Seminar 6

1. To check if $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$ we need to check:

   (a) $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$ which we've seen that it's true from last seminar

   (b) $\forall A \in GL_n(\mathbb{R}), \forall B \in SL_n(\mathbb{R})$ we have $A^{-1} \cdot B \cdot A \in SL_n(\mathbb{R})$
   We take $A \in GL_n(\mathbb{R}) \iff det(A) \neq 0$. Also we take $B \in SL_n(\mathbb{R}) \iff det(B) = 1$. Then $det(A^{-1} \cdot B \cdot A) = det(A^{-1}) \cdot det(B) \cdot det(A) = det(A^{-1}) \cdot det(A) \cdot det(B) = det(A^{-1} \cdot A) \cdot det(B) = det(I_n) \cdot det(B) = 1 \Rightarrow A^{-1} \cdot B \cdot A \in SL_n(\mathbb{R})$

2. Homework.

3. From seminar 3, exercise 8, we know that $Z(G) \leq G$. So we only need to prove that $\forall g \in G$ and $\forall z \in Z(G) \Rightarrow g^{-1}zg \in Z(G)$. For this, we take any $z \in Z(G) \Rightarrow z \in G$ and $zg = gz, \forall g \in G$. If we multiply by g^{-1} on the left $\Rightarrow \forall z, g \in G$ we have $g^{-1}zg = g^{-1}gz = z \in Z(G)$. So, in conclusion $Z(G) \trianglelefteq G$.

4. The subgroups of $\mathbb{Z}$ are all sets $n\mathbb{Z}$ with $n \in \mathbb{N}$. We know that $(\mathbb{Z}, +)$ is abelian, so any subgroup of $(\mathbb{Z}, +)$ is a normal subgroup.

   The factor groups of $\mathbb{Z}$ are $\mathbb{Z}/n\mathbb{Z} = \{x + n\mathbb{Z} \mid x \in \mathbb{Z}\}$ with $n \in \mathbb{N}$. Consider $x = n \cdot q + r$, with $0 \leq r < n$. Then

   $$\mathbb{Z}/n\mathbb{Z} = \{nq + r + n\mathbb{Z} \mid 0 \leq r < n\} = \{r + n\mathbb{Z} \mid 0 \leq r < n\} = \mathbb{Z}_n.$$

5. As $(\mathbb{Z}_6, +)$ is an abelian group, every subgroup is normal. As $(\mathbb{Z}_6, +)$ is a cyclic group, we know that every subgroup is cyclic. We compute all subgroups $< x >$ with $x \in \mathbb{Z}_6$.

   We have $< \hat{0} >= \{\hat{0}\}$, $< \hat{1} >=< \hat{5} >= \mathbb{Z}_6$, $< \hat{2} >=< \hat{4} >= \{\hat{0}, \hat{2}, \hat{4}\}$ and $< \hat{3} >= \{\hat{0}, \hat{3}\}$. Hence the (normal) subgroups of $\mathbb{Z}_6$ are: $\{\hat{0}\}$, $\{\hat{0}, \hat{3}\}$, $\{\hat{0}, \hat{2}, \hat{4}\}$ and $\mathbb{Z}_6$.

   Now the factor groups of $\mathbb{Z}_6$ (all of them must be partitions of $\mathbb{Z}_6$) are:

   (a) $\mathbb{Z}_6/ < \hat{0} >= \{\hat{x} + \{\hat{0}\} \mid \hat{x} \in \mathbb{Z}_6\} = \{\{\hat{x}\} \mid \hat{x} \in \mathbb{Z}_6\}\} \cong \mathbb{Z}_6$,

(b) $\mathbb{Z}_6/ <\hat{1}>= \{\hat{x}+\mathbb{Z}_6 \mid \hat{x}\in\mathbb{Z}_6\} = \{\mathbb{Z}_6\} \cong \{\hat{0}\}$,

(c) $\mathbb{Z}_6/ <\hat{2}>= \{\hat{x}+\{\hat{0},\hat{2},\hat{4}\} \mid \hat{x}\in\mathbb{Z}_6\} = \{\{\hat{0},\hat{2},\hat{4}\},\{\hat{1},\hat{3},\hat{5}\}\}$,

(d) $\mathbb{Z}_6/ <\hat{3}>= \{\hat{x}+\{\hat{0},\hat{3}\} \mid \hat{x}\in\mathbb{Z}_6\} = \{\{\hat{0},\hat{3}\},\{\hat{1},\hat{4}\},\{\hat{2},\hat{5}\}\}$.

6. We know $(K,\cdot)\simeq(\mathbb{Z}_2\times\mathbb{Z}_2,+)$, where the operation is commutative. So any subgroup is a normal subgroup. As the order of the group is 4, by Lagrange's theorem, any subgroup has order 1, 2 or 4. The only subgroup with order 1 is the subgroup with the identity element and the only subgroup of order 4 is the whole group. So we only need to discuss about the subgroups of order 2.

   For H to be a subgroup, the identity element has to be in H. All non-zero elements of $\mathbb{Z}_2\times\mathbb{Z}_2$ have order 2, hence they are their own inverses. We find 3 subgroups with this properties: $H_1=\{(0,0),(0,1)\}$, $H_2=\{(0,0),(1,0)\}$ and $H_3=\{(0,0),(1,1)\}$.

   The factor groups are:

   (a) $(\mathbb{Z}_2\times\mathbb{Z}_2)/\{(0,0)\}\cong\mathbb{Z}_2\times\mathbb{Z}_2$

   (b) $(\mathbb{Z}_2\times\mathbb{Z}_2)/(\mathbb{Z}_2\times\mathbb{Z}_2)\cong\{(0,0)\}$

   (c) $(\mathbb{Z}_2\times\mathbb{Z}_2)/H_1 = \{(x,y)+\{(0,0),(0,1)\} \mid (x,y)\in\mathbb{Z}_2\times\mathbb{Z}_2\} = \{\{(0,0),(0,1)\},\{(1,0),(1,1)\}\}$

   (d) $(\mathbb{Z}_2\times\mathbb{Z}_2)/H_2 = \{(x,y)+\{(0,0),(1,0)\} \mid (x,y)\in\mathbb{Z}_2\times\mathbb{Z}_2\} = \{\{(0,0),(1,0)\},\{(0,1),(1,1)\}\}$

   (e) $(\mathbb{Z}_2\times\mathbb{Z}_2)/H_3 = \{(x,y)+\{(0,0),(1,1)\} \mid (x,y)\in\mathbb{Z}_2\times\mathbb{Z}_2\} = \{\{(0,0),(1,1)\},\{(1,0),(0,1)\}\}$

7. As the order of S_3 is 6, the subgroups of S_3 can have the order 1, 2, 3, 6. For the subgroup of order 1, we know it is the subgroup with only the identity element. As for the subgroup of order 6, it is the group S_3.

   We can easily see that the subgroups of order 2 have the form $H_i=\{e,\sigma_i\}$ for some transposition σ_i of S_3, as the inverse of a transposition is itself. Hence $H_1=\{e,\sigma_1\}$ with $\sigma_1=(2\ 3)$, $H_2=\{e,\sigma_2\}$ with $\sigma_2=(1\ 3)$ and $H_3=\{e,\sigma_3\}$ with $\sigma_3=(1\ 2)$.

   The only subgroup of order 3 must be $N=\{e,\sigma_4,\sigma_5\}$, where $\sigma_4=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ and $\sigma_5=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$.

We have:

$$S_3/r_{H_1} = \{\sigma \circ H_1 \mid \sigma \in S_3\} = \{\{e, \sigma_1\}, \{\sigma_2, \sigma_5\}, \{\sigma_3, \sigma_4\}\},$$

$$S_3/r'_{H_1} = \{H_1 \circ \sigma \mid \sigma \in S_3\} = \{\{e, \sigma_1\}, \{\sigma_2, \sigma_4\}, \{\sigma_3, \sigma_5\}\}.$$

Then H_1 is not a normal subgroup of S_3, because $S_3/r_{H_1} \neq S_3/r'_{H_1}$. Similarly, H_2 and H_3 are not normal subgroups of S_3.

We also have:

$$S_3/r_N = \{\sigma \circ N \mid \sigma \in S_3\} = \{\{e, \sigma_4\,\sigma_5\}, \{\sigma_1, \sigma_2, \sigma_3\}\},$$

$$S_3/r'_N = \{N \circ \sigma \mid \sigma \in S_3\} = \{\{e, \sigma_4\,\sigma_5\}, \{\sigma_1, \sigma_2, \sigma_3\}\}.$$

Hence $S_3/r_N = S_3/r'_N$, and so N is a normal subgroup of S_3.

We have $S_3/N = \{\{e, \sigma_4\,\sigma_5\}, \{\sigma_1, \sigma_2, \sigma_3\}\} = \{N, \sigma_1 \circ N\}$.

The operation table for the factor group S_3/N is:

$\circ$	N	$\sigma_1 \circ N$
N	N	$\sigma_1 \circ N$
$\sigma_1 \circ N$	$\sigma_1 \circ N$	N

8. We know that $(Q, \cdot) = (Q_8, \cdot) = \{\pm 1, \pm i, \pm j, \pm k\}$, where 1 is the identity element $i^2 = j^2 = k^2 = -1$, $i^4 = 1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$ and the signs rule holds.

   The order of the group is 8, so the subgroups must have the order 1, 2, 4 or 8. Clearly, $\{1\}$ and Q are (normal) subgroups of Q. It is easy to find the other subgroups of Q, namely $N_i = \{\pm 1, \pm i\}$, $N_j = \{\pm 1, \pm j\}$ and $N_k = \{\pm 1, \pm k\}$ of order 4, and $\{\pm 1\} = Z(Q)$ of order 2.

   For $Z(Q)$ we know from exercise 3 that is a normal subgroup of $(Q, \cdot)$. For the others, we know that if $\mid Q : N \mid = 2$, then $N \trianglelefteq Q$, for $N \leq Q$. Hence, all the subgroups we found are normal subgroups.

   The factor groups are easy to find:

   (a) $Q/\{1\} \cong Q$
   (b) $Q/Q \cong \{1\}$
   (c) $Q/Z(Q) = \{\{\pm 1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}\}$
   (d) $Q/N_i = \{\{\pm 1, \pm i\}, \{\pm j, \pm k\}\}$
   (e) $Q/N_j = \{\{\pm 1, \pm j\}, \{\pm i, \pm k\}\}$

(f) $Q/N_k = \{\{\pm 1, \pm k\}, \{\pm i, \pm j\}\}$

The operation table for the factor group $Q/Z(Q)$ is:

$\cdot$	$\{\pm 1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
$\{\pm 1\}$	$\{\pm 1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
$\{\pm i\}$	$\{\pm i\}$	$\{\pm 1\}$	$\{\pm k\}$	$\{\pm j\}$
$\{\pm j\}$	$\{\pm j\}$	$\{\pm k\}$	$\{\pm 1\}$	$\{\pm i\}$
$\{\pm k\}$	$\{\pm k\}$	$\{\pm j\}$	$\{\pm i\}$	$\{\pm 1\}$