1. Determine the intersection of the hyperboloid

$$\mathcal{H}^1_{4,3,1} : \frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \langle \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

2. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1} : \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

3. Determine the generators of the hyperboloid

$$\frac{x^2}{36} + \frac{y^2}{9} - \frac{x^2}{4} = 1$$

which are parallel to the plane $x + y + z = 0$.

4. Determine the intersection of the hyperboloid

$$\mathcal{H}^2_{2,1,3} : \frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{9} = -1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \langle \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

5. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad \ell = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

6. Determine the tangent plane of

1. the elliptic paraboloid $\frac{x^2}{5} + \frac{y^2}{3} = z$ and of
2. the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = z$

which are parallel to the plane $x - 3y + 2z - 1 = 0$.

7. Determine the plane which contains the line

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \rangle \quad \text{and is tangent to the quadric} \quad x^2 + 2y^2 - z^2 + 1 = 0.$$

8. Show that the parabolid $\mathcal{P}^e_{p,p}$ is the locus of points for which the distance from a point equals the distance to a plane. Such a surface is called *elliptic paraboloid of revolution*.

9. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution.

10. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}:\begin{cases} x=\sqrt{1+t^2}\cos(s)\\ y=\sqrt{1+t^2}\sin(s)\\ z=2t\end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,2)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

11. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}:\begin{cases} x=s\\ y=t\\ z=s^2-t^2\end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,0)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

12. Determine the generators of the paraboloid

$$4x^2-9y^2=36z$$

containing the point $P(3\sqrt{2},2,1)$.

13. Determine the generators of the paraboloid

$$\frac{x^2}{16}-\frac{y^2}{4}=z$$

which are parallel to the plane $3x+2y-4z=0$.

14. Which of the following is a hyperboliod?

1. $\mathcal{S}: 2xz+2xy+2yz=1$
2. $\mathcal{S}: 5x^2+3y^2+xz=1$
3. $\mathcal{S}: 2xy+2yz+y+z=2$

1. Determine the intersection of the hyperboloid

$$\mathcal{H}^1_{4,3,1} : \frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \langle \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

$$\frac{(4+4t)^2}{16} + \frac{(-2)^2}{9} - \frac{(1+t)^2}{1} = 1.$$

$$\frac{16 + 32t + 16t^2}{16} + \frac{4}{9} - 1 - 2t - t^2 = 1.$$

$$\cancel{1} + \cancel{2t} + \cancel{t^2} + \frac{4}{9} - \cancel{1} - \cancel{2t} - \cancel{t^2} = 1.$$

$\therefore$ Thus no solutions;

$$\mathcal{H}^1_{4,3,1} \cap \ell = \emptyset.$$

2. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1} : \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

$$T_M(\mathcal{H}_{2,3,1}) : \frac{2 \cdot x}{4} + \frac{3 \cdot y}{9} - z = 1.$$

$$\boxed{T_M(\mathcal{H}_{2,3,1}) : \quad 3x + 2y - 6z - 6 = 0}$$

First, we determine $\mathcal{H}^1_{2,3,1} \cap T_M(\mathcal{H}^1_{2,3,1})$:

$$\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 1 \\[2ex] z = 1 - \dfrac{x}{2} - \dfrac{y}{3}. \end{cases}$$

$$\therefore \quad \frac{x^2}{4} + \frac{y^2}{9} - \left(1 - \frac{x}{2} - \frac{y}{3}\right)^2 = 1.$$

$$\Leftrightarrow \quad \cancel{\frac{x^2}{4}} + \cancel{\frac{y^2}{9}} - \left(1^2 + \cancel{\frac{x^2}{4}} + \cancel{\frac{y^2}{9}} - x - \frac{2y}{3} + \frac{xy}{3}\right) = 1.$$

$$\Leftrightarrow \quad \frac{xy}{3} - x - \frac{2y}{3} = -2$$

$$xy - 3x - 2y + 6 = 0$$

$$x(y-3) - 2y + 6 = 0$$

$$x(y-3) - 2(y-3) = 0$$

$$\Rightarrow (x-2)\cdot(y-3) = 0.$$

$$\Rightarrow \quad x - 2 = 0 \quad \text{or} \quad y - 3 = 0.$$

$$d_1 : \begin{cases} x - 2 = 0 \\ 3x + 2y - 6z - 6 = 0 \end{cases} \leftarrow \text{First line.}$$

$$d_2 : \begin{cases} y - 3 = 0 \\ 3x + 2y - 6z - 6 = 0 \end{cases} \leftarrow \text{Second line.}$$

$$\mathcal{H} \cap T_M(\mathcal{H}) = d_1 \cup d_2.$$

3. Determine the generators of the hyperboloid

$$\frac{x^2}{36} + \frac{y^2}{9} - \frac{\cancel{x^2}\ z^2}{4} = 1$$

which are parallel to the plane $x + y + z = 0$.

$$\mathcal{H} : \quad \frac{x^2}{36} - \frac{z^2}{4} = 1 - \frac{y^2}{9}$$

$$\mathcal{H}: \left(\frac{x}{6}-\frac{z}{2}\right)\cdot\left(\frac{x}{6}+\frac{z}{2}\right)=\left(1-\frac{y}{3}\right)\left(1+\frac{y}{3}\right)$$

$$l_\alpha: \begin{cases} \frac{x}{6}-\frac{z}{2}=\alpha\cdot\left(1-\frac{y}{3}\right) \\ \alpha\cdot\left(\frac{x}{6}+\frac{z}{2}\right)=1+\frac{y}{3}. \end{cases}, \alpha\in\mathbb{R}.$$

First family of generators.

$$l_\beta: \begin{cases} \frac{x}{6}-\frac{z}{2}=\beta\cdot\left(1+\frac{y}{3}\right) \\ \beta\left(\frac{x}{6}+\frac{z}{2}\right)=1-\frac{y}{3}. \end{cases}, \beta\in\mathbb{R}.$$

Second family of generators.

$$\ell_\alpha : \begin{cases} \frac{x}{6} - \frac{z}{2} = \alpha \cdot \left(1 - \frac{y}{3}\right) \\ \alpha \cdot \left(\frac{x}{6} + \frac{z}{2}\right) = 1 + \frac{y}{3} \quad \Big| \alpha \end{cases}, \alpha \in \mathbb{R}$$

$z = t$

$$\frac{x}{6} - \frac{z}{2} + \alpha^2 \left(\frac{x}{6} + \frac{z}{2}\right) = 2\alpha.$$

$$x \cdot \left(\frac{1}{6} + \frac{\alpha^2}{6}\right) = 2\alpha + \frac{t}{2} + \frac{\alpha^2}{2} \cdot t.$$

$$x = \frac{12\alpha}{1 + \alpha^2} + \frac{\left(\frac{1}{2} + \frac{\alpha^2}{2}\right)}{\left(\frac{1}{6} + \frac{\alpha^2}{6}\right)} \cdot t.$$

$$x = \frac{12\alpha}{1 + \alpha^2} + 3 \cdot t.$$

$$y = a_\alpha + b_\alpha \cdot t.$$

$$\vec{d}_\alpha = \begin{bmatrix} 3 \\ b_\alpha \\ 1 \end{bmatrix}$$

$$\ell_\alpha \parallel \pi \iff$$

$$\underline{1 \cdot 3 + 1 \cdot b_\alpha + 1 \cdot 1 = 0}.$$

$$\rightarrow \boxed{\text{Find } \alpha.}$$

5. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad \ell = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

$$(2+2t)^2 - 4 \cdot (0+t)^2 = 4(3-2t)$$

$$\cancel{4t^2} + 8t + 4 - \cancel{4t^2} = 12 - 8t$$

$$16t = 8 \iff t = \frac{1}{2}.$$

$P \begin{bmatrix} 3 \\ 1/2 \\ 2 \end{bmatrix}$ $\boxed{: \frac{x^2}{2} - 2y^2 = 2z}$ $x \cdot x$ $z+z$

$$T_P(\mathcal{P}^h_{2,\frac{1}{2}}) : \frac{3x}{2} - 2 \cdot \frac{1}{2} \cdot y = z + 2.$$

$$\frac{x^2}{100} - \frac{y^2}{8} = z \quad \Big| \cdot 2.$$

$$\mathcal{P}.\ \frac{x^2}{50} - \frac{y^2}{4} = 2z.$$

$$T_P(\mathcal{P}) : \frac{x_0 \cdot x}{5} - \frac{y_0 \cdot y}{4} = z + z_0$$

elliptic paraboloid of revolution.

10. For the surface $\mathcal{S}$ with parametrization

$[(1+t^2)^{\frac{1}{2}}]'$

$$\mathcal{S} : \begin{cases} x = \sqrt{1+t^2}\cos(s) \\ y = \sqrt{1+t^2}\sin(s) \\ z = 2t \end{cases}$$

$, s \in [0, 2\pi]$

$= \frac{1}{\cancel{2}} \cdot (1+t^2)^{-\frac{1}{2}} \cdot \cancel{2}t$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,2)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

$$\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\sigma(t, s) = \begin{bmatrix} \sqrt{1+t^2}\cos(s) \\ \sqrt{1+t^2}\sin(s) \\ 2t \end{bmatrix}.$$

$$x^2 + y^2 = 1 + t^2$$

$$z^2 = 4t^2$$

$$4(x^2+y^2) - z^2 = 4. \quad \Big| :$$

$$H : x^2 + y^2 - \frac{z^2}{2^2} = 1.$$

$$P(1, 1, 2) \in H.$$

$$2t = 2 \Rightarrow t = 1.$$

$$\therefore \cos(s) = \sin(s) = \frac{\sqrt{2}}{2}.$$

$$\boxed{s = \frac{\pi}{4}.}$$

$$\mathcal{H} : x^2 + y^2 - \frac{z^2}{2^2} = 1.$$

$$P(1, 1, 2) \in \mathcal{H}.$$

$$\therefore \; T_P(\mathcal{H}) : \quad x + y - \frac{2z}{4} = 1.$$

$$x + y - \frac{z}{2} = 1.$$

- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.$(\mathcal{H})$

$$T_P(\mathcal{H}) : \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \left\langle \frac{\partial \sigma}{\partial t}\left(1, \frac{\pi}{4}\right), \frac{\partial \sigma}{\partial s}\left(1, \frac{\pi}{4}\right) \right\rangle$$

$$\frac{\partial \sigma}{\partial t} = \begin{bmatrix} \frac{t}{\sqrt{1+t^2}} \cdot \cos(s) \\ \frac{t}{\sqrt{1+t^2}} \cdot \sin(s) \\ 2 \end{bmatrix} \Rightarrow \frac{\partial \sigma}{\partial t}\left(1, \frac{\pi}{4}\right) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}$$

$$\frac{\partial \sigma}{\partial s} = \begin{bmatrix} -\sqrt{1+t^2} \cdot \sin(s) \\ \sqrt{1+t^2} \cdot \cos(s) \\ 0 \end{bmatrix} \Rightarrow \frac{\partial \sigma}{\partial s}\left(1, \frac{\pi}{4}\right) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$T_P(\mathcal{H}) : \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \left\langle \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle .$$

$$T_P(\mathcal{H}) : \begin{vmatrix} x-1 & y-1 & z-2 \\ 1/2 & 1/2 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 0.$$

11. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S} : \begin{cases} x = s \\ y = t \\ z = s^2 - t^2 \end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,0)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \quad \sigma(t,s) = \begin{bmatrix} s \\ t \\ s^2 - t^2 \end{bmatrix}.$$

$$S : \quad x^2 - y^2 = z.$$

The parameters of $P(1,1,0)$ are $\underline{\underline{(1,1)}}$.

$$\frac{\partial \sigma}{\partial t} = \begin{bmatrix} 0 \\ 1 \\ -2t \end{bmatrix}, \quad \frac{\partial \sigma}{\partial s} = \begin{bmatrix} 1 \\ 0 \\ 2s \end{bmatrix}$$

$$T_P(S) : \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\rangle.$$

An equation for $T_P(S)$ is:

$$T_P(S) : \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{vmatrix} = 0.$$

This should be the same as:

$S: \quad x^2 - y^2 = z \quad \Big| \cdot 2$

$\Leftrightarrow S: 2x^2 - 2y^2 = 2z.$

$P(1, 1, 0)$

$T_P(S): 2 \cdot x - 2 \cdot y = z.$