SEMINAR 11

1) a) Let $a, b, c \in \mathbb{R}$ and $f_1 = (X-b)(X-c),\ f_2 = (X-c)(X-a),\ f_3 = (X-a)(X-b)$. Show that:
i) f_1, f_2, f_3 are linearly independent in ${}_{\mathbb{R}}\mathbb{R}[X]$ if and only if

$$(a-b)(b-c)(c-a) \neq 0;$$

ii) if $(a-b)(b-c)(c-a) \neq 0$ then for any $f \in \mathbb{R}[X]$ with $\deg f \leq 2$ there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, uniquely determined, such that $f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$.
b) Determine $\lambda_1, \lambda_2, \lambda_3$ when $f = 1 + 2X - X^2$, $a = 1$, $b = 2$ and $c = 3$.

2) Let $p \in \mathbb{N}$ be a prime. Show that the usual addition and multiplication determine on

$$V = \{a + b\sqrt[3]{p} + c\sqrt[3]{p^2} \mid a, b, c \in \mathbb{Q}\}$$

a $\mathbb{Q}$-vector space structure and determine a basis and the dimension of this vector space.

3) Let V be a K-vector space with the dimension $n \in \mathbb{N}^*$ and let $A, B \leq_K V$ with $\dim A = n-1$ and $B \nsubseteq A$. Show that

$$\dim(A \cap B) = \dim B - 1 \text{ and } A + B = V.$$

4) Let V be a finite dimensional K-vector space and $A, B \leq_K V$ such that

$$\dim(A+B) = \dim(A \cap B) + 1.$$

Show that $A \subseteq B$ or $B \subseteq A$.

5) Let f and g be two endomorphisms of a finite dimensional K-vector space V. If $f+g$ is an automorphism of V and $f \circ g$ is the zero endomorphism then

$$\dim V = \dim f(V) + \dim g(V).$$

6) a) Let $\varphi \in \mathbb{R}$. Show that the plane rotation

$$h : \mathbb{R}^2 \to \mathbb{R}^2,\ h(x,y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi),$$

is an automorphism of $\mathbb{R}^2$. Write the matrix of h of $\mathbb{R}^2$ (i.e. the basis $E = (e_1, e_2)$, with $e_1 = (1,0)$, $e_2 = (0,1)$).
b) Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x, -y)$ (the symmetry with respein the standard basis ct to Ox) and $g : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (-x, y)$ (the symmetry with respect to Oy) are automorphisms of $\mathbb{R}^2$. Find the matrices of f, g, $f-g$, $f+2g$ and $g \circ f$ in the standard basis.

7) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x+y, 2x-y, 3x+2y)$. Show that f is an $\mathbb{R}$-linear map, that $B = ((1,2),(-2,1))$ and $B' = ((1,-1,0),(-1,0,1),(1,1,1))$ are bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, then determine the matrix of f in the pair of bases (B, B').