Seminar 1

Operation: $* : A \times A \to A$ with $x, y \in A \Rightarrow x * y \in A$.
Grupoid: $(A, *)$
Semigroup: $(A, *)$ grupoid + associativity
Monoid: $(A, *)$ semigroup + identity element
Group: $(A, *)$ monoid + all elements have a symmetric
Abelian group: $(A, *)$ group + commutativity
Subgroupoid = stable part: $\forall a, b \in A \Rightarrow a * b \in A$
Subgroup: $H \leq (G, *)$ if H is a sable part in G ($H \subseteq G$) and $(H, *)$ is a group.

1. Addition: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

   Substraction: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

   Multiplication: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

   Division: $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

2. Monoid: $(\mathbb{N}, +)$, $(\mathbb{N}, \cdot)$, $(\mathbb{Z}, \cdot)$

   Group: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $(\mathbb{Q}^*, \cdot)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$.

3. (i) $(\mathbb{C}, :)$
   (ii) $(\mathbb{Z}^*, +)$
   (iii) $(\mathbb{N}, +)$

4. If we take the set $A = \{a, b, c, e\}$ where e is the identity element of the operation $*$, then we define the operation such that $a * b = e$ and $c * a = e$, and for the rest it can be however we like. Then a has two different symmetrical elements.

5. (i) 3 elements in 3 spaces $\Rightarrow 3^9$

	a	b	c
a			
b			
c			

(ii) 3^3 (3 elements in 3 free spaces) and 3^3 (3 commutative elements in 3 spaces) $\Rightarrow 3^6$.

	a	b	c
a		c	b
b	c		a
c	b	a	

(iii) 3^4 (3 elemnts in 4 free spaces) and 3 elements, which can be e $\Rightarrow 3^5$

	e	b	c
e	e	b	c
b	b		
c	c		

Generalization:

(i) n^{n2}

(ii) $n^n \cdot n^{\frac{n(n-1)}{2}}$

(iii) $n^{(n-1)^2+1}$

6. (i) Stable part: $\forall x, y \in \mathbb{R} \Rightarrow x*y = x+y+xy = (x+1)(y+1)-1 \in \mathbb{R}$
Associativity: $\forall x, y \in \mathbb{R} \Rightarrow (x*y)*z = x*(y*z)$
Identity element: $\exists e \in \mathbb{R}$ such that $\forall x \in \mathbb{R} \Rightarrow x*e = e*x = x$
Commutativity: $\forall x, y \in \mathbb{R} \Rightarrow x*y = y*x$.

(ii) Let A be our interval. Then A is a stable subset of $(\mathbb{R}, *)$ $\iff$ $\forall x, y \in A \Rightarrow x*y \in A$.
$x, y \in A \Rightarrow -1 \leq x, -1 \leq y \Rightarrow 0 \leq x+1, 0 \leq y+1 \Rightarrow 0 \leq (x+1)(y+1) \Rightarrow -1 \leq (x+1)(y+1)-1 \Rightarrow x*y \in A$

7. (i) Here is interesting to see the associativity: $\forall x, y, z \in \mathbb{N} \Rightarrow (x*y)*z = gcd(x,y)*z = gcd(gcd(x,y),z) = \alpha \Rightarrow \alpha \mid gcd(x,y)$ and $\alpha \mid z$. From $gcd(x,y) = d \Rightarrow x = dx_1$ and $y = dy_1$, but $\alpha \mid d \Rightarrow \alpha \mid x$ and $\alpha \mid y \Rightarrow \alpha \mid x, y, z \Rightarrow \alpha \mid gcd(y,z) \Rightarrow \alpha \mid gcd(x, gcd(y,z))$. Analogus for $gcd(x, gcd(y,z)) \mid \alpha$.

(ii) $\forall x, y \in D_n \Rightarrow x \mid n$ and $y \mid n \Rightarrow n = xd_1$ and $n = yd_2$. We compute $x * y = gcd(x, y) = \alpha \Rightarrow x = \alpha x_1$ and $y = \alpha y_1 \Rightarrow n = \alpha x_1 d_1$ and $n = \alpha y_1 d_2 \Rightarrow \alpha \mid n \Rightarrow gcd(x, y) \mid n \Rightarrow x * y \in D_n$. Associativity, commutativity and identity element are easy to prove.

(iii) $D_6 = \{1, 2, 3, 6\}$

	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

8. $H \subseteq \mathbb{Z}$ and H stable part of $\mathbb{Z} \Rightarrow \exists x \in H \Rightarrow \forall n \in \mathbb{N}^*$ we have $x^n \in H$, but H is finite $\Rightarrow \exists n \in \mathbb{N}^*$such that $x^i = x^j, i, j \in \mathbb{N}^*$ and $0 < i < j \Rightarrow x \in \{-1, 0, 1\} \Rightarrow H$ can be $\emptyset, \{0\}, \{1\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}$.

9. The power set of A together with reunion is a monoid, as: it is associative, the identity element is $\emptyset$, but the only element which has a symmetric is $\emptyset$.

   The power set of A together with intersection is a monoid, as: it is associative, the identity element is A, but the only element which has a symmetric is A.

10. (i) $(A, \cdot)$ commutative, i.e. $\forall a, b \in A$ we have $a \cdot b = b \cdot a$.

    We know that $X, Y \subseteq A \Rightarrow \forall x \in X, y \in Y$ we have $x \cdot y = y \cdot x$. So, for $X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\} = \{y \cdot x \mid y \in Y, x \in X\} = Y \cdot X \Rightarrow (P(A), \cdot)$ is commutative.

    (ii) $(A, \cdot)$ semigroup, i.e. $\forall a, b, c \in A$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

    We know that $X, Y \in A \Rightarrow X \cup Y \in A \Rightarrow \forall x \in X, y \in Y, z \in X \cup Y$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. So, for $(X \cdot Y) \cdot X \cup Y = \{x \cdot y \mid x \in X, y \in Y\} \cdot X \cup Y = \{(x \cdot y) \cdot z \mid x \in X, y \in Y, z \in X \cup Y\} = \{x \cdot (y \cdot z) \mid x \in X, y \in Y, z \in X \cup Y\} = X \cdot \{y \cdot z \mid y \in Y, z \in X \cup Y\} = X \cdot (Y \cdot Z) \Rightarrow (P(A), \cdot)$ is a semigroup.

(iii) Here we may talk about the identity element. So, if $(A, \cdot)$ has e as the identity element, then $\forall x \in A$, we have $x \cdot e = e \cdot x = x$. Then $\{e\} \cdot X = \{e \cdot x \mid x \in X\} = \{x \cdot e \mid x \in X\} = \{x \mid x \in X\} = X$. Hence, $(P(A), \cdot)$ is a monoid.

(iv) Here, the problem is with the symmetrical elements. So, if $x_1, x_2 \in A$, then $\exists y_1, y_2 \in A$ such that $x_1 \cdot y_1 = y_1 \cdot x_1 = e$ and $x_2 \cdot y_2 = y_2 \cdot x_2 = e$. But, for $X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}$, where we can find even $x_1 \cdot y_2, x_1 \in X, y_2 \in Y$, which are not symmetric elements. Hence $(P(A), \cdot)$ is not always a group.