Analytic Geometry

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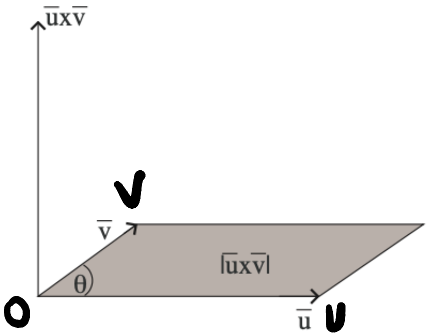
Del capítulo anterior...



- If the vectors $\overline{u}, \overline{v}$ are not collinear, then if $\overrightarrow{OU} \in \overline{u}$ and $\overrightarrow{OV} \in \overline{v}$, then $||\overline{u} \times \overline{v}||$ is the area of the parallelogram formed by $\overrightarrow{OU}$ and $\overrightarrow{OV}$.
- The area of the triangle $\triangle OUV$ can be computed as

$$\mathrm{Area}_{\triangle OUV} = \frac{||\overline{u} \times \overline{v}||}{2}.$$

Algebraically

If $\overline{u} = u_1\overline{i} + u_2\overline{j} + u_3\overline{k}$ and $\overline{v} = v_1\overline{i} + v_2\overline{j} + v_3\overline{k}$ are vectors in V_3, then

$$\overline{u} \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{1}$$

Some observations

$$\overline{u} \times \overline{v} = -\overline{v} \times \overline{u}$$

The cross product shares a few similarities with the dot product. However, there are some differences which you have to remember:

1. The cross product is not commutative. In fact, it is anti-commutative.
2. The cross product of two vectors is a vector, not a scalar (as it is the case for the result of a dot product). Therefore, it makes sense to consider products with multiple factors. One should be very careful with those, since the cross product is not associative either :)

A closer look at a high-school formula

In high-school, you probably learned how to compute the area of a triangle determined by $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$.

- Let see these in $3D$ and assume WLOG they line in the plane xOy.
- We therefore have $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$ and $C(x_C, y_C, 0)$. These points determine the vectors $\overline{AB}(x_B - x_A, y_B - y_A, 0)$ and $\overline{AC}(x_C - x_A, y_C - y_A, 0)$.
- Computing, we have

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix} = \bar{k} \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix},$$

$$\overline{AB} \times \overline{AC} = \bar{k} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

It follows that

$$||\overline{AB} \times \overline{AC}|| = \pm \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix},$$

hence

$$\mathrm{Area}_{\triangle ABC} = \pm \frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

Double cross product

Let $\overline{u}, \overline{v}$ and $\overline{w}$ be vectors in V_3. The double cross product of these three vectors is, by definition, the vector $(\overline{u} \times \overline{v}) \times \overline{w}$. The following relation holds

$$\boxed{(\overline{u} \times \overline{v}) \times \overline{w} = \underbrace{(\overline{u} \cdot \overline{w})}_{\in \mathbb{R}}\overline{v} - \underbrace{(\overline{v} \cdot \overline{w})}_{\in \mathbb{R}}\overline{u}}.$$

On the other hand,

$$\overline{u} \times (\overline{v} \times \overline{w}) = -(\overline{v} \times \overline{w}) \times \overline{u} = (\overline{v} \cdot \overline{u})\overline{w} - (\overline{w} \cdot \overline{u})\overline{v}$$

Comparing $(\overline{u} \times \overline{v}) \times \overline{w}$ and $\overline{u} \times (\overline{v} \times \overline{w})$ we find that these are equal if

$$-(\overline{v} \cdot \overline{w})\overline{u} + 2(\overline{u} \cdot \overline{w})\overline{v} - (\overline{u} \cdot \overline{v})\overline{w} = \overline{0}.$$

We notice that $\overline{u}, \overline{v}$ and $\overline{w}$ being coplanar is a necessary condition for associativity. However, this is not sufficient.

Using the equality

$$\boxed{(\overline{u} \times \overline{v}) \times \overline{w} = (\overline{u} \cdot \overline{w})\overline{v} - (\overline{v} \cdot \overline{w})\overline{u}},$$

one can easily show that the "Jacobi's identity"

$$(\overline{u} \times \overline{v}) \times \overline{w} + (\overline{v} \times \overline{w}) \times \overline{u} + (\overline{w} \times \overline{u}) \times \overline{v} = \overline{0}$$

holds for any $\overline{u}, \overline{v}, \overline{w} \in V_3$.

Triple scalar product

Given three vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ from V_3, one defines their *triple scalar product* to be the real number $(\overline{a}, \overline{b}, \overline{c}) = \overline{a} \cdot (\overline{b} \times \overline{c})$.
If $\overline{a} = (a_1, a_2, a_3)$, $\overline{b} = (b_1, b_2, b_3)$ and $\overline{c} = (c_1, c_2, c_3)$, then the triple scalar product can be calculated as

$$(\bar a, \bar b, \bar c) = -(\bar b, \bar a, \bar c)$$
$$= (b, \bar c, \bar a)$$

$$(\overline{a}, \overline{b}, \overline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{2}$$

Indeed,

$$(\overline{a}, \overline{b}, \overline{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) =$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Remark: It can be seen easily that the triple scalar product can be also seen as $(\overline{a}, \overline{b}, \overline{c}) = (\overline{a} \times \overline{b}) \cdot \overline{c}$.

Theorem

If $\overline{a}$, $\overline{b}$ and $\overline{c}$ are vectors in V_3, then:

a) $(\overline{a}, \overline{b}, \overline{c}) = (\overline{c}, \overline{a}, \overline{b}) = (\overline{b}, \overline{c}, \overline{a})$;

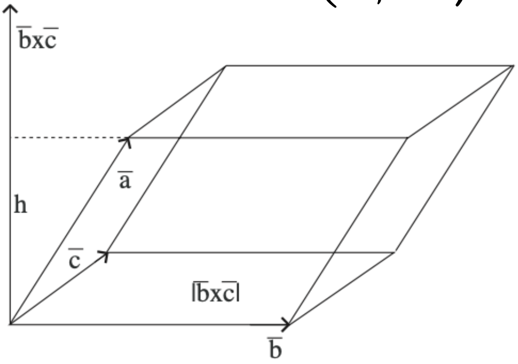
b) $(\overline{a}, \overline{b}, \overline{c}) = 0$ *if and only if $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent (i.e. they have representatives situated on the same plane).*

<u>Defⁿ</u>: Say that $\overline{a}, \overline{b}, \overline{c}$ have the same orientation if $(\overline{a}, \overline{b}, \overline{c}) > 0$.

Different orientation if $(\overline{a}, \overline{b}, \overline{c}) < 0$.

The triple scalar product has a geometric meaning. Suppose that the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly independent and choose a representer for each, having the same original point. These form the adjacent sides of a parallelepiped, as below.

$(\overline{a}, \overline{b}, \overline{c}) = \overline{a} \cdot (\overline{b} \times \overline{c})$



Suppose that the base of this parallelepiped is the parallelogram constructed on $\overline{b}$ and $\overline{c}$. The height of the parallelepiped is the length of the orthogonal projection of the vector $\overline{a}$ on the direction of the vector $\overline{b} \times \overline{c}$,

$$h = |\mathrm{pr}_{\overline{b}\times\overline{c}}\overline{a}| = \left| \frac{\overline{a} \cdot (\overline{b} \times \overline{c})}{|\overline{b} \times \overline{c}|} \right| = \frac{|(\overline{a}, \overline{b}, \overline{c})|}{|\overline{b} \times \overline{c}|}.$$

Then, the volume of the parallelepiped whose adjacent sides are the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ is the absolute value of the triple scalar product $(\overline{a}, \overline{b}, \overline{c})$:

$$V = h \cdot \mathrm{Area}(\overline{b}, \overline{c}) = \frac{|(\overline{a}, \overline{b}, \overline{c})|}{|\overline{b} \times \overline{c}|} |\overline{b} \times \overline{c}| = |(\overline{a}, \overline{b}, \overline{c})|. \tag{3}$$

The volume of a tethrahedron

Suppose we have a tetrahedron $OABC$ such that $\overrightarrow{OA} \in \overline{a}$, $\overrightarrow{OB} \in \overline{b}$ and $\overrightarrow{OC} \in \overline{c}$. Then, the volume of the tetrahedron can be computed as

$$\mathrm{Vol}_{OABC} = \frac{1}{3}\mathrm{d}(A, OBC) \cdot \mathrm{Area}_{\triangle OBC}$$

$$\boxed{\mathrm{Vol}_{OABC} = \frac{1}{6}|\mathrm{pr}_{\overline{b}\times\overline{c}}(\overline{a})| \cdot |\overline{b} \times \overline{c}| = \frac{1}{6}\frac{|(\overline{a}, \overline{b}, \overline{c})|}{|\overline{b} \times \overline{c}|}|\overline{b} \times \overline{c}| = \frac{1}{6}|(\overline{a}, \overline{b}, \overline{c})|}$$

An easy application

We are given a tetrahedron $ABCD$ of volume 5 with three of its vertices $A(2,1,-1)$, $B(3,0,1)$ and $C(2,-1,3)$. Its fourth vertex D is situated on somewhere on the Oy axis. Find the coordinates of the point D.

$$5 = V_{ABCD} = \frac{1}{6} \cdot \left| \left(\overline{AB}, \overline{AC}, \overline{AD} \right) \right|$$

$D(0, d, 0)$ for some $d \in \mathbb{R}$

(since $D \in Oy$).

$\overline{AB}(1,-1,2)$, $\overline{AC}(0,-2,4)$ and

$\overline{AD}(-2, d-1, 1)$

Replacing in the volume formula,

$$5 = \frac{1}{6} \cdot \left| \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & d-1 & 1 \end{vmatrix} \right| \iff$$

$$30 = \left| -4d + 2 \right| \iff$$

$$15 = \left| -2d + 1 \right|$$

$2d \in \{16, -14\}$, so $d \in \{8, -7\}$.

$$D \in \{(0,-7,0), (0,8,0)\}.$$

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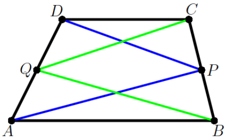
An application of the cross product

Consider a trapezoid $ABCD$ with $AB \parallel CD$. Let P and Q be the midpoints of $[BC]$ and $[DA]$. Prove that the triangles APD and CQB have the same area.



Let $\overline{AB} = \overline{b}$ and $\overline{AD} = \overline{d}$.

Since $DC \parallel AB$, $\overline{DC} = t \cdot \overline{AB} = t \cdot \overline{b}$ for some $t \in \mathbb{R}_{>0}$.

$$\overline{AP} = \frac{1}{2}(\overline{AB} + \overline{AC}) = \frac{1}{2}\cdot\overline{b} + \frac{1}{2}(\overline{AD} + \overline{DC}).$$

$$= \frac{1}{2}\cdot\overline{b} + \frac{1}{2}\cdot(\overline{d} + t\cdot\overline{b})$$

$$= \frac{1+t}{2}\cdot\overline{b} + \frac{1}{2}\cdot\overline{d}.$$

Now we can compute

$$\overline{AP}\times\overline{AD} = \left[\frac{1+t}{2}\cdot\overline{b} + \frac{1}{2}\overline{d}\right]\times\overline{d}$$

$$= \frac{1+t}{2}(\overline{b}\times\overline{d}) + \frac{1}{2}(\underbrace{\overline{d}\times\overline{d}}_{=\overline{0}})$$

$$= \frac{1}{2}\cdot(1+t)\cdot(\overline{b}\times\overline{d})$$

$$\text{Area}_{\triangle APD} = \frac{1}{2}\,\|\overline{AP} \times \overline{AD}\| = \underline{\underline{\frac{1}{4}\cdot|1+t|\cdot\|\overline{b}\times\overline{d}\|}}. \quad (1)$$

$$\overline{CQ} = \overline{CD} + \overline{DQ} = -t\cdot\overline{b} - \frac{1}{2}\cdot\overline{d}.$$

$$\overline{CB} = \overline{CA} + \overline{AB} = -\overline{AC} + \overline{AB}.$$

$$= -\overline{d} - t\cdot\overline{b} + \overline{b}$$

$$= (1-t)\cdot\overline{b} - \overline{d}$$

$$\overline{CQ} \times \overline{CB} = \left(-t\cdot\overline{b} - \frac{1}{2}\overline{d}\right) \times \left((1-t)\overline{b} - \overline{d}\right)$$

$$= t\cdot(\overline{b}\times\overline{d}) - \frac{1}{2}(1-t)\cdot(\overline{d}\times\overline{b})$$

$$= \frac{1}{2}(1+t)(\bar{b} \times \bar{d})$$

$$\text{Area}_{\triangle CQB} = \frac{1}{4} \cdot |1+t| \cdot \|\bar{b} \times \bar{d}\| . \qquad (2)$$

Comparing (1) and (2), we get the conclusion.

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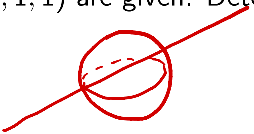
The vectors $\overline{a}(8,4,1)$, $\overline{b}(2,2,1)$ and $\overline{c}(1,1,1)$ are given. Determine the vector $\overline{d}$ such that:

1. $\angle(\overline{d},\overline{a}) = \angle(\overline{d},\overline{b})$;
2. $\overline{d} \perp \overline{c}$;
3. $||\overline{d}|| = 1$;
4. The triples $\{\overline{a},\overline{b},\overline{c}\}$ and $\{\overline{a},\overline{b},\overline{d}\}$ have the same orientation.



Look for $\overline{d}(d_1, d_2, d_3)$.

$$(1) \iff \cos(\sphericalangle(\overline{d},\overline{a})) = \cos(\sphericalangle(\overline{d},\overline{b}))$$

$$\iff \frac{\overline{d}\cdot\overline{a}}{\cancel{||\overline{d}||}\cdot||\overline{a}||} = \frac{\overline{d}\cdot\overline{b}}{\cancel{||\overline{d}||}\cdot||\overline{b}||}$$

$$\Leftrightarrow \frac{\bar{a}}{\|\bar{a}\|} \cdot \bar{d} = \frac{\bar{b}}{\|\bar{b}\|} \cdot \bar{d}$$

$\|\bar{a}\| = 9$ and $\|\bar{b}\| = 3$

$$\Leftrightarrow \left(\frac{8}{9}, \frac{4}{9}, \frac{1}{9}\right) \cdot (d_1, d_2, d_3) =$$

$$= \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \cdot (d_1, d_2, d_3)$$

$$\Leftrightarrow \boxed{d_1 - d_2 + d_3 = 0} \quad (1) \qquad x - y + z = 0$$

2. $\bar{d} \perp \bar{c} \iff \bar{d} \cdot \bar{c} = 0 \iff$

$$\boxed{d_1 + d_2 + d_3 = 0} \quad (2)$$

3. $\|\bar{d}\|^2 = 1 \iff \bar{d} \cdot \bar{d} = 1.$

$$d_1^2 + d_2^2 + d_3^2 = 0 \quad (3)$$

Solving the system (1), (2), (3) we get 2 solutions.

$$\left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right).$$

(4.) $\Leftrightarrow$ $\operatorname{sgn}\left((\bar{a}, \bar{b}, \bar{c})\right) = \operatorname{sgn}\left((\bar{a}, \bar{b}, \bar{d})\right)$

$$(\bar{a}, \bar{b}, \bar{c}) = \begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 4 > 0.$$

Check which solution gives.

$$(\bar{a}, \bar{b}, \bar{d}) > 0.$$

The problem set for this week will be posted soon. Ideally you would think about it before the seminar on Friday.

Thank you very much for your attention!