Analytic Geometry

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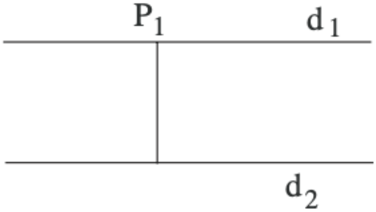
Recap...

- Last time we discussed "metric" problems in space.

The distance between two lines

Let d_1 and d_2 be two lines in the 3-space.

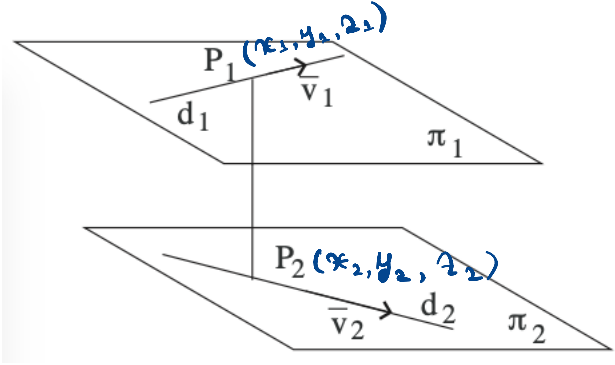
- If the lines are identical or concurrent, then $d(d_1, d_2) = 0$.
- If the lines are parallel, it is enough to choose an arbitrary point $P_1 \in d_1$ and $d(d_1, d_2) = d(P_1, d_2)$.



- If d_1 and d_2 are skew, there exists a unique line which is orthogonal on both d_1 and d_2 and intersects both d_1 and d_2. The length of the segment determined by these intersection points is the distance between the skew lines.
  Suppose that

$$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t \\ z = z_1 + r_1 t \end{cases}, t \in \mathbb{R} \text{ and } d_2 : \begin{cases} x = x_2 + p_2 s \\ y = y_2 + q_2 s \\ z = z_2 + r_2 s \end{cases}, s \in \mathbb{R}$$

  are, respectively, the parametric equations of the lines of director vectors $\overline{v}_1(p_1, q_1, r_1) \neq \overline{0}$, respectively $\overline{v}_2(p_2, q_2, r_2) \neq \overline{0}$.
  One can determine the equations of two parallel planes $\pi_1 \parallel \pi_2$, such that $d_1 \subset \pi_1$ and $d_2 \subset \pi_2$. The normal vector $\overline{n}$ of these planes has to be orthogonal on both $\overline{v}_1$ and $\overline{v}_2$, hence $\overline{n} = \overline{v}_1 \times \overline{v}_2$.


P1 (x1, y1, z1)
v1
d1
π1
P2 (x2, y2, z2)
v2
d2
π2

Then $\overline{n}(A, B, C)$, with $A = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}$, $B = \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}$ and $C = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$.
The equations of the planes π_1 and π_2 are:

$$\pi_1 : A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

$$\pi_2 : A(x - x_2) + B(y - y_2) + C(z - z_2) = 0.$$

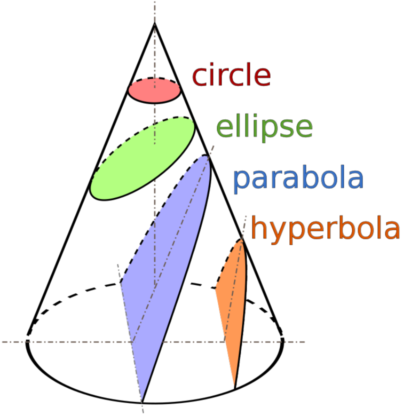
Now, the distance between d_1 and d_2 is the distance between the parallel planes π_1 and π_2; $d(d_1, d_2) = d(\pi_1, \pi_2)$, and one has the following theorem.

Theorem

The distance between two skew lines d_1 and d_2 is given by

$$d(d_1, d_2) = \frac{|A(x_1 - x_2) + B(y_1 - y_2) + C(z_1 - z_2)|}{\sqrt{A^2 + B^2 + C^2}}. \tag{1}$$

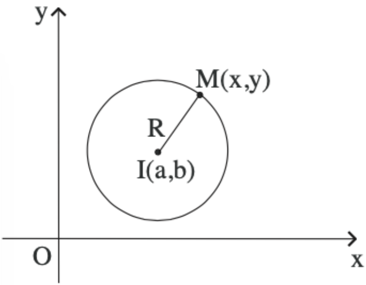
Conic sections



The circle

A *circle* is a closed plane curve, defined as the geometric locus of the points at a given distance R from a point I. The point I is the *center* of the circle and the number R is the *radius* of the circle. We shall denote the circle of center I and radius R by $\mathcal{C}(I, R)$.

In order to determine the equation of the circle, suppose that xOy is an associated Cartesian system of coordinates in $\mathcal{E}_2$, and $I(a, b)$. An arbitrary point $M(x, y)$ belongs to $\mathcal{C}(I, R)$ if and only if $|MI| = R$.



Hence, $\sqrt{(x-a)^2+(y-b)^2}=R$, or

$$(x-a)^2+(y-b)^2=R^2. \tag{2}$$

The equation (2) represents the equation of the circle centered at $I(a,b)$ and of radius R.

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The equation (2) represents the equation of the circle centered at $I(a,b)$ and of radius R.
Remark: In a Cartesian system of coordinates, the equation

$$x^2+y^2-2ax-2by+c=0 \tag{3}$$

represents either a circle, or a point, or the empty set.

How do we see this?

→ By completing the square in (3)

$$(x^2 - 2ax + a^2) + (y^2 - 2bx + b^2) + c - a^2 - b^2 = 0 \iff$$

$$(x-a)^2 + \underbrace{(y-b)^2}_{} = a^2 + b^2 - c.$$

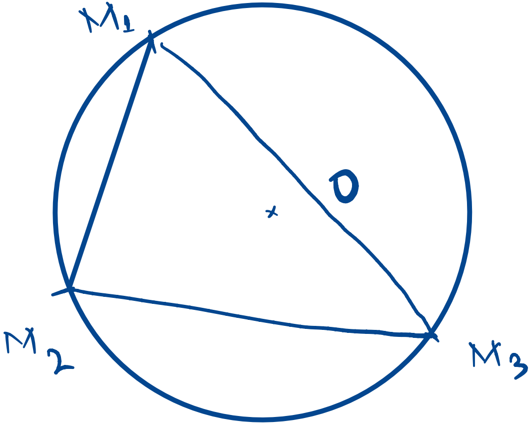
$$\parallel$$

$$d^2\big((x,y),(a,b)\big)$$

$$\begin{cases} a^2+b^2-c < 0, & \text{then (3) describes } \emptyset. \\ a^2+b^2-c = 0, & \text{then (3) describes a point } (a,b). \\ a^2+b^2-c > 0, & \text{then (3) is a circle with center } (a,b) \text{ and radius } \sqrt{a^2+b^2-c}. \end{cases}$$

The Circle Determined by Three Points

Given three noncollinear points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ and $M_3(x_3, y_3)$, there exists a unique circle passing through them.



Suppose that the circle determined by $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ and $M_3(x_3, y_3)$ has the general equation

$$x^2 + y^2 - 2ax - 2by + c = 0,$$

with $a^2 + b^2 - c > 0$. Since the three points are on the circle, one obtains the system of equations (with variables a, b and c)

linear.

$$\begin{cases} x^2 + y^2 - 2ax - 2by + c = 0 \\ x_1^2 + y_1^2 - 2ax_1 - 2by_1 + c = 0 \\ x_2^2 + y_2^2 - 2ax_2 - 2by_2 + c = 0 \\ x_3^2 + y_3^2 - 2ax_3 - 2by_3 + c = 0 \end{cases},$$

which has to be compatible, so that

$$\begin{pmatrix} -2x & -2y & 1 \\ -2x_1 & -2y_1 & 1 \\ -2x_2 & -2y_2 & 1 \\ -2x_3 & -2y_3 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -x^2 - y^2 \\ -x_1^2 - y_1^2 \\ -x_2^2 - y_2^2 \\ -x_3^2 - y_3^2 \end{pmatrix}$$

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{4}$$

The equation (4) is the equation of the circle determined by three points. It follows immediately that four points $M_i(x_i, y_i)$, $i = \overline{1,4}$, belong to a circle if and only if

$$\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0. \tag{5}$$

Intersection of a Circle and a Line

Let $\mathcal{C}$ be a circle and d be a line on $\mathcal{E}_2$. One may choose a system of coordinates having the center at the center of the circle, so that the equation of $\mathcal{C}$ is $x^2 + y^2 - R^2 = 0$. Let $d : y = mx + n$.

The intersection between $\mathcal{C}$ and d is given by the solutions of the system of equations

$$\begin{cases} x^2 + y^2 - R^2 = 0 \\ y = mx + n \end{cases}.$$

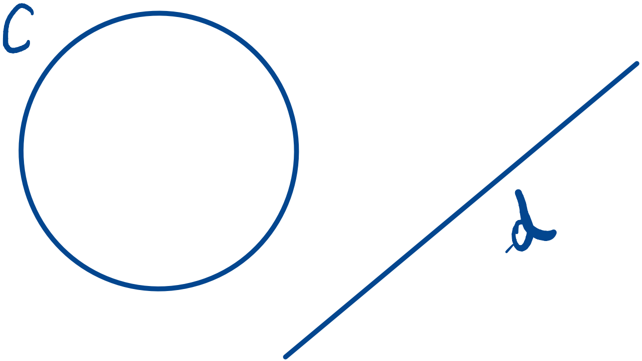
By substituting y in the equation of the circle, one obtains

$$(1 + m^2)x^2 + 2mnx + n^2 - R^2 = 0. \quad , x \in \mathbb{R}.$$

The discriminant of this second degree equation is

$$\Delta = 4(R^2 + m^2R^2 - n^2).$$

- If $R^2 + m^2R^2 - n^2 < 0$, then there are no intersection points between $\mathcal{C}$ and d. The line is *exterior* to the circle;



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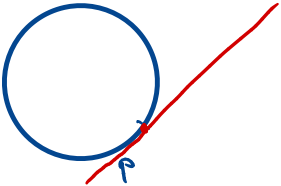
- If $R^2 + m^2R^2 - n^2 = 0$, then there is a double point (a *tangency* point) between $\mathcal{C}$ and d. The line is *tangent* to the circle. The coordinates of the tangency point are $\left(-\dfrac{mn}{1+m^2}, \dfrac{n}{1+m^2}\right)$;



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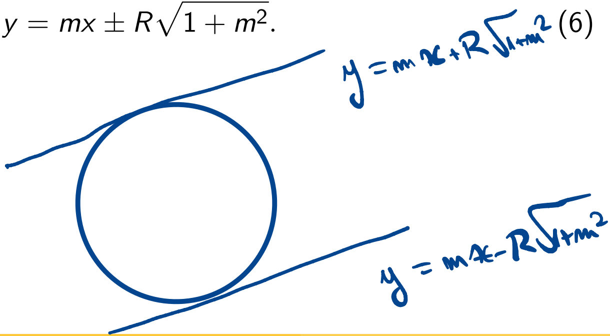
- If $R^2 + m^2R^2 - n^2 > 0$, then there are two intersection points between $\mathcal{C}$ and d. The line is *secant* to the circle. If x_1 and x_2 are the roots of the above equation, then the intersection points between $\mathcal{C}$ and d are $P_1(x_1, mx_1 + n)$ and $P_2(x_2, mx_2 + n)$.

The tangent of slope m to a given circle

Let $\mathcal{C}$ be the circle of equation $x^2 + y^2 - R^2 = 0$ and $m \in \mathbb{R}$ a given real number. There are two lines, having the angular coefficient m, and which are tangent to $\mathcal{C}$.

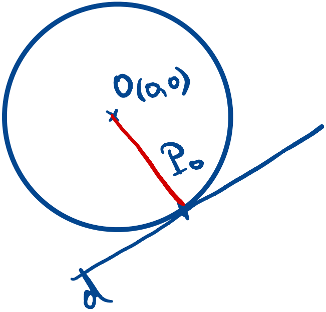
We saw, in the previous paragraph, that a line $d : y = mx + n$ is tangent to $\mathcal{C}$ if and only if $R^2 + m^2R^2 - n^2 = 0$. Then, the equations of the two tangent lines of ~~direction~~ slope m are

$$y = mx \pm R\sqrt{1 + m^2}. \tag{6}$$



The tangent to a circle at a point of the circle

Let $\mathcal{C} : x^2 + y^2 - r^2 = 0$ be a circle and $P_0(x_0, y_0)$ be a point on $\mathcal{C}$.



Slope of OP_0 is $\frac{y_0}{x_0}$.

Slope of d is $-\frac{x_0}{y_0}$.

So $OP_0 \perp d$.

The tangent at P_0 to $\mathcal{C}$ is a line from the bundle of lines $y - y_0 = m(x - x_0)$, $m \in \mathbb{R}$, having the vertex P. (missing $x = x_0$)

On the other hand, the tangent has to be of the form (6): $y = mx \pm R\sqrt{1+m^2}$. Then, the angular coefficient m must verify

$$\begin{cases} y - y_0 = m(x - x_0) \\ y = mx \pm R\sqrt{1+m^2} \end{cases},$$

hence

$$(y_0 - mx_0)^2 = R^2(1+m^2).$$

But $x_0^2 + y_0^2 = R^2$ (since $P_0 \in \mathcal{C}$) and one obtains $(mx_0 - y_0)^2 = 0$.

Therefore $m = -\dfrac{x_0}{y_0}$ (one may suppose that $y_0 \neq 0$; otherwise, one gets the tangent at the point $(R, 0)$, which is of equation $x = R$). Replacing m in the equation of the bundle, one obtains

$$y - y_0 = -\frac{x_0}{y_0} \cdot (x - x_0)$$

or

$$\boxed{x_0 x + y_0 y - (x_0^2 + y_0^2) = 0.}$$

Again, $x_0^2 + y_0^2 = R^2$, and the equation of the tangent line to $\mathcal{C}$ at the point $P_0 \in \mathcal{C}$ is

$$x_0 x + y_0 y - R^2 = 0. \tag{7}$$

Remark: The equation of the line OP_0 is $y = \dfrac{y_0}{x_0} x$. Then, the product of the angular coefficients of OP_0 and of the tangent at P_0 is -1, meaning that the tangent at a point to a circle is orthogonal on the radius which corresponds to the point.

Intersection of Two Circles

Given two circles,

$$\mathcal{C}_1 : x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0$$

and

$$\mathcal{C}_2 : x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0,$$

the system of equations

$$\begin{cases} x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0 \\ x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0 \end{cases}$$

gives informations about the intersection of the two circles.

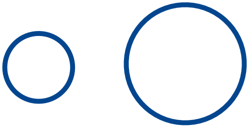
The previous system is equivalent to

$$\begin{cases} x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0 \\ 2(a_2 - a_1)x + 2(b_2 - b_1)y - (c_2 - c_1) = 0 \end{cases}$$

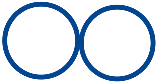
which will give rise to a second degree equation, of discriminant Δ.

I. $\Delta < 0$

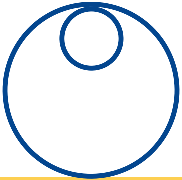
$\Rightarrow$



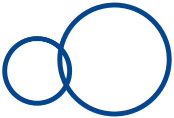
II. $\Delta = 0$



or



III. $\Delta > 0$



III. If $\Delta > 0$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are *secant* (they have two intersection points);

II. If $\Delta = 0$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are *tangent* (they have one *tangency* point);

I. If $\Delta < 0$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ have no intersection points.

Plane isometries

A map $f : \mathcal{E}_2 \to \mathcal{E}_2$ is said to be an *isometry* of the plane $\mathcal{E}_2$ if f conserves the distances, i.e.

$$|f(A)f(B)| = |AB|, \quad \forall\, A, B \in \mathcal{E}_2.$$

(One denotes $|AB| = d_2(A, B)$).

We briefly list a few properties of isometries. These are all proved in Chapter 4 of our textbook.

(Read in the textbook)

1) The image of a segment through an isometry is a segment.

2) The image of a half-line is a half-line;

3) The image of a line is a line;

4) If A, B and C are three noncollinear points on $\mathcal{E}_2$, then so are their images $f(A)$, $f(B)$ and $f(C)$;

5) The image of a triangle ΔABC is triangle $\Delta f(A)f(B)f(C)$, such that

$$\Delta ABC \equiv \Delta f(A)f(B)f(C);$$

6) The image of an angle $\widehat{AOB}$ is an angle $f(A)\widehat{f(O)}f(B)$ having the same measure;

7) Two orthogonal lines are transformed into two orthogonal lines;

8) Two parallel lines are transformed into two parallel lines.

9) Any isometry $f : \mathcal{E}_2 \to \mathcal{E}_2$ is surjective.

Denote the set of isometries of the plane by $\mathsf{Iso}(\mathcal{E}_2)$;

$$\mathsf{Iso}(\mathcal{E}_2) = \{f : \mathcal{E}_2 \to \mathcal{E}_2, f \text{ isometry}\}.$$

Theorem

$(\mathrm{Iso}(\mathcal{E}_2), \circ)$ *is a group, called the group of isometries of the plane.*

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Theorem

$(\mathrm{Iso}(\mathcal{E}_2), \circ)$ *is a group, called the group of isometries of the plane.*

- A point $A \in \mathcal{E}_2$ is a *fixed point* for the isometry f if $f(A) = A$;
- A line $d \in \mathcal{E}_2$ is said to be *invariant* with respect to f if $f(d) = d$ (obviously, a line whose points are all fixed is invariant, while the converse is not necessarily true).

Examples. Symmetries (reflections)

Let d be a line in $\mathcal{E}_2$. The map $s_d : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$s_d(P) = P', \quad \text{where } P' \text{ is the symmetrical of } P \text{ with respect to the line } d,$$

is called *axial symmetry*. The line d is the *axis* of the symmetry.

Let be given a point O in the plane. The map $s_O : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$s_O(P) = P', \quad \text{where } P' \text{ is the symmetrical of } P \text{ with respect to the point } P,$$

is called *central symmetry*. The point O is the *center* of the symmetry.

Another example. Translations

Let $\overline{v}$ be a vector in V_2. The map $t_{\overline{v}} : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$t_{\overline{v}}(M) = M', \qquad \text{where} \quad \overline{MM'} = \overline{v},$$

is called *translation* of vector $\overline{v}$.

Rotations

An angle $\widehat{AOB}$ is said to be *oriented* if the pair of half-lines $\{[OA, [OB\}$ is ordered. The angle $\widehat{AOB}$ is *positively oriented* if $[OA$ gets over $[OB$ counterclockwisely. Otherwise, $\widehat{AOB}$ is *negatively oriented*. If the measure of the *nonoriented* angle $\widehat{AOB}$ is θ, then the measure of the oriented angle $\widehat{AOB}$ is either θ, or $-\theta$, depending on the orientation of $\widehat{AOB}$.

Let $O \in \mathcal{E}_2$ be a point and $\theta \in [-2\pi, 2\pi]$ be a number. The map $r_{O,\theta} : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$r_{O,\theta}(M) = M', \quad \text{where} \quad \begin{cases} |OM| = |OM'| \\ m(\widehat{MOM'}) = \theta \end{cases},$$

is called *rotation* of center O and oriented angle θ.

Analytic form of isometries

Theorem

Let $P(x_0, y_0)$ be the center of the central symmetry s_P. The map s_P can be expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto -I_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2x_0 \\ 2y_0 \end{pmatrix}$$

Proof.

Let $M(x, y)$ be an arbitrary point on $\mathcal{E}_2$ and $M' = s_P(M)$ its symmetrical with respect to P, $M' = (x', y')$.

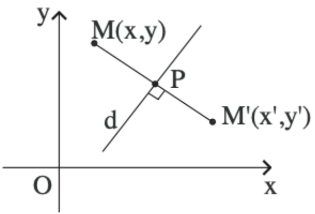
Since P is the midpoint of the segment $[MM']$, then $x_0 = \dfrac{x + x'}{2}$ and $y_0 = \dfrac{y + y'}{2}$, and the conclusion follows. □

Let us now see the analytic form of an axial symmetry.

Theorem

Let $d : ax + by + c = 0$, $a^2 + b^2 > 0$, be a line in $\mathcal{E}_2$. The axial symmetry s_d can be expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & -\frac{b^2-a^2}{a^2+b^2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{2ac}{a^2+b^2} \\ -\frac{2bc}{a^2+b^2} \end{pmatrix}.$$



Proof.

- One may suppose that $b \neq 0$.
- Let $M(x, y)$ be an arbitrary point and $M' = s_d(M)$, $M'(x', y')$.
- The points M and M' are symmetric with respect to d if and only if the line passing through M and M' is orthogonal on d and the midpoint P of the segment $[MM']$ belongs to d.
- The equation of the line determined by M and M' is $\dfrac{X - x}{x' - x} = \dfrac{Y - y}{y' - y}$. The orthogonality condition gives $a(y' - y) = b(x' - x)$.
- The midpoint of $[MM']$ is a point of d if and only if

$$a\left(\frac{x + x'}{2}\right) + b\left(\frac{y + y'}{2}\right) + c = 0.$$

$\square$

Continuation of the proof.

Then, the coordinates (x', y') of M' are the solution of the system of equation

$$\begin{cases} ax' + by' & = & -(ax + by + 2c) \\ bx' - ay' & = & bx - ay \end{cases}$$

and one obtains

$$\begin{cases} x' = \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y - \dfrac{2ac}{a^2 + b^2} \\ y' = -\dfrac{2ab}{a^2 + b^2}x - \dfrac{b^2 - a^2}{a^2 + b^2}y - \dfrac{2bc}{a^2 + b^2} \end{cases}.$$

In vector form, this can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & -\frac{b^2-a^2}{a^2+b^2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{2ac}{a^2+b^2} \\ -\frac{2bc}{a^2+b^2} \end{pmatrix}.$$

□

A few remarks

- If the line d passes through the origin O, then $c = 0$ and the coordinates of M' become

$$\begin{cases} x' = \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y \\ y' = -\dfrac{2ab}{a^2 + b^2}x - \dfrac{b^2 - a^2}{a^2 + b^2}y \end{cases} . \tag{8}$$

- If the line d is parallel to Ox, then $a = 0$ and the coordinates of M' become

$$\begin{cases} x' = x \\ y' = -y - \dfrac{2c}{b} \end{cases} . \tag{9}$$

- If the line d is parallel to Oy, then $b = 0$ and the coordinates of M' become

$$\begin{cases} x' = -x - \dfrac{2c}{a} \\ y' = y \end{cases} . \tag{10}$$

Translations

Let $\overline{v}(x_0, y_0)$ be a vector. The translation $t_{\overline{v}}$ of vector $\overline{v}$ can be expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Theorem

If f is an arbitrary isometry of $\mathcal{E}_2$, then its analytic form is given by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -\epsilon b \\ b & \epsilon a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

where $a^2 + b^2 = 1$ and $\epsilon = \pm 1$.

The problem set for this week is already posted. Ideally you would think about some of them before the seminar.

Thank you very much for your attention!