COURSE 10

The sum of two subspaces. Linear maps

Let $(K, +, \cdot)$ be a field and let V be a vector space over K.

We remind from the previous course:

• For $S \subseteq V$ the following conditions are equivalent:

1) $S \leq_K V$.

2) The following conditions hold for S:

α) $0 \in S$;

β) $\forall x, y \in S\,,\ x + y \in S$;

γ) $\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$

3) The following conditions hold for S:

α) $0 \in S$;

δ) $\forall k_1, k_2 \in K\,,\ \forall x, y \in S\,,\ k_1 x + k_2 y \in S\,.$

• If $X \subseteq V$ then $\langle X \rangle = \bigcap\{S \leq_K V \mid X \subseteq S\}$ is the **subspace generated by** X.

• We have:

(i) If $S \leq_K V$ then $\langle S \rangle = S$.

(ii) If $X \subseteq V$ then $\langle\langle X \rangle\rangle = \langle X \rangle$.

(iii) If $X \subseteq Y \subseteq V$ then $\langle X \rangle \subseteq \langle Y \rangle$.

• If $\emptyset \neq X \subseteq V$ then $\langle X \rangle$ is the set of all finite linear combinations of vectors of X. In particular, if $x_1, \dots, x_n \in V$ then

$$\langle x_1, \dots, x_n \rangle = \{k_1 x_1 + \dots + k_n x_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \dots, n\}\,.$$

Remark 1. Notice that a linear combination of linear combinations is again a linear combination.

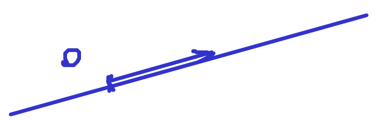
Examples 2. (a) Consider the real vector space $\mathbb{R}^3$. Then

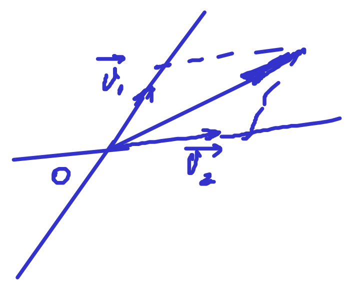
$$\langle (1,0,0), (0,1,0), (0,0,1) \rangle = \{k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \mid k_1, k_2, k_3 \in \mathbb{R}\} =$$

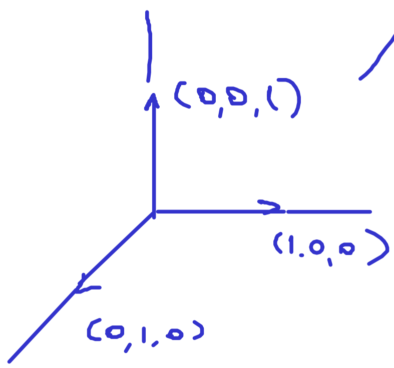
$$= \{(k_1,0,0) + (0,k_2,0) + (0,0,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} = \{(k_1,k_2,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} = \mathbb{R}^3\,.$$

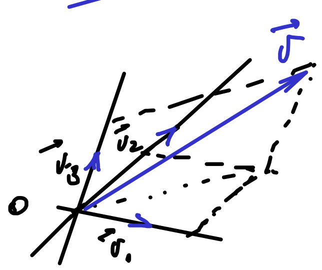
Hence $\mathbb{R}^3$ is generated by the three vectors $(1,0,0), (0,1,0), (0,0,1)$.

(b) More generally, the subspaces of $\mathbb{R}^3$ are the trivial subspaces, the lines containing the origin and the planes containing the origin. $\mathbb{R}$ $\{(0,0,0)\}$, $\mathbb{R}^3$









If $S, T \leq_K V$, the smallest subspace of V which contains the union $S \cup T$ is $\langle S \cup T \rangle$. We will show that this subspace is the sum of the given subspaces.

Definition 3. Let V be a vector space over K and let $S, T \leq_K V$. Then we define the **sum** of the subspaces S and T as the set

$$S + T = \{s + t \mid s \in S,\ t \in T\}.$$

If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the **direct sum** of the subspaces S and T.

Remarks 4. a) If V is a K-vector space, $V_1, V_2 \leq_K V$, then $V = V_1 \oplus V_2$ if and only if

$$V = V_1 + V_2 \text{ and } V_1 \cap V_2 = \{0\}.$$

Under these circumstances, we say that V_i $(i = 1, 2)$ is a **direct summand** of V.

b) If $V_1, V_2, V_3 \leq_K V$ and $V = V_1 \oplus V_2 = V_1 \oplus V_3$, we cannot deduce that $V_2 = V_3$.

Ex: $V = \mathbb{R}^2$, $K = \mathbb{R}$, $V_1 = \{(a,0) / a \in \mathbb{R}\} \leq_{\mathbb{R}} \mathbb{R}^2$

$V_2 = \{(0,b) / b \in \mathbb{R}\} \leq_{\mathbb{R}} \mathbb{R}^2$

$V_3 = \{(c,c) / c \in \mathbb{R}\} \leq_{\mathbb{R}} \mathbb{R}^2$

$V_1 + V_2 = \{(a,0)+(0,b) / a,b \in \mathbb{R}\} = \{(a,b) / a,b \in \mathbb{R}\} = \mathbb{R}^2$ $\Big\} \Rightarrow \mathbb{R}^2 = V_1 \oplus V_2$

$V_1 \cap V_2 = \{(0,0)\}$

$a = b - c$

$V_1 + V_3 = \{(a,0)+(c,c) / a,c \in \mathbb{R}\} = \{(a+c,c) / a,c \in \mathbb{R}\} = \{(b,c) / b,c \in \mathbb{R}\} = \mathbb{R}^2$ $\Big\} \Rightarrow \mathbb{R}^2 = V_1 \oplus V_3$

$V_1 \cap V_3 = \{(0,0)\}$ Yet, $V_2 \neq V_3$, $(1,1) \in V_3$, $(1,1) \notin V_2$.

c) The property of a subspace of being a direct summand is transitive. (during the seminar)

Theorem 5. Let V be a vector space over K and let $S, T \leq_K V$. Then

$$\langle S \cup T \rangle = S + T = \{s + t \mid s \in S,\ t \in T\}.$$

Proof. I) $S+T \overset{?}{\leq_K} V$

II) $S \cup T \subseteq S+T$ (?)

III) $A \leq_K V$, $S \cup T \subseteq A \Longrightarrow S+T \overset{?}{\subseteq} A$ □

I) $0 = \underset{\in S}{0} + \underset{\in T}{0} \in S+T$

Let $x, y \in S+T$, $x + y \overset{?}{\in} S+T$

$\Downarrow$

$\exists s, s' \in S, \exists t, t' \in T$ s.t. $x = s+t$, $y = s'+t'$ $\Longrightarrow$

$\Longrightarrow x+y = s+t+s'+t' = \underbrace{(s+s')}_{\in S} + \underbrace{(t+t')}_{\in T} \in S+T$

Let $x \in S$ ($x = s+t$, $s \in S$, $t \in T$), $\alpha \in K$, $\alpha x \overset{?}{\in} S+T$

$\alpha x = \alpha(s+t) = \underbrace{\alpha s}_{\in S} + \underbrace{\alpha t}_{\in T} \in S+T$

$$\text{II)}\quad \left.\begin{array}{l} \forall s\in S,\ \underset{\in S}{s} = s + \underset{\in T}{0} \in S+T \Rightarrow S\subseteq S+T \\ \forall t\in T,\ t = \underset{\in S}{0}+\underset{\in T}{t} \in S+T \Rightarrow T\subseteq S+T \end{array}\right\} \Rightarrow S\cup T\subseteq S+T$$

$$\text{III)}\quad \forall x\in S+T,\ \exists s\in S \subseteq S\cup T\subseteq A,\ \exists t\in T\subseteq S\cup T\subseteq A:\ x = \underset{\in A}{s} + \underset{\in A}{t} \in A \quad (A\leq_K V)$$

$S_1+S_2+\dots+S_n \leq_K V$ (homework)

a part of ...

a ∈ X *

Remarks 6. (1) Actually, a more general result can be proved: if $S_1, \ldots, S_n \leq_K V$ then

homework

$$S_1 + \cdots + S_n = \langle S_1 \cup \cdots \cup S_n \rangle.$$

(2) Moreover, if $X_i \subseteq V$ $(i = 1, \ldots, n)$, then $\langle X_1 \cup \cdots \cup X_n \rangle = \langle X_1 \rangle + \cdots + \langle X_n \rangle$.

proof: "$\subseteq$" $\quad X_i \subseteq \langle X_i\rangle \subseteq \langle X_1\rangle+\dots+\langle X_n\rangle,\ i=\overline{1,n} \Longrightarrow$

$$\Longrightarrow X_1\cup\dots\cup X_n \subseteq \langle X_1\rangle+\dots+\langle X_n\rangle \ (\leq_K V)$$

$$\Longrightarrow \langle X_1\cup\dots\cup X_n\rangle \subseteq \langle X_1\rangle+\dots+\langle X_n\rangle$$

"$\supseteq$" $\quad X_i \subseteq X_1\cup\dots\cup X_n \ (i=\overline{1,n}) \Longrightarrow \langle X_i\rangle \subseteq \langle X_1\cup\dots\cup X_n\rangle,\ i=\overline{1,n}$

any subspace is closed w.r.t. +

$$\Longrightarrow \langle X_1\rangle+\dots+\langle X_n\rangle \subseteq \langle X_1\cup\dots\cup X_n\rangle+\dots+\langle X_1\cup\dots\cup X_n\rangle \subseteq$$
$$\subseteq \langle X_1\cup\dots\cup X_n\rangle.$$

isomorphism = bijective linear map

endomorphism = linear map from a v.s. into itself

automorphism = bijective endom.

Definition 7. Let V and V' be vector spaces over K. The map $f : V \to V'$ is called a **(vector space) homomorphism** or a **linear map** (or a **linear transformation**) if

$$f(x+y) = f(x) + f(y)\,,\ \forall x, y \in V\,,$$

$$f(kx) = kf(x),\ \ \forall k \in K,\ \forall x \in V\,.$$

The **(vector space) isomorphism**, **endomorphism** and **automorphism** are defined as usual.

We will mainly use the name *linear map* or *K-linear map*.

Remarks 8. (1) When defining a linear map, we consider vector spaces over the same field K.
(2) If $f : V \to V'$ is a K-linear map, then the first condition from its definition tells us that f is a group homomorphism between $(V, +)$ and $(V', +)$. Thus we have

$$f(0) = 0' \text{ and } f(-x) = -f(x),\ \forall x \in V.$$

We denote by $V \simeq V'$ the fact that two vector spaces V and V' are isomorphic and

$$Hom_K(V, V') = \{f : V \to V' \mid f \text{ is a } K\text{-linear map}\}\,,$$

$V=V' \Longrightarrow Hom_K(V,V)=$ $End_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-linear map}\}\,,$ endom. ${}_K V$

$$Aut_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-isomorphism}\}\,.$$

autom. ${}_K V$.

Theorem 9. Let V, V' be K-vector spaces. Then $f : V \to V'$ is a linear map if and only if

$$f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2)\,,\ \forall k_1, k_2 \in K,\ \forall v_1, v_2 \in V. \quad (*)$$

Proof. "$\Longrightarrow$" Let $k_1, k_2 \in K$, $v_1, v_2 \in V$

$f(k_1v_1+k_2v_2) = f(k_1v_1)+f(k_2v_2) = k_1f(v_1)+k_2f(v_2)$.

"$\Longleftarrow$" Let us take $k_1=k_2=1$ in $(*)$

$(*) \Longrightarrow f(v_1+v_2) = f(v_1)+f(v_2)$, $\forall v_1, v_2 \in V$.

Let us take $k_2=0$ in $(*)$ $(k_1=k,\ v_1=v)$

$(*) \Longrightarrow f(kv) = kf(v)$, $\forall k\in K$, $\forall v \in V$. □

One can easily prove by way of induction the following:

Corollary 10. If $f : V \to V'$ is a linear map, then

homework

$$f(k_1v_1 + \cdots + k_nv_n) = k_1f(v_1) + \cdots + k_nf(v_n),\ \forall v_1, \ldots, v_n \in V,\ \forall k_1, \ldots, k_n \in K. \quad (n \in \mathbb{N}^*)$$

$\overset{0}{\text{''}}$

Examples 11. (a) Let V and V' be K-vector spaces and let $f : V \to V'$ be defined by $f(x) = 0'$, for any $x \in V$. Then f is a K-linear map, called the **trivial linear map**.
(b) Let V be a vector space over K. Then the identity map $1_V : V \to V$ is an automorphism of V. $1_V(x) = x$
(c) Let V be a vector space and $S \leq_K V$. Define $i : S \to V$ by $i(x) = x$, for any $x \in S$. Then i is a K-linear map, called the **inclusion linear map**.
(d) Let us consider $\varphi \in \mathbb{R}$. The map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi),$$

i.e. the plane rotation with the rotation angle φ, is a linear map. (during the seminar)
(e) If $a, b \in \mathbb{R}$, $a < b$, $I = [a, b]$, and $C(I, \mathbb{R}) = \{f : I \to \mathbb{R} \mid f \text{ continuous on } I\}$, then

$$F : C(I, \mathbb{R}) \to \mathbb{R},\ F(f) = \int_a^b f(x)dx$$

is a linear map. between ${}_{\mathbb{R}}C(I,\mathbb{R})$ and ${}_{\mathbb{R}}\mathbb{R}$.

As in the case of group homomorphisms, we have the following:

Theorem 12. Let V, V', V'' be K-vector spaces.
(i) If $f : V \to V'$ and $g : V' \to V''$ are K-linear maps (isomorphisms) then $g \circ f : V \to V''$ is a K-linear map (isomorphism).
(ii) If $f : V \to V'$ is an isomorphism of K-vector spaces then $f^{-1} : V' \to V$ is again an isomorphism of K-vector spaces.

Proof. □

(i) Let $v_1, v_2 \in V$, $k_1, k_2 \in K$,

$$(g \circ f)(k_1v_1 + k_2v_2) \overset{?}{=} k_1(g \circ f)(v_1) + k_2(g \circ f)(v_2).$$

$$(g \circ f)(k_1v_1 + k_2v_2) = g(f(k_1v_1 + k_2v_2)) \underset{\text{linear map}}{\overset{f}{=}} g(k_1f(v_1) + k_2f(v_2)) \underset{\text{map}}{\overset{g\text{ linear}}{=}} k_1g(f(v_1)) + k_2g(f(v_2)) = k_1(g \circ f)(v_1) + k_2(g \circ f)(v_2).$$

(ii) $\forall v_1', v_2' \in V'$, $\forall k_1, k_2 \in K$

$$f^{-1}(k_1v_1' + k_2v_2') \overset{?}{=} k_1f^{-1}(v_1') + k_2f^{-1}(v_2')$$

$v_1' \in V'$, f bijective $\Longrightarrow \exists!\, v_1 \in V : f(v_1) = v_1' \Longrightarrow v_1 = f^{-1}(v_1')$
$v_2' \in V'$, ——"—— $\Longrightarrow \exists!\, v_2 \in V : f(v_2) = v_2' \Rightarrow v_2 = f^{-1}(v_2')$

$$f^{-1}(k_1v_1' + k_2v_2') = f^{-1}(k_1f(v_1) + k_2f(v_2)) \overset{f \text{ linear map.}}{=}$$

$= f^{-1}(f(k_1 v_1 + k_2 v_2)) = k_1 v_1 + k_2 v_2 = \underbrace{k_1 f^{-1}(v_1) + k_2 f^{-1}(v_2)}$.

Obviously f^{-1} is bijective $\Rightarrow$ f^{-1} isom.

Definition 13. Let $f : V \to V'$ be a K-linear map. Then the set

$$\mathrm{Ker} f = \{x \in V \mid f(x) = 0'\}$$

is called the **kernel** of the K-linear map f and the set

$$\mathrm{Im} f = \{f(x) \mid x \in V\} = f(V) \leq V'$$

is called the **image** of the K-linear map f.

Theorem 14. Let $f : V \to V'$ be a K-linear map. Then we have

1) $\mathrm{Ker} f \leq_K V$ and $\mathrm{Im} f \leq_K V'$.

2) f is injective if and only if $\mathrm{Ker} f = \{0\}$.

Proof. $\square$

1) $\mathrm{Im} f = f(V) \leq_K V'$ (homework)

$A = V \leq_K V$ seminar: $A \leq_K V \longrightarrow f(A) = \{f(a) \mid a \in A\} \overset{?}{\leq_K} V'$

$\mathrm{Ker} f \overset{?}{\leq_K} V$ $\quad f(0) = 0' \Rightarrow 0 \in \mathrm{Ker} f$.

Let $x, y \in \mathrm{Ker} f$, $x + y \overset{?}{\in} \mathrm{Ker} f$

$f(x+y) \underset{\text{linear map}}{\overset{f}{=}} f(x) + f(y) = 0' + 0' = 0'$

Let $\alpha \in K$, $x \in \mathrm{Ker} f$, $\alpha x \overset{?}{\in} \mathrm{Ker} f$

$f(\alpha x) = \alpha f(x) = \alpha \cdot 0' = 0'$

2) f inj. $\overset{?}{\Longleftrightarrow}$ $\mathrm{Ker} f = \{0\}$

"$\Longrightarrow$" $\underbrace{\text{Let } x \in \mathrm{Ker} f} \Longrightarrow \underbrace{f(x)} = 0' = \underbrace{f(0)} \Longrightarrow$

$\underset{f \text{ inj.}}{\Longrightarrow}$ $\underline{x = 0}$.

"$\Leftarrow$" Let $x, y \in V$, $f(x) = f(y) \iff f(x) - f(y) = 0' \overset{f \text{ linear map}}{\iff} f(x-y) = 0' \implies x - y \in \operatorname{Ker} f = \{0\} \implies x - y = 0$

$\implies x = y$. Thus f inj.

Theorem 15. Let $f : V \to V'$ be a K-linear map and let $X \subseteq V$. Then

$$f(\langle X \rangle) = \langle f(X) \rangle .$$

Proof. I) $X = \emptyset \implies$ $\square$

$f(\langle \emptyset \rangle) = f(\{0\}) = \{f(0)\} = \{0'\} = \underset{\text{in } V'}{\langle \emptyset \rangle} = \langle f(\emptyset) \rangle$

II) $X \neq \emptyset$

$f(\langle X \rangle) = f(\{k_1 x_1 + \dots + k_n x_n \mid k_i \in K, x_i \in X, i = \overline{1,n}, n \in \mathbb{N}^*\}) =$

$= \{f(k_1 x_1 + \dots + k_n x_n) \mid k_i \in K, x_i \in X, i = \overline{1,n}, n \in \mathbb{N}^*\} =$

$= \{k_1 \underbrace{f(x_1)}_{= y_1} + \dots + k_n \underbrace{f(x_n)}_{= y_n} \mid k_i \in K, x_i \in X, i = \overline{1,n}, n \in \mathbb{N}^*\}$

$= \{k_1 y_1 + \dots + k_n y_n \mid k_i \in K, y_i \in f(X), i = \overline{1,n}, n \in \mathbb{N}^*\} =$

$= \langle f(X) \rangle$.

↓ COURSE 11

Theorem 16. Let V and V' be vector spaces over K. For any $f, g \in Hom_K(V, V')$ and for any $k \in K$, we consider $f + g, k \cdot f \in Hom_K(V, V')$,

$$(f+g)(x) = f(x) + g(x),\ \forall x \in V\ ,$$

$$(kf)(x) = kf(x),\ \forall x \in V\ .$$

These equalities define an addition and a scalar multiplication on $Hom_K(V, V')$ and $Hom_K(V, V')$ is a vector space over K.

Proof. □

Corollary 17. If V is a K-vector space, then $End_K(V)$ is a vector space over K.

Remarks 18. a) Let V be a K-vector space. From Theorem 12 one deduces that $End_K(V)$ is a subgroupoid of $(V^V, \circ)$ and from Example 11 (b) it follows that $(End_K(V), \circ)$ is a monoid. Moreover, the endomorphism composition $\circ$ is distributive with respect to endomorphism addition $+$, thus $End_K(V)$ also has a unitary ring structure, $(End_K(V), +, \circ)$.

b) The set $Aut_K(V)$ is the group of the units of $(End_K(V), \circ)$.