CHAPTER 1

Affine spaces

Contents

1.1 Affine spaces

Definition 1.1. Let $\mathbf{V}$ be a vector space over the field $\mathbf{K}$. An *affine space over* $\mathbf{V}$ is a non-empty set $\mathbf{A}$, whose elements are called *points* of $\mathbf{A}$, together with a map

$$\mathbf{A} \times \mathbf{A} \to \mathbf{V} \tag{1.1}$$

that associates to every ordered pair $(P,Q) \in \mathbf{A} \times \mathbf{A}$ a vector in $\mathbf{V}$ which is denoted $\overrightarrow{PQ}$, in such a way that the following two axioms are satisfied:

(AS1) For every point $P \in \mathbf{A}$ and every $\mathbf{v} \in \mathbf{V}$ there is a unique point $Q \in \mathbf{A}$ such that

$$\overrightarrow{PQ} = \mathbf{v}.$$

(AS2) For every triple of points $P, Q, R \in \mathbf{A}$ the following identity holds in $\mathbf{V}$:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}.$$

- We say that $\mathbf{V}$ is the vector space associated to the affine space $\mathbf{A}$ or that $\mathbf{A}$ is an *affine space with associated vector space* $\mathbf{V}$.
- If $\mathbf{v} = \overrightarrow{PQ}$ we say that *the vector* $\mathbf{v}$ *is represented by the base point* P *and the end point* Q, or, shorter, we say that *the vector* $\mathbf{v}$ *is represented by* P *and* Q.
- If $\mathbf{K}$ is the field of real numbers $\mathbb{R}$, we call $\mathbf{A}$ a *real affine space*.
- If $\mathbf{K}$ is the field of complex numbers $\mathbb{C}$, we call $\mathbf{A}$ a *complex affine space*.
- The map in (1.1) is said to define on the set $\mathbf{A}$ *the structure of an affine space*, or, shorter, it defines an *affine structure* on $\mathbf{A}$.

Definition 1.2. Let $\mathbf{A}$ be an affine space over the $\mathbf{K}$-vector space $\mathbf{V}$. The *dimension of the affine space* $\mathbf{A}$ is by definition the dimension $\dim(\mathbf{V})$ of $\mathbf{V}$. We denote the dimension of $\mathbf{A}$ by $\dim(\mathbf{A})$.

Example 1.3. The Euclidean line $\mathbf{E}^1$, the Euclidean plane $\mathbf{E}^2$ and the Euclidean space $\mathbf{E}^3$ are examples of real affine spaces of dimension 1, 2 and 3 respectively. The map which defines an affine structure on $\mathbf{E}^2$ is

$$\mathbf{E}^2 \times \mathbf{E}^2 \to \mathbf{V} \quad (A,B) \mapsto \overrightarrow{AB}$$

where $\overrightarrow{AB}$ is the equipollence class of the pair (A,B) as in Lecture 2 and 3 of the Geometry course last semester (there you used the notation $\overline{AB}$). Thus affine spaces are generalizations of the ordinary line, plane and space.

Example 1.4. Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbf{K}$. The map

$$\mathbf{V} \times \mathbf{V} \to \mathbf{V} \quad (\mathbf{a},\mathbf{b}) = \mathbf{b} - \mathbf{a}$$

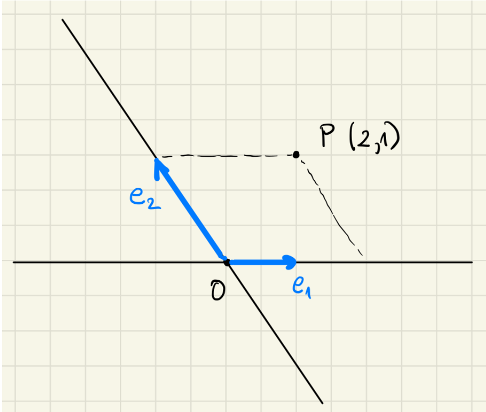
defines the structure of an affine space on $\mathbf{V}$ over itself. Thus, every vector space can be considered as an affine space over itself. If we want to view $\mathbf{V}$ as an affine space in this way, we denote it by $\mathbf{V}_a$.

Example 1.5. A particular case of the previous example is $\mathbf{V} = \mathbf{K}^n$. The affine space $\mathbf{K}^n_a$ is sometimes called the *affine numerical n-space over* $\mathbf{K}$. It is usually denoted by $\mathbf{A}^n(\mathbf{K})$ or just $\mathbf{A}^n$ if $\mathbf{K}$ is clear from the context. Every vector space over the field $\mathbf{K}$ which has dimension n is isomorphic to $\mathbf{K}^n$ so one can show that any affine space over a vector space $\mathbf{V}$ is isomorphic to $\mathbf{A}^n(\mathbf{K})$ for some integer n and some field $\mathbf{K}$.

Definition 1.6. Let $\mathbf{A}$ be an affine space over the $\mathbf{K}$-vector space $\mathbf{V}$. An affine system of coordinates in the space $\mathbf{A}$ is given by a point $O \in A$ and a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ of $\mathbf{V}$. We denote such a coordinate system by $O\mathbf{e}_1\dots\mathbf{e}_n$.

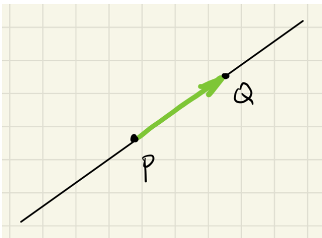
For each point $P \in \mathbf{A}$ we have

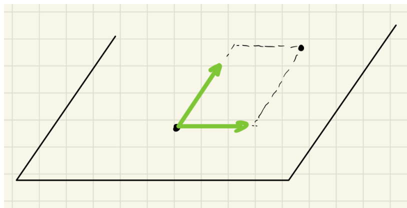
$$\overrightarrow{OP} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n \quad \text{for unique} \quad a_1,\dots,a_n \in \mathbf{K}.$$



The scalars $a_1, \dots, a_n$ are called *affine coordinates* (or just *coordinates*) of the point P with respect to the coordinate system $O\mathbf{e}_1 \dots \mathbf{e}_n$. Given a coordinate system, we write $P(a_1, \dots, a_n)$ when we want to indicate the coordinates of the point P. The point O of the coordinate system $O\mathbf{e}_1 \dots \mathbf{e}_n$ is called the *origin of the coordinate system* and has coordinates $(0, 0, \dots, 0)$.

1.2 Affine subspaces





Definition 1.7. Let $\mathbf{A}$ be an affine space over the $\mathbf{K}$-vector space $\mathbf{V}$. Given a point Q and a vector subspace $\mathbf{W}$ of $\mathbf{V}$, the *affine subspace of* $\mathbf{A}$ *passing through* Q *and parallel to* $\mathbf{W}$ is the subset $S \subseteq \mathbf{A}$ consisting of points $P \in \mathbf{A}$ with $\overrightarrow{QP} \in \mathbf{W}$.

- The subspace $\mathbf{W} \subseteq V$ is called the *vector subspace associated to* S.
- $\mathbf{A}$ is an affine subspace of itself.
- An affine subspace S of $\mathbf{A}$ is called *proper* if $S \subsetneq \mathbf{A}$.
- We define the *dimension* of S to be the dimension of $\mathbf{W}$ and denote it by $\dim(\mathbf{W})$.
- If $\dim(S) = 1$ then S is said to be a *line* in $\mathbf{A}$ and $\mathbf{W}$ is called the *direction* of S. Any non-zero vector $\mathbf{a} \in \mathbf{W}$ is called a *direction vector* for the line. It follows from the definition that

$$S = \{P \in \mathbf{A} : \text{ for which } \overrightarrow{QP} = t\mathbf{a} \text{ for some } t \in \mathbf{K}\}.$$

- If $\dim(S) = 2$ then S is said to be a *plane* in $\mathbf{A}$.
- If S is an affine subspace of the affine space $\mathbf{A}$ then $\dim(S) \leq \dim(\mathbf{A})$.
- If $\dim(S) = \dim(\mathbf{A})$ then $S = \mathbf{A}$.
- If $\dim(S) = \dim(\mathbf{A}) - 1$ then S is said to be a *hyperplane* of $\mathbf{A}$.

Example 1.8. The proper affine subspaces of $\mathbf{E}^1$ are the points of $\mathbf{E}^1$. The proper affine subspaces of $\mathbf{E}^2$ are the points and the lines in $\mathbf{E}^2$. The proper affine subspaces of $\mathbf{E}^3$ are the points, the lines and the planes of $\mathbf{E}^3$.

Example 1.9. Let $\mathbf{V}$ be a non-zero finite dimensional $\mathbf{K}$-vector space. Consider a vector subspace $\mathbf{W} \subseteq \mathbf{V}$ and a point $\mathbf{q} \in \mathbf{V}_a$. The affine subspace of $\mathbf{V}_a$ passing through $\mathbf{q}$ and parallel to $\mathbf{W}$ is the set

$$\mathbf{q} + \mathbf{W} = \{\mathbf{q} + \mathbf{w} : \mathbf{w} \in \mathbf{W}\}.$$

In the particular case when $\mathbf{q} \in \mathbf{W}$ we have $\mathbf{q} + \mathbf{W} = \mathbf{W}$. Thus, any vector subspace of a vector space $\mathbf{V}$ is an affine subspace of $\mathbf{V}_a$ and every affine subspace of $\mathbf{V}_a$ is of the form $\mathbf{q} + \mathbf{W}$ for some $\mathbf{q} \in \mathbf{V}_a$ and $\mathbf{W} \subseteq \mathbf{V}$. In other words, affine subspaces are *translates* of vector subspaces.

Definition 1.10. Given $n + 1 \geq 2$ points $P_0, \ldots, P_n$ in an affine space $\mathbf{A}$, the affine subspace passing through P_0 and having associated vector subspace $\langle \overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \ldots, \overrightarrow{P_0P_n} \rangle$ is called the *subspace generated by* (or *span of*) $P_0, P_1, \ldots, P_n$. We denote it by $\langle P_0, P_1, \ldots, P_n \rangle$.

- By definition it follows that $\dim\langle P_0, P_1, \ldots, P_n \rangle \leq n$.
- If $\dim\langle P_0, P_1, \ldots, P_n \rangle = n$ we say that the points $P_0, P_1, \ldots, P_n$ are *independent*, otherwise we say that they are *dependent*. By definition, therefore, the points $P_0, P_1, \ldots, P_n$ are independent if and only if the vectors $\overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \ldots, \overrightarrow{P_0P_n}$ are linearly independent.
- If $P_0, P_1, \ldots, P_n$ are independent then $n \leq \dim(\mathbf{A})$.
- Two points $P_0, P_1 \in \mathbf{A}$ are independent if and only if they are distinct, in which case $\langle P_0, P_1 \rangle$ is a line.
- Three points $P_0, P_1, P_2 \in \mathbf{A}$ are independent if and only if they do not belong to a line, in which case $\langle P_0, P_1, P_2 \rangle$ is a plane
- The points $P_0, P_1, \ldots, P_n \in \mathbf{A}$ are said to be *collinear* if there is a line containing them, or equivalently if $\dim\langle P_0, P_1, \ldots, P_n \rangle \leq 1$.
- The points $P_0, P_1, \ldots, P_n \in \mathbf{A}$ are said to be *coplanar* if there is a plane containing them, or equivalently if $\dim\langle P_0, P_1, \ldots, P_n \rangle \leq 2$.

Proposition 1.11.

1. An affine subspace is determined by its associated vector subspace and any one of its points.
2. Let S be an affine subspace of $\mathbf{A}$ with associated vector subspace $\mathbf{W}$. Associating to any pair of points $P, Q \in S$ the vector $\overrightarrow{PQ}$ defines on S the structure of an affine space over $\mathbf{W}$.

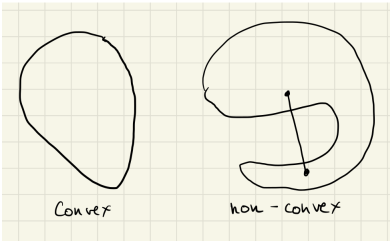
1.3 Convexity

In this section $\mathbf{A}$ will denote a *real* affine space.

Definition 1.12. Let P and Q be two points of $\mathbf{A}$. The *line segment* (or simply *segment*) AB is the set of points

$$\left\{X \in \mathbf{A} : \overrightarrow{PX} = t\overrightarrow{PQ} \text{ for some } t \in [0,1]\right\}.$$

Definition 1.13. A subset S of $\mathbf{A}$ is said to be *convex* if for every $A, B \in S$ the segment AB lies in S.

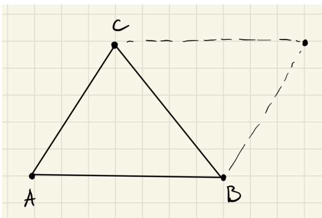


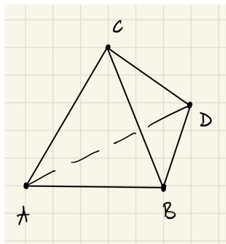
Example 1.14. It is easy to check that every affine subspace of a real affine space is convex.

Example 1.15. Consider a point $Q \in \mathbf{A}$ and a non-zero vector $\mathbf{a} \in \mathbf{V}$. The *half line with origin Q and direction* $\mathbf{a}$ is the set of points

$$\left\{X \in \mathbf{A} : \overrightarrow{PX} = t\mathbf{a} \text{ for some } t \in \mathbb{R} \text{ with } t \geq 0\right\}.$$

It is a convex set.





Example 1.16. Consider three points $A, B, C \in \mathbf{A}$ which are not collinear. The *triangle with vertices A, B, C* is the set of points

$$\left\{X \in \mathbf{A} : \overrightarrow{AX} = s\overrightarrow{AB} + t\overrightarrow{AC} \text{ for some } s, t \in \mathbb{R} \text{ with } s, t \geq 0 \text{ and } s + t \leq 1\right\}.$$

A *parallelogram* with vertices A, B, C is the set of points

$$\left\{X \in \mathbf{A} : \overrightarrow{AX} = s\overrightarrow{AB} + t\overrightarrow{AC} \text{ for some } s, t \in \mathbb{R} \text{ with } 0 \leq s, t \leq 1\right\}.$$

Notice that there are two more parallelograms determined by A, B, C and they have a similar description.

Example 1.17. Consider four points $A, B, C, D \in \mathbf{A}$ which are not coplanar. The *tetrahedron with vertices* A, B, C, D is the set of points

$$\left\{X \in \mathbf{A} : \overrightarrow{AX} = s\overrightarrow{AB} + t\overrightarrow{AC} + u\overrightarrow{AD} \text{ for some } s,t,u \in \mathbb{R} \text{ with } s,t \geq 0 \text{ and } s+t+u \leq 1\right\}.$$

A *parallelepiped* with vertices A, B, C, D is the set of points

$$\left\{X \in \mathbf{A} : \overrightarrow{AX} = s\overrightarrow{AB} + t\overrightarrow{AC} + u\overrightarrow{AD} \text{ for some } s,t,u \in \mathbb{R} \text{ with } 0 \leq s,t,u \leq 1\right\}.$$

Notice that there are three more parallelepipeds determined by A, B, C, D and they have a similar description.

Example 1.18. Consider $k+1$ independent points $P_0, \dots, P_k \in \mathbf{A}$. The *$k$-simplex with vertices* $P_0, \dots, P_k$ is the set of points

$$\left\{X \in \mathbf{A} : \overrightarrow{P_0X} = \sum_{i=1}^{k} t_i \overrightarrow{P_0P_i} \text{ for some } t_i \in \mathbb{R} \text{ with } t_i \geq 0 \text{ and } \sum_{i=1}^{k} t_i \leq 1\right\}.$$

Definition 1.19. Let S be a subset of $\mathbf{A}$. The *convex hull* of S is defined to be the smallest convex set containing S. It is thus the intersection of all convex subsets containing S.

1.4 A characterization of affine spaces

Let $\mathbf{A}$ be an affine space with associated vector space $\mathbf{V}$. Axiom AS1 defines a map

$$t : \mathbf{A} \times \mathbf{V} \to \mathbf{A}$$

that associates to each pair $(A, \mathbf{a})$ the point $B = t(A, \mathbf{a})$ with the property that $\overrightarrow{AB} = \mathbf{a}$. The map t has the following properties:

(AS1') For every $A, B \in \mathbf{A}$ there is a unique $\mathbf{a} \in \mathbf{V}$ such that

$$B = t(A, \mathbf{a}).$$

(AS2') For every $A \in \mathbf{A}$ and $\mathbf{a}, \mathbf{b} \in \mathbf{V}$ we have

$$t(t(A, \mathbf{a}), \mathbf{b}) = t(A, \mathbf{a} + \mathbf{b}).$$

Proposition 1.20. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$. A set $\mathbf{A}$ is an affine space if and only if there is a map $t : \mathbf{A} \times \mathbf{V} \to \mathbf{A}$ satisfying (AS1') and (AS2').

1.5 Connections to reality

Apparently there are people who believe that the Earth is flat. So, they are probably living in $\mathbf{A}^2(\mathbb{R})$. Everybody else understands that the theories which we develop to understand the world around us need to be constantly refined. The 3-dimensional space $\mathbf{A}^3(\mathbb{R})$ is a good model for the observed space, good enough to formulate classical mechanics, but even there it is customary to work with $\mathbf{A}^n(\mathbb{R})$ in general. New theories use higher dimensional theoretical models to explain different physical phenomena.

But geometry is not only about the observed space. Sometimes you just have a lot of data and in order to make sense of it you think about it as points in $\mathbf{A}^{10000}(\mathbb{R})$ or some higher dimensional space. This is common when doing machine learning for instance. In such situations, most of the time your data sits in a very high dimensional spaces and this gives everybody a headache. People call this 'the curse of dimensionality' and try to develope methods for 'dimensionality reduction'. When you are in a situation like this you need to have some understanding of what an n-dimensional affine space is.

Cryptography is another area where geometric intuition is useful. In this context one can view codes as points in the affine space $\mathbf{A}^n(\mathbb{F}_p)$ where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p elements. Some methods in cryptography use the geometric intuition that we have from curves in the Euclidean plane and work with curves in $\mathbf{A}^2(\mathbb{F}_p)$ (or a space that contains this 2-dimensional affine space).