Exercises Predicate Resolution

Exercise 1

Transform the following formulas into prenex, Skolem and clausal normal forms.

1. $U_1 = (\forall x)(\forall y)((\exists z)P(y,z) \wedge (\exists u)(Q(x,u) \rightarrow (\exists z)R(u,z,x)))$;
2. $U_2 = (\exists x)(\forall y)((\exists z)\neg P(z) \vee (\exists u)(R(x,u) \rightarrow (\forall z)\neg Q(u,z)))$;
3. $U_3 = (\forall x)(\exists y)((\exists z)P(z,x) \rightarrow (\forall u)(Q(x,u) \wedge (\exists z)\neg R(y,z)))$;
4. $U_4 = (\exists x)(\exists y)((\forall z)P(z) \rightarrow (\forall u)(\neg Q(y,u) \rightarrow (\forall z)R(y,z)))$;
5. $U_5 = (\forall x)(\exists y)((\exists z)Q(z,y) \wedge (\exists u)(P(x,u) \rightarrow (\exists z)R(u,z)))$;
6. $U_6 = (\forall x)(\forall y)((\exists z)\neg Q(z,x) \rightarrow (\forall u)(R(y,u) \rightarrow (\exists z)\neg P(x,z)))$;
7. $U_7 = (\forall x)(\forall y)((\exists z)R(z,y) \vee (\exists u)(\neg P(x,u) \rightarrow (\forall z)\neg Q(y,z)))$;
8. $U_8 = (\exists x)(\forall y)((\exists z)R(z,x) \wedge (\forall u)(Q(x,u) \rightarrow (\exists z)P(y,z)))$.

Exercise 2

Are the literals from the following pairs unifiable? If yes, find their most general unifier.

$x,y,z \in Var,\ a,b \in Const$, $f,g \in F_1, h \in F_2,\ P \in P_3$.

9. $P(a,x,g(g(y)))$ and $P(y,f(z),f(z))$;
$P(x,g(f(a)),f(x))$ and $P(f(y),z,y)$;
$P(a,x,g(g(y)))$ and $P(z,h(z,u),g(u))$

10. $P(a,x,f(g(y)))$ and $P(y,f(z),f(z))$;
$P(x,g(f(a)),f(b))$ and $P(f(y),z,z)$;
$P(a,x,f(g(y)))$ and $P(z,h(z,u),f(b))$;

11. $P(a,f(x),g(h(y)))$ and $P(y,f(z),g(z))$;
$P(x,g(f(a)),h(x,y))$ and $P(f(z),g(z),y)$;
$P(g(y),x,f(g(y)))$ and $P(z,h(z,u),f(u))$;

12. $P(a,g(x),f(g(y)))$ and $P(y,z,f(z))$;
$P(b,g(f(a)),z)$ and $P(f(y),z,g(y))$;
$P(a,h(x,b),f(g(y)))$ and $P(z,h(z,u),f(u))$;

13. $P(a,x,g(f(y)))$ and $P(f(z),z,g(x))$;
$P(a,x,g(f(y)))$ and $P(x,y,g(f(b)))$;
$P(a,h(x,u),g(f(z)))$ and $P(y,h(u,f(z)),g(x))$;

14. $P(a,y,g(f(z)))$ and $P(z,f(z),x)$;
$P(y,f(x),z)$ and $P(y,f(y),f(y))$;
$P(h(x,y),x,y)$ and $P(h(y,x),f(z),z)$;

15. $P(a,x,g(f(y)))$ and $P(f(y),z,x)$;
$P(x,a,g(b))$ and $P(f(y),f(y),g(x))$;
$P(h(x,z),f(z),y)$ and $P(h(f(y),x),f(x),a)$;

16. $P(a,x,g(f(y)))$ and $P(f(y),f(z),g(z))$;
$P(x,g(f(a)),x)$ and $P(f(y),z,h(y,f(y)))$;
$P(a,h(x,u),f(g(y)))$ and $P(z,h(z,u),f(u))$

Exercise 3

Prove the inconsistency of the following set of clauses using lock resolution.
Try two different indexings for the literals.

17. $S_1 = \{\neg P(x) \vee Q(x), P(a), \neg Q(x) \vee \neg R(x), \neg W(a), R(y) \vee W(y)\}$;

18. $S_2 = \{P(x) \vee \neg Q(x), \neg P(a) \vee R(x), Q(x), W(z), \neg R(y) \vee \neg W(y)\}$;

19. $S_3 = \{P(x) \vee Q(x) \vee R(x), \neg P(a), \neg Q(x), \neg W(a), \neg R(y) \vee W(y)\}$;

20. $S_4 = \{P(x) \vee Q(x), \neg P(x) \vee R(x), \neg Q(y) \vee R(y), \neg R(x) \vee W(x), \neg W(f(z))\}$;

21. $S_5 = \{P(x) \vee Q(x), \neg P(a) \vee W(x), \neg Q(y) \vee R(y), \neg R(x) \vee W(x), \neg W(a)\}$;

22. $S_6 = \{\neg P(x) \vee \neg Q(x), P(z) \vee W(x), Q(y) \vee W(y) \vee \neg R(y), \neg R(x) \vee \neg W(x), R(g(a,b))\}$;

23. $S_7 = \{P(x) \vee Q(x), \neg P(x), \neg Q(f(a)) \vee R(z), \neg W(z), \neg R(y) \vee W(y)\}$

24. $S_8 = \{\neg P(x) \vee Q(x) \vee \neg R(x), P(f(b)), \neg Q(x), \neg W(y), R(y) \vee W(y)\}$.

Exercise 4
Using a refinement of predicate resolution prove:

25. the semidistributivity of '$\forall$' over '$\vee$':
$\vdash (\forall x)P(x) \vee (\forall x)Q(x) \rightarrow (\forall x)(P(x) \vee Q(x))$
$\nvdash (\forall x)(P(x) \vee Q(x)) \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$

26. the semidistributivity of '$\exists$' over '$\wedge$':
$\vdash (\exists x)(P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$
$\nvdash (\exists x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)(P(x) \wedge Q(x))$

27. $\vdash (\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\forall x)P(x) \rightarrow (\exists x)Q(x))$
$\nvdash ((\exists x)P(x) \wedge (\exists x)Q(x)) \rightarrow (\forall x)(P(x) \wedge Q(x))$

28. the semidistributivity of '$\exists$' over '$\rightarrow$':
$\vdash ((\exists x)P(x) \rightarrow (\exists x)Q(x)) \rightarrow (\exists x)(P(x) \rightarrow Q(x))$
$\nvdash (\exists x)(P(x) \rightarrow Q(x)) \rightarrow ((\exists x)P(x) \rightarrow (\exists x)Q(x))$

29. the distributivity of '$\forall$' over '$\wedge$':
$\vdash (\forall x)P(x) \wedge (\forall x)Q(x) \leftrightarrow (\forall x)(P(x) \wedge Q(x))$.

30. $\vdash (\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\exists x)P(x) \rightarrow (\exists x)Q(x))$
$\nvdash ((\exists x)P(x) \rightarrow (\exists x)Q(x)) \rightarrow (\forall x)(P(x) \rightarrow Q(x))$

7. the distributivity of '$\exists$' over '$\vee$':
$\vdash (\exists x)(P(x) \vee Q(x)) \leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$;

8. $\vdash (\exists x)(P(x) \rightarrow Q(x)) \leftrightarrow ((\forall x)P(x) \rightarrow (\exists x)Q(x))$

Exercise 5
Check whether the following formulas are theorems or not using predicate resolution.

31. $U_1 = (\forall y)(\exists x)P(x,y) \leftrightarrow (\exists y)(\exists x)P(x,y)$

32. $U_2 = (\forall y)(\forall x)P(x,y) \leftrightarrow (\exists x)(\exists y)P(x,y)$;

33. $U_3 = (\forall x)(\forall y)P(x,y) \leftrightarrow (\exists x)(\forall y)P(x,y)$;

35. $U_5 = (\exists y)(\exists x)P(x,y) \leftrightarrow (\forall x)(\exists y)P(x,y)$;

36. $U_6 = (\forall y)(\forall x)P(x,y) \leftrightarrow (\forall x)(\exists y)P(x,y)$;

37. $U_7 = (\exists y)(\exists x)P(x,y) \leftrightarrow (\exists x)(\forall y)P(x,y)$;

34. $U_4 = (\exists x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\exists x)P(x,y)$, 38. $U_8 = (\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x)P(x,y)$.

Exercise 6. Succession to the British throne

Hypotheses:

H_1. If x is the king and y is his oldest son, then y can become the king.

H_2. If x is the king and y defeats x, then y will become the king.

H_3. $RichardIII$ is the king.

H_4. $HenryVII$ defeated $RichardIII$.

H_5. $HenryVIII$ is $HenryVII$'s oldest son.

Conclusion:

C. Can $HenryVIII$ become the king?

Check whether the conclusion C is derivable from the set of hypotheses { H_1, H_2, H_3, H_4, H_5 } using linear predicate resolution. $H_1, H_2, H_3, H_4, H_5 \overset{?}{\vdash} C$.

Exercise 7.

Hypotheses:

H_1: All hummingbirds are richly colored.

H_2: No large birds live on honey.

H_3: Birds that do not live on honey are dull in color.

H_4: *Piky* is a hummingbird.

Conclusion:

C: *Piky* is a small bird and lives on honey.

Using linear predicate resolution check whether the following deduction holds:

$H_1, H_2, H_3, H_4 \overset{?}{\vdash} C$.

Exercise 1
Transform the following formulas into prenex, Skolem and clausal normal forms.

$$U = (\exists x)(\forall y)\left((\exists z)\neg P(z) \vee (\exists u)\left(R(x,u) \rightarrow (\forall z)\neg Q(u,z)\right)\right)$$
$$\equiv (\exists x)(\forall y)\left((\exists z)\neg P(z) \vee (\exists u)\left(\neg R(x,u) \vee (\forall z)\neg Q(u,z)\right)\right)$$

There are 5! Prenex Normal Forms for U

$U_1^P = (\exists x)(\forall y)(\exists z)(\exists u)(\forall t)\neg P(z) \vee \neg R(x,u) \vee \neg Q(u,t)$
$[x \leftarrow a, z \leftarrow f(y), u \leftarrow g(y)]$
$U_1^S = (\forall y)(\forall t)\neg P(f(y)) \vee \neg R(a, g(y)) \vee \neg Q(g(y), t)$
$U_1^C = \neg P(f(y)) \vee \neg R(a, g(y)) \vee \neg Q(g(y), t)$

$U_2^P = (\exists x)(\forall y)(\exists u)(\forall t)(\exists z)\neg P(z) \vee \neg R(x,u) \vee \neg Q(u,t)$
$[x \leftarrow a, u \leftarrow f(y), z \leftarrow g(y,t)]$
$U_2^S = (\forall y)(\forall t)\ \neg P(g(y,t)) \vee \neg R(a, f(y)) \vee \neg Q(f(y), t)$
$U_2^C = \neg P(g(y,t)) \vee \neg R(a, f(y)) \vee \neg Q(f(y), t)$

Exercise 2
Are the literals from the following pairs unifiable? If yes, find their most general unifier.

$x, y, z \in Var \quad a, b \in Const \quad f, g \in F_1 \quad h \in F_2 \quad p \in P_3$

$l_1 = P(a, x, f(g(y)))$
$l_2 = P(y, f(z), f(z))$
$\theta := \varepsilon$

$\theta(l_1) = P(\underline{a}, x, f(g(y)))$
$\theta(l_2) = P(\underline{y}, f(z), f(z))$

RULES:
$x, y \in Var$
$a \in Const$
$x \leftarrow y$ ✓
$x \leftarrow a$ ✓
$x \leftarrow f(y)$ ✓
$x \leftarrow f(x)$ ✗
$x \leftarrow f(a)$ ✓
$a \leftarrow x$ ✗
$a \leftarrow f(x)$ ✗
$f(x) \leftarrow a$ ✗
$x \leftarrow f(x, y)$ ✗

First iteration
$\lambda = [y \leftarrow a]$
$\theta := \theta\lambda = [y \leftarrow a]$
$\theta(l_1) = P(a, \underline{x}, f(g(a)))$
$\theta(l_2) = P(a, \underline{f(z)}, f(z))$

Second iteration
$\lambda := [x \leftarrow f(z)]$
$\theta := \theta\lambda = [y \leftarrow a, x \leftarrow f(z)]$
$\theta(l_1) = P(a, f(z), f(\underline{g(a)}))$
$\theta(l_2) = P(a, f(z), f(\underline{z}))$

Third iteration
$\lambda := [z \leftarrow g(a)]$
$\theta := \theta\lambda = [y \leftarrow a, x \leftarrow f(g(a)), z \leftarrow g(a)]$
$\theta(l_1) = \theta(l_2) = P(a, f(z), f(g(a)))$
$\Rightarrow mgu(l_1, l_2) = \theta$

Exercise 3
Prove the inconsistency of the following set of clauses using lock resolution.
Try two different indexings for the literals.

$$S = \{ \underset{(2)}{P(x)} \vee \underset{(1)}{\neg Q(x)},\ \underset{(4)}{\neg P(a)} \vee \underset{(3)}{R(x)},\ \underset{(5)}{Q(x)},\ \underset{(6)}{W(z)},\ \underset{(7)}{\neg R(y)} \vee \underset{(8)}{\neg W(y)} \}$$

$\boxed{I}$

$C_1 = \underset{(2)}{P(x)} \vee \underset{(1)}{\neg Q(x)}$

$C_2 = \underset{(4)}{\neg P(a)} \vee \underset{(3)}{R(x)}$

$C_3 = \underset{(5)}{Q(x)}$

$C_4 = \underset{(6)}{W(z)}$

$C_5 = \underset{(7)}{\neg R(y)} \vee \underset{(8)}{\neg W(y)}$

$C_6 = Res^{pred}_{[x \leftarrow a]}(C_1, C_3) = \underset{(2)}{P(a)}$

$C_7 = Res^{pred}_{[y \leftarrow x]}(C_2, C_5) = \underset{(4)}{\neg P(a)} \vee \underset{(8)}{\neg W(x)}$

$C_8 = Res^{pred}(C_6, C_7) = \underset{8}{\neg W(x)}$

$C_9 = Res^{pred}_{[z \leftarrow x]}(C_4, C_8) = \square \Rightarrow$ inconsistent

Exercise 4
Using a refinement of predicate resolution prove:

$U_1 = \vdash (\forall x)(P(x) \to Q(x)) \to ((\forall x)P(x) \to (\exists x)Q(x))$

$U_2 = ((\exists x)P(x) \wedge (\exists x)Q(x)) \to (\forall x)(P(x) \wedge Q(x))$

$\neg U_1 = (\forall x)(\neg P(x) \vee Q(x)) \wedge ((\forall x)(P(x) \to (\exists x)\ Q(x))$

$\equiv (\forall x)(\neg P(x) \vee Q(x)) \wedge (\neg(\forall y)P(y) \vee (\exists z)\ Q(z))$

$\equiv (\forall x)(\neg P(x) \vee Q(x)) \wedge (\forall y)P(y) \wedge \neg(\exists z)Q(z))$

$\equiv (\forall x)(\neg P(x) \vee Q(x)) \wedge (\forall y)P(y) \wedge (\forall z)\neg Q(z))$

$(U_1)^P = (\forall x)(\forall y)(\forall z)(\neg P(x) \vee Q(x)) \wedge P(y) \wedge \neg Q(z)$

$(U_1)^C = \neg P(x) \vee Q(x) \wedge P(y) \wedge \neg Q(z)$

$S_1 = \{ \neg P(x) \vee Q(x),\ P(y),\ \neg Q(z) \}$

$C_1 = \underset{(1)}{\neg P(x)} \vee \underset{(2)}{Q(x)}$

$C_2 = \underset{(3)}{P(y)}$

$C_3 = \underset{(4)}{\neg Q(z)}$

$C_4 = Res^{pred}_{[y \leftarrow x]}(C_1, C_2) = \underset{(2)}{Q(x)}$

$C_5 = Res^{pred}_{[z \leftarrow x]}(C_3, C_4) = \square \Rightarrow U_1$ valid

$\neg U_2 = ((\exists x)P(x) \wedge (\exists x)Q(x) \wedge \neg(\forall x)(P(x) \wedge Q(x))$

$\equiv (\exists x)P(x) \wedge (\exists y)Q(y) \wedge (\exists z)(\neg P(z) \vee \neg Q(z))$

$(\neg U_2)^P = (\exists x)(\exists y)(\exists z)\ (P(x) \wedge Q(y) \wedge (\neg P(z) \vee \neg Q(z)))$

$(\neg U_2)^C = P(a) \wedge Q(b) \wedge (\neg P(c) \vee \neg Q(c))$

$S_2 = \{ P(a),\ Q(b),\ (\neg P(c) \vee \neg Q(c)) \}$

a, b, c - distinct constants → not unifiable → cannot apply resolution

$\Rightarrow$ cannot derive $\square \Rightarrow \neg U_2^C \nvdash^{Pr}_{Res} \square \Rightarrow\ \nvDash U_2$

Exercise 5

Check whether the following formulas are theorems or not using predicate resolution.

$U = (\forall x)(\forall y)\, P(x,y) \leftrightarrow (\exists x)(\forall y)\, P(x,y)$

$\equiv \underbrace{((\forall x)(\forall y)\, P(x,y) \rightarrow (\exists x)(\forall y)\, P(x,y))}_{U_1} \wedge \underbrace{((\exists x)(\forall y)\, P(x,y) \rightarrow (\forall x)(\forall y)\, P(x,y))}_{U_2}$

$\neg U = \neg U_1 \vee \neg U_2$

$\boxed{U \text{ theorem if } (\neg U)^c \vdash \square}$

① $\neg U_1 = (\forall x)(\forall y)\, P(x,y) \wedge \neg(\exists x)(\forall y)\, P(x,y)$

$\equiv (\forall x)(\forall y)\, P(x,y) \wedge (\forall u)(\exists t)\neg P(u,t)$

$(\neg U_1)^P = (\forall u)(\exists t)(\forall x)(\forall y)\, P(x,y) \wedge \neg P(u,t)$

$(\neg U_1)^c = P(x,y) \wedge \neg P(u, f(u))$

$C_1 = P(x,y)$ $\quad \theta_1 = [x \leftarrow u,\ y \leftarrow f(u)]$

$C_2 = \neg P(u, f(u))$

$C_3 = \mathrm{Res}^{\mathrm{pred}}_{\theta_1}(C_1, C_2) = \square \Rightarrow \neg U_1$ inconsistent

② $\neg U_2 = (\exists x)(\forall y)\, P(x,y) \wedge (\exists u)(\exists t)\neg P(u,t)$

$(\neg U_2)^P = (\exists u)(\exists t)(\exists x)(\forall y)\, P(x,y) \wedge \neg P(u,t)$

$\{u \leftarrow a,\ t \leftarrow b,\ x \leftarrow c\}$

$(\neg U_2)^c = P(c,y) \wedge \neg P(a,b)$

$\left.\begin{array}{l} C_1 = P(c,y) \\ C_2 = \neg P(a,b) \end{array}\right\} ?$

Exercise 6. Succession to the British throne

Hypotheses:

H_1. If x is the king and y is his oldest son, then y can become the king.

H_2. If x is the king and y defeats x, then y will become the king.

H_3. $RichardIII$ is the king.

H_4. $HenryVII$ defeated $RichardIII$.

H_5. $HenryVIII$ is $HenryVII$'s oldest son.

Conclusion:

C. Can $HenryVIII$ become the king?

Check whether the conclusion C is derivable from the set of hypotheses $\{H_1, H_2, H_3, H_4, H_5\}$ using linear predicate resolution. $H_1, H_2, H_3, H_4, H_5 \overset{?}{\vdash} C$.

$C_1 : (\forall x)(\forall y)\,(k(x) \wedge o(x, y) \rightarrow k(y))$

$C_2 : (\forall x)(\forall y)\,(k(x) \wedge d(y, x) \rightarrow k(y))$

$C_3 : k(R_3)$

$C_4 : d(H_7, R_3)$

$C_5 : o(H_7, H_8)$

$C_6 : \neg k(H_8)$

$(C_1)^c = \neg k(x) \vee \neg o(x, y) \vee k(y)$

$(C_2)^c = \neg k(x) \vee \neg d(y, x) \vee k(y)$

PREDICATES & CONSTANTS

k : king

o : oldest son

d : defeats

R_3 : Richard III

H_7 : Henry VII

H_8 : Henry VIII

$C_7 = C_2 \underset{\{x \leftarrow R_3,\, y \leftarrow H_7\}}{\overset{fact}{\vdash}} = \neg k(R_3) \vee \neg d(H_7, R_3) \vee k(H_7)$

$C_{10} = C_1 \underset{\{x \leftarrow H_7,\, y \leftarrow H_8\}}{\overset{fact}{\vdash}} = \neg k(H_7) \vee \neg o(H_7, H_8) \vee k(H_8)$

C_3: $k(R_3)$; C_7: $\neg k(R_3) \vee \neg d(H_7, R_3) \vee k(H_7)$ — $k(R_3)$

C_8: $\neg d(H_7, R_3) \vee k(H_7)$; C_4: $d(H_7, R_3)$ — $d(H_7, R_3)$

C_9: $k(H_7)$; C_{10}: $\neg k(H_7) \vee \neg o(H_7, H_8) \vee k(H_8)$ — $k(H_7)$

C_{11}: $\neg o(H_7, H_8) \vee k(H_8)$; C_5: $o(H_7, H_8)$ — $o(H_7, H_8)$

C_{12}: $k(H_8)$; C_6: $\neg k(H_8)$

C_{13}: $\square$ $\Rightarrow \{H_1, H_2, H_3, H_4, H_5\} \vdash C$

Exercise 7.

Hypotheses:

H_1: All hummingbirds are richly colored.

H_2: No large birds live on honey.

H_3: Birds that do not live on honey are dull in color.

H_4: *Piky* is a hummingbird.

Conclusion:

C: *Piky* is a small bird and lives on honey.

Using linear predicate resolution check whether the following deduction holds:

$H_1, H_2, H_3, H_4 \overset{?}{\vdash} C$.

$H_1 : (\forall x)\,(hb(x) \rightarrow rc(x))$

$H_2 : \neg(\exists x)\,(\neg sb(x) \wedge lh(x))$

$H_3 : (\forall x)\,(\neg lh(x) \rightarrow \neg rc(x))$

$H_4 : hb(P)$

$C : sb(P) \wedge lh(P)$

$C_1 = (H_1)^c = \neg hb(x) \vee rc(x)$

$C_2 = (H_2)^c = sb(x) \vee \neg lh(x)$

$C_3 = (H_3)^c = lh(x) \vee \neg rc(x)$

$C_4 = hb(P)$

$C_5 = \neg C = \neg sb(P) \vee \neg lh(P)$

$C_6 = C_1 \vdash^{fact}_{[x \leftarrow P]} \neg hb(P) \vee rc(P)$

$C_8 = C_3 \vdash^{fact}_{[x \leftarrow P]} lh(P) \vee \neg rc(P)$

$C_{10} = C_2 \vdash^{fact}_{[x \leftarrow P]} sb(P) \vee \neg lh(P)$

C_4: $hb(P)$ and C_6: $\neg hb(P) \vee rc(P)$ $\xrightarrow{hb(P)}$ C_7: $rc(P)$

C_7: $rc(P)$ and C_8: $lh(P) \vee \neg rc(P)$ $\xrightarrow{rc(P)}$ C_9: $lh(P)$

C_9: $lh(P)$ and C_{10}: $sb(P) \vee \neg lh(P)$ $\xrightarrow{lh(P)}$ C_{11}: $sb(P)$

C_{11}: $sb(P)$ and C_5: $\neg sb(P) \vee \neg lh(P)$ $\xrightarrow{sb(P)}$ C_{12}: $\neg lh(P)$

C_{12}: $\neg lh(P)$ and C_9: $lh(P)$ $\xrightarrow{lh(P)}$ $\square$

$\square \Rightarrow \{H_1, H_2, H_3, H_4\} \vdash C$