COURSE 3

Some important examples of rings

We remind that $(R, +, \cdot)$ is a **ring** if $(R, +)$ is an Abelian group, $\cdot$ is associative and the distributive laws hold (that is, $\cdot$ is distributive with respect to $+$). The ring $(R, +, \cdot)$ is a **unitary ring** if it has a multiplicative identity element.

The polynomial ring over a field

Let $(K, +, \cdot)$ be a field and let us denote by $K^{\mathbb{N}}$ the set

$$K^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to K\}.$$

If $f : \mathbb{N} \to K$ then, denoting $f(n) = a_n$, we can write

$$f = (a_0, a_1, a_2, \dots).$$

For $f = (a_0, a_1, a_2, \dots)$, $g = (b_0, b_1, b_2, \dots) \in K^{\mathbb{N}}$ one defines:

$$f + g = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots) \tag{1}$$

$$f \cdot g = (c_0, c_1, c_2, \dots) \tag{2}$$

where

$$\begin{aligned} c_0 &= a_0 b_0, \\ c_1 &= a_0 b_1 + a_1 b_0, \\ &\vdots \\ c_n &= a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \sum_{i+j=n} a_i b_j, \\ &\vdots \end{aligned}$$

Theorem 1. $K^{\mathbb{N}}$ forms a commutative unitary ring with respect to the operations defined by (1) and (2) called **the ring of formal power series over** K.

Proof. HOMEWORK $(0, 0, \dots)$, $(1, 0, 0, \dots)$ the id. elements □

Let $f = (a_0, a_1, a_2, \dots) \in K^{\mathbb{N}}$. The **support of** f is the subset of $\mathbb{N}$ defined by

$$supp\, f = \{k \in \mathbb{N} \mid a_k \neq 0\}.$$

We denote by $K^{(\mathbb{N})}$ the subset consisting of all the sequences from $K^{\mathbb{N}}$ with a finite support. Then

$$f \in K^{(\mathbb{N})} \Leftrightarrow \exists\, n \in \mathbb{N} \text{ such that } a_i = 0 \text{ for } i \geq n \Leftrightarrow f = (a_0, a_1, a_2, \dots, a_{n-1}, 0, 0, \dots).$$

For instance, $f = (0, 1, 2, 3, 0, 5, 0, 0, \dots) \in \mathbb{R}^{\mathbb{N}}$ then

$supp\, f = \{1, 2, 3, 5\}$ $(a_i = 0, \forall i \geq 6)$

$supp\, (0, 1, 0, 0, \dots) = \{1\}$ $(a_i = 0, \forall i \geq 2)$

Remark : $\varphi : K \to K^{(\mathbb{N})}$ inj. morphism $\Rightarrow \varphi' : K \to \varphi(K)$, $\varphi'(k) = \varphi(k)$
$\varphi(K) = \{\varphi(a) \mid a \in K\}$ subring of $K^{(\mathbb{N})}$ isom.

$\Rightarrow K \simeq \varphi(K)$ and this allows us to identify $a \in K$ with $(a, 0, 0, \dots)$

Theorem 2. i) $K^{(\mathbb{N})}$ is a subring of $K^{\mathbb{N}}$ which contains the multiplicative identity element.
ii) The mapping $\varphi : K \to K^{(\mathbb{N})}$, $\varphi(a) = (a, 0, 0, \dots)$ is an injective unitary ring morphism.

Proof. i) Let $f = (a_0, a_1, \dots, a_n, 0, 0, \dots) = (a_0, a_1, \dots, a_k, 0, 0, \dots)$
$g = (b_0, b_1, \dots, b_m, 0, 0, \dots) = (b_0, b_1, \dots, b_k, 0, 0, \dots)$

$k = \max\{n, m\}$

$f + g = (a_0 + b_0, a_1 + b_1, \dots, a_k + b_k, 0, 0, \dots) \xRightarrow[\subseteq \{0,1,\dots,k\}]{\operatorname{supp}(f+g) \subseteq} f + g \in K^{(\mathbb{N})}$.

Let $f = (a_0, a_1, \dots, a_n, 0, 0, \dots)$, $g = (b_0, b_1, \dots, b_m, 0, 0, \dots)$

$fg = (c_0, c_1, \dots, c_k, \dots)$. Let $k \geq m + n + 1$. Then

$$c_k = \sum_{i+j=k} a_i b_j = a_0 \underbrace{b_k}_{=0} + a_1 \underbrace{b_{k-1}}_{=0} + \dots + a_n \underbrace{b_{k-n}}_{\geq m+1}^{=0} + \underbrace{a_{n+1}}_{=0} \underbrace{b_{k-n-1}}_{\geq m} + \dots$$
$$+ \dots + \underbrace{a_k}_{=0} b_0 = 0$$

$\Rightarrow fg = (c_0, c_1, \dots, c_{m+n}, 0, 0, \dots) \in K^{(\mathbb{N})}$

$(K^{(\mathbb{N})}, +, \cdot)$ unitary ring which preserves the multipl. id. elem (?)
ind. op. $+$ assoc., comm.

$\operatorname{supp}(0, 0, \dots 0, \dots) = \emptyset$ finite $\Rightarrow (0, 0, \dots) \in K^{(\mathbb{N})}$

$\forall f = (a_0, a_1, \dots, a_n, 0, 0, \dots) \in K^{(\mathbb{N})}$, $-f = (-a_0, \dots, -a_n, 0, 0, \dots) \in K^{(\mathbb{N})}$

$\Rightarrow (K^{(\mathbb{N})}, +)$ Abelian group.

· assoc., comm. , $(1, 0, 0, \dots) \in K^{(\mathbb{N})}$ $(\operatorname{supp}(1, 0, 0, \dots) = \{0\})$.
· is distr. w.r.t. $+$ in $K^{(\mathbb{N})}$

Thus $K^{(\mathbb{N})}$ is, indeed, a subring of $K^{\mathbb{N}}$ which preserves $(1, 0, \dots)$

ii) $\varphi : K \to K^{(\mathbb{N})}$, $\varphi(a) = (a, 0, 0, \dots)$ $\forall a, b \in K$

$\varphi(a+b) = (a+b, 0, 0, \dots) = (a, 0, \dots) + (b, 0, \dots) = \varphi(a) + \varphi(b)$
$\varphi(ab) = (ab, 0, 0, \dots) = (a, 0, \dots)(b, 0, \dots) = \varphi(a) \cdot \varphi(b)$ $\Rightarrow \varphi$ ring morphism

$\varphi(1) = (1, 0, 0, \dots)$ ← unitary

$a, b \in K$ $\varphi(a) = \varphi(b) \Leftrightarrow (a, 0, 0, \dots) = (b, 0, \dots) \Rightarrow a = b$. Thus φ inj. □

The ring $(K^{(\mathbb{N})}, +, \cdot)$ is called **polynomial ring** over K. How can we make this ring look like the one we know from high school?

The injective morphism φ allows us to identify $a \in K$ with $(a, 0, 0, \dots)$. This way K can be seen as a subring of $K^{(\mathbb{N})}$. The polynomial

$$X = (0, 1, 0, 0, \dots)$$

is called **indeterminate** or **variable**. From (2) one deduces that:

$$X^2 = (0, 0, 1, 0, 0, \dots)$$
$$X^3 = (0, 0, 0, 1, 0, 0, \dots)$$
$$\vdots$$
$$X^m = (\underbrace{0, 0, \dots, 0}_{m\ \cancel{ori}\ \text{times}}, 1, 0, 0, \dots)$$
$$\vdots$$

$f = (a_0, a_1, \dots, a_k, 0, 0, \dots) =$
$= (a_0, 0, 0, \dots) +$
$(0, a_1, 0, \dots) + \dots$
$+ (0, \dots, 0, a_n, 0, \dots) =$
$= a_0 + a_1 X + \dots + a_n X^n$

Since we identified $a \in K$ with $(a, 0, 0, \dots)$, from (2) it follows:

$$aX^m = (\underbrace{0, 0, \dots, 0}_{m\ \cancel{ori}\ \text{times}}, a, 0, 0, \dots) \tag{3}$$

This way we have

Theorem 3. Any $f \in K^{(\mathbb{N})}$ which is not zero can be uniquely written as

$$f = a_0 + a_1 X + a_2 X^2 + \dots + a_n X^n \tag{4}$$

where $a_i \in K$, $i \in \{0, 1, \dots, n\}$ and $a_n \neq 0$.

We can rewrite

$$K^{(\mathbb{N})} = \{f = a_0 + a_1 X + \dots + a_n X^n \mid a_0, a_1, \dots, a_n \in K,\ n \in \mathbb{N}\} \stackrel{\text{not}}{=} K[X].$$

The elements of $K[X]$ are called **polynomials over** K, and if $f = a_0 + a_1 X + \dots + a_n X^n$ then $a_0, \dots, a_n \in K$ are **the coefficients of** f, $a_0, a_1 X \dots, a_n X^n$ are called **monomials**, and a_0 is **the constant term of** f. Now, we can rewrite the operations from $(K[X], +, \cdot)$ as we did in high school (during the seminar).

If $f \in K[X]$, $f \neq 0$ and f is given by (4), then n is called **the degree of** f, and if $f = 0$ we say that the degree of f is $-\infty$. We will denote the degree of f by $\deg f$. Thus we have

$$\deg f = 0 \Leftrightarrow f \in K^*.$$

$(\Updownarrow\ f = a_0 \neq 0 \Longleftrightarrow f \in K \setminus \{0\})$

By definition

$$-\infty + m = m + (-\infty) = -\infty,\ -\infty + (-\infty) = -\infty,\ -\infty < m,\ \forall\, m \in \mathbb{N}.$$

Therefore:

i) $\deg(f + g) \leq \max\{\deg f, \cancel{\mathrm{grad}}\ \mathrm{deg}\ g\}, \forall f, g \in K[X]$;

ii) $\deg(fg) = \deg f + \deg g, \forall f, g \in K[X]$;

iii) $K[X]$ is an integral domain (during the seminar);

iv) a polynomial $f \in K[X]$ este is a unit in $K[X]$ if and only if $f \in K^*$ (during the seminar).

Here are some useful notions and results concerning polynomials:
If $f, g \in K[X]$ then

$$f \mid g \Leftrightarrow \exists\ h \in \overset{K[X]}{\cancel{K}},\ g = fh.$$

The divisibility $\mid$ is reflexive and transitive. The polynomial 0 satisfies the following relations

$$f \mid 0,\ \forall f \in K[X] \text{ and } \nexists f \in K[X] \setminus \{0\} :\ 0 \mid f.$$

Two polynomials $f, g \in K[X]$ are **associates** (we write $f \sim g$) if

the units from K[X]

$$\exists\ a \in K^* :\ f = ag.$$

The relation $\sim$ is reflexive, transitive and symmetric.

A polynomial $f \in K[X]^*$ is **irreducible** if $\deg f \geq 1$ and

$$f = gh\ (g, h \in K[X]) \Rightarrow g \in K^* \text{ or } h \in K^*.$$

The gcd and lcm are defined as for integers, the product of a gcm and lcm ~~af~~ of two polynomials f, g and the product fg are associates and the polynomials divisibility acts with respect to sum and product in the way we are familiar with from the integers case.

If $f = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n \in K[X]$ and $c \in K$, then

$$f(c) = a_0 + a_1c + a_2c\cancel{X}^2 + \cdots + a_nc^n \in K$$

is called **the evaluation of** f **at** c. The element $c \in K$ is **a root of** f if $f(c) = 0$.

Theorem 4. (The Division Algorithm in $K[X]$) For any polynomials $f, g \in K[X]$, $g \neq 0$, there exist $q, r \in K[X]$ uniquely determined such that

$$f = gq + r \text{ and } \deg r < \deg g. \tag{5}$$

Proof. (optional) Let $a_0, \ldots, a_n, b_0, \ldots, b_m \in K$, $b_m \neq 0$ and

$$f = a_0 + a_1X + \cdots + a_nX^n \text{ şi } g = b_0 + b_1X + \cdots + b_mX^m.$$

The existence of q and r: If $f = 0$ then $q = r = 0$ satisfy (5).

For $f \neq 0$ we prove by induction that that the property holds for any $n = \deg f$. If $n < m$ (since $m \geq 0$, there exist polynomials f which satisfy this condition), then (5) holds for $q = 0$ and $r = f$.

Let us assume the statement proved for any polynomials with the degree $n \geq m$. Since a_nX^n is the maximum degree monomial of the polynomial $a_nb_m^{-1}X^{n-m}g$, for $h = f - a_nb_m^{-1}X^{n-m}g$, we have $\deg h < n$ and, according to ourassumption, there exist $q', r \in R[X]$ such that

$$h = gq' + r \text{ and } \deg r < \deg g.$$

Thus, we have $f = h + a_nb_m^{-1}X^{n-m}g = (a_nb_m^{-1}X^{n-m} + q')g + r = gq + r$ where $q = a_nb_m^{-1}X^{n-m} + q'$. Now, the existence of q and r from (5) is proved.

The uniqueness of q and r: If we also have

$$f = gq_1 + r_1 \text{ and } \deg r_1 < \deg g,$$

then $gq + r = gq_1 + r_1$. It follows that $r - r_1 = g(q_1 - q)$ and $\deg(r - r_1) < \deg g$. Since $g \neq 0$ we have $q_1 - q = 0$ and, consequently, $r - r_1 = 0$, thus $q_1 = q$ and $r_1 = r$. $\square$

We call the polynomials q and r from (5) **the quotient** and **the remainder** of f when dividing by g, respectively.

Corollary 5. Let K be a field and $c \in K$. The remainder of a polynomial $f \in K[X]$ when dividing by $X - c$ is $f(c)$.

Indeed, from (5) one deduces that $r \in K$, and since $f = (X - c)q + r$, one finds that $r = f(c)$. For $r = 0$ we obtain:

Corollary 6. Let K be a field. The element $c \in K$ is a root of f if and only if $(X - c) \mid f$.

Corollary 7. If K is a field and $f \in K[X]$ has the degree $k \in \mathbb{N}$, then the number of the roots of f from K is at most k.

homework

Indeed, the statement is true for zero-degree polynomials, since they have no roots. We consider $k > 0$ and we assume the property valid for any polynomialwith the degree smaller than k. If $c_1 \in K$ is a root of f then $f = (X - c_1)q$ and $\deg q = k - 1$. According to our assumption, q has at most $k - 1$ roots in K. Since K is a field, $K[X]$ is an integral domain and from $f = (X - c_1)q$ it follows that $c \in K$ is a root of f if and only if $c = c_1$ or c is a root of q. Thus f has at most k roots in K.

The ring of square matrices over a field

Let K be a set and $m, n \in \mathbb{N}^*$. A mapping

$$A : \{1, \ldots, m\} \times \{1, \ldots, n\} \to K$$

is called $m \times n$ **matrix** over K. When $m = n$, we call A a **square matrix of size** n. For each $i = 1, \ldots, m$ and $j = 1, \ldots, n$ we denote $A(i, j)$ by $a_{ij} (\in K)$ and we represent A as a rectangular array with m rows and n columns in which the image of each pair (i, j) is written in the i'th row and the j'th column

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

We also denote this array by

$$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

or, simpler, $A = (a_{ij})$. We denote the set of all $m \times n$ matrices over K by $M_{m,n}(K)$ and, when $m = n$, by $M_n(K)$.

Let $(K, +, \cdot)$ be a field. Then $+$ from K determines an operation $+$ on $M_{m,n}(K)$ defined as follows: if $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices, then

$$A + B = (a_{ij} + b_{ij}).$$

One can easily check that this operation is associative, commutative, it has an identity element which is the matrix $O_{m,n}$ consisting only of 0 (called **the** $m \times n$ **zero matrix**) and each matrix $A = (a_{ij})$ from $M_{m,n}(K)$ has an opposite (the matrix $-A = (-a_{ij})$). Therefore,

Theorem 8. $(M_{m,n}(K), +)$ is an Abelian group.

The scalar multiplication of a matrix $A = (a_{ij}) \in M_{m,n}(K)$ and a scalar $\alpha \in K$ is defined by

$$\alpha A = (\alpha a_{ij}).$$

One can easily check that:

i) $\alpha(A + B) = \alpha A + \alpha B,\ \forall \alpha \in K,\ \forall A, B \in M_{m,n}(K)$;

ii) $(\alpha + \beta)A = \alpha A + \beta A,\ \forall \alpha, \beta \in K,\ \forall A \in M_{m,n}(K)$;

iii) $(\alpha\beta)A = \alpha(\beta A),\ \forall \alpha, \beta \in K,\ \forall A \in M_{m,n}(K)$;

iv) $1 \cdot A = A,\ \forall A \in M_{m,n}(K)$. 1 is the multipl. id. of K.

The matrix multiplication is defined as follows: if $A = (a_{ij}) \in M_{m,n}(K)$ and $B = (b_{ij}) \in M_{n,p}(K)$, then

$$AB = (c_{ij}) \in M_{m,p},\ \text{cu } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},\ (i,j) \in \{1, \dots, m\} \times \{1, \dots, p\}.$$

$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj} \in K$

For $n \in \mathbb{N}^*$ we consider the $n \times n$ square matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

If $m, n, p, q \in \mathbb{N}^*$, then:

1) $(AB)C = A(BC)$, for any matrices $A \in M_{m,n}(K),\ B \in M_{n,p}(K),\ C \in M_{p,q}(K)$;

2) $I_m A = A = A I_n,\ \forall A \in M_{m,n}(K)$;

3) $A(B + C) = AB + AC$ for any matrices $A \in M_{m,n}(K),\ B, C \in M_{n,p}(K)$;

3') $(B + C)D = BD + CD$, for any matrices $B, C \in M_{n,p}(K),\ D \in M_{p,q}(K)$;

4) $\alpha(AB) = (\alpha A)B = A(\alpha B),\ \forall \alpha \in K,\ \forall A_{m,n}(K),\ \forall B \in M_{n,p}(K)$.

We prove 1). To prove the other properties is easier, so we consider this your HOMEWORK.

Indeed, if $A = (a_{ij}) \in M_{m,n}(K),\ B = (b_{ij}) \in M_{n,p}(K),\ C = (c_{ij}) \in M_{p,q}(K)$, the element from the row $i \in \{1, \dots, m\}$ and the column $l \in \{1, \dots, q\}$ of the product $(AB)C$ is

$$\sum_{j=1}^{p} \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right) c_{jl} = \sum_{j=\overline{1,p}, k=\overline{1,n}} (a_{ik} b_{kj})\, c_{jl} = \sum_{j=\overline{1,p}, k=\overline{1,n}} a_{ik} (b_{kj} c_{jl}) = \sum_{k=1}^{n} a_{ik} \left(\sum_{j=1}^{p} b_{kj} c_{jl} \right).$$

Rings of matrices

We notice that this is also the element from the row $i \in \{1, \dots, m\}$ and column $l \in \{1, \dots, q\}$ of the product $A(BC)$.

If we work with $n \times n$ square matrices the matrix multiplication becomes a binary (internal) operation $\cdot$ on $M_n(K)$, and the equalities 1)–3') show that $\cdot$ is associative, I_n is a multiplicative identity element (called **the identity matrix** of size n) and $\cdot$ is distributive with respect to $+$. Hence,

Theorem 9. $(M_n(K), +, \cdot)$ is a unitary ring called **the ring of the square matrices of size** n **over** K.

Remarks 10. a) If $n \geq 2$ then $M_n(K)$ is not commutative and it has zero divisors. If $a, b \in K^*$, the non-zero matrices

$$\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \dots & b \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

can be used to prove this.
b) Using the properties of the addition, multiplication and scalar multiplication, one can easily prove that

$$f : K \to M_n(K),\ f(a) = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix} = aI_n$$

is a unitary injective ring homomorphism.

The transpose of an $m \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})$ is the $n \times m$ matrix

$$^tA = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} = (a_{ji}).$$

The way the transpose acts with respect to the matrix addition, matrix multiplication and scalar multiplication is given bellow:

$^t(A+B) =^t A + {}^tB,\ \forall A, B \in M_{m,n}(K);$
$^t(AB) =^t B \cdot {}^tA,\ \forall A \in M_{m,n}(K),\ \forall B \in M_{n,p}(K);$
$^t(\alpha A) = \alpha \cdot {}^tA,\ \forall A \in M_{m,n}(K).$

Let K be a field. The set of the units of $M_n(K)$ is

$$GL_n(K) = \{A \in M_n(K) \mid \exists B \in M_n(K) : AB = BA = I_n\}. = U(M_n(K))$$

The set $GL_n(K)$ is closed in $(M_n(K), \cdot)$ and $(GL_n(K), \cdot)$ is a group called **the general linear group of degree** n over K. We know from high school that if K is one of the number fields ($\mathbb{Q}$, $\mathbb{R}$ sau $\mathbb{C}$) then $A \in M_n(K)$ is invertible if and only if $\det A \neq 0$. Thus,

$$\rightarrow \quad GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det A \neq 0\},$$

COURSE 4

and analogously we can rewrite $GL_n(\mathbb{R})$ and $GL_n(\mathbb{Q})$. We will see next that this recipe works for any matrix ring $M_n(K)$ with K field. This is why our next course topic will be **the determinant of a square matrix over a field** K.

Determinants

Let $(K, +, \cdot)$ be a field, $n \in \mathbb{N}^*$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \in M_n(K).$$

Definition 11. The determinant of (the squre matrix) A is

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} (\in K).$$

The map $M_n(K) \to K$, $A \mapsto \det A$ is also called **determinant**.

Remark 12. None of the products from the above definition contains 2 elements from the same row or the same column.

We also denote the determinant of A by $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$.

Examples 13. a) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$

Lemma 14. The determinant of A and the determinant of the transpose matrix ${}^tA$ are equal.

Proof. □