SEMINAR 3

1) Compute:

a) $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$; b) $\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$;

c) $\begin{vmatrix} -1 & a & a & \dots & a & a \\ a & -1 & a & \dots & a & a \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a & a & a & \dots & a & -1 \end{vmatrix}$ (determinant of a size n matrix, $n \in \mathbb{N}$, $n \geq 2$);

d) $\begin{vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{vmatrix}$, where $x_1, x_2, x_3 \in \mathbb{C}$ are the roots of the polynomial $X^3 - 2X^2 + 2X + 17 \in \mathbb{Q}[X]$;

e) $\begin{vmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ x_2 & x_3 & \dots & x_n & x_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & x_1 & \dots & x_{n-2} & x_{n-1} \end{vmatrix}$, where $n \in \mathbb{N},\ n \geq 2$ and $x_1, x_2, \dots, x_n \in \mathbb{C}$ are the roots of the polynomial $X^n + a_{n-2}X^{n-2} + \cdots + a_1X + a_0 \in \mathbb{R}[X]$.

2) Solve in $\mathbb{C}$ the following equations:

a) $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = 0 \quad (a \in \mathbb{C})$; b) $\begin{vmatrix} x & 0 & -1 & 1 & 0 \\ 1 & x & -1 & 1 & 0 \\ 1 & 0 & x-1 & 0 & 1 \\ 0 & 1 & -1 & x & 1 \\ 0 & 1 & -1 & 0 & x \end{vmatrix} = 0.$

3) Let $n \in \mathbb{N},\ n \geq 2$ and $a_1, a_2, \dots, a_n \in \mathbb{C}$. Show that:

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (a_j - a_i).$$

4) Are these matrices invertible? If yes, find their inverses:

a) $\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 1 & 4 \end{pmatrix}$; b) $\begin{pmatrix} 3 & 4 & 2 \\ 6 & 8 & 5 \\ 9 & 12 & 10 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.