SEMINAR 10

1) Show that the vectors $(1, 2, -1)$, $(3, 2, 4)$, $(-1, 2, -6)$ from ${}_{\mathbb{R}}\mathbb{R}^3$ are linearly dependendent and find a dependency relation between them.

2) Give a necessary and sufficient condition for the vectors $v_1 = (a_1, b_1)$, $v_2 = (a_2, b_2)$ to form a basis for the $\mathbb{R}$-vector space $\mathbb{R}^2$. What does this condition mean from geometrical point of view? Using the condition established, find infinitely many bases for ${}_{\mathbb{R}}\mathbb{R}^2$. Is there any basis of $\mathbb{R}^2$ for which the coordinates of a vector $v = (x, y)$ are exactly x and y? Show that $v_1 = (1, 0)$ and $v_2 = (1, 1)$ form a basis of $\mathbb{R}^2$ and find the coordinates of $v = (x, y)$ in this basis.
Homework: Formulate and solve a similar problem for the $\mathbb{R}$-vector space $\mathbb{R}^3$.

3) Determine the values of $a \in \mathbb{R}$ for which the vectors $v_1 = (a, 1, 1)$, $v_2 = (1, a, 1)$, $v_3 = (1, 1, a)$ form a basis of ${}_{\mathbb{R}}\mathbb{R}^3$.

4) Which of the following systems of vectors from $\mathbb{R}^3$:
a) $((1, 0, -1), (2, 5, 1), (0, -4, 3))$;
b) $((2, -4, 1), (0, 3, -1), (6, 0, 1))$;
c) $((1, 2, -1), (1, 0, 3), (2, 1, 1))$;
d) $((-1, 3, 1), (2, -4, -3), (-3, 8, 2))$;
e) $((1, -3, -2), (-3, 1, 3), (-2, -10, -2))$
are bases for the $\mathbb{R}$-vector space $\mathbb{R}^3$?

5) Let V be an $\mathbb{R}$-vector space and $v_1, v_2, v_3 \in V$. Show that the vectors v_1, v_2, v_3 are linearly independent if and only if the vectors $v_2 + v_3, v_3 + v_1, v_1 + v_2$ are linearly independent.
Extra: i) Show that $\langle v_1, v_2, v_3\rangle = \langle v_2 + v_3, v_3 + v_1, v_1 + v_2\rangle$.
ii) Is the property valid in a vector space V over an arbitrary field K?

6) Show that in the $\mathbb{R}$-vector space $M_2(\mathbb{R})$ the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis and determine the coordinates of $A = \begin{pmatrix} -2 & 3 \\ 4 & -2 \end{pmatrix}$ in this basis.

7) a) Let $a, b, c \in \mathbb{R}$ and $f_1 = (X - b)(X - c)$, $f_2 = (X - c)(X - a)$, $f_3 = (X - a)(X - b)$. Show that:
i) f_1, f_2, f_3 are linearly independent in ${}_{\mathbb{R}}\mathbb{R}[X]$ if and only if

$$(a - b)(b - c)(c - a) \neq 0;$$

ii) if $(a - b)(b - c)(c - a) \neq 0$ then for any $f \in \mathbb{R}[X]$ with $\deg f \leq 2$ there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, uniquely determined, such that $f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$.
b) Determine $\lambda_1, \lambda_2, \lambda_3$ when $f = 1 + 2X - X^2$, $a = 1$, $b = 2$ and $c = 3$.

8) Let $n \in \mathbb{N}$ and $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \sin^n x$. Show that $L = \{f_n \mid n \in \mathbb{N}\}$ is a linearly independent subset of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$.