Let $a>0$, $\gamma : [0,2\pi] \to \mathbb{R}^2$ be the parameterized path defined by

$$\gamma(t) = (a(t-\sin t),\ a(1-\cos t)),$$

and let $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\vec{F}(x,y) = \underbrace{(2a-y)}_{F_1(x,y)}\,\vec{i} + \underbrace{x}_{F_2(x,y)}\,\vec{j}.$$

Evaluate $\int_\gamma \vec{F}\cdot d\vec{r}$.

<u>Solution</u>.

Theorem (computation of line integrals of vector fields by means of Riemann integrals)

Let A be a nonempty subset of $\mathbb{R}^n$, let $\gamma : [a,b] \to A$ be a parameterized path of class C^1, and let $F := (F_1, \dots, F_n) : A \to \mathbb{R}^n$ be a vector field in A. Then one has

$$\int_\gamma \vec{F}\cdot d\vec{r} = \sum_{i=1}^{n} \int_a^b (F_i \circ \gamma)(t)\, \gamma_i'(t)\, dt.$$

$$I := \int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{2\pi} [2a - a(1-\cos t)]\,(a(t-\sin t))'\,dt + \int_0^{2\pi} a(t-\sin t)\,(a(1-\cos t))'\,dt$$

$$= \int_0^{2\pi} a\underline{(1+\cos t)}\cdot a\underline{(1-\cos t)}\,dt + \int_0^{2\pi} a(t-\sin t)\cdot a\sin t\,dt$$

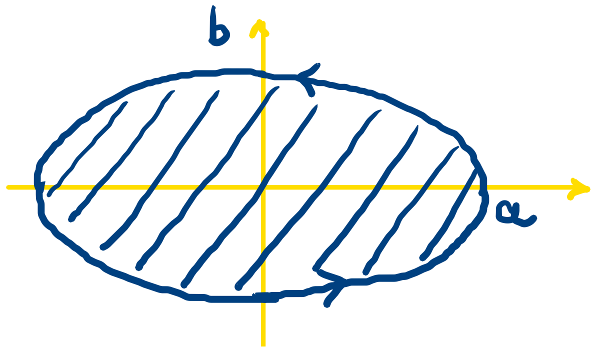
$$= a^2\int_0^{2\pi} (\cancel{1} - \cancel{\cos^2} t + t\sin t - \cancel{\sin^2} t)\,dt$$

$$= a^2 \int_0^{2\pi} t\sin t\,dt = \dots$$

Determine the area of the plane region bounded by the ellipse

$$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

<u>Solution</u> $D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$



$$\partial D : \begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad t \in [0, 2\pi]$$

$$m(D) = \frac{1}{2}\left(\int_0^{2\pi} a\cos t \cdot (b\sin t)' \, dt - \int_0^{2\pi} b\sin t\,(a\cos t)' \, dt \right)$$

$$= \frac{1}{2} \cdot ab \underbrace{\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt}_{2\pi}$$

$$\boxed{m(D) = \pi a b}$$

Corollary

Let D be a subset of $\mathbb{R}^2$ that is a normal domain with respect to both the x-axis and the y-axis, such that the boundary of D oriented in the positive sense is a rectifiable curve. Then D is Jordan measurable and its Jordan measure is given by

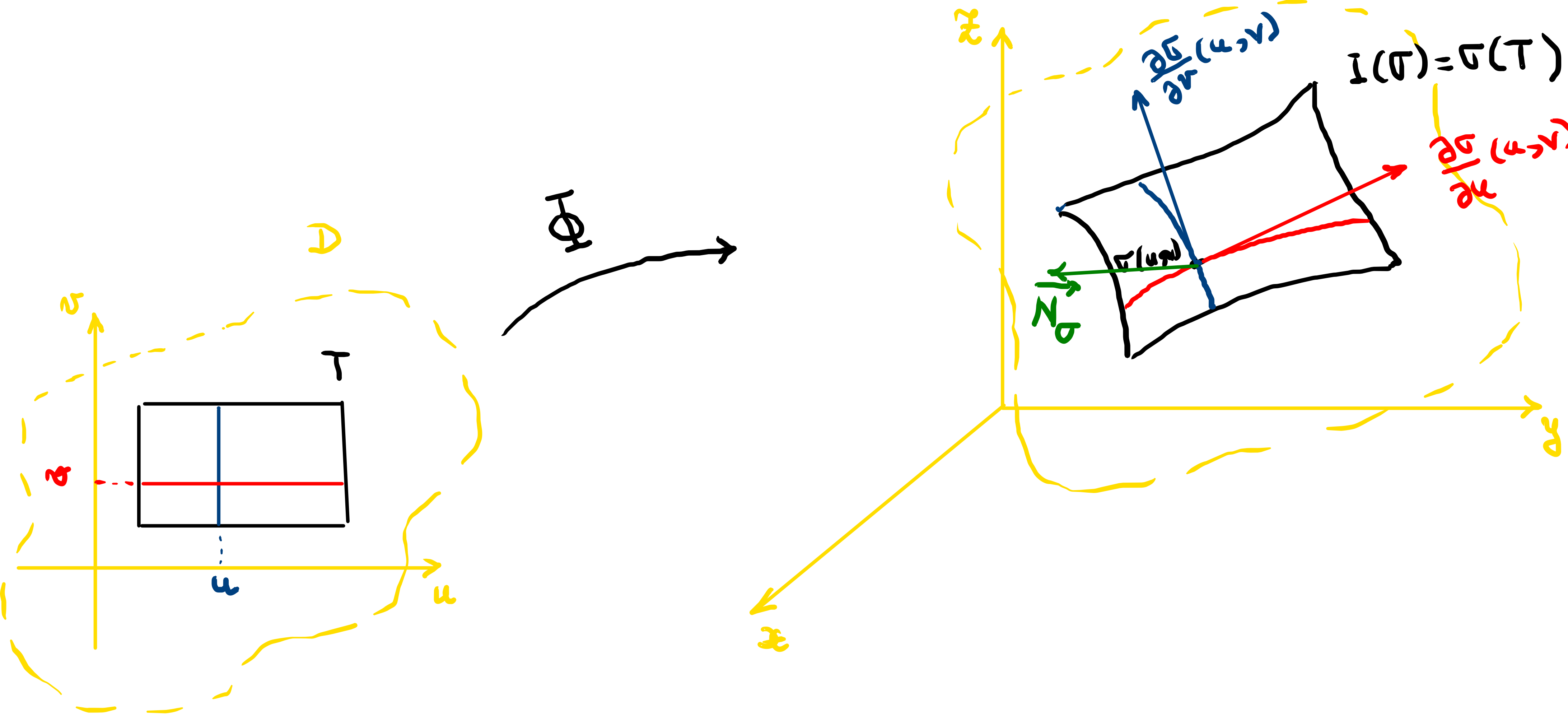
$$m(D) = \frac{1}{2} \oint_{\partial D} x dy - y dx.$$

$$m(D) = \iint_D dx\,dy$$

We use generalized polar coordinates

$$\begin{cases} \dfrac{x}{a} = \rho\cos\theta \\ \dfrac{y}{b} = \rho\sin\theta \end{cases} \qquad \begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases} \qquad \begin{array}{l} \rho\in[0,1] \\ \theta\in[0,2\pi] \end{array} \qquad \frac{D(x,y)}{D(\rho,\theta)} = ab\rho$$

$$m(D) = \int_0^1\int_0^{2\pi} ab\rho\,d\rho\,d\theta = ab\left(\int_0^1 \rho\,d\rho\right)\left(\int_0^{2\pi} d\theta\right) = ab\cdot\frac{1}{2}\cdot 2\pi = \boxed{\pi ab}$$



D
T
v
u
Φ
z
y
x
∂σ/∂v (u,v)
∂σ/∂u (u,v)
I(σ)=σ(T)
σ(u,v)
N_σ