Bonus projects for the course Algebra 2

- The solutions will consist of **commented source code** and at least 5 relevant **input and output files**, and will be sent to the e-mail address: **scrivei@gmail.com**.
- Any programming language may be used.
- If necessary, you will be asked to explain your solution.
- The first 5 solutions for each project will be rewarded.
- You may submit improvements of your solutions, but they will be considered only if they are still in the first 5 solutions for each project.
- The final deadline is Week 14.

Project 1 (0.2 points)

- *Input*: non-zero natural number n
- *Output*:
  1. the number of associative operations on a set $A = \{a_1, \ldots, a_n\}$
  2. the operation table of each associative operation (for $n \leq 4$)

Example:

- *Input*: $n = 2$
- *Output*:
  1. the number of associative operations on a set $A = \{a_1, a_2\}$ is 8
  2. identifying an operation table $\begin{array}{c|cc} & a_1 & a_2 \\ \hline a_1 & x & y \\ a_2 & z & t \end{array}$ by the matrix $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(A)$, the operation tables of the associative operations on $A = \{a_1, a_2\}$ are given by the matrices:

$$\begin{pmatrix} a_1 & a_1 \\ a_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_1 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_1 \\ a_2 & a_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_2 & a_2 \end{pmatrix}, \begin{pmatrix} a_2 & a_1 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_2 & a_2 \\ a_2 & a_2 \end{pmatrix}.$$

Project 2 (0.2 points)

- *Input*: non-zero natural number n
- *Output*:
  1. the number of abelian group structures which can be defined on a set $A = \{a_1, \ldots, a_n\}$
  2. the operation table of each such abelian group (for $n \leq 7$)

Example: The operation table of a group G has the property that each element of G appears exactly once on each row and on each column. The operation table of an abelian group is symmetric with respect to the main diagonal. We may identify an operation table by a matrix. Make sure that the operations are associative and have identity element.

- *Input*: $n = 4$
- *Output*:
  1. the number of abelian group structures on a set $G = \{a_1, a_2, a_3, a_4\}$ is 16
  2. the abelian group structures on G with identity element a_1 are given by the matrices:

$$\begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_1} & a_4 & a_3 \\ \mathbf{a_3} & a_4 & a_1 & a_2 \\ \mathbf{a_4} & a_3 & a_2 & a_1 \end{pmatrix}, \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_1} & a_4 & a_3 \\ \mathbf{a_3} & a_4 & a_2 & a_1 \\ \mathbf{a_4} & a_3 & a_1 & a_2 \end{pmatrix}, \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_3} & a_4 & a_1 \\ \mathbf{a_3} & a_4 & a_1 & a_2 \\ \mathbf{a_4} & a_1 & a_2 & a_3 \end{pmatrix}, \begin{pmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} \\ \mathbf{a_2} & \boxed{a_4} & a_1 & a_3 \\ \mathbf{a_3} & a_1 & a_4 & a_2 \\ \mathbf{a_4} & a_3 & a_2 & a_1 \end{pmatrix}.$$

There are 4 similar abelian group structures for each possible identity element.

Project 3 (0.2 points)

- *Input*: natural numbers $m, n \geq 2$
- *Output*:
  1. the number of subgroups of the abelian group $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$
  2. the subgroups of the abelian group $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$ (for $2 \leq m, n \leq 10$)

Example: For a finite group $(G, +)$, a non-empty subset H of G is a subgroup of $(G, +)$ if and only if H is a stable subset of $(G, +)$.

- *Input*: $m = 2$, $n = 4$
- *Output*:
  1. the number of subgroups of the abelian group $(\mathbb{Z}_2 \times \mathbb{Z}_4, +)$ is 8
  2. the subgroups of the abelian group $(\mathbb{Z}_2 \times \mathbb{Z}_4, +)$ are:

$H_1 = \{(0,0)\}$
$H_2 = \{(0,0), (1,0)\}$
$H_3 = \{(0,0), (0,2)\}$
$H_4 = \{(0,0), (1,2))\}$
$H_5 = \{(0,0), (0,1), (0,2), (0,3)\}$
$H_6 = \{(0,0), (1,0), (0,2), (1,2)\}$
$H_7 = \{(0,0), (1,1), (0,2), (1,3)\}$
$H_8 = \mathbb{Z}_2 \times \mathbb{Z}_4$

Project 4 (0.2 points)

- *Input*: natural number $n \geq 2$
- *Output*:
  1. the order of the group $(GL_n(\mathbb{Z}_2), \cdot)$ and its elements
  2. the orders of the elements of the group $(GL_n(\mathbb{Z}_2), \cdot)$ (for $n \leq 5$)

Example: $GL_n(\mathbb{Z}_2) = \{A \in M_n(\mathbb{Z}_2) \mid det(A) \neq 0\}$.

- *Input*: $n = 2$
- *Output*:
  1. the order of the group $(GL_2(\mathbb{Z}_2), \cdot)$ is 6, and its elements are:

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right).$$

  2. the orders of the elements of the group $(GL_2(\mathbb{Z}_2), \cdot)$ are:

$ord\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1$
$ord\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = 2$
$ord\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) = 3$
$ord\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = 2$
$ord\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = 2$
$ord\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) = 3$

Project 5 (0.2 points)

- *Input*: natural number $n \geq 2$
- *Output*:
  1. the elements of the group $(S_n, \circ)$
  2. the orders of the elements of the group $(S_n, \circ)$ (for $n \leq 6$)

Example:

- *Input*: $n = 3$
- *Output*:
  1. the elements of the group $(S_3, \circ)$ are:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2\ 3), \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3),$$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\ 2), \quad \sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3), \quad \sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ 3\ 2).$$

  2. the orders of the elements of the group $(S_3, \circ)$ are:

$$ord(e) = 1, ord(\sigma_1) = 2, ord(\sigma_2) = 2, ord(\sigma_3) = 2, ord(\sigma_4) = 3, ord(\sigma_5) = 3.$$