CHAPTER 5

Changing coordinates

Contents

5.1 Linear maps and matrices

Definition 5.1 (Algebra 1, Lecture 11)**.** Let $\mathbf{V}$ and $\mathbf{W}$ be two $\mathbf{K}$-vector spaces. A *map* $\phi : \mathbf{V} \to \mathbf{W}$ is said to be *linear* if for every $\mathbf{v}, \mathbf{u} \in \mathbf{V}$ and $a, b \in \mathbf{K}$ one has

$$\phi(a\mathbf{v} + b\mathbf{u}) = a\phi(\mathbf{v}) + b\phi(\mathbf{u}).$$

Theorem 5.2 (Algebra 1, Lecture 12, Theorem 3)**.** Let $\mathbf{V}$ and $\mathbf{W}$ be two $\mathbf{K}$-vector spaces. Let $\boldsymbol{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbf{V}$ and let $\mathbf{w}_1, \ldots, \mathbf{w}_k$ be arbitrary vectors of $\mathbf{W}$. Then there is a unique linear map $\phi : \mathbf{V} \to \mathbf{W}$ such that

$$\phi(\mathbf{v}_i) = \mathbf{w}_i \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

Definition 5.3. Let $\mathbf{V}$ and $\mathbf{W}$ be two $\mathbf{K}$-vector spaces with basis $\boldsymbol{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\boldsymbol{w} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ respectively. Let $\phi : \mathbf{V} \to \mathbf{W}$ be a linear map. The $k \times n$ matrix whose j-th column ($j = 1, \ldots, n$) consists of the coordinates of the vector $\phi(\mathbf{v}_j) \in \mathbf{W}$ with respect to the basis $\boldsymbol{w}$ is called *the matrix of ϕ with respect to the basis $\boldsymbol{v}$ and $\boldsymbol{w}$*, and is denoted by $\mathrm{M}_{\boldsymbol{w},\boldsymbol{v}}(\phi)$. Explicitly,

$$\mathrm{M}_{\boldsymbol{w},\boldsymbol{v}}(\phi) = \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{k1} & m_{m2} & \ldots & m_{kn} \end{bmatrix}$$

where
$$\phi(\mathbf{v}_j) = m_{1j}\mathbf{w}_1 + m_{2j}\mathbf{w}_2 + \cdots + m_{kj}\mathbf{w}_k.$$

- In your Algebra 1 course you used the notation $[\phi]_{v,w}$ for $\mathrm{M}_{w,v}(\phi)$. The indices v, w are *reversed*.
- Notice that $\mathrm{M}_{w,v}(\phi)$ depends not only on ϕ but also on the bases v and w.
- The importance of the matrix $\mathrm{M}_{w,v}(\phi)$ lies in the fact that, once the basis v and w are fixed, the map can be reconstructed from the matrix.

Theorem 5.4 (Algebra 1, Lecture 13, Theorem 9)**.** Let $\mathbf{V}$ and $\mathbf{W}$ be two $\mathbf{K}$-vector spaces with basis $v = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $w = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ respectively. The map
$$\mathrm{M}_{w,v} : \mathrm{Hom}(\mathbf{V}, \mathbf{W}) \to \mathrm{M}_{k,n}(\mathbf{K}) \quad \text{defined by} \quad \phi \mapsto \mathrm{M}_{w,v}(\phi)$$
is an isomorphism of $\mathbf{K}$-vector spaces.

Proposition 5.5 (Algebra 1, Lecture 13, Theorem 8)**.** Let $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ be $\mathbf{K}$-vector spaces with basis $u = \{\mathbf{u}_1, \ldots, \mathbf{u}_s\}$, $v = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $w = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ respectively. Let $\phi : \mathbf{U} \to \mathbf{V}$ and $\psi : \mathbf{V} \to \mathbf{W}$ be linear maps. Then
$$\mathrm{M}_{w,u}(\psi \circ \phi) = \mathrm{M}_{w,v}(\psi)\mathrm{M}_{v,u}(\phi).$$

Definition 5.6. A particularly important case arises when v and w are two distinct basis of $\mathbf{V}$ and $\phi = \mathrm{Id}_{\mathbf{V}}$, the identity map. In this case, $\mathrm{M}_{w,v}(\mathrm{Id}_{\mathbf{V}})$ is called the *change of basis matrix from the basis v to the basis w*.

- By definition the j-th column of $\mathrm{M}_{w,v}(\mathrm{Id}_{\mathbf{V}})$ consists of the components of the basis vector $\mathbf{v}_j$ with respect to the basis w.
- For every $\mathbf{v} \in \mathbf{V}$ one has
$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = x'_1\mathbf{w}_1 + \cdots + x'_n\mathbf{w}_n$$
and, letting $\mathbf{x} = (x_1, \ldots, x_n)^t$, $\mathbf{x}' = (y_1, \ldots, y_n)^t$, one has
$$\mathbf{x}' = \mathrm{M}_{w,v}(\mathrm{Id}_{\mathbf{V}})\mathbf{x}.$$
- Thus, with the matrix $\mathrm{M}_{w,v}(\mathrm{Id}_{\mathbf{V}})$ one can find the components $\mathbf{x}'$ of any vector $\mathbf{v}$ with respect to the basis w given the components $\mathbf{x}$ of $\mathbf{v}$ with respect to the basis v.
- Note that by Proposition 5.5,
$$\mathrm{M}_{v,w}(\mathrm{Id}_{\mathbf{V}})\mathrm{M}_{w,v}(\mathrm{Id}_{\mathbf{V}}) = \mathrm{M}_{v,v}(\mathrm{Id}_{\mathbf{V}}) = \mathrm{Id}_n$$
and so
$$\mathrm{M}_{w,v}(\mathrm{Id}_{\mathbf{V}}) = \mathrm{M}_{v,w}(\mathrm{Id}_{\mathbf{V}})^{-1}.$$

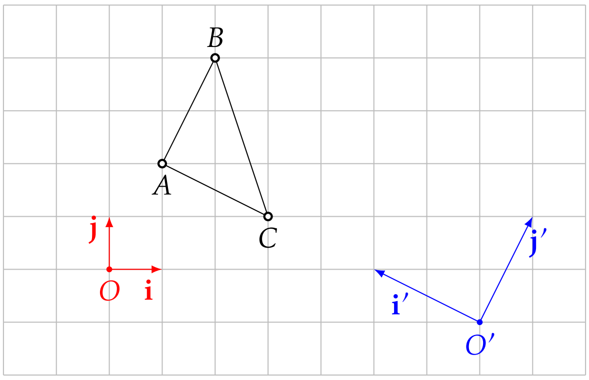
Definition 5.7. Suppose $\mathbf{V}$ is a *real* vector space. Two bases $e = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $f = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ of $\mathbf{V}$ are said to *have the same orientation* if $\det(\mathrm{M}_{e,f}(\mathrm{Id}_{\mathbf{V}})) \geq 0$, and one writes $e \sim_{\mathrm{or}} f$. Otherwise the bases have *different orientations*.

5.2 Changing reference frames

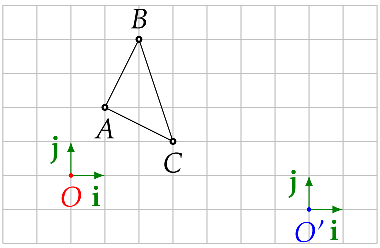
Example 5.8 (In dimension 2)**.** Let $\mathbf{A}$ be the two-dimensional real affine space $\mathbf{A}^2(\mathbb{R})$. Let $\mathcal{K} = O\mathbf{ij}$ and $\mathcal{K}' = O'\mathbf{i}'\mathbf{j}'$ be two coordinate systems (reference frames). Suppose that we know O', $\mathbf{i}'$ and $\mathbf{j}'$ relative to $\mathcal{K}$:

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}_{\mathcal{K}}, \quad \mathbf{i}' = -2\mathbf{i} + \mathbf{j} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad \mathbf{j}' = \mathbf{i} + 2\mathbf{j} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{K}}.$$

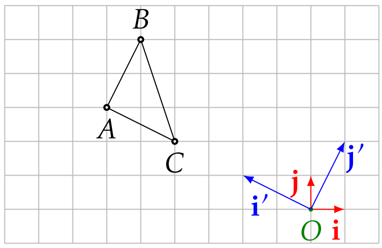
How can we translate the coordinates of points from $\mathcal{K}$ to $\mathcal{K}'$?



We can do this in two steps: (a) first we change the origin, i.e. we go from $O\mathbf{ij}$ to $O'\mathbf{ij}$ and (b) we change the directions of the coordinate axes, i.e. we go from $O'\mathbf{ij}$ to $O'\mathbf{i}'\mathbf{j}'$. The first step is just a translation and the second step corresponds to the usual base change from linear algebra.



(a) Change the origin.



(b) Change the direction of the axes.

For the first step

$$[\overrightarrow{O'A}]_{\mathcal{K}'} = [\overrightarrow{O'A}]_{\mathcal{K}} = [\overrightarrow{OA}]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{K}}.$$

For the second step let $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$ denote the base change matrix from the basis in $\mathcal{K}$ to the basis in $\mathcal{K}'$. Then

$$[\overrightarrow{OA}]_{\mathcal{K}} = \mathrm{M}_{\mathcal{K},\mathcal{K}'}[\overrightarrow{OA}]_{\mathcal{K}'} \quad \text{and} \quad [\overrightarrow{OA}]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OA}]_{\mathcal{K}}.$$

Hence composing the two operations, (a) and (b), we obtain

$$[\overrightarrow{OA}]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}([\overrightarrow{OA}]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OA}]_{\mathcal{K}} - \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OO'}]_{\mathcal{K}} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[\overrightarrow{OA}]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{K}'}$$

Hence, the formula for changing coordinates from the system $\mathcal{K}$ to the system $\mathcal{K}'$ is

$$[A]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}([A]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[A]_{\mathcal{K}} + [O]_{\mathcal{K}'} \tag{5.1}$$

Suppose now that a point A is given and that the coordinates of A in the frame $\mathcal{K}$ (relative to the coordinate frame/coordinate system $\mathcal{K}$) are $(1,2)$. Then the coordinates of A relative to $\mathcal{K}'$ are

$$[A]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}\begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{K}} + [O]_{\mathcal{K}'}$$

Since we know $\mathbf{i}'$ and $\mathbf{j}'$ with respect to $\mathbf{i}$ and $\mathbf{j}$, we can write down the matrix $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$ and then $\mathrm{M}_{\mathcal{K}',\mathcal{K}} = \mathrm{M}^{-1}_{\mathcal{K},\mathcal{K}'}$. Since we know the coordinates of O' with respect to $\mathcal{K}$, it is more convenient to use the first equality in (5.1)

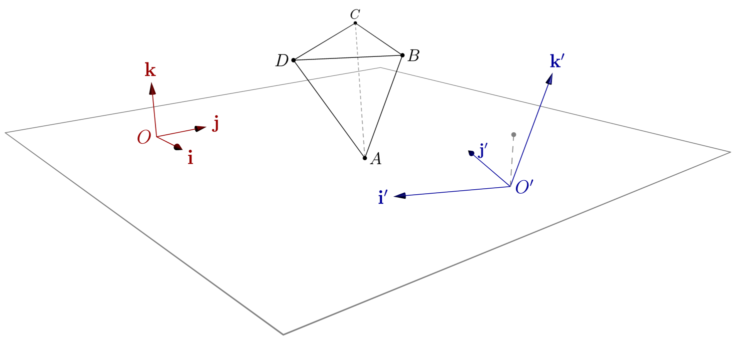
$$[A]_{\mathcal{K}'} = \begin{bmatrix}-2&1\\1&2\end{bmatrix}^{-1}\cdot\left(\begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{K}} - \begin{bmatrix}7\\-1\end{bmatrix}_{\mathcal{K}}\right) = \frac{1}{-5}\begin{bmatrix}2&-1\\-1&-2\end{bmatrix}\cdot\left(\begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{K}} + \begin{bmatrix}7\\-1\end{bmatrix}_{\mathcal{K}}\right) = \begin{bmatrix}3\\0\end{bmatrix}.$$

Theorem 5.9. Let $\mathbf{A}$ be an affine space and let $\mathbf{W}$ be the associated $\mathbf{K}$-vector space. Let $\mathcal{K} = O\mathbf{e}_1\ldots\mathbf{e}_n$ and $\mathcal{K}' = O'\mathbf{f}_1,\ldots,\mathbf{f}_n$ be two coordinate systems of $\mathbf{A}$. Denote by $\boldsymbol{e} = \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ and by $\boldsymbol{f} = \{\mathbf{f}_1,\ldots,\mathbf{f}_n\}$ the two bases of $\mathbf{V}$. Let $\mathrm{M}_{\mathcal{K}',\mathcal{K}}$ be the change of basis matrix from the basis $\boldsymbol{e}$ to the basis $\boldsymbol{f}$ and let $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$ be the change of basis matrix from the basis $\boldsymbol{f}$ to the basis $\boldsymbol{e}$. For any point $P \in A$ we have

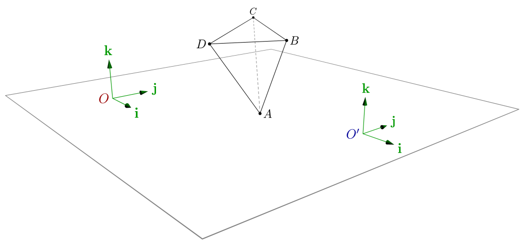
$$[P]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}^{-1}_{\mathcal{K},\mathcal{K}'}\cdot([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot[P]_{\mathcal{K}} + [O]_{\mathcal{K}'}. \tag{5.2}$$

Example 5.10 (In dimension 3)**.** Let $\mathbf{A}$ be the three-dimensional real affine space $\mathbf{A}^3(\mathbb{R})$. Let $\mathcal{K} = O\mathbf{ijk}$ and $\mathcal{K}' = O'\mathbf{i}'\mathbf{j}'\mathbf{k}'$ be two coordinate systems (reference frames). Suppose that we know O', $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$ relative to $\mathcal{K}$:

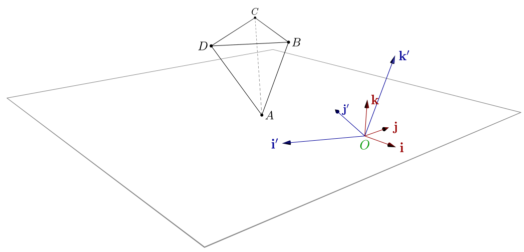
$$[O']_{\mathcal{K}} = \begin{bmatrix}4\\5\\-1\end{bmatrix}_{\mathcal{K}}, \quad \mathbf{i}' = -\mathbf{i} - 2\mathbf{j} = \begin{bmatrix}-1\\-2\\0\end{bmatrix}_{\mathcal{K}}, \quad \mathbf{j}' = -2\mathbf{i} + \mathbf{j} = \begin{bmatrix}-2\\1\\0\end{bmatrix}_{\mathcal{K}}, \quad \mathbf{k}' = \mathbf{j} + 2\mathbf{k} = \begin{bmatrix}0\\1\\2\end{bmatrix}_{\mathcal{K}}.$$



The coordinates with respect to $\mathcal{K}'$ can be obtained from the coordinates with respect to $\mathcal{K}$ in two steps:



(a) Change the origin.



(b) Change the direction of the axes.

If B is the point with coordinates $(1,5,1)$ with respect to $\mathcal{K}$, then

$$[B]_{\mathcal{K}'}=\mathrm{M}^{-1}_{\mathcal{K},\mathcal{K}'}\cdot([B]_{\mathcal{K}}-[O']_{\mathcal{K}})=\begin{bmatrix}-1&-2&0\\-2&1&1\\0&0&2\end{bmatrix}^{-1}\left(\begin{bmatrix}1\\5\\1\end{bmatrix}-\begin{bmatrix}4\\5\\-1\end{bmatrix}\right)=\frac{1}{10}\begin{bmatrix}-2&-4&2\\-4&2&-1\\0&0&5\end{bmatrix}\left(\begin{bmatrix}1\\5\\1\end{bmatrix}-\begin{bmatrix}4\\5\\-1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

5.3 Equations of affine subspaces in different reference frames

Let $\mathbf{A}$ be an affine space with associated $\mathbf{K}$-vector space $\mathbf{V}$. Let S be a d-dimensional affine subspace of $\mathbf{V}$ with associated vector subspace $\mathbf{W}$. Let $\mathcal{K}=O\mathbf{e}_1\dots\mathbf{e}_n$ and $\mathcal{K}'=O'\mathbf{f}_1,\dots,\mathbf{f}_n$ be two coordinate systems of $\mathbf{A}$. How do the parametric equations and the Cartesian equations of S change from one reference frame to another?

- [Parametric equations] Suppose that S is a line passing through Q and having $\mathbf{v}$ as direction vector. Let $(q_1,\dots,q_n)$ be the coordinates of Q with respect to $\mathcal{K}$ and let $(v_1,\dots,v_n)$ be the components of $\mathbf{v}$ with respect to $\mathcal{K}$. A set of parametric equations for S is:

$$\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}q_1\\q_2\\\vdots\\q_n\end{bmatrix}+t\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}.$$

Therefore, if we want to translate the coordinates $(x_1,\dots,x_2)$ relative to $\mathcal{K}$, of a point in S, to coordinates $(x'_1,\dots,x'_2)$ relative to $\mathcal{K}'$ we can directly apply (5.2):

$$\begin{bmatrix}x'_1\\x'_2\\\vdots\\x'_n\end{bmatrix}=\mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}-[O']_{\mathcal{K}})=\mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot(\begin{bmatrix}q_1\\q_2\\\vdots\\q_n\end{bmatrix}+t\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}-[O']_{\mathcal{K}})=\mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot(\begin{bmatrix}q_1\\q_2\\\vdots\\q_n\end{bmatrix}-[O']_{\mathcal{K}})+t\cdot\mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}.$$

Therefore, it suffices to find $[Q]_{\mathcal{K}'}$ and $[\mathbf{v}]_{\mathcal{K}'}$:

$$[Q]_{\mathcal{K}'}=\mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot(\begin{bmatrix}q_1\\q_2\\\vdots\\q_n\end{bmatrix}-[O']_{\mathcal{K}})\quad\text{and}\quad[\mathbf{v}]_{\mathcal{K}'}=\mathrm{M}_{\mathcal{K}',\mathcal{K}}\cdot\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}.$$

If S has dimension $d > 1$, the method is similar.

- [Cartesian equations] Suppose that S is a hyperplane with Cartesian equation

$$S : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad \Leftrightarrow \quad \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

in the reference frame $\mathcal{K}$. Then, since

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathrm{M}_{\mathcal{K},\mathcal{K}'} \cdot \left(\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} - [O]_{\mathcal{K}'} \right),$$

the corresponding equation of the hyperplane S with respect to $\mathcal{K}'$ is

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \cdot \mathrm{M}_{\mathcal{K},\mathcal{K}'} \cdot \left(\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} - [O]_{\mathcal{K}'} \right) = b,$$

an equation in $x_1', \dots, x_n'$.

If S is smaller than a hyperplane, it is an intersection of several hyperplanes. One obtains the corresponding system of equations for S by translating each equation as in the above case of the hyperplane.

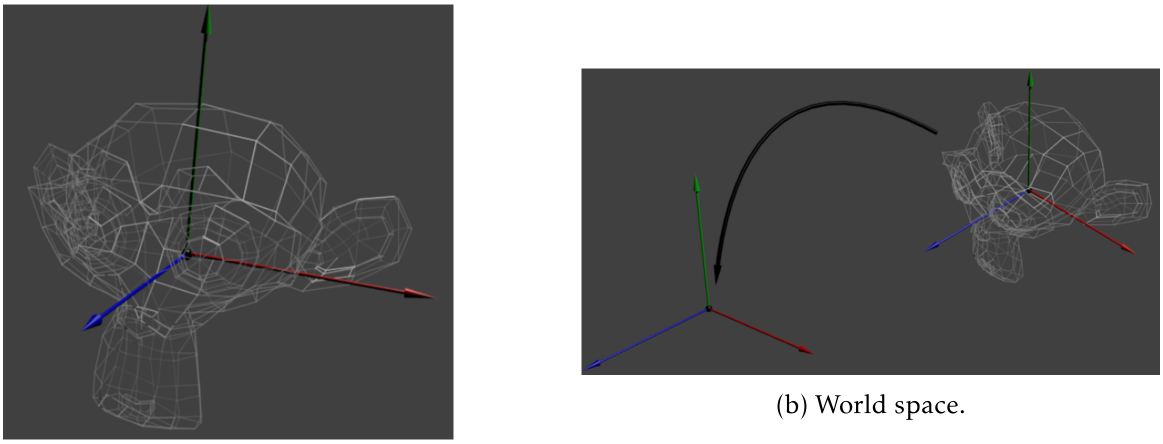
- Notice that for parametric equations we use $\mathrm{M}_{\mathcal{K}',\mathcal{K}}$ but for Cartesian equations we use $\mathrm{M}_{\mathcal{K},\mathcal{K}'}$.

5.4 Connections to reality

Changing coordinates is so common in real life applications that it is impossible to do any meaningful modeling without coordinate changes. Take for example the OpenGL-pipeline (a process under which objects are rendered on the screen in order to simulate a 3D scene). In this process there are two dimensional changes of coordinates for instance when a PNG-image is transformed into an OpenGL-texture, or when an OpenGL texture is translated into device coordinates before being displayed on the screen.

Three dimensional changes of coordinates are needed in the OpenGL-pipeline as well as in any modeling software (like Blender or Maya). Complex 3D-models are constructed by several people, even several teams. The different pieces are constructed separately, each in its own coordinate system. When they are ready, the whole object is assembled by putting all pieces in a common coordinate system. In this context one calls the coordinate system of a single piece 'model space' and the coordinate system where the whole object is placed in, is called the 'world space'. This is important when modeling machines such as spacecrafts or cars. It is also important when constructing computer games.

As an example, here is Suzanne, the Blender monkey:



(a) Model space.

(b) World space.