Seminar 3

1. $(\mathbb{C},+)$ has the following subgroups: $(\mathbb{Z},+),(\mathbb{Q},+),(\mathbb{R},+),(\mathbb{C},+)$.

   $(\mathbb{C}^*,\cdot)$ has the following subgroups: $(\mathbb{Q}^*,\cdot),(\mathbb{R}^*,\cdot),(\mathbb{C}^*,\cdot)$.

2. $H \neq \emptyset$, as $z = 1 \in H$.

   $\forall z \in H, z^{-1} \in H$ we get: $\mid z \cdot z^{-1} \mid = \mid z \mid \cdot \mid z^{-1} \mid = 1$, hence $z \cdot z^{-1} \in H$.

   So, $(H,\cdot)$ is a subgroup of $(\mathbb{C}^*,\cdot)$.

   But, for $(H,+)$, it is not true, as $z = 1 \in H$, but $z - z \notin H$, because $\mid z - z \mid = \mid 0 \mid = 0 \neq 1$.

3. For (i) and (ii) see Seminar 2, exercise 2.

   (iii) We know that $I_n \in SL_n(\mathbb{C})$, so $SL_n(\mathbb{C}) \neq \emptyset$.

   And $\forall A, B \in SL_n(\mathbb{C})$, we have $det(A \cdot B^{-1}) = det(A) \cdot det(B^{-1}) = det(A) \cdot (det(B))^{-1} = 1$, as $B \cdot B^{-1} = I_n$. So $(SL_n(\mathbb{C}),\cdot)$ is a subgroup of $(GL_n(\mathbb{C}),\cdot)$.

4. We can easily prove this using Seminar 2, exercise 3.

5. (i)

   $\boxed{\Rightarrow}$ $n\mathbb{Z} \subseteq m\mathbb{Z} \Rightarrow n \in m\mathbb{Z} \Rightarrow \exists l \in \mathbb{Z}$ such that $n = ml \Rightarrow m \mid n$.

   $\boxed{\Leftarrow}$ $m \mid n \Rightarrow \exists l \in \mathbb{Z}$ such that $n = ml$.
   $\forall y \in n\mathbb{Z} \Rightarrow \exists k \in \mathbb{Z}$ such that $y = nk$.
   So, we get that $y = m \cdot k \cdot l \in m\mathbb{Z} \Rightarrow n\mathbb{Z} \subseteq m\mathbb{Z}$.

   (ii)

   $\boxed{\subseteq}$ Let $x \in m\mathbb{Z} \cap n\mathbb{Z} \Rightarrow x \in n\mathbb{Z}$ and $x \in m\mathbb{Z} \Rightarrow (n \mid x$ and $m \mid x) \Rightarrow [m,n] \mid x \Rightarrow x \in [m,n]\mathbb{Z}$.

   $\boxed{\subseteq}$ Let $x \in [m,n]\mathbb{Z} \Rightarrow [m,n] \mid x \Rightarrow (m \mid x$ and $n \mid x) \Rightarrow x \in m\mathbb{Z}, x \in n\mathbb{Z} \Rightarrow x \in m\mathbb{Z} \cap n\mathbb{Z}$.

(iii) $m\mathbb{Z} + n\mathbb{Z} = \{a + b \mid a \in m\mathbb{Z}, b \in n\mathbb{Z}\} = \{ml + nk \mid l, k \in \mathbb{Z}\}$.

We know that: $m\mathbb{Z} + n\mathbb{Z} \leq (\mathbb{Z}, +)$ and $S(\mathbb{Z}, +) = \{a\mathbb{Z} \mid a \in \mathbb{N}\}$, so we get that: $\exists d \in \mathbb{N}$ such that $m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$.

Take $m = m \cdot 1 + n \cdot 0$, which is from $m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z} \Rightarrow d \mid m$. The same goes for $n = m \cdot 0 + n \cdot 1 \in m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z} \Rightarrow d \mid n$.

Let $d_1 \in \mathbb{Z}$ be such that $d_1 \mid m$ and $d_1 \mid n$. Then $m\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $n\mathbb{Z} \subseteq d_1\mathbb{Z}$. So, we get that: $m\mathbb{Z} + n\mathbb{Z} \subseteq d_1\mathbb{Z} \Rightarrow d\mathbb{Z} \subseteq d_1\mathbb{Z}$. And, from (i): $d_1 \mid d$.

In conclusion, $d = (m, n)$.

6. $\boxed{\Rightarrow}$ Let $H \subseteq K \Rightarrow H \cup K = K \leq G$.

   Same $K \subseteq H \Rightarrow H \cup K = H \leq G$.

   $\boxed{\Leftarrow}$ Suppose that $H \nleq K$ and $K \nleq H$. We must prove: $H \cup K \nleq G$.

   $H \nleq K \Rightarrow \exists h \in H, h \notin K$.

   $K \nleq H \Rightarrow \exists k \in K, k \notin H$.

   But $h, k \in H \cup K$.

   Suppose $H \cup K \leq G$. Then $h \cdot k \in H \cup K$. This means $h \cdot k \in H$ or $h \cdot k \in K$.

   If $h \cdot k \in H \Rightarrow h^{-1} \cdot h \cdot k \in H$, as $H \leq G \Rightarrow 1 \cdot k \in H$, which is a contradiction.

   The same goes for $h \cdot k \in K$.

   In the end $H \cup K \nleq G$.

7. $\boxed{\Rightarrow}$ Clear.

   $\boxed{\Leftarrow}$ Let H be a stable subset of $(G, \cdot)$. Let $h \in H \neq \emptyset$. Then $h^2 = h \cdot h \in H$ and, inductively, $h^k \in H$, $\forall k \in \mathbb{N}^*$. Since H is finite, $\exists m, n \in \mathbb{N}^*$ with $m > n$ such that $h^m = h^n$. Then $1 = h^{m-n} \in H$, and so $h^k \in H$, $\forall k \in \mathbb{N}$. Then $h^{-1} = h^{m-n-1} \in H$. Hence H is a subgroup of $(G, \cdot)$.

8. $Z(G) \neq \emptyset$ as $x = 1 \in Z(G)$, the identity element in $Z(G)$.

   We show that $\forall x, y \in Z(G) \Rightarrow x \cdot y \in Z(G)$ and $x^{-1} \in Z(G)$.

$xg = gx$ and $yg = gy, \forall g \in G \Rightarrow (xy)g = xyg = xgy = gxy = g(xy) \Rightarrow xy \in Z(G)$.

$xg = gx, \forall g \in G \Rightarrow x^{-1}xg = x^{-1}gx \Rightarrow g = x^{-1}gx \Rightarrow gx^{-1} = x^{-1}gxx^{-1} \Rightarrow gx^{-1} = x^{-1}g \Rightarrow x^{-1} \in Z(G)$.

$Z(G) = G \Leftrightarrow G$ is abelian.

9. $Z(GL_2(\mathbb{R})) = \{A \in GL_2(\mathbb{R}) \mid A \cdot B = B \cdot A, \forall B \in GL_2(\mathbb{R})\}$.

   $A \in GL_2(\mathbb{R}) \Rightarrow detA \neq 0$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   Take $B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, with $B_1, B_2 \in GL_2(\mathbb{R})$.

   $A \cdot B_1 = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $B_1 \cdot A = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$.

   But $A \cdot B_1 = B_1 \cdot A$. So we get the next equalities: $a = a + c$ and $a + b = d + b \Rightarrow c = 0$ and $a = d$.

   We do the same things for B_2 and get that $b = 0$.

   In the end, $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a \cdot I_2 \in GL_2(\mathbb{R})$. One checks that these verify the condition.

10. We know that $e \circ \sigma = \sigma \circ e$, $\forall \sigma \in S_3$.

    Suppose that $\exists \sigma \in S_3$ with $\sigma \neq e$ such that $\sigma \in Z(S_3, \circ)$. Then $\exists a, b \in \{1, 2, 3\}$ with $a \neq b$ such that $\sigma(a) = b$. Let $c \in \{1, 2, 3\} \setminus \{a, b\}$. Consider $\tau \in S_3$ defined by $\tau(b) = c$, $\tau(c) = b$ and $\tau(a) = a$. We must have $\sigma \circ \tau = \tau \circ \sigma$. But $\sigma(\tau(a)) = b$ and $\tau(\sigma(a)) = c$, which is a contradiction.

    So $Z(S_3, \circ) = \{e\}$. And similarly goes for S_n.