1. Which of the following are affine subspace?

1. a line in $\mathbf{E}^2$
2. a circle in $\mathbf{E}^2$ centered at $(0,0)$
3. $(0,1)$ in $\mathbf{E}^2$
4. the half plane $x > y$ in $\mathbf{E}^2$
5. a line in $\mathbf{E}^3$
6. a triangle in $\mathbf{E}^2$ or $\mathbf{E}^3$
7. a circle in $\mathbf{E}^2$ centered at $(-1,1)$
8. a halfline in $\mathbf{E}^2$ or $\mathbf{E}^3$
9. a parabola in $\mathbf{E}^2$
10. a plane in $\mathbf{E}^3$
11. a disk in $\mathbf{E}^3$
12. a sphere in $\mathbf{E}^3$

2. Have a look at the third problem set of the geometry cours from last semester. Which problems on that list are purely affine, i.e. involve only vectors and vector relations in a vector space?

3. Let $\mathbf{A}$ be an affine space over the $\mathbf{K}$-vector space $\mathbf{V}$. Show that $\overrightarrow{PP} = 0$ for every $P \in \mathbf{A}$ and $\overrightarrow{PQ} = -\overrightarrow{QP}$ for every $P, Q \in \mathbf{A}$.

4. Give two distinct affine structures for the Euclidean plane $\mathbf{E}^2$. (*hint:* $\mathbb{R}$ and $\mathbb{C}$.)

5. Let S be an affine subspace of the affine space $\mathbf{A}$. Show that if $\dim(S) = \dim(\mathbf{A})$ then $S = \mathbf{A}$.

6. Which of the following admits the structure of an affine space? (explain why)

1. $\mathcal{C}(\mathbb{R})$ = the set of continuous functions $\mathbb{R} \to \mathbb{R}$.
2. $\{P \in \mathbf{K}[x] : \deg(P) \leq n\}$ = the set of polynomials of degree at most n.
3. $\{P \in \mathbf{K}[x] : \deg(P) = n\}$ = the set of polynomials of degree n.

7. Let p be a prime number.

1. Show that $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is a field.
2. How many vectors does an $\mathbb{F}_p$-vector space have?
3. How many points does an affine space over an $\mathbb{F}_p$-vector space have?

8. Let $\mathbf{A}$ be an affine space and consider four points $A, B, C, D \in \mathbf{A}$. Show that if $\overrightarrow{AB} = \overrightarrow{CD}$ then $\overrightarrow{AC} = \overrightarrow{BD}$.

9. In the affine space $\mathbf{A}^2(\mathbb{C})$ consider the line passing through the point $A(4, -2\mathbf{i})$ and having direction vector $\mathbf{a}(7+\mathbf{i}, 1)$. Give several parametric equations for this line.

10. In the affine space $\mathbf{A}^3(\mathbb{C})$ consider the plane passing through the point $A(2+\mathbf{i}, 5, -\mathbf{i})$ and parallel to the vectors $\mathbf{a}(2,3,1)$ and $\mathbf{b}(-1,-11,3)$. Give several parametric equations for this plane.

11. In the affine space $\mathbf{A}^4(\mathbb{R})$ consider

$$\text{the plane} \quad \alpha = \left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\4\\1\\2\end{bmatrix} \quad \text{and the line} \quad \beta = \left\langle \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\3\\-1\\1\end{bmatrix}.$$

Determine $\alpha \cap \beta$.

12. Let $\mathbf{V}$ be a vector space of dimension at least 5 and let $\mathbf{A}$ be an affine space with associated vector space $\mathbf{V}$. Consider three distinct points $a, b, c \in \mathbf{A}$ and a plane $\pi = \langle v_1, v_2 \rangle + a$. Determine an affine subspace in $\mathbf{A}$ of dimension at most 4 which contains a, b, c and π.

13. Show that the definition of the affine subspace $\langle P_0, \ldots, P_n \rangle$ generated by $P_0, \ldots, P_n$ does not depend on the point P_0.

14. Let $\mathbf{A}$ be an affine space over the vector subspace $\mathbf{V}$ and let $C \in \mathbf{A}$. For each $P \in \mathbf{A}$, the *reflection* of P in C is the point $\mathrm{Ref}_C(P)$ satisfying the vector identity

$$\overrightarrow{C\,\mathrm{Ref}_C(P)} = -\overrightarrow{CP}.$$

This defines a map $\mathrm{Ref}_C : \mathbf{A} \to \mathbf{A}$. Show that $\mathrm{Ref}_C(P)$ maps affine subspaces to affine subspaces.

15. Show that the definition of a k-simplex with vertices $P_0, \ldots, P_k$ doesn't depend on the choice of P_0.

16. Show that a k-simplex is convex.