Seminar 2

1. For $(\mathbb{R}, *)$ associativity holds, the identity element is 6, but the symmetrical element, $x' = \frac{e-5}{x-5} + 5$ does not exist if $x = 5$. So $(\mathbb{R}, *)$ is not a group. But $(\mathbb{R} - \{5\}, *)$ is a group (with the same reasoning).

2. $GL_n(\mathbb{R})$ is a stable subset of $M_n(\mathbb{R})$ as, $\forall A, B \in GL_n(\mathbb{R})$, with $det(A) \neq 0$ and $det(B) \neq 0 \Rightarrow det(A \cdot B) = det(A) \cdot det(B) \neq 0$. So, $A \cdot B \in GL_n(\mathbb{R})$.

   Now, $(GL_n(\mathbb{R}), \cdot)$ is a group, as the multiplication of matrices is associative, the identity element is I_n, which is in $GL_n(\mathbb{R})$ as $det(I_n) = 1 \neq 0$, and $\forall A \in GL_n(\mathbb{R}) \Rightarrow det(A) \neq 0 \Rightarrow \exists A^{-1}$ the inverse of A, which is in $GL_n(\mathbb{R})$ as $det(A \cdot A^{-1}) = det(I_n) \neq 0$ and $det(A) \cdot det(A^{-1}) \neq 0$ with $det(A) \neq 0 \Rightarrow det(A^{-1}) \neq 0$.

3. U_n is a stable subset of C^*, because $\forall z_1, z_2 \in U_n \Rightarrow z_1^n = 1$ and $z_2^n = 1 \Rightarrow (z_1 \cdot z_2)^n = z_1^n \cdot z_2^n = 1$ (Remember, we can say this because the multiplication of complex numbers is commutative) $\Rightarrow z_1 \cdot z_2 \in U_n$.

   Now, $(U_n, \cdot)$ is a group, because the multiplication of complex numbers is associative, the identity element is 1, which is in U_n and $\forall z \in U_n, \exists z^{-1}$ its symmetric, which is in U_n, as $z^{-1} = \frac{1}{z}$ (which is true, as 0 is not in U_n) $\Rightarrow (z^{-1})^n = \frac{1^n}{z^n} = \frac{1}{1} = 1$.

   For $n = 1 \Rightarrow U_1 = \{1\}$.

   For $n = 2 \Rightarrow U_2 = \{-1, 1\}$.

   For $n \geq 3 \Rightarrow U_n = \{\cos(\frac{2k\pi}{n}) + i \sin(\frac{2k\pi}{n}) \mid k \in \{0, \ldots, n-1\}\}$.

4. Associativity: $\forall \hat{x} = x + nk, \hat{y} = y + nl, \hat{z} = z + nj$ in $\mathbb{Z}_n$, with $k, l, j \in \mathbb{Z}$, we have: $(\hat{x} + \hat{y}) + \hat{z} = (x + y + n(k + l)) + z + nj = x + y + z + n(k + l + j) = x + nk + (y + z + n(l + j)) = \hat{x} + (\hat{y} + \hat{z})$.

   Commutativity: $\forall \hat{x} = x + nk, \hat{y} = y + nl \in \mathbb{Z}_n$, with $k, l \in \mathbb{Z}$, we have: $\hat{x} + \hat{y} = x + y + n(k + l) = y + x + n(k + l) = \hat{y} + \hat{x}$.

   Identity element: $\exists \hat{e} \in \mathbb{Z}_n, \forall \hat{x} \in \mathbb{Z}_n$ such that: $\hat{x} + \hat{e} = \hat{x} \Rightarrow x + nk + e + nl = n + nk \Rightarrow e + nl = 0 + n \cdot 0 \Rightarrow \hat{e} = \hat{0} \in \mathbb{Z}_n$.

   Symmetric elements: $\forall \hat{x} \in \mathbb{Z}_n, \exists \hat{x'} \in \mathbb{Z}_n$ such that: $\hat{x} + \hat{x'} = \hat{0} \Rightarrow x + nk + x' + nl = 0 + n \cdot 0 \Rightarrow x' = -x, l = -k \Rightarrow \hat{x'} = \widehat{-x} \in \mathbb{Z}_n$.

Now, $\forall \hat{x} \in \mathbb{Z}_n \Rightarrow \hat{x} = x + n\mathbb{Z}_n$. If $x = n \Rightarrow \hat{x} = \hat{n} = n + n\mathbb{Z}_n$, but $n \in n\mathbb{Z}_n$, so $\hat{n} = n\mathbb{Z}_n = 0 + n\mathbb{Z}_n = \hat{0}$. The same goes for any multiple of n. So, $\hat{0}$ is a representative for all multiples of n.

If $x = n + 1 \Rightarrow \hat{x} = \widehat{n+1} = n + 1 + n\mathbb{Z}_n = 1 + n\mathbb{Z}_n = \hat{1}$. So, $\hat{1}$ is a representative for any $kn + 1, k \in \mathbb{Z}$.

And we do the same, until we get another multiple of n. So, in the end, $\mathbb{Z}_n = \{\hat{0}, \hat{1}, \hat{2}, \ldots, \widehat{n-1}\}$. Hence, $card(\mathbb{Z}_n) = n$.

5. (i) $(S_M, \circ)$ is a group, as the composition of functions is associative, the identity element is the identity function $1_M(x) = x$, which is a bijective function and, $\forall f \in S_M$, with f bijective $\Rightarrow \exists f^{-1}$ its symmetric, which is also bijective.

   (ii) Here, S_3 is the set of all bijective functions with domain and codomain of three elements. So, S_3 is actually the set of permutations of three elements.

6. Let $D_3 = \{r_0, r_1, r_2, s_1, s_2, s_3\}$, where:

   r_0 is a rotation by $360°$

   r_1 is a rotation by $120°$

   r_2 is a rotation by $240°$

   s_1, s_2, s_3 are symmetries across the 3 axis of the triangle.

   We know that: $r_i r_j = r_{i+j}$, $r_i s_j = s_{i+j}$ and $s_i r_j = s_i$. So, we may complete the table.

   We can also see the group as the group of permutations of three elements, where the elements are the points of the triangle.

7. The same as *Exercise 6*, but D_4 is the group of permutations of four elements.

8. This is true, as the associativity is easy to prove, the identity elements if the pair (e, e'), where e is the identity element in G and e' is the identity element in G', and the symmetric of any element of the form (g_1, g_1') is the pair $(g_1^*, g_1'^*)$, where g_1^* is the symmetric of g_1 in G and $g_1'^*$ is the symmetric of g_1' in G'.

9. For $(\mathbb{N}, +)$ we have $(\{0\}, +)$.

   For $(\mathbb{N}, \cdot)$ we have $(\{1\}, \cdot)$.

   For $(\mathbb{Z}, \cdot)$ we have $(\{-1, 1\}, \cdot)$.

   For $(\mathbb{Q}, \cdot)$ we have $(\mathbb{Q}^*, \cdot)$.

   For $(\mathbb{R}, \cdot)$ we have $(\mathbb{R}^*, \cdot)$.

   For $(\mathbb{C}, \cdot)$ we have $(\mathbb{C}^*, \cdot)$.

   For $(M_n(\mathbb{R}), \cdot)$ we have $(GL_n(\mathbb{R}), \cdot)$. *See exercise 2*

   For $(M^M, \circ)$ we have $(S_M, \circ)$. *See exercise 5*

10. (i) If G is Abelian $\Rightarrow$ the operation is commutative $\Rightarrow$ $(xy)^2 = (xy)(xy) = xyxy = xxyy = x^2y^2$.

    If $(xy)^2 = x^2y^2 \Rightarrow xyxy = xxyy$. We multiply of the right with y^{-1} and on the left with x^{-1} (we have them as G is a group) $\Rightarrow yx = xy \Rightarrow G$ is Abelian.

    (ii) $\forall x, y \in G \Rightarrow x^2 = 1, y^2 = 1$ and $xy \in G \Rightarrow (xy)^2 = 1 \Rightarrow xyxy = 1$. We multiply on the right with y and on the left with $x \Rightarrow x^2yxy^2 = xy \Rightarrow yx = xy \Rightarrow G$ is Abelian.