


Theorem 9.2 (Schwarz Inequality). Let \mathbf{V} be a Euclidean vector space. If $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ then

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle$$

with equality if and only if \mathbf{v} and \mathbf{w} are parallel.

If $\mathbf{w} = 0$ the inequality is obvious since both sides are zero.

Suppose $\mathbf{w} \neq 0$

$$\text{If } a, b \in \mathbb{R} \quad 0 \leq \langle a\mathbf{v} + b\mathbf{w}, a\mathbf{v} + b\mathbf{w} \rangle = a^2 \langle \mathbf{v}, \mathbf{v} \rangle + 2ab \langle \mathbf{v}, \mathbf{w} \rangle + b^2 \langle \mathbf{w}, \mathbf{w} \rangle$$

with equality if and only if $a\mathbf{v} + b\mathbf{w} = 0$ since $\langle \cdot, \cdot \rangle$ is positive definite

$$a\mathbf{v} + b\mathbf{w} = 0 \quad \text{if and only if} \quad \begin{cases} \mathbf{v}, \mathbf{w} \text{ are parallel} \\ \text{or} \\ a = b = 0 \end{cases}$$

Choose $a = \langle \mathbf{w}, \mathbf{w} \rangle \neq 0$ and $b = -\langle \mathbf{v}, \mathbf{w} \rangle$, then

$$0 \leq \langle \mathbf{w}, \mathbf{w} \rangle^2 \langle \mathbf{v}, \mathbf{v} \rangle - 2 \langle \mathbf{w}, \mathbf{w} \rangle \langle \mathbf{v}, \mathbf{w} \rangle^2 + \langle \mathbf{v}, \mathbf{w} \rangle^2 \langle \mathbf{w}, \mathbf{w} \rangle \quad | : \langle \mathbf{w}, \mathbf{w} \rangle \neq 0$$

$$\Rightarrow 0 \leq \langle \mathbf{w}, \mathbf{w} \rangle \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle^2$$

Proposition 9.3. The norm of a vector enjoys the following three properties

(N1) $\|\mathbf{v}\| \geq 0$ for any $\mathbf{v} \in \mathbf{V}$, with equality if and only if $\mathbf{v} = 0$.

(N2) $\|r\mathbf{v}\| = |r| \cdot \|\mathbf{v}\|$ for every $\mathbf{v} \in \mathbf{V}$ and $r \in \mathbb{R}$.

(N3) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$, for every $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ with equality if and only if \mathbf{v} and \mathbf{w} are parallel.

(N1) \Leftrightarrow scalar product is positive definite

(N2) $\Leftrightarrow \langle r\mathbf{v}, r\mathbf{v} \rangle = r^2 \langle \mathbf{v}, \mathbf{v} \rangle$ by taking square roots

$$(N3) \Leftrightarrow \|\mathbf{v} + \mathbf{w}\|^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle = \|\mathbf{v}\|^2 + 2 \langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 = (\|\mathbf{v}\| + \|\mathbf{w}\|)^2$$

called triangle inequality

Schwarz

Proposition 9.4. Any set of mutually orthogonal vectors is linearly independent. In particular, if $\dim(\mathbf{V}) = n$ is finite, any set of mutually orthogonal vectors is a basis of \mathbf{V} .

Let v_1, \dots, v_t be an orthogonal set of vectors

If we have a linear relation $a_1 v_1 + \dots + a_t v_t = 0$

Then, for each $i \in \{1, \dots, t\}$

$$\begin{aligned} 0 &= \langle v_i, a_1 v_1 + \dots + a_t v_t \rangle \\ &= a_1 \langle v_i, v_1 \rangle + \dots + a_t \langle v_i, v_t \rangle \\ &= a_i \langle v_i, v_i \rangle \end{aligned}$$

Since $\langle v_i, v_i \rangle \geq 0$ it follows that $a_i = 0$

So for all i $a_i = 0 \Rightarrow v_1, \dots, v_t$ are linearly independent.

Proposition 9.5. Let $e = \{e_1, \dots, e_n\}$ and $f = \{f_1, \dots, f_n\}$ be two bases of the Euclidean vector space \mathbf{V} , and suppose that e is orthonormal. The basis f is orthonormal if and only if the base change matrix $M_{e,f}$ is orthogonal.

- The columns of $M = M_{e,f} (\text{Id}_V)$ are the components of the vectors f_1, \dots, f_n with respect to the basis e

- The basis f is orthogonal if and only if

$$M_i^T M_j = \langle f_i, f_j \rangle = \delta_{ij} \quad \begin{matrix} i \text{th column} \\ j \text{th column} \end{matrix}$$

- The above condition translates exactly to $M^T M = I_n \Leftrightarrow M \in O(n)$.

Proposition 9.8. Let $\phi \in \text{AGL}(\mathbb{E}^n)$ be an affine transformation given by $\phi(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ with respect to some orthonormal coordinate system. The following are equivalent:

1. ϕ is an isometry
2. $A = A^t$.

1. \Rightarrow 2. Let $\psi(\mathbf{x}) = \phi(\mathbf{x}) - \mathbf{b}$ be the linear map given by $\psi(\mathbf{x}) = A\mathbf{x}$

ψ is the composition of the isometric ϕ with a translation $\Rightarrow \psi$ is an isometric
translations are isometries

$$\Rightarrow \|\psi(\mathbf{v})\| = \|\mathbf{v}\| \quad \forall \mathbf{v} \in V \quad (\text{vector space associated to } \mathbb{E}^n)$$

Let $0, e_1, \dots, e_n$ be the orthonormal coordinate system w.r.t ϕ is given

$$\text{let } f_i = \psi(e_i) = Ae_i, \quad f_1 = \psi(e_1) = Ae_1, \quad f_2 = \psi(e_2) = Ae_2, \dots, \quad f_n = \psi(e_n) = Ae_n$$

$$\text{Then } \langle f_i, f_j \rangle = \frac{1}{4} (\|f_i + f_j\|^2 - \|f_i - f_j\|^2)$$

$$= \frac{1}{4} (\|e_i + e_j\|^2 - \|e_i - e_j\|^2)$$

$$= \langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The components of $f_i = Ae_i$ is the i -th column of the matrix A

\Rightarrow The (i,j) -entry of the matrix $A^t A$ is $\langle f_i, f_j \rangle = \delta_{ij}$

$$\Rightarrow A^t A = I_n$$

2. \Rightarrow 1. If points $x, y \in \mathbb{E}^n$ we have $\|f(x) - f(y)\| = \|(Ax+b) - (Ay+b)\| = \|A(x-y)\|$

$$\Rightarrow \|f(x) - f(y)\|^2 = \|A(x-y)\|^2 = \langle A(x-y), A(x-y) \rangle$$

$$= (A(x-y))^t A(x-y)$$

$$= (x-y)^t A^t A (x-y) = (x-y)^t (x-y)$$

$$= \|x-y\|^2$$

$$\Rightarrow d(f(x), f(y)) = d(x, y)$$

Proposition 9.9. Let $A \in \text{Mat}_{n \times n}(\mathbb{R})$ be a matrix such that $A^t A = I_n$. Then $\det(A) \in \{\pm 1\}$.

$$1 = \det(I_n) = \det(A^t A) = \det(A^t) \det(A) = \det(A)^2$$

Proposition 9.11. A matrix A is in $SO(2)$ if and only if A has the form

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some $\theta \in \mathbb{R}$.

\Leftarrow It is an easy check to see that $R_\theta \in SO(2)$

\Rightarrow Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2)$

$$\text{then } \det A = ad - bc = 1.$$

$$\text{and } A^t A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \left\{ \begin{array}{l} a^2 + c^2 = 1 \\ b^2 + d^2 = 1 \\ ab + cd = 0 \\ ad - bc = 1 \end{array} \right\}$$

$$a^2 + c^2 = 1 \Rightarrow \exists \theta \text{ s.t. } a = \cos \theta \text{ and } c = \sin \theta$$

$$b^2 + d^2 = 1 \Rightarrow \exists \tilde{\theta} \text{ s.t. } d = \cos \tilde{\theta} \text{ and } b = \sin \tilde{\theta}$$

$$\text{So } ab + cd = \cos \theta \sin \tilde{\theta} + \sin \theta \cos \tilde{\theta} = \sin(\theta + \tilde{\theta}) = 0 \quad \left\{ \begin{array}{l} \Rightarrow \tilde{\theta} = -\theta + 2k\pi \end{array} \right.$$

$$ad - bc = \cos \theta \cos \tilde{\theta} - \sin \theta \sin \tilde{\theta} = \cos(\theta + \tilde{\theta}) = 1$$

$$\Rightarrow A = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Corollary 9.12. A direct isometry ϕ of E^2 that fixes a point is either the identity or a rotation. Moreover, the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\text{tr}(\text{lin}(\phi))}{2}.$$

Let $\phi(z) = Az + b$. It is an direct isometry if $A \in SO(2) \Leftrightarrow A = R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

so $\phi(z) = R_\theta z + b$

If $\theta = 0$ then ϕ has a fixed point only if $b = 0$ in which case $\phi = \text{Id}$

Assume that $\theta \neq 0$

Let p be a fixed point for ϕ

$$\phi(p) = p \Leftrightarrow R_\theta p + b = p \Leftrightarrow b = (I_2 - R_\theta)p \quad (\star)$$

Moreover

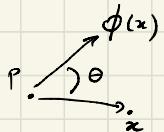
$$\det(I - R_\theta) = \det \begin{pmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{pmatrix} = (1 - \cos\theta)^2 - \sin^2\theta = 2 - 2\cos\theta$$

So, if $\theta \neq 0$ then $\det(I - R_\theta) \neq 0 \Rightarrow$ eq. (\star) has a unique solution p

So, in this case ϕ has a unique fixed point

To finish the proof we show that ϕ is a rotation around the point p

$$\begin{aligned}\phi(x) - \phi(p) &= \phi(x) - p = R_\theta x + b - p \\ &= R_\theta x + (I - R_\theta)p - p \\ &= R_\theta(x - p)\end{aligned}$$



this means that ϕ rotates \overrightarrow{px} with θ and maps it to $\overrightarrow{\phi(p)\phi(x)} = \overrightarrow{p\phi(x)}$

. Finally $\text{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$ so $\text{tr}(\text{lin } \phi) = \text{tr } R_\theta = 2\cos\theta$.

Theorem 9.13 (Euler). A direct isometry ϕ of E^3 that fixes a point is either the identity or a rotation around an axis that passes through that point. Moreover the angle θ of the rotation is such that

$$\cos(\theta) = \frac{\text{tr}(\text{lin } \phi) - 1}{2}.$$

. We may suppose that ϕ fixes the origin. $\Rightarrow \phi$ is linear given by $\phi(x) = Ax$
with respect to some coordinate system

. Since ϕ is a direct isometry $A \in SO(3)$ $A^T A = I_3, \det A = 1$

. A rotation around an axis fixes the axis. We show that A fixes a line.

For this it suffices to show that A has an eigenvector v for the eigenvalue 1 since then $\phi(tv) = A(tv) = t(Av) = t v \quad \forall t \in \mathbb{R}$

• So, we show that $\det(A - I) = 0$

$$\begin{aligned} \text{We have } \det(A - I_3) &= \underbrace{\det(A^t)}_1 \det(A - I_3) = \det(A^t(A - I_3)) \\ &= \det(A^t A - A^t) = \det(I_3 - A^t) = \det((I_3 - A)^t) \\ &= \det(I_3 - A) \end{aligned}$$

since $A - I$ is a 3×3 matrix $\det(A - I_3) = -\det(I_3 - A)$

$$\Rightarrow \det(I_3 - A) = -\det(I_3 - A)$$

$$\Rightarrow \det(I_3 - A) = 0$$

• Next, we show that ϕ is a rotation around the axis Ra

• Choose a vector $u_1 \perp a$ with $\|u_1\|=1$ and let $u_2 := a \times u_1$

Then Au_1 and Au_2 are also orthogonal to a (to the axis Ra) since

$$\langle Au_1, a \rangle = \langle Au_2, a \rangle = \langle u_1, a \rangle = 0$$

$\Rightarrow Au_1$ and Au_2 are linear combinations of u_1 and u_2

since $\|Au_1\| = \|Au_2\| = 1$ we have

$$Au_1 = \cos(\theta) u_1 + \sin(\theta) u_2$$

$$Au_2 = -\sin(\theta) u_1 + \cos(\theta) u_2$$

where θ is the angle between u_1 and Au_1

\Rightarrow the matrix of ϕ with respect to the basis u_1, u_2, a is

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a rotation around the axis Ra

• Finally the trace of ϕ is the trace of the above matrix $= 2\cos\theta + 1$.

Lemma 9.16. The characteristic polynomial of a symmetric matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ has only real roots.

- We may consider A as a matrix over \mathbb{C} , $A \in \text{Mat}_{n \times n}(\mathbb{R}) \subseteq \text{Mat}_{n \times n}(\mathbb{C})$
- Consider the operator $\phi_A: \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by A
- Let $\lambda \in \mathbb{C}$ be a root of the characteristic polynomial of A and let $x \in \mathbb{C}^n$ be an eigenvector for λ

$$Ax = \lambda x$$

• Taking complex conjugates we have $\bar{A}x = \bar{A}\bar{x} = A\bar{x} = \bar{\lambda}\bar{x}$

Then $\bar{x}^t A x = \bar{x}^t (Ax) = \bar{x}^t \lambda x = \lambda \bar{x}^t x$

and $\bar{x}^t A x = (\bar{x}^t A)x = (A\bar{x})^t x = (\bar{\lambda}\bar{x}^t)x = \bar{\lambda}\bar{x}^t x$

• Note that $\bar{x}^t x = \bar{x}_1 x_1 + \dots + \bar{x}_n x_n \geq 0$ since $x \neq 0$

$$\Rightarrow \bar{\lambda} = \lambda$$

$$\Rightarrow \lambda \in \mathbb{R}$$

Theorem 9.17. (Spectral Theorem) Let V be the associated vector space of the Euclidean space E^n and let $T: V \rightarrow V$ be a symmetric operator. There is an orthonormal basis of V with respect to which the matrix of T is diagonal.

. Proof by induction on $n = \dim(V)$

- If $n=1$ there is nothing to prove
- Suppose $n \geq 2$ and that the theorem holds for a space of dimension $n-1$
- Lem. 9.16 \Rightarrow all eigenvalues for T are real
 $\Rightarrow T$ has n real eigenvalues

- Let λ be an eigenvalue of T and let e_1 be an eigenvector
we may choose e_1 of length 1.

- Consider $U = e_1^\perp$. For each $u \in U$ we have

$$\langle T(u), e_1 \rangle = \langle u, T(e_1) \rangle = \langle u, \lambda e_1 \rangle = \lambda \langle u, e_1 \rangle = \lambda \cdot 0 = 0$$

$$\Rightarrow T(u) \in U$$

$\Rightarrow T$ induces an operator $T_U : U \rightarrow U$

Since $T_U(u) = T(u)$ & $u \in U$ the operator T_U is symmetric

- By induction, U has an orthonormal basis $\{e_2, \dots, e_n\}$ which diagonalizes T_U
- Thus $\{e_1, \dots, e_n\}$ is an orthonormal basis of V which diagonalizes T .

Theorem 9.18. For every real symmetric matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ there is an orthogonal matrix $M \in O(n)$ such that $M^{-1}AM$ is diagonal.

A is the matrix of a symmetric operator ϕ_A on \mathbb{R}^n
w.r.t the canonical basis e_1, \dots, e_n

By the Spectral Theorem, there is an orthonormal basis f_1, \dots, f_n
w.r.t which the matrix of T_A is diagonal

$$\begin{matrix} \{T_A\}_f & = & M_{ef}^{-1} & \{T_A\}_e & M_{ef} \\ \parallel & & \parallel & & \parallel \\ \text{diagonal} & & M^{-1} & A & M \end{matrix} \quad M_{ef} \in O(n)$$

Theorem 9.19. For every quadratic form $q : V \rightarrow \mathbb{R}$ on a finite dimensional Euclidean space, there is a diagonalizing orthonormal basis.

The proof is similar to the proof of Theorem 9.18.

Proposition 9.20. Let $T : V \rightarrow V$ be a symmetric operator on a Euclidean vector space. If λ and μ are two distinct eigenvalues of T then every eigenvector with eigenvalue λ is orthogonal to every eigenvector with eigenvalue μ .

. Let v be an eigenvector for λ

$$w \quad \perp \perp \quad \mu$$

$$\text{. Then } \langle T(v), w \rangle = \langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

$$\langle v, T(w) \rangle = \langle v, \mu w \rangle = \mu \langle v, w \rangle$$

$$\text{Since } T \text{ is symmetric } \langle T(v), w \rangle = \langle v, T(w) \rangle$$

$$\Rightarrow \lambda \langle v, w \rangle = \mu \langle v, w \rangle$$

$$\Rightarrow \langle v, w \rangle = 0 \quad \text{since } \lambda \neq \mu.$$