Course 12 – Green's Theorem. Integration of Conservative Vector Fields

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May 18, 2021

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$$D = \{\, (x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x) \,\}, \qquad (1)$$

where $a, b \in \mathbb{R}$, $a < b$, while $\varphi, \psi : [a, b] \to \mathbb{R}$ are continuous functions satisfying $\varphi \le \psi$.

Definition (the boundary of a normal domain in the plane, oriented in the positive sense)

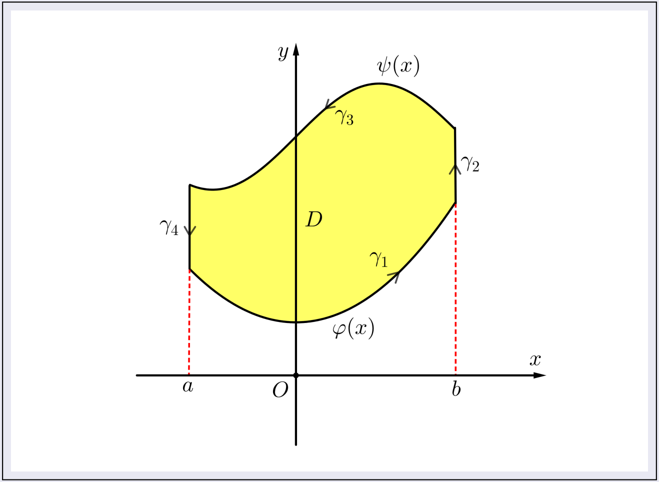

Figure 1: The boundary of a normal domain oriented in the positive sense.

Definition (the boundary of a normal domain in the plane, oriented in the positive sense)

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\gamma_1 &: [a,b] \to \mathbb{R}^2, & \gamma_1(t) &:= (t, \varphi(t)),\\
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Let $\gamma := \gamma_1 \vee \gamma_2 \vee \bar{\gamma}_3 \vee \gamma_4$. Note that $I(\gamma) = \operatorname{bd} D$, the tracing sense being counterclockwise. The oriented curve in $\mathbb{R}^2$, containing the parameterized path γ, is called *the boundary of D oriented in the positive sense* and it will be denoted by ∂D.

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Analogously one can define the boundary oriented in the positive sense of a normal domain with respect to the y-axis $D \subseteq \mathbb{R}^2$.

We denote by $\oint_{\partial D} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} := \int_{\gamma} \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}$ the integral of the vector field $\overrightarrow{F}$ along the oriented curve ∂D.

Lemma

Let $A \subseteq \mathbb{R}^2$ *be an open set, let* $D \subseteq A$ *be a normal domain with respect to the* x*-axis, such that the boundary of* D *oriented in the positive sense is a rectifiable curve, and let* $F_1 : A \to \mathbb{R}$ *be a function of class* C^1 *on* A*. Then the vector field* $\overrightarrow{F} := F_1 \cdot \overrightarrow{i}$ *is integrable along* ∂D *and it holds*

$$\oint_{\partial D} \overrightarrow{F} \cdot d\overrightarrow{r} = \oint_{\partial D} F_1(x,y)\, dx = -\iint_D \frac{\partial F_1}{\partial y}(x,y)\, dxdy. \qquad (2)$$

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Assume that D is defined by (1). Let γ be the parameterized path defined on the previous slide. Since ∂D is a rectifiable curve, it follows that

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Proof.

Assume that D is defined by (1). Let γ be the parameterized path defined on the previous slide. Since ∂D is a rectifiable curve, it follows that γ is rectifiable, too. By virtue of a result in the previous course it follows that $\overrightarrow{F}$ is integrable along γ, hence along ∂D.

Proof.

$$\oint_{\partial D} F_1 \,\mathrm{d}x = \int_{\gamma} F_1 \,\mathrm{d}x = \int_{\gamma_1} F_1 \,\mathrm{d}x + \int_{\gamma_2} F_1 \,\mathrm{d}x - \int_{\gamma_3} F_1 \,\mathrm{d}x + \int_{\gamma_4} F_1 \,\mathrm{d}x.$$

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$$= \int_a^b F_1(x, y) \Big|_{y=\varphi(x)}^{y=\psi(x)} \mathrm{d}x = \int_a^b \left[F_1(x, \psi(x)) - F_1(x, \varphi(x)) \right] \mathrm{d}x.$$

The above equalities ensures that (2) holds.

Lemma

Let $A \subseteq \mathbb{R}^2$ be an open set, let $D \subseteq A$ be a normal domain with respect to the y-axis, such that the boundary of D oriented in the positive sense is a rectifiable curve, and let $F_2 : A \to \mathbb{R}$ be a function of class C^1 on A. Then the vector field $\overrightarrow{F} := F_2 \cdot \overrightarrow{j}$ is integrable along ∂D and it holds

$$\oint_{\partial D} \overrightarrow{F} \cdot d\overrightarrow{r} = \oint_{\partial D} F_2(x, y)\, dy = \iint_D \frac{\partial F_2}{\partial x}(x, y)\, dxdy.$$

Theorem (G. Green)

Let $A \subseteq \mathbb{R}^2$ be an open set, let $D \subseteq A$ be a normal domain with respect to both the x-axis and the y-axis, such that the boundary of D oriented in the positive sense is a rectifiable curve, and let $F_1, F_2 : A \to \mathbb{R}$ be functions of class C^1 on A. Then the vector field $\overrightarrow{F} := F_1 \cdot \overrightarrow{i} + F_2 \cdot \overrightarrow{j}$ is integrable along ∂D and it holds

$$\oint_{\partial D} \overrightarrow{F} \cdot d\overrightarrow{r} = \oint_{\partial D} F_1\, dx + F_2\, dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

Figure 2: George Green (1793 – 1841).

Green's Theorem

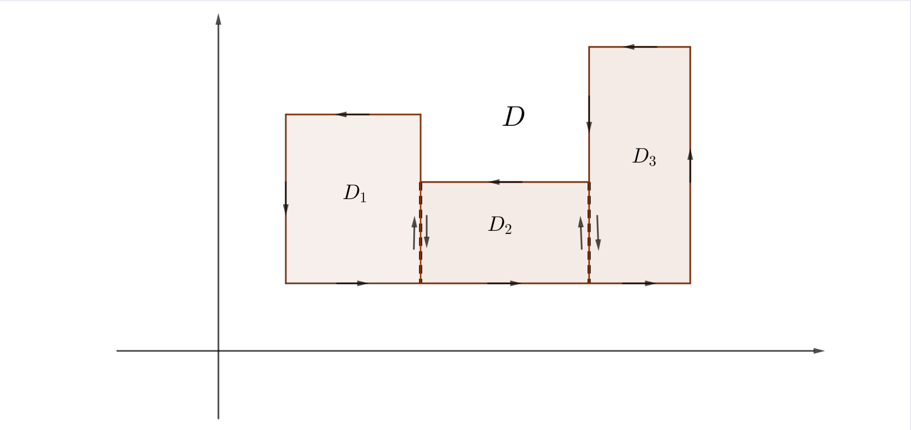

Figure 3: A set that is not a normal domain with respect to both axes, but can be decomposed into several sub-domains with this property.

Corollary

Let D be a subset of $\mathbb{R}^2$ that is a normal domain with respect to both the x-axis and the y-axis, such that the boundary of D oriented in the positive sense is a rectifiable curve. Then D is Jordan measurable and its Jordan measure is given by

$$m(D) = \frac{1}{2} \oint_{\partial D} x dy - y dx.$$

Theorem (G. W. Leibniz – I. Newton)

Let $A \subseteq \mathbb{R}^n$ be an open set, let $\gamma : [a, b] \to A$ be a C^1 parameterized path, and let $F := (F_1, \ldots, F_n) : A \to \mathbb{R}^n$ be a conservative vector field in A. Then F is integrable along γ and for every scalar potential $U : A \to \mathbb{R}$ of F it holds

$$\int_\gamma \overrightarrow{F} \cdot d\overrightarrow{r} = U(\gamma(b)) - U(\gamma(a)) \overset{not}{=:} U\Big|_{\gamma(a)}^{\gamma(b)}.$$

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$$\begin{aligned}\int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} &= \sum_{i=1}^{n} \int_a^b (F_i \circ \gamma)(t)\, \gamma_i'(t)\, \mathrm{d}t \\ &= \end{aligned}$$

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&= \int_a^b (U \circ \gamma)'(t)\, \mathrm{d}t = U \circ \gamma \Big|_a^b = U(\gamma(b)) - U(\gamma(a)).
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The additivity of the Riemann integral with respect to the interval ensures that the previous theorem holds also in the case when γ is a piecewise C^1 parameterized path.

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Corollary

If $A \subseteq \mathbb{R}^n$ is an open set, $\gamma : [a, b] \to A$ and $\rho : [c, d] \to A$ are piecewise C^1 parameterized paths having the same endpoints (i.e., $\gamma(a) = \rho(c)$ and $\gamma(b) = \rho(d)$), and F is a conservative vector field in A, then

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Definition

Let A be a subset of $\mathbb{R}^n$ and let $F : A \to \mathbb{R}^n$ be a vector field in A. One says that the work of F does not depend on the path of integration if for every pair of piecewise C^1 parameterized paths $\gamma : [a, b] \to A$ and $\rho : [c, d] \to A$, having the same endpoints, it holds that $\int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} = \int_\rho \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}$.

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is used in order to emphasize the fact that $\int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r}$ does not depend on γ, but only on the endpoints of γ. The previous corollary shows that if F is a conservative vector field, then the work of F does not depend on the path of integration. It is natural to ask if the converse of this assertion holds, too.

Definition (star domains)

A subset A of $\mathbb{R}^n$ is called a *star domain* or a *star-shaped set* if there exists a point $a \in A$ such that $[a, x] \subseteq A$ for every other point $x \in A$, where $[a, x] := \{(1-t)a + tx \mid t \in [0, 1]\}$ is the line segment from a to x.

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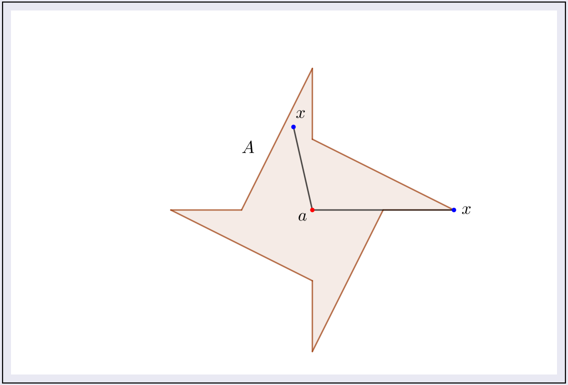

Figure 4: A star domain that is not convex.

Theorem (H. Poincaré)

Let $A \subseteq \mathbb{R}^n$ be an open star domain, and let $F : A \to \mathbb{R}^n$ be a vector field in A, such that all functions $F_i : A \to \mathbb{R}$ are of class C^1 on A. Then the following assertions are equivalent:

1° *F is a conservative vector field.*

2° *For all $i, j \in \{1, \dots, n\}$, $i \neq j$ it holds*

$$\frac{\partial F_i}{\partial x_j}(x) = \frac{\partial F_j}{\partial x_i}(x) \quad \text{for all } x \in A.$$

3° *The work of F does not depend on the path of integration.*

Figure 5: Henri Poincaré (1854 – 1912).

"Le savant doit ordonner; on fait la science avec des faits comme une maison avec des pierres; mais une accumulation de faits n'est pas plus une science qu'un tas de pierres n'est une maison."

Work done in Earth's gravitational field

Our goal is to determine the work that has to be done to move a point particle M of mass m in Earth's gravitational field.

Work done in Earth's gravitational field

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where $G \approx 6,672 \times 10^{-11}$ N m^2/ kg^2 is the universal gravitational constant, while $\overrightarrow{r} = \overrightarrow{OM}$. If the Cartesian coordinates of M are (x, y, z), then $\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$. Setting $k := G\,m\,M_P$, we have

$$\overrightarrow{F}(x, y, z) = -k\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/}}\right.$$

Work done in Earth's gravitational field

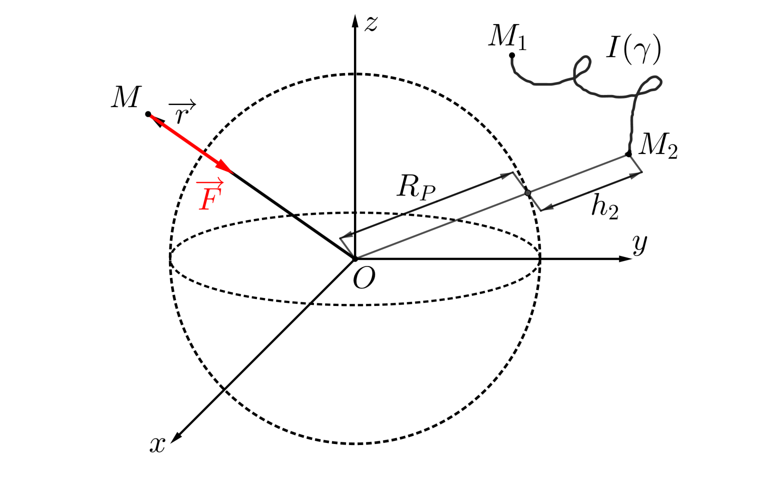

Figure 6: Earth's gravitational field.

Work done in Earth's gravitational field

Set

$$\begin{aligned} P(x,y,z) &:= \frac{-kx}{(x^2+y^2+z^2)^{3/2}}, \\ Q(x,y,z) &:= \frac{-ky}{(x^2+y^2+z^2)^{3/2}}, \\ R(x,y,z) &:= \frac{-kz}{(x^2+y^2+z^2)^{3/2}}. \end{aligned}$$

It is easily seen that the function $U : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$, defined by

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Work done in Earth's gravitational field

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It is easily seen that the function $U : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$, defined by $U(x,y,z) := \dfrac{k}{\sqrt{x^2+y^2+z^2}}$, satisfies

$$\frac{\partial U}{\partial x} = P, \quad \frac{\partial U}{\partial y} = Q, \quad \frac{\partial U}{\partial z} = R.$$

Consequently, U is a scalar potential of $\overrightarrow{F}$.

Work done in Earth's gravitational field

Assume that the point particle M is moving under the action of the gravitational force $\overrightarrow{F}$ on a trajectory whose shape is $I(\gamma)$, where γ is a parameterized path in $\mathbb{R}^3 \setminus \{(0,0,0)\}$ having the starting point $M_1(x_1, y_1, z_1)$ and the terminal point $M_2(x_2, y_2, z_2)$.

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$$
\begin{aligned}
W &= \int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} = \int_\gamma P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz \\
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$$\begin{aligned} W &= \int_\gamma \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{r} = \int_\gamma P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz \\ &= U(x_2, y_2, z_2) - U(x_1, y_1, z_1) \\ &= \end{aligned}$$

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&= k\,\frac{\sqrt{x_1^2+y_1^2+z_1^2} - \sqrt{x_2^2+y_2^2+z_2^2}}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}} \\
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&= k\,\frac{OM_1 - OM_2}{OM_1 \cdot OM_2} =
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&= k\,\frac{\sqrt{x_1^2+y_1^2+z_1^2} - \sqrt{x_2^2+y_2^2+z_2^2}}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}} \\
&= k\,\frac{OM_1 - OM_2}{OM_1 \cdot OM_2} = m\,\frac{GM_P}{R_P^2}\,\frac{OM_1 - OM_2}{\frac{OM_1}{R_P} \cdot \frac{OM_2}{R_P}}\,.
\end{aligned}$$

Work done in Earth's gravitational field

Let $h_1 := OM_1 - R_P$ and $h_2 := OM_2 - R_P$ denote the heights at which the points M_1 and M_2 are with respect to Earth's surface.

Work done in Earth's gravitational field

Let $h_1 := OM_1 - R_P$ and $h_2 := OM_2 - R_P$ denote the heights at which the points M_1 and M_2 are with respect to Earth's surface. Under the assumption that the two heights are much smaller than R_P, we have

$$\frac{OM_1}{R_P} = \frac{R_P + h_1}{R_P} = 1 + \frac{h_1}{R_P} \approx 1$$

and analogously $\dfrac{OM_2}{R_P} \approx 1$.

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and analogously $\dfrac{OM_2}{R_P} \approx 1$. Taking into account that $\dfrac{K\,M_P}{R_P^2} = g \approx 9,81$ m / s^2 represents the gravitational acceleration near the surface of the Earth, we get

$$W \approx mg(h_1 - h_2).$$

Work done in Earth's gravitational field

The exterior work W', that has to be done to move the point particle M along the path $I(\gamma)$ is

$$W' = -W \approx mgh_2 - mgh_1 = E_p(M_2) - E_p(M_1).$$

We recover a classical result in physics: the work W' does not depend on the trajectory $I(\gamma)$, but only on its endpoints M_1 and M_2. Moreover, W' can be approximated with the difference of the potential energies at the points M_2 and M_1, respectively.

Definition (parameterized surfaces)

Let D be a domain in $\mathbb{R}^2$ (i.e., an open connected subset of $\mathbb{R}^2$) and let $\Phi : D \to \mathbb{R}^3$ be a continuous function. If $T := [a_1, b_1] \times [a_2, b_2]$ is a rectangle such that $T \subset D$, then the restriction $\sigma := \Phi\Big|_T$ is called *parameterized surface*. The set

$$I(\sigma) := \{\sigma(u, v) \mid u \in [a_1, b_1],\ v \in [a_2, b_2]\}$$

is called *the image of the parameterized surface* σ. A subset of $\mathbb{R}^3$ is called a *geometric surface* provided that it is the image of a parameterized surface.

Definition

Let $D \subseteq \mathbb{R}^2$ be a domain and let $\Phi : D \to \mathbb{R}^3$ be a function of class C^1 on D,

$$\forall\ (u,v) \in D \ \longmapsto\ \Phi(u,v) := \big(x(u,v),\, y(u,v),\, z(u,v)\big) \in \mathbb{R}^3.$$

Further let $T := [a_1, b_1] \times [a_2, b_2]$ be a rectangle included in D and let σ be the parameterized surface $\sigma := \Phi\Big|_T$. Consider the vector defined by

$$\overrightarrow{N}_\sigma \ := \ \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} =$$

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$$\begin{aligned}\overrightarrow{N}_\sigma \ &:= \ \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \\ &= \ \frac{D(y,z)}{D(u,v)} \cdot \overrightarrow{i} + \frac{D(z,x)}{D(u,v)} \cdot \overrightarrow{j} + \frac{D(x,y)}{D(u,v)} \cdot \overrightarrow{k},\end{aligned}$$

Definition

where $\dfrac{D(y,z)}{D(u,v)}$ denotes the Jacobian determinant

$$\frac{D(y,z)}{D(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

The Jacobian determinants $\dfrac{D(z,x)}{D(u,v)}$ and $\dfrac{D(x,y)}{D(u,v)}$ are defined analogously.

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The Jacobian determinants $\dfrac{D(z,x)}{D(u,v)}$ and $\dfrac{D(x,y)}{D(u,v)}$ are defined analogously. You will see at the Differential Geometry course that the set defined by

$$P := \left\{ \sigma(u,v) + s\,\frac{\partial \sigma}{\partial u}(u,v) + t\,\frac{\partial \sigma}{\partial v}(u,v) \;\middle|\; s,t \in \mathbb{R} \right\}$$

is a plane that is tangent to $I(\sigma)$ at the point $\sigma(u,v)$. The vector $\overrightarrow{N}_\sigma$ is perpendicular to the plane P. Due to this fact, $\overrightarrow{N}_\sigma$ is said to be *the normal to the parameterized surface* σ.

Examples

a) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\Phi(\varphi, \theta) := (a \sin\varphi \cos\theta,\ a \sin\varphi \sin\theta,\ a \cos\varphi),$$

let $T := [0, \pi] \times [0, 2\pi]$, and let $\sigma := \Phi\Big|_T$. Then σ is a parameterized surface. Its image is

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let $T := [0, \pi] \times [0, 2\pi]$, and let $\sigma := \Phi\Big|_T$. Then σ is a parameterized surface. Its image is the sphere

$$I(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \ \mid\ x^2 + y^2 + z^2 = a^2\}.$$

b) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\Phi(\theta, z) := (a \cos\theta,\ a \sin\theta,\ z),$$

let $T := [0, 2\pi] \times [0, h]$, and let $\sigma := \Phi\Big|_T$. Then σ is a parameterized surface. Its image is

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let $T := [0, 2\pi] \times [0, h]$, and let $\sigma := \Phi\Big|_T$. Then σ is a parameterized surface. Its image is the cylinder

$$I(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2, 0 \le z \le h\}.$$

Examples

c) Let $a, c > 0$, let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\Phi(u, v) := \big(u \cos v,\ u \sin v,\ cv\big),$$

let $T := [0, a] \times [0, 2\pi]$, and let $\sigma := \Phi\Big|_T$.

Examples

c) Let $a, c > 0$, let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

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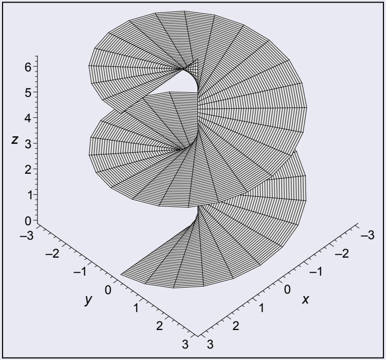

Figure 7: The image of σ for $a = 3$ and $c = 0.5$.

Examples

d) Let $R, r > 0$, let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\Phi(u, v) := \big((R + r\cos v)\cos u,\ (R + r\cos v)\sin u,\ r\sin v\big),$$

let $T := [0, 2\pi] \times [0, 2\pi]$, and let $\sigma := \Phi\Big|_T$.

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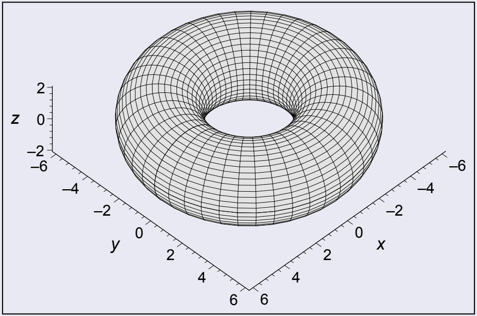

Figure 8: The image of σ for $r = 2$ and $R = 4$.

Definition (the integral of a scalar function on a parameterized surface)

Let $D \subseteq \mathbb{R}^2$ be a domain, let $\Phi = (x, y, z) : D \to \mathbb{R}^3$ be a function of class C^1 on D, let T be a rectangle included in D, and let σ be the parameterized surface $\sigma := \Phi\Big|_T$.

Definition (the integral of a scalar function on a parameterized surface)

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Definition (the integral of a scalar function on a parameterized surface)

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$$\begin{aligned} f_\sigma(u, v) &:= (f \circ \sigma)(u, v) \left\| \overrightarrow{N}_\sigma(u, v) \right\| \\ &= f\big(x(u, v), y(u, v), z(u, v)\big) \left\| \overrightarrow{N}_\sigma(u, v) \right\|. \end{aligned}$$

Definition (the integral of a scalar function on a parameterized surface)

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If f_σ is Riemann integrable over T, then the real number $\iint_T f_\sigma(u, v)\,\mathrm{d}u\mathrm{d}v$ is called *the surface integral of f over the parameterized surface σ* and it will be denoted by

$$\int_\sigma f\,\mathrm{d}S \qquad \text{or by} \qquad \int_\sigma f(x, y, z)\,\mathrm{d}S.$$

Definition (the integral of a scalar function on a parameterized surface)

Therefore, we have the formula

$$\int_{\sigma} f(x,y,z)\,\mathrm{d}S = \iint_{T} f(x(u,v),y(u,v),z(u,v))\,\|\overrightarrow{N}_{\sigma}(u,v)\|\,\mathrm{d}u\mathrm{d}v.$$

Definition (the integral of a scalar function on a parameterized surface)

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In the special case when $f = 1$ we get

$$\int_{\sigma} dS = \iint_T \|\overrightarrow{N}_{\sigma}(u,v)\|\,\mathrm{d}u\mathrm{d}v =: S(\sigma),$$

the area of the parameterized surface σ.

The area of a parameterized surface

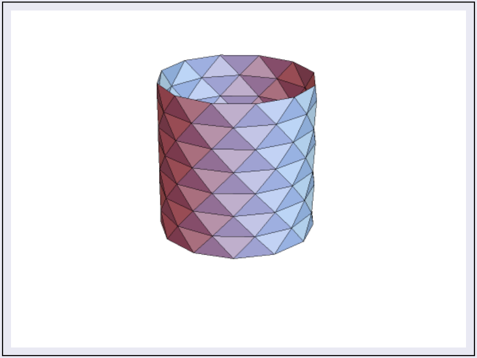

Figure 9: The Schwarz lantern.

Setting

$$A_\sigma(u,v) = A(u,v) := \frac{D(y,z)}{D(u,v)}\,(u,v),$$
$$B_\sigma(u,v) = B(u,v) := \frac{D(z,x)}{D(u,v)}\,(u,v),$$
$$C_\sigma(u,v) = C(u,v) := \frac{D(x,y)}{D(u,v)}\,(u,v),$$

we have

$$\|\overrightarrow{N}_\sigma(u,v)\| = \sqrt{A^2(u,v) + B^2(u,v) + C^2(u,v)}.$$

$$
\begin{aligned}
E_\sigma(u,v) &= E(u,v) = \left\| \frac{\partial \sigma}{\partial u}(u,v) \right\|^2 \\
&= \left(\frac{\partial x}{\partial u}\right)^2 (u,v) + \left(\frac{\partial y}{\partial u}\right)^2 (u,v) + \left(\frac{\partial z}{\partial u}\right)^2 (u,v), \\
G_\sigma(u,v) &= G(u,v) = \left\| \frac{\partial \sigma}{\partial v}(u,v) \right\|^2 \\
&= \left(\frac{\partial x}{\partial v}\right)^2 (u,v) + \left(\frac{\partial y}{\partial v}\right)^2 (u,v) + \left(\frac{\partial z}{\partial v}\right)^2 (u,v), \\
F_\sigma(u,v) &= F(u,v) = \left\langle \frac{\partial \sigma}{\partial u}(u,v), \frac{\partial \sigma}{\partial v}(u,v) \right\rangle \\
&= \left(\frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \cdot \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v} \right)(u,v).
\end{aligned}
$$

Using the Gauss coefficients, $\|\overrightarrow{N}_\sigma(u,v)\|$ can be expressed by the formula

$$\|\overrightarrow{N}_\sigma(u,v)\| = \sqrt{E(u,v)G(u,v) - F^2(u,v)}.$$

$$S(\sigma) = \iint_T \sqrt{E(u,v)G(u,v) - F^2(u,v)}\,\mathrm{d}u\mathrm{d}v$$