SRT

$\sum x_n$ SRT $\iff x_n \neq 0 \ \forall n \in \mathbb{N}$

Abel-Dirichlet criterion

$(a_n), (b_n)$ two sequences $\subseteq \mathbb{R}$ s.t. $\begin{cases} (a_n) \text{ decreasing} \\ \lim\limits_{n\to\infty} a_n = 0 \quad , \quad a_n > 0 \ \forall n \in \mathbb{N} \\ (b_n) \text{ has } (s_n) \text{ bounded} \end{cases}$

Then the series $\sum u_n \cdot a_n$ is C.

The Leibniz theorem

Consider $(a_n) \subseteq \mathbb{R}$ $\begin{cases} (a_n) \text{ decreasing} \\ \lim\limits_{n\to\infty} a_n = 0 \end{cases}$

Then $\sum\limits_{n\geq 1} (-1)^n \cdot a_n$ is C

Absolutely convergent

the series $\sum x_n$ is said to be absolutely convergent (A.C.) if the series $\sum |x_n|$ is convergent.

$$\sum |x_n| \text{ is C} \iff \sum x_n \text{ is A.C.}$$

The link between AC and C

If $\sum x_n$ is A.C. $\Longrightarrow \sum x_n$ is C
$\qquad\qquad\qquad\quad \not\Longleftarrow$

In exercises

Analyse:

- $\sum |x_n|$ (SPT $\exists$ plenty of conv. criteria)
- If $\sum |x_n|$ is C $\overset{def}{\iff} \sum x_n$ is A.C. $\overset{T}{\Longrightarrow} \sum x_n$ is C
- If $\sum |x_n|$ is D $\longrightarrow \exists$ several approaches for $\sum x_n$ $\rightarrow$ Abel-Dirichlet
  $\rightarrow$ Leibniz
  $\rightarrow$ direct (s_n)

Examples:

a) Study the nature of the series $\sum_{n\geq 1} \frac{(-1)^n}{n}$.

Solution:

$\forall n\in\mathbb{N}$ $a_n := \frac{1}{n}$ $\Rightarrow$ • (a_n) is decreasing $\frac{1}{n+1} < \frac{1}{n}$

• $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\xRightarrow{\text{Leibniz th}} \sum (-1)^n a_n = \boxed{= \sum \frac{(-1)^n}{n} \text{ is C.}}$

b) Study the nature of the series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n(n^2+3)}}$

$\forall n\in\mathbb{N}$ $a_n := \frac{1}{\sqrt{n(n^2+3)}}$ $\Rightarrow$ • (a_n) decreasing. • $\lim_{n\to\infty} a_n = 0$ $\xRightarrow{\text{Leibniz th}} \sum (-1)^n \cdot \frac{1}{\sqrt{n(n^2+3)}}$ is C.

Remark: Study the nature of the series $\sum |x_n|$ from the above examples (a) and b))

a) $\sum |x_n| = \sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ → the harmonic series D.

b) $\sum |x_n| = \sum \left| \frac{(-1)^n}{\sqrt{n(n^2+3)}} \right| = \sum \frac{1}{\sqrt{n(n^2+3)}} \sim \sum \frac{1}{n^{3/2}}$ $\boxed{3/2 > 1}$ C.

Starting with a SRT C $\sum x_n$, the SPT generated $\sum |x_n|$ is uncertain.

Def: The series $\sum x_n$ is said to be ABSOLUTELY CONVERGENT (A.C.) if the series $\sum |x_n|$ is C.

T (the link between AC and C)

If $\sum x_n$ is A.C. $\Longrightarrow$ $\sum x_n$ is C.

Proof - with the general convergence criterion of Cauchy.

We know:

(1) $\sum x_n$ is A.C. $\xLeftrightarrow{\text{def}}$ $\sum |x_n|$ is C.

$\xLeftrightarrow{\text{Cauchy}}$ $\forall \varepsilon>0, \exists n_\varepsilon\in\mathbb{N}$ s.t. $\forall n\geq n_\varepsilon$, $\forall p\in\mathbb{N}$

$\big|\, |x_{n+1}| + |x_{n+2}| + \dots + |x_{n+p}| \,\big| < \varepsilon$

We want:

(2) $\sum x_n$ is C. $\xLeftrightarrow{\text{Cauchy}}$ $\forall\varepsilon>0, \exists n_\varepsilon\in\mathbb{N}$ s.t. $\forall n\geq n_\varepsilon$

$|x_{n+1} + \dots + x_{n+p}| < \varepsilon$

Choose $\varepsilon>0$ random

$\text{Jump?} \; |x_{n+1} + \dots + x_{n+p}| < \varepsilon$

$|x_{n+1} + \dots + x_{n+p}| \leq |x_{n+1}| + \dots + |x_{n+p}| \overset{(1)}{<} \varepsilon$

so $\exists n'\in\mathbb{N}$ s.t. $\forall n\geq n'$, $\forall p\in\mathbb{N}$ $|x_{n+1} + \dots + x_{n+p}| < \varepsilon$ ✓

Remark: A.C. $\Longrightarrow$ C.

$\not\Longleftarrow$ (See Ex. 1a) $\sum \frac{(-1)^n}{n}$ is C but not A.C.

Remark:

• In exercises, we should start by analysing $\sum |x_n|$ (because it is a SPT ($\exists$ plenty of conv. criteria))

If • $\sum |x_n|$ is C $\xLeftrightarrow{\text{def}}$ $\sum x_n$ is A.C. $\xRightarrow{T}$ $\sum x_n$ is C.

• $\sum |x_n|$ is D $\Longrightarrow$? $\exists$ several approaches for $\sum x_n$: • Abel-Dirichlet • Leibniz • direct (s_n)

Example: Study both the C and the A.C. of the following series:

$$\sum_{n\geq 1} \frac{\sin n}{n(n+1)}$$

$x_n = \dfrac{\sin n}{n(n+1)} \neq 0 \quad \forall n \in \mathbb{N}$ SRT

$\boxed{\text{Step 1}}$ Start with analyzing $\sum |x_n| = \sum \dfrac{|\sin n|}{n(n+1)}$ which is a SPT

hint $|\sin x| \leq 1$, $|\cos x| \leq 1$

$$|x_n| = \frac{|\sin n|}{n(n+1)} \leq \underbrace{\frac{1}{n(n+1)}}_{y_n}$$

$$\sum y_n = \sum \frac{1}{n(n+1)} \sim \sum \frac{1}{n^{\alpha}} \quad \boxed{C2C} \Rightarrow \sum y_n \sim \sum \frac{1}{n^2}$$

$t_n = \dfrac{1}{n^{\alpha}}$ $\quad \lim\limits_{n\to\infty} \dfrac{y_n}{t_n} = \lim\limits_{n\to\infty} \dfrac{n^{\alpha}}{n(n+1)} = 1 \in (0,\infty)$ $\hookrightarrow \boxed{\alpha=2}$

$2>1$ C.

$\left.\begin{array}{l}\sum y_n \text{ is C.}\\ |x_n| \leq y_n\end{array}\right\}$ $\xRightarrow{\boxed{CIC}}$ $\sum |x_n|$ is C. $\overset{\text{def}}{\Longleftrightarrow}$ $\sum x_n$ is A.C. $\overset{T}{\Downarrow}$ $\boxed{\sum x_n \text{ is C.}}$

~~We can~~ In conclusion $\sum x_n$ is both C and A.C.

Remark: For the study of the C, we could have used the Abel-Dirichlet theorem, but it is difficult to prove that $(\underbrace{\sin 1 + \dots + \sin n}_{s_n})$ is bounded

Hw Study the nature of the following series (both C and A.C.)

$$\sum_{n\geq 2} \frac{(-1)^n}{\ln n}, \quad \sum_{n\geq 1} \frac{(-1)^n}{\ln(\ln n)}, \quad \sum_{n\geq 1} \frac{\cos\left(2n\pi + \frac{\pi}{2}\right)}{n^2+1}, \quad \sum_{n\geq 1} \frac{\cos(2n)}{n^3+n}$$

$$\sum_{n\geq 2} (-1)^{n+1} \frac{n+1}{a^n+3^n}$$

$$x_n = (-1)^{n+1} \frac{n+1}{a^n+3^n} \neq 0 \quad \forall_{\substack{n\in\mathbb{N}\\ n\geq 2}} \Rightarrow SRT$$

$$x_n = (-1)^n \cdot (-1) \cdot \frac{n+1}{a^n+3^n} = (-1)^n \underbrace{\frac{1-n}{a^n+3^n}}$$

$$a_n = \frac{1-n}{a^n+3^n}$$

$$a_1 = 0$$

$$a_2 = \frac{-1}{a^2+3^2} < 0$$

$$a_3 = \frac{-2}{a^3+3^3} < 0$$

a_n decreasing

$$\lim_{n\to\infty} \frac{\overset{\to 0}{1-n}}{a^n+3^n} = 0$$

Leibniz $\Rightarrow \sum x_n$ C.

$$x_n = (-1)^{n+1} \frac{n+1}{a^n+3^n}$$

$$x_n = \left| (-1)^{n+1} \frac{n+1}{a^n+3^n} \right|$$

$$x_n = \left| \frac{n+1}{a^n+3^n} \right| > 0 \quad \forall_n \in \mathbb{N}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n+1}{a^n+3^n} = \frac{\infty}{\infty} = \lim_{n\to\infty} \frac{1}{(a^n+3^n)'} = 0 \in \mathbb{R}$$

L'H.