CHAPTER 2

Affine subspaces

Contents

2.1 Parametric equations

Let $\mathbf{A}$ be an affine subspace over the $\mathbf{K}$-vector space $\mathbf{V}$. Recall that an affine subspace S of $\mathbf{A}$ passing through $Q \in \mathbf{A}$ and parallel to the vector subspace $\mathbf{W} \subseteq \mathbf{V}$ is the set of points $P \in \mathbf{A}$ such that

$$\overrightarrow{QP} \in \mathbf{W}.$$

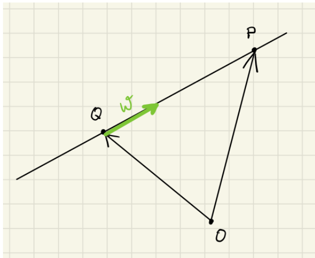
If $\mathbf{w}_1, \dots, \mathbf{w}_s$ is a basis of $\mathbf{W}$ then the above condition is equivalent to

$$\overrightarrow{QP} = t_1\mathbf{w}_1 + \dots + t_s\mathbf{w}_s \quad \text{for some scalars} \quad t_1, \dots, t_s \in \mathbf{K}.$$

Let $O\mathbf{e}_1 \dots \mathbf{e}_n$ be a coordinate system of $\mathbf{A}$. With respect to this coordinate system $P = P(x_1, \dots, x_n)$, $Q = Q(q_1, \dots, q_n)$ and (with respect to the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $\mathbf{V}$) $\mathbf{w}_i = \mathbf{w}_i(w_{1i}, \dots, w_{ni})$ for each $i = 1, \dots, s = \dim S$. Therefore

$$\begin{array}{rcl} x_1 &=& q_1 + t_1 w_{11} + \dots + t_s w_{1s} \\ x_2 &=& q_2 + t_1 w_{21} + \dots + t_s w_{2s} \\ &\vdots& \\ x_n &=& q_2 + t_1 w_{n1} + \dots + t_s w_{ns} \end{array} \quad \text{or, in matrix notation,} \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + t_1 \begin{bmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{n1} \end{bmatrix} + \dots + t_s \begin{bmatrix} w_{1s} \\ w_{2s} \\ \vdots \\ w_{ns} \end{bmatrix}. \quad (2.1)$$

The scalars $t_1, \dots, t_s$ are unique for each point P and by varying these parameters one obtains every point in S.



Definition 2.1. The set of equations in (2.1) are called *parametric equations of S with respect to Q and with respect to $\mathbf{w}_1, \dots, \mathbf{w}_s$ in the coordinate system $O\mathbf{e}_1 \dots \mathbf{e}_n$*, or, shorter, *parametric equations for S*.

- Notice that the parametric equations are not unique. If we replace $\mathbf{w}_1, \dots, \mathbf{w}_s$ by another basis of $\mathbf{W}$ then we get another set of parametric equations. If we choose a different point Q in S we get another set of parametric equations.

- Notice that the parametric equations depend on the coordinate system $O\mathbf{e}_1 \dots \mathbf{e}_n$. The same set of equations can be used to describe a different affine subspace if we change the coordinate system.

Example 2.2. If ℓ is a line given by two distinct points $Q(q_1, \dots, q_n)$ and $Q'(q_1', \dots, q_n')$ then $\overrightarrow{QQ'}$ is a vector in the direction of ℓ and a set of parametric equations for ℓ is

$$\begin{array}{rcl} x_1 & = & q_1 + t(q_1' - q_1) \\ x_2 & = & q_2 + t(q_2' - q_2) \\ & \vdots & \\ x_n & = & q_n + t(q_n' - q_n) \end{array} \quad \text{or, in matrix notation,} \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + t \begin{bmatrix} q_1' - q_1 \\ q_2' - q_2 \\ \vdots \\ q_n' - q_n \end{bmatrix}.$$

2.2 Cartesian equations

Another way of representing an affine subspace by equations is the following.

Theorem 2.3. Let $\mathbf{A}$ be an affine space with coordinate system $O\mathbf{e}_1 \dots \mathbf{e}_n$. Let

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ & \vdots & \\ a_{t1}x_1 + \cdots + a_{tn}x_n & = & b_t \end{array} \tag{2.2}$$

be a system of linear equations in the unknowns $x_1, \dots, x_n$. The set S of points of $\mathbf{A}$ whose coordinates are solutions of (2.2), if there are any, is an affine space of dimension $n - r$ where r is the rank of the matrix of coefficients of the system. The vector subspace associated to S is the vector subspace $\mathbf{W}$ of $\mathbf{V}$ whose equations are given by the associated homogeneous system

$$\begin{array}{rcl} a_{11}x_1+\cdots+a_{1n}x_n & = & 0 \\ & \vdots & . \\ a_{t1}x_1+\cdots+a_{tn}x_n & = & 0 \end{array} \tag{2.3}$$

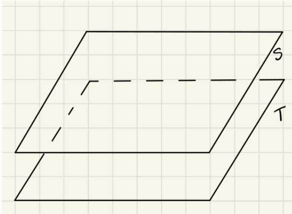
Conversely, for every affine subspace S of $\mathbf{A}$ of dimension s there is a system of $n-s$ linear equations in n unknowns whose solutions correspond precisely to the coordinates of the points in S.

Definition 2.4. The set of equations in (2.2) are called *Cartesian equations of the subspace S with respect to the coordinate system $O\mathbf{e}_1\dots\mathbf{e}_n$*, or, shorter, *equations for S*.

- Notice that an affine subspace has several systems of equations with respect to the same coordinate system. Two systems of equations determine the same affine subspace of $\mathbf{A}$ (with respect to the same coordinate system) if and only if they are equivalent, i.e. if and only if they have the same solution set.
- Notice that the equations depend on the coordinate system $O\mathbf{e}_1\dots\mathbf{e}_n$. The same set of equations can be used to describe a different affine subspace if we change the coordinate system.
- The subspace S with equations (2.2) contains the origin if and only if $b_1=\cdots=b_t=0$, i.e. if and only if the equations are homogeneous. Thus, every homogeneous system of equations (2.3) defines not only a vector subspace $\mathbf{W}$ of $\mathbf{V}$ but also an affine subspace S of $\mathbf{A}$. There is a one-to-one correspondence between affine subspaces of $\mathbf{A}$ containing the origin and vector subspaces of $\mathbf{V}$.
- From Theorem 2.3 it follows that every hyperplane H of $\mathbf{A}$ is represented by an equation of the form
$$a_1x_1+\cdots+a_nx_n=b$$
with $a_1,\dots,a_n\in\mathbf{K}$ not all zero. The hyperplane contains the origin if and only if $b=0$. The j-th coordinate hyperplane has equation
$$x_j=0.$$
- From Theorem 2.3 it also follows that each affine subspace of $\mathbf{A}$ is the intersection of several hyperplanes of $\mathbf{A}$.

2.3 Relative positions

Definition 2.5. Let S and T be two affine subspaces of $\mathbf{A}$ of positive dimension, with associated vector subspaces $\mathbf{W}$ and $\mathbf{U}$ respectively. Then S and T are said to be *parallel* if $\mathbf{W}\subseteq\mathbf{U}$ or $\mathbf{U}\subseteq\mathbf{W}$. If they are parallel we write $S\parallel T$.



- If $S \subseteq T$ then S is parallel to T.
- If $\dim(S) = \dim(T)$, then S and T are parallel if and only if $\mathbf{U} = \mathbf{W}$.
- If S and T are lines, they are parallel if they have the same direction, i.e. any two of their direction vectors are proportional.
- If S and T are hyperplanes then they are parallel if the coefficients of the unknowns in their equations are proportional.

Proposition 2.6. Let S and T be parallel affine subspaces of $\mathbf{A}$ with $\dim(S) \le \dim(T)$.

1.) If S and T have a point in common then $S \subseteq T$.

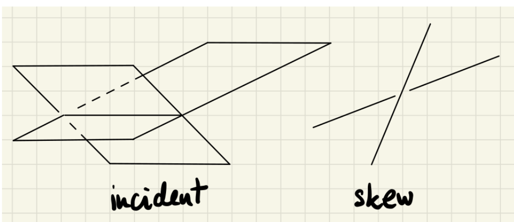
2.) If $\dim(S) = \dim(T)$, and S and T have a point in common then $S = T$.

Corollary 2.7. If S is an affine subspace of $\mathbf{A}$ and $P \in \mathbf{A}$, there is a unique affine subspace T of $\mathbf{A}$ which contains P, is parallel to S and has the same dimension as S.

- Corollary 2.7 is equivalent to the 'parallel postulate' of Euclidean geometry (in the plane). The axioms of affine spaces therefore imply the validity of this postulate in affine planes.

Example 2.8. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space and let $\mathbf{W}$ be a vector subspace of $\mathbf{V}$. From Proposition (2.6) it follows that, if $\mathbf{v}, \mathbf{u} \in \mathbf{V}$, the two affine subspaces $\mathbf{v} + \mathbf{W}$ and $\mathbf{u} + \mathbf{W}$ of $\mathbf{V}_a$ either coincide or are disjoint. From this it follows that the family of all affine subspaces of $\mathbf{V}_a$ having associated vector subspace $\mathbf{W}$ form a partition of $\mathbf{V}$. The quotient set of $\mathbf{V}$ by this partition is the set whose elements are the affine subspaces of $\mathbf{V}_a$ having associated vector space $\mathbf{W}$. We denote this quotient by $\mathbf{V}/\mathbf{W}$.

Definition 2.9. If two affine subspace S and T of $\mathbf{A}$ are not parallel, they are said to be either *skew* if they do not meet, or *incident* if they have a point in common.



2.4 Intersections of affine subspaces

Consider two subspaces S and T with $\dim(S) = s$ and $\dim(T) = t$. Suppose they have equations

$$S : \sum_{j=1}^{n} m_{ij} x_j = b_i \quad \text{for} \quad i = 1, \ldots, n-s \tag{2.4}$$

$$T : \sum_{j=1}^{n} n_{kj} x_j = c_k \quad \text{for} \quad k = 1, \ldots, n-t. \tag{2.5}$$

The intersection $S \cap T$ is the locus of points in $\mathbf{A}$ whose coordinates are simultaneously solutions of both (2.4) and (2.5), i.e. they are solutions of the system

$$S \cap T : \begin{cases} \sum_{j=1}^{n} m_{ij} x_j = b_i & \text{for} \quad i = 1, \ldots, n-s, \\ \sum_{j=1}^{n} n_{kj} x_j = c_k & \text{for} \quad k = 1, \ldots, n-t. \end{cases} \tag{2.6}$$

By Theorem 2.3, if the system (2.6) has a solution, then it represents an affine subspace. Thus, if $S \cap T$ is non-empty it is an affine subspace of $\mathbf{A}$.

Proposition 2.10. If the intersection $S \cap T$ of two affine subspaces of $\mathbf{A}$ is non-empty it is an affine subspace satisfying

$$\dim(S) + \dim(T) - \dim(\mathbf{A}) \leq \dim(S \cap T) \leq \min\{\dim(S), \dim(T)\}. \tag{2.7}$$

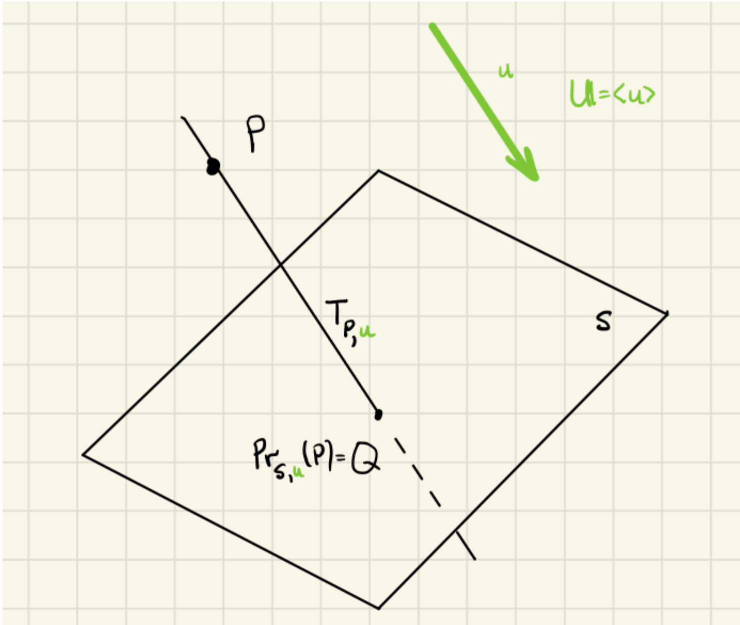
- The second inequality is an equality if $S \subseteq T$ or $T \subseteq S$.
- When is the first inequality an equality?

Proposition 2.11. Let S and T be two affine subspaces of $\mathbf{A}$ with associated vector subspaces $\mathbf{W}$ and $\mathbf{U}$ respectively. Then $\mathbf{V} = \mathbf{W} + \mathbf{U}$ if and only if $S \cap T \neq \emptyset$ *and*

$$\dim(S \cap T) = \dim(S) + \dim(T) - \dim(\mathbf{A}). \tag{2.8}$$

Example 2.12. A particularly important case is that in which the associated vector subspaces $\mathbf{W}$ and $\mathbf{U}$ of the affine subspaces are supplementary, i.e. are such that $\mathbf{V} = \mathbf{W} \oplus \mathbf{U}$. In this case $\dim(S) + \dim(T) = \dim(\mathbf{A})$ and therefore S and T have a unique point in common (by (2.8)). By Corollary 2.7, for each point $P \in \mathbf{A}$ there is a unique affine subspace $T_{P,\mathbf{U}}$ passing through P and parallel to $\mathbf{U}$. Then $S \cap T_{P,\mathbf{U}}$ is a unique point Q denoted by $\mathrm{Pr}_{S,\mathbf{U}}(P)$.

Definition 2.13. The map $\mathrm{Pr}_{S,\mathbf{U}} : \mathbf{A} \to S$ defined above is called the *projection of* $\mathbf{A}$ *onto* S *parallel to* $\mathbf{U}$, or *projection of* $\mathbf{A}$ *onto* S *along* T.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U=<u>
P
$T_{P,u}$
S
$Pr_{S,u}(P)=Q$