# ANALYTIC GEOMETRY, PROBLEM SET 2

## More polar coordinates

**1.** Let $ABCDEF$ be a regular hexagon with size length $l$. Find the polar coordinates of its vertices in each of the following cases:

a) The center of the hexagon $O$ is chosen as *the pole* and the half-line $[OA$ is set as *the polar axis*.

b) The vertex $A$ is chosen as *the pole* and the half-line $[AB$ is set as *the polar axis*.

**2.** Find the polar equation corresponding to the given Cartesian equation: a) $y = 5$; b) $x + 1 = 0$; c) $y = 7x$; d) $3x + 8y + 6 = 0$; e) $y^2 = -4x + 4$; f) $x^2 - 12y - 36 = 0$; g) $x^2 + y^2 = 36$; h) $x^2 - y^2 = 25$. Briefly give a geometric interpretation for the solutions to these equations.

**3.** Find the polar coordinates of the point $P \in \mathcal{E}_2$, whose rectangular (Cartesian) coordinates are $(1 + \cos\alpha, \sin\alpha)$, where $\alpha \in (0, 2\pi)$ is fixed.

## Cylindrical and spherical (everything is in 3D here)

**Warmp-up 1.** In the cylindrical coordinate system, what do the following equations represent in $\mathcal{E}_3$? $(r, \theta, z)$

a) $r = r_0$, where $r_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed; ← half-plane.

c) $z = z_0$, where $z_0 \in \mathbb{R}$ is fixed. ← a plane.

**Warm-up 2.** In the spherical coordinate system, what do the following equations represent in $\mathcal{E}_3$? $(\rho, \theta, \varphi) \in \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi]$.

a) $\rho = \rho_0$, where $\rho_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed;

c) $\varphi = \varphi_0$, where $\varphi_0 \in [0, \pi]$ is fixed.

**4.** Let $P_1(r_1, \theta_1, z_1)$ and $P_2(r_2, \theta_2, z_2)$ be points in $\mathcal{E}_3$ expressed using their cylindrical coordinates. Find the distance $P_1P_2$, as an expression of $r_i, \theta_i, z_i$, where $i \in \{1, 2\}$.

**5.** Let $P_1(r_1, \theta_1, \varphi_1)$ and $P_2(r_2, \theta_2, \varphi_2)$ be points in $\mathcal{E}_3$, expressed using their spherical coordinates. Find the distance $P_1P_2$, as an expression of $r_i, \theta_i, \varphi_i$, where $i \in \{1, 2\}$.

**6.** Determine, in cylindrical coordinates, the equation of the surface whose equation in rectangular coordinates is $z = x^2 + y^2 - 2x + y$.

**7.** Find the equation, in rectangular coordinates, of the surface whose equation in cylindrical coordinates is $r = 4\cos(\theta)$. Explain what the equation describes geometrically.

**8.** (Non-examinable) Three spheres are pairwise exterior tangent; a plane is tangent to these spheres at points $A$, $B$ and $C$. Find the radii of the spheres in terms of $a$, $b$, $c$, representing the lengths of the sides of triangle $ABC$.

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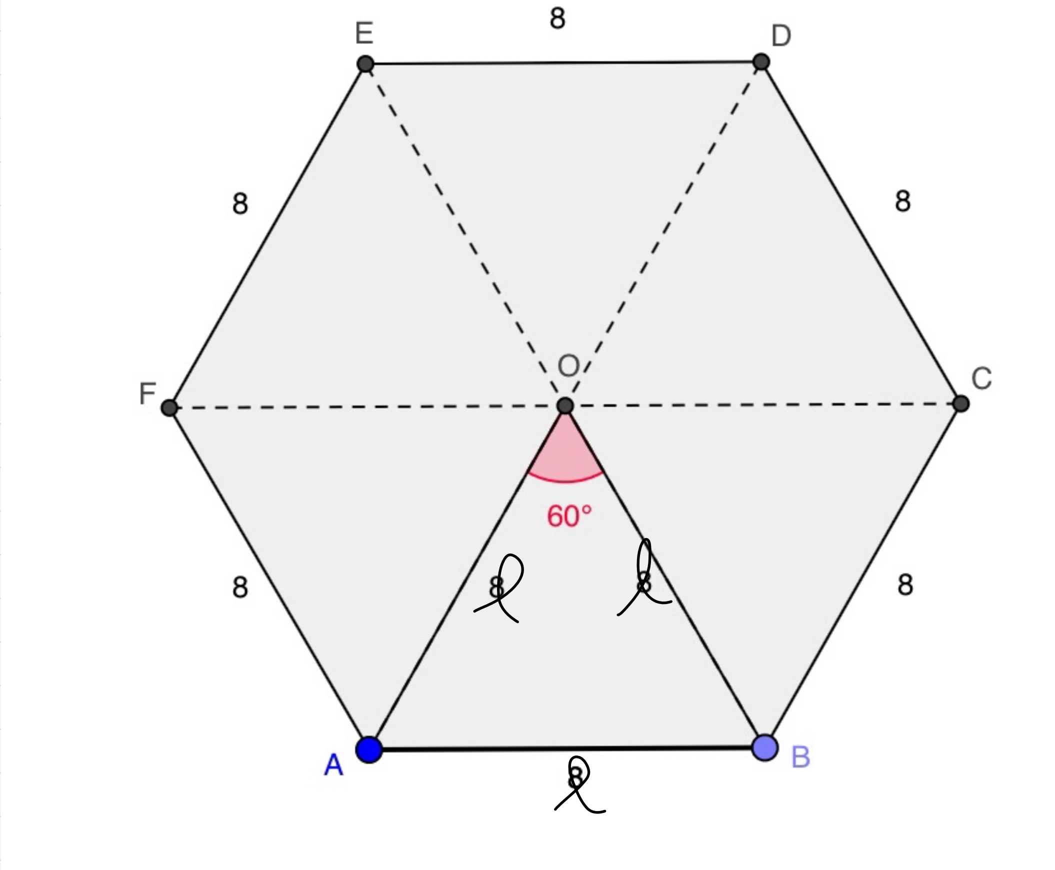
*Date*: October 2, 2021.

1.




O - pole , [OA - polar axis.

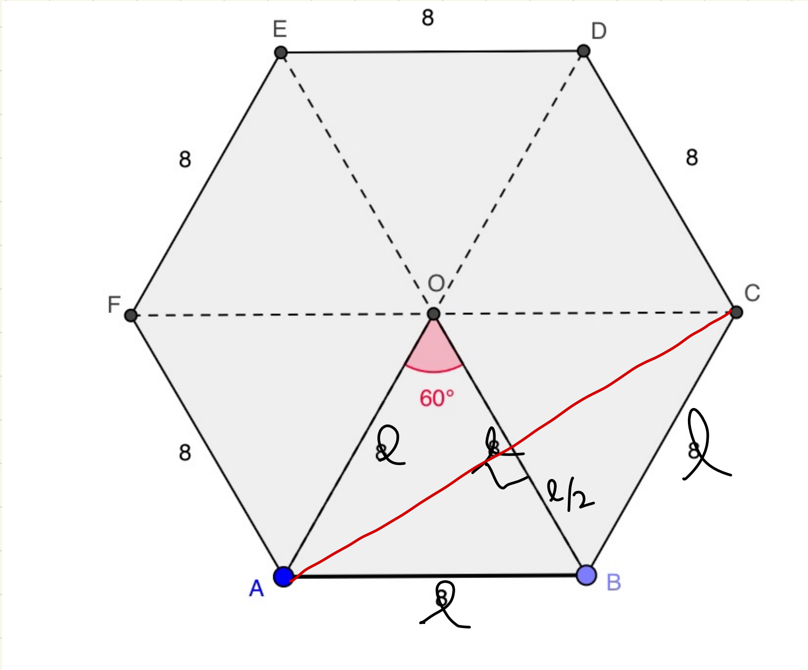
Polar coordinates:

$A(\ell, 0^\circ)$ , $D(\ell, \pi)$

$B(\ell, \frac{\pi}{3})$ , $E(\ell, \frac{4\pi}{3})$

$C(\ell, \frac{2\pi}{3})$ , $F(\ell, \frac{5\pi}{3})$

b) $A$ - pole , $[AB$ - polar axis




$A(0,0)$ , $B(\ell, 0)$

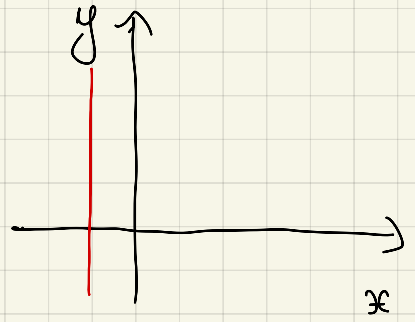
$C\left(\ell\sqrt{3}, \frac{\pi}{6}\right)$ , $D\left(2\ell, \frac{\pi}{3}\right)$

$E\left(\ell\sqrt{3}, \frac{\pi}{2}\right)$ , $F\left(\ell, \frac{2\pi}{3}\right)$.

2. Find the polar equation corresponding to the given Cartesian equation: a) $y = 5$; b) $x + 1 = 0$; c) $y = 7x$; d) $3x + 8y + 6 = 0$; e) $y^2 = -4x + 4$; f) $x^2 - 12y - 36 = 0$; g) $x^2 + y^2 = 36$; h) $x^2 - y^2 = 25$. Briefly give a geometric interpretation for the solutions to these equations.

b) $x = r \cdot \cos\theta$

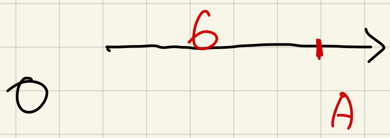
$\therefore \quad x + 1 = 0 \iff r \cdot \cos\theta = -1.$




g) $x^2 + y^2 = r^2\cos^2\theta + r^2 \cdot \sin^2\theta = r^2.$

$r^2 = 36 \quad (\text{but } r > 0)$

$\therefore \quad r = 6$



h) $$x^2 - y^2 = 25.$$

In polar

$$r^2(\cos^2\theta - \sin^2\theta) = 25.$$

$$r^2 \cdot \cos(2\theta) = 25.$$

**3.** Find the polar coordinates of the point $P \in \mathcal{E}_2$, whose rectangular (Cartesian) coordinates are $(1 + \cos\alpha, \sin\alpha)$, where $\alpha \in (0, 2\pi)$ is fixed.

$$\theta = \arctan\left(\frac{\sin\alpha}{1+\cos\alpha}\right)$$

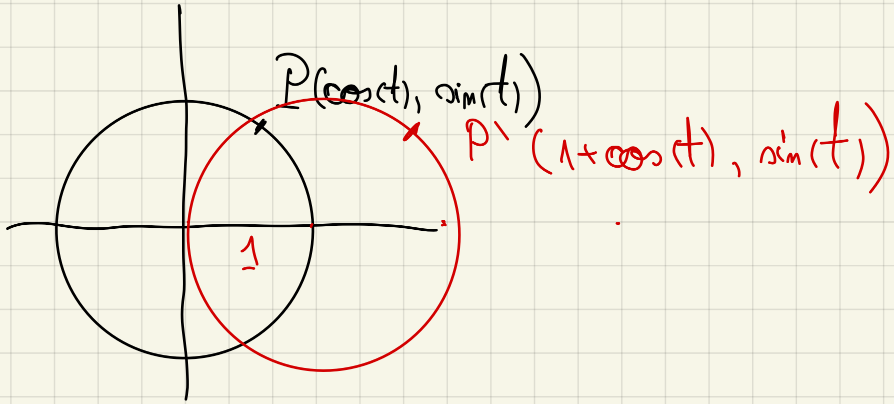
If $\alpha = \pi$, then $P(0, 0) = O$

$\uparrow$

the pole.

$$t \in [0, 2\pi] \longmapsto (\cos(t), \sin(t)).$$

$$P(x, \quad y).$$




$(1 + \cos t, \ \sin t)$

**4.** Let $P_1(r_1, \theta_1, z_1)$ and $P_2(r_2, \theta_2, z_2)$ be points in $\mathcal{E}_3$ expressed using their cylindrical coordinates. Find the distance $P_1P_2$, as an expression of $r_i, \theta_i, z_i$, where $i \in \{1, 2\}$.

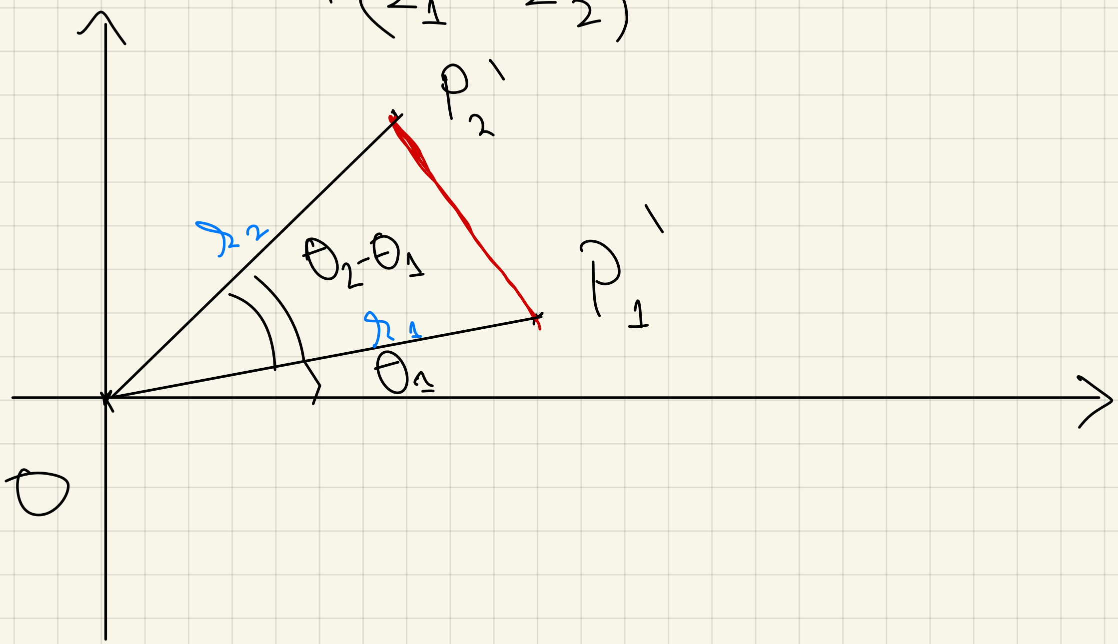
$P_1(x_1, y_1, z_1), \quad P_2(x_2, y_2, z_2).$

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

$$\begin{cases} x_i = r_i \cdot \cos(\theta_i) \\ y_i = r_i \cdot \sin(\theta_i) \\ z_i = z_i \end{cases}, \quad \forall\, i \in \{1, 2\}.$$

In polar coordinates,

$$d(P_1, P_2) = \sqrt{(r_1 \cdot \cos\theta_1 - r_2 \cdot \cos\theta_2)^2 + (r_1 \cdot \sin\theta_1 - r_2 \cdot \sin\theta_2)^2 + (z_1 - z_2)^2}$$




Let $P_i'$ be the projection of $P_i$ in $xOy$.

From the cosine theorem:

$$|P_1' P_2'|^2 = r_1^2 + r_2^2 - 2 \cdot r_1 \cdot r_2 \cdot \cos(\theta_2 - \theta_1).$$

$$d(P_1, P_2) = \sqrt{r_1^2 + r_2^2 - 2r_1 \cdot r_2 \cdot \cos(\theta_2 - \theta_1) + (z_1 - z_2)^2}.$$

**6.** Determine, in cylindrical coordinates, the equation of the surface whose equation in rectangular coordinates is $z = x^2 + y^2 - 2x + y$.

$$z = r^2 \cos^2\theta + r^2 \cdot \sin^2\theta - 2r\cos\theta + r \cdot \sin\theta$$

$$= r^2 - 2r \cdot \cos\theta + r \cdot \sin\theta.$$

$$= r^2 - 2r\left(\cos\theta - \frac{\sin\theta}{2}\right)$$

**7.** Find the equation, in rectangular coordinates, of the surface whose equation in cylindrical coordinates is $r = 4\cos(\theta)$. Explain what the equation describes geometrically.

**8.** (Non-examinable) Three spheres are pairwise exterior tangent; a plane is tangent to these

$$r = 4 \cdot \cos\theta \quad \Big| \cdot r$$

$$r^2 = 4r \cdot \cos\theta \quad \Leftrightarrow$$

$$x^2 + y^2 = 4x \quad \Leftrightarrow$$

$$x^2 - 4x + 4 - 4 + y^2 = 0 \quad \Leftrightarrow$$

$$(x-2)^2 + y^2 = 4. \quad (*)$$

The points $P(x, y, z)$ satisfying $(*)$ form an infinite cilinder.