COURSE 13

The matrix of a linear map

First, we define the matrix of a vector in a basis of a vector space. For certain reasons, it is presented as a column-matrix, but it must be said that this is rather a convention than a constraint. But if one changes the convention, the form of the next notions and results must be properly changed.

Let K be a field.

Definition 1. Let V be a K-vector space, $v \in V$ and $B = (v_1, \dots, v_n)$ a basis of V. If

$$v = k_1 v_1 + \cdots + k_n v_n \ (k_1, \dots, k_n \in K) \qquad n \in \mathbb{N}^*$$

is the unique representation of v as a linear combination of the vectors of B, then the **matrix of the vector** v in the basis B is

$$[v]_B = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$$

Definition 2. Let $f : V \to V'$ be a K-linear map, let $B = (v_1, \dots, v_n)$ be a basis of V and let $B' = (v'_1, \dots, v'_m)$ be a basis of V'. Then we can uniquely write the vectors of $f(B)$ as linear combinations of the vectors of B', i.e. there exist $a_{ij} \in K$ ($i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$) uniquely determined such that $\quad m, n \in \mathbb{N}^*$

$$\begin{cases} f(v_1) = a_{11}v'_1 + a_{21}v'_2 + \cdots + a_{m1}v'_m \\ f(v_2) = a_{12}v'_1 + a_{22}v'_2 + \cdots + a_{m2}v'_m \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f(v_n) = a_{1n}v'_1 + a_{2n}v'_2 + \cdots + a_{mn}v'_m \end{cases}$$

Then the **matrix of the K-linear map** f in the pair of bases (B, B') (or, simply, in the bases B and B') is the matrix whose columns consist of the coordinates of the vectors of $f(B)$ in the basis B', that is,

$$\begin{array}{cc} & \begin{array}{cccc} \text{coord. of } f(v_1) & f(v_2) & & f(v_n) \ \text{in } B' \end{array} \\ [f]_{BB'} = & \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in M_{m,n}(K) \end{array}$$

For $V = V'$ and $B = B'$, we denote $[f]_B = [f]_{BB}$ and we call it the **matrix of** f in the basis B.

Remarks 3. (1) We complete the matrix of a linear map by columns. This is also a part of the convention we mentioned at the beginning of this section. !

(2) As we will see next, the matrix of a linear map depends on the map, on the considered bases, but also by the order of the elements in each basis.

Examples 4. a) For any n-dimensional K-vector space V and any basis B of V, we have

$$[1_V]_B = I_n.$$

! $[1_V]_{B,B'} \neq I_n$

$B \neq B'$

b) Consider the $\mathbb{R}$-linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

h.w. (f $\mathbb{R}$-linear map)

$$f(x, y, z, t) = (x + y + z, y + z + t, z + t + x)\,,\ \forall (x, y, z, t) \in \mathbb{R}^4\,.$$

Let $E = (e_1, e_2, e_3, e_4)$ and $E' = (e'_1, e'_2, e'_3)$ be the standard bases in $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Since

$$\begin{cases} f(e_1) = f(1,0,0,0) = (1,0,1) = e'_1 + e'_3 \\ f(e_2) = f(0,1,0,0) = (1,1,0) = e'_1 + e'_2 \\ f(e_3) = f(0,0,1,0) = (1,1,1) = e'_1 + e'_2 + e'_3 \\ f(e_4) = f(0,0,0,1) = (0,1,1) = e'_2 + e'_3 \end{cases}$$

it follows that the matrix of f in the bases E and E' is

$$[f]_{EE'} = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \\ \begin{array}{cccc} f(e_1) & f(e_2) & f(e_3) & f(e_4) \end{array} \end{array}.$$

c) Let $\mathbb{R}_n[X]$ be the $\mathbb{R}$ - vector space of the polynomials with the degree at most n and real coefficients. The map

$$\varphi : \mathbb{R}_3[X] \to \mathbb{R}_2[X],\ \varphi(a_0 + a_1X + a_2X^2 + a_3X^3) = a_1 + 2a_2X + 3a_3X^2$$

(which associates a polynomial f its formal derivative f') is a linear map. Let us write the matrix of φ in the pair of basis $B = (1, X, X^2, X^3)$, $B' = (1, X, X^2)$, and then in the pair of basis $B = (1, X, X^2, X^3)$, $B'' = (X^2, 1, X)$. We have

$$\begin{aligned} \varphi(1) &= 0 \cdot 1 + 0 \cdot X + 0 \cdot X^2 = 0 \cdot X^2 + 0 \cdot 1 + 0 \cdot X \\ \varphi(X) &= 1 \cdot 1 + 0 \cdot X + 0 \cdot X^2 = 0 \cdot X^2 + 1 \cdot 1 + 0 \cdot X \\ \varphi(X^2) &= 0 \cdot 1 + 2 \cdot X + 0 \cdot X^2 = 0 \cdot X^2 + 0 \cdot 1 + 2 \cdot X \\ \varphi(X^3) &= 0 \cdot 1 + 0 \cdot X + 3 \cdot X^2 = 3 \cdot X^2 + 0 \cdot 1 + 0 \cdot X \end{aligned}$$

thus,

$$[\varphi]_{BB'} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \text{ and } [\varphi]_{BB''} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

d) Let K be a field, $m, n \in \mathbb{N}^*$, $A \in M_{m,n}(K)$, E the standard basis of K^n and E' the standard basis of K^m. Then

(a_{ij})

$$f_A : K^n \to K^m,\ f_A(x_1, \ldots, x_n) = \left[A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right]^t = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{pmatrix}^t$$

h.w.

is a linear map and $[f_A]_{EE'} = A$.

Theorem 5. Let $f : V \to V'$ be a K-linear map, $B = (v_1, \ldots, v_n)$ a basis of V, $B' = (v'_1, \ldots, v'_m)$ a basis of V' and $v \in V$. Then

$$[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B\,.$$

Proof. Let $[f]_{BB'} = (a_{ij}) \in M_{mn}(K)$. Let $v = \sum_{j=1}^{n} k_j v_j$ and $f(v) = \sum_{i=1}^{m} k'_i v'_i$ with $k_i, k'_i \in K$. On the other hand, using the definition of the matrix of f in the bases B and B', we have

$$f(v) = f\Big(\sum_{j=1}^{n} k_j v_j\Big) = \sum_{j=1}^{n} k_j f(v_j) = \sum_{j=1}^{n} k_j \Big(\sum_{i=1}^{m} a_{ij} v'_i\Big) = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij} k_j\Big) v'_i\,.$$

But there is only one way to write $f(v)$ as a linear combination of the vectors of the basis B', hence we have

$$k'_i = \sum_{j=1}^{n} a_{ij} k_j, \forall i \in \{1, \dots, m\}.$$

Therefore, $[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$. □

For a K-linear map $f : V \to V'$ the dimension $\dim(\mathrm{Im} f)$ is also called **the rank of** f. We denote it by $\mathrm{rank}(f)$. The rank of a linear map and the rank of its matrix in a given pair of bases are strongly connected.

Theorem 6. Let $f : V \to V'$ be a K-linear map. Then

$$\mathrm{rank}(f) = \mathrm{rank}[f]_{BB'}\,,$$

given

where B and B' are arbitrary bases of V and V' respectively.

Proof. Let $B = (v_1, \dots, v_n)$ and $[f]_{BB'} = A$. We have

$$\mathrm{rank}(f) = \dim(\mathrm{Im} f) = \dim f(V) = \dim f(\langle v_1, \dots, v_n\rangle) = \dim\langle f(v_1), \dots, f(v_n)\rangle =$$
$$= \mathrm{rank}({}^t A) = \mathrm{rank}(A) = \mathrm{rank}[f]_{BB'}\,.$$

□

Remark 7. The matrices of a linear map in different pairs of bases have the same rank.

We continue this section by presenting one of the key results in Linear Algebra, connecting linear maps and matrices.

Theorem 8. Let V, V' and V'' be vector spaces over K with $\dim V = n$, $\dim V' = m$ and $\dim V'' = p$ and let B, B' and B'' be bases of V, V' and V'' respectively. If $f, g \in Hom_K(V, V')$, $h \in Hom_K(V', V'')$ and $k \in K$, then

$$[f+g]_{BB'} = [f]_{BB'} + [g]_{BB'}\,, \quad [kf]_{BB'} = k \cdot [f]_{BB'}\,,$$

$$[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'}\,.$$

Proof. If $[f]_{BB'} = (a_{ij}) \in M_{mn}(K)$, $[g]_{BB'} = (b_{ij}) \in M_{mn}(K)$ and $[h]_{B'B''} = (c_{ki}) \in M_{pm}(K)$ then

$$f(v_j) = \sum_{i=1}^{m} a_{ij} v'_i\,, \quad g(v_j) = \sum_{i=1}^{m} b_{ij} v'_i\,, \quad h(v'_i) = \sum_{k=1}^{p} c_{ki} v''_k$$

for any $j \in \{1, \dots, n\}$ and for any $i \in \{1, \dots, m\}$.

Then for any $k \in K$ and for any $j \in \{1, \dots, n\}$ we have

$$(f+g)(v_j) = f(v_j) + g(v_j) = \sum_{i=1}^{m} a_{ij} v'_i + \sum_{i=1}^{m} b_{ij} v'_i = \sum_{i=1}^{m} (a_{ij} + b_{ij}) v'_i\,,$$

$$(kf)(v_j) = kf(v_j) = k \cdot \left(\sum_{i=1}^{m} a_{ij} v'_i \right) = \sum_{i=1}^{m} (k a_{ij}) v'_i \,,$$

hence $[f+g]_{BB'} = [f]_{BB'} + [g]_{BB'}$ and $[kf]_{BB'} = k \cdot [f]_{BB'}$.

Finally, for any $j \in \{1, \dots, n\}$ we have

$$(h \circ f)(v_j) = h(f(v_j)) = h \left(\sum_{i=1}^{m} a_{ij} v'_i \right) = \sum_{i=1}^{m} a_{ij} h(v'_i) = \sum_{i=1}^{m} a_{ij} \left(\sum_{k=1}^{p} c_{ki} v''_k \right) =$$

$$= \sum_{k=1}^{p} \sum_{i=1}^{m} (c_{ki} a_{ij}) v''_k = \sum_{k=1}^{p} \left(\sum_{i=1}^{m} c_{ki} a_{ij} \right) v''_k \,,$$

hence $[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'}$. ☐

Theorem 9. Let V and V' be vector spaces over K with $\dim V = n$ and $\dim V' = m$ and let B and B' be bases of V and V' respectively. Then the map $\varphi : Hom_K(V, V') \to M_{mn}(K)$ defined by

$$\varphi(f) = [f]_{BB'} \,, \ \forall f \in Hom_K(V, V')$$

is an isomorphism of vector spaces.

Proof. Let us prove first that φ is bijective.

Let $f, g \in Hom_K(V, V')$ such that $\varphi(f) = \varphi(g)$. Then $[f]_{BB'} = [g]_{BB'} = (a_{ij})$ and

$$f(v_j) = a_{1j} v'_1 + a_{2j} v'_2 + \dots + a_{mj} v'_m = g(v_j) \,, \ \forall j \in \{1, \dots, n\}.$$

Then $f = g$ by the universal property of vector spaces. Thus, φ is injective.

Now let $A = (a_{ij}) \in M_{mn}(K)$, seen as a list of column-vectors $(a^1, \dots, a^n)$, where $a^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

Consider $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_m)$ and consider the K-linear map $f : V \to V'$ defined on the basis B of V by

$$f(v_j) = a_{1j} v'_1 + \dots + a_{mj} v'_m \,, \ \forall j \in \{1, \dots, n\}.$$

Then

$$\varphi(f) = [f]_{BB'} = (a_{ij}) = A \,.$$

Thus, φ is surjective.

The proof is completed by Theorem 8. ☐

!

Remark 10. The extremely important isomorphism given in Theorem 9 allows us to work with matrices instead of linear maps, which is much simpler from a computational point of view.

As we previously saw, $(End_K(V), +, \circ)$ is a unitary ring.

$V' = V$
$B' = B$

Theorem 11. Let V be a vector space over K with $\dim V = n$ and let B be a basis of V. Then the map $\varphi : End_K(V) \to M_n(K)$ defined by

$$\varphi(f) = [f]_B \,, \ \forall f \in End_K(V)$$

is an isomorphism of vector spaces and of rings.

Proof. It follows by Theorem 8 and Theorem 9. ☐

Corollary 12. Let V be a K-vector space, B an arbitrary basis of V and $f \in End_K(V)$. Then

$$f \in Aut_K(V) \Leftrightarrow \det[f]_B \neq 0\,.$$

Indeed, $f \in Aut_K(V)$ (i.e. f is a unit in the ring $(End_K(V), +, \circ)$) if and only if $[f]_B$ is a unit in $(M_n(K), +, \cdot)$ which means that $\det[f]_B \neq 0$.

Definition 13. Let [illegible] $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_n)$ be bases of V. Then we can write

$$\begin{cases} v'_1 = t_{11}v_1 + t_{21}v_2 + \cdots + t_{n1}v_n \\ v'_2 = t_{12}v_1 + t_{22}v_2 + \cdots + t_{n2}v_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ v'_n = t_{1n}v_1 + t_{2n}v_2 + \cdots + t_{nn}v_n \end{cases}$$

for some $t_{ij} \in K$. Then the matrix $(t_{ij}) \in M_n(K)$, having as columns the coordinates of the vectors of the basis B' in the basis B, is called the **transition matrix from** B **to** B' and is denoted by $T_{BB'}$.

Remarks 14. 1) Sometimes the basis B is referred to as the "old" basis and the basis B' is referred to as the "new" basis.

2) The j-th column of $T_{BB'}$ $(j = 1, \cdots, n)$ consists of the coordinates of $v'_j = 1_V(v'_j)$ in the basis B, hence $T_{BB'} = [1_V]_{B'B}$.

Theorem 15. Let $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_n)$ be bases of V. Then the transition matrix $T_{BB'}$ is invertible and its inverse is the transition matrix $T_{B'B}$.

Proof. Since $T = T_{BB'}$ is the transition matrix from the basis B to the basis B' we have

$$v'_j = \sum_{i=1}^{n} t_{ij}v_i\,, \ \forall j \in \{1, \dots, n\}.$$

Shorter proof:
$1_V \circ 1_V = 1_V$
$\overset{T8}{\Rightarrow} \underset{T_{BB'}}{[1_V]_{B'B}} \cdot \underset{T_{B'B}}{[1_V]_{BB'}} = [1_V]_B = I_n$
Similar $T_{B'B} \cdot T_{BB'} = I_n$

Denote $S = (s_{ij}) \in M_{mn}(K)$ the transition matrix from the basis B' to the basis B. Then

$$v_i = \sum_{k=1}^{n} s_{ki}v'_k\,, \ \forall i \in \{1, \dots, n\}.$$

It follows that

$$v'_j = \sum_{i=1}^{n} t_{ij}\left(\sum_{k=1}^{n} s_{ki}v'_k\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} s_{ki}t_{ij}\right) v'_k\,.$$

By the uniqueness of writing of each v'_j as linear combination of the vectors of the basis B', it follows that

$$\sum_{i=1}^{n} s_{ki}t_{ij} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases},$$

that is, $S \cdot T = I_n$.

Similarly, one can show that $T \cdot S = I_n$. Thus, T is invertible and its inverse is S. ☐

Theorem 16. Let $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_n)$ be bases of V and let $v \in V$. Then

$$[v]_B = T_{BB'} \cdot [v]_{B'}\,.$$

Proof. Let $v \in V$ and let us write v in the two bases B and B'. Then

$$v = \sum_{i=1}^{n} k_i v_i \text{ and } v = \sum_{j=1}^{n} k'_j v'_j$$

for some $k_i, k'_j \in K$. Since $T_{BB'} = (t_{ij}) \in M_n(K)$, we have

$$v'_j = \sum_{i=1}^{n} t_{ij} v_i\,, \ \forall j \in \{1, \dots, n\}.$$

It follows that

$$v = \sum_{j=1}^{n} k'_j \left(\sum_{i=1}^{n} t_{ij} v_i \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} t_{ij} k'_j \right) v_i\,.$$

By the uniqueness of writing of v as a linear combination of the vectors of B, it follows that

$T_{B'B} = T^{-1}_{BB'}$

$$k_i = \sum_{j=1}^{n} t_{ij} k'_j\,,$$

hence $[v]_B = T_{BB'} \cdot [v]_{B'}$. □

Remark 17. Usually, we are interested in computing the coordinates of a vector v in the new basis B', knowing the coordinates of the same vector v in the old basis B and the transition matrix from B to B'. Then by Theorem 16, we have

$$[v]_{B'} = T^{-1}_{BB'} \cdot [v]_B = T_{B'B} \cdot [v]_B\,.$$

Theorem 18. Let $f \in Hom_K(V, V')$, let B_1 and B_2 be bases of V and let B'_1 and B'_2 be bases of V'. Then

$$[f]_{B_2B'_2} = T^{-1}_{B'_1B'_2} \cdot [f]_{B_1B'_1} \cdot T_{B_1B_2}\,.$$

Proof. We have $T_{B_1B_2} = [1_V]_{B_2B_1}$ and $T_{B'_1B'_2} = [1_{V'}]_{B'_2B'_1}$ (see Remark 14 2)). Of course, we also vave $T^{-1}_{B'_1B'_2} = [1_{V'}]_{B'_1B'_2}$ and by applying Theorem 8 to the equality $f = 1_{V'} \circ f \circ 1_V$ we get

$$[f]_{B_2B'_2} = [1_{V'}]_{B'_1B'_2} \cdot [f]_{B_1B'_1} \cdot [1_V]_{B_2B_1} = T^{-1}_{B'_1B'_2} \cdot [f]_{B_1B'_1} \cdot T_{B_1B_2}\,,$$

hence the expected conclusion. □

If we take $V' = V$, $B_1 = B'_1$ and $B_2 = B'_2$, we deduce:

Corollary 19. Let $f \in End_K(V)$ and let B_1 and B_2 be bases of V. Then

$$[f]_{B_2} = T^{-1}_{B_1B_2} \cdot [f]_{B_1} \cdot T_{B_1B_2}\,.$$

Exercise h.w.

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x + y, 2x - y, 3x + 2y)$. Show that f is an $\mathbb{R}$-linear map, that $B = ((1, 2), (-2, 1))$ and $B' = ((1, -1, 0), (-1, 0, 1), (1, 1, 1))$ are bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, then determine the matrix of f in the pair of bases (B, B').

Solution 1: $\quad \mathcal{B}$ basis in$_{\mathbb{R}}$ $\mathbb{R}^2 \Leftarrow \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \neq 0$

$\mathcal{B}'$ basis in$_{\mathbb{R}}$ $\mathbb{R}^3 \Longleftarrow \begin{vmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \neq 0 \leftarrow$ t.w.

$$[f]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} \frac{10}{3} & \frac{5}{3} \\ \frac{11}{3} & -\frac{2}{3} \\ \frac{10}{3} & -\frac{10}{3} \end{pmatrix} \in M_{3,2}(\mathbb{R})$$

$$(3,0,7) = f(1,2) = a_{11}(1,-1,0) + a_{21}(-1,0,1) + a_{31}(1,1,1) \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} a_{11} - a_{21} + a_{31} = 3 \\ -a_{11} \qquad + a_{31} = 0 & \Longrightarrow a_{11} = \frac{10}{3} \\ \qquad a_{21} + a_{31} = 7 & \Longrightarrow a_{21} = \frac{11}{3} \end{cases}$$

$$3a_{31} = 10 \Longrightarrow a_{31} = \frac{10}{3}$$

$$(-1,-5,-4) = f(-2,1) = a_{12}(1,-1,0) + a_{22}(-1,0,1) + a_{32}(1,1,1) \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} a_{12} - a_{22} + a_{32} = -1 \\ -a_{12} \qquad + a_{32} = -5 & \Longrightarrow a_{12} = \frac{5}{3} \\ \qquad a_{22} + a_{32} = -4 & \Longrightarrow a_{21} = -\frac{2}{3} \end{cases}$$

$$3a_{32} = -10 \Longrightarrow a_{32} = -\frac{10}{3}$$

Solution 2: $\quad E, E'$ standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively

$$[f]_{EE'} = \begin{array}{cc} \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 2 \end{pmatrix} \\ f(e_1) \;\; f(e_2) \end{array}$$

$f(e_1) = f(1,0) = (1,2,3)$

$f(e_2) = f(0,1) = (1,-1,2)$

$E \xrightarrow[T_{E\mathcal{B}}]{} \mathcal{B}$, $\quad T_{E\mathcal{B}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ invertible $\Longrightarrow \mathcal{B}$ basis for $\mathbb{R}^2$

$E' \Longrightarrow \mathcal{B}'$, $\quad T_{E'\mathcal{B}'} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ —— " —— $\Longrightarrow \mathcal{B}'$ basis for $\mathbb{R}^3$

$$[f]_{BB'} = T_{E'B'}^{-1} \cdot [f]_{EE'} \cdot T_{EB} = \dots$$