SEMINAR 13+14

1) a) Let $\varphi \in \mathbb{R}$. Show that the plane rotation

$$h : \mathbb{R}^2 \to \mathbb{R}^2,\ h(x, y) = (x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi),$$

is an automorphism of $\mathbb{R}^2$. Write the matrix of h of $\mathbb{R}^2$ in the standard basis (i.e. the basis $E = (e_1, e_2)$, with $e_1 = (1, 0)$, $e_2 = (0, 1)$).
b) Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, -y)$ (the symmetry with respect to Ox) and $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (-x, y)$ (the symmetry with respect to Oy) are automorphisms of $\mathbb{R}^2$. Find the matrices of f, g, $f - g$, $f + 2g$ and $g \circ f$ in the standard basis.

2) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x + y, 2x - y, 3x + 2y)$. Show that f is an $\mathbb{R}$-linear map, that $B = ((1, 2), (-2, 1))$ and $B' = ((1, -1, 0), (-1, 0, 1), (1, 1, 1))$ are bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, then determine the matrix of f in the pair of bases (B, B').

Extra: Show that (v_1, v_2, v_3) and (v'_1, v'_2, v'_3) with

$$v_1 = (1, 2, 1), v_2 = (2, 3, 3), v_3 = (3, 7, 1) \text{ şi } v'_1 = (3, 1, 4), v'_2 = (5, 2, 1), v'_3 = (1, 1, -6)$$

are bases of $\mathbb{R}^3$ and find a connection between the matrices of a given vector in these two bases.

3) Let $B = (v_1, v_2, v_3, v_4)$ be a basis of the $\mathbb{R}$-vector space $\mathbb{R}^4$, the vectors

$$u_1 = v_1,\ u_2 = v_1 + v_2,\ u_3 = v_1 + v_2 + v_3,\ u_4 = v_1 + v_2 + v_3 + v_4$$

and $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with

$$[f]_B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

Show that $B' = (u_1, u_2, u_3, u_4)$ is a basis of $\mathbb{R}^4$ and find the matrix $[f]_{B'}$.

4) Let V be a real vector space, $B = (v_1, v_2, v_3)$ a basis of V, the vectors

$$u_1 = v_1 + 2v_2 + v_3,\ u_2 = v_1 + v_2 + 2v_3,\ u_3 = v_1 + v_2$$

and $f \in End_{\mathbb{R}}(V)$. Show that $B' = (u_1, u_2, u_3)$ is a basis of V and determine the matrix $[f]_B$ provided that

$$[f]_{B'} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}.$$

5) Let V, V' be $\mathbb{R}$-vector spaces, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ bases in V and V', respectively and $f : V \to V'$ a $\mathbb{R}$-linear map with

$$[f]_{a,b} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Determine:
i) $f(v)$ for an arbitrary $v \in V$;
ii) the dimensions of $\operatorname{Im} f$ and $\operatorname{Ker} f$;
iii) the matrix $[f]_{a',b'}$ when $a' = (a_1, a_1+a_2, a_1+a_2+a_3)$ and $b' = (b_1, b_1+b_2, b_1+b_2+b_3)$.

6) Let V, V' be $\mathbb{R}$-vector spaces, $B = (v_1, v_2, v_3)$ a basis for V, $B' = (v_1', v_2', v_3')$ a basis for V' and $f : V \to V'$ a $\mathbb{R}$-linear map with

$$[f]_{B,B'} = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & -5 \end{pmatrix}.$$

Determine:
i) the dimension and a basis for each of the spaces $\operatorname{Im} f$ and $\operatorname{Ker} f$;
ii) $[f]_{B,E'}$ when $V' = \mathbb{R}^3$, $v_1' = (1,0,0)$, $v_2' = (0,1,1)$, $v_3' = (0,0,1)$ and E' is the standard basis of $\mathbb{R}^3$;
iii) $f(x)$ for $x = 2v_1 - v_2 + 3v_3$, under the circumstances of ii).

7) Let $f \in End_{\mathbb{Q}}(\mathbb{Q}^4)$ with the matrix in the standard basis

$$a) \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix}; \quad b) \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 2 & 1 & 0 \\ 3 & 0 & -1 & -2 \\ 5 & -3 & -1 & 1 \end{pmatrix}.$$

Determine a basis for each of the $\mathbb{Q}$-vector spaces $\operatorname{Ker} f$, $\operatorname{Im} f$, $\operatorname{Ker} f + \operatorname{Im} f$ and $\operatorname{Ker} f \cap \operatorname{Im} f$.

HOMEWORK: 1) Let $S = \{(t, 2t, 3t) \mid t \in \mathbb{R}\}$ and $T = \{(x, y, z) \mid x + y + z = 0\}$.
i) Show that S and T are subspaces of ${}_{\mathbb{R}}\mathbb{R}^3$.
ii) Determine a basis for each of the subspaces S and T.
iii) Determine $S \cap T$ and $S + T$.

2) Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ the $\mathbb{R}$-linear map defined on the standard basis of $\mathbb{R}^3$ by:

$$f(e_1) = (1,2,3,4),\ f(e_2) = (4,3,2,1),\ f(e_3) = (-2,1,4,1).$$

Determine:
i) $f(v)$ for any $v \in \mathbb{R}^3$;
ii) the matrix of f in the standard bases;
iii) a basis for $\operatorname{Im} f$ and a basis for $\operatorname{Ker} f$.