CHAPTER 4

Affine subspaces in dimension 3

Contents

4.1 Affine lines and planes

- An affine line is a 1-dimensional affine space and an affine plane is a 2 dimensional affine space. Here we look at how affine lines and affine planes appear as subspaces of a three dimensional affine space. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space of dimension 3. Let $\mathbf{A}$ be an affine space over $\mathbf{V}$.

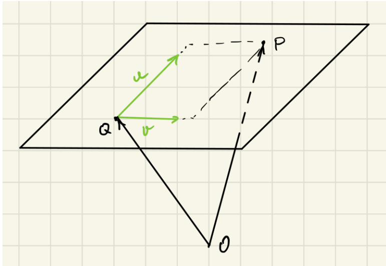
- While most of the content should look familiar to you, bear in mind that we are working with some arbitrary field $\mathbf{K}$.

- Let $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ be a reference frame (another term for coordinate system) of $\mathbf{A}$.

- Every plane π has parametric equations of the form

$$\pi : \begin{cases} x = x_Q + sv_x + tu_x \\ y = y_Q + tv_y + tu_y \\ z = z_Q + tv_z + tu_z \end{cases} \quad \text{or, in matrix form,} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_Q \\ y_Q \\ z_Q \end{bmatrix} + s \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + t \cdot \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}. \tag{4.1}$$

  where $Q(x_Q, y_Q, z_Q)$ is a point in π and where $\{\mathbf{v}(v_x, v_y, v_z), \mathbf{u}(u_x, u_y, u_z)\}$ is a basis for the vector space $\mathbf{W}$ associated to π. The equations describe the points $P(x, y, z)$ in π obtained for the different values of the parameters $s, t \in \mathbf{K}$.

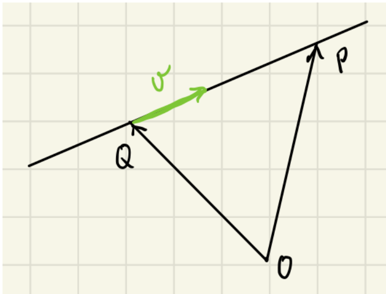


- Clearly, for different choices of $Q \in \pi$ and different choices of the basis $\{\mathbf{v}, \mathbf{u}\}$ of $\mathbf{W}$ one obtains different parametric equations for π.
- Since π has codimension 1, it is a hyperplane in $\mathbf{A}$. Hence, π is described by a single equation
$$\pi : ax + by + cz + d = 0 \tag{4.2}$$
for some $a, b, c, d \in \mathbf{K}$ with $(a, b, c) \neq (0, 0, 0)$.
- The constants a, b, c, d are determined by π only up to a non-zero common factor, so a plane has in general many equations, each proportional to the others.
- The planes with equations $x = 0$, $y = 0$ or $z = 0$ are the *coordinate planes*: the yz-plane, the xz-axis and the xy-plane respectively.
- [From parametric equations to Cartesian equations] Suppose we are given parametric equations (4.1) for the plane π, i.e. Q, $\mathbf{v}$ and $\mathbf{u}$ are given. To obtain a Cartesian equation for π one should view (4.1) as saying that $P(x_P, y_P, z_P)$ lies in π if and only if the vectors $\overrightarrow{QP}$, $\mathbf{v}$ and $\mathbf{u}$ are linearly dependent. This last condition can also be expressed by requiring that (x_P, y_P, z_P) is a solution to the equation
$$\begin{vmatrix} x - x_Q & y - y_Q & z - z_Q \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$$
which is equivalent to
$$ax + by + cz + d = 0.$$
where
$$a = \begin{vmatrix} v_y & v_z \\ u_y & u_z \end{vmatrix}, \quad b = -\begin{vmatrix} v_x & v_z \\ u_x & u_z \end{vmatrix}, \quad c = \begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix} \quad \text{and} \quad d = -ax_Q - by_Q - cz_Q.$$
- [From Cartesian equations to parametric equations] Suppose we are given the equation (4.2) for the plane π. If $a \neq 0$ then, viewing y and z as 'free variables', (4.2) is equivalent to
$$\begin{cases} x = -\frac{b}{a}y - \frac{c}{a}z - \frac{d}{a} \\ y = y \\ z = z \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{d}{a} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix}$$
which are parametric equations for π in the parameters $y, z \in \mathbf{K}$. The cases when $b \neq 0$ or $c \neq 0$ can be treated similarly.

- Every line ℓ has parametric equations of the form

$$\ell : \begin{cases} x = x_Q + t v_x \\ y = y_Q + t v_y \\ z = z_Q + t v_z \end{cases} \quad \text{or, in matrix form,} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_Q \\ y_Q \\ z_Q \end{bmatrix} + t \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}. \tag{4.3}$$

where $Q(x_Q, y_Q, z_Q)$ is a point in ℓ and $\mathbf{v}(v_x, v_y, v_z)$ is a direction vector of ℓ. The equations describe the points $P(x, y, z)$ on ℓ obtained for the different values of the parameter $t \in \mathbf{K}$.



- Clearly, for different choices of $Q \in \ell$ and different direction vectors $\mathbf{v}$ of ℓ one obtains different parametric equations for ℓ.
- Since ℓ has codimension 2 in $\mathbf{A}$, it can be described by two Cartesian equation

$$\ell : \begin{cases} ax + by + cz + d &= 0 \\ a'x + b'y + c'z + d' &= 0 \end{cases} \tag{4.4}$$

for some $a, b, c, d, a', b', c', d' \in \mathbf{K}$ with $(a, b, c) \neq (0, 0, 0)$ and $(a', b', c') \neq (0, 0, 0)$.

- The equations in (4.4) just say that ℓ is the intersection of two planes.
- The two equations in (4.4) describe the same line ℓ if they are multiplied with non-zero scalars, so a line has in general many such descriptions as a system of two equations.
- The *coordinate axes* have equations

$$Ox : \begin{cases} y = 0 \\ z = 0 \end{cases}, \quad Oy : \begin{cases} x = 0 \\ z = 0 \end{cases} \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ y = 0 \end{cases}.$$

- [From parametric equations to Cartesian equations] Suppose we are given parametric equations (4.3) for the line ℓ, i.e. Q and $\mathbf{v}$ are given. To obtain a Cartesian equation for ℓ one should view (4.3) as saying that $P(x_P, y_P, z_P)$ lies in ℓ if and only if the vectors $\overrightarrow{QP}$ and $\mathbf{v}$ are linearly dependent. This last condition can also be expressed by requiring that the matrix

$$\begin{bmatrix} x - x_Q & y - y_Q & z - z_Q \\ v_x & v_y & v_z \end{bmatrix}$$

has all minors equal to zero.

- [From Cartesian equations to parametric equations] Suppose we are given the equations (4.3) for the line ℓ. The direction of ℓ is the 1-dimensional vector subspace of $\mathbf{V}$ given by the associated homogenous system
$$\begin{cases} ax+by+cz &= 0\\ a'x+b'y+c'z' &= 0 \end{cases} \tag{4.5}$$
It follows that the vector $\mathbf{v}(v_x, v_y, v_z)$ with coordinates
$$v_x = \begin{vmatrix} b & c\\ b' & c' \end{vmatrix}, \quad v_y = -\begin{vmatrix} a & c\\ a' & c' \end{vmatrix} \quad \text{and} \quad v_z = \begin{vmatrix} a & b\\ a' & b' \end{vmatrix}.$$
is a direction vector for ℓ, because the coordinates of $\mathbf{v}$ are a non-zero solution of the homogenous system (4.5). This gives a practical method for calculating a direction vector of a line if we know Cartesian equations. It *looks* very much like the vector product that you learned last semester but *it is not the vector product* because in the context of affine geometry we didn't define such a product.

4.2 Relative positions of lines and planes

Proposition 4.1. Let π and π' be two planes in $\mathbf{A}$ with Cartesian equations
$$\begin{aligned} ax+by+cz+d &= 0\\ a'x+b'y+c'z+d' &= 0 \end{aligned} \tag{4.6}$$
respectively. Then

1. π and π' are parallel if and only if the matrix
$$\begin{bmatrix} a & b & c\\ a' & b' & c' \end{bmatrix} \tag{4.7}$$
has rank 1.

2. If the matrix (4.7) has rank 1 then π and π' are disjoint if the augmented matrix
$$\begin{bmatrix} a & b & c & d\\ a' & b' & c' & d' \end{bmatrix} \tag{4.8}$$
has rank 2; if its rank is 1 then they coincide.

3. If π and π' are not parallel then they intersect, and $\pi\cap\pi'$ is a line; this occurs if and only if the matrix (4.7) has rank 2.

Proposition 4.2. Let ℓ be a line with parametric equations (4.3) and Cartesian equations (4.4). Let π'' be a plane with equation
$$a''x+b''y+c''z+d''=0. \tag{4.9}$$
Then

1. ℓ and π'' are parallel if and only if
$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0 \tag{4.10}$$
or equivalently, if and only if
$$a''v_x + b''v_y + c''v_z = 0. \tag{4.11}$$

2. If (4.10) is satisfied, then $\ell \subseteq \pi''$ if and only if the matrix
$$\begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{bmatrix} \tag{4.12}$$
has rank 2, otherwise they are disjoint (and the matrix has rank 3).

3. If ℓ and π'' are not parallel, then they are incident, and $\ell \cap \pi''$ consists of only one point; this occurs if and only if
$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \neq 0 \tag{4.13}$$
or equivalently, if and only if,
$$a''v_x + b''v_y + c''v_z \neq 0. \tag{4.14}$$

Definition 4.3. Two lines ℓ_1 and ℓ_2 are said to be *coplanar* if there is a plane which contains them both.

Proposition 4.4. Two lines ℓ_1 and ℓ_2 in $\mathbf{A}$ are coplanar if and only if one of the following conditions is satisfied:

1. ℓ_1 and ℓ_2 are parallel.
2. ℓ_1 and ℓ_2 are incident.

In particular, two lines ℓ_1 and ℓ_2 in $\mathbf{A}$ are coplanar if and only if they are not skew.

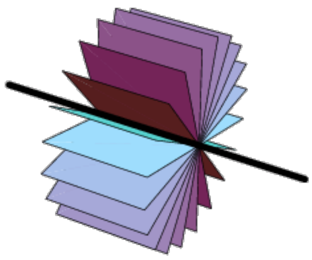
Proposition 4.5. Let ℓ and ℓ_1 be two lines in $\mathbf{A}$ and suppose that ℓ has Cartesian equations (4.4) and ℓ_1 has equations
$$\ell_1 : \begin{cases} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_1'x + b_1'y + c_1'z + d_1' &= 0 \end{cases}. \tag{4.15}$$
Let $Q(x_0, y_0, z_0) \in \ell$ and $Q_1(x_1, y_1, z_1) \in \ell_1$, let $\mathbf{v}(v_x, v_y, v_z)$ and $\mathbf{u}(u_x, u_y, u_z)$ be direction vectors for ℓ and ℓ_1 respectively. The following conditions are equivalent

1. ℓ and ℓ' are coplanar;
2. $\begin{vmatrix} x_0 - x_1 & y_0 - y_1 & z_0 - z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0;$

3. $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a_1 & b_1 & c_1 & d_1 \\ a_1' & b_1' & c_1' & d_1' \end{vmatrix} = 0.$

4.3 Pencils of planes

Definition 4.6. Let $\ell \subseteq \mathbf{A}$ be a line. The set Π_ℓ of all planes in $\mathbf{A}$ containing ℓ is called a *pencil of planes* and ℓ is called the *axis* of the pencil Π_ℓ.



Proposition 4.7. If $\pi_1 : ax + by + cz + d = 0$ and $\pi_2 : a'x + b'y + c'z + d' = 0$ are two distinct planes in the pencil Π_ℓ, then Π_ℓ consists of planes having equations of the form

$$\pi_{\lambda,\mu} : \lambda(ax + by + cz + d) + \mu(a'x + b'y + c'z + d') = 0.$$

where $\lambda, \mu \in \mathbf{K}$ not both zero.

- Pencils of planes are useful in praxis when a line ℓ is given as the intersection of two planes, and one wants to find the equation of a plane containing ℓ and satisfying some other condition. For example, the condition that it contain some point P which does not belong to ℓ or that it is parallel to a given line.
- There is redundancy in the two parameters λ, μ, meaning that there are not two independent parameters here. If $\lambda \neq 0$ then one can divide the equation of $\pi_{\lambda,\mu}$ by λ to obtain

  $$\pi_{1,t} : (ax + by + cz + d) + t(a'x + b'y + c'z + d') = 0.$$

  where $\frac{\mu}{\lambda} = t \in \mathbf{K}$. So $\pi_{1,\frac{\mu}{\lambda}}$ and $\pi_{\lambda,\mu}$ are in fact the same planes.
- A *reduced pencil* is the set of all planes Π_ℓ with axis ℓ from which *we remove one plane*, i.e. it is $\Pi_\ell \setminus \{\pi_2\}$ for some $\pi_2 \in \Pi_\ell$. With the above notation and discussion, it is the set

  $$\{\pi_{1,t} : (ax + by + cz + d) + t(a'x + b'y + c'z + d') = 0 : t \in \mathbf{K}\}.$$

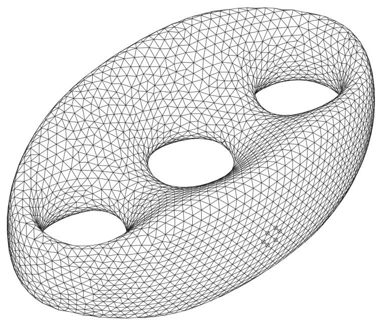
  The fact that we use one parameter instead of two, to describe almost all planes containing ℓ greatly simplifies calculations.

Definition 4.8. Let $\mathbf{W} \subseteq \mathbf{V}$ be a vector subspace of dimension 2. The set $\Pi_{\mathbf{W}}$ of all planes in $\mathbf{A}$ having associated vector subspace $\mathbf{W}$ is called an *improper pencil of planes*, and $\mathbf{W}$ is called the vector subspace associated to the pencil $\mathcal{L}_{\mathbf{v}}$.

- The connection between pencils of planes and improper pencils of planes is best understood through projective geometry, where we can think about the improper pencil of planes as the set of all planes intersecting in a line at infinity.

4.4 Connections to reality

Many of the beautiful 3D graphics that we enjoy today are generated with so-called ray tracing algorithms. A 3D-model, however complicated, can be decomposed into triangles. So, we can think about a 3D-scene as a large collection of triangles.



The coordinates of the vertices of these triangles are specified relative to a coordinate system. When we want to project the scene on a screen, we think about the screen as being a plane and the pixels of the screen as being points in that plane. In order to decide what color to choose for a given pixel, we cast a ray from the camera through that pixel and see which of the triangles in the scene it hits. For this one needs to intersect the ray with the planes determined by each of the triangles. So, one needs to intersect many lines (rays) with very many planes (determined by the triangles). This is the essence of ray tracing algorithms.