Seminar 10 – Linearly independence. Bases

Let V be a K-v.s., $x_1, \dots, x_n \in V$ are linearly independent if
$$k_1, \dots, k_n \in K, \quad k_1 x_1 + \dots + k_n x_n = 0 \implies k_1 = \dots = k_n = 0.$$

$B = (v_1, \dots, v_n)$ is a system of vectors from V

$$B \text{ basis for } {}_K V \overset{def}{\iff} \begin{cases} V = \langle B \rangle \\ B \text{ l. indep.} \end{cases} \iff$$

$\iff \forall v \in V, \ \exists k_1, \dots, k_n \in K$ uniquely determined s.t.
$$v = k_1 v_1 + \dots + k_n v_n$$

$k_1, \dots, k_n \in K$ are the coordinates of v in B.

1) $v_1 = (1,2,-1)$, $v_2 = (3,2,4)$, $v_3 = (-1,2,-6)$ are l. dependent in ${}_{\mathbb{R}}\mathbb{R}^3$

Find a dependency relation between them (?).

Solution: $v_1, v_2, v_3 \in \mathbb{R}^3$ are l. dep. $\iff$

$\iff \exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ not all of them 0 such that
$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = (0,0,0) \qquad (1)$$

$(1) \iff (0,0,0) = \alpha_1(1,2,-1) + \alpha_2(3,2,4) + \alpha_3(-1,2,-6) \iff$

$\iff (0,0,0) = (\alpha_1 + 3\alpha_2 - \alpha_3, \ 2\alpha_1 + 2\alpha_2 + 2\alpha_3, \ -\alpha_1 + 4\alpha_2 - 6\alpha_3) \iff$

$$\iff \begin{cases} \alpha_1 + 3\alpha_2 - \alpha_3 = 0 \\ 2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 \\ -\alpha_1 + 4\alpha_2 - 6\alpha_3 = 0 \end{cases} \qquad (2)$$

$\exists (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ s.t. $(1) (\iff (2)) \iff$

$\iff$ (2) is consistent with more than one solution $\iff$

$\iff$ the determinant of the system matrix is 0 (?)

$$d = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ -1 & 4 & -6 \end{vmatrix} \underset{c_3 - c_2}{\overset{c_1 - c_2}{=\!=}} \begin{vmatrix} -2 & 3 & -4 \\ 0 & 2 & 0 \\ -5 & 4 & -10 \end{vmatrix} = 2 \begin{vmatrix} -2 & -4 \\ -5 & -10 \end{vmatrix} = 0$$

We must find a nonzero triple $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ which satisfies $(1) (\iff (2))$

$\shortparallel$

nonzero solution of (2) (?)

Since $d = 0$ and $\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \neq 0 \implies$ we can consider α_3 as a side unknown.

Let us take $\alpha_3 = 1$

$$(2) \implies \begin{cases} \alpha_1 + 3\alpha_2 = 1 \\ 2\alpha_1 + 2\alpha_2 = -2 \end{cases} \iff \begin{cases} \alpha_1 = -2 \\ \alpha_2 = 1 \end{cases}$$

Thus a nonzero solution of (2) is $(-2,1,1) \Longrightarrow$

$\overset{(1)}{\Longrightarrow} \; -2v_1+v_2+v_3=(0,0,0) \; \left(\Longleftrightarrow \; v_1=\frac{1}{2}v_2+\frac{1}{2}v_3\right).$

2) $v_1=(a_1,b_1),\ v_2=(a_2,b_2)$

(v_1,v_2) basis for ${}_{\mathbb{R}}\mathbb{R}^2 \Longleftrightarrow \forall\, v=(x,y),\ \exists\, \alpha_1,\alpha_2\in\mathbb{R}$ uniquely determined s.t.

$$v=\alpha_1 v_1+\alpha_2 v_2 \quad (1)$$

$(1) \Longleftrightarrow (x,y)=\alpha_1(a_1,b_1)+\alpha_2(a_2,b_2) \Longleftrightarrow \begin{cases}\alpha_1 a_1+\alpha_2 a_2=x\\ \alpha_1 b_1+\alpha_2 b_2=y\end{cases} \quad (2)$

(v_1,v_2) basis ${}_{\mathbb{R}}\mathbb{R}^2 \Longleftrightarrow$ (2) is consistent, with a unique solution $\Longleftrightarrow$

$\Longleftrightarrow \begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix} \neq 0.$

$$\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix} \neq 0 \Longleftrightarrow \begin{vmatrix} 1 & 1 & 1\\ a_1 & a_2 & 0\\ b_1 & b_2 & 0\end{vmatrix} \neq 0 \Longleftrightarrow (a_1,b_1),(a_2,b_2),(0,0) \text{ are not colinear}$$

($\Longleftrightarrow$ the vectors $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$ are not colinear ($M_i(a_i,b_i),\ i=1,2$)).

$((1,0),(0,a))$ is a basis for ${}_{\mathbb{R}}\mathbb{R}^2$ for any $a\in\mathbb{R}^*$ since

$\begin{vmatrix} 1 & 0\\ 0 & a\end{vmatrix} = a \neq 0$

$\Longrightarrow$ an infinity of bases of ${}_{\mathbb{R}}\mathbb{R}^2$.

$e_1=(1,0),\ e_2=(0,1)$ $\quad$ (e_1,e_2) is the standard basis of ${}_{\mathbb{R}}\mathbb{R}^2$

$v=(x,y)=(x,0)+(0,y)=x(1,0)+y(0,1)=xe_1+ye_2$

$v_1=(1,0),\ v_2=(1,1)$ form a basis for ${}_{\mathbb{R}}\mathbb{R}^2$ because

$\begin{vmatrix} 1 & 1\\ 0 & 1\end{vmatrix} = 1 \neq 0$

The coordinates of $v=(x,y)$ in the basis $\overset{B}{(v_1,v_2)}$ result by solving the system (2) which becomes in our case:

$\begin{cases}\alpha_1+\alpha_2=x\\ \alpha_2=y\end{cases} \Longleftrightarrow \begin{cases}\alpha_1=x-y\\ \alpha_2=y\end{cases}$ $\rangle$ the coordinates of v in B

$(\Longleftrightarrow v=(x-y)v_1+yv_2).$

3) $v_1=(a,1,1),\ v_2=(1,a,1),\ v_3=(1,1,a)$

$a \in \mathbb{R}$? s.t. (v_1, v_2, v_3) basis for $_{\mathbb{R}}\mathbb{R}^3$.

Solution: (v_1, v_2, v_3) basis for $_{\mathbb{R}}\mathbb{R}^3 \iff$

$$\iff \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \neq 0 \iff a \in \mathbb{R} \setminus \{-2, 1\}.$$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = d$$

$$d = (a+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \underset{r_3 - r_1}{\overset{r_2 - r_1}{=}} (a+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & a-1 & 0 \\ 0 & 0 & a-1 \end{vmatrix} = (a+2)(a-1)^2$$

R: (v_1, v_2, v_3) l. indep. in $_{\mathbb{R}}\mathbb{R}^3 \iff (v_1, v_2, v_3)$ basis in $_{\mathbb{R}}\mathbb{R}^3$

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3$.

4) homework

5) $v_1, v_2, v_3 \in V$ (is a $\mathbb{R}$-v.s.)

$$\left.\begin{aligned} u_1 &= v_2 + v_3 \\ u_2 &= v_1 + v_3 \\ u_3 &= v_1 + v_2 \end{aligned}\right\} \Rightarrow u_1 + u_2 + u_3 = 2(v_1 + v_2 + v_3) \Rightarrow$$

$$\Rightarrow v_1 + v_2 + v_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3 \Rightarrow \begin{cases} v_1 = -\frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3 \\ v_2 = \frac{1}{2}u_1 - \frac{1}{2}u_2 + \frac{1}{2}u_3 \\ v_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2 - \frac{1}{2}u_3 \end{cases}$$

v_1, v_2, v_3 l. indep. in $_{\mathbb{R}}V \underset{?}{\iff} u_1, u_2, u_3$ l. indep. in $_{\mathbb{R}}V$

Solution:

„$\Rightarrow$" Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0 \overset{?}{\Rightarrow} \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \alpha_1(v_2 + v_3) + \alpha_2(v_1 + v_3) + \alpha_3(v_1 + v_2) =$$

$$= \alpha_1 v_2 + \alpha_1 v_3 + \alpha_2 v_1 + \alpha_2 v_3 + \alpha_3 v_1 + \alpha_3 v_2 =$$

$$= (\alpha_2 + \alpha_3) v_1 + (\alpha_1 + \alpha_3) v_2 + (\alpha_1 + \alpha_2) v_3$$

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0 \iff (\alpha_2 + \alpha_3) v_1 + (\alpha_1 + \alpha_3) v_2 + (\alpha_1 + \alpha_2) v_3 = 0 \xrightarrow[\text{l. indep}]{v_1, v_2, v_3}$$

$$\Rightarrow \begin{cases} \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

„$\Leftarrow$" Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \overset{?}{\Rightarrow} \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \iff \alpha_1\left(-\frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3\right) + \alpha_2\left(\frac{1}{2}u_1 - \frac{1}{2}u_2 + \frac{1}{2}u_3\right) +$$

$$+ \alpha_3\left(\frac{1}{2}u_1 + \frac{1}{2}u_2 - \frac{1}{2}u_3\right) = 0 \iff -\alpha_1 u_1 + \alpha_1 u_2 + \alpha_1 u_3 + \alpha_2 u_1 - \alpha_2 u_2 +$$

$$+ \alpha_2 u_3 + \alpha_3 u_1 + \alpha_3 u_2 - \alpha_3 u_3 = 0 \iff$$

$$\iff (-\alpha_1 + \alpha_2 + \alpha_3) u_1 + (\alpha_1 - \alpha_2 + \alpha_3) u_2 + (\alpha_1 + \alpha_2 - \alpha_3) u_3 = 0 \xrightarrow[\text{l. indep.}]{u_1, u_2, u_3}$$

$$\Longrightarrow \begin{cases} -\alpha_1+\alpha_2+\alpha_3=0 \\ \alpha_1-\alpha_2+\alpha_3=0 \\ \alpha_1+\alpha_2-\alpha_3=0 \end{cases} \Longrightarrow \underbrace{\alpha_1=\alpha_2=\alpha_3=0}$$

6) $E_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $E_2=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $E_3=\begin{pmatrix}1&1\\1&0\end{pmatrix}$, $E_4=\begin{pmatrix}1&1\\1&1\end{pmatrix}$.

Show that $B=(E_1,E_2,E_3,E_4)$ is a basis $\overset{?}{}$ for ${}_{\mathbb{R}}M_2(\mathbb{R})$

Find the coordinates of $A=\begin{pmatrix}-2&3\\4&-2\end{pmatrix}$ in B (?)

<u>Solution</u>: $\forall\ X=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, $\overset{?}{\exists}\ \alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}$ <u>uniquely determined</u> s.t.

$$X=\alpha_1E_1+\alpha_2E_2+\alpha_3E_3+\alpha_4E_4 \quad (1)$$

$$(1)\Longleftrightarrow \begin{pmatrix}a&b\\c&d\end{pmatrix}=\alpha_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+\alpha_2\begin{pmatrix}1&1\\0&0\end{pmatrix}+\alpha_3\begin{pmatrix}1&1\\1&0\end{pmatrix}+\alpha_4\begin{pmatrix}1&1\\1&1\end{pmatrix}\Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases}\alpha_1+\alpha_2+\alpha_3+\alpha_4=a\\ \alpha_2+\alpha_3+\alpha_4=b\\ \alpha_3+\alpha_4=c\\ \alpha_4=d\end{cases}\Longleftrightarrow\begin{cases}\alpha_1=a-b\\ \alpha_2=b-c\\ \alpha_3=c-d\\ \alpha_4=d\end{cases}$$

For A, $a=-2, b=3, c=4, d=-2 \Longrightarrow \alpha_1=-5, \alpha_2=-1, \alpha_3=6, \alpha_4=-2$ the coordinates of A in B

$$(A=-5E_1-E_2+6E_3-2E_4).$$

8) $n\in\mathbb{N}$, $f_n:\mathbb{R}\to\mathbb{R}$, $f_n(x)=\sin^n x$

$L=\{f_n \mid n\in\mathbb{N}\}$ l. indep. $\overset{?}{\text{in}}$ ${}_{\mathbb{R}}\mathbb{R}^{\mathbb{R}}$

<u>Solution</u>: L l. indep. in ${}_{\mathbb{R}}\mathbb{R}^{\mathbb{R}} \Longleftrightarrow \forall\, n\in\mathbb{N}, \forall\, \underbrace{i_1,\dots,i_n}_{\text{mutually different}}\in\mathbb{N}$, $f_{i_1},\dots,f_{i_n}$ $\overset{?}{\text{are}}$ l. indep. in ${}_{\mathbb{R}}\mathbb{R}^{\mathbb{R}}$

$\underbrace{\text{Let } \alpha_1,\dots,\alpha_n\in\mathbb{R},\ \alpha_1f_{i_1}+\alpha_2f_{i_2}+\dots+\alpha_nf_{i_n}=\theta}$ (1)

$$(1)\Longleftrightarrow \forall x\in\mathbb{R},\ \alpha_1f_{i_1}(x)+\dots+\alpha_nf_{i_n}(x)=0\Longleftrightarrow$$

$$\Longleftrightarrow \forall x\in\mathbb{R},\ \alpha_1\sin^{i_1}x+\dots+\alpha_n\sin^{i_n}x=0\Longleftrightarrow$$

$$\underset{?}{\Longleftrightarrow} \forall t\in[-1,1],\ \alpha_1t^{i_1}+\alpha_2t^{i_2}+\dots+\alpha_nt^{i_n}=0\Longrightarrow$$

$\sin x=t$ $\Longrightarrow \forall t\in[-1,1]$, t is a root for the polynomial

$$\alpha_1X^{i_1}+\alpha_2X^{i_2}+\dots+\alpha_nX^{i_n}\in\mathbb{R}[X]$$

$$\Longrightarrow \alpha_1X^{i_1}+\alpha_2X^{i_2}+\dots+\alpha_nX^{i_n}=0\Longrightarrow\underbrace{\alpha_1=\alpha_2=\dots=\alpha_n=0}$$

<u>Another solution</u>: $m=\max\{i_1,\dots,i_n\}$

$$f_{i_1},\dots,f_{i_n} \text{ l. indep} \overset{?}{\Longleftarrow} f_0,f_1,\dots,f_m \text{ l. indep.}^{?} \quad (\forall m\in\mathbb{N})$$

$\alpha_0, \alpha_1, \dots, \alpha_m \in \mathbb{R}$, $\alpha_0 f_0 + \alpha_1 f_1 + \dots + \alpha_m f_m = \theta$ (2)

(2) $\Longleftrightarrow \forall x \in \mathbb{R}$, $\underline{\alpha_0} \cdot 1 + \underline{\alpha_1} \sin x + \underline{\alpha_2} \sin^2 x + \dots + \underline{\alpha_m} \sin^m x = 0$ (3)

Let $x_0, x_1, \dots, x_m \in \mathbb{R}$ such that $\sin x_i \neq \sin x_j$, $\forall i, j \in \{0, \dots, m\}$ $i \neq j$.

$$(3) \Longrightarrow \begin{cases} \alpha_0 + \alpha_1 \sin x_0 + \alpha_2 \sin^2 x_0 + \dots + \alpha_m \sin^m x_0 = 0 \\ \alpha_0 + \alpha_1 \sin x_1 + \alpha_2 \sin^2 x_1 + \dots + \alpha_m \sin^m x_1 = 0 \\ \dots \\ \alpha_0 + \alpha_1 \sin x_m + \alpha_2 \sin^2 x_m + \dots + \alpha_m \sin^m x_m = 0 \end{cases} \quad (4)$$

(4) has the system matrix det.

$$\begin{vmatrix} 1 & \sin x_0 & \sin^2 x_0 & \dots & \sin^m x_0 \\ 1 & \sin x_1 & \sin^2 x_1 & \dots & \sin^m x_1 \\ \dots \\ 1 & \sin x_m & \sin^2 x_m & \dots & \sin^m x_m \end{vmatrix} = \prod_{0 \le i < j \le m} \overbrace{(\sin x_j - \sin x_i)}^{\neq 0} \neq 0$$

$$\Longrightarrow \underline{\alpha_0 = \alpha_1 = \dots = \alpha_m = 0}$$