Seminar 12 – The dimension of generated subspaces

1) Let $a=(-2,1,3)$, $b=(3,-2,-1)$, $c=(1,-1,2)$, $d=(-5,3,4)$, $e=(-9,5,10)$ in ${}_{\mathbb{Q}}\mathbb{Q}^3$.

$$\langle a,b\rangle \overset{?}{=} \langle c,d,e\rangle.$$

Solution: $\dim\langle a,b\rangle = \operatorname{rank}\begin{pmatrix} -2 & 1 & 3 \\ 3 & -2 & -1 \end{pmatrix} = 2 \Longrightarrow (a,b)$ basis for $\langle a,b\rangle$

$\dim\langle c,d,e\rangle = 2 \Longrightarrow (c,d)$ basis for $\langle c,d,e\rangle (=\langle c,d\rangle)$

$$\begin{array}{c} c \\ d \\ e \end{array}\begin{pmatrix} 1 & -1 & 2 \\ -5 & 3 & 4 \\ -9 & 5 & 10 \end{pmatrix} \underset{c_3-2c_1}{\overset{c_2+c_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 14 \\ 0 & -4 & 28 \end{pmatrix}$$

$$\langle a,b\rangle \overset{?}{=} \langle c,d\rangle$$
$$\Updownarrow$$
$$\langle a,b\rangle \leq_{\mathbb{Q}} \langle c,d\rangle$$
$$\Updownarrow$$
$$\{a,b\} \overset{?}{\subseteq} \langle c,d\rangle \Longrightarrow \langle a,b\rangle \subseteq \langle c,d\rangle$$

$a \overset{?}{\in} \langle c,d\rangle \Longleftrightarrow \begin{vmatrix} -2 & 1 & 3 \\ 1 & -1 & 2 \\ -5 & 3 & 4 \end{vmatrix} \overset{?}{=} 0 \quad (\overset{\text{0-P}}{\longleftarrow}\text{ yes}).$

$b \overset{?}{\in} \langle c,d\rangle \Longleftrightarrow \begin{vmatrix} 3 & -2 & -1 \\ 1 & -1 & 2 \\ -5 & 3 & 4 \end{vmatrix} \overset{?}{=} 0 \quad (\overset{\text{R-Z}}{\longleftarrow}\text{ yes}).$

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2) a) $S=\langle u_1,u_2\rangle$, $u_1=(1,1,0,0)$, $u_2=(1,0,1,1)$
$T=\langle v_1,v_2\rangle$, $v_1=(0,0,1,1)$, $v_2=(0,1,1,0)$ in ${}_{\mathbb{R}}\mathbb{R}^4$

Find bases and the dimension for S, T, $S+T$ and $S\cap T$ (?)

$\dim S = \operatorname{rank}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 2$ and (u_1,u_2) is a basis for S

$\dim T = \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = 2$ and (v_1,v_2) — " — " — T

$S+T = \langle S\cup T\rangle = \langle u_1,u_2,v_1,v_2\rangle$

$\dim(S+T) = 4$ and (u_1,u_2,v_1,v_2) basis for $S+T$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \overset{r_2-r_1}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underset{r_4+r_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \overset{c_4-c_3}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$(S+T \leq_{\mathbb{R}} \mathbb{R}^4,\ \dim(S+T) = \dim \mathbb{R}^4 = 4 \Longrightarrow S+T=\mathbb{R}^4)$

$\dim(S+T) + \dim(S\cap T) = \dim S + \dim T \implies$

$\implies \dim(S\cap T) = \dim S + \dim T - \dim(S+T) = 2+2-4=0$

$\implies S\cap T = \{(0,0,0,0)\} \implies \emptyset$ is a basis for $S\cap T$

b) $S = \langle u_1, u_2, u_3\rangle$, $u_1 = (1,2,-1,-2)$, $u_2 = (3,1,1,1)$, $u_3 = (-1,0,1,-1)$

$T = \langle v_1, v_2\rangle$, $v_1 = (-1,2,-7,-3)$, $v_2 = (2,5,-6,-5)$

$\dim S = 3$ and (u_1,u_2,u_3) basis for S (<u>homework</u>)

$\dim T = 2$ and (v_1, v_2) —"— T (—"—)

$\dim(S+T) = \dim\langle u_1,u_2,u_3,v_1,v_2\rangle = 3$

$$\begin{array}{c} u_1\\ u_2\\ u_3\\ v_1\\ v_2\end{array}\begin{pmatrix} 1 & 2 & -1 & -2\\ 3 & 1 & 1 & 1\\ -1 & 0 & 1 & -1\\ -1 & 2 & -7 & -3\\ 2 & 5 & -6 & -5\end{pmatrix} \underset{\substack{c_3+c_1\\ c_4+2c_1}}{\overset{c_2-2c_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -5 & 4 & 7\\ 0 & 2 & 0 & -3\\ 0 & 4 & -8 & -5\\ 0 & 1 & -4 & -1\end{pmatrix} \underset{r_2\leftrightarrow r_5}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -4 & -1\\ 0 & 2 & 0 & -3\\ 0 & 4 & -8 & -5\\ 0 & -5 & 4 & 7\end{pmatrix} \underset{c_4+c_2}{\overset{c_3+4c_2}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 8 & -1\\ 0 & 0 & 8 & -1\\ 0 & 0 & -16 & 2\end{pmatrix} \overset{c_4+\frac{1}{8}c_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 8 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$\implies (u_1, v_2, u_3)$ is a basis for $S+T$

(R: $\dim(S+T) = \dim S$, $S \leq_{\mathbb{R}} S+T \implies S = S+T \implies$

$\implies (u_1,u_2,u_3)$ a basis for $S+T$).

$\dim(S\cap T) = \dim S + \dim T - \dim(S+T) = 3+2-3 = 2 = \dim T$

$\xRightarrow[S\cap T \leq_{\mathbb{R}} T]{} S\cap T = T \implies (v_1, v_2)$ is a basis for $S\cap T$.

c) $S = \langle u_1, u_2\rangle$, $u_1 = (1,2,1,0)$, $u_2 = (-1,1,1,1)$

$T = \langle v_1, v_2\rangle$, $v_1 = (2,-1,0,1)$, $v_2 = (1,-1,3,7)$

$\dim S = 2$ and (u_1,u_2) basis for S (<u>homework</u>)

$\dim T = 2$ —"— (v_1, v_2) —"— T (—"—)

$\dim(S+T) = \dim\langle u_1,u_2,v_1,v_2\rangle = 3$ and (u_1,u_2,v_1) basis for $S+T$.

$$\begin{array}{c} u_1\\ u_2\\ v_1\\ v_2\end{array}\begin{pmatrix} 1 & 2 & 1 & 0\\ -1 & 1 & 1 & 1\\ 2 & -1 & 0 & 1\\ 1 & -1 & 3 & 7\end{pmatrix} \underset{c_3-c_1}{\overset{c_2-2c_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 3 & 2 & 1\\ 0 & -5 & -2 & 1\\ 0 & -3 & 2 & 7\end{pmatrix} \underset{c_2\leftrightarrow c_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 2 & 3 & 1\\ 0 & -2 & -5 & 1\\ 0 & 2 & -3 & 7\end{pmatrix} \underset{r_3-r_2}{\overset{r_3+r_2}{\sim}}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ \hline 0 & 0 & -6 & 6 \end{array}\right)$$

$\dim(S\cap T) = \dim S + \dim T - \dim(S+T) = 2+2-3 = \underline{\underline{1}}$

A basis for $S\cap T$ is given by a nonzero vector of $S\cap T$.

$$S\cap T = \{x = (x_1,x_2,x_3,x_4)\in \mathbb{R}^4 \mid \exists \alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{R} : x = \alpha_1 u_1 + \alpha_2 u_2 = \beta_1 v_1 + \beta_2 v_2\}$$

$$\alpha_1 u_1 + \alpha_2 u_2 - \beta_1 v_1 - \beta_2 v_2 = (0,0,0,0) \qquad (1)$$

$$(1) \Longleftrightarrow \alpha_1(1,2,1,0) + \alpha_2(-1,1,1,1) - \beta_1(2,-1,0,1) - \beta_2(1,-1,3,7) = (0,0,0,0) \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \alpha_1 - \alpha_2 - 2\beta_1 - \beta_2 = 0 \\ 2\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 0 \\ \alpha_1 + \alpha_2 - 3\beta_2 = 0 \\ \alpha_2 - \beta_1 - 7\beta_2 = 0 \end{cases} \qquad (2)$$

(2) is a homogeneous system with the system matrix

$$\begin{array}{cccc} \multicolumn{4}{c}{\begin{pmatrix} 1 & -1 & -2 & -1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \\ 0 & 1 & -1 & -7 \end{pmatrix}} \\ u_1 & u_2 & -v_1 & -v_2 \end{array}$$

which has the rank 3; a 3-size nonzero minor of this matrix can be "cutted" from the first 3 rows and first 3 columns.

$\Longrightarrow \beta_2$ can be considered a side unknown.

$$\underset{\sim}{\beta_2 = 1} \xRightarrow{(2)} \begin{cases} \alpha_1 - \alpha_2 - 2\beta_1 = 1 \\ 2\alpha_1 + \alpha_2 + \beta_1 = -1 \\ \alpha_1 + \alpha_2 = 3 \end{cases} \Longrightarrow \beta_1 = \ldots$$

$$\beta_1 = \frac{\Delta_3}{\Delta}, \text{ where}$$

$$\left.\begin{array}{l} \Delta = \begin{vmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -4 \\ \Delta_3 = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 3 \end{vmatrix} = 12 \end{array}\right\} \Longrightarrow \underset{\sim}{\beta_1 = -3}$$

Thus $\underbrace{x_0}_{\text{nonzero}} = -3v_1 + v_2 = (\ ,\ ,\ ,\ ) \in S\cap T \Longrightarrow x_0$ forms (alone) a basis for $S\cap T$.

d) $S = \langle u_1, u_2, u_3\rangle$, $u_1 = (1,2,1,-2)$, $u_2 = (2,3,1,0)$, $u_3 = (1,2,2,-3)$

$T = \langle v_1, v_2, v_3\rangle$, $v_1 = (1,1,1,1)$, $v_2 = (1,0,1,-1)$, $v_3 = (1,3,0,-3)$

$\dim S = 3$, $\dim T = 3$,

$\dim(S+T) = 4$ and (u_1, u_2, u_3, v_2) basis for $S+T$

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{pmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & -3 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & -3 \end{pmatrix} \underset{\substack{c_3 - c_1 \\ c_4 + 2c_1}}{\overset{c_2 - 2c_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 3 \\ 0 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \underset{c_4 + 4c_2}{\overset{c_3 - c_2}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & -2 & 3 \end{pmatrix}$$

(Even if $S+T = \mathbb{R}^4$ it is more convenient for determining a basis for $S \cap T$ to work with the basis (u_1, u_2, u_3, v_2) instead of the standard basis).

$\dim(S \cap T) = 3 + 3 - 4 = \underline{\underline{2}}$

We are searching for 2 l. indep. vectors from $S \cap T$.

$S \cap T = \{x \in \mathbb{R}^4 \mid \exists\, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R} : x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3\}$

$\alpha_1 \underline{u_1} + \alpha_2 \underline{u_2} + \alpha_3 \underline{u_3} - \beta_1 \underline{v_1} - \beta_2 \underline{v_2} - \beta_3 \underline{v_3} = (0,0,0,0)$

The rank of the system matrix is 4 and

β_1 and β_3 can be considered side unknowns.

$\underline{\beta_1 = 1, \beta_3 = 0} \Longrightarrow \beta_2' = \dots \Longrightarrow x' = v_1 + \beta_2' v_2 = (\;,\;,\;,\;)$

$\underline{\beta_1 = 0, \beta_3 = 1} \Longrightarrow \beta_2'' = \dots \Longrightarrow x'' = \beta_2'' v_2 + v_3 = (\;,\;,\;,\;)$

2 l. indep. in $S \cap T$

$\Longrightarrow (x', x'')$ basis for $S \cap T$.

$$\left(\begin{array}{cc|c} 1 & 0 & \beta_2' \\ 0 & 1 & \beta_2'' \end{array}\right) \Longrightarrow x', x'' \text{ l. indep.}, \quad \dim(S \cap T) = 2$$

<u>Homework</u>: complete the solution of d).