis for ${}_{\mathbb{R}}\mathbb{C}$.

(d) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ and $T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$ are subspaces of ${}_{\mathbb{R}}\mathbb{R}^3$ with $\dim S = 2$ and $\dim T = 1$. More general, the subspaces of $\mathbb{R}^3$ are $\{(0, 0, 0)\}$, any line containing the origin, any plane containing the origin and ${}_{\mathbb{R}}\mathbb{R}^3$. Their dimensions are 0, 1, 2 and 3, respectively.

(e) Let K be a field and $n \in \mathbb{N}$. Then $\dim K_n[X] = n + 1$.

(f) If V_1 and V_2 are K-vector spaces and $B_1 = (x_1, \ldots, x_m)$ and $B_2 = (y_1, \ldots, y_n)$ are bases for V_1 and V_2, respectively, then $\dim(V_1 \times V_2) = m + n = \dim V_1 + \dim V_2$.

Theorem 10. Let V be a vector space over K. Then the following statements are equivalent:

(i) $\dim V = n$;

(ii) The maximum number of linearly independent vectors in V is n;

(iii) The minimum number of generators for V is n.

Proof. (i)$\Rightarrow$(ii) Assume $\dim V = n$. Let $B = (v_1, \ldots, v_n)$ be a basis of V. Since B is a system of generators for V, any linearly independent list in V must have at most n elements by Theorem 6.
(ii)$\Rightarrow$(i) Let $B = (v_1, \ldots, v_m)$ be a basis of V and let $(u_1, \ldots, u_n)$ be a linearly independent list in V. Since B is linearly independent, we have $m \le n$ by hypothesis. Since B is a system of generators for V, we have $n \le m$ by Theorem 6. Hence $m = n$ and consequently $\dim V = n$.
(i)$\Rightarrow$(iii) Assume $\dim V = n$. Let $B = (v_1, \ldots, v_n)$ be a basis of V. Since B is a linearly independent list in V, any system of generators for V must have at least n elements by Theorem 6.
(iii)$\Rightarrow$(i) Let $B = (v_1, \ldots, v_m)$ be a basis of V and let $(u_1, \ldots, u_n)$ be a system of generators for V. Since B is a system of generators for V, we have $n \le m$ by hypothesis. Since B is linearly independent, we have $m \le n$ by Theorem 6. Hence $m = n$ and consequently $\dim V = n$. $\square$

Theorem 11. Let V be a vector space over K with $\dim V = n$ and $X = (u_1, \ldots, u_n)$ a list of vectors in V. Then X is linearly independent in V if and only if X is a system of generators for V.

Proof. Let $B = (v_1, \ldots, v_n)$ be a basis of V.

Let us assume that X is linearly independent. Since B is a system of generators for V, we know by Theorem 6 that n vectors of B, i.e., all the vectors of B, can be replaced by the vectors of X and we get another system of generators for V. Hence $\langle X \rangle = V$. Thus, X is a system of generators for V.

Conversely, let us suppose that X is a system of generators for V. Assume by contradiction that X is linearly dependent. Then there exists $j \in \{1, \ldots, n\}$ such that

$$u_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i u_i$$

for some $k_i \in K$. It follows that $V = \langle X \rangle = \langle u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n \rangle$. This contradicts the fact that the minimum number of generators for V is n (see Theorem 10). Thus our assumption must have been false. So X is linearly independent. $\square$

Theorem 12. Any linearly independent list of vectors in a vector space can be completed to a basis of the vector space.

Proof. Let V be a K-vector space, let $B = (v_1, \ldots, v_n)$ be a basis of V and $(u_1, \ldots, u_m)$ be a linearly independent list in V. Since B is a system of generators for V, we know by Theorem 6 that $m \leq n$ and m vectors of B can be replaced by the vectors $(u_1, \ldots, u_m)$ obtaining again a system of generators for V, say $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$. But by Theorem 11, this is also linearly independent in V and consequently a basis of V. $\square$

Remark 13. The completion of a linearly independent list to a basis of the vector space is not unique. For instance, the vector $(1, 0)$ can be completed either with $(0, 1)$ or with $(1, 1)$ to a basis of ${}_{\mathbb{R}}\mathbb{R}^2$ (see Example 12 (c) of the previous course).

Corollary 14. Let V be a vector space over K and $S \leq_K V$. Then:

(i) Any basis of S is a part of some basis of V.

(ii) $\dim S \leq \dim V$.

(iii) $\dim S = \dim V \Leftrightarrow S = V$.

Proof. (i) Let $(u_1, \ldots, u_m)$ be a basis of S. Since the list is linearly independent, it can be completed to a basis $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ of V by Theorem 12.
(ii) follows from (i).
(iii) Assume that $\dim S = \dim V = n$. Let $(u_1, \ldots, u_n)$ be a basis of S. Then it is linearly independent in V, hence it is a basis of V by Theorem 11. Thus, if $v \in V$, then $v = k_1 u_1 + \cdots + k_n u_n$ for some $k_1, \ldots, k_n \in K$, hence $v \in S$. Therefore, $S = V$. $\square$

Remark 15. For the equivalence (iii) from the previous corollary the fact that we are working in a finitely generated space is essential.

Theorem 16. Let V and V' be vector spaces over K. Then

$$V \simeq V' \Leftrightarrow \dim V = \dim V'.$$

Proof. $\Rightarrow$. Let $f : V \to V'$ be a K-isomorphism. If $(v_1, \dots, v_n)$ is a basis of V, then by Theorem 5, $(f(v_1), \dots, f(v_n))$ is a basis of V'. Hence $\dim V = \dim V'$.
$\Leftarrow$. Assume that $\dim V = \dim V' = n$. Let $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_n)$ be bases of V and V' respectively. We know by Theorem 3 that a K-linear map $f : V \to V'$ is determined by its values on the vectors of the basis B. Define $f(v_i) = v'_i$, for any $i \in \{1, \dots, n\}$. Then it is easy to check that f is a K-isomorphism. □

Corollary 17. Any vector space V over K with $\dim V = n (\in \mathbb{N}^*)$ is isomorphic to the canonical vector space K^n over K.

Remarks 18. Corollary 17 is a very important structure result, saying that, up to an isomorphism, any finite dimensional vector space over K is, actually, the canonical vector space K^n over K. Thus, we have an explanation why we have used so often this kind of vector spaces: not only because the operations are very nice and easily defined, but they are, up to an isomorphism, the only types of finite dimensional vector spaces.

Let consider $f : K^n \to V$ the above-mentioned K-isomorphism $\Rightarrow f^{-1} : V \to K^n$ K-isomorphism and for any $v \in V$ which is represented as $v = x_1 b_1 + \dots + x_n b_n$, $x_1, \dots, x_n \in K$ ← the coord. of v in B.
$f^{-1}(v) = (x_1, \dots, x_n) \in K^n$.

Remarks: a) Let V be a K-v.s., $v_1, \dots, v_m \in V$
$\dim \langle v_1, \dots, v_m \rangle =$ the maximum number of l.indep. vectors taken from $v_1, \dots, v_m$ $=$
$=$ the maximum number of vectors taken from $v_1, \dots, v_m$ s.t. none of them can be expressed as a linear combination of the others.
b) Let K be a field, $m, n \in \mathbb{N}^*$; $v_1, \dots, v_m \in K^n$
$\dim \langle v_1, \dots, v_m \rangle =$
$=$ the maximum number of n-tuples taken from $v_1, \dots, v_m$ s.t. none of them can be expressed as a linear combination of the others $=$ the rank of the $m \times n$ matrix formed with $v_1, \dots, v_m$ as rows ($=$ the rank of the $n \times m$ matrix formed with $v_1, \dots, v_m$ as columns).

Using the above K-isomorphism $f^{-1} : V \to K^n$ we have: for any $v_1, \dots, v_m \in V$
$$f^{-1}(\langle v_1, \dots, v_m \rangle) = \langle f^{-1}(v_1), \dots, f^{-1}(v_m) \rangle$$

$$\overset{f^{-1}\text{ isom}}{\Longrightarrow} \dim \langle v_1, \dots, v_m \rangle = \dim f^{-1}(\langle v_1, \dots, v_m \rangle) = \dim \langle \underbrace{f^{-1}(v_1)}_{\in K^n}, \dots, \underbrace{f^{-1}(v_m)}_{\in K^n} \rangle \overset{\text{Remark b)}}{=}$$
$=$ the rank of the $m \times n$ matrix which has as rows the coordinates of $v_1, \dots, v_m$ in the basis B.

We end this section with some important formulas involving vector space dimension.

Theorem 19. Let $f : V \to V'$ be a K-linear map. Then
$$\dim V = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f).$$

Proof. *(optional)* Let $(u_1, \dots, u_m)$ be a basis of the subspace $\operatorname{Ker} f$ of V. Then by Corollary 14, it can be completed to a basis $B = (u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ of V. We are going to prove that $B' = (f(v_{m+1}), \dots, f(v_n))$ is a basis of $\operatorname{Im} f$.

First, we prove that B' is linearly independent in $\operatorname{Im} f$. Let us take $k_{m+1}, \dots, k_n \in K$. By the K-linearity of f we have:

$$\sum_{i=m+1}^{n} k_i f(v_i) = 0 \Rightarrow f\Big(\sum_{i=m+1}^{n} k_i v_i\Big) = 0 \Rightarrow \sum_{i=m+1}^{n} k_i v_i \in \operatorname{Ker} f\,.$$

Since $(u_1, \dots, u_m)$ is a basis of $\operatorname{Ker} f$, there exist $k_1, \dots, k_m \in K$ such that

$$\sum_{i=m+1}^{n} k_i v_i = \sum_{i=1}^{m} k_i u_i \Leftrightarrow \sum_{i=1}^{m} k_i u_i - \sum_{i=m+1}^{n} k_i v_i = 0\,.$$

But $B = (u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ is a basis of V, hence it follows that $k_i = 0$, for any $i \in \{1, \dots, n\}$. Therefore, B' is linearly independent in $\operatorname{Im} f$.

Let us now show that B' is a system of generators for $\operatorname{Im} f$. Let $v' \in \operatorname{Im} f$. Then $v' = f(v)$ for some $v \in V$. Since B is a basis of V, there exist $k_1, \dots, k_n \in K$ such that

$$v = \sum_{i=1}^{m} k_i u_i + \sum_{i=m+1}^{n} k_i v_i\,.$$

By the K-linearity of f and the fact that $u_1, \dots, u_m \in \operatorname{Ker} f$, it follows that

$$v' = f(v) = f\Big(\sum_{i=1}^{m} k_i u_i + \sum_{i=m+1}^{n} k_i v_i\Big) = \sum_{i=1}^{m} k_i f(u_i) + \sum_{i=m+1}^{n} k_i f(v_i) = \sum_{i=m+1}^{n} k_i f(v_i)\,.$$

Hence B' is a system of generators for $\operatorname{Im} f$.

Therefore, B' is a basis of $\operatorname{Im} f$ and consequently,

$$\dim V = n = m + (n - m) = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f)\,.$$ □

Corollary 20. a) Let V be a K-vector space and let S, T be subspaces of V. Then

$$\dim S + \dim T = \dim(S \cap T) + \dim(S + T).$$

Indeed, $f : S \times T \to S + T$, $f(x, y) = x - y$ is a surjective linear map with the kernel $\operatorname{Ker} f = \{(x, x) \mid x \in S \cap T\}$. Hence,

k.w.

$$\dim(S \times T) = \dim(\operatorname{Ker} f) + \dim(S + T).$$

Since $g : S \cap T \to \operatorname{Ker} f$, $g(x) = (x, x)$ is an isomorphism, we have

$$\dim(\operatorname{Ker} f) = \dim(S \cap T),$$

and by Example 9 g) we have $\dim(S \times T) = \dim S + \dim T$, which completes the proof of the statement.

b) If V is a K-vector space and $S, T \leq_K V$, then

$$\dim(S + T) = \dim S + \dim T \Leftrightarrow S + T = S \oplus T.$$

c) Let V be a K-vector space and $f \in End_K(V)$. The following statements are equivalent:

(i) f is injective;

(ii) f is surjective;

(iii) f is bijective.

$\dim V = \dim \mathrm{Ker} f + \dim \mathrm{Im} f.$

Of course, it is enough to show that (i)$\Leftrightarrow$ (ii).

(i)$\Rightarrow$(ii) If f is injective, then $\mathrm{Ker} f = \{0\}$, hence $\dim(\mathrm{Ker} f) = 0$. By Theorem 19, it follows that $\dim(\mathrm{Im} f) = \dim V$. But $\mathrm{Im} f \leq_K V$, so $\mathrm{Im} f = V$ by Corollary 14.

(ii)$\Rightarrow$(i) Let us assume that f is surjective. Since $\mathrm{Im} f = V$, it follows by Theorem 19 that $\dim(\mathrm{Ker} f) = 0$, hence $\mathrm{Ker} f = \{0\}$. Thus f is injective.