CHAPTER 3

Affine subspaces in dimension 2

Contents

3.1 Affine lines

- An affine line is a 1-dimensional affine space. Here we look at how affine lines appear as subspaces of a two dimensional affine space. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space of dimension 2. Let $\mathbf{A}$ be an affine space over $\mathbf{V}$.

- While most of the content should look familiar to you, bear in mind that we are working with some arbitrary field $\mathbf{K}$.

- Let $O\mathbf{e}_1\mathbf{e}_2$ be a reference frame (another term for coordinate system) of $\mathbf{A}$.

- Every line ℓ has parametric equations of the form

$$\ell : \begin{cases} x = x_Q + tv_x \\ y = y_Q + tv_y \end{cases} \quad \text{or, in matrix form,} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_Q \\ y_Q \end{bmatrix} + t \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix}. \tag{3.1}$$

  where $Q(x_Q, y_Q)$ is a point in ℓ and $\mathbf{v}(v_x, v_y)$ is a direction vector of ℓ. The equations describe the points $P(x,y)$ on ℓ obtained for the different values of the parameter $t \in \mathbf{K}$.

- Clearly, for different choices of $Q \in \ell$ and different direction vectors $\mathbf{v}$ of ℓ one obtains different parametric equations for ℓ.
- Since ℓ is a hyperplane in $\mathbf{A}$, it is described by a single equation
$$\ell : ax + by + c = 0 \tag{3.2}$$
for some $a, b, c \in \mathbf{K}$ with $(a, b) \neq (0, 0)$.
- The constants a, b, c are determined by ℓ only up to a non-zero common factor, so a line has in general many equations, each proportional to the others.
- The lines with equations $x = 0$ and $y = 0$ are the *coordinate axes*: the y-axis and the x-axis respectively.
- [From parametric equations to Cartesian equations] Suppose we are given a parametric equation (3.1) for the line ℓ, i.e. Q and $\mathbf{v}$ are given. To obtain a cartesian equation for ℓ one should view (3.1) as saying that $P(x_P, y_P)$ lies in ℓ if and only if the vector $\overrightarrow{QP}$ is parallel to $\mathbf{v}$. This last condition can also be expressed by requiring that (x_P, y_P) is a solution of the equation
$$\begin{vmatrix} x - x_Q & y - y_Q \\ v_x & v_y \end{vmatrix} = 0$$
which is equivalent to
$$v_y x - v_x y + v_x y_Q - v_y x_Q = 0.$$
- [From Cartesian equations to parametric equations] Suppose we are given the equation (3.1) for the line ℓ. If $b \neq 0$ then, viewing x as a 'free variable', (3.1) is equivalent to
$$\begin{cases} x = x \\ y = -\frac{a}{b}x - \frac{c}{b} \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{c}{b} \end{bmatrix} + x \begin{bmatrix} 1 \\ -\frac{a}{b} \end{bmatrix}$$
which are parametric equations for ℓ in the parameter $x \in \mathbf{K}$. The case when $a \neq 0$ can be treated similarly.

3.2 Relative positions of two lines

Proposition 3.1. Let ℓ and ℓ' be two lines in $\mathbf{A}$ with Cartesian equations
$$\ell : ax + by + c = 0 \quad \text{and} \quad \ell' : a'x + b'y + c = 0.$$
Then

1. ℓ and ℓ' are parallel if and only if the matrix
$$\begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \tag{3.3}$$
has rank 1, that is, if and only if $ab' - a'b = 0$.

2. If ℓ and ℓ' are parallel, then they are disjoint or they coincide accordingly as the matrix

$$\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} \tag{3.4}$$

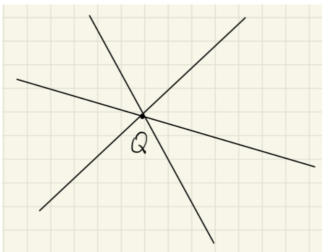
has rank 2 or 1.

3. ℓ and ℓ' have precisely one point in common if and only if the matrix (3.3) has rank 2. In this case the point $\ell \cap \ell'$ has coordinates

$$x_0 = \frac{cb' - c'b}{ab' - a'b}, \quad y_0 = \frac{ac' - a'c}{ab' - a'b}.$$

- The above proposition covers all possibilities of relative positions of two lines in $\mathbf{A}$, because they correspond to all possibilities for two equations in two unknowns.
- In particular, the only possibility for two lines in an affine plane not to intersect is for them to be parallel. Thus, two lines in an affine plane cannot be skew.

3.3 Pencils of lines



Definition 3.2. Let $Q \in \mathbf{A}$. The set $\mathcal{L}_Q$ of all lines in $\mathbf{A}$ passing through Q is called a *pencil of lines* and Q is called the *center* of the pencil $\mathcal{L}_Q$.

Proposition 3.3. If $\ell_1 : ax + by + c = 0$ and $\ell_2 : a'x + b'y + c' = 0$ are two distinct lines in the pencil $\mathcal{L}_Q$, then $\mathcal{L}_Q$ consists of lines having equations of the form

$$\ell_{\lambda,\mu} : \lambda(ax + by + c) + \mu(a'x + b'y + c') = 0.$$

where $\lambda, \mu \in \mathbf{K}$ not both zero.

- In particular, if $Q = Q(x_0, y_0)$, $\ell_1 : x = x_0$ and $\ell_2 : y = y_0$ then

$$\mathcal{L}_Q = \left\{ \ell_{\lambda,\mu} : \lambda(x - x_0) + \mu(y - y_0) = 0 : \lambda, \mu \in \mathbf{K} \text{ not both zero} \right\}.$$

- Pencils of lines are useful in praxis when a point Q is given as the intersection of two lines, but its coordinates are not known explicitly, and one wants to find the equation of a line passing through Q and satisfying some other condition. For example, the condition that it contain some point P distinct from Q or that it is parallel to a given line.

- There is redundancy in the two parameters λ,μ, meaning that there are not two independent parameters here. If $\lambda\neq 0$ then one can divide the equation of $\ell_{\lambda,\mu}$ by λ to obtain

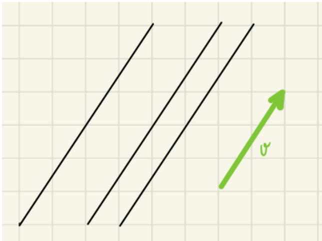
$$\ell_{1,t}:(ax+by+c)+t(a'x+b'y+c')=0.$$

where $\frac{\mu}{\lambda}=t\in\mathbf{K}$. So $\ell_{1,\frac{\mu}{\lambda}}$ and $\ell_{\lambda,\mu}$ are in fact the same lines.

- A *reduced pencil* is the set of all lines $\mathcal{L}_Q$ passing through a common point Q from which we remove one line, i.e. it is $\mathcal{L}_Q\setminus\{\ell_2\}$ for some $\ell_2\in\mathcal{L}_Q$. With the above notation and discussion, it is the set

$$\{\ell_{1,t}:(ax+by+c)+t(a'x+b'y+c')=0:t\in\mathbf{K}\}$$

The fact that we use one parameter instead of two, to describe almost all lines passing through Q greatly simplifies calculations.



Definition 3.4. Let $\mathbf{v}\in\mathbf{V}$. The set $\mathcal{L}_{\mathbf{v}}$ of all lines in $\mathbf{A}$ with direction $\langle\mathbf{v}\rangle$ is called an *improper pencil of lines*, and $\langle\mathbf{v}\rangle$ is called the direction of the pencil $\mathcal{L}_{\mathbf{v}}$.

- The connection between pencils of lines and improper pencils of lines is best understood through projective geometry, where the improper pencil of lines is the set of all lines intersecting in the same point at infinity.

3.4 Classical theorems

How does a line intersect parallel lines? How does a line lie relative to an improper pencil? How do two lines intersect an improper pencil?

Example 3.5. Let $Q(x_0,y_0)$ be a point in $\mathbf{A}$ and consider the reflection $\mathrm{Ref}_Q:\mathbf{A}\to\mathbf{A}$ in the point Q. If $\ell\subseteq\mathbf{A}$ is a line with parametric equations

$$\ell:\begin{cases}x=x_A+tv_x\\ y=y_A+tv_y\end{cases}\quad\Leftrightarrow\quad\text{any point on }\ell\text{ is}\quad P(x_A+tv_x,y_A+tv_y)\quad\text{for some }t\in\mathbf{K},$$

then, for the points on ℓ we have $\mathrm{Ref}_Q(P)=(2x_0-x_A-tv_x,2y_0-y_A-tv_y)$. In other words, the image of ℓ under the reflection is a line which has parametric equations

$$\mathrm{Ref}_Q(\ell):\begin{cases}x=2x_0-x_A-tv_x\\ y=2y_0-y_A-tv_y\end{cases}.$$

- The reflection Ref_Q maps a line in the pencil $\mathcal{L}_Q$ to itself.
- The point P, Q and $\mathrm{Ref}_Q(P)$ are collinear (for $P \neq Q$) and Q divides the segment $[P\,\mathrm{Ref}_Q(P)]$ in ratio $1:2$.
- The reflection Ref_Q maps parallel lines to parallel lines, i.e. it preserves all improper pencils.

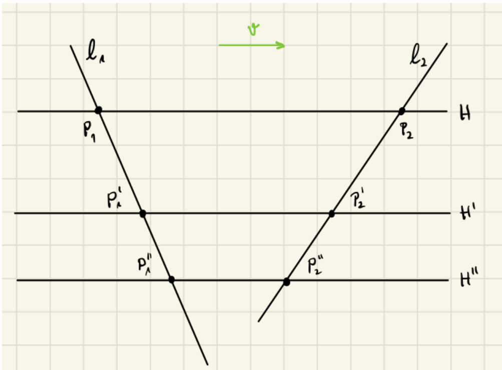
Theorem 3.6 (Thales)**.** Let H, H' and H'' be three distinct parallel lines in an affine plane $\mathbf{A}$, and let ℓ_1 and ℓ_2 be two lines not parallel to H, H', H''. For $i = 1,2$ let

$$\begin{array}{rcl} P_i & = & \ell_i \cap H \\ P_i' & = & \ell_i \cap H' \\ P_i'' & = & \ell_i \cap H'' \end{array}$$

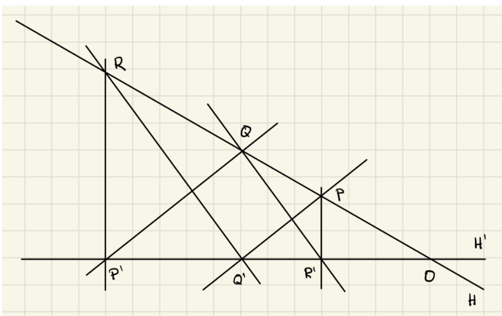
and let $k_1, k_2 \in \mathbf{K}$ be such that

$$\overrightarrow{P_iP_i''} = k_i\,\overrightarrow{P_iP_i'}$$

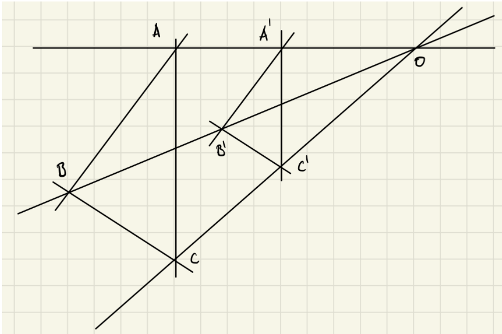
Then $k_1 = k_2$.



Theorem 3.7 (Pappus)**.** Let H, H' be two distinct lines in an affine plane $\mathbf{A}$. Let $P, Q, R \in H$ and $P', Q', R' \in H$ be distinct points, none of which lies at the intersection $H \cap H'$. If $\langle P, Q' \rangle \parallel \langle P', Q \rangle$ and $\langle Q, R' \rangle \parallel \langle Q', R \rangle$ and then $\langle P, R' \rangle \parallel \langle P', R \rangle$.

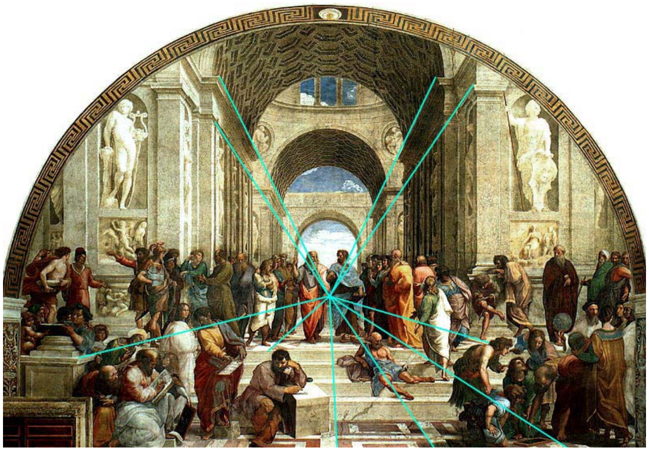


Theorem 3.8 (Desargues)**.** Let $A, B, C, A', B', C' \in \mathbf{A}$ be points such that no three are collinear, and such that $\langle A, B\rangle \parallel \langle A', B'\rangle$, $\langle B, C\rangle \parallel \langle B', C'\rangle$ and $\langle A, C\rangle \parallel \langle A', C'\rangle$. Then the three lines $\langle A, A'\rangle$, $\langle B, B'\rangle$ and $\langle C, C'\rangle$ are either parallel or have a point in common.



3.5 Connections to reality

Thales' theorem gives a way of organizing perspective in a painting. This is especially useful for large scenes. It can also be observed in picture depicting or simulating reality.



(a) Scuola di Atene, Raphael, 1510ish.

(b) Assassin's Creed.

Clearly, it would be a bit difficult to simulate reality frame by frame with paintings. It is nevertheless a first encounter with the notion of perspective, which is needed to accurately simulate reality.