Analytic Geometry

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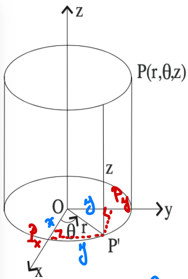
More motivation and a little recap...

- Cartesian (or rectangular) coordinates are the simplest type of coordinate system, where the reference axes are orthogonal (at right angles) to each other. In most everyday applications, such as drawing a graph or reading a map, you would use the principles of Cartesian coordinate systems. In these situations, the exact, unique position of each data point or map reference is defined by a pair of (x,y) coordinates (or (x,y,z) in three dimensions). The coordinates are the point's 'address', its location relative to a known position called the origin, within a two- or three-dimensional grid on a flat surface or rectangular 3D space.
- However, some applications involve curved lines, surfaces and spaces. Here, rectangular coordinates are difficult to use and it is convenient to use a system derived from circular shapes, such as polar, spherical or cylindrical coordinate systems.

The cylindrical coordinate system

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated to a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point.

Let $P(x, y, z)$ be a point in a rectangular system of coordinates $Oxyz$ and P' be the orthogonal projection of P on xOy. One can associate to the point P the triple (r, θ, z), where (r, θ) are the polar coordinates of P'.



$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}.$

The triple (r, θ, z) gives the *cylindrical coordinates* of the point P. There is the bijection

$$h_1 : \mathcal{E}_3 \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R},\ P \to (r, \theta, z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* (CS) in $\mathcal{E}_3$.

In the following table, the conversion formulas relative to the cylindrical coordinate system (CS) and the rectangular coordinate system (RS) are presented.

Conversion	Formulas
CS$\to$RS $(r,\theta,z)\to(x,y,z)$	$x=r\cos\theta,\ y=r\sin\theta,\ z=z$
RS$\to$CS $(x,y,z)\to(r,\theta,z)$	$r=\sqrt{x^2+y^2}$, $z=z$ and θ is given as follows: Case 1. If $x\neq 0$, then $\theta=\arctan\frac{y}{x}+k\pi$, where $k=\begin{cases}0, \text{if } P\in I\cup(Ox\\ 1, \text{ if } P\in II\cup III\cup(Ox'\\ 2, \text{ if } P\in IV\end{cases}$ Case 2. If $x=0$ and $y\neq 0$, then $\theta=\begin{cases}\frac{\pi}{2} \text{ when } P\in(Oy\\ \frac{3\pi}{2} \text{ when } P\in(Oy'\end{cases}$ Case 3. If $x=0$ and $y=0$, then $\theta=0$.

Some examples

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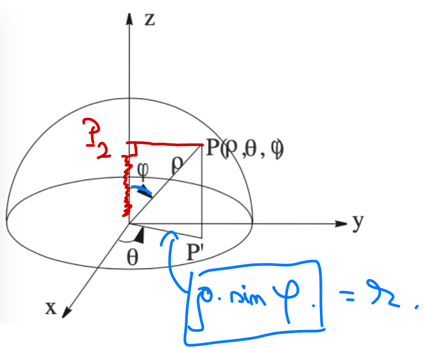
2. The equation $\theta = \theta_0$ describes a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.
3. The equation $z = z_0$ defines a plane which is parallel to the coordinate plane xOy.

The Spherical Coordinate system

Another way to associate to each point P in $\mathcal{E}_3$ a triple of real numbers is illustrated below. If $P(x, y, z)$ is a point in a rectangular system of coordinates $Oxyz$ and P' its orthogonal projection on Oxy, let ρ be the length of the segment $[OP]$, θ be the oriented angle determined by $[Ox$ and $[OP'$ and φ be the oriented angle between $[Oz$ and $[OP$.



The triple (ρ, θ, φ) gives the *spherical coordinates* of the point P. This way, one obtains the bijection

phi

$$h_2 : \mathcal{E}_3 \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \to (\rho, \theta, \varphi),$$

theta

which defines a new coordinate system in $\mathcal{E}_3$, called the *spherical coordinate system* (SS).

The conversion formulas involving the spherical coordinate system (SS) and the rectangular coordinate system (RS) are presented in the following table.

$\cos : [0, \pi] \to [-1, 1]$

$\arccos : [-1, 1] \to [0, \pi].$

Conversion	Formulas
SS→RS $(\rho,\theta,\varphi)\to(x,y,z)$	$x=\rho\cos\theta\sin\varphi,\ y=\rho\sin\theta\sin\varphi,\ z=\rho\cos\varphi$
RS→SS $(x,y,z)\to(\rho,\theta,\varphi)$	$\rho=\sqrt{x^2+y^2+z^2},\ \varphi=\arccos\dfrac{z}{\sqrt{x^2+y^2+z^2}}$ θ is given as follows: Case 1. If $x\neq 0$, then $\theta=\arctan\dfrac{y}{x}+k\pi,$ where $k=\begin{cases}0, P'\in I\cup(Ox\\ 1, P'\in II\cup III\cup(Ox'\\ 2, P'\in IV\end{cases}$ Case 2. If $x=0$ and $y\neq 0$, then $\theta=\begin{cases}\dfrac{\pi}{2}, P'\in(Oy\\ \dfrac{3\pi}{2}, P'\in(Oy'\end{cases}$ Case 3. If $x=0$ and $y=0$, then $\theta=0$

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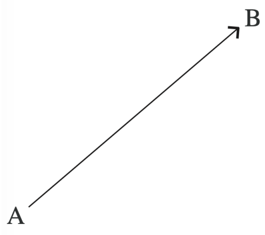
2. As in the cylindrical coordinates, the equation $\theta = \theta_0$ defines a half-plane attached along the z-axis, making an angle θ_0 with the positive x-axis.
3. The equation $\varphi = \varphi_0$ describes the points P for which the angle determined by $[OP$ and $[Oz$ is φ_0. If $\varphi_0 \neq \dfrac{\pi}{2}$ and $\varphi_0 \neq \pi$, this is a right circular cone, having the vertex at the origin and centered on the z-axis. The equation $\varphi = \dfrac{\pi}{2}$ defines the coordinate plane xOy. The equation $\varphi = \pi$ describes the negative axis $(Oz'$.

Both knew their vectors pretty well...

W. Hamilton

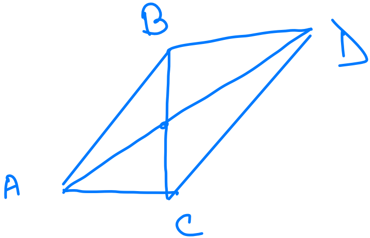
M. Hamilton

- Let $\mathcal{E}$ denote the Euclidean plane $\mathcal{E}_2$ or the Euclidean 3-space $\mathcal{E}_3$. A pair $(A, B) \in \mathcal{E} \times \mathcal{E}$ is called an *ordered pair* of points or a *vector at the point* A. Such a pair is, shortly, denoted by $\overrightarrow{AB}$. The point A is the *original point*, while B is the *terminal point* and the line AB (if $A \neq B$) gives the direction of $\overrightarrow{AB}$. A vector $\overrightarrow{AB}$ at A has the *orientation* from A to B, i.e. from its original to its terminal point.
- The length of the segment $[AB]$ represents the *length* of the vector $\overrightarrow{AB}$ and is denoted by $||\overrightarrow{AB}||$ or by $|\overrightarrow{AB}|$. Usually, the vector $\overrightarrow{AB}$ at A is represented as



An equivalence relation on pairs of points...

- Let consider the relation $\mathcal{E} \times \mathcal{E}$: $(A, B) \sim (C, D)$ if and only if the segments $[AD]$ and $[BC]$ have the same midpoint.



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- When the points A, B, C and D are not collinear, this means that $(A, B) \sim (C, D)$ if and only if $ABCD$ is a parallelogram.
- It is not difficult to check that " $\sim$ " is an equivalence relation.

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- When the points A, B, C and D are not collinear, this means that $(A, B) \sim (C, D)$ if and only if $ABCD$ is a parallelogram.
- It is not difficult to check that " $\sim$ " is an equivalence relation.
- Let us denote by V_3 the set $(\mathcal{E}_3 \times \mathcal{E}_3)/_\sim$ of equivalence classes and by V_2 the set $(\mathcal{E}_2 \times \mathcal{E}_2)/_\sim$.

- If $\overrightarrow{AB} \in \mathcal{E} \times \mathcal{E}$, its equivalence class is denoted by $\overline{AB}$ and is called a *vector* in $\mathcal{E}$ ($\mathcal{E}_2$ or $\mathcal{E}_3$). In this case, $\overrightarrow{AB}$ is a *representative* of $\overline{AB}$.
- Suppose that $A \neq B$. The line AB defines the *direction* of the vector $\overline{AB}$. The *length* of $\overline{AB}$ is given by

$$||\overline{AB}|| = ||\overrightarrow{AB}|| = AB,$$

  the length of the segment $[AB]$. The *orientation* of $\overline{AB}$, from A to B, is given by the orientation of $\overrightarrow{AB}$.

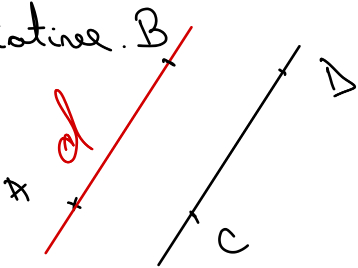
We shall denote the vectors in V_2 or V_3 by small letters: $\overline{a}$, $\overline{b}$,... $\overline{u}$, $\overline{v}$, $\overline{w}$.

Proposition

Given a vector $\overline{a}$ in V_2 (or V_3) and a fixed point A, there exists a unique representative of $\overline{a}$, having the original point at A.

Proof. If $\overline{a} = \overline{0}$, then $\overrightarrow{AA} \in \overline{a}$.

Suppose $\overline{a} \neq \overline{0}$. Let $\overrightarrow{CD} \in \overline{a}$ be any representative.

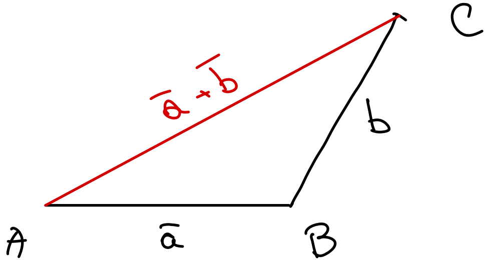


There is a unique point B such that $ACDB$ is a parallelogram.

[This point B lies on d, the unique parallel to CD which passes through A. Moreover $AB = CD$ and $\overrightarrow{AB}$, $\overrightarrow{CD}$ have the same direction.].

Vector operations

Let $\overline{a}$ and $\overline{b}$ be two vectors in V_3 (or V_2). The *sum* of $\overline{a}$ and $\overline{b}$ is the vector denoted by $\overline{a}+\overline{b}$, so that, if $\overrightarrow{AB} \in \overline{a}$ and $\overrightarrow{BC} \in \overline{b}$, then $\overrightarrow{AC}$ is the representative of $\overline{a}+\overline{b}$.



• We ended here !

- f $\overline{v}$ is a vector in V_3 (or V_2), then the *opposite vector* of $\overline{v}$ is denoted by $-\overline{v}$, so that, if $\overrightarrow{AB}$ is a representative of $\overline{v}$, then $\overrightarrow{BA}$ is a representative of $-\overline{v}$.
- The sum $\overline{a} + (-\overline{b})$ will be, shortly, denoted by $\overline{a} - \overline{b}$ and it will be called the *difference* of the vectors $\overline{a}$ and $\overline{b}$.
- Let $\overline{a}$ be a vector in V_3 (or V_2) and k be a real number. The *product* $k \cdot \overline{a}$ is the vector defined as follows:
  1. $\overline{0}$ if $\overline{a} = \overline{0}$ or $k = 0$;
  2. if $k > 0$, then $k \cdot \overline{a}$ has the same direction and orientation as $\overline{a}$ and $||k \cdot \overline{a}|| = k \cdot ||\overline{a}||$;
  3. if $k < 0$, then $k \cdot \overline{a}$ has the same direction as $\overline{a}$, opposite orientation to $\overline{a}$ and $||k \cdot \overline{a}|| = -k \cdot ||\overline{a}||$.

The components of a vector

- Let $\overline{a}$ be a vector in V_2 and xOy be a rectangular coordinates system in $\mathcal{E}_2$. There is a unique point $A \in \mathcal{E}_2$, such that $\overrightarrow{OA} \in \overline{a}$. The coordinates of the point A are called the *components* of the vector $\overline{a}$ and write $\overline{a}(a_1, a_2)$.

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- Similarly, $\overline{a}$ a vector in V_3 and a rectangular coordinate system $Oxyz$ in $\mathcal{E}_3$, there exists a unique point $A(a_1, a_2, a_3)$, such that $\overrightarrow{OA} \in \overline{a}$. The triple (a_1, a_2, a_3) gives the *components* of $\overline{a}$ and we denote it by $\overline{a}(a_1, a_2, a_3)$.
- Since $\overline{0}(0, 0)$ in V_2 and $\overline{0}(0, 0, 0)$ in V_3, then two vectors are equal if and only if they have the same components.

Theorem

Let $\overline{a}(a_1, a_2)$ and $\overline{b}(b_1, b_2)$ be two vectors in V_2 and $k \in \mathbb{R}$. Then:

(1) *the components of $\overline{a} + \overline{b}$ are $(a_1 + b_1, a_2 + b_2)$;*

(2) *the components of $k \cdot \overline{a}$ are (ka_1, ka_2).*

Proof.

An analogous theorem for 3D

Theorem

Let $\overline{a}(a_1, a_2, a_3)$ and $\overline{b}(b_1, b_2, b_3)$ be two vectors in V_3 and $k \in \mathbb{R}$. Then:

(1) *the components of $\overline{a} + \overline{b}$ are $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$;*

(2) *the components of $k \cdot \overline{a}$ are (ka_1, ka_2, ka_3).*

Theorem

(1) *If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points in $\mathcal{E}_2$, then*

$$\overline{P_1 P_2}(x_2 - x_1, y_2 - y_1).$$

(2) *If $Q_1(x_1, y_1, z_1)$ and $Q_2(x_2, y_2, z_2)$ are two points in $\mathcal{E}_3$, then*

$$\overline{Q_1 Q_2}(x_2 - x_1, y_2 - y_2, z_2 - z_1).$$

Proof.

The set of vectors is a very structured one

Theorem (Prop. of the summation)

Let $\overline{a}$, $\overline{b}$ and $\overline{c}$ be vectors in V_3 (or V_2) and $\alpha, \beta \in \mathbb{R}$. Then:

1) $\overline{a} + \overline{b} = \overline{b} + \overline{a}$ *(commutativity);*

2) $(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c})$ *(associativity);*

3) $\overline{a} + \overline{0} = \overline{0} + \overline{a} = \overline{a}$ *($\overline{0}$ is the neutral element for summation);*

4) $\overline{a} + (-\overline{a}) = (-\overline{a}) + \overline{a} = \overline{0}$ *($-\overline{a}$ is the inverse of $\overline{a}$);*

5) $\alpha(\beta\overline{a}) = (\alpha\beta)\overline{a}$;

6) $\alpha \cdot (\overline{a} + \overline{b}) = \alpha \cdot \overline{a} + \beta \cdot \overline{b}$ *(multiplication by real scalars is distributive with respect to the summation of vectors);*

7) $(\alpha + \beta) \cdot \overline{a} = \alpha \cdot \overline{a} + \beta \cdot \overline{a}$ *(multiplication by real scalars is distributive with respect to the summation of scalars);*

8) $1 \cdot \overline{a} = \overline{a}$.

Proof.

Proposition

(1) *Let* $\overline{a}(a_1, a_2)$ *be a vector in* V_2. *The length of* $\overline{a}$ *is given by*

$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2}.$$

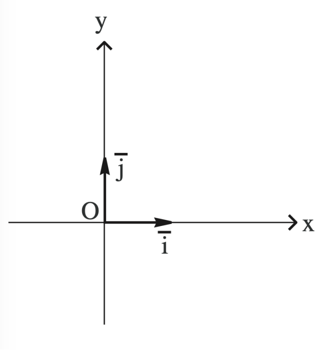
(2) *Let* $\overline{a}(a_1, a_2, a_3)$ *be a vector in* V_3. *The length of* $\overline{a}$ *is given by*

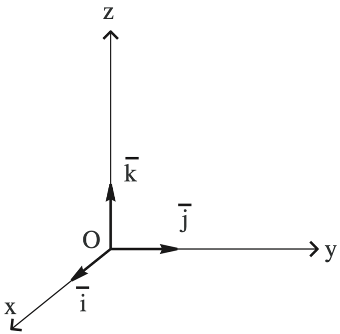
$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^3}.$$

Proof.

- The vectors $\bar{i}(1,0)$ and $\bar{j}(0,1)$ in V_2 are called the *unit vectors* (or *versors*) of the coordinate axes Ox and Oy.
- The vectors $\bar{i}(1,0,0)$, $\bar{j}(0,1,0)$ and $\overline{k}(0,0,1)$ are called the *unit vectors* (or *versors*) of the coordinate axes Ox, Oy and Oz.
- It is clear that

$$||\bar{i}|| = ||\bar{j}|| = ||\overline{k}|| = 1.$$





The problem set for this week will be posted soon. Ideally you would think about it before the seminar on Friday.

Thank you very much for your attention!