Seminar 8

1. $\sigma = (2\ 3\ 4) = (2\ 3)(3\ 4) \Rightarrow sgn(\sigma) = (-1)^2 = +1$, where 2 is number of transpositions of σ.

   It is easy to see that the inverse on the left is the same as the one on the right, which is $\sigma^{-1} = (2\ 4\ 3)$

2. $\sigma = (1\ 2\ 4)(3\ 5) \Rightarrow orb(\sigma) = \{\{1, 2, 4\}, \{3, 5\}\}$.

3. $$\sigma_1 = (1\ 2\ 4)(3\ 6\ 7) = (1\ 2)(2\ 4)(3\ 6)(6\ 7)$$
   $$\sigma_2 = (1\ 6)(2\ 4)(3\ 7\ 8\ 9) = (1\ 6)(2\ 4)(3\ 7)(7\ 8)(8\ 9)$$

4. We know that $S_3 = \{e, (2\ 3), (1\ 2), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. So the only even permutations are the ones that have an even number of transpositions $\Rightarrow A_3 = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$.

   Same goes for $A_4 = \{e, (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 4\ 3), (2\ 3\ 4),$ $(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

5. For H to be a subgroup of $(S_5, \circ)$ it needs to have the next properties:

   (a) $H \neq \emptyset$
   (b) $\sigma_i, \sigma_j \in H \Rightarrow \sigma_i \circ \sigma_j \in H$
   (c) $\sigma_i \in H \Rightarrow \sigma_i^{-1} \in H$

   For the first condition, we have the permutation $e \in H$, as $e(1) = 1$ and $e(5) = 5$.

   Let $\sigma_i \in H$, with $\sigma_i(1) = 1$ and $\sigma_i(5) = i \neq 5$. Also, let $\sigma_j \in H$, with $\sigma_j(5) = 5$ and $\sigma_j(1) = j \neq 1$. By computing $(\sigma_i \circ \sigma_j)(5) = \sigma_i(\sigma_j(5)) = \sigma_i(5) = i \neq 5$. The same goes for $(\sigma_j \circ \sigma_i)(1) = \sigma_j(\sigma_i(1)) = \sigma_j(1) = j \neq 1$. And so, the second condition does not hold. Hence, H is not a subgroup of $(S_5, \circ)$.

6. We know about the elements of D_3 from seminar 2. Remember the triangle ABC. Now, if we denote $A = 1$, $B = 2$ and $C = 3$, we can write the elements of D_3 as permutations:

$$r_0 = e,\ r_1 = (1\ 2\ 3),\ r_2 = (1\ 3\ 2)$$

$$s_1 = (2\ 3),\ s_2 = (1\ 3),\ s_3 = (1\ 2)$$

So, we have the same operation as in S_3 and all the elements of S_3 in D_3. It is obvious that $(D_3, \cdot) \simeq (S_3, \circ)$.

7. We use the fact that $ord(i_1 \ldots i_r) = r$ for $(i_1 \ldots i_r)$ a cycle of r elements in S_n, $r \leq n$.

   The cyclic subgroups of S_3 are: $\{e\}, < e, (1\ \ 2) >, < e, (1\ \ 3) >, < e, (2\ 3) >, < e, (1\ 2\ 3), (1\ 3\ 2) >, S_3$.

8. Using last seminar and exercise 7.