Course 2

March 2, 2021

2.6. Example. Every open ball in $\mathbb{R}^n$ is an open set, while every closed ball in $\mathbb{R}^n$ is a closed set.

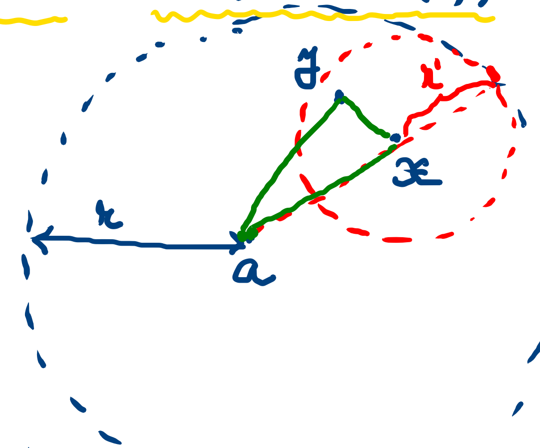
Proof. Let $a \in \mathbb{R}^n$, and let $r > 0$.

$\cancel{?}$ $B(a,r)$ is an open set $\iff$ $\forall x \in B(a,r)$ $\exists r' > 0$ s.t. $B(x,r') \subseteq B(a,r)$

Let $x \in B(a,r) \Rightarrow \|x-a\| < r$

Let $r' := r - \|x-a\| \Rightarrow r' > 0$.

We claim that $B(x,r') \subseteq B(a,r)$ $\quad (*)$



Pick an arbitrary $y \in B(x,r')$

$$\|y-a\| = \|(y-x)+(x-a)\| \le \underbrace{\|y-x\|}_{<r'} + \|x-a\| < r' + \|x-a\| = r$$

because $y \in B(x,r')$

$\Rightarrow y \in B(a,r)$, whence $(*)$ holds.

HW: $\overline{B}(a,r)$ is closed.

2.7. Theorem. If $A \subseteq \mathbb{R}^n$, then $\begin{cases} \text{int } A & \text{is an open set} \\ \text{cl } A & \text{is a closed set} \end{cases}$

(if and only if)

2.8. Theorem (characterization of open sets). A set $A \subseteq \mathbb{R}^n$ is open iff $A = \text{int } A$.

2.9. Theorem (characterization of closed sets). A set $A \subseteq \mathbb{R}^n$ is closed iff $A = \text{cl } A$.

3. Sequences in $\mathbb{R}^n$

3.1. Definition (convergent sequences in $\mathbb{R}^n$) Any function $f : \mathbb{N} \to \mathbb{R}^n$ is called a sequence of points in $\mathbb{R}^n$. If $f(k) = x_k$ $(k \in \mathbb{N})$, then the sequence is denoted by $(x_k)_{k \in \mathbb{N}}$, $(x_k)_{k \geq 1}$, (x_k).

Let (x_k) be a seq. of points in $\mathbb{R}^n$ $\Rightarrow$ $\forall k \in \mathbb{N}$: $x_k \in \mathbb{R}^n$, so it is of the form $x_k = (x_{k1}, x_{k2}, \ldots, x_{kn})$

Example $x_k = \left(\frac{k}{k+1}, \left(1+\frac{1}{k}\right)^k, k \sin\frac{\pi}{k} \right) \quad k \geq 1$

$\Rightarrow (x_k)$ is a sequence in $\mathbb{R}^3$

Let (x_k) be a seq. in $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. One says that (x_k) converges to x if

$\Leftrightarrow \big| \|x_k - x\| - 0 \big| < \varepsilon$

$$\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \text{ s.t. } \forall k \geq k_0 : \quad \|x_k - x\| < \varepsilon$$

If (x_k) is convergent, then its limit is unique.

Notation: $\lim_{k\to\infty} x_k = x$ or $(x_k) \longrightarrow x$

Remark. $\lim_{k\to\infty} x_k = x \iff \lim_{k\to\infty} \|x_k - x\| = 0$

Let (x_k) be a seq. in $\mathbb{R}^n$

$$\begin{aligned}
x_1 &= (x_{11}, x_{12}, \dots, x_{1n}) \\
x_2 &= (x_{21}, x_{22}, \dots, x_{2n}) \\
&\dots \\
x_k &= (x_{k1}, x_{k2}, \dots, x_{kn}) \\
&\dots \\
\downarrow \quad & \quad \downarrow \quad \downarrow \quad \quad \downarrow \\
\bar{x} &= (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)
\end{aligned}$$

3.2. Theorem. Let (x_k) be a seq. in $\mathbb{R}^n$, $x_k = (x_{k1}, x_{k2}, \dots, x_{kn})$ $(k \in \mathbb{N})$, and let $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in \mathbb{R}^n$. Then

$$\lim_{k\to\infty} x_k = \bar{x} \iff \lim_{k\to\infty} x_{kj} = \bar{x}_j \quad \forall j = \overline{1,n}$$

Proof $\boxed{\Rightarrow}$ Assume that $(x_k) \longrightarrow \bar{x}$. Let $j \in \{1, \dots, n\}$. We have

$$0 \le |x_{kj} - \bar{x}_j| \le \sqrt{(x_{k1}-\bar{x}_1)^2 + \dots + (x_{kn}-\bar{x}_n)^2} = \|x_k - \bar{x}\| \xrightarrow{k\to\infty} 0$$

$$\Rightarrow \lim_{k\to\infty} |x_{kj} - \bar{x}_j| = 0 \quad \Rightarrow \quad \lim_{k\to\infty} x_{kj} = \bar{x}_j$$

$\boxed{\Leftarrow}$ Assume that $\lim\limits_{k\to\infty} x_{kj} = \bar{x}_j \quad \forall j \in \{1, \dots, n\}$. We have

$$0 \le \|x_k - \bar{x}\| = \sqrt{(x_{k1}-\bar{x}_1)^2 + \dots + (x_{kn}-\bar{x}_n)^2} \le \underbrace{|x_{k1} - \bar{x}_1| + \dots + |x_{kn} - \bar{x}_n|}_{\xrightarrow{k\to\infty} 0}$$

$$\Rightarrow \lim_{k\to\infty} \|x_k - \bar{x}\| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} x_k = \bar{x}.$$

3.3. Definition (Cauchy sequences). A seq. (x_k) of points in $\mathbb{R}^n$ is said to be a Cauchy sequence if

$$\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \text{ s.t. } \begin{matrix}\forall k \ge k_0 \\ \forall p \ge 1\end{matrix} : \quad \|x_{k+p} - x_k\| < \varepsilon.$$

$$\lim_{k\to\infty}\left(\frac{k}{k+1}, \left(1+\frac{1}{k}\right)^k, k\sin\frac{\pi}{k}\right) = (1, e, \pi)$$

$\frac{k}{k+1} \to 1$, $\left(1+\frac{1}{k}\right)^k \to e$, $k\sin\frac{\pi}{k} \to \pi$

$$k\sin\frac{\pi}{k} = \frac{\sin\frac{\pi}{k}}{\frac{\pi}{k}}\cdot\pi \xrightarrow{k\to\infty} \pi$$

<u>3.4. Theorem</u>. A seq (x_k) of points in $\mathbb{R}^n$, $x_k = (x_{k1}, \dots, x_{kn})$ $(k\in\mathbb{N})$ is a Cauchy seq. iff $(x_{kj})_{k\geq 1}$ is a Cauchy seq. $\forall j\in\{1,\dots,n\}$.

<u>3.5. Theorem</u> (A. L. Cauchy). A sequence in $\mathbb{R}^n$ converges iff it is a Cauchy sequence

<u>Proof</u>. Let (x_k) be a seq. in $\mathbb{R}^n$, $x_k = (x_{k1}, \dots, x_{kn})$

(x_k) converges $\overset{\text{T3.2}}{\Longleftrightarrow}$ $(x_{kj})_{k\geq 1}$ converges $\forall j\in\{1,\dots,n\}$

$\overset{\text{Cauchy Thm. for seq's of real numbers}}{\Longleftrightarrow}$ $(x_{kj})_{k\geq 1}$ is a Cauchy seq. $\forall j=\overline{1,n}$

$\Longleftrightarrow$ (x_k) is a Cauchy sequence.

3.6 Theorem (characterization of adherent points by using sequences).

Let $A \subseteq \mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Then

$$x \in \operatorname{cl} A \iff \exists\, (x_k) \text{ seq. of points in } A \text{ s.t. } (x_k) \longrightarrow x$$

Proof $\boxed{\Rightarrow}$ Assume that $x \in \operatorname{cl} A \Rightarrow \forall V \in \mathcal{V}(x) : V \cap A \neq \emptyset$

Choosing $V := B(x, \frac{1}{k}) \Rightarrow \forall k \geq 1 : B(x, \frac{1}{k}) \cap A \neq \emptyset$

$\Rightarrow \forall k \geq 1 \quad \exists\, x_k \in B(x, \frac{1}{k}) \cap A \Rightarrow x_k \in A$

and

$x_k \in B(x, \frac{1}{k}) \Rightarrow 0 \leq \|x_k - x\| < \frac{1}{k}$

$k \to \infty$

0

$\Rightarrow (x_k)$ is a seq. in A s.t. $\lim\limits_{k \to \infty} \|x_k - x\| = 0$

$$\Downarrow$$

$$(x_k) \longrightarrow x$$

$\boxed{\Leftarrow}$ Assume that $\exists\, (x_k)$ seq. in A s.t. $(x_k) \longrightarrow x$.

Let $V \in \mathcal{V}(x)$ arbitrarily chosen. $\Rightarrow \exists\, \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq V$

Since $(x_k) \longrightarrow x \Rightarrow \exists\, k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0 : \|x_k - x\| < \varepsilon \iff x_k \in B(x, \varepsilon) \subseteq V$

$\Rightarrow \forall k \geq k_0 : x_k \in V \cap A \Rightarrow V \cap A \neq 0 \Rightarrow x \in \operatorname{cl} A$

3.7. Theorem (characterization of limit points by using sequences)

Let $A \subseteq \mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Then

$$x \in A' \iff \exists (x_k) \text{ seq. in } A \setminus \{x\} \text{ s.t. } (x_k) \longrightarrow x$$

Proof. Note that $x \in A' \iff \forall V \in \mathcal{V}(x) : V \cap \big(A \setminus \{x\}\big) \neq \phi$

$$\iff x \in cl(A \setminus \{x\})$$

$$\overset{T3.6}{\iff} \exists (x_k) \text{ seq. in } A \setminus \{x\} \text{ s.t. } (x_k) \to x.$$

3.8. Theorem (characterization of closed sets by using sequences)

A set $A \subseteq \mathbb{R}^n$ is closed $\iff$ the limit of every convergent seq. of points in A does belong to A, too

Proof $\boxed{\Rightarrow}$ Assume that A is closed

Let (x_k) be an arbitrary convergent seq. of points in A, and let $x := \lim\limits_{k \to \infty} x_k$.

By T3.6 $\Rightarrow x \in cl A = A$

$\boxed{\Leftarrow}$ A is closed $\iff A = cl A \iff cl A \subseteq A$ ✓

Pick $x \in cl A \overset{T3.6}{\Longrightarrow} \exists (x_k) \text{ seq. in } A \text{ s.t. } (x_k) \longrightarrow x \xrightarrow{\text{our hyp.}} x \in A$

4. Compact sets in $\mathbb{R}^n$

4.1. Definition (compact sets). Let $A \subseteq \mathbb{R}^n$.

- Any family $(A_i)_{i\in I}$ of subsets of $\mathbb{R}^n$ with the property that

$$A \subseteq \bigcup_{i\in I} A_i$$

is called a covering of A

- A covering $(G_i)_{i\in I}$ of A is said to be open if G_i is open $\forall i \in I$
- The set A is called compact if every open covering of A has a finite subcovering

$$A \text{ is compact} \iff \forall (G_i)_{i\in I} \text{ open covering of } A \ \exists J \subseteq I,\ J \text{ finite s.t. } A \subseteq \bigcup_{i\in J} G_i$$

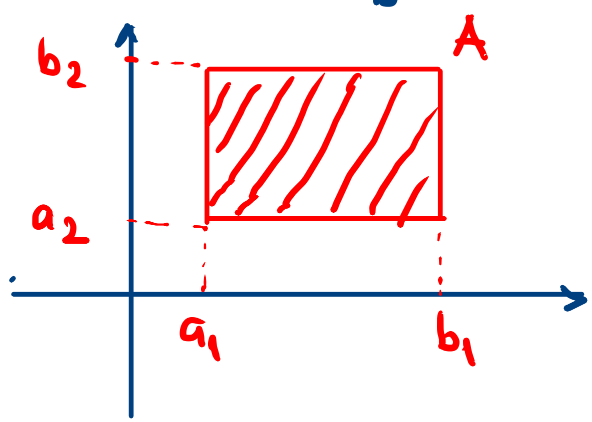
Obviously, every finite set A is compact

4.2 Example A set $A \subseteq \mathbb{R}^n$ is called a <u>closed cell</u> if it is of the form

$$A = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$$
$$= \{ (x_1, x_2, \dots, x_n) \mid x_1 \in [a_1, b_1],\ x_2 \in [a_2, b_2],\ \dots,\ x_n \in [a_n, b_n] \}$$

$n = 1$

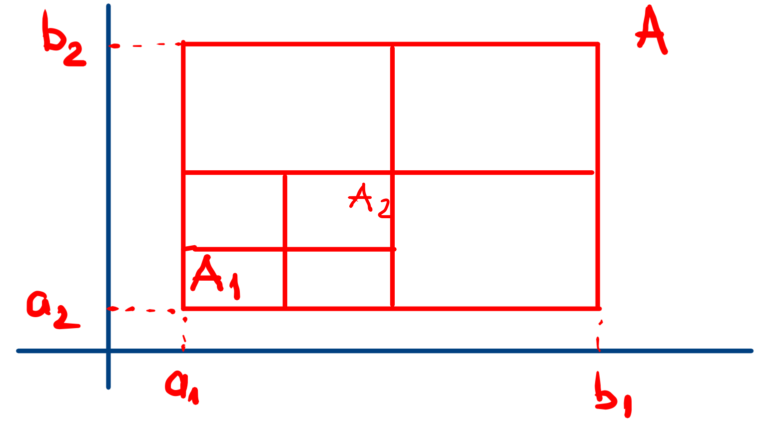
$n = 2$



Every closed cell in $\mathbb{R}^n$ is a compact set.

$n=2$ $\quad A = [a_1,b_1] \times [a_2,b_2]$ is a rectangle

Let $\quad \ell_1 = b_1 - a_1$, $\quad \ell_2 = b_2 - a_2$



Assume, by reductio ad absurdum, that

A is <u>NOT</u> compact

$\Rightarrow \exists (G_i)_{i\in I}$ an open covering of A which does not have a finite subcovering

By means of its centre we split the rectangle A into 4 congruent rectangles

Since A cannot be covered by finitely many sets $(G_i)_{i\in I}$ $\Rightarrow$

$\Rightarrow$ at least one of the 4 smaller rectangles cannot be covered by finitely many sets $(G_i)_{i\in I}$

Let A_1 be this rectangle, $\quad A_1 = I_1^{(1)} \times I_2^{(1)}$, $\quad I_1^{(1)}, I_2^{(1)}$ are closed intervals

$\ell(I_1^{(1)}) = \dfrac{\ell_1}{2} \qquad \ell(I_2^{(1)}) = \dfrac{\ell_2}{2}$

By its centre we split A_1 into 4 congruent rectangles.

Since A_1 cannot be covered by finitely many sets $(G_i)_{i\in I}$ $\Rightarrow$

$\Rightarrow$ at least one of the 4 smaller rectangles cannot be covered by finitely many sets $(G_i)_{i\in I}$.

Let A_2 be this rectangle $A_2 = I_1^{(2)} \times I_2^{(2)}$

where $I_1^{(2)}$, $I_2^{(2)}$ are closed intervals $I_1^{(2)} \subseteq I_1^{(1)}$, $I_2^{(2)} \subseteq I_2^{(1)}$

$$\ell(I_1^{(2)}) = \frac{\ell_1}{2^2} \qquad \ell(I_2^{(2)}) = \frac{\ell_2}{2^2}$$

Continuing the above reasoning we obtain a sequence of rectangles

$$(*) \quad A \supseteq A_1 \supseteq A_2 \supseteq \dots \supseteq A_k \supseteq \dots$$ s.t. every A_k cannot be covered by finitely many sets $(G_i)_{i\in I}$

$A_k = I_1^{(k)} \times I_2^{(k)}$, $I_1^{(k)}$, $I_2^{(k)}$ closed intervals

$$\ell(I_1^{(k)}) = \frac{\ell_1}{2^k} \qquad \ell(I_2^{(k)}) = \frac{\ell_2}{2^k}$$

From $(*)$ $\Rightarrow$

$$\left.\begin{aligned} &[a_1, b_1] \supseteq I_1^{(1)} \supseteq I_1^{(2)} \supseteq \dots \supseteq I_1^{(k)} \supseteq \dots \\ &[a_2, b_2] \supseteq I_2^{(1)} \supseteq I_2^{(2)} \supseteq \dots \supseteq I_2^{(k)} \supseteq \dots \end{aligned}\right\} \xRightarrow{\text{The nested closed intervals theorem}}$$

$$\Rightarrow \bigcap_{k=1}^{\infty} I_1^{(k)} = \{x_1^*\} \text{ and } \bigcap_{k=1}^{\infty} I_2^{(k)} = \{x_2^*\}.$$

Let $x^* := (x_1^*, x_2^*) \Rightarrow x^* \in A_k \quad \forall k \geq 1$

$x^* \in A \subseteq \bigcup_{i \in I} G_i \quad \Rightarrow \quad \exists i_0 \in I$ s.t. $x^* \in G_{i_0}$, G_{i_0} is open $\Rightarrow \exists r > 0$ s.t. $B(x^*, r) \subseteq G_{i_0}$

Let $k \in \mathbb{N}$ s.t. $2^k > \frac{l_1 + l_2}{r}$

Let $x = (x_1, x_2) \in A_k$

$x^* = (x_1^*, x_2^*) \in A_k$

$$\|x - x^*\| = \sqrt{(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2} \leq \sqrt{\frac{l_1^2 + l_2^2}{2^{2k}}} < \frac{l_1 + l_2}{2^k} < r$$

$$\Downarrow$$

$x \in B(x^*, r) \subseteq G_{i_0}$

$x_1, x_1^* \in I_1^{(k)} \Rightarrow |x - x_1^*| \leq l(I_1^{(k)}) = \frac{l_1}{2^k}$

$$\Rightarrow (x_1 - x_1^*)^2 \leq \frac{l_1^2}{2^{2k}}$$

similarly $(x_2 - x_2^*)^2 \leq \frac{l_2^2}{2^{2k}}$

Since x was an arbitrary point in $A_k \Rightarrow A_k \subseteq G_{i_0}$ $\Rightarrow\!\!\not\Leftarrow$ A_k cannot be covered by finitely many sets $(G_i)_{i \in I}$

Therefore A is a compact set.

4.3. Definition A set $A \subseteq \mathbb{R}^n$ is said to be <u>bounded</u> if

$\exists a \in \mathbb{R}^n \ \exists r > 0$ s.t. $A \subseteq \overline{B}(a, r)$

$\iff \exists r > 0$ s.t. $A \subseteq \overline{B}(0_n, r)$

$\iff \exists C$ closed cell in $\mathbb{R}^n$ s.t. $A \subseteq C$.

4.4. Theorem (characterization of compact sets in $\mathbb{R}^n$). Given $A \subseteq \mathbb{R}^n$, the following statements are equivalent

1° A is compact

2° A is sequentially compact (i.e., every seq. in A has a subsequence converging to some point in A)

3° A is bounded and closed.