COURSE 6

Systems of linear equations

Let K be a field and let us consider the system of m linear equations with n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

where $a_{ij}, b_i \in K$, $i = 1, \dots, m$; $j = 1, \dots, n$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \ \overline{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

We remind that $A \in M_{m,n}(K)$ is **the matrix of the system** (1), B is **the matrix of constant terms** and $\overline{A}$ is **the augmented matrix of the system**. If all the constant terms are zero, i.e. $b_1 = b_2 = \cdots = b_m = 0$, the system (1) is a **homogeneous linear system**. By denoting

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the system (1) can be written as a matrix equation

$$AX = B \tag{2}$$

The system $AX = O_{m,1}$ is **the homogeneous system associated** to the system $AX = B$.

Definition 1. An n-tuple $(\alpha_1, \dots, \alpha_n) \in K^n$ is a **solution of the system** (1) if the all the equalities resulted by replacing x_i with α_i $(i = 1, \dots, n)$ in (1) are true. The system (1) is called **consistent** if it has at least one solution. Otherwise, the system (1) is **inconsistent**. Two systems of linear equations with n unknowns are **equivalent** if they have the same solution set.

Remarks 2. a) Cramer's Theorem states that for $m = n$ and $\det A \neq 0$ the system (1) is consistent, with a unique solution, and its solution is given by Cramer's formulas.
b) If (1) is a homogeneous system, then $(0, 0, \dots, 0) \in K^n$ is a solution of (1), called **the trivial solution**, so any homogeneous linear system is consistent.

The following result is a very important tool for the study of the consistency of the general linear systems.

Theorem 3. (Kronecker-Cappelli) The linear system (1) is consistent if and only if the rank of its matrix is the same as the rank of its augmented matrix, i.e. $\operatorname{rank} A = \operatorname{rank} \overline{A}$.

the columns of A

Let us denote by $\overbrace{c_1, \ldots, c_n}^{\text{the columns of } A}, c_{n+1}$ the columns of $\bar{A}$.

Proof. "$\Rightarrow$" (1) is consistent $\Rightarrow \exists (\alpha_1, \ldots, \alpha_n) \in K^n$ s.t.

$\rightarrow c_{n+1} = B = \alpha_1 c_1 + \ldots + \alpha_n c_n$

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$\Rightarrow c_{n+1}$ is a linear a comb. of all the other columns of $\bar{A} \Rightarrow$

$\Rightarrow$ when computing rank $\bar{A}$ we can omit the last column of $\bar{A}$

$\Rightarrow$ rank A = rank $\bar{A}$.

"$\Leftarrow$" rank A = rank $\bar{A} \Rightarrow c_{n+1}$ is a linear combination of the other columns of $\bar{A} \Rightarrow \exists \alpha_1, \ldots, \alpha_n \in K$ s.t.

$B = c_{n+1} = \alpha_1 c_1 + \ldots + \alpha_n c_n$

$\Rightarrow (\alpha_1, \ldots, \alpha_n) \in K^n$ is a solution for (1) $\Rightarrow$ (1) is consistent.

Assuming that we know the rank r of A, in order to find rank $\bar{A}$ we have to complete an r-size non-zero minor of A with elements from the last column of $\bar{A}$.

$\square$

Let us consider that rank$A = r$. Based on how one can determine the rank of a matrix one can restate the previous theorem as follows:

Theorem 4. **(Rouché)** Let d_p be a nonzero $r \times r$ minor of the matrix A. The system (1) is consistent if and only if all the $(r+1) \times (r+1)$ minors of $\overline{A}$ obtained by completing d_p with a column of constant terms and the corresponding row are zero (if such $(r+1) \times (r+1)$ minors exist).

We end this section by presenting an algorithm for solving arbitrary systems of linear equations based on Rouché Theorem.

with the size $= r =$ rank A.

An algorithm for solving systems of linear equations:
We begin by studying the consistency of (1) by using Rouché's theorem and let us consider that we have found a minor d_p of A. If one finds a nonzero $(r+1) \times (r+1)$ minor which completes d_p as in Rouché Theorem, then (1) is inconsistent and the algorithm ends. If $r = m$ or all the Rouché Theorem $(r+1) \times (r+1)$ minor completions of d_p are 0, then (1) is consistent.

We call the unknowns corresponding to the the entries of d_p **main unknowns** and the other unknowns **side unknowns**. For simpler notations, we consider that the minor d_p was "cut" from the first r rows and the first r columns of A. One considers only the r equations which determined

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the rows of d_p. Since $\operatorname{rank}\overline{A} = \operatorname{rank} A = r$, ~~Corollary ??~~ tells us that all the other equations are linear combinations" of these r equations, hence (1) is equivalent to

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n = b_r \end{cases} \tag{3}$$

(⟺ (1))

If $n = r$, i.e. all the unknowns are main unknowns, then (3) is a Cramer system. The Cramer's rule gives us its unique solution, hence the unique solution of (1).

Otherwise, $n > r$, and $x_{r+1}, \dots, x_n$ are side unknowns. We can assign them arbitrary "values" from K, $\alpha_{r+1}, \dots, \alpha_n$, respectively. Then (3) becomes

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r}x_r = b_1 - a_{1,r+1}\alpha_{r+1} - \cdots - a_{1n}\alpha_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2r}x_r = b_2 - a_{2,r+1}\alpha_{r+1} - \cdots - a_{2n}\alpha_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rr}x_r = b_r - a_{r,r+1}\alpha_{r+1} - \cdots - a_{rn}\alpha_n \end{cases} \tag{4}$$

The determinant of the matrix of (4) is $d_p \neq 0$, hence we can express the main unknowns using the side unknowns, by solving the Cramer system (4).

Elementary operations on matrices. Applications

Let K be a field, $m, n \in \mathbb{N}^*$ and $A = (a_{ij}) \in M_{m,n}(K)$.

Definition 5. By an **elementary operation on the rows (columns)** of a matrix we understand one of the following:

(I) the interchange of two rows (columns).

(II) multiplying a row (column) by a non-zero element $\alpha \in K$.

(III) multiplying a row (column) by an element $\alpha \in K$ (also called scalar) and adding the result to another row (column).

Application 1. Computing determinants: see the seminars.

We remind that these operations can be used for computing determinants and none of them turns a non-zero determinant into a zero and, also, none of them turns a zero determinant into a non-zero one. One can easily deduce from this fact that performing these operations on a matrix A we get a matrix B for which maximum number of columns (rows) we can choose from the columns (rows) of such that none of them is a linear combination of the others is the same, hence A and B have the same rank.

This remark can be used for developing another algorithm for finding the rank of a matrix.

Application 2. Computing the rank of a matrix:

For finding the rank of a matrix $A \in M_{m,n}(K)$ we may proceed by performing certain elementary operations on the rows and columns of A and on the matrices that we successively get this way. Our purpose is to transform A (by using elementary operations) into a matrix B for which the only non-zero elements are on the main diagonal, preferably at the beginning (if such elements exist). This way, from A we get a matrix

$$B = \begin{pmatrix} \underline{a_{11}} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \underline{a_{22}} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \underline{a_{rr}} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

with all $a_{11}, a_{22}, \dots, a_{rr}$ non-zero. The matrices A and B have the same rank which is r.

Example 6. Let us use this algorithm to compute the rank of

$$A = \begin{pmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & -3 \end{pmatrix} \underset{\substack{r_2-2r_1 \\ r_3-r_1}}{\sim} \begin{pmatrix} 1 & 2 & 1 & -2 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \underset{\substack{c_2-2c_1 \\ c_3-c_1 \\ c_4+2c_1}}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim$$

we perform these steps at once

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \underset{c_4+c_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \operatorname{rank} A = 3.$$

Application 3. Solving systems of linear equations by using **Gauss elimination** algorithm. Let K be a field and let us consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

over K with tha augmented matrix $\overline{A}$. This algorithm is based on the fact that

(i) interchanging of two equations of (1),

(ii) multiplying an equation of (1) by a non-zero element $\alpha \in K$,

(iii) multiplying an equation of (1) by $\alpha \in K$ and adding the resulted equation to another one,

are operations which lead us to systems whic are equivalent to (1). Since all these operations act on the coefficients and constant terms of the system, it is quite obvious that these operations can be performed as elementary row operations on the system augmented matrix.

Thus, we can infer that providing elementary row operations on the augmented matrix of (1), we get tha augmented matrix of an equivalent system. **Gaussian elimination** (also known as **row reduction**) is an algorithm which uses row elementary operations on some matrices resulted from $\overline{A}$ in order to get a matrix with a number zero entries at the beginning of each row which strictly increases while we descend in the matrix (matrix known as **echelon matrix** or **echelon form**). This procedure corresponds to a partial elimination of some unknowns to get an equivalent system which can be easier solved.

Definition 7. A matrix $A \in M_{mn}(K)$ is in an **echelon form** with $k \geq 1$ non-zero rows if:

(1) the rows $1, \dots, k$ are non-zero and the rows $k+1, \dots, m$ consists only of 0;

(2) if $N(i)$ is the number of zeros at the beginning of the row i $(i \in \{1, \dots, k\})$, then

$$0 \leq N(1) < N(2) < \cdots < N(k).$$

A k non-zero rows echelon form with $N(1) = 0$, $N(2) = 1$, $N(3) = 2, \dots$, $N(k) = k-1$ is called **trapezoidal form**.

Ex: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ trapezoidal form with 3 non-zero rows

echelon form with 3 non-zero rows ~~trapezoidal~~

trapezoidal forms

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is an echelon form ~~trapezoidal~~

Remarks 8. a) Any matrix can be brought to an echelon form by elementary row operations.

Let $A \in M_{m,n}(K)$. We can perform rows-switching on A s.t. the first row of the resulted matrix has the minimum number of zeros at the beginning

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

We can assume that $a_{11} \neq 0$. $\exists a_{11}^{-1} \in K$. We perform the following row-operations $r_2 - a_{11}^{-1} a_{21} r_1$, $r_3 - a_{11}^{-1} a_{31} r_1, \dots, r_m - a_{11}^{-1} a_{m1} r_1$

$\Longrightarrow$ all the elements of the first column, except for a_{11} are 0.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \dots & a'_{3n} \\ 0 & a'_{m2} & a'_{m3} & \dots & a'_{mn} \end{pmatrix} \sim \dots \begin{pmatrix} a''_{11} & a''_{12} & \dots & a''_{1n} \\ 0 & a''_{22} & \dots & a''_{2n} \\ - & - & - & \dots \end{pmatrix}$$

we apply the previous alg. to this matrix.

echelon form (not necessarily trapezoidal)

b) A square matrix A of size n is invertible if and only it can be brought to a trapezoidal form with n non-zero rows by using only elementary row operations (such a matrix is called **triangular form (matrix)**).

"$\Rightarrow$" $\det A \neq 0 \xrightarrow[\text{the previously indicated row operations starting from } A]{\text{after performing}}$ we get an echelon form B which is a square matrix and $\det B = \underset{K^*}{\alpha} \cdot \det A \neq 0$

$\det B$ — the product of the elem. on its diagonal

$\Longrightarrow$ B has no zero on its diagonal

$\Longrightarrow$ B is a triangular form.

"$\Leftarrow$" $A \overset{\text{row. operations}}{\sim} \dots \sim B \leftarrow$ trapezoidal form with n non-zero rows (triangular form)

$\Longrightarrow \det B \neq 0 \Longrightarrow \det A = \underset{K^*}{\alpha} \cdot \det B \neq 0 \Longrightarrow$

$\Longrightarrow \exists A^{-1} \in M_n(K)$.

$\det A \neq 0$, $A \sim \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$ $a_{11}, a_{22}, \dots, a_{nn} \in K^*$

c) A square matrix of size n is invertible if and only it can be brought to the identity matrix I_n by using only elementary row operations.

"$\Leftarrow$" straightforward.

"$\Rightarrow$" $a_{22} \neq 0 \xRightarrow{r_1 - a_{12}a_{22}^{-1}r_2}$ a matrix with all the elements in the second column zero, except for a_{22}.

$\begin{pmatrix} a_{11} & 0 & a'_{13} & \dots & a'_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix} \sim$ a matrix with all the third column zero, except for a_{33}

$r_1 - a'_{13}a_{33}^{-1} \cdot r_3$, $r_2 - a_{23}a_{33}^{-1}r_3$ and so on ...

$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \underset{\substack{a_{22}^{-1} r_2 \\ \vdots \\ a_{nn}^{-1} r_n}}{\overset{a_{11}^{-1} r_1}{\sim}} I_n$.

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Thus, the purpose of Gauss elimination is to successively use elementary operations on the rows of the augmented matrix $\overline{A}$ of (1) in order to bring it to an echelon form B. If we manage to do this, then B is the augmented matrix of an equivalent system. In some forms of Gauss elimination, and we plan to use this form, the purpose is to bring $\overline{A}$ to a trapezoidal form

$$B = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & \dots & a'_{1k} & a'_{1,k+1} & \dots & a'_{1n} & b'_1 \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2k} & a'_{2,k+1} & \dots & a'_{2n} & b'_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a'_{kk} & a'_{k,k+1} & \dots & a'_{kn} & b'_k \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Some information on the given system can be easily read from this form. E.g. the rank of $\overline{A}$ is (the rank of B which is) the number of the nonzero elements on the diagonal of B and these nonzero elements on the diagonal of B provide us with the main unknowns.

Remarks 9. a) Finding a trapezoidal form is not always possible by using only row elementary operations. Sometimes, we have to interchange two columns of the first n columns, hence columns corresponding to the matrix of a certain equivalent system. This is, obviously, allowed since this

means that we commute the two corresponding terms in each equation of this system and this is possible based on the commutativity of the addition in K.
b) If, during this algorithm, one finds a row for which all the elements are 0, except for the last one, which is $a \in K^*$, then (1) is inconsistent since it is equivalent to a system which contains the equality $0 = a$ which is not possible.

Assume that we brought A to the above mentioned trapezoidal form B. This means that (1) is equivalent to a system of the form

$$\begin{cases} a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1,k-1}x_{k-1} + a'_{1k}x_k + a'_{1,k+1}x_{k+1} + \cdots + a'_{1n}x_n = b'_1 \\ \qquad a'_{22}x_2 + \cdots + a'_{2,k-1}x_{k-1} + a'_{2k}x_k + a'_{2,k+1}x_{k+1} + \cdots + a'_{2n}x_n = b'_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad\qquad a'_{k-1,k-1}x_{k-1} + a'_{k-1,k}x_k + a'_{k-1,k+1}x_{k+1} + \cdots + a'_{k-1,n}x_n = b'_{k-1} \\ \qquad\qquad\qquad a'_{kk}x_k + a'_{k,k+1}x_{k+1} + \cdots + a'_{kn}x_n = b'_k \end{cases}$$

(possibly with the unknowns succeeding in a different way, if we permuted columns) The main unknowns $x_1, \ldots, x_k$ can be easily computed starting from the last equation of this system.

Remarks 10. a) A few more steps in Gauss elimination allow us to bring $\overline{A}$ by elementary row operations and, if necessary, by switching columns different from the last one to the following trapezoidal form

$$B = \begin{pmatrix} a''_{11} & 0 & 0 & \ldots & 0 & a''_{1,k+1} & \ldots & a''_{1n} & b''_1 \\ 0 & a''_{22} & 0 & \ldots & 0 & a''_{2,k+1} & \ldots & a''_{2n} & b''_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a''_{kk} & a''_{k,k+1} & \ldots & a''_{kn} & b''_k \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

with $a''_{11}, a''_{22}, \ldots, a''_{kk}$ non-zero (of course, this is possible only if the system (1) is consistent, otherwise, some non-zero elements may appear in the last column, bellow b''_k). One can easily notice the advantage we have when we form the equivalent system of (1) provided by B. This algorithm is known as **Gauss-Jordan elimination**.

b) Moreover, we can bring the augmented matrix of a consistent system to the following form:

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & a'''_{1,k+1} & \dots & a'''_{1n} & b'''_1 \\ 0 & 1 & 0 & \dots & 0 & a'''_{2,k+1} & \dots & a'''_{2n} & b'''_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & a'''_{k,k+1} & \dots & a'''_{kn} & b'''_k \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Now, it is straightforward to express the main unknowns as linear combinations of the side unknowns.

Application 4. Computing the inverse of a matrix: Let K be a field, $n \in \mathbb{N}^*$ and let us consider $A = (a_{ij}) \in M_n(K)$ a matrix with $d = \det A \neq 0$. We rimind that the matrix equation

$$A \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \tag{2}$$

is an equivalent form of a (consistent) Cramer system and that its unique solution is

$$\begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = A^{-1} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}.$$

Let us take $j = 1$ and $\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then $\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$ is the first column of the matrix A^{-1}, i.e.

$$\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} d^{-1}\alpha_{11} \\ d^{-1}\alpha_{12} \\ \vdots \\ d^{-1}\alpha_{1n} \end{pmatrix}$$

(we remind that in our previous courses we denoted by α_{ij} the cofactor of a_{ij}). Of course,

$$\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = I_n \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$$

By means of Gauss-Jordan algorithm, one deduces that the augmented matrix of the system (2) can be brought by elementary row operations to the following form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & d^{-1}\alpha_{11} \\ 0 & 1 & 0 & \dots & 0 & d^{-1}\alpha_{12} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & d^{-1}\alpha_{1n} \end{pmatrix}.$$

Taking, successively, $j = 2$ and $\begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then $j = 3$ and $\begin{pmatrix} b_{13} \\ b_{23} \\ \vdots \\ b_{n3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,..., $j = n$ and $\begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, we form the corresponding systems (2) and we use the Gauss-Jordan algorithm to solve them. We perform exactly the same elementary operations as in the case $j = 1$ on the rows of each augmented matrix of a resulted system in order to bring the system matrix to the form I_n. We get:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{21} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{22} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{2n} \end{pmatrix}, \begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{31} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{32} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{3n} \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 & \dots & 0 & d^{-1}\alpha_{n1} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{n2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d^{-1}\alpha_{nn} \end{pmatrix},$$

respectively. The constant terms column and, consequently, the solution of each system we solved is the column 2 of A^{-1}, column 3 of A^{-1},..., column n of A^{-1}, respectively.

Since we performed the same row operations on each of the previously mentioned n systems, we can solve all of them using the same algorithm. This way one can finds an algorithm for computing the inverse of the matrix A: we start from the $n \times 2n$ matrix resulted by attaching the matrices A

and I_n

$$(A \mid I_n) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array}\right) \in M_{n,2n}(K)$$

and we perform successive elementary row operations (and only row operations) on this matrix and on the matrices successively resulted from this in order to transform the left size block into I_n. Remark 8 c) ensures us that this is possible (if and only if A is invertible). The resulted matrix is:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & d^{-1}\alpha_{11} & d^{-1}\alpha_{21} & \dots & d^{-1}\alpha_{n1} \\ 0 & 1 & \dots & 0 & d^{-1}\alpha_{12} & d^{-1}\alpha_{22} & \dots & d^{-1}\alpha_{n2} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & last d^{-1}\alpha_{1n} & d^{-1}\alpha_{2n} & \dots & d^{-1}\alpha_{nn} \end{array}\right) = (I_n \mid A^{-1})$$

Thus, the right side block of the resulted matrix is the exactly the inverse matrix of A.