Seminar 12

1. (i) $O_2 \in A \Rightarrow A \neq \emptyset$. Let $X = \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} y & y \\ 0 & 0 \end{bmatrix} \Rightarrow X - Y = \begin{bmatrix} x-y & x-y \\ 0 & 0 \end{bmatrix} \in A$. Now, consider $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If we compute $R \cdot X = \begin{bmatrix} ax & ax \\ cx & cx \end{bmatrix}$ and $X \cdot R = \begin{bmatrix} xa + xc & xb + xd \\ 0 & 0 \end{bmatrix}, \forall X \in A$, we see that the results are not in A. So, A is not an ideal, and neither left ideal nor right ideal.

   (ii) $O_2 \in B \Rightarrow B \neq \emptyset$. One can easily see that the difference of two matrices from B is still in B. Let $X = \begin{bmatrix} x & x \\ 0 & y \end{bmatrix} \in B$ and $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If we compute $R \cdot X$ and $X \cdot R$ the results are not in B. In the end, B is not an ideal, and neither left ideal nor right ideal.

   (iii) $O_2 \in C \Rightarrow C \neq \emptyset$. Again, the difference of two matrices from C is still in C. Now, let $X = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \in C$ and $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Computing $X \cdot R = \begin{bmatrix} xa + yb & xc + yd \\ 0 & 0 \end{bmatrix}$ which is in C, but $R \cdot X$ is not in C. In the end, C is a right ideal.

2. No. For instance, we have seen that $\mathcal{C} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$ is a right ideal of $M_2(\mathbb{R})$. Similarly, one shows that $\mathcal{D} = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$ is a left ideal of $M_2(\mathbb{R})$. But $C \cap D = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$ is not an ideal of $M_2(\mathbb{R})$.

3. (i) We know from seminar 10, exercise 4, that $(A, +)$ is a subgroup of $(\mathbb{R}[X], +)$. Now take $f = a_1X + a_2X^2 + \cdots \in A$ and $g = b_0 + b_1X + b_2X^2 + \cdots \in \mathbb{R}[X]$. We know that multiplication is commutative for polynomials, so we only compute $f \cdot g = b_0a_1X + (b_0a_2 + b_1a_1)X^2 + \cdots \in A \Rightarrow A$ is an ideal of $\mathbb{R}[X]$.

   (ii) Also, from seminar 10, exercise 4, we know that $(B, +)$ is not a subgroup of $(\mathbb{R}[X], +)$, so B can't be an ideal.

(iii) From the same exercise, we know that $(C, +)$ is a subgroup of $(\mathbb{R}[X], +)$. Now take $f = a_0 + a_2X^2 + \cdots \in C$ and $g = b_0 + b_1X + b_2X^2 + \cdots \in \mathbb{R}[X]$. ompute $f \cdot g = a_0b_0 + a_0b_1X + \ldots$ which is not in C. In the end, C is not an ideal.

4. It is easy to prove that R is a subring of $\mathbb{Q}$, as addition is associative and commutative, the identity element for addition is $\frac{0}{1}$ and the inverse for any element $\frac{a}{b}$ is $-\frac{a}{b}$, multiplication is associative and distributivity holds.

   Now, for the ideal part, we see that $\frac{2}{1} \in U \Rightarrow U \neq \emptyset$. Take $\forall \frac{a}{b}, \frac{c}{d} \in U \Rightarrow \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$. For this to be in U, $ad - cb$ has to be an even number, which is obvious as the product of two even numbers is an even number and the same goes for difference. Now, take $\frac{m}{n} \in R$ and $\frac{a}{b} \in U \Rightarrow \frac{m}{n} \cdot \frac{a}{b} = \frac{a}{b} \cdot \frac{m}{n} = \frac{ma}{nb}$, which is in U as n, b are odd, so nb is also odd and a even, so ma is also even.

   In the end, U is an ideal of R.

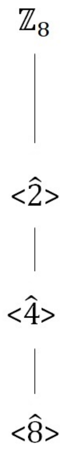
5. We will discuss about Ra as the proof is the same for aR.

   $Ra \neq \emptyset$, as $0 = 0 \cdot a \in Ra$. Take $ra, qa \in Ra : ra - qa = (r - q)a \in Ra$, as $r - q \in R(ring)$. Also, $\forall r \in R, \forall ra \in Ra$, we have $r \cdot ra = r^2a \in Ra$, as $r^2 \in R$. In the end, Ra is a left ideal.

6. $Ann(R) \neq \emptyset$, as $0 \in Ann(R)$. For any $a, b \in Ann(R) \Rightarrow (a - b) \cdot x = ax - bx = 0 - 0 = 0$ and the same goes for $x \cdot (a - b) = xa - xb = 0$, hence $a - b \in Ann(R)$. Let $r \in R, a \in Ann(R)$. We show that $ra, ar \in Ann(R)$. $\forall x \in R \Rightarrow rax = r \cdot 00$ and $xar = 0 \cdot r = 0$, so $ra, ar \in Ann(R)$. Hence, $Ann(R)$ is an ideal of R.

7. We know that an ideal is a subgroup. So, we have to look for ideals between the generated subgroups of $\mathbb{Z}_n$. As any generated subgroup of $\mathbb{Z}_n$, call it U, is cyclic, then $\forall r \in \mathbb{Z}_n$, we have that $rx, xr \in U, \forall x \in U$. So, all generated subgroups of $\mathbb{Z}_n$ are ideals.

   We know that $\forall d > 0$ with $d \mid n \Rightarrow (\hat{d})$ generated subgroup of $\mathbb{Z}_n$. So, in our case we need to find the divisors of 8, in order to find the ideals. Hence, $(\hat{1}) = \mathbb{Z}_8, (\hat{2}) = \{\hat{0}, \hat{2}, \hat{4}, \hat{6}\}, (\hat{4}) = \{\hat{0}, \hat{4}\}, (\hat{8}) = \{\hat{0}\}$ are the ideals of $\mathbb{Z}_8$.



8. The same reasoning as above and also, using seminar 7, exercise 6. We find that the ideals of $\mathbb{Z}_{12}$ are $(\hat{0}), (\hat{2}), (\hat{3}), (\hat{4}), (\hat{6}), (\hat{12})$.