Seminar 11 – Bases. Dimension

1) a) $f_1=(X-b)(X-c)$, $f_2=(X-a)(X-c)$, $f_3=(X-a)(X-b)$, $a,b,c\in\mathbb{R}$

i) f_1,f_2,f_3 l. indep. in ${}_{\mathbb{R}}\mathbb{R}[X] \overset{?}{\iff} (a-b)(b-c)(c-a)\overset{?}{\neq}0$

$\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$, $\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3=0 \overset{?}{\Longrightarrow} \alpha_1=\alpha_2=\alpha_3=0$

$0=\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3=\alpha_1\underline{(X-b)(X-c)}+\alpha_2(X-a)(X-c)+\alpha_3(X-a)(X-b)=$

$=\alpha_1[X^2-(b+c)X+bc]+\alpha_2[X^2-(a+c)X+ac]+\alpha_3[X^2-(a+b)X+ab]\iff$

$$\iff\begin{cases}\alpha_1+\alpha_2+\alpha_3=0\\ -(b+c)\alpha_1-(a+c)\alpha_2-(a+b)\alpha_3=0\\ bc\,\alpha_1+ac\,\alpha_2+ab\,\alpha_3=0\end{cases}\qquad(1)$$

f_1,f_2,f_3 l. indep. in ${}_{\mathbb{R}}\mathbb{R}[X]\iff$ the system (1) has only one solution $\iff$

$\iff$ the determinant of the system (1) matrix is not zero $\iff (a-b)(b-c)(c-a)\neq 0$

$$\begin{vmatrix}1&1&1\\-b-c&-a-c&-a-b\\bc&ac&ab\end{vmatrix}\ \begin{matrix}c_2-c_1\\ =\\ c_3-c_1\end{matrix}\ \begin{vmatrix}1&0&0\\-b-c&b-a&c-a\\bc&c(a-b)&b(a-c)\end{vmatrix}=(a-b)(c-a)\begin{vmatrix}-1&1\\c&-b\end{vmatrix}=$$

$=(a-b)(b-c)(c-a)$

ii) $\mathbb{R}_2[X]=\{f\in\mathbb{R}[X]\mid \deg f\leq 2\}\leq_{\mathbb{R}}\mathbb{R}[X]\Longrightarrow {}_{\mathbb{R}}\mathbb{R}_2[X]$

$\dim_{\mathbb{R}}\mathbb{R}_2[X]=3$ ($\Longleftarrow (1,X,X^2)$ basis for ${}_{\mathbb{R}}\mathbb{R}_2[X]$).

$(a-b)(b-c)(c-a)\neq 0 \overset{i)}{\iff} f_1,f_2,f_3$ l. indep. in ${}_{\mathbb{R}}\mathbb{R}_2[X] \underset{\dim_{\mathbb{R}}\mathbb{R}_2[X]=3}{\iff}$

$\iff (f_1,f_2,f_3)$ basis for ${}_{\mathbb{R}}\mathbb{R}_2[X]\iff \forall f\in\mathbb{R}_2[X]$, $\exists\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$ uniquely determined s.t.

$f=\lambda_1 f_1+\lambda_2 f_2+\lambda_3 f_3$ (2)

b) homework In the given particular case you will write (2). This will lead you to a system very similar to (1) which must be solved.

R: In a previous seminar there was an exercise which asked us if

$f_1=\ldots$, $f_2=\ldots$, $f_3=\ldots$ generate ${}_{\mathbb{R}}\mathbb{R}_3[X]$.

Answer: NO! since $\dim_{\mathbb{R}}\mathbb{R}^3[X]=4$ ($\Longleftarrow (1,X,X^2,X^3)$ basis for ${}_{\mathbb{R}}\mathbb{R}^3[X]$)

$\Longrightarrow$ any generating set of ${}_{\mathbb{R}}\mathbb{R}_3[X]$ has at least 4 elements.

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2). $p\in\mathbb{N}$ is a prime ($\sqrt[3]{p}\in\mathbb{R}\setminus\mathbb{Q}$, $\sqrt[3]{p^2}\in\mathbb{R}\setminus\mathbb{Q}$)

$V=\{a+b\sqrt[3]{p}+c\sqrt[3]{p^2}\mid a,b,c\in\mathbb{Q}\}\subseteq\mathbb{R}$

$V \overset{?}{\ Q - v.s.}$ (?) and find a basis and the dimension of ${}_{\mathbb{Q}}V$ (?)

Solution: $\mathbb{Q}$ is a subfield of $\mathbb{R} \Longrightarrow \mathbb{R}$ is a $\mathbb{Q}$-v.s.

$V \overset{?}{\leq}_{\mathbb{Q}} \mathbb{R} \Longrightarrow V$ is the requested $\mathbb{Q}$-v.s.

– homework.

$V = \{a \cdot 1 + b\sqrt[3]{p} + c\sqrt[3]{p^2} \mid a, b, c \in \mathbb{Q}\} = \langle 1, \sqrt[3]{p}, \sqrt[3]{p^2} \rangle$

$1, \sqrt[3]{p}, \sqrt[3]{p^2}$ l. indep. in ${}_{\mathbb{Q}}V \overset{?}{\ } \Longrightarrow (1, \sqrt[3]{p}, \sqrt[3]{p^2})$ basis for ${}_{\mathbb{Q}}V$ and

$\dim_{\mathbb{Q}} V = 3$.

Let $a, b, c \in \mathbb{Q}$ arbitrary, $a + b\sqrt[3]{p} + c\sqrt[3]{p^2} = 0 \,\Big|\cdot\sqrt[3]{p} \overset{?}{\Longrightarrow} a = b = c = 0$

$$\begin{array}{l|l} a + b\sqrt[3]{p} + c\sqrt[3]{p^2} = 0 & \cdot b \\ cp + a\sqrt[3]{p} + b\sqrt[3]{p^2} = 0 & \cdot(-c) \end{array}$$

$(ab - c^2p) + (b^2 - ac)\sqrt[3]{p} = 0$

Assume by contradiction that $b^2 - ac \neq 0$ $\Big\} \Longrightarrow \sqrt[3]{p} = \dfrac{c^2p - ab}{b^2 - ac} \in \mathbb{Q}$ impossible

Hence $\begin{cases} b^2 - ac = 0 \Longrightarrow ac = b^2 \\ ab - c^2p = 0 \mid \cdot c \Longrightarrow abc - c^3p = 0 \Longleftrightarrow acb = c^3p \Longrightarrow b^3 = c^3p \end{cases}$

Assume by contradiction that $c \neq 0$

$\Longrightarrow p = \dfrac{b^3}{c^3} \Longrightarrow \sqrt[3]{p} = \dfrac{b}{c} \in \mathbb{Q}$ imp.

Thus $c = 0 \Longrightarrow b = 0 \Longrightarrow a = 0$.

Let V, V' be K-v.s., $f: V \to V'$ K-linear map. Then

$$\dim V = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

Let V be K-v.s., $A, B \leq_K V$. Then

$$\dim A + \dim B = \dim(A+B) + \dim(A \cap B).$$

Let V be K-v.s., $A \leq_K V \Longrightarrow \dim A \leq \dim V$

$A \leq_K V$, $\dim A = \dim V \Longrightarrow A = V$.

? $\dim V < \infty$.

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3) V K-v.s., $\dim V = n\ (\in \mathbb{N}^*)$, $A, B \leq_K V$, $\dim A = n-1$, $B \nsubseteq A$.

$A + B \overset{?}{=} V$ and $\dim(A \cap B) \overset{?}{=} \dim B - 1$

Solution: $\dim A + \dim B = \dim(A+B) + \dim(A \cap B) \Longrightarrow$

$\Longrightarrow \dim(A+B) = \underbrace{\dim A}_{= n-1} + \underbrace{\dim B - \dim(A \cap B)}_{\geq 1}$ (1)

$B \not\subseteq A \iff \underbrace{A\cap B \subsetneq B}_{\text{subspaces of } {}_K V} \implies A\cap B \underset{\neq}{\leq_K} B \implies \dim(A\cap B) \underset{\neq}{<} \dim B \implies$

$\implies \dim B - \dim(A\cap B) > 0 \iff \dim B - \dim(A\cap B) \geq 1 \qquad (2)$

From (1) and (2) we deduce that

$$\left.\begin{array}{l} \dim(A+B) \geq n-1+1 = n = \dim V \\ A+B = \langle A\cup B\rangle \leq_K V \implies \dim(A+B) \leq \dim V \end{array}\right\} \implies \left.\begin{array}{l}\dim(A+B) = \dim V \\ A+B \leq_K V\end{array}\right\}$$

$\implies A+B = V.$

$\dim(A\cap B) = \underbrace{\dim A}_{=n-1} + \dim B - \underbrace{\dim(A+B)}_{=\dim V = n} = \dim B - 1.$

4) V K-v.s. ($\dim V < \infty$), $A, B \leq_K V$

$\dim(A+B) = \dim(A\cap B) + 1 \overset{?}{\implies} A \subseteq B$ or $B \subseteq A$

Solution: $A \not\subseteq B$ and $B \not\subseteq A \implies \dim(A+B) \neq \dim(A\cap B)+1$

$\iff \dim(A+B) - \dim(A\cap B) \overset{?}{\neq} 1$

$B \not\subseteq A \iff A\cap B \subsetneq B \implies \dim(A\cap B) < \dim B \iff \dim B - \dim(A\cap B) \geq 1 \quad (1)$

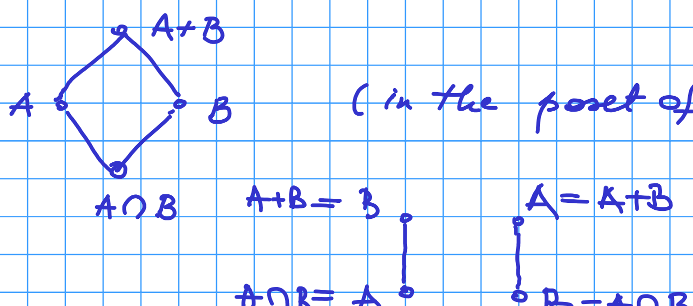
$A \subseteq B \iff B = \langle A\cup B\rangle = A+B$

$A \not\subseteq B \iff \underbrace{B \subsetneq A+B}_{\text{subspaces of } {}_K V} \implies \dim B < \dim(A+B) \iff \dim(A+B) - \dim B \geq 1 \quad (2)$

From (1) and (2) we deduce that $\dim(A+B) - \dim(A\cap B) \geq 2 \implies$

$\implies \dim(A+B) - \dim(A\cap B) \neq 1.$

R :



(in the poset of the subspaces of $_K V$ ordered with „$\subseteq$")

5) V K-v.s. ($\dim V < \infty$), $f, g \in \mathrm{End}_K(V)$, $f+g \in \mathrm{Aut}_K(V)$, $f\circ g = \theta$

$\dim V \overset{?}{=} \dim f(V) + \dim g(V)$

Solution: $\dim V = \dim f(V) + \dim \operatorname{Ker} f \quad (1)$

$f\circ g = \theta \iff \forall x \in V, (f\circ g)(x) = \theta(x) \iff \forall x\in V, f(g(x)) = 0 \iff$

$\iff \forall x \in V, g(x) \in \operatorname{Ker} f \implies \underbrace{g(V) \subseteq \operatorname{Ker} f}_{\text{subspaces of } {}_K V} \implies$

$\Longrightarrow \dim g(V) \leq \dim \operatorname{Ker} f \qquad (2)$

From (1) and (2) we deduce

$$\dim V \geq \dim f(V) + \dim g(V) \qquad (3)$$

$\leq ?$

$f+g \in \operatorname{Aut}_K(V) \Leftrightarrow V = (f+g)(V) = \{(f+g)(x) \mid x \in V\} = \{f(x) + g(x) \mid x \in V\} \subseteq$

$f+g$ surj.

$$\subseteq f(V) + g(V) \subseteq V$$

$$f(V) + g(V) = \{f(x) + g(y) \mid x, y \in V\}$$

$\Longrightarrow V = (f+g)(V) = f(V) + g(V) \Longrightarrow$

$\Longrightarrow \dim V = \dim(f(V) + g(V)) = \dim f(V) + \dim g(V) - \dim(f(V) \cap g(V))$

$\Longrightarrow \dim V \leq \dim f(V) + \dim g(V) \qquad (4)$

From (3) and (4) we deduce that

$$\dim V = \dim f(V) + \dim g(V).$$

R: $V = f(V) + g(V)$.

$\dim f(V) + \dim g(V) = \dim V = \dim f(V) + \dim g(V) - \dim(f(V) \cap g(V))$

$\Longrightarrow \dim(f(V) \cap g(V)) = 0 \Longrightarrow f(V) \cap g(V) = 0$

Thus $V = f(V) \oplus g(V)$.