Analytic Geometry

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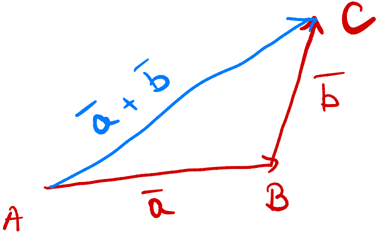
Vector operations

Let $\overline{a}$ and $\overline{b}$ be two vectors in V_3 (or V_2). The *sum* of $\overline{a}$ and $\overline{b}$ is the vector denoted by $\overline{a} + \overline{b}$, so that, if $\overrightarrow{AB} \in \overline{a}$ and $\overrightarrow{BC} \in \overline{b}$, then $\overrightarrow{AC}$ is the representative of $\overline{a} + \overline{b}$.



- If $\overline{v}$ is a vector in V_3 (or V_2), then the *opposite vector* of $\overline{v}$ is denoted by $-\overline{v}$, so that, if $\overrightarrow{AB}$ is a representative of $\overline{v}$, then $\overrightarrow{BA}$ is a representative of $-\overline{v}$.
- The sum $\overline{a} + (-\overline{b})$ will be, shortly, denoted by $\overline{a} - \overline{b}$ and it will be called the *difference* of the vectors $\overline{a}$ and $\overline{b}$.
- Let $\overline{a}$ be a vector in V_3 (or V_2) and k be a real number. The *product* $k \cdot \overline{a}$ is the vector defined as follows:
  1. $\overline{0}$ if $\overline{a} = \overline{0}$ or $k = 0$;
  2. if $k > 0$, then $k \cdot \overline{a}$ has the same direction and orientation as $\overline{a}$ and $||k \cdot \overline{a}|| = k \cdot ||\overline{a}||$;
  3. if $k < 0$, then $k \cdot \overline{a}$ has the same direction as $\overline{a}$, opposite orientation to $\overline{a}$ and $||k \cdot \overline{a}|| = -k \cdot ||\overline{a}||$.

The components of a vector

- Let $\overline{a}$ be a vector in V_2 and xOy be a rectangular coordinates system in $\mathcal{E}_2$. There is a unique point $A \in \mathcal{E}_2$, such that $\overrightarrow{OA} \in \overline{a}$. The coordinates a_1, a_2 of the point $A(a_1, a_2)$ are called the *components* of the vector $\overline{a}$ and write $\overline{a}(a_1, a_2)$.

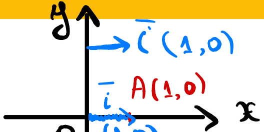
coordinates.

$$A(a_1, a_2)$$

$$\overrightarrow{OA} \in \overline{a}(a_1, a_2)$$

components.

The components of a vector



- Let $\overline{a}$ be a vector in V_2 and xOy be a rectangular coordinates system in $\mathcal{E}_2$. There is a unique point $A \in \mathcal{E}_2$, such that $\overrightarrow{OA} \in \overline{a}$. The coordinates a_1, a_2 of the point $A(a_1, a_2)$ are called0 the *components* of the vector $\overline{a}$ and write $\overline{a}(a_1, a_2)$.
- Similarly, $\overline{a}$ a vector in V_3 and a rectangular coordinate system $Oxyz$ in $\mathcal{E}_3$, there exists a unique point $A(a_1, a_2, a_3)$, such that $\overrightarrow{OA} \in \overline{a}$. The triple (a_1, a_2, a_3) gives the *components* of $\overline{a}$ and we denote it by $\overline{a}(a_1, a_2, a_3)$.
- Since $\overline{0}(0, 0)$ in V_2 and $\overline{0}(0, 0, 0)$ in V_3, then two vectors are equal if and only if they have the same components.

Theorem

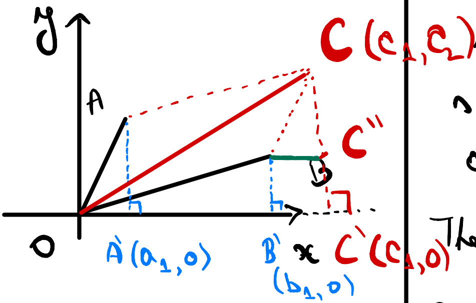
Let $\overline{a}(a_1, a_2)$ and $\overline{b}(b_1, b_2)$ be two vectors in V_2 and $k \in \mathbb{R}$. Then:

(1) *the components of $\overline{a} + \overline{b}$ are $(a_1 + b_1, a_2 + b_2)$;*

(2) *the components of $k \cdot \overline{a}$ are (ka_1, ka_2).*

Proof.

(1)



Let $A, B \in \mathcal{E}_2$ s.t. $A(a_1, a_2)$ and $B(b_1, b_2)$.

Then $\overrightarrow{OA} \in \overline{a}$ and $\overrightarrow{OB} \in \overline{b}$.

Let $C \in \mathcal{E}_2$ be s.t. $OBCA$ is a parallelogram. Then $\overrightarrow{OC} \in \bar{a} + \bar{b}$.

- One can prove that $\Delta AOA' \cong \Delta CBC''$.

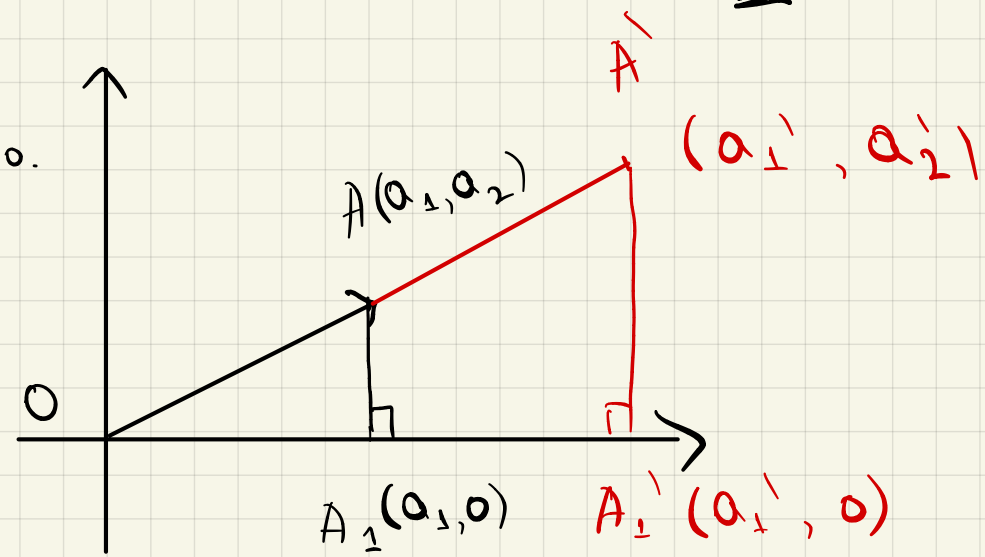
$$\therefore |OA'| = |BC''| \Rightarrow |OA'| = |B'C'|$$

$$\therefore \quad c_1 = a_1 + b_1 .$$

Similarly, one can show that

$$c_2 = a_2 + b_2 . \qquad \boxed{}$$

(2) Assume $k > 0$.



Let $\overrightarrow{OA'} \in k \cdot \bar{a}$. From similar Δ, we get $\boxed{a_1' = k \cdot a_1.}$

An analogous theorem for 3D

Theorem

Let $\overline{a}(a_1, a_2, a_3)$ *and* $\overline{b}(b_1, b_2, b_3)$ *be two vectors in* V_3 *and* $k \in \mathbb{R}$. *Then:*

(1) *the components of* $\overline{a} + \overline{b}$ *are* $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$;

(2) *the components of* $k \cdot \overline{a}$ *are* (ka_1, ka_2, ka_3).

Theorem

(1) *If* $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ *are two points in* $\mathcal{E}_2$*, then*

$$\overline{P_1P_2}(x_2 - x_1, y_2 - y_1).$$

(2) *If* $Q_1(x_1, y_1, z_1)$ *and* $Q_2(x_2, y_2, z_2)$ *are two points in* $\mathcal{E}_3$*, then*

$$\overline{Q_1Q_2}(x_2 - x_1, y_2 - y_2, z_2 - z_1).$$

Proof.

(1) Let $O \in \mathcal{E}_2$ be the origin.

$$\overline{P_1P_2} = \overline{P_1O} + \overline{OP_2} = (-\overline{OP_1}) + \overline{OP_2} = \overline{OP_2} - \overline{OP_1}$$

The components of $\overline{OP_2} - \overline{OP_1}$ are $(x_2 - x_1, y_2 - y_1)$.

The set of vectors is a very structured one

Theorem (Prop. of the summation) (Axioms of a vector space)

Let $\overline{a}$, $\overline{b}$ and $\overline{c}$ be vectors in V_3 (or V_2) and $\alpha, \beta \in \mathbb{R}$. Then:

$\overline{a}(a_1, a_2)$, $\overline{b}(b_1, b_2)$

$\overline{a} + \overline{b}(a_1 + b_1, a_2 + b_2)$

1) $\overline{a} + \overline{b} = \overline{b} + \overline{a}$ *(commutativity);*

2) $(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c})$ *(associativity);*

3) $\overline{a} + \overline{0} = \overline{0} + \overline{a} = \overline{a}$ *($\overline{0}$ is the neutral element for summation);*

4) $\overline{a} + (-\overline{a}) = (-\overline{a}) + \overline{a} = \overline{0}$ *($-\overline{a}$ is the inverse of $\overline{a}$);*

5) $\alpha(\beta\overline{a}) = (\alpha\beta)\overline{a}$;

6) $\alpha \cdot (\overline{a} + \overline{b}) = \alpha \cdot \overline{a} + \beta \cdot \overline{b}$ *(multiplication by real scalars is distributive with respect to the summation of vectors);*

7) $(\alpha + \beta) \cdot \overline{a} = \alpha \cdot \overline{a} + \beta \cdot \overline{a}$ *(multiplication by real scalars is distributive with respect to the summation of scalars);*

8) $1 \cdot \overline{a} = \overline{a}$.

Proof. Compare components of the LHS and RHS of each equality. ∎

Proposition

(1) *Let $\overline{a}(a_1, a_2)$ be a vector in V_2. The length of $\overline{a}$ is given by*

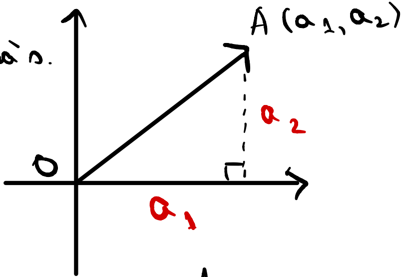
$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2}.$$

(2) *Let $\overline{a}(a_1, a_2, a_3)$ be a vector in V_3. The length of $\overline{a}$ is given by*

$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Proof.

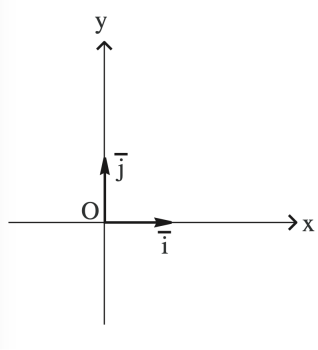
(1) Use Pythagora's.

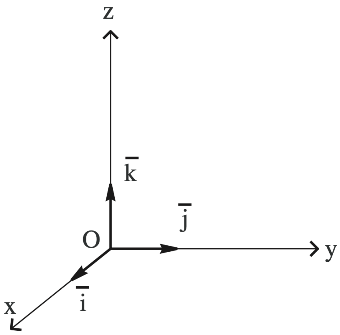


(2) Use Pythagora's twice. ∎

- The vectors $\bar{i}(1,0)$ and $\bar{j}(0,1)$ in V_2 are called the *unit vectors* (or *versors*) of the coordinate axes Ox and Oy.
- The vectors $\bar{i}(1,0,0)$, $\bar{j}(0,1,0)$ and $\bar{k}(0,0,1)$ are called the *unit vectors* (or *versors*) of the coordinate axes Ox, Oy and Oz.
- It is clear that

$$||\bar{i}|| = ||\bar{j}|| = ||\bar{k}|| = 1.$$





Interlude... not really related to the course

- In general, if we are given an equivalence relation $\sim$ on a set X, then the set of equivalence classes $X/\sim$ is "smaller" than the whole set X.

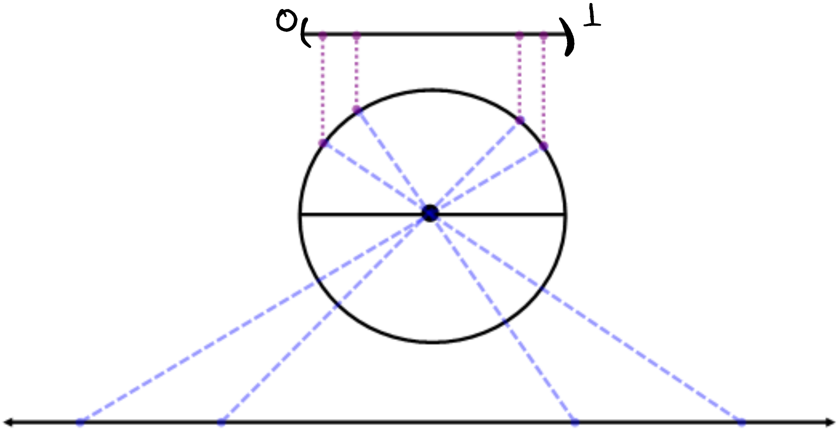
Examples: $X = \mathbb{Z}$ and

$x \sim y \iff x - y$ is even.

$X/\sim = \{\bar{0}, \bar{1}\}$.

Interlude... not really related to the course

- In general, if we are given an equivalence relation $\sim$ on a set X, then the set of equivalence classes $X/\sim$ is "smaller" than the whole set X.
- Always smaller?... Take $X = \mathbb{R}$ and say that for $x, y \in \mathbb{R}$ we have $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Then, every real number has a representative in $[0, 1)$, so we can think of $\mathbb{R}/\sim$ as of the interval $[0, 1)$. But is this really "smaller" than $\mathbb{R}$?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So far we have defined the operations

$$+ : V_2 \times V_2 \to V_2, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

$$\cdot : \mathbb{R} \times V_2 \to V_2, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}$$

and, of course,

$$+ : V_3 \times V_3 \to V_3, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

$$\cdot : \mathbb{R} \times V_3 \to V_3, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}.$$

V_2 the same thing as $\mathbb{R}^2$, or V_3 the same thing as $\mathbb{R}^3$?

$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$

Theorem

1. $(V_2, +)$ *is a vector space over* $\mathbb{R}$*, which is isomorphic to* $(\mathbb{R}^2, +)$*. The set* $\{\bar{i}, \bar{j}\}$ *is a base of* V_2*, therefore* $\dim_{\mathbb{R}} V_2 = 2$*.*
2. $(V_3, +)$ *is a vector space over* $\mathbb{R}$*, which is isomorphic to* $(\mathbb{R}^3, +)$*. The set* $\{\bar{i}, \bar{j}, \bar{k}\}$ *is a base of* V_3*, therefore* $\dim_{\mathbb{R}} V_3 = 3$*.*

Proof. (1) Fixe a system of coordinates in $\mathcal{E}_2$. Then, we define

$$\phi : V_2 \to \mathbb{R}^2$$
$$\overline{a} \mapsto (a_1, a_2)$$

where a_1, a_2 are the components of V_2.

. In the previous theorem we saw that $(V_2, +)$ is an $\mathbb{R}$- vector space.

It can be easily checked that ϕ is an isomorphism of vector spaces.

(2) The proof is analogous.

$$\varphi : V_3 \longrightarrow \mathbb{R}^3$$

$$\bar{a} \longmapsto (a_1, a_2, a_3) \quad , \text{ where}$$

a_1, a_2, a_3 are the components of V_3 is an isomorphism.

$\blacksquare$

A few definitions

- Let $\overline{a}$ and $\overline{b}$ be two nonzero vectors in V_3 (or V_2). They are *linearly dependent* if there exist the scalars $\alpha, \beta \in \mathbb{R}^*$ such that $\alpha\overline{a} + \beta\overline{b} = \overline{0}$.
- Let set $\overline{a}$, $\overline{b}$ and $\overline{c}$ be three nonzero vectors in V_3. They are *linearly dependent* if there exist the scalars $\alpha, \beta, \gamma \in \mathbb{R}$, not all equal to zero, such that $\alpha\overline{a} + \beta\overline{b} + \gamma\overline{c} = \overline{0}$.
- The vectors $\overline{a}$ and $\overline{b}$ in V_3 (or V_2), $\overline{a}, \overline{b} \neq \overline{0}$, are *collinear* if they have representatives situated on the same line.
- The vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ in V_3, $\overline{a}, \overline{b}, \overline{c} \neq \overline{0}$ are *coplanar* if they have representatives situated in the same plane.

Theorem

1. *The vectors $\overline{a}$ and $\overline{b}$ are linearly dependent if and only if they are collinear.*
2. *The vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent in V_3 if and only if they are coplanar.*

Proof.

1. If the vectors $\overline{a}$ and $\overline{b}$ are collinear, then there exists a scalar $\alpha \in \mathbb{R}^*$ such that $\overline{a} = \alpha \cdot \overline{b}$, i.e.

$$1 \cdot \overline{a} + (-\alpha) \cdot \overline{b} = \overline{0},$$

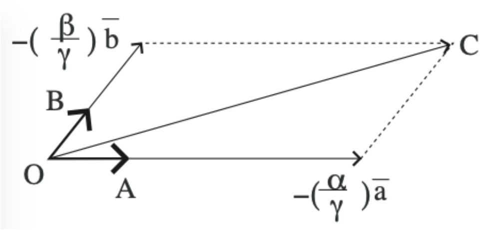
so, by definition, $\overline{a}$ and $\overline{b}$ are linearly dependent.

Conversely, if $\alpha\overline{a} + \beta\overline{b} = \overline{0}$ for some scalars $\alpha, \beta \in \mathbb{R}^*$, then we can write $\overline{a} = \left(-\dfrac{\beta}{\alpha}\right)\overline{b}$. By definition, $\overline{a}$ and $\overline{b}$ are collinear.

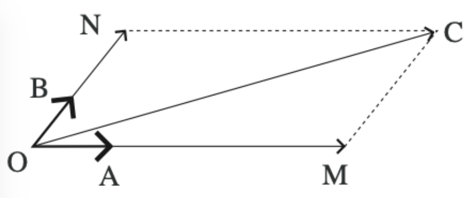
2. Suppose that the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent. Then, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ not all zero, such that $\alpha\overline{a} + \beta\overline{b} + \gamma\overline{c} = \overline{0}$. Suppose that $\gamma \neq 0$. One obtains

$$\overline{c} = \left(-\frac{\alpha}{\gamma}\right)\overline{a} + \left(-\frac{\beta}{\gamma}\right)\overline{b}.$$

If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are representative of $\overline{a}$ respectively $\overline{b}$, then the representative $\overrightarrow{OC}$ of $\overline{c}$, constructed as below, is coplanar with $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



Conversely, if $\overline{a}$, $\overline{b}$ and $\overline{c}$ are coplanar, let us consider the representatives $\overrightarrow{OA} \in \overline{a}$, $\overrightarrow{OB} \in \overline{b}$ and $\overrightarrow{OC} \in \overline{c}$, situated in the same plane. In the diagram below, $OMCN$ is a parallelogram.



Then, there exist $\alpha, \beta \in \mathbb{R}$ such that $\overrightarrow{OM} = \alpha \cdot \overrightarrow{OA}$ and $\overrightarrow{ON} = \beta \cdot \overrightarrow{OB}$. Hence $\overline{OC} = \overline{OM} + \overline{ON} = \alpha \cdot \overline{OA} + \beta \cdot \overline{OB}$ and $\overline{c} = \alpha \cdot \overline{a} + \beta \cdot \overline{b}$, so that $\alpha \cdot \overline{a} + \beta \cdot \overline{b} + (-1) \cdot \overline{c} = \overline{0}$ and the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent.

$\square$

To keep in mind...

- The set $\{\overline{a}, \overline{b}\}$ is a base in V_2 if and only if the vectors $\overline{a}$, $\overline{b}$ are not collinear.
- The set $\{\overline{a}, \overline{b}, \overline{c}\}$ is a base in V_3 if and only if the vectors $\overline{a}$, $\overline{b}$, $\overline{c}$ are not coplanar.

The problem set for this week will be posted soon. Ideally you would think about some of them before the seminar.

Thank you very much for your attention!