Exercises – Semantic Tableaux Method

Exercise 1

Using the semantic tableaux method decide what kind (consistent, inconsistent, valid) of formula is $U_j, j \in \{1,2,...,8\}$.

If $U_j, j \in \{1,2,...,8\}$ is consistent, find all its models.

1. $U_1 = (p \wedge q) \vee (\neg p \wedge \neg r) \rightarrow (q \leftrightarrow r)$;
2. $U_2 = (p \vee q \rightarrow r) \rightarrow (p \vee r \rightarrow q)$;
3. $U_3 = (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r) \wedge q$;
4. $U_4 = (q \vee r \rightarrow p) \rightarrow (p \rightarrow r) \wedge q$;
5. $U_5 = (r \vee q) \vee (p \rightarrow \neg r) \rightarrow (p \leftrightarrow q)$;
6. $U_6 = (r \wedge q) \vee (\neg p \vee \neg r) \rightarrow (p \leftrightarrow q)$;
7. $U_7 = (q \wedge r \rightarrow p) \rightarrow (p \rightarrow r) \wedge q$;

Exercise 2

Prove that the following formulas are tautologies using the semantic tableaux method:

1.distribution of '$\rightarrow$' over '$\wedge$': $(p \rightarrow q \wedge r) \leftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$;

2.separation of the premises law: $(p \wedge q \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$;

3.distribution of '$\rightarrow$' over '$\vee$': $(p \rightarrow q \vee r) \leftrightarrow (p \rightarrow q) \vee (p \rightarrow r)$;

4.distribution of '$\vee$' over '$\leftrightarrow$': $(p \vee (q \leftrightarrow r)) \leftrightarrow ((p \vee q) \leftrightarrow (p \vee r))$;

5.reunion of the premises law: $(p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r)$;

6.distribution of implication: $(p \rightarrow (q \rightarrow r)) \leftrightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$;

7.distribution of '$\rightarrow$' over '$\leftrightarrow$': $(p \rightarrow (q \leftrightarrow r)) \leftrightarrow ((p \rightarrow q) \leftrightarrow (p \rightarrow r))$.

8.permutation of the premises law: $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$;

Exercise 3

Using the semantic tableaux method, decide whether the following logical consequences hold or not.
If a logical consequence does not hold find an anti-model of it.

1. $p \rightarrow (\neg q \vee r \wedge s), p, \neg s \models \neg q$
2. $\neg p \rightarrow (\neg q \rightarrow r), r \vee q \models (\neg p \rightarrow q) \vee r$
3. $p \rightarrow (q \vee r \wedge s), p, \neg r \models q$
4. $p \rightarrow q, r \rightarrow t, p \wedge r \models q \wedge t$
5. $p \wedge (q \rightarrow r), q \vee r \models p \rightarrow (q \rightarrow r)$
6. $p \rightarrow q \models (r \rightarrow t) \rightarrow (p \wedge r \rightarrow q \wedge t)$
7. $p \wedge (q \rightarrow r), q \vee r \models p \rightarrow (q \rightarrow r)$
8. $p \rightarrow q \vee r \models (p \rightarrow q) \vee (p \rightarrow r)$

Exercise 4

Write all the anti-models of the propositional formulas $U_1,...,U_8$ using the semantic tableaux method.

1. $U_1 = (p \vee q) \wedge \neg r \rightarrow p \wedge q \wedge r$;
2. $U_2 = q \wedge \neg p \wedge r \rightarrow \neg p \vee \neg(q \wedge r)$;
3. $U_3 = p \rightarrow (q \wedge r) \vee q \wedge \neg p$;
4. $U_4 = \neg p \vee (\neg q \vee r) \rightarrow q \vee \neg p \vee r$;
5. $U_5 = \neg p \vee (\neg q \vee \neg r) \rightarrow q \wedge \neg p$;
6. $U_6 = \neg p \vee (\neg q \wedge \neg r) \rightarrow q \wedge \neg p \wedge r$;
7. $U_7 = \neg p \vee \neg(q \wedge r) \rightarrow q \wedge \neg p$;

Exercise 5.

Check whether the conclusion C is a logical consequence of the set of hypotheses using the semantic tableaux method.

Hypotheses:

H_1. All hummingbirds are richly colored.

H_2. No large birds live on honey.

H_3. Birds that do not live on honey are dull in color.

Conclusion: C. All hummingbirds are small.

Exercise 6

Check whether the conclusion C is a logical consequence of the set of hypotheses using the semantic tableaux method. $H_1, H_2, H_3, H_4 \models C$?

Hypotheses:

H_1. Any Computer Science student likes *logic* and likes any programming language.

H_2. Someone who likes *logic* is a Computer Science student or a Philosophy student.

H_3. *Java* is a programming language.

H_4. *John* does not like *Java* but he likes *logic*.

Conclusion: *C*. John is a Philosophy student but he is not a Computer Science student.

Exercise 7

Using the semantic tableaux method, prove the following properties in predicate logic:

1.' $\exists$ ' is semi-distributive over ' $\wedge$ ':

$\models (\exists x)(A(x) \wedge B(x)) \rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$ and

$\not\models (\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x))$

2.' $\forall$ ' is semi-distributive over ' $\vee$ ':

$\models (\forall x)A(x) \vee (\forall x)B(x) \rightarrow (\forall x)(A(x) \vee B(x))$ and

$\not\models (\forall x)(A(x) \vee B(x)) \rightarrow (\forall x)A(x) \vee (\forall x)B(x)$

3.' $\exists$ ' is semi-distributive over ' $\rightarrow$ ':

$\models ((\exists x)A(x) \rightarrow (\exists x)B(x)) \rightarrow (\exists x)(A(x) \rightarrow B(x))$ and

$\not\models (\exists x)(A(x) \rightarrow B(x)) \rightarrow ((\exists x)A(x) \rightarrow (\exists x)B(x))$

4.' $\forall$ ' is semi-distributive over ' $\rightarrow$ ':

$\models (\forall x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\forall x)B(x))$ and

$\not\models ((\forall x)A(x) \rightarrow (\forall x)B(x)) \rightarrow (\forall x)(A(x) \rightarrow B(x))$

5. $\models (\exists x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\exists x)B(x))$ and

$\not\models ((\exists x)A(x) \rightarrow (\exists x)B(x)) \rightarrow (\forall x)(A(x) \rightarrow B(x))$

6.' $\exists$ ' is distributive over ' $\vee$ '

$\models (\exists x)(A(x) \vee B(x)) \leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

7.' $\forall$ ' is distributive over ' $\wedge$ '

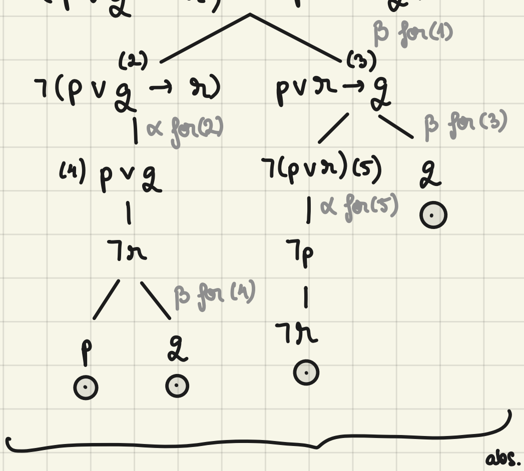
$\models (\forall x)(A(x) \wedge B(x)) \leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$

Exercise 1

Using the semantic tableaux method decide what kind (consistent, inconsistent, valid) of formula is $U_j, j \in \{1,2,\ldots,8\}$.

If $U_j, j \in \{1,2,\ldots,8\}$ is consistent, find all its models.

$U_2 = (p \vee q \to r) \to (p \vee r \to q)$ (1)



abs. ⇒ q

$$DNF(U_2) = (\neg r \wedge p) \vee (\neg r \wedge q) \vee (\neg p \wedge \neg r) \vee q$$
$$= (\neg r \wedge p) \vee (\neg p \wedge \neg r) \vee q$$

$i : \{p, q, r\} \to \{T, F\}$

	i_1	i_2	i_3	i_4	i_5	i_6
p	T	T	F	F	T	F
q	T	F	T	F	T	T
r	F	F	F	F	T	T

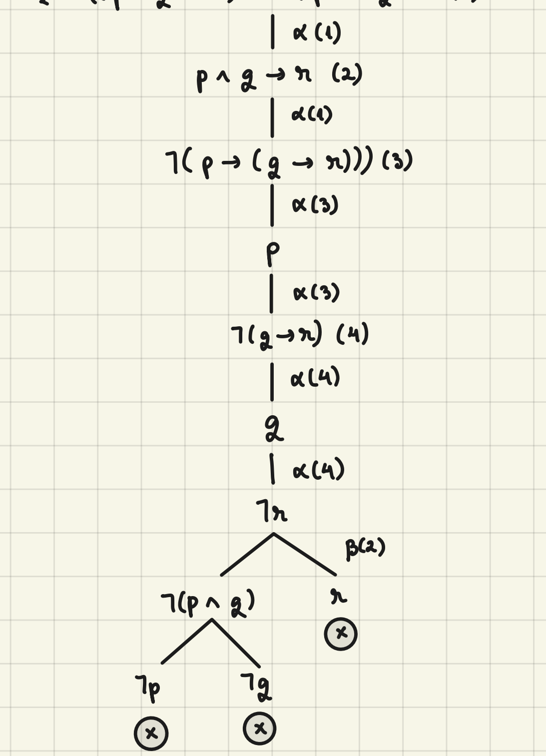
cube 1 (i_1, i_2)   cube 2 (i_3, i_4)   cube 3 + i_1, i_3 (i_5, i_6)

Exercise 2

Prove that the following formulas are tautologies using the semantic tableaux method:

$U_2 = (p \wedge q \to r) \to (p \to (q \to r))$

$\neg U_2 = \neg((p \wedge q \to r) \to (p \to (q \to r)))$ (1)



⇒ U_2 is a tautology

Exercise 3
Using the semantic tableaux method, decide whether the following logical consequences hold or not. If a logical consequence does not hold find an anti-model of it.

$U_2 = \neg p \to (\neg q \to r),\ r \vee q \models (\neg p \to q) \vee r$ $\qquad$ U_2 holds if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C$ is closed

$(\neg p \to (\neg q \to r)) \wedge (r \vee q) \wedge \neg((\neg p \to q) \vee r)$ (1)

$| \ \alpha(1)$

$\neg p \to (\neg q \to r)$ (2)

$| \ \alpha(1)$

$r \vee q$ (3)

$| \ \alpha(1)$

$\neg((\neg p \to q) \vee r)$ (4)

$| \ \alpha(4)$

$\neg(\neg p \to q)$ (5)

$| \ \alpha(4)$

$\neg r$

$| \ \alpha(5)$

$\neg p$

$| \ \alpha(5)$

$\neg q$

$\beta(2)$

p $\otimes$ $\qquad$ $\neg q \to r$ (6)

q $\otimes$ $\qquad$ r $\otimes$

$\Rightarrow$ the deduction holds

Exercise 4
Write all the anti-models of the propositional formulas $U_1,\ldots,U_8$ using the semantic tableaux method.

$U_2 = q \wedge \neg p \wedge r \to \neg p \vee \neg(q \wedge r)$

$\neg U_2 = \neg(q \wedge \neg p \wedge r \to \neg p \vee \neg(q \wedge r))$ (1)

$| \ \alpha(1)$

$q \wedge \neg p \wedge r$ (2)

$|$

$\neg(\neg p \vee \neg(q \wedge r))$ (3)

$| \ \alpha(2)$

$q, \neg p, r$ $\qquad$ $\Rightarrow$ no anti-models

$| \ \alpha(3)$

p

$|$

$q \wedge r$

$\otimes$

Exercise 5.

Check whether the conclusion C is a logical consequence of the set of hypotheses using the semantic tableaux method.

Hypotheses:

H_1. All hummingbirds are richly colored.

H_2. No large birds live on honey.

H_3. Birds that do not live on honey are dull in color.

Conclusion: C. All hummingbirds are small.

PREDICATES

$rc(x)$: "x is richly colored"

$hb(x)$: "x hummingbird"

$sb(x)$: "x small bird"

$lh(x)$: "x lives on honey"

D= all birds

$H_1: (\forall x)(hb(x) \to rc(x)) \equiv (\forall x)(\neg hb(x) \vee rc(x))$

$H_2: \neg(\exists x)(\neg sb(x) \wedge lh(x)) \equiv (\forall x)(sb(x) \vee \neg lh(x))$

$H_3: (\forall x)(\neg lh(x) \to \neg rc(x)) \equiv (\forall x)(lh(x) \vee \neg rc(x))$

$C: (\forall x)(hb(x) \to sb(x)) \equiv (\forall x)(\neg hb(x) \vee sb(x))$

$\neg C: \neg(\forall x)(\neg hb(x) \vee \neg sb(x)) \equiv (\exists x)\neg(\neg hb(x) \vee sb(x))$

(1)

$\boxed{H_1 \wedge H_2 \wedge H_3 \wedge \neg C}$

| $\alpha(1)$

$H_1: (\forall x)(\neg hb(x) \vee rc(x))$ (2)

$H_2: (\forall x)(sb(x) \vee \neg lh(x))$ (3)

$H_3: (\forall x)(lh(x) \vee \neg rc(x))$ (4)

$\neg C: (\exists x)\neg(\neg hb(x) \vee sb(x))$ (5)

| $\delta(5)$ - c (new constant)

$\neg(\neg hb(c) \vee sb(c))$ (6)

| $\alpha(6)$

$hb(c)$

$\neg sb(c)$

| $\gamma(2)$, c used for instantiation

$\neg hb(c) \vee rc(c)$ (7)

(2) copy

$\beta(7)$

$\neg hb(c)$ ⊗ | $rc(c)$

| $\gamma(3)$, c used

$sb(c) \vee \neg lh(c)$ (8)

(3) copy

$sb(c)$ ⊗ | $\neg lh(c)$

| $\gamma(4)$, c used

$lh(c) \vee \neg rc(c)$ (9)

$\beta(9)$

$lh(c)$ ⊗ | $\neg rc(c)$ ⊗

δ RULES

$(\exists x)\,A(x)$	$\neg(\forall x)\,A(x)$
$A(c)$	$\neg A(c)$

γ RULES

$(\forall x)\,A(x)$	$\neg(\exists x)\,A(x)$
$A(c_1)$	$\neg A(c_1)$
...	...
$A(c_n)$	$\neg A(c_n)$
$(\forall x)\,A(x)$	$\neg(\exists x)\,A(x)$

Exercise 6

Check whether the conclusion C is a logical consequence of the set of hypotheses using the semantic tableaux method. $H_1, H_2, H_3, H_4 \models C$?

Hypotheses:

- H_1. Any Computer Science student likes *logic* and likes any programming language.
- H_2. Someone who likes *logic* is a Computer Science student or a Philosophy student.
- H_3. *Java* is a programming language.
- H_4. *John* does not like *Java* but he likes *logic*.

Conclusion: *C*. John is a Philosophy student but he is not a Computer Science student.

cs(x): "x cs student"
l(x,y): "x likes y"
ps(x): "x phil. student"
p(x): "x progr. language"

$H_1 \wedge H_2 \wedge H_3 \wedge H_4 \wedge \neg C$ (1)

| α(1)

$H_1: (\forall x)(\forall y)(cs(x) \to l(x, logic) \wedge (\forall y)\, p(y) \to l(x, y))$ (2)

$H_2: (\forall x)(l(x, logic) \to cs(x) \vee ps(x))$ (3)

$H_3: p(Java)$ (4)

$H_4: \neg l(John, Java) \wedge l(John, logic)$ (5)

$\neg C: \neg ps(John) \vee cs(John)$ (6)

| α(5)

$\neg l(John, Java)$

$l(John, logic)$

| γ(2) John

$cs(John) \to l(John, logic) \wedge (\forall y)(p(y) \to l(John, y))$ (7)

| (2) copy
| γ(3) John

$l(John, logic) \to cs(John) \vee ps(John)$ (8)

| (3) copy

β(6):

- $\neg ps(John)$
  - β(8):
    - $\neg l(John, logic)$ $\otimes$
    - $cs(John) \vee ps(John)$ (9)
      - β(9):
        - $cs(John)$
          - β(7):
            - $\neg cs(John)$ $\otimes$
            - $l(John, logic) \wedge (\forall y)(p(y) \to l(John, y))$ (10)
              - | α(10)
              - $l(John, logic)$
              - $(\forall y)(p(y) \to l(John, y))$ (11)
              - | γ(11), Java
              - $p(Java) \to l(John, Java)$ (12)
              - | (11) copy
              - β(12):
                - $\neg p(Java)$ $\otimes$
                - $l(John, Java)$ $\otimes$
        - $ps(John)$ $\otimes$
- $cs(John)$

Exercise 7

Using the semantic tableaux method, prove the following properties in predicate logic:

2. ' $\forall$ ' is semi-distributive over ' $\vee$ ':

$\models (\forall x)A(x) \vee (\forall x)B(x) \rightarrow (\forall x)(A(x) \vee B(x))$ and

$\not\models (\forall x)(A(x) \vee B(x)) \rightarrow (\forall x)A(x) \vee (\forall x)B(x)$

$U_1 = ((\forall x)\ A(x) \vee (\forall x)\ B(x)) \rightarrow (\forall x)(A(x) \vee B(x))$ $\Rightarrow$ prove $\neg U_1$ closed tableaux

$U_2 = (\forall x)(A(x) \vee B(x)) \rightarrow ((\forall x)\ A(x) \vee (\forall x)\ B(x))$ $\Rightarrow$ prove $\neg U_2$ open $\Rightarrow$ models for $\neg U_2$ $\Rightarrow$ anti-models for U_2

$\neg U_1 = \neg(((\forall x)\ A(x) \vee (\forall x)\ B(x)) \rightarrow (\forall x)(A(x) \vee B(x)))$ (1)

| $\alpha(1)$

$(\forall x)\ A(x) \vee (\forall x)\ B(x)$ (2)

|

$\neg(\forall x)(A(x) \vee B(x))$ (3)

| $\delta(3)$, a

$\neg(A(a) \vee B(a))$ (4)

| $\alpha(4)$

$\neg A(a)$

|

$\neg B(a)$

$\beta(2)$:

- $(\forall x)\ A(x)$ (5) | $\gamma(5)$ → $A(a)$ ⊗
- $(\forall x)\ B(x)$ (6) | $\gamma(5)$ → $B(a)$ ⊗

$\neg U_2 = \neg((\forall x)(A(x) \vee B(x)) \rightarrow ((\forall x)\ A(x) \vee (\forall x)\ B(x)))$ (1)

| $\alpha(1)$

$(\forall x)(A(x) \vee B(x))$ (2)

|

$\neg((\forall x)\ A(x) \vee (\forall x)\ B(x))$ (3)

| $\alpha(3)$

$\neg(\forall x)\ A(x)$ (4)

|

$\neg(\forall x)\ B(x)$ (5)

| $\delta(4)$, a

$\neg A(a)$

| $\delta(5)$, b

$\neg B(b)$

| $\gamma(2)$ - a, b used

$A(a) \vee B(a)$ (6)

|

$A(b) \vee B(b)$ (7)

| (2) copy

$\beta(6)$:

- $A(a)$ ⊗
- $B(a)$
  - $\beta(7)$:
    - $A(b)$ ⊙
    - $B(b)$ ⊗

$\Rightarrow \neg A(a) \wedge \neg B(b) \wedge A(b) \wedge B(a)$ anti-model for U_2

$I = \langle D, m \rangle$ $\quad D = \{a, b\}$

$m(A)(a) = F$ $\quad m(B)(a) = T$

$m(A)(b) = T$ $\quad m(B)(b) = F$