Seminar 11

1. By computing the square of each elemenet in $\mathbb{Z}_{12}$ we find the idempotents: $\hat{0}$, $\hat{1}$, $\hat{4}$, $\hat{9}$.

   For $M_2(\mathbb{Z})$, it is obvious that O_2 and I_2 are idempotents. Also, the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

2. As $\mathbb{Z}_{12}$ is cyclic, we have to compute the powers of each element until we find the cycle. So, the nilpotent elements are: $\hat{0}$ and $\hat{6}$. As a remark: all idempotents of $\mathbb{Z}_{12}$ are not nilpotents, except $\hat{0}$.

   As for $M_2(\mathbb{Z})$, it is obvious that O_2 is nilpotent and also, using the same reasoning as in exercise 1, we find the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is nilpotent.

3. (a) $\forall x, y \in \mathbb{Z} \Rightarrow f(x+y) = \widehat{x+y} = \hat{x} + \hat{y} = f(x) + f(y)$. Also, $f(x \cdot y) = \widehat{x \cdot y} = \hat{x} \cdot \hat{y} = f(x) \cdot f(y)$. So f is a ring homomorphism.

   (b) $\forall x, y \in \mathbb{R} \Rightarrow f(x+y) = \begin{bmatrix} x+y & 0 \\ 0 & x+y \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = f(x) + f(y)$. As for the multiplication we have: $f(x) \cdot f(y) = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \cdot \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} xy & 0 \\ 0 & xy \end{bmatrix} = f(x \cdot y)$. So f is a ring homomorphism.

   (c) As $det(A+B) \neq det(A) + det(B)$, for any $A, B \in M_2(\mathbb{R}) \Rightarrow g$ is not a ring homomorphism.

4. In order to prove that f is well-defined (function), we need to show that $\forall \hat{x}, \hat{y} \in \mathbb{Z}_{12}$ such that $\hat{x} = \hat{y}$, we have $\bar{x} = \bar{y}$. If $\hat{x} = \hat{y}$, then $12 | x - y$, which implies that $4 | x - y$, and so $\bar{x} = \bar{y}$.

   Now, we have to prove that f is a ring homomorphism. So, $\forall \hat{x}, \hat{y} \in \mathbb{Z}_{12} \Rightarrow f(\hat{x} + \hat{y}) = f(\widehat{x+y}) = \overline{x+y} = \bar{x} + \bar{y} = f(\hat{x}) + f(\hat{y})$, and $f(\hat{x} \cdot \hat{y}) = f(\widehat{x \cdot y}) = \overline{x \cdot y} = \bar{x} \cdot \bar{y} = f(\hat{x}) \cdot f(\hat{y})$. In the end, f is a ring homomorphism.

5. For those fields to be isomorphic, we need to find a function between them, such that the function is a ring isomorphism. Take $f : \mathbb{Q}(\sqrt{2}) \to \mathcal{M}$ with $f(a + b\sqrt{2}) = \begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$. By simple computations, we find that f is a ring homomorphism. But, f also has to be bijective. We can easily see that, $\forall \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} \in \mathcal{M}, \exists! a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ such that $f(a+b\sqrt{2}) = \begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$. In the end, we find that f is a ring isomorphism.

6. Take $f : \mathbb{C} \to \mathcal{M}$ with $f(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. By the same reasoning as in exercise 5, we find that f is a ring isomorphism.

7. First, note that $End(\mathbb{Z}, +, \cdot) \subseteq End(\mathbb{Z}, +)$. From $End(\mathbb{Z}, +) = \{t_a \mid a \in \mathbb{Z}\}$ we know that $t_a(x + y) = t_a(x) + t_a(y)$. So, we look for t_a such that $t_a(x \cdot y) = t_a(x) \cdot t_a(y)$.

   $\forall x, y \in \mathbb{Z} \Rightarrow t_a(x \cdot y) = a \cdot x \cdot y$ and $t_a(x) \cdot t_a(y) = a \cdot x \cdot a \cdot y = a^2 \cdot x \cdot y \Rightarrow a^2 \cdot x \cdot y = a \cdot x \cdot y \iff a^2 = a, \forall a \in \mathbb{Z} \Rightarrow a = 1$ or $a = 0$. One checks that these are ring endomorphisms, and so $End(\mathbb{Z}, +, \cdot) = \{t_0, t_1\}$. Now, to find the automorphisms, we look in $End(\mathbb{Z}, +, \cdot)$ for the bijective ones. First, let's look at t_1 and $\forall x \in \mathbb{Z}$ we have $t_1(x) = x \Rightarrow t_1(x) = 1_{End\mathbb{Z}}(x) \in Aut(\mathbb{Z}, +, \cdot)$ (we already know this). Now, $\forall x \in \mathbb{Z}$ we take $t_0(x) = 0$. But, take $1 \in \mathbb{Z}$, then there is no $x \in \mathbb{Z}$ such that $t_0(x) = 1$. So t_0 is not bijective. In the end, $Aut(\mathbb{Z}, +, \cdot) = \{t_1\}$.

8. Let $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ be an automorphism.

   $\mathbb{Q}(\sqrt{2})$ has unity 1 and if f is an automorphism $\Rightarrow f(1) = 1$. Then one shows that $f(n) = nf(1) = n$, whence we deduce that $f(-n) = -f(n) = -n$ for every $n \in \mathbb{N}$. So $f(n) = n$ for every $n \in \mathbb{Z}$.

   Then $1 = f(1) = f(n \cdot \frac{1}{n}) = f(n)f(\frac{1}{n}) = nf(\frac{1}{n})$, hence $f(\frac{1}{n}) = \frac{1}{n}$ for every $n \in \mathbb{Z}$.

   Next let $\frac{m}{n} \in \mathbb{Q}$. We may assume that $m \in \mathbb{N}$. Then

$$f\left(\frac{m}{n}\right) = f\left(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ times}}\right) = mf\left(\frac{1}{n}\right) = \frac{m}{n}.$$

Hence $f(x) = x$ for every $x \in \mathbb{Q}$.

It follows that $\forall a, b \in \mathbb{Q} : f(a + b\sqrt{2}) = f(a) + f(b\sqrt{2}) = f(a) + f(b) \cdot f(\sqrt{2}) = a + b \cdot f(\sqrt{2})$. But, $2 = \sqrt{2} \cdot \sqrt{2}$ and $f(2) = 2 \Rightarrow 2 = f(2) = f(\sqrt{2} \cdot \sqrt{2}) = f(\sqrt{2}) \cdot f(\sqrt{2}) = [f(\sqrt{2})]^2 \Rightarrow f(\sqrt{2}) = \pm\sqrt{2}$.

If $f(\sqrt{2}) = \sqrt{2} \Rightarrow f = 1_{\mathbb{Q}(\sqrt{2})}$, which is an automorphism (we know it). If $f(\sqrt{2}) = -\sqrt{2} \Rightarrow f(a + b\sqrt{2}) = a - b\sqrt{2}$. It is easy to see that, in this case, f is also an automorphism. In the end, $Aut(\mathbb{Q}(\sqrt{2}), +, \cdot) = \{1_{\mathbb{Q}(\sqrt{2})}, f\}$, where $f(a + b\sqrt{2}) = a - b\sqrt{2}$.