Seminar 13+14 – Matrices and linear maps

1) a) h $\mathbb{R}$-linear map ($=$ endom. ${}_{\mathbb{R}}\mathbb{R}^2$) — homework

$h(e_1) = h(1,0) = (\cos\varphi, \sin\varphi)$

$h(e_2) = h(0,1) = (-\sin\varphi, \cos\varphi)$

$$[h]_E = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in M_2(\mathbb{R})$$

$$\begin{array}{cc} \parallel & \parallel \\ h(e_1) & h(e_2) \end{array}$$

h autom ${}_{\mathbb{R}}\mathbb{R}^2 \overset{?}{\iff} [h]_E$ is invertible $\overset{?}{\iff} \det [h]_E \neq 0$

$\left(\begin{cases} End_{\mathbb{R}}(\mathbb{R}^2) \longrightarrow M_2(\mathbb{R}), \quad f \longmapsto [f]_E \\ \mathbb{R}\text{-isom. and } \underline{\text{ring isom.}} \end{cases} \right)$

$\det [h]_E = \cos^2\varphi + \sin^2\varphi = 1 \neq 0$

b) $\begin{cases} f, g : \mathbb{R}^2 \to \mathbb{R}^2, \ f(x,y) = (x,-y), \ g(x,y) = (-x,y) \\ \text{endom. of } {}_{\mathbb{R}}\mathbb{R}^2 \text{ (homework)} \end{cases}$

$f(e_1) = f(1,0) = (1,0)$

$f(e_2) = f(0,1) = (0,-1)$

$\Longrightarrow [f]_E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\det [f]_E = -1 \neq 0 \Longrightarrow [f]_E \in GL_2(\mathbb{R}) \Longrightarrow f$ autom ${}_{\mathbb{R}}\mathbb{R}^2$

$g(e_1) = g(1,0) = (-1,0)$

$g(e_2) = g(0,1) = (0,1)$

$\Longrightarrow [g]_E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det [g]_E = -1 \neq 0 \Longrightarrow [g]_E \in GL_2(\mathbb{R}) \Longrightarrow g$ autom ${}_{\mathbb{R}}\mathbb{R}^2$

$$[f-g]_E = [f]_E - [g]_E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

$$[f+2g]_E = [f]_E + 2\cdot[g]_E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$[g\circ f]_E = [g]_E \cdot [f]_E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\underline{R}$: $\begin{cases} g\circ f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (g\circ f)(x,y) = {}^t\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right] = (-x,-y). \\ \text{the symmetry w.r.t. } O(0,0). \end{cases}$

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2) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x+y, 2x-y, 3x+2y)$ $\mathbb{R}$-linear map (homework)

$B = (v_1, v_2)$, $v_1 = (1,2)$, $v_2 = (-2,1)$, $B \overset{?}{\text{basis}}$ for ${}_{\mathbb{R}}\mathbb{R}^2$

$B' = (v_1', v_2', v_3')$, $v_1' = (1,-1,0)$, $v_2' = (-1,0,1)$, $v_3' = (1,1,1)$, B' $\overset{?}{\text{basis}}$ for$_{\mathbb{R}}$ $\mathbb{R}^3$

$[f]_{BB'} = ?$

Solution 1: B basis for$_{\mathbb{R}}$ $\mathbb{R}^2 \iff \begin{array}{cc} v_1 & v_2 \\ \end{array}$ $\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \neq 0$

B' $\overset{?}{\text{basis}}$ for$_{\mathbb{R}}$ $\mathbb{R}^3 \iff \begin{vmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \dots \neq 0$ homework

$$[f]_{BB'} = \begin{pmatrix} \frac{10}{3} & \frac{5}{3} \\ \frac{11}{3} & -\frac{2}{3} \\ \frac{10}{3} & -\frac{10}{3} \end{pmatrix} \in M_{32}(\mathbb{R}).$$
$$\begin{array}{cc} \uparrow & \uparrow \\ f(v_1) & f(v_2) \end{array}$$

$$(3,0,7) = f(1,2) = f(v_1) = \alpha_{11} \cdot v_1' + \alpha_{21} \cdot v_2' + \alpha_{31} \cdot v_3' =$$
$$= \alpha_{11}(1,-1,0) + \alpha_{21}(-1,0,1) + \alpha_{31}(1,1,1) \iff$$

$$\iff \begin{cases} \alpha_{11} - \alpha_{21} + \alpha_{31} = 3 \\ -\alpha_{11} + \alpha_{31} = 0 \\ \alpha_{21} + \alpha_{31} = 7 \end{cases} \quad \begin{array}{l} \Rightarrow \alpha_{11} = \frac{10}{3} \\ \Rightarrow \alpha_{21} = \frac{11}{3} \end{array}$$

$$3\alpha_{31} = 10 \Longrightarrow \alpha_{31} = \frac{10}{3}$$

$$(-1,-5,-4) = f(-2,1) = f(v_2) = \alpha_{12} v_1' + \alpha_{22} v_2' + \alpha_{32} v_3' =$$
$$= \alpha_{12}(1,-1,0) + \alpha_{22}(-1,0,1) + \alpha_{32}(1,1,1) \iff$$

$$\iff \begin{cases} \alpha_{12} - \alpha_{22} + \alpha_{32} = -1 \\ -\alpha_{12} + \alpha_{32} = -5 \\ \alpha_{22} + \alpha_{32} = -4 \end{cases} \quad \begin{array}{l} \Rightarrow \alpha_{12} = \frac{5}{3} \\ \Rightarrow \alpha_{22} = -\frac{2}{3} \end{array}$$

$$3\alpha_{32} = -10 \Longrightarrow \alpha_{32} = -\frac{10}{3}$$

Solution 2: $E = (e_1, e_2)$, $E' = (e_1', e_2', e_3')$ standard bases of $_{\mathbb{R}}\mathbb{R}^2$, $_{\mathbb{R}}\mathbb{R}^3$ $\quad [f]_{EE'} = ?$

$$\left.\begin{array}{l} f(e_1) = f(1,0) = (1,2,3) \\ f(e_2) = f(0,1) = (1,-1,2) \end{array}\right\} \Longrightarrow [f]_{EE'} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = T_{EB}$$
$$\begin{array}{cc} v_1 & v_2 \end{array}$$

B $\overset{?}{\text{basis}}$ for$_{\mathbb{R}}$ $\mathbb{R}^2 \iff S$ is invertible $\iff$

$\iff \det S = 5 \overset{?}{\neq} 0$

$$T=\begin{pmatrix}1&-1&1\\-1&0&1\\0&1&1\end{pmatrix}\overset{?}{=\!=}T_{E'B'}$$

B' basis for $_{\mathbb{R}}\mathbb{R}^3 \iff T$ is invertible $\iff \det T\neq 0$

$$[f]_{BB'}=T_{E'B'}^{-1}\cdot[f]_{EE'}\cdot T_{EB}=T^{-1}\cdot[f]_{EE'}\cdot S=\ldots$$

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3) $f\in End_{\mathbb{R}}(\mathbb{R}^4)$, $B=(v_1,v_2,v_3,v_4)$ basis for $_{\mathbb{R}}\mathbb{R}^4$, $[f]_B$ is given

$u_1=v_1$

$u_2=v_1+v_2$

$u_3=v_1+v_2+v_3$

$u_4=v_1+v_2+v_3+v_4$

B' basis for $_{\mathbb{R}}\mathbb{R}^4$ $\overset{?}{}$, $[f]_{B'}$?

$$\begin{array}{cc} & \begin{array}{cccc}&&&\end{array}\\ S= & \begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}\overset{?}{=\!=}T_{BB'}\\ & \begin{array}{cccc}\uparrow&\uparrow&\uparrow&\uparrow\\u_1&u_2&u_3&u_4\end{array}\end{array}$$

$\det S=1\neq 0 \implies S\in GL_4(\mathbb{R}) \implies B'$ basis of $_{\mathbb{R}}\mathbb{R}^4$

$$[f]_{B'}=T_{BB'}^{-1}\cdot[f]_B\cdot T_{BB'}=S^{-1}\cdot[f]_B\cdot S=\ldots \text{ homework.}$$

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4) V $\mathbb{R}$-v.s., $f\in End_{\mathbb{R}}(V)$, $B=(v_1,v_2,v_3)$ basis for $_{\mathbb{R}}V$

$u_1=v_1+2v_2+v_3$, $u_2=v_1+v_2+2v_3$, $u_3=v_1+v_2$

$B'=(u_1,u_2,u_3)$ basis$^?$ for $_{\mathbb{R}}V$, $[f]_B$? provided that $[f]_{B'}$ is given.

$$S=\begin{pmatrix}1&1&1\\2&1&1\\1&2&0\end{pmatrix}\overset{?}{=\!=}T_{BB'}$$

$$\det S \underset{r_2-r_1}{=} \begin{vmatrix}1&1&1\\1&0&0\\1&2&0\end{vmatrix}=2\neq 0 \implies S \text{ is invertible} \implies$$

$\implies B'$ is a basis for $_{\mathbb{R}}V$

$$[f]_{B'}=T_{BB'}^{-1}\cdot[f]_B\cdot T_{BB'}=S^{-1}\cdot\overset{?}{[f]_B}\cdot S$$

$$S\cdot\Big|\,[f]_{B'}=S^{-1}\cdot[f]_B\cdot S\,\Big|\cdot S^{-1} \implies [f]_B=S\cdot[f]_{B'}\cdot S^{-1}=\ldots$$

$\underline{\underline{\text{OR}}}$: $[f]_B=T_{B'B}^{-1}\cdot[f]_{B'}\cdot T_{B'B}=T_{BB'}\cdot[f]_{B'}\cdot T_{BB'}^{-1}=S\cdot[f]_{B'}\cdot S^{-1}=\ldots$

5) V, V' $\mathbb{R}$-v.s., $f: V \to V'$ $\mathbb{R}$-linear map

$a = (a_1, a_2, a_3)$ basis of $\mathbb{R}^V$, $\quad [f]_{a,b} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$b = (b_1, b_2, b_3)$ — " — $\mathbb{R}^{V'}$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ f(a_1) & f(a_2) & f(a_3) \end{array}$$

a) $f(v)$, $v \in V$ arbitrary

b) $\dim \operatorname{Im} f$, $\dim \operatorname{Ker} f = ?$

c) $[f]_{a',b'} = ?$, $\quad a' = (a_1, a_1 + a_2, a_1 + a_2 + a_3)$

$\quad b' = (b_1, b_1 + b_2, b_1 + b_2 + b_3)$

<u>Solution</u>: $\quad f(a_1) = -b_1 + b_2$

$f(a_2) = 0$

$f(a_3) = b_1 - b_2$

a) $v \in V \iff \exists\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ uniquely determined s.t.

$$v = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

$$\underline{f(v)} = f(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) = \alpha_1 f(a_1) + \alpha_2 f(a_2) + \alpha_3 f(a_3) =$$

$$= \alpha_1(-b_1 + b_2) + \alpha_3(b_1 - b_2) = -\alpha_1 b_1 + \alpha_1 b_2 + \alpha_3 b_1 - \alpha_3 b_2 =$$

$$= \underline{(-\alpha_1 + \alpha_3) b_1 + (\alpha_1 - \alpha_3) b_2}.$$

c) $S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \overset{?}{=} T_{aa'}$, $\quad \det S = 1 \neq 0 \Longrightarrow a'$ basis for $\mathbb{R}^V$

$T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = T_{bb'}$, $\quad \det T = 1 \neq 0 \Longrightarrow b'$ — " — $\mathbb{R}^{V'}$.

$$[f]_{a'b'} = T_{bb'}^{-1} \cdot [f]_{ab} \cdot T_{aa'} = T^{-1} \cdot [f]_{ab} \cdot S = \dots$$

b) $\operatorname{Im} f = f(V) = f(\langle a_1, a_2, a_3 \rangle) = \langle f(a_1), f(a_2), f(a_3) \rangle$

$$\left.\begin{array}{l} \dim \operatorname{Im} f = \operatorname{rank} [f]_{ab} = 1 \\ \underbrace{\dim \operatorname{Im} f}_{=1} + \dim \operatorname{Ker} f = \underbrace{\dim V}_{=3} \end{array}\right\} \Longrightarrow \dim \operatorname{Ker} f = 3 - 1 = 2.$$

<u>OR</u>: $\quad v \in V$, $v = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 \in \operatorname{Ker} f \iff f(v) = 0$

$$\iff [f]_{ab} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\iff \begin{cases} -\alpha_1 \quad +\alpha_3 = 0 \\ \alpha_1 \quad -\alpha_3 = 0 \end{cases} \iff \begin{cases} \alpha_1 = \alpha_3 \\ \alpha_2 \in \mathbb{R} \\ \alpha_3 \in \mathbb{R} \end{cases}$$

$$\operatorname{Ker} f = \{ \underline{\alpha_3 a_1} + \alpha_2 a_2 + \underline{\alpha_3 a_3} \mid \alpha_2, \alpha_3 \in \mathbb{R} \} =$$

$$= \{ \alpha_3(\underline{a_1 + a_3}) + \alpha_2 \underline{a_2} \mid \alpha_2, \alpha_3 \in \mathbb{R} \} = \langle \underline{a_1 + a_3}, \underline{a_2} \rangle$$

$$\operatorname{rank} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) = 2 \implies a_1 + a_3, a_2 \text{ l. indep}$$

$$\implies \dim \operatorname{Ker} f = 2$$

$$\dim \operatorname{Im} f = \dim V - \dim \operatorname{Ker} f = 3 - 2 = 1$$

6) V, V' $\mathbb{R}$-v.s., $\mathcal{B} = (v_1, v_2, v_3)$ basis for V, $\mathcal{B}' = (v_1', v_2', v_3')$ basis for V'

$f: V \to V'$ linear map,

$$\begin{array}{c} \\ [f]_{\mathcal{B}\mathcal{B}'} = \\ \\ \end{array} \begin{array}{c} \left(\begin{array}{cc|c} 0 & -1 & 5 \\ 1 & 0 & 0 \\ \hline 0 & 1 & -5 \end{array} \right) \\ \begin{array}{ccc} f(v_1) & f(v_2) & f(v_3) \end{array} \end{array} \text{ in } \mathcal{B}'$$

i) Find a basis and the dimension for $\operatorname{Im} f$ and $\operatorname{Ker} f$.

ii) $[f]_{\mathcal{B}\mathcal{E}'}$ when $V' = \mathbb{R}^3$, $v_1' = (1,0,0)$, $v_2' = (0,1,1)$, $v_3' = (0,0,1)$ and $\mathcal{E}'$ is the standard basis of $_\mathbb{R}\mathbb{R}^3$

iii) $f(x) = ?$ when $x = 2v_1 - v_2 + 3v_3$ (in the situation ii)).

<u>Solution:</u> i) $f(v_1) = v_2'$

$$f(v_2) = -v_1' + v_3'$$

$$f(v_3) = 5v_1' - 5v_3'$$

$$\operatorname{Im} f = f(V) = f(\langle v_1, v_2, v_3 \rangle) = \langle f(v_1), f(v_2), f(v_3) \rangle \implies$$

$\implies \dim \operatorname{Im} f = \operatorname{rank} [f]_{\mathcal{B}\mathcal{B}'} = 2$ and $(f(v_1), f(v_2))$ basis for $\operatorname{Im} f$

Thus $(v_2', -v_1' + v_3')$ basis for $\operatorname{Im} f$.

$$\dim \operatorname{Ker} f + \underbrace{\dim \operatorname{Im} f}_{=2} = \underbrace{\dim V}_{=3} \implies \dim \operatorname{Ker} f = 3 - 2 = 1$$

A basis in a 1-dimensional vector space is given by a non-zero vector.

$$c_3 = -5c_2 \iff f(v_3) = -5f(v_2) \iff 5f(v_2) + f(v_3) = 0 \iff$$

$$\iff f(\underbrace{5v_2 + v_3}_{= x_0 \in \operatorname{Ker} f,\ x_0 \neq 0}) = 0 \implies x_0 = 5v_2 + v_3 \text{ forms (alone) a basis for } \operatorname{Ker} f.$$

<u>Another way to find a basis for $\operatorname{Ker} f$:</u>

$$x = x_1 v_1 + x_2 v_2 + x_3 v_3 \in \operatorname{Ker} f \iff \underline{[f]_{\mathcal{B}\mathcal{B}'}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$(\iff f(x) = 0)$$

$$\Longleftrightarrow \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} -x_2 + 5x_3 = 0 \\ x_1 = 0 \\ x_2 - 5x_3 = 0 \end{cases} \Longleftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 5x_3 \\ x_3 \in \mathbb{R} \end{cases}$$

$$\operatorname{Ker} f = \{5x_3 v_2 + x_3 v_3 \mid x_3 \in \mathbb{R}\} = \{x_3(5v_2 + v_3) \mid x_3 \in \mathbb{R}\} =$$

$$\left.= \langle \underbrace{5v_2 + v_3}_{= x_0 \in \operatorname{Ker} f} \rangle \quad \checkmark \right\} \Longrightarrow (x_0) \text{ is a basis for } \operatorname{Ker} f.$$

ii) E', B' bases in ${}_{\mathbb{R}}\mathbb{R}^3 \Longrightarrow T_{E'B'} = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ \quad v_1' \quad v_2' \quad v_3' \end{array} = T$

$(\det T = 1 \neq 0 \Longrightarrow B'$ basis for ${}_{\mathbb{R}}\mathbb{R}^3)$

$$[f]_{BE'} = \underbrace{T^{-1}_{B'E'}}_{= T_{E'B'}} \cdot [f]_{BB'} \cdot \underbrace{T_{BB}}_{= I_3} = T \cdot [f]_{BB'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & -5 \end{pmatrix} =$$

$$= \begin{array}{c} \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 0 \\ 1 & 1 & -5 \end{pmatrix} \\ \shortparallel \quad \shortparallel \quad \shortparallel \\ f(v_1)\ f(v_2)\ f(v_3) \end{array}.$$

$\underline{\underline{R}}$: $\left.\begin{array}{l} f(v_1) = (0,1,1) \\ f(v_2) = (-1,0,1) \\ f(v_3) = (5,0,-5) \end{array}\right\}$

iii) $f(x) = f(2v_1 - v_2 + 3v_3) = 2f(v_1) - f(v_2) + 3f(v_3) =$

$= 2(0,1,1) - (-1,0,1) + 3(5,0,-5) = (16, 2, -14).$

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8) a) $f \in \operatorname{End}_{\mathbb{Q}}(\mathbb{Q}^4)$, $E = (e_1, e_2, e_3, e_4)$ the standard basis of ${}_{\mathbb{Q}}\mathbb{Q}^4$

$$[f]_E = \begin{array}{c} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix} \\ \shortparallel \quad \shortparallel \quad \shortparallel \quad \shortparallel \\ f(e_1)\ f(e_2)\ f(e_3)\ f(e_4) \end{array}$$

Find a basis for each of the spaces $\operatorname{Im} f$, $\operatorname{Ker} f$, $\operatorname{Im} f + \operatorname{Ker} f$ and $\operatorname{Im} f \cap \operatorname{Ker} f$.

<u>Solution</u>: $f(e_1) = (1,3,-1,0)$
$f(e_2) = (2,2,-3,4)$
$f(e_3) = (1,3,0,-1)$
$f(e_4) = (2,2,4,-3)$

$\dim \operatorname{Im} f = \operatorname{rank} [f]_E = 3$ and $(f(e_1), f(e_2), f(e_3))$ basis for $\operatorname{Im} f$.

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix} \underset{r_3+r_1}{\overset{r_2-3r_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & -4 \\ 0 & -1 & 1 & 6 \\ 0 & 4 & -1 & -3 \end{pmatrix} \underset{c_4+c_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & -1 & -7 \end{pmatrix}$$

$\dim \operatorname{Ker} f = \dim \mathbb{Q}^4 - \dim \operatorname{Im} f = 4 - 3 = 1$

$$c_1 - c_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}; \quad c_2 - c_4 = \begin{pmatrix} 0 \\ 0 \\ -7 \\ 7 \end{pmatrix} = 7(c_1 - c_3) \iff f(e_2) - f(e_4) = 7f(e_1) - 7f(e_3)$$

$\iff 7f(e_1) - f(e_2) - 7f(e_3) + f(e_4) = (0,0,0,0) \iff$

$\iff f(\underbrace{7e_1 - e_2 - 7e_3 + e_4}_{= x' \in \operatorname{Ker} f}) = (0,0,0,0)$, $x' = (7, -1, -7, 1) \in \operatorname{Ker} f$ forms (alone) a basis in $\operatorname{Ker} f$.

OR:

$$x = (x_1, x_2, x_3, x_4) \in \operatorname{Ker} f \iff f(x) = (0,0,0,0) \iff [f]_E \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\iff \begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 0 \\ 3x_1 + 2x_2 + 3x_3 + 2x_4 = 0 \\ -x_1 - 3x_2 + 4x_4 = 0 \\ 4x_2 - x_3 - 3x_4 = 0 \end{cases} \implies \begin{array}{l} x_1 + x_3 = 0 \iff x_1 = -x_3 \\ \implies x_2 + x_4 \iff x_2 = -x_4 \end{array} \implies x_3 + 7x_4 = 0$$

$$\begin{cases} x_1 = 7x_4 \\ x_2 = -x_4 \\ x_3 = -7x_4 \\ x_4 \in \mathbb{R} \end{cases}$$

(x_4 side unknown.

$x_4 = 1 \implies x_1, x_2, x_3 \ldots$)

$$\operatorname{Ker} f = \{(7x_4, -x_4, -7x_4, x_4) / x_4 \in \mathbb{R}\} = \{x_4(7, -1, -7, 1) / x_4 \in \mathbb{R}\} =$$

$$= \langle \underbrace{(7, -1, -7, 1)}_{= x'} \rangle$$

$= x' \longrightarrow$ forms (alone) a basis for $\operatorname{Ker} f$.

$$\left.\begin{array}{l} \operatorname{Im} f = \langle f(e_1), f(e_2), f(e_3) \rangle \\ \operatorname{Ker} f = \langle x' \rangle \end{array}\right\} \implies \operatorname{Im} f + \operatorname{Ker} f = \langle f(e_1), f(e_2), f(e_3), x' \rangle$$

$\dim(\operatorname{Im} f + \operatorname{Ker} f) = 4$ and $(f(e_1), f(e_2), f(e_3), x')$ basis in $\operatorname{Im} f + \operatorname{Ker} f$.

$$\begin{pmatrix} 1 & 2 & 1 & 7 \\ 3 & 2 & 3 & -1 \\ -1 & -3 & 0 & -7 \\ 0 & 4 & -1 & 1 \end{pmatrix} \underset{r_3+r_1}{\overset{r_2-3r_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -4 & 0 & -22 \\ 0 & 4 & -1 & 1 \end{pmatrix} \underset{c_3+c_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & -22 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

$\dim(\operatorname{Im} f \cap \operatorname{Ker} f) + \underbrace{\dim(\operatorname{Im} f + \operatorname{Ker} f)}_{=4} = \dim \operatorname{Im} f + \dim \operatorname{Ker} f = \underbrace{\dim \mathbb{R}^4}_{=4}$

$\Longrightarrow \dim(\operatorname{Im} f \cap \operatorname{Ker} f) = 0 \longrightarrow \operatorname{Im} f \cap \operatorname{Ker} f = \{(0,0,0,0)\}$

and $\emptyset$ is a basis for $\operatorname{Im} f \cap \operatorname{Ker} f$.

BONUS:

Compute $A^4 - 11A^2 + 22A$ for

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix} \in M_3(\mathbb{R}).$$

Solution: $f = X^4 - 11X^2 + 22X$, $f(A) = ?$

$$p_A(\lambda) = \begin{vmatrix} -\lambda & 0 & 2 \\ 2 & 1-\lambda & 0 \\ -1 & -1 & 3-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - 5\lambda - 2$$

$$p_A = -X^3 + 4X^2 - 5X - 2 \in \mathbb{R}[X], \quad f \in \mathbb{R}[X].$$

The Division Algorithm for polynomials: Let K be a field.

$\forall f, g \in K[X]$, $g \neq 0$, $\exists!\, q, r \in K[X]$ s.t.

$$f = g\underbrace{q}_{\text{the quotient}} + \underbrace{r}_{\text{the remainder}}, \quad \text{and } \deg r < \deg g.$$

$f, p_A \in \mathbb{R}[X] \Longrightarrow \exists\, q, r \in \mathbb{R}[X]$: $f = p_A \cdot q + r$, and $\deg r \leq 2$

$$f(A) = \underbrace{p_A(A)}_{\overset{\text{C.-H. Thm.}}{=} O_3} \cdot q(A) + r(A) = r(A)$$

$$\begin{array}{l|l}
X^4 \quad -11X^2 + 22X & -X^3 + 4X^2 - 5X - 2 \\ \cline{2-2}
-\;X^4 + 4X^3 - 5X^2 - 2X & -X - 4 \\ \cline{1-1}
\quad 4X^3 - 16X^2 + 20X & = q \\
-\;4X^3 + 16X^2 - 20X - 8 & \\ \cline{1-1}
\quad\quad\quad -8 = r &
\end{array}$$

$$f(A) = r(A) = -8 \cdot I_3 = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}.$$