CHAPTER 7

Eigenvectors

Contents

7.1 Linear operators

Definition 7.1. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space. The set of linear maps $\phi : \mathbf{V} \to \mathbf{V}$ is denoted by $\mathrm{End}(\mathbf{V})$. The elements of $\mathrm{End}(\mathbf{V})$ are called *linear operators*.

- By definition of affine morphism, a linear map is a an affine morphism

  $$\mathrm{End}(\mathbf{V}) \subseteq \mathrm{End}_{\mathrm{aff}}(\mathbf{V}_a)$$

  so, linear maps are particular cases of affine maps.

- Recall that if we fix a basis $\boldsymbol{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ for $\mathbf{V}$ then any linear map $\phi : \mathbf{V} \to \mathbf{V}$ is represented by a matrix

  $$[\phi]_{\boldsymbol{e}} \in \mathrm{Mat}_{n\times n}(\mathbf{K}).$$

Definition 7.2. Two matrices $A, B \in \mathrm{Mat}_n(\mathbf{K})$ are said to be *similar* if there is a matrix $M \in \mathrm{GL}_n(\mathbf{K})$ such that $B = M^{-1}AM$.

- Similarity of matrices is an equivalence relation in $\mathrm{Mat}_n(\mathbf{K})$.

Proposition 7.3. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$ of dimension n and let $A, B \in \mathrm{M}_n(\mathbf{K})$. Then A and B are similar if and only if there is a linear operator $\phi \in \mathrm{End}(\mathbf{V})$ and bases $\boldsymbol{e}$ and $\boldsymbol{f}$ of $\mathbf{V}$ such that $[\phi]_{\boldsymbol{e}} = A$ and $[\phi]_{\boldsymbol{f}} = B$.

Definition 7.4. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space of dimension n. An operator $\phi \in \mathrm{End}(\mathbf{V})$ is said to be *diagonalizable* if there is a basis $\boldsymbol{e}$ of $\mathbf{V}$ such that $[\phi]_{\boldsymbol{e}}$ is a diagonal matrix, i.e. $[\phi]_{\boldsymbol{e}}$ is of the form

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \tag{7.1}$$

for some $\lambda_1, \dots, \lambda_n \in \mathbf{K}$. In this case $\boldsymbol{e}$ is said to be a *diagonalizing basis* for ϕ. A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbf{K})$ is said to be *diagonalizable* if it is similar to a diagonal matrix.

- If $\phi \in \mathrm{End}(\mathbf{V})$ and $\boldsymbol{e}$ is a basis of $\mathbf{V}$ then ϕ is diagonalizable if and only if $[\phi]_{\boldsymbol{e}}$ is a diagonalizable matrix.

- If $\phi : \mathbf{V} \to \mathbf{V}$ is a diagonalizable linear operator and $\boldsymbol{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ is a diagonalizable basis for ϕ (as in the above definition) then

$$\phi(\mathbf{e}_1) = \lambda_i \mathbf{e}_i \quad \text{for each } i = 1, \dots, n.$$

- Conversely, if there exists a basis $\boldsymbol{e}$ satisfying the above equations then the matrix $[\phi]_{\boldsymbol{e}}$ is diagonal and so ϕ is diagonalizable and $\boldsymbol{e}$ is a diagonalizing basis for ϕ.

- If $\dim(\mathbf{V}) = 1$ then every $\phi \in \mathrm{End}(\mathbf{V})$ is diagonalizable and every basis of $\mathbf{V}$ is diagonalizing for ϕ.

- If $\dim(\mathbf{V}) > 1$ then there are operators which are not diagonalizable.

7.2 Eigenvalues and eigenvectors

Definition 7.5. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space and let $\phi \in \mathrm{End}(\mathbf{V})$. A non-zero vector $\mathbf{v} \in \mathbf{V}$ is called an *eigenvector of* ϕ if there is a scalar $\lambda \in \mathbf{K}$ such that $\phi(\mathbf{v}) = \lambda \mathbf{v}$. The scalar λ is then called *the eigenvalue associated to the eigenvector* $\mathbf{v}$. The set of eigenvalues of the operator ϕ is called the *spectrum of* ϕ.

For $A \in \mathrm{Mat}_{n\times n}(\mathbf{K})$ an *eigenvector of* A is an eigenvector $\mathbf{x} \in \mathbf{K}^n$ for the operator $\phi_A : \mathbf{K}^n \to \mathbf{K}^n$ defined by A, and an *eigenvalue* of A is an eigenvalue for ϕ_A.

- If $\phi = \mathrm{Id}_{\mathbf{V}}$, then every non-zero vector $\mathbf{v}$ is an eigenvector of ϕ with eigenvalue $\lambda = 1$.

- If ϕ is an operator with $\ker(\phi) \neq \langle 0 \rangle$, then every non-zero vector in $\ker(\phi)$ is an eigenvector for ϕ with eigenvalue $\lambda = 0$.

Proposition 7.6. The eigenvalue associated to an eigenvector is uniquely determined.

Proposition 7.7. If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$ are eigenvectors with the same eigenvalue λ, then for every $c_1, c_2 \in \mathbf{K}$ the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, if it is non-zero, is also an eigenvector with eigenvalue λ.

Definition 7.8. Form the above proposition it follows that for each $\lambda \in \mathbf{K}$ the set

$$\mathbf{V}_\lambda(\phi) = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} \text{ is an eigenvector of } \phi \text{ with eigenvalue } \lambda\} \cup \{0\}$$

is a vector subspace of $\mathbf{V}$, called the *eigenspace for the eigenvalue* λ. For a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbf{K})$ the *eigenspace for the eigenvalue* λ is defined to be the subspace $\mathbf{V}_\lambda(A) := \mathbf{V}_\lambda(\phi_A)$ in $\mathbf{K}^n$.

Proposition 7.9. If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbf{V}$ are eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, and these λ_i are pairwise distinct, then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Proposition 7.10. If every $\mathbf{v} \in \mathbf{V} \setminus \{0\}$ is an eigenvector of ϕ then there exists $\lambda \in \mathbf{K}$ such that $\phi = \lambda \,\mathrm{Id}_\mathbf{V}$.

7.3 Characteristic polynomial

In order to find the eigenvalues of a linear operator or a matrix one uses the so-called 'characteristic polynomial'.

Proposition 7.11. Let $\mathbf{V}$ be a finite dimensional vector space and let $\phi \in \mathrm{End}(\mathbf{V})$. A scalar $\lambda \in \mathbf{K}$ is an eigenvalue of ϕ if and only if the operator

$$\phi - \lambda\,\mathrm{Id}_\mathbf{V} : \mathbf{V} \to \mathbf{V} \quad \text{defined by} \quad (\phi - \lambda\,\mathrm{Id}_\mathbf{V})(\mathbf{v}) = \phi(\mathbf{v}) - \lambda\mathbf{v}$$

is not an isomorphism, that is, if and only if $\det(\phi - \lambda\,\mathrm{Id}_\mathbf{V}) = 0$.

- Let $\boldsymbol{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be a basis of $\mathbf{V}$. The matrix associated to the operator $\lambda\,\mathrm{Id}_\mathbf{V}$ is

$$[\lambda\,\mathrm{Id}_\mathbf{V}]_{\boldsymbol{e}} = \begin{bmatrix} \lambda & 0 & \ldots & 0 \\ 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda \end{bmatrix}$$

  if $A = (a_{ij}) = [\phi]_{\boldsymbol{e}}$ then

$$[\phi - \lambda\,\mathrm{Id}_\mathbf{V}]_{\boldsymbol{e}} = \begin{bmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{bmatrix}.$$

Definition 7.12. Let $A \in \mathrm{Mat}_{n\times n}(\mathbf{K})$. The determinant

$$P_A(T) = \det(A - T\,\mathrm{Id}_n) = \begin{vmatrix} a_{11} - T & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - T & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - T \end{vmatrix}$$

is a polynomial of degree n in T, called *the characteristic polynomial of* A. If $\phi \in \mathrm{End}(\mathbf{V})$ and $\boldsymbol{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a basis of $\mathbf{V}$ then the *characteristic polynomial of* ϕ is $P_\phi := P_{[\phi]_{\boldsymbol{e}}}$.

Proposition 7.13. The definition of P_ϕ is independent of the basis $\boldsymbol{e}$.

Corollary 7.14. Let $\mathbf{V}$ be a vector space of dimension n, and let $\phi \in \mathrm{End}(\mathbf{V})$. Then $\lambda \in \mathbf{K}$ is an eigenvalue of ϕ if and only if λ is a root of the polynomial P_ϕ. In particular, ϕ has at most n eigenvalues.

Proposition 7.15. Let $\mathbf{V}$ be a finite dimensional vector space. An operator $\phi \in \mathrm{End}(\mathbf{V})$ is diagonalizable if and only if there is a basis of $\mathbf{V}$ consisting entirely of eigenvectors of ϕ.

Theorem 7.16. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space of dimension n, and let $\phi \in \mathrm{End}(\mathbf{V})$. If $\{\lambda_1, \dots, \lambda_k\} \subseteq \mathbf{K}$ is the spectrum of ϕ, then

$$\dim(V_{\lambda_1}(\phi)) + \cdots + \dim(V_{\lambda_k}(\phi)) \leq n$$

with equality if and only if ϕ is diagonalizable.

Corollary 7.17. If $\dim(\mathbf{V}) = n$ and $\phi \in \mathrm{End}(\mathbf{V})$ has n distinct eigenvalues then it is diagonalizable.

- We have a practical method for finding eigenvalues and eigenvectors of an operator or a matrix. By choosing a basis for $\mathbf{V}$ an operator becomes a matrix.

- Suppose we are given $A \in \mathrm{Mat}_{n \times n}(\mathbf{K})$.

  1. calculate the characteristic polynomial P_A.
  2. find the eigenvalues of A by calculating the roots of P_A in $\mathbf{K}$.
  3. For each eigenvalue $\lambda \in \mathbf{K}$, the homogenous system of n equations in n unknowns:

  $$(A - \lambda\,\mathrm{Id}_n)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

  has rank $r < n$ and therefore has nontrivial solutions. The space of solutions is the eigenspace $\mathbf{V}_\lambda(A)$.

- If the sum of the dimensions of the eigenspaces found by letting λ vary over the roots of P_A is equal to n then A is diagonalizable (by Theorem 7.16)

- An operator need not have any eigenvalues nor eigenvectors.

- If $\mathbf{K} = \mathbb{C}$, by the fundamental theorem of algebra, P_A has roots in $\mathbb{C}$. Thus, every operator on a finite dimensional complex vector space has at least one eigenvalue, and so at least one eigenvector. Of course, this does not mean that the operator is diagonalizable.

- If $\mathbf{K} = \mathbb{R}$ and $\dim(\mathbf{V})$ is odd, then the characteristic polynomial has odd degree and so has at least one real root. Thus, every operator on an odd dimensional real vector space has at least one eigenvalue, and so at least one eigenvector.

Definition 7.18. Let $\mathbf{V}$ be a finite dimensional $\mathbf{K}$-vector space and let $\phi \in \mathrm{End}(\mathbf{V})$ with λ an eigenvalue of ϕ. The number $\dim(\mathbf{V}_\lambda(\phi))$ is called the *geometric multiplicity of λ for ϕ*. The *algebraic multiplicity of λ for ϕ* is instead the multiplicity of λ as a root of the characteristic polynomial P_ϕ; this is denoted by $h_\phi(\lambda)$.

Proposition 7.19. For any operator $\phi \in \mathrm{End}(\mathbf{V})$ and $\lambda \in \mathbf{K}$ one has

$$\dim(\mathbf{V}_\lambda(\phi)) \leq h_\phi(\lambda),$$

that is, the geometric multiplicity is not larger than the algebraic multiplicity.

7.4 Connection to reality

Similar to different methods of projecting, eigenvectors and eigenvalues are fundamental in mathematics, physics and computer science. Check out the Applications section on the Wikipedia page for eigenvalues and eigenvectors.

Everyone is nowadays familiar with the overhyped topic of machine learning, at least by name. One useful tool in machine learning is *principal component analysis*. In this context, the principal components are the eigenvectors of the data's covariance matrix. Such a matrix is huge in general.