Calculus on $\mathbb{R}$ Lect. dr. Anca Grad
Homework 7 **2021**

Limits of Functions

Recall the following:

$$\lim_{x\to\infty} q^x = \begin{cases} +\infty & : q > 1 \\ 1 & : q = 1 \\ 0 : & |q| < 1 \\ \nexists & : q \leq 1 \end{cases}$$

$$\lim_{x\to\infty} q^x = q^{x_0}, \forall q \in (0,\infty) \quad and \quad x_0 \in \mathbb{R}$$

$$\lim_{x\to x_0} \log_a x = \log_a x_0, \forall a \in (0,\infty) \setminus \{1\}, x_0 > 0.$$

$$\lim_{x\to\infty} \log_a x = \begin{cases} +\infty & : a > 1 \\ -\infty & : 0 < a < 1 \end{cases}$$

$$\lim_{x\to 0} \frac{q^x - 1}{x} = \ln q, \forall q > 0 \quad and \quad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

Exercise 1: Compute the limits of the following functions at the specified points:

$$a)\lim_{x\to\infty} x\cos^2\frac{x+2}{x} \quad b)\lim_{x\to 1}\frac{x}{x^2+1} \quad c)\lim_{x\to-\infty}\frac{x^2+5}{x^3} \quad d)\lim_{x\to\infty}\frac{(x+2)(2x+1)}{x^2+3x+5}$$

$$e)\lim_{x\to 1}\frac{x^2-1}{x^3-1} \quad f)\lim_{x\to 2}\left(\frac{1}{2-x}-\frac{2x}{4-x^2}\right)$$

$$g)\lim_{x\to 1}\frac{1+x+x^2+...+x^n-(n+1)}{x-1}, n\in\mathbb{N} \quad h)\lim_{x\to 1}\frac{x+x^2+...+x^n-n}{x+x^2+...+x^m-m}, \forall m,n\in\mathbb{N}.$$

$$i)\lim_{x\to 27}\frac{x-27}{\sqrt[3]{x}-3} \quad j)\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

$$k)\lim_{x\to\infty}\left(\sqrt[3]{ax^3+x^2+bx+c}-(bx+c)\right)\forall a,b,c>0.$$

Exercise 2: Compute the limits of the following functions at the specified points:

$$a)\lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}} \quad b)\lim_{x\to 0}\left(\frac{3\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}}$$

$$c)\ \lim_{x\to 0}(1+\cos x)^{\frac{1}{x^2}} \qquad d)\ \lim_{x\to 0}(e^x - x + 1)^{\frac{1}{1-\cos x}}$$

$$e)\ \lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} \qquad f)\ \lim_{x\to\infty}\left(\frac{x+7}{x}\right)^x$$

Exercise 3:

$$a)\ \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sin^2 x+\sin^2 2x+\ldots+\sin^2 nx\right)^{\frac{1}{n^3x^2}}\right]$$

$$a)\ \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx)\right)^{\frac{1}{n^2x}}\right]$$

Exercise 4: Compute the following limits:

$$a)\ \lim_{x\to 0}\frac{e^{2x}-1}{3x}; \quad b)\ \lim_{x\to 0}\frac{e^x-\cos x}{3x}.$$

Exercise 1: Compute the limits of the following functions at the specified points:

$$a)\ \lim_{x\to\infty} x\cos^2\frac{x+2}{x}\quad b)\ \lim_{x\to 1}\frac{x}{x^2+1}\quad c)\ \lim_{x\to-\infty}\frac{x^2+5}{x^3}\quad d)\ \lim_{x\to\infty}\frac{(x+2)(2x+1)}{x^2+3x+5}$$

$$e)\ \lim_{x\to 1}\frac{x^2-1}{x^3-1}\quad f)\ \lim_{x\to 2}\left(\frac{1}{2-x}-\frac{2x}{4-x^2}\right)$$

$$g)\ \lim_{x\to 1}\frac{1+x+x^2+\ldots+x^n-(n+1)}{x-1}, n\in\mathbb{N}\quad h)\ \lim_{x\to 1}\frac{x+x^2+\ldots+x^n-n}{x+x^2+\ldots+x^m-m}, \forall m,n\in\mathbb{N}.$$

$$i)\ \lim_{x\to 27}\frac{x-27}{\sqrt[3]{x}-3}\quad j)\ \lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

$$k)\ \lim_{x\to\infty}\left(\sqrt[3]{ax^3+x^2+bx+c}-(bx+c)\right)\ \forall a,b,c>0.$$

a) $\lim\limits_{x\to\infty} x\cos^2\dfrac{x+2}{x} = \infty\cdot\cos^2 1 = \infty$

b) $\lim\limits_{x\to 1}\dfrac{x}{x^2+1} = \dfrac{1}{2}$

c) $\lim\limits_{x\to-\infty}\dfrac{x^2+5}{x^3} = \lim\limits_{x\to-\infty}\dfrac{\cancel{x^2}\left(1+\frac{5}{x^2}\right)}{x^{\cancel{3}}} = 0$

d) $\lim\limits_{x\to\infty}\dfrac{(x+2)(2x+1)}{x^2+3x+5} = \lim\limits_{x\to\infty}\dfrac{2x^2+5x+2}{x^2+3x+5} = 2$

e) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^3-1} = \lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \lim\limits_{x\to 1}\dfrac{x+1}{x^2+x+1} = \dfrac{2}{3}$

f) $\lim\limits_{x\to 2}\left(\dfrac{^{2+x)}1}{2-x}-\dfrac{2x}{4-x^2}\right) = \lim\limits_{x\to 2}\dfrac{2+x-2x}{(2-x)(2+x)} = \lim\limits_{x\to 2}\dfrac{\cancel{2-x}}{\cancel{(2-x)}(2+x)} = \lim\limits_{x\to 2}\dfrac{1}{x+2} = \dfrac{1}{4}$

g) $\lim\limits_{x\to 1}\dfrac{1+x+x^2+\ldots+x^n-(n+1)}{x-1} = \lim\limits_{x\to 1}\dfrac{\cancel{(1-1)}+(x-1)+(x^2-1)+\ldots+(x^n-1)}{x-1} = \lim\limits_{x\to 1}\dfrac{(x-1)+(x-1)(x+1)+(x-1)(x^2+x+1)+\ldots+(x-1)(1+x+x^2+\ldots+x^{n-1})}{x-1} =$

$= \lim\limits_{x\to 1}\dfrac{\cancel{(x-1)}(1+(1+x)+(1+x+x^2)+\ldots+(1+x+x^2+\ldots+x^{n-1}))}{\cancel{x-1}} = 1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$, $n\in\mathbb{N}$

h) $\lim\limits_{x\to 1}\dfrac{x+x^2+\ldots+x^n-n}{x+x^2+\ldots+x^m-m} = \lim\limits_{x\to 1}\dfrac{(x-1)+(x^2-1)+\ldots+(x^n-1)}{(x-1)+(x^2-1)+\ldots+(x^m-1)} \overset{\text{as in g)}}{=} \lim\limits_{x\to 1}\dfrac{(x-1)(1+(1+x)+\ldots+(1+x+\ldots+x^{n-1}))}{(x-1)(1+(1+x)+\ldots+(1+x+\ldots+x^{m-1}))} =$

$= \dfrac{1+2+\ldots+n}{1+2+\ldots+m} = \dfrac{\frac{n(n+1)}{\cancel{2}}}{\frac{m(m+1)}{\cancel{2}}} = \dfrac{n(n+1)}{m(m+1)}$, $\forall m,n\in\mathbb{N}$

i) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$

$x = t^{12}$

$x\to 1 \Rightarrow t\to 1 \Rightarrow \lim\limits_{t\to 1}\dfrac{t^4-1}{t^3-1} = \lim\limits_{t\to 1}\dfrac{\cancel{(t-1)}(t^3+t^2+t+1)}{\cancel{(t-1)}(t^2+t+1)} = \dfrac{4}{3}$

k) $\lim\limits_{x\to\infty}\left(\sqrt[3]{ax^3+x^2+bx+c}-(bx+c)\right) = -c+\lim\limits_{x\to\infty}\left(\sqrt[3]{ax^3+x^2+bx+c}-bx\right) =$

$= -c+\lim\limits_{x\to\infty}\dfrac{ax^3+x^2+bx+c-b^3x^3}{\sqrt[3]{(ax^3+x^2+bx+c)^2}+bx\sqrt[3]{ax^3+x^2+bx+c}+b^2x^2} =$

$= -c+\lim\limits_{x\to\infty}\dfrac{(a-b^3)x^3+x^2+bx+c}{x^2\left(\sqrt[3]{\left(a+\frac{1}{x}+\frac{b}{x^2}+\frac{c}{x^3}\right)^2}+b\sqrt[3]{a+\frac{1}{x}+\frac{b}{x^2}+\frac{c}{x^3}}+b^2\right)} =$

We denote the limit by J

If $a-b^3=0 \Rightarrow a=b^3$: $J = -c+\dfrac{1}{3b^2}$, $\forall a,b,c>0$

If $a-b^3\neq 0$: $J = \begin{cases}-c-\infty, & \text{if } a-b^3<0\\ -c+\infty, & \text{if } a-b^3>0\end{cases} = \begin{cases}-\infty, & a-b^3<0\\ \infty, & a-b^3>0\end{cases}$

Exercise 4: Compute the following limits:

$$a)\lim_{x\to 0}\frac{e^{2x}-1}{3x};\quad b)\lim_{x\to 0}\frac{e^{x}-\cos x}{3x}.$$

$$a)\ \lim_{x\to 0}\frac{e^{2x}-1}{3x}=\frac{1}{3}\lim_{x\to 0}\frac{(e^{x}-1)(e^{x}+1)}{x}=\frac{1}{3}\lim_{x\to 0}\frac{e^{x}-1}{x}\cdot(e^{x}+1)=\frac{1}{3}\cdot 1\cdot 2=\frac{2}{3}$$

$$b)\ \lim_{x\to 0}\frac{e^{x}-\cos x}{3x}=\frac{1}{3}\lim_{x\to 0}\frac{e^{x}-1-\cos x+1}{x}=\frac{1}{3}\lim_{x\to 0}\left(\frac{e^{x}-1}{x}-\frac{\cos x-1}{x}\right)=\frac{1}{3}\cdot 1+\frac{1}{3}\lim_{x\to 0}\frac{1-\cos x}{x}=$$

$$=\frac{1}{3}+\frac{1}{3}\lim_{x\to 0}\frac{\cancel{1}-\cancel{1}+2\sin^{2}\frac{x}{2}}{x}=\frac{1}{3}+\frac{2}{3}\lim_{x\to 0}\frac{\sin^{2}\frac{x}{2}}{\frac{x^{2}}{4}}\cdot\frac{x^{2}}{4}\cdot\frac{1}{x}=\frac{1}{3}+\frac{2}{3}\cdot 1\cdot\frac{0}{4}=\frac{1}{3}$$

Exercise 3:

$$a)\lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sin^{2}x+\sin^{2}2x+\ldots+\sin^{2}nx\right)^{\frac{1}{n^{3}x^{2}}}\right]$$

$$a)\lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx)\right)^{\frac{1}{n^{2}x}}\right]$$

$$a)\ \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sin^{2}x+\sin^{2}2x+\ldots+\sin^{2}nx\right)^{\frac{1}{n^{3}x^{2}}}\right]=$$

$$=\lim_{n\to\infty}\left(\lim_{x\to 0}\left(\left(1+\sin^{2}x+\sin^{2}2x+\ldots+\sin^{2}nx\right)^{\frac{1}{\sin^{2}x+\ldots+\sin^{2}nx}}\right)^{\frac{\sin^{2}x+\sin^{2}2x+\ldots+\sin^{2}nx}{n^{3}x^{2}}}\right)=$$

$$=\lim_{n\to\infty}e^{\lim_{x\to 0}\frac{1}{n^{3}}\left(\frac{\sin^{2}x}{x^{2}}+\frac{\sin^{2}2x}{2^{2}\cdot x^{2}}\cdot 2^{2}+\ldots+\frac{\sin^{2}nx}{n^{2}x^{2}}\cdot n^{2}\right)}=$$

$$=\lim_{n\to\infty}e^{\frac{1+2^{2}+\ldots+n^{2}}{n^{3}}}=e^{\lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}}{n^{3}}}=e^{\frac{2}{6}}=e^{\frac{1}{3}}=\sqrt[3]{e}$$

$$b)\ \lim_{n\to\infty}\left(\lim_{x\to 0}\left(1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx)\right)^{\frac{1}{n^{2}x}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{x\to 0}\left(\left(1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx)\right)^{\frac{1}{\ln(1+x)+\ldots+\ln(1+nx)}}\right)^{\frac{\ln(1+x)+\ldots+\ln(1+nx)}{n^{2}x}}\right)=$$

$$=\lim_{n\to\infty}e^{\lim_{x\to 0}\frac{1}{n^{2}}\left(\frac{\ln(1+x)}{x}+\frac{\ln(1+2x)}{2x}\cdot 2+\ldots+\frac{\ln(1+nx)}{nx}\cdot n\right)}=$$

$$=e^{\lim_{n\to\infty}\frac{1}{n^{2}}(1+2+\ldots+n)}=e^{\lim_{n\to\infty}\frac{n(n+1)}{2n^{2}}}=e^{\frac{1}{2}}=\sqrt{e}$$

Exercise 2: Compute the limits of the following functions at the specified points:

$$a)\ \lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}} \qquad b)\ \lim_{x\to 0}\left(\frac{3\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}}$$

$$c)\ \lim_{x\to 0}(1+\cos x)^{\frac{1}{x^2}} \qquad d)\ \lim_{x\to 0}(e^x-x+1)^{\frac{1}{1-\cos x}}$$

$$e)\ \lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} \qquad f)\ \lim_{x\to\infty}\left(\frac{x+7}{x}\right)^{x}$$

a) $\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}} = 0^{\frac{5}{2}} = 0$

b) $\lim\limits_{x\to\infty}\left(\frac{3\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}} = \lim\limits_{x\to 0}\left(3\cdot\underbrace{\frac{\sin x}{x}}_{=1}-\underbrace{\frac{\tan x}{x}}_{=1}\right)^{\frac{\sin x}{x}+2} = (3-1)^{1+2} = 2^3 = 8$

c) $\lim\limits_{x\to 0}(1+\cos x)^{\frac{1}{2}} = (1+1)^{\frac{1}{0+}} = 2^{\infty} = \infty$

d) $\lim\limits_{x\to 0}(e^x-x+1)^{\frac{1}{1-\cos x}} = \lim\limits_{x\to 0}(e^x-x+1)^{\frac{1}{1-1+2\sin^2\frac{x}{2}}} = \lim\limits_{x\to 0}(e^x-x+1)^{\frac{1}{2\sin^2\frac{x}{2}}} =$

$= \lim\limits_{x\to 0}(e^x-x+1)^{\frac{1}{\cancel{2}\frac{\sin^2\frac{x}{2}}{\frac{x^2}{2}}\cdot\frac{x^2}{\cancel{2}}}} = \lim\limits_{x\to 0}(e^x-x+1)^{\frac{1}{x^2}} = (1-0+1)^{\frac{1}{0+}} = 2^{\infty} = \infty$

e) $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}} = \lim\limits_{x\to 0}\left((1+\sin x)^{\frac{1}{\sin x}}\right)^{\frac{\sin x}{x}} = e^{\lim\limits_{x\to 0}\frac{\sin x}{x}} = e^1 = e$

f) $\lim\limits_{x\to\infty}\left(\frac{x+7}{x}\right)^{x} = \lim\limits_{x\to\infty}\left(1+\frac{7}{x}\right)^{x} = \lim\limits_{x\to\infty}\left(\left(1+\frac{7}{x}\right)^{\frac{x}{7}}\right)^{\frac{7x}{x}} = e^{7}$