Analytic Geometry

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Recap...

- A plane π in the 3-dimensional space can be uniquely determined by specifying a point $P_0(x_0, y_0, z_0)$ in the plane and a nonzero vector $\overline{n}(a, b, c)$, orthogonal to the plane. $\overline{n}$ is called the *normal vector* to the plane π.
- An arbitrary point $P(x, y, z)$ is contained into the plane π if and only if
$$\overline{n} \perp \overline{P_0P},$$
or
$$\overline{n} \cdot \overline{P_0P} = 0.$$
- But $\overline{P_0P}(x - x_0, y - y_0, z - z_0)$ and one obtains the *normal* equation of the plane π containing the point $P_0(x_0, y_0, z_0)$ and of normal vector $\overline{n}(a, b, c)$.
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{1}$$
Remark: The equation (1) can be written in the form $ax + by + cz + d = 0$.

Theorem

Given $a, b, c, d \in \mathbb{R}$, with $a^2 + b^2 + c^2 > 0$, the equation

$$ax + by + cz + d = 0 \tag{2}$$

describes a plane in $\mathcal{E}_3$. This plane has $\overline{n}(a, b, c)$ as a normal vector.

The analytic equation of the plane determined by a point and two nonparallel directions

π a plane. $A(x_A, y_A, z_A) \in \pi$.

$\overline{v}_1(p_1, q_1, r_1)$, $\overline{v}_2(p_2, q_2, r_2) \parallel \pi$.

$\overline{v}_1 \nparallel \overline{v}_2$.

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \tag{3}$$

$P(x, y, z) \in \mathcal{E}_3$. $P \in \pi$ if and only if $\overline{AP}$, $\overline{v}_1$, $\overline{v}_2$ are coplanar (linearly dependent)

$\overline{AP}\ (x - x_A,\ y - y_A,\ z - z_A)$

$\overline{v_1}\ (p_1\ ,\ q_1\ ,\ r_1)$

$\overline{v_2}\ (p_2,\ q_2\ ,\ r_2)$.

$$\Delta := \begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}$$

one of the lines can be expressed as a linear combination of the other 2.

$\therefore \quad \boxed{\Delta = 0.}$

The analytic equation of the plane determined by three noncollinear points

$A, B, C \in \pi$. (use $A \in \pi$, $\overline{AB}$, $\overline{AC}$)

$P(x, y, z) \in \mathcal{E}_3$

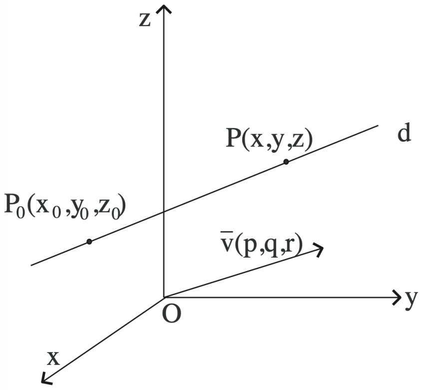
$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0, \tag{4}$$

Use previous formula.

$$= \begin{vmatrix} x - x_A & y - y_A & z - z_A & 0 \\ \cancel{x_A} & \cancel{y_A} & \cancel{z_A} & \cancel{1} \\ x_B - x_A & y_B - y_A & z_B - z_A & 0 \\ x_C - x_A & y_C - y_A & z_C - z_A & 0 \end{vmatrix} = 0$$

The line in space

- As in the 2-space, a line d in the 3-space is completely determined by a point $P_0(x_0, y_0, z_0)$ of the line and a nonzero vector $\overline{v}(p, q, r)$, parallel to d.



The parametric equations of a line

- If $P(x, y, z)$ is an arbitrary point on the line d, then the vectors $\overline{P_0P}$ and $\overline{v}$ are linearly dependent in V_3 and there exists $t \in \mathbb{R}$, such that
(parallel)
$$\overline{P_0P} = t\overline{v}. \tag{5}$$

- Since $\overline{P_0P}(x - x_0, y - y_0, z - z_0)$, by decomposing (5) in components, one obtains the *parametric* equations of the line passing through $P_0(x_0, y_0, z_0)$ and parallel to $\overline{v}(p, q, r)$:

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \quad t \in \mathbb{R}. \tag{6}$$

$t = \frac{x-x_0}{p}$

$t = \frac{y-y_0}{q}$

$t = \frac{z-z_0}{r}$.

- The vector $\overline{v}(p, q, r)$ is called the *director* vector of the line d.

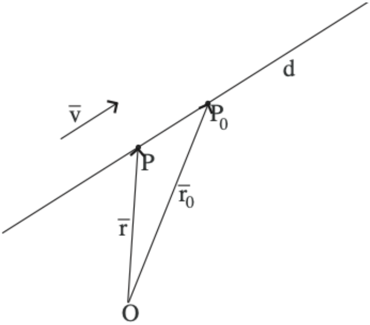
The vector equation of a line

If we fix an origin in the space, the vector $\overline{P_0P}$ can be expressed as the difference $\overline{r} - \overline{r}_0$ and the equation (5) becomes

$\overline{r} = \overline{OP}$
$\overline{r}_0 = \overline{OP_0}$

$$\overline{r} = \overline{r}_0 + t\overline{v}, \qquad t \in \mathbb{R}, \tag{7}$$

said to be the *vector* equation of the line in 3-space.



The symmetric equation

If $p, q, r \in \mathbb{R}^*$.

- Expressing t three times in (6), one obtains the *symmetric* equations of the line d:
$$\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}. \tag{8}$$
- *Remark*: The director vector $\overline{v}$ is a nonzero vector, i.e. at least one of its components is different from zero. As in the 2-dimensional case, if $p=0$, for instance, the meaning of $\dfrac{x-x_0}{0}$ is that $x=x_0$.

The equation of a line determined by two points

- A line d can be determined by two different points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ which belong to the line. In this case, the director vector of d is
$$\overline{P_1P_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1) \neq \bar{0}$$
and the equations of the line determined by two points are
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{9}$$

(Only if $x_2 \neq x_1$, $y_2 \neq y_1$, $z_2 \neq z_1$).

The lines as intersection of two planes

- Given two distinct and nonparallel planes $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$ (the planes π_1 and π_2 are parallel when their normal vectors $\overline{n}_1(A_1, B_1, C_1)$ and $\overline{n}_2(A_2, B_2, C_2)$ are parallel, i.e. the rank of the matrix $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ is 1), they have an entire line d in common.
- Then, a line in 3-space can be determined as the intersection of two nonparallel planes:

$$d : \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \tag{10}$$

with

$$\operatorname{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$$

The relative positions of two lines

- Let d_1 and d_2 be two lines in $\mathcal{E}_3$, of director vectors $\overline{v}_1(p_1, q_1, r_1) \neq \overline{0}$, respectively $\overline{v}_2(p_2, q_2, r_2) \neq \overline{0}$. The parametric equations of these lines are

$$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t \\ z = z_1 + r_1 t \end{cases}, \quad t \in \mathbb{R};$$

and

$$d_2 : \begin{cases} x = x_2 + p_2 s \\ y = y_2 + q_2 s \\ z = z_2 + r_2 s \end{cases}, \quad s \in \mathbb{R}.$$

- The set of the intersection points of d_1 and d_2 is given by the set of the solutions of the system of equations

We search for (t, s).

$$\begin{cases} x_1 + p_1 t = x_2 + p_2 s \\ y_1 + q_1 t = y_2 + q_2 s \\ z_1 + r_1 t = z_2 + r_2 s \end{cases} . \tag{11}$$

- If the system (11) has a unique solution (t_0, s_0), then the lines d_1 and d_2 have exactly one intersection point P_0, corresponding to t_0 (or s_0). One says that the lines are *concurrent* (or *incident*); $\{P_0\} = d_1 \cap d_2$.
- The vectors $\overline{v}_1$ and $\overline{v}_2$ are in that case linearly independent.

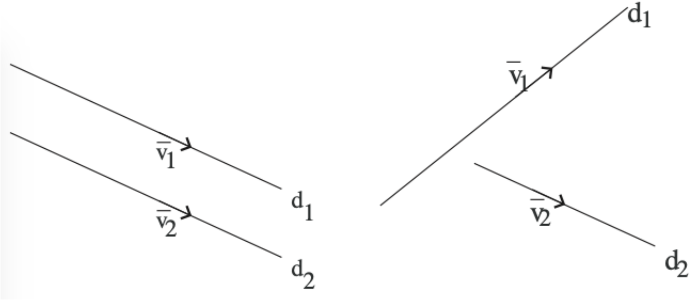
- If the system (11) has infinitely many solutions, then the two lines have infinitely many points in common, so they coincide. We say that these lines are *identical*; $d_1 = d_2$. There exists $\alpha \in \mathbb{R}^*$ such that $\overline{v}_1 = \alpha \overline{v}_2$ (their director vectors are linearly dependent) and any arbitrary point of d_1 belongs to d_2 (and vice-versa).

Suppose the system

$$\begin{cases} x_1 + p_1 t = x_2 + p_2 s \\ y_1 + q_1 t = y_2 + q_2 s \\ z_1 + r_1 t = z_2 + r_2 s \end{cases} \tag{12}$$

is not compatible.

- If the director vectors are linearly dependent (there exists $\alpha \in \mathbb{R}^*$ such that $\overline{v}_1 = \alpha \overline{v}_2$ or, equivalently, $\dfrac{p_1}{p_2} = \dfrac{q_1}{q_2} = \dfrac{r_1}{r_2}$), then the lines are *parallel*; $d_1 \parallel d_2$.
- If the director vectors are linearly independent, then one deals with *skew* lines (nonparallel and nonincident); $d_1 \cap d_2 = \emptyset$ and $d_1 \nparallel d_2$.


$\bar{v}_1$
d_1
$\bar{v}_2$
d_2
d_1
$\bar{v}_1$
$\bar{v}_2$
d_2

Relative position of two planes

- Let

$$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0, \qquad \overline{n}_1(a_1, b_1, c_1) \neq \overline{0}$$

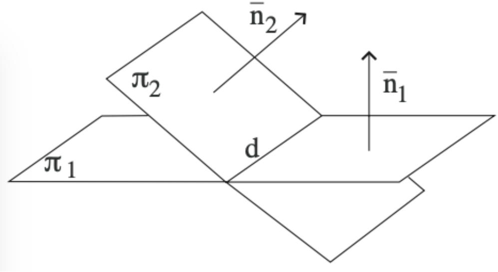
and

$$\pi_2 : a_2x + b_2y + c_2z + d_2 = 0, \qquad \overline{n}_2(a_2, b_2, c_2) \neq \overline{0}$$

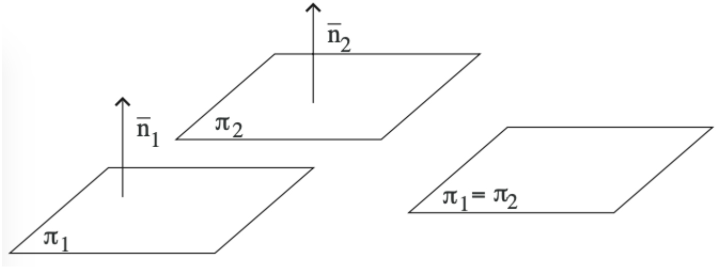
be two planes, having the normal vectors $\overline{n}_1$, respectively $\overline{n}_2$.

- The intersection of these planes is given by the solution of the system of equations

$$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \end{cases} . \tag{13}$$


$\bar{n}_2$
π_2
$\bar{n}_1$
d
π_1

- If $\operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$, then the system (13) is compatible and the planes have a line in common. They are *incident*; $\pi_1 \cap \pi_2 = d$.
- If $\operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 1$, then the rows of the matrix are proportional, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, which means that the normal vectors of the planes are linearly dependent. There are two possible situations:


$\bar{n}_2$
$\bar{n}_1$
π_2
π_1
$\pi_1 = \pi_2$

- If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2}$, then the system (13) is not compatible, and the planes are *parallel*; $\pi_1 \parallel \pi_2$.
- If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = \dfrac{d_1}{d_2}$, then the planes are *identical*; $\pi_1 = \pi_2$.

The problem set for this week will be posted soon. Ideally you would think about it before the seminar on Friday.

Thank you very much for your attention!