COURSE 9

Subspaces. The generated subspace

Let $(K,+,\cdot)$ be a field. Throughout this course this condition on K will always be valid.

We remind that:

- A **K-vector space** is an Abelian group $(V,+)$ with an external operation

$$\cdot : K\times V\to V\,,\quad (k,v)\mapsto k\cdot v\,,$$

satisfying the following axioms: for any $k,k_1,k_2\in K$ and any $v,v_1,v_2\in V$,

(L_1) $k\cdot(v_1+v_2)=k\cdot v_1+k\cdot v_2$;
(L_2) $(k_1+k_2)\cdot v=k_1\cdot v+k_2\cdot v$;
(L_3) $(k_1\cdot k_2)\cdot v=k_1\cdot(k_2\cdot v)$;
(L_4) $1\cdot v=v$.

- If V is a vector space over K, a subset $S\subseteq V$ is a **subspace** of V (and we write $S\le_K V$) if:

(1) S is closed with respect to the addition of V and to the scalar multiplication, that is,

$$\forall x,y\in S\,,\quad x+y\in S\,,$$

$$\forall k\in K\,,\ \forall x\in S\,,\ kx\in S\,.$$

(2) S is a vector space over K with respect to the induced operations of addition and scalar multiplication. ← from ${}_K V$

- If $S\le_K V$ then S contains the zero vector of V, i.e. $0\in S$.

We have the following **characterization theorem for subspaces**.

Theorem 1. Let V be a vector space over K and let $S\subseteq V$. The following conditions are equivalent:

1) $S\le_K V$.
2) The following conditions hold for S:
α) $0\in S$;
β) $\forall x,y\in S\,,\ x+y\in S$;
γ) $\forall k\in K\,,\ \forall x\in S\,,\ kx\in S\,.$
3) The following conditions hold for S:
α) $0\in S$;
δ) $\forall k_1,k_2\in K\,,\ \forall x,y\in S\,,\ k_1x+k_2y\in S\,.$ ← linear comb. of x_1, x_2

Proof. 1) $\Rightarrow$ 2) is obvious. □

2) $\Rightarrow$ 1) β) and γ) $\Rightarrow$ S is closed in V with respect to the vector addition and the scalar multiplication

$(S,+)$ ← ind. op. Abelian group (?) and the induced $\cdot : K\times S\to S$ satisfies $L_1)-L_4)$ (?) ← obviously.

+ assoc., comm.

α) → $0\in S \Rightarrow 0$ is the additive id. elem. of $(S,+)$. $\Big\}\Longrightarrow$

$\forall x\in S$, $-x=(-1)\cdot x \overset{\gamma)}{\in} S \Rightarrow -x$ is the opp. of x in the monoid $(S,+)$

the opp. of x in $(V,+)$; $\in K$; $S\subseteq V$

$\Rightarrow$ $(S,+)$ Abelian group
$\cdot : K\times S \to S$ verif. L1)–L4) $\Big\} \Longrightarrow S \leq_K V.$

2) $\Rightarrow$ 3) $\forall\, k_1, k_2 \in K,\ \forall\, x, y \in S \xRightarrow{\gamma)} k_1x,\ k_2y \in S \Longrightarrow$

$\Longrightarrow k_1x + k_2y \in S \Rightarrow \delta)$

3) $\to$ 2) If we take $k_1 = k_2 = 1$ in $\delta) \Longrightarrow \beta)$.
If we take $k_2 = 0$ in δ (and $k_1 = k$) $\Longrightarrow \gamma)$.

Let us denote by $\alpha')$ $S \neq \emptyset$

$\underline{\alpha)}, \beta), \gamma) \Longleftrightarrow \alpha'), \beta), \gamma)$
$\Longrightarrow$
$\Longleftarrow$?

$S \neq \emptyset \Rightarrow \exists\, x_0 \in S \xRightarrow{\gamma)} -x_0 = \underset{\in K}{(-1)}\, \underset{\in S}{x_0} \in S \Longrightarrow$

$\xRightarrow{\beta)} \underset{\in V}{0} = \underset{\in S}{x_0} + \underset{\in S}{(-x_0)} \in S \Rightarrow \alpha)$

Homework: Rewrite the previous thm. by replacing $\alpha)$ with $\alpha')$.

Remark 2. (1) One can replace $\alpha)$ in the previous theorem with $\underline{S \neq \emptyset}$.
$\to$ (2) If $S \leq_K V$, $k_1, \ldots, k_n \in K$ and $x_1, \ldots, x_n \in S$ then $k_1x_1 + \cdots + k_nx_n \in S$. (hw – by way of induction on $n \in \mathbb{N}^*$)

Examples 3. (a) Every non-zero vector space V over K has two subspaces, namely $\{0\}$ and V. They are called the **trivial subspaces**.
(b) Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

homework!

Then S and T are subspaces of the real vector space $\mathbb{R}^3$. $({}_{\mathbb{R}}\mathbb{R}^3)$

$\alpha)$ $(0,0,0) \in S$ because $0+0+0=0$

$\beta)$ $(x,y,z), (x',y',z') \in S$, $(x,y,z)+(x',y',z') \in S$?

$(x,y,z)+(x',y',z') = (x+x', y+y', z+z') \overset{?}{\in} S$

$(x+x') + (y+y') + (z+z') = \underbrace{(x+y+z)}_{=0} + \underbrace{(x'+y'+z')}_{=0} = 0$

$\delta)$ $\alpha \in \mathbb{R}$, $(x,y,z) \in S$, $\alpha \cdot (x,y,z) \overset{?}{\in} S$

$\alpha(x,y,z) = (\alpha x, \alpha y, \alpha z) \overset{?}{\in} S$

$\alpha x + \alpha y + \alpha z = \alpha(\underbrace{x+y+z}_{=0}) = 0$

Thus $S \leq_{\mathbb{R}} \mathbb{R}^3$.

(c) Let $n \in \mathbb{N}$ and let

$$K_n[X] = \{f \in K[X] \mid \deg f \leq n\}.$$

Then $K_n[X]$ is a subspace of the polynomial vector space $K[X]$ over K. (during the seminar)

(d) Let $I \subseteq \mathbb{R}$ be an interval. The set $\mathbb{R}^I = \{f \mid f : I \to \mathbb{R}\}$ is a $\mathbb{R}$-vector space with respect to the following operations

$$(f+g)(x) = f(x) + g(x), \ (\alpha f)(x) = \alpha f(x)$$

$\theta : I \to \mathbb{R}$
$\theta(x) = 0$ is the additive id. elem. (the zero vector of $\mathbb{R}^I$)

with $f, g \in \mathbb{R}^I$ and $\alpha \in \mathbb{R}$. The subsets

$$C(I,\mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ continuous on } I\}, \ D(I,\mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ derivable on } I\}$$

are subspaces of $\mathbb{R}^I$ since they are nonempty and

$\theta \in C(I,\mathbb{R}) \cap D(I,\mathbb{R})$ and

$$\alpha, \beta \in \mathbb{R}, \ f, g \in C(I,\mathbb{R}) \Rightarrow \alpha f + \beta g \in C(I,\mathbb{R}); \leftarrow$$

$$\alpha, \beta \in \mathbb{R}, \ f, g \in D(I,\mathbb{R}) \Rightarrow \alpha f + \beta g \in D(I,\mathbb{R}). \leftarrow$$

Theorem 4. Let I be a nonempty set, V be a vector space over K and let $(S_i)_{i\in I}$ be a family of subspaces of V. Then $\bigcap_{i\in I} S_i \leq_K V$.

Proof. □

$\alpha)$ $0 \overset{?}{\in} \bigcap_{i\in I} S_i \Leftarrow 0 \in S_i$, $\forall i \in I$ since $S_i \leq_K V$

$\beta)$ $\forall k_1, k_2 \in K$, $\forall x, y \in \bigcap_{i\in I} S_i$, $k_1 x + k_2 y \overset{?}{\in} \bigcap_{i\in I} S_i$.

$\forall i \in I$, $x, y \in S_i$, $k_1, k_2 \in K \underset{S_i \leq_K V}{\Longrightarrow} k_1 x + k_2 y \in S_i$, $\forall i \in I \Longrightarrow$

$\Longrightarrow k_1 x + k_2 y \in \bigcap_{i\in I} S_i$.

$\alpha), \beta)$ hold for $\bigcap_{i\in I} S_i \Longrightarrow \bigcap_{i\in I} S_i \leq_K V$.

Remark 5. In general, the union of two subspaces is not a subspace.

For instance, ... Let us consider the $\mathbb{R}$-v.s. $\mathbb{R}^2$ and.

$S = \{(x,0) \mid x \in \mathbb{R}\} \leq_{\mathbb{R}} \mathbb{R}^2$, $S \cup T \overset{?}{\leq_{\mathbb{R}}} \mathbb{R}^2$

$T = \{(0,y) \mid y \in \mathbb{R}\} \leq_{\mathbb{R}} \mathbb{R}^2$

$\underset{\in S}{(1,0)} + \underset{\in T}{(0,1)} = (1,1) \notin S \cup T$.

$S \subseteq S \cup T \supseteq T$

Next, we will see how to complete a subset of a vector space to a subspace in a minimal way.

Definition 6. Let V be a vector space and let $X \subseteq V$. We denote

Thm 4.

$$\langle X \rangle = \bigcap \{S \leq_K V \mid X \subseteq S\} \leq_K V$$

and we call it the **subspace generated (or spanned) by** X. The set X is the **generating set** of $\langle X \rangle$. If $X = \{x_1, \ldots, x_n\}$, we denote $\langle x_1, \ldots, x_n \rangle = \langle \{x_1, \ldots, x_n\} \rangle$.

Remarks 7. (1) $\langle X \rangle$ is the smallest subspace of V (with respect to $\subseteq$) which contains X.
(2) Notice that $\langle \emptyset \rangle = \{0\}$. $\langle \emptyset \rangle = \{0\} = \langle 0 \rangle$
(3) If V is a K-vector space, then:
(i) If $S \leq_K V$ then $\langle S \rangle = S$.
(ii) If $X \subseteq V$ then $\langle \langle X \rangle \rangle = \langle X \rangle$.
(iii) If $X \subseteq Y \subseteq V$ then $\langle X \rangle \subseteq \langle Y \rangle$. ← homework.

Definition 8. A K-vector space V is **finitely generated** if there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in V$ such that $V = \langle x_1, \ldots, x_n \rangle$. The set $\{x_1, \ldots, x_n\}$ is also called **system of generators** for V.

Definition 9. Let V be a K-vector space. A finite sum of the form

$$k_1 x_1 + \cdots + k_n x_n\,,$$

with $k_1, \ldots, k_n \in K$ and $x_1, \ldots, x_n \in V$, is called a **linear combination** of the vectors $x_1, \ldots, x_n$.

Let us show how the elements of a generated subspace look like.

Theorem 10. Let V be a vector space over K and let $\emptyset \neq X \subseteq V$. Then

$$\langle X \rangle = \{k_1 x_1 + \cdots + k_n x_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \ldots, n\,,\ n \in \mathbb{N}^*\}\,,$$

i.e. $\langle X \rangle$ is the set of all finite linear combinations of vectors of X.

Proof. Let $M = \{k_1x_1 + \ldots + k_nx_n \mid k_i \in K, x_i \in X, i = \overline{1,n},\ n \in \mathbb{N}^*\}$ □

I) $M \leq_K V$ (?)

II) $X \subseteq M$ (?)

III) $S \leq_K V,\ X \subseteq S \Longrightarrow M \subseteq S$ (?)

II) Let $x \in X$, $\quad x = 1 \cdot x \in M$ ($1 \in K$, $x \in X$)

I) $X \neq \emptyset \Rightarrow \exists x_0 \in X \Longrightarrow \underset{\in V}{0} = \underset{\in K}{0} \cdot \underset{\in X}{x_0} \in M. \Longrightarrow \alpha)$ holds for M.

Let $k_1x_1 + k_2x_2 + \ldots + k_nx_n,\ k_1'x_1 + \ldots + k_n'x_n \in M$

$(x_1, \ldots, x_n \in X,\ k_1, \ldots, k_n, k_1', \ldots, k_n' \in K)$

(R: We can use exactly the same x_i's by considering 0 scalars in the repres. of one term or another, or both, if necessary).

Then $\underline{k_1}x_1 + \dots + k_n x_n + \underline{k_1'}x_1 + \dots + k_n' x_n =$

$= (k_1 + k_1')x_1 + \dots + (k_n + k_n')x_n \in M$. Thus β) holds for M.

If $k \in K$, $k \cdot (k_1 x_1 + \dots + k_n x_n) = (k k_1) \underset{X}{x_1} + \dots + (k k_n) \underset{X}{x_n} \in M$.

Thus γ) holds for M.

iii) Let $k_1, \dots, k_n \in K$, $x_1, \dots, x_n \in X \subseteq S$

$\underbrace{k_1 \underset{S}{x_1} + \dots + k_n \underset{S}{x_n} \in M}$ $\xrightarrow[S \leq_K V]{}$ $\underbrace{k_1 x_1 + \dots + k_n x_n \in S}$

Corollary 11. Let V be a vector space over K and $x_1, \dots, x_n \in V$. Then

$$\langle x_1, \dots, x_n \rangle = \{k_1 x_1 + \dots + k_n x_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \dots, n\}\,.$$

Remark 12. Notice that a linear combination of linear combinations is again a linear combination.

Examples 13. (a) Consider the real vector space $\mathbb{R}^3$. Then

$$\langle (1,0,0), (0,1,0), (0,0,1) \rangle = \{k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \mid k_1, k_2, k_3 \in \mathbb{R}\} =$$

$$= \{(k_1,0,0) + (0,k_2,0) + (0,0,k_3) \mid k_1,k_2,k_3 \in \mathbb{R}\} = \{(k_1,k_2,k_3) \mid k_1,k_2,k_3 \in \mathbb{R}\} = \mathbb{R}^3\,.$$

Hence $\mathbb{R}^3$ is generated by the three vectors $(1,0,0), (0,1,0), (0,0,1)$.
(b) More generally, the subspaces of $\mathbb{R}^3$ are the trivial subspaces, the lines containing the origin and the planes containing the origin.

If $S, T \leq_K V$, the smallest subspace of V which contains the union $S \cup T$ is $\langle S \cup T \rangle$. We will show that this subspace is the sum of the given subspaces.

Definition 14. Let V be a vector space over K and let $S, T \leq_K V$. Then we define the **sum** of the subspaces S and T as the set

$$S + T = \{s + t \mid s \in S\,,\ t \in T\}\,.$$

If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the **direct sum** of the subspaces S and T.

Remarks 15. a) If V is a K-vector space, $V_1, V_2 \leq_K V$, then $V = V_1 \oplus V_2$ if and only if

$$V = V_1 + V_2 \text{ and } V_1 \cap V_2 = \{0\}.$$

Under these circumstances, we say that V_i $(i = 1,2)$ is a **direct summand** of V.
b) If $V_1, V_2, V_3 \leq_K V$ and $V = V_1 \oplus V_2 = V_1 \oplus V_3$, we cannot deduce that $V_2 = V_3$.

c) The property of a subspace of being a direct summand is transitive. (during the seminar)

Theorem 16. Let V be a vector space over K and let $S, T \leq_K V$. Then

$$S + T = \langle S \cup T \rangle .$$

Proof. □

Remarks 17. (1) Actually, a more general result can be proved: if $S_1, \dots, S_n$ are subspaces of a K-vector space V then

$$S_1 + \cdots + S_n = \langle S_1 \cup \cdots \cup S_n \rangle.$$

(2) Moreover, if $X_i \subseteq V$ $(i = 1, \ldots, n)$, then $\langle X_1 \cup \cdots \cup X_n \rangle = \langle X_1 \rangle + \cdots + \langle X_n \rangle$.