CHAPTER 9

Euclidean spaces

Contents

In these notes we consider only real vector spaces, so $\mathbf{K} = \mathbb{R}$.

9.1 Euclidean spaces

Definition 9.1. Let $\mathbf{V}$ be a real vector space. A positive definite symmetric bilinear form on $\mathbf{V}$ is called a *scalar product*, or *inner product*. If $\mathbf{V}$ is endowed with a scalar product then the affine space with associated vector space $\mathbf{V}$ is called an *Euclidean space*. Moreover, we call $\mathbf{V}$ a *Euclidean vector space*.

- If $\mathbf{V}$ is finite dimensional, of dimension n, then it is unique up to isomorphism, which is why we denote an n-dimensional Euclidean space by $\mathbf{E}^n$.
- $\mathbf{E}^2$ is the Euclidean plane and $\mathbf{E}^3$ is the 3-dimensional Euclidean space which you considered last semester.
- Moreover, scalar products are more often denoted by $\langle \_, \_ \rangle : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$.

- The standard bilinear form on $\mathbb{R}^n$ is a scalar product called the *standard scalar product* on $\mathbb{R}^n$. Recall, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then
$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n = \mathbf{x}^t \mathbf{y}.$$

Theorem 9.2 (Schwarz Inequality)**.** Let $\mathbf{V}$ be a Euclidean vector space. If $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ then
$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle$$
with equality if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.

- For $\mathbf{v} \in \mathbf{V}$ the *length* or *norm* of the vector $\mathbf{v}$ is
$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- Thus, the Schwartz inequality can be expressed as
$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq ||\mathbf{v}|| \cdot ||\mathbf{w}||.$$

Proposition 9.3. The norm of a vector enjoys the following three properties

(N1) $||\mathbf{v}|| \geq 0$ for any $\mathbf{v} \in \mathbf{V}$, with equality if and only if $\mathbf{v} = 0$.

(N2) $||r\mathbf{v}|| = |r| \cdot ||\mathbf{v}||$ for every $\mathbf{v} \in \mathbf{V}$ and $r \in \mathbb{R}$.

(N3) $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$, for every $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ with equality if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.

- These properties are generally taken as axioms for norms and normed spaces.
- They induce a distance between points. If A, B are two points then the distance between A and B is
$$d(A, B) = ||\overrightarrow{AB}||.$$
- Everything that you deduced last semester using only the scalar product is valid for $\mathbf{E}^2$ and $\mathbf{E}^3$ as we defined it here.

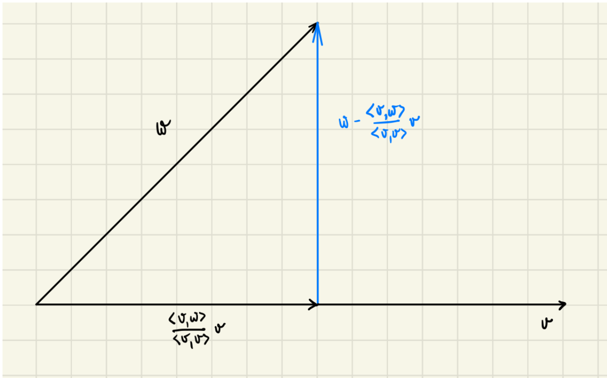
Proposition 9.4. Any set of mutually orthogonal vectors is linearly independent. In particular, if $\dim(\mathbf{V}) = n$ is finite, any set of mutually orthogonal vectors is a basis of $\mathbf{V}$.

Proposition 9.5. Let $\boldsymbol{e} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\boldsymbol{f} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ be two bases of the Euclidean vector space $\mathbf{V}$, and suppose that $\boldsymbol{e}$ is orthonormal. The basis $\boldsymbol{f}$ is orthonormal if and only if the base change matrix $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}}$ is orthogonal.

9.1.1 Gram-Schmidt process

Definition 9.6. Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathbf{V}$. We define the *orthogonal projection of* $\mathbf{w}$ *on* $\mathbf{v}$ to be
$$\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}.$$

Then $\mathbf{w}' = \mathbf{w} - \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$ is orthogonal on $\mathbf{v}$ since
$$\langle \mathbf{w}', \mathbf{v} \rangle = \langle \mathbf{w} - \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle - \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \langle \mathbf{v}, \mathbf{v} \rangle = 0.$$



If we have a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbf{V}$, how do we obtain an orthonormal basis? We can do this in two steps:

1. Construct an orthogonal basis $\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n$ as follows

$$\begin{aligned} \mathbf{e}'_1 &= \mathbf{e}_1 \\ \mathbf{e}'_2 &= \mathbf{e}_2 - \frac{\langle \mathbf{e}'_1, \mathbf{e}_2 \rangle}{\langle \mathbf{e}'_1, \mathbf{e}'_1 \rangle} \mathbf{e}'_1 \\ \mathbf{e}'_3 &= \mathbf{e}_3 - \frac{\langle \mathbf{e}'_1, \mathbf{e}_3 \rangle}{\langle \mathbf{e}'_1, \mathbf{e}'_1 \rangle} \mathbf{e}'_1 - \frac{\langle \mathbf{e}'_2, \mathbf{e}_3 \rangle}{\langle \mathbf{e}'_2, \mathbf{e}'_2 \rangle} \mathbf{e}'_2 \\ &\vdots \end{aligned}$$

2. Normalize the vectors to obtain the basis

$$\left\{ \frac{1}{\|\mathbf{e}'_1\|} \mathbf{e}'_1, \ldots, \frac{1}{\|\mathbf{e}'_n\|} \mathbf{e}'_n \right\}.$$

This process of obtaining an orthonormal basis from a given basis is called the *Gram-Schmidt process*.

9.2 Isometries

Definition 9.7. An *isometry* is a map $\phi : \mathbf{E}^n \to \mathbf{E}^n$ which preserves distances, i.e.

$$d(\phi(P), \phi(Q)) = d(P, Q)$$

for any points $P, Q \in \mathbf{E}^n$.

- One can show that isometries are affine transformations, i.e. they are elements in $\mathrm{AGL}(\mathbf{E}^n)$.

Proposition 9.8. Let $\phi \in \mathrm{AGL}(\mathbf{E}^n)$ be an affine transformation given by $\phi(\mathbf{x}) = A\mathbf{x} + b$ with respect to some orthonormal coordinate system. The following are equivalent:

1. ϕ is an isometry

2. $A = A^t$.

Proposition 9.9. Let $A \in \text{Mat}_{n\times n}(\mathbb{R})$ be a matrix such that $A^t A = I_n$. Then $\det(A) \in \{\pm 1\}$.

Definition 9.10. A matrix $A \in \text{Mat}_{n\times n}(\mathbb{R})$ such that $A^t A = I_n$ is called *orthogonal*. The set of all such matrices are denoted by $O(n)$. The set of matrices in $O(n)$ with determinant 1 is denoted by $SO(n)$. Such matrices are called *special orthogonal*.

The set $O(n)$ is a subgroup of $\text{AGL}(\mathbb{R}^n)$ and $SO(n)$ is a normal subgroup of $O(n)$:

$$SO(n) \trianglelefteq O(n) \leq \text{AGL}(\mathbb{R}^n).$$

Let $\phi \in \text{AGL}(\mathbf{E}^n)$ be given by $\phi(\mathbf{x}) = A\mathbf{x} + b$ with respect to some orthonormal coordinate system. Then ϕ is called a *displacement*, or a *direct isometry*, if $A \in SO(n)$.

9.2.1 Rotations in dimension 2

Proposition 9.11. A matrix A is in $SO(2)$ if and only if A has the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some $\theta \in \mathbb{R}$.

Corollary 9.12. A direct isometry ϕ of $\mathbf{E}^2$ that fixes a point is either the identity or a rotation. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\text{tr}(\text{lin}(\phi))}{2}.$$

9.2.2 Rotations in dimension 3

Theorem 9.13 (Euler)**.** A direct isometry ϕ of $\mathbf{E}^3$ that fixes a point is either the identity or a rotation around an axis that passes through that point. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\text{tr}(\text{lin}(\phi)) - 1}{2}.$$

9.2.3 Classification of isometries

Theorem 9.14 (Chasles)**.** An isometry of the plane $\mathbf{E}^2$ is either a direct isometry, in which case it is

- the identity, or
- a translation, or
- a rotation;

or an indirect isometry, in which case it is

- a reflection, or
- a glidereflection.

Theorem 9.15 (Euler)**.** Any isometry of the 3-dimensional Euclidean space $\mathbf{E}^3$ is either a direct isometry, in which case it is

- the identity, or
- a translation, or
- a rotation around an axis, or
- a gliderotation (also called helical displacement);

or an indirect isometry, in which case it is

- a reflection, or
- a glidereflection, or
- a rotation-reflection.

9.3 Spectral theorem

- We saw in Proposition 9.5, that if we change the coordinate system from an orthonormal basis $\boldsymbol{e}$ to an orthonormal basis $\boldsymbol{f}$ then the base change matrix $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}}$ is in $O(n)$.
- For a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and a matrix $M \in O(n)$, since $M^{-1} = M^t$ we have

$$M^{-1}AM = M^t AM.$$

  So, the above matrix is simultaneously congruent and similar to A.
- Notice that if we change the order of two vectors in the basis $\boldsymbol{f}$ then we change the sign of $\det(M_{\boldsymbol{e},\boldsymbol{f}})$. Notice also that if we change the sign of one vector in $\boldsymbol{f}$ we change the sign of $\det(M_{\boldsymbol{e},\boldsymbol{f}})$. So, changing orthonormal coordinate systems can be performed with matrices in $SO(n)$ if we give ourselves the freedom of interchanging two axes or of changing the direction of one axis.
- The above observation is of interest because coordinate changes performed with matrices in $SO(n)$ are displacements. Such transformations are close to our intuition since they correspond to the usual movements that we do/see in our surroundings.
- The next statements show how a symmetric operator can be diagonalized with matrices in $SO(n)$.

Lemma 9.16. The characteristic polynomial of a symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ has only real roots.

Theorem 9.17. (Spectral Theorem) Let $\mathbf{V}$ be the associated vector space of the Euclidean space $\mathbf{E}^n$ and let $T : \mathbf{V} \to \mathbf{V}$ be a symmetric operator. There is an orthonormal basis of $\mathbf{V}$ with respect to which the matrix of T is diagonal.

Theorem 9.18. For every real symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ there is an orthogonal matrix $M \in O(n)$ such that $M^{-1}AM$ is diagonal.

Theorem 9.19. For every quadratic form $q : \mathbf{V} \to \mathbb{R}$ on a finite dimensional Euclidean space, there is a diagonalizing orthonormal basis.

Proposition 9.20. Let $T : \mathbf{V} \to \mathbf{V}$ be a symmetric operator on a Euclidean vector space. If λ and μ are two distinct eigenvalues of T then every eigenvector with eigenvalue λ is orthogonal to every eigenvector with eigenvalue μ.