Seminar 8 - The generated subspace

Let K be a field, V be a K-v.s., $X \subseteq V$, $X \neq \emptyset$

$v \in \langle X \rangle \iff \exists n \in \mathbb{N}^*, \exists x_1, \dots, x_n \in X, \exists \alpha_1, \dots, \alpha_n \in K$ such that
$v = \alpha_1 x_1 + \dots + \alpha_n x_n$.

$A, B \leq_K V$, $V = A \oplus B$ $\Big|$ $V \overset{\subseteq}{=} A + B = \{a + b \mid a \in A, b \in B\}$, $A \cap B \underset{\subseteq}{=} \{0\}$

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1) V K-v.s., $S \leq_K V$, $x, y \in V$, $x \in V \setminus S$.

$$x \in \langle S, y \rangle \implies y \overset{?}{\in} \langle S, x \rangle.$$

Solution: $x \in \langle S, y \rangle \iff \exists n \in \mathbb{N}^*, \exists s_1, \dots, s_n \in S, \exists \alpha_1, \dots, \alpha_n, \beta \in K$ such that

$$x = \alpha_1 s_1 + \dots + \alpha_n s_n + \beta y \qquad (1)$$

Assume by contradiction that $\beta = 0 \underset{(1)}{\implies} x = \underset{\in S}{\alpha_1 s_1} + \dots + \underset{\in S}{\alpha_n s_n} \overset{S \leq_K V}{\in} S$ contrad.

Hence $\beta \neq 0 \implies \exists \beta^{-1} \in K : \beta^{-1} \cdot \beta = 1$

$(1) \underset{\beta^{-1} \cdot}{\iff} \beta y = x - \alpha_1 s_1 - \dots - \alpha_n s_n \implies y = \beta^{-1} x - (\beta^{-1} \alpha_1) s_1 - \dots - (\beta^{-1} \alpha_n) s_n \in \langle S, x \rangle$.

2) V K-v.s., $\alpha, \beta, \gamma \in K$, $x, y, z \in V$, $\alpha\gamma \neq 0$

$$\alpha x + \beta y + \gamma z = 0 \implies \langle x, y \rangle \overset{?}{=} \langle y, z \rangle$$

Solution: $\langle x, y \rangle \overset{?}{\subseteq} \langle y, z \rangle \Longleftarrow x, \cancel{y} \overset{?}{\in} \langle y, z \rangle \longleftarrow$

$\langle x, y \rangle \overset{?}{\supseteq} \langle y, z \rangle \Longleftarrow \cancel{y}, z \overset{?}{\in} \langle x, y \rangle$ (homework)

$\alpha\gamma \neq 0 \implies \alpha \neq 0$ and $\gamma \neq 0 \implies \exists \alpha^{-1} \in K : \alpha^{-1} \cdot \alpha = 1$

$\alpha x + \beta y + \gamma z = 0 \underset{\alpha^{-1} \cdot}{\implies} \alpha x = -\beta y - \gamma z \implies x = -(\alpha^{-1}\beta) y - (\alpha^{-1}\gamma) z \in \langle y, z \rangle$

3) Let $f_1 = 3X + 2$, $f_2 = 4X^2 - X + 1$, $f_3 = X^3 - X^2 + 3$ in ${}_{\mathbb{R}}\mathbb{R}_3[X]$

$$\mathbb{R}_3[X] = \{f \in \mathbb{R}[X] \mid \deg f \leq 3\} \; (\leq_{\mathbb{R}} \mathbb{R}[X])$$

$$\mathbb{R}_3[X] \overset{?}{=} \langle f_1, f_2, f_3 \rangle$$

Solution: $\forall f = a + bX + cX^2 + dX^3 \in \mathbb{R}_3[X]$, $\overset{?}{\exists} \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$f = \underbrace{\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3}_{\text{is either 0, or it has the degree 1 or 2 or 3.}} \qquad (1)$$

(the zero degree polynomials are missing)

$a, b, c, d \in \mathbb{R}$ arbitrary

$(1) \iff a + bX + cX^2 + dX^3 = f = \alpha_1(3X + 2) + \alpha_2(4X^2 - X + 1) + \alpha_3(X^3 - X^2 + 3) = \dots$

$$\iff \begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = a \\ 3\alpha_1 - \alpha_2 = b \\ 4\alpha_2 - \alpha_3 = c \\ \alpha_3 = d \end{cases} \qquad (2)$$

We investigate the consistency of (2).

Let $f = 1$ ($\iff a=1,\ b=c=d=0$). Then (2) becomes

$$\left.\begin{array}{l} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 1 \\ 3\alpha_1 - \alpha_2 = 0 \implies \alpha_1 = 0 \\ 4\alpha_2 - \alpha_3 = 0 \implies \alpha_2 = 0 \\ \alpha_3 = 0 \end{array}\right| \implies 0 = 1 \text{ imp.}$$

Thus, for $f=1$, the system (2) is inconsistent $\implies 1 \notin \langle f_1, f_2, f_3\rangle \implies$ ($1 \in \mathbb{R}_3[X]$)

$\implies \mathbb{R}_3[X] \neq \langle f_1, f_2, f_3\rangle$.

V, V' K-v.s., $f: V \to V'$ is a linear map $\left|\begin{array}{l} f(x+y) = f(x)+f(y),\ \forall x,y\in V \\ f(\alpha x) = \alpha\cdot f(x),\ \forall \alpha\in K, \forall x\in V\end{array}\right. \iff$

$\iff f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),\ \forall \alpha,\beta\in K,\ \forall x,y\in V.$

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4) $f: V\to V'$ K-linear map.

a) $A \leq_K V \implies f(A) = \{f(x) \mid x\in A\} \overset{?}{\leq_K} V'$

b) $A' \leq_K V' \implies f^{-1}(A') = \{x\in V \mid f(x)\in A'\} \overset{?}{\leq_K} V$

Solution:

a) $0\in A \implies 0' = f(0) \in f(A)$

Let $\alpha,\beta\in K$, $x', y' \in f(A)$, $\alpha x' + \beta y' \overset{?}{\in} f(A)$

$x' \in f(A) \iff \exists\, x\in A : x' = f(x)$

$y' \in f(A) \iff \exists\, y\in A : y' = f(y)$

$\implies \alpha x' + \beta y' = \alpha f(x) + \beta f(y) = f(\alpha x + \beta y) \in f(A)$

$x,y\in A \xRightarrow[A\leq_K V]{} \alpha x + \beta y \in A$

b) $f(0) = 0' \in A' \implies 0 \in f^{-1}(A')$

$\forall \alpha,\beta\in K$, $\forall x,y\in f^{-1}(A')$, $\alpha x + \beta y \overset{?}{\in} f^{-1}(A')$

$f(\alpha x + \beta y) = \alpha \underbrace{f(x)}_{\in A'} + \beta \underbrace{f(y)}_{\in A'} \underset{A'\leq_K V'}{\in} A'$

Particular cases: i) $A = V \implies \operatorname{Im} f = f(V) \leq_K V'$

ii) $A' = \{0'\} \implies f^{-1}(\{0'\}) \leq_K V$

$= \{x \in V \mid f(x) = 0'\} = \operatorname{Ker} f.$

5) $\mathbb{R}^{\mathbb{R}} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ $\mathbb{R}$-v.s.

$\mathbb{R}_o^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is odd}\}$; $\mathbb{R}_e^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is even}\}$

a) $\mathbb{R}_o^{\mathbb{R}} \overset{?}{\leq}_{\mathbb{R}} \mathbb{R}^{\mathbb{R}} \leftarrow$

$\mathbb{R}_e^{\mathbb{R}} \overset{?}{\leq}_{\mathbb{R}} \mathbb{R}^{\mathbb{R}}$ (homework)

b) $\mathbb{R}^{\mathbb{R}} \overset{?}{=} \mathbb{R}_o^{\mathbb{R}} \oplus \mathbb{R}_e^{\mathbb{R}}$.

Solution: $f : \mathbb{R} \to \mathbb{R}$, f is odd : $f(-x) = -f(x)$, $\forall x \in \mathbb{R}$

f is even : $f(-x) = f(x)$, $\forall x \in \mathbb{R}$

a) $\theta : \mathbb{R} \to \mathbb{R}$, $\theta(x) = 0$, $\forall x \in \mathbb{R}$, θ is odd

$\forall \alpha, \beta \in \mathbb{R}$, $\forall f, g : \mathbb{R} \to \mathbb{R}$ odd, $\alpha f + \beta g : \mathbb{R} \to \mathbb{R}$ odd ?

$\forall x \in \mathbb{R}$, $(\alpha f + \beta g)(-x) = (\alpha f)(-x) + (\beta g)(-x) = \alpha \cdot f(-x) + \beta \cdot g(-x) =$

$= \alpha \cdot (-f(x)) + \beta(-g(x)) = -(\alpha \cdot f(x) + \beta \cdot g(x)) = -((\alpha f)(x) + (\beta g)(x)) =$

$= -(\alpha f + \beta g)(x)$

b) $\mathbb{R}^{\mathbb{R}} \overset{?}{=} \mathbb{R}_o^{\mathbb{R}} + \mathbb{R}_e^{\mathbb{R}}$ (1)

$\mathbb{R}_o^{\mathbb{R}} \cap \mathbb{R}_e^{\mathbb{R}} \overset{?}{=} \{\theta\}$ (2)

(1) $\forall f : \mathbb{R} \to \mathbb{R}$, $\overset{?}{\exists} g, h : \mathbb{R} \to \mathbb{R}$, g odd$^?$, h even$^?$ such that $f \overset{?}{=} g + h$

Assume that such g, h exist. Then

$\forall x \in \mathbb{R}$, $f(x) = (g+h)(x) = g(x) + h(x)$

$f(-x) = g(-x) + h(-x) = -g(x) + h(x)$

$f(x) - f(-x) = 2g(x) \implies g(x) = \frac{1}{2}(f(x) - f(-x))$

$f(x) + f(-x) = 2h(x) \implies h(x) = \frac{1}{2}(f(x) + f(-x))$

g odd (homework)

h even (homework)

$f = g + h$ (homework)

(2) Let $f \in \mathbb{R}_o^{\mathbb{R}} \cap \mathbb{R}_e^{\mathbb{R}} \iff f : \mathbb{R} \to \mathbb{R}$ is even : $f(-x) = f(x)$, $\forall x \in \mathbb{R}$

and

f is odd : $f(-x) = -f(x)$, $\forall x \in \mathbb{R}$

$0 = 2f(x)$, $\forall x \in \mathbb{R}$

$\Longrightarrow f(x)=0,\ \forall x\in\mathbb{R} \Longrightarrow f=\theta$

6) Let V K-v.s., $A, B, C, D \leq_K V$ $\quad (C, D \leq_K A \overset{A\leq_K V}{\underset{C,D\subseteq A}{\Longleftrightarrow}} C, D \leq_K V)$

$$\left.\begin{array}{l} V = A\oplus B \\ A = C\oplus D \end{array}\right\} \Longrightarrow \overset{?}{\exists}\, S \leq_K V : \ V = C\oplus S$$

<u>Solution</u>:

$$\left.\begin{array}{l} V = A+B \\ A\cap B=\{0\} \\ A = C+D \\ C\cap D=\{0\} \end{array}\right. \Longrightarrow V = (C+D)+B \overset{\text{assoc. "+" in } (V,+)}{=} C+\underbrace{(D+B)}_{\overset{?}{=}S}$$

$$C\cap(D+B) \overset{?}{=} \{0\}$$

Let $c\in C\cap(D+B) \Longleftrightarrow c\in C$ and $\exists\, d\in D, \exists\, b\in B : c = d+b \quad (1)$

$(1) \Longleftrightarrow \underset{B}{\overset{\cap}{b}} = c-d = \underbrace{\overset{\in C}{c}+(\overset{\in D}{-d})}_{\in C+D=A} \in A\cap B=\{0\} \Longrightarrow b=0 \overset{(1)}{\Longrightarrow}$

$\Longrightarrow \underset{C}{\overset{\cap}{c}} = \underset{D}{\overset{\cap}{d}} \in C\cap D=\{0\} \Longrightarrow c=0.$

Thus $\left.\begin{array}{l} A \text{ is a direct summand in } V \\ C \text{ — " — " — } A \end{array}\right\} \Longrightarrow C$ is a direct summand in V.