Seminar 7

1. Take $f : GL_n(\mathbb{R}) \to \mathbb{R}^*$, with $f(A) = det(A)$, which is a homomorphism between multiplicative groups, $Ker\, f = SL_n(\mathbb{R})$ and $Im\, f = \mathbb{R}^*$. Then use the first isomorphism theorem.

2. Take $f : \mathbb{C} \to \mathbb{R}$, with $f(z) = Re(z)$. So, if $z = a+bi$, then $f(a+bi) = a$. It is easy to see that f is a homomorphism between additive groups, $Ker\, f = \mathbb{R}$ and $Im\, f = \mathbb{R}$. Then use the first isomorphism theorem.

3. Take $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$, with $f(x \mod mn) = (x \mod m, x \mod n)$. One shows first that f is a function. This means that if $x \mod mn = y \mod mn$, then $(x \mod m, x \mod n) = (y \mod m, y \mod n)$. As $(x + y) \mod m = (x \mod m) + (y \mod m)$, it is easy to see that f is a group homomorphism. Also, $Ker\, f = \{0 \mod mn\}$, which means that f is an injective function. But the domain and the codomain have the same number of elements, so f must be bijective. Hence, f is an isomorphism.

   Alternatively, one may take $f : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$, with $f(x) = (x \mod m, x \mod n)$, and show that f is a group homomorphism, $Ker\, f = mn\mathbb{Z}$ and $Im\, f = \mathbb{Z}_m \times \mathbb{Z}_n$. Then the conclusion follows by the first isomorphism theorem.

   Or, one may notice that the element $(1 \mod m, 1 \mod n) \in \mathbb{Z}_m \times \mathbb{Z}_n$ has order $mn = |\mathbb{Z}_m \times \mathbb{Z}_n|$. This means that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ must be cyclic. But being a cyclic group of order mn, we must have the isomorphism $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$.

4.
$$H = \{\hat{0}, \hat{4}, \hat{8}, \widehat{12}, \widehat{16}, \widehat{20}\} \leq \mathbb{Z}_{24}$$
$$N = \{\hat{0}, \hat{6}, \widehat{12}, \widehat{18}\} \leq \mathbb{Z}_{24}$$
$$H \cap N = \{\hat{0}, \widehat{12}\} (\trianglelefteq H)$$
$$H + N = \{\hat{0}, \hat{2}, \hat{4}, \hat{6}, \hat{8}, \widehat{10}, \widehat{12}, \widehat{14}, \widehat{16}, \widehat{18}, \widehat{20}, \widehat{22}\}$$
$$H \cup N = \{\hat{0}, \hat{4}, \hat{6}, \hat{8}, \widehat{12}, \widehat{16}, \widehat{18}, \widehat{20}\}$$

   We can see that $< H \cup N >= H + N$, so $N \trianglelefteq < H \cup N >$. And, as "$+$" is commutative, we can apply the second isomorphism theorem.

5. The subgroups of $\mathbb{Z}_{12} = \mathbb{Z}/12\mathbb{Z}$ are of the form $d\mathbb{Z}/12\mathbb{Z}$ with $12\mathbb{Z} \subseteq d\mathbb{Z}$, that is, $d|12$. Hence they are $\mathbb{Z}/12\mathbb{Z}$, $2\mathbb{Z}/12\mathbb{Z}$, $3\mathbb{Z}/12\mathbb{Z}$, $4\mathbb{Z}/12\mathbb{Z}$, $6\mathbb{Z}/12\mathbb{Z}$ and $12\mathbb{Z}/12\mathbb{Z}$. Alternatively, one may use the fact that the group $(\mathbb{Z}_{12}, +)$ is cyclic, hence all its subgroups are cyclic. So, they can be computed as $< x >$ with $x \in \mathbb{Z}_{12}$.

   Let $d \mid 12 \Rightarrow d\mathbb{Z}/12\mathbb{Z} \trianglelefteq \mathbb{Z}_{12}$. Then the factor groups of $\mathbb{Z}_{12}$ are, for every divisor d of 12: $\mathbb{Z}_{12}/(d\mathbb{Z}/12\mathbb{Z}) \simeq (\mathbb{Z}/12\mathbb{Z})/(d\mathbb{Z}/12\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z} = \mathbb{Z}_d$.

6. For n prime $\Rightarrow$ $n \in \{2, 3, 5, 7, 11\}$ and $\forall x \in \mathbb{Z}_n$, we have $< \hat{x} >= \mathbb{Z}_n \iff \mathbb{Z}_n$ has only 2 subgroups $\{\hat{0}\}$ and $\mathbb{Z}_n$.

   For $n = 1 \Rightarrow \mathbb{Z}_1 = \{\hat{0}\} \leq \mathbb{Z}_1$.

   Now, for the others, we will use Lagrange's theorem to find the subgroups.

   For $n = 4 \Rightarrow < \hat{0} >, < \hat{1} >, < \hat{2} > \leq \mathbb{Z}_4$.

   For $n = 6 \Rightarrow < \hat{0} >, < \hat{1} >, < \hat{2} >, < \hat{3} > \leq \mathbb{Z}_6$.

   For $n = 8 \Rightarrow < \hat{0} >, < \hat{1} >, < \hat{2} >, < \hat{4} > \leq \mathbb{Z}_8$.

   For $n = 9 \Rightarrow < \hat{0} >, < \hat{1} >, < \hat{3} >, < \hat{6} > \leq \mathbb{Z}_9$.

   For $n = 10 \Rightarrow < \hat{0} >, < \hat{1} >, < \hat{2} >, < \hat{5} > \leq \mathbb{Z}_{10}$.

   For $n = 12 \Rightarrow < \hat{0} >, < \hat{1} >, < \hat{2} >, < \hat{3} >, < \hat{4} >, < \hat{6} > \leq \mathbb{Z}_{12}$.

   The Hasse diagram of $\mathbb{Z}_n$, with n prime is: $\mathbb{Z}_n \to \{\hat{0}\}$.

   Now, let's take the exemples for $\mathbb{Z}_{12}$. The Hasse diagram of the subgroup lattice is:

$$
\begin{tikzcd}
 & & \mathbb{Z}_{12} \arrow[dl] \arrow[dr] & \\
 & \langle\hat{2}\rangle \arrow[dl] \arrow[dr] & & \langle\hat{3}\rangle \arrow[dl] \\
\langle\hat{4}\rangle \arrow[dr] & & \langle\hat{6}\rangle \arrow[dl] & \\
 & \langle\hat{0}\rangle & &
\end{tikzcd}
$$

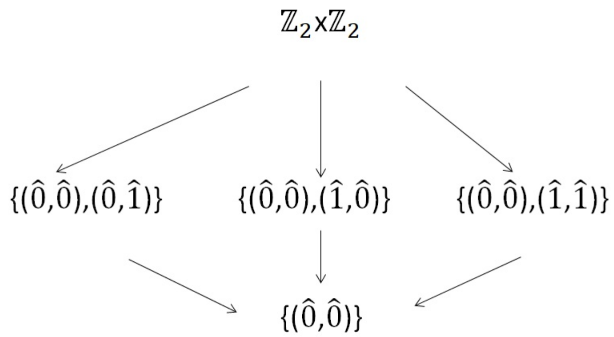
   And if we talk about $\mathbb{Z}_8$, we have the next Hasse diagram:

$$\mathbb{Z}_8 \to < \hat{2} > \to < \hat{4} > \to < \hat{0} >$$

$$
\begin{tikzcd}
\mathbb{Z}_8 \arrow[d, dash] \\
\langle\hat{2}\rangle \arrow[d, dash] \\
\langle\hat{4}\rangle \arrow[d, dash] \\
\langle\hat{0}\rangle
\end{tikzcd}
$$

7. We know the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ from last seminar.

   So we have the next Hasse diagram:



8. We know the subgroups of Q from last seminar.