Analytic Geometry

George Țurcaș

Maths & Comp. Sci., UBB Cluj-Napoca

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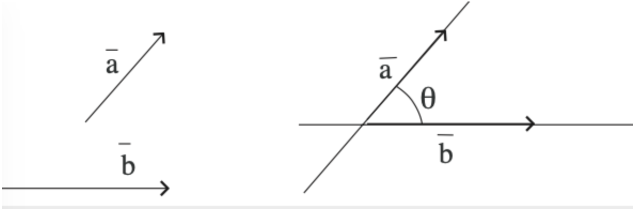
Dot product (= „produs scalar")

The angle between two nonzero vectors $\overline{a}$ and $\overline{b}$ from V_2 or V_3 is defined as the angle $\theta = \widehat{(\overline{a}, \overline{b})} \in [0, \pi]$ determined by their directions, taking into account their orientations.



Given the vectors $\overline{a}$ and $\overline{b}$ in V_2 (or V_3), their **dot product** is the real number defined through

$$\overline{a} \cdot \overline{b} = \begin{cases} |\overline{a}||\overline{b}| \cos\theta, & \text{if } \overline{a} \neq 0 \text{ and } \overline{b} \neq 0 \\ 0, & \text{otherwise} \end{cases} .$$

Does $\mathbb{R}^3$ (or $\mathbb{R}^n$) "know" any geometry?

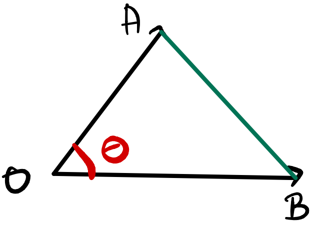
(a_1, a_2, a_3) $(a_1, \ldots, a_n)$

Theorem

1. *If* $\overline{a}(a_1, a_2)$ *and* $\overline{b}(b_1, b_2)$ *are two vectors in* V_2*, then*
$$\overline{a} \cdot \overline{b} = a_1 b_1 + a_2 b_2; \tag{1}$$
2. *If* $\overline{a}(a_1, a_2, a_3)$ *and* $\overline{b}(b_1, b_2, b_3)$ *are two vectors in* V_3*, then*
$$\overline{a} \cdot \overline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{2}$$

Remark: $\overline{b} \cdot \overline{b} = b_1^2 + b_2^2 + b_3^2 = |\overline{b}|^2.$

Proof. Let $\overrightarrow{OA} \in \bar{a}$ and $\overrightarrow{OB} \in \bar{b}$.



From the cosine thm. in ΔOAB, we have:

$$AB^2 = |\bar{a}|^2 + |\bar{b}|^2 - 2 \cdot |\bar{a}| \cdot |\bar{b}| \cdot \cos\theta.$$

Hence

$$\bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cdot \cos\theta = \frac{1}{2}\left(|\bar{a}|^2 + |\bar{b}|^2 - |AB|^2\right)$$

~~Proof.~~

Suppose we are in V_3 (case 2) and O in the origin of the system of coordinates

$A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$

$$\bar{a} \cdot \bar{b} = \frac{1}{2}\Big((a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - (b_1 - a_1)^2 - (b_2 - a_2)^2 - (b_3 - a_3)^2 \Big)$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$\boxtimes$

Since $\cos\theta = \dfrac{\overline{a}\cdot\overline{b}}{|\overline{a}||\overline{b}|}$, then, for two nonzero vectors $\overline{a}$ and $\overline{b}$, one has

$$\cos\widehat{(\overline{a},\overline{b})} = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2+a_2^2}\cdot\sqrt{b_1^2+b_2^2}}, \text{ for } \overline{a},\overline{b}\in V_2; \tag{3}$$

$$\cos\widehat{(\overline{a},\overline{b})} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}, \text{ for } \overline{a},\overline{b}\in V_3. \tag{4}$$

<u>Remark</u>: If $\overline{a}(a_1,\dots,a_n)$, $\overline{b}(b_1,\dots,b_n)$ then we can define θ, the $\measuredangle$ between $\overline{a}$ and $\overline{b}$: $\begin{cases} \theta\in[0,\pi] \text{ and} \\ \cos\theta = \dfrac{\overline{a}\cdot\overline{b}}{|\overline{a}|\cdot|\overline{b}|}. \end{cases}$

Theorem

If $\overline{u}$ and $\overline{v}$ are nonzero vectors in V_2 (or V_3) and θ is the angle between them, then

$\overline{u} \cdot \overline{v} = |\overline{u}| \cdot |\overline{v}| \cdot \cos\theta.$

a) *θ is acute if and only if $\overline{u} \cdot \overline{v} > 0$;*

b) *θ is obtuse if and only if $\overline{u} \cdot \overline{v} < 0$;*

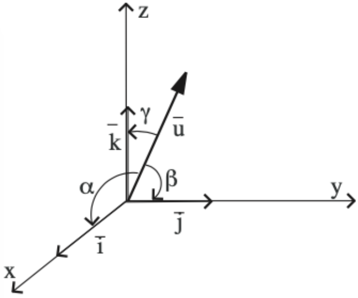
c) *$\theta = \dfrac{\pi}{2}$ if and only if $\overline{u} \cdot \overline{v} = 0$.*

Proof. The sign of the cosine of θ coincides with the sign of the dot product $\overline{u} \cdot \overline{w}$. The assertions follow trivially. □

The notions of "acute", "obtuse" or orthogonal (perpendicular) can be generalized to vectors with more than 3 components using the algebraic form of the dot product, even if there's no obvious "geometrical" interpretation.

The 3 axes determine 3 angles

Given an arbitrary vector $\overline{u} \in V_3$ and an associated Cartesian system of coordinates, one defines the *director angles* of $\overline{u}$ to be the three angles determined by $\overline{u}$ with the versors of the system of coordinates $\alpha = \widehat{(\overline{u}, \overline{i})}$, $\beta = \widehat{(\overline{u}, \overline{j})}$ and $\gamma = \widehat{(\overline{u}, \overline{k})}$, respectively.



The values $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are sometimes called *director cosines* of the vector $\overline{u}$.

Theorem

The director cosines of a vector $\overline{u}(u_1, u_2, u_3) \in V_3$, $\overline{u} \neq \overline{0}$, are

$$\cos\alpha = \frac{u_1}{|\overline{u}|},\ \cos\beta = \frac{u_2}{|\overline{u}|},\ \cos\gamma = \frac{u_3}{|\overline{u}|}. \tag{5}$$

Proof.

Take $\overline{i}$.

(1) $\overline{i} \cdot \overline{u} = |\overline{i}| \cdot |\overline{u}| \cdot \cos\alpha = |\overline{u}| \cdot \cos\alpha$.

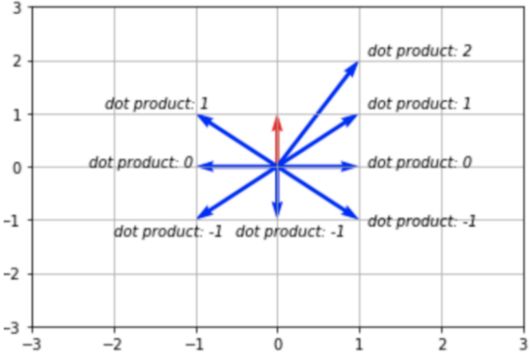
(2) $\underset{(1,0,0)}{\overline{i}} \cdot \underset{(u_1,u_2,u_3)}{\overline{u}} = 1 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 = u_1$.

From (1) and (2), $\cos\alpha = \dfrac{u_1}{|\overline{u}|}$.

Exercise

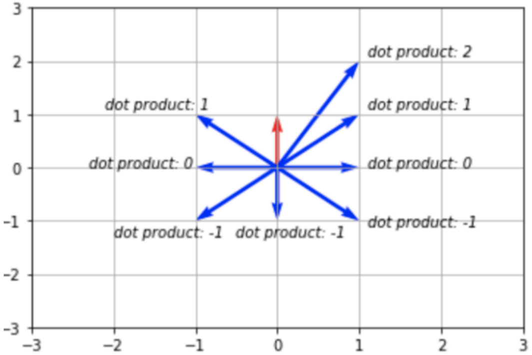
↑ - fixed .

$|\bar{u}| \cdot |\bar{v}| \cdot \cos\theta$.

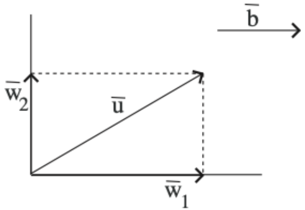


- Dot product is larger when the direction of the blue vectors is similar to the direction of the red one.

Exercise



- Dot product is larger when the direction of the blue vectors is similar to the direction of the red one.
- Dot product is larger when the magnitude of the blue vector is larger.



Let $\overline{u}$ and $\overline{b}$ be two nonzero vectors and project (orthogonally) a representative of the vector $\overline{u}$ on a line passing through the original point of this representative and parallel to the direction of $\overline{b}$. One gets the vector $\overline{w}_1$, having the direction of $\overline{b}$ and, by making the difference $\overline{u} - \overline{w}_1$, another vector $\overline{w}_2$, orthogonal on the direction of $\overline{b}$; $\overline{u} = \overline{w}_1 + \overline{w}_2$.

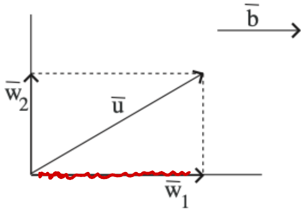
Projections

- The vector $\overline{w}_1$ is called the *orthogonal projection of $\overline{u}$ on $\overline{b}$* and it is denoted by $\mathrm{pr}_{\overline{b}}\overline{u}$.
- The vector $\overline{w}_2$ is called the *vector component of $\overline{u}$ orthogonal to $\overline{b}$* and $\overline{w}_2 = \overline{u} - \mathrm{pr}_{\overline{b}}\overline{u}$.

Theorem

If $\overline{u}$ and $\overline{b}$ are vectors in V_2 or V_3 and $\overline{b} \neq 0$, then

- *the orthogonal projection of $\overline{u}$ on $\overline{b}$ is* $pr_{\overline{b}}\overline{u} = \dfrac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$;
- *the vector component of $\overline{u}$ orthogonal to $\overline{b}$ is* $\overline{u}\text{-}pr_{\overline{b}}\overline{u} = \overline{u} - \dfrac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$.



$\overline{w}_1 = pr_{\overline{b}}(\overline{u})$.

Proof of the previous theorem

$\bar{w}_1$ is parallel to $\bar{b}$. $\exists\, k \in \mathbb{R}$ such that

$\bar{w}_1 = k \cdot \bar{b}$.

dot product

$$\bar{u} = k \cdot \bar{b} + \bar{w}_2 \quad \Big| \cdot \bar{b}$$

$$\bar{u} \cdot \bar{b} = (k \cdot \bar{b} + \bar{w}_2) \cdot \bar{b} \iff$$

$$\bar{u} \cdot \bar{b} = k \cdot \bar{b} \cdot \bar{b} \Rightarrow k = \frac{\bar{u} \cdot \bar{b}}{|b|^2}.$$

$\blacksquare$

The length of the orthogonal projection of the vector $\overline{u}$ on $\overline{b}$ can be obtained as following:

$$|\mathrm{pr}_{\overline{b}}\overline{u}| = \left|\frac{\overline{u}\cdot\overline{b}}{|\overline{b}|^2}\cdot\overline{b}\right| = \left|\frac{\overline{u}\cdot\overline{b}}{|\overline{b}|^2}\right||\overline{b}|,$$

which yields

$$|\mathrm{pr}_{\overline{b}}\overline{u}| = \frac{|\overline{u}\cdot\overline{b}|}{|\overline{b}|}, = \frac{\left| |\overline{u}|\cdot\cancel{|\overline{b}|}\cdot\cos\theta \right|}{\cancel{|\overline{b}|}}.$$

and if θ is the angle between $\overline{u}$ and $\overline{b}$, then

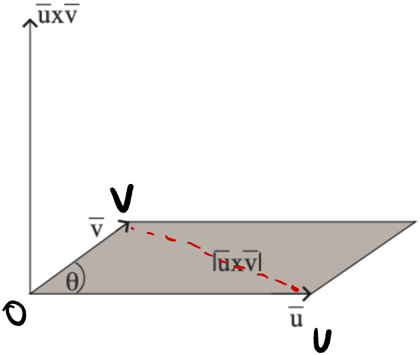
$$|\mathrm{pr}_{\overline{b}}\overline{u}| = |\overline{u}||\cos\theta|.$$

The cross product

Definition

The cross product of two vectors $\overline{u}$ and $\overline{v}$ is another vector $\overline{u} \times \overline{v}$, which can be determined by the following conditions:

- If $\overline{u}$ and $\overline{v}$ are colinear, then $\overline{u} \times \overline{v} := \overline{0}$;
- Else, let $0 < \theta < \pi$ be the angle between them. The vector $\overline{u} \times \overline{v}$ is such that:
  1. $||\overline{u} \times \overline{v}|| = ||\overline{u}|| \cdot ||\overline{v}|| \cdot \sin(\theta)$;
  2. $\overline{u} \times \overline{v}$ is perpendicular on $\overline{u}$ and on $\overline{v}$; (we need to be in V_3)
  3. the orientation of $\overline{u} \times \overline{v}$ is given by the right-hand rule.



- If the vectors $\overline{u}, \overline{v}$ are not collinear, then if $\overrightarrow{OU} \in \overline{u}$ and $\overrightarrow{OV} \in \overline{v}$, then $||\overline{u} \times \overline{v}||$ is the area of the parallelogram formed by $\overrightarrow{OU}$ and $\overrightarrow{OV}$.
- The area of the triangle $\triangle OAB$ can be computed as

$$\mathrm{Area}_{\triangle OAB} = \frac{||\overline{u} \times \overline{v}||}{2}.$$

The algebraic form of the cross product

If $\overline{u} = u_1\overline{i} + u_2\overline{j} + u_3\overline{k}$ and $\overline{v} = v_1\overline{i} + v_2\overline{j} + v_3\overline{k}$ are vectors in V_3, then their *cross product* $\overline{u} \times \overline{v}$ is the vector

$$\overline{u} \times \overline{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \overline{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \overline{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \overline{k}, \tag{6}$$

or, shortly,

$$\overline{u} \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{7}$$

$\overline{u} \times \overline{v} = -\overline{v} \times \overline{u}$, since one permutes lines 2 & 3 in the det.

Did we defined the same thing?

Let $\overline{u}(u_1, u_2, u_3)$ and $\overline{v}(v_1, v_2, v_3)$. Using the algebraic definition, we get $\overline{u} \times \overline{v}(u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$.

- $\overline{u} \cdot (\overline{u} \times \overline{v}) = 0$, so $\overline{u} \times \overline{v}$ is orthogonal on $\overline{u}$;

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- $\overline{u} \cdot (\overline{u} \times \overline{v}) = 0$, so $\overline{u} \times \overline{v}$ is orthogonal on $\overline{u}$; Indeed, notice that

$$\overline{u} \cdot (\overline{u} \times \overline{v}) = u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) = 0.$$

- Similarly, $\overline{v} \cdot (\overline{u} \times \overline{v}) = 0$.

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$$\overline{u} \cdot (\overline{u} \times \overline{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0.$$

- Similarly, $\overline{v} \cdot (\overline{u} \times \overline{v}) = 0$.
- We have that

$$|\overline{u} \times \overline{v}|^2 = |\overline{u}|^2|\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2$$

(Lagrange's identity).

To prove Lagrange's identity, one just has to open the brackets and check that

$$|\overline{u} \times \overline{v}|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

equals to

$$|\overline{u}|^2 |\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2.$$

Using Lagrange's identity,

$$|\overline{u} \times \overline{v}|^2 = \underbrace{|\overline{u}|^2 |\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2}_{\geq 0} = |\overline{u}|^2 |\overline{v}|^2 - |\overline{u}|^2 |\overline{v}|^2 \cos^2 \theta = |\overline{u}|^2 |\overline{v}|^2 \sin^2 \theta.$$

Are you convinced that the cross product defined geometrically and the cross product defined algebraically are one and the same?

$$(u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \leq (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$$

An immediate consequence of the Lagrange's identity is that $|\overline{u}|^2|\overline{v}|^2 - (\overline{u}\cdot\overline{v})^2 \geq 0$, or $|\overline{u}\cdot\overline{v}| \leq |\overline{u}||\overline{v}|$, which leads, after replacing the components of the vectors, to the Cauchy-Schwartz inequality. The equality $|\overline{u}\cdot\overline{v}| = |\overline{u}||\overline{v}|$ holds if and only if the vector $\overline{u}\times\overline{v}$ is the zero vector, i.e. its components are all zero, which happens if and only if $\dfrac{v_1}{u_1} = \dfrac{v_2}{u_2} = \dfrac{v_3}{u_3} = \lambda$, or $\overline{v} = \lambda\overline{u}$, $\lambda \in \mathbb{R}^*$. In summary, one has:

Theorem

If $\overline{u}$ and $\overline{v}$ are nonzero vectors in V_3, then $\overline{u}\times\overline{v} = \overline{0}$ if and only if $\overline{u}$ and $\overline{v}$ are parallel.

More properties of the cross product

For any vectors $\overline{u}$, $\overline{v}$ and $\overline{w}$ from V_3 and any scalar $\lambda \in \mathbb{R}$, the following equalities hold:

a) $\overline{u} \times \overline{v} = -\overline{v} \times \overline{u}$; (not commutative)

b) $\overline{u} \times (\overline{v} + \overline{w}) = \overline{u} \times \overline{v} + \overline{u} \times \overline{w}$;

c) $(\overline{u} + \overline{v}) \times \overline{w} = \overline{u} \times \overline{w} + \overline{v} \times \overline{w}$;

d) $\lambda(\overline{u} \times \overline{v}) = (\lambda \overline{u}) \times \overline{v} = \overline{u} \times (\lambda \overline{v})$;

e) $\overline{u} \times \overline{0} = \overline{0} \times \overline{u} = \overline{0}$;

f) $\overline{u} \times \overline{u} = \overline{0}$.

Some easy examples

It is very easy to compute the cross products of the versors of the axes:

$$\begin{array}{ccc} \bar{i} \times \bar{j} = \bar{k} & \bar{j} \times \bar{k} = \bar{i} & \bar{k} \times \bar{i} = \bar{j} \\ \bar{j} \times \bar{i} = -\bar{k} & \bar{k} \times \bar{j} = -\bar{i} & \bar{i} \times \bar{k} = -\bar{j} \\ \bar{i} \times \bar{i} = \bar{0} & \bar{j} \times \bar{j} = \bar{0} & \bar{k} \times \bar{k} = \bar{0} \end{array}.$$

Some observations

The cross product shares a few similarities with the dot product. However, there are some differences which you have to remember:

1. The cross product is not commutative. In fact, it is anti-commutative.
2. The cross product of two vectors is a vector, not a scalar (as it is the case for the result of a dot product). Therefore, it makes sense to consider products with multiple factors. One should be very careful with those, since the cross product is not associative either :)

A closer look at a high-school formula

In high-school, you probably learned how to compute the area of a triangle determined by $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$.

- Let see these in $3D$ and assume WLOG they line in the plane xOy.
- We therefore have $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$ and $C(x_C, y_C, 0)$. These points determine the vectors $\overline{AB}(x_B - x_A, y_B - y_A, 0)$ and $\overline{AC}(x_C - x_A, y_C - y_A, 0)$.
- Computing, we have

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix} = \bar{k} \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix},$$

$$\overline{AB} \times \overline{AC} = \bar{k} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

It follows that

$$||\overline{AB} \times \overline{AC}|| = \pm \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix},$$

hence

$$\mathrm{Area}_{\triangle ABC} = \pm\frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

The problem set for this week will be posted soon. Ideally you would think about it before the seminar on Friday.

Thank you very much for your attention!