Şequences of functions

Study the pointwise convergence (by specifying the convergence set and the pointwise limit function) and the uniform convergence for the following sequences of functions:

1. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{\cos nx}{n^\alpha}$ unde $\alpha > 0$;
2. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
3. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{x^2}{x^4+n^2}$;
4. $f_n : [0,\infty) \to \mathbb{R},\ f_n(x) = \dfrac{1}{1+nx}$;
5. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$;
6. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{nx}{1+n^2x^2}$;
7. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \sqrt{x^2+\dfrac{1}{n^2}}$;
8. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = n\left(\sqrt{x+\dfrac{1}{n}} - \sqrt{x}\right)$;
9. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{nx}{e^{nx^2}}$;
10. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
11. $f_n : [-1,1] \to \mathbb{R},\ f_n(x) = \dfrac{x}{1+n^2x^2}$;

Theory

Fie $\emptyset \neq D \subseteq \mathbb{R}$. We denote by

$$\mathcal{F}(D) = \{f| \quad f : D \to \mathbb{R}\}$$

the set of all the functions defined on the set D. A **sequence of functions** is each function $x : \mathbb{N}_k \to \mathcal{F}(D)$, which associates uniquely to each natureal number $n \geq k$, a function. Thus

$$x(n) := f_n, \quad \forall n \in \mathbb{N}_k.$$

Recall that $\mathbb{N}_k = \{n \in \mathbb{N} : n \geq k\}$, for a given $k \in \mathbb{N}$.
The usual notations for sequences of functions are

$$(f_n) = (f_n)_{n \in \mathbb{N}_k} = (f_n)_{n \geq k}.$$

We will further use the following framework:

$$(f_n) \subseteq \mathcal{F}(D) \text{ is a sequence of functions defined on } \emptyset \neq D \subseteq \mathbb{R}.$$

A point $x_0 \in D$ is called a (pointwise) **convergence point** if the sequence of the real numbers obtained by applying the sequence of functions to that given point x, is convergent. Namely,

$$\exists \lim_{n \to \infty} f_n(x_0) \in \mathbb{R}.$$

The set of all of the convergence points is called **the convergence set of the sequence of functions** and is denoted by

$$\mathcal{C} = \left\{ x \in D : \lim_{n \to \infty} f_n(x) \in \mathbb{R} \right\}.$$

Whenever the convergence set associated to a sequence of functions is nonempty, to it, we may associated, naturally, a function called the **pointwise limit function**,

$$f : \mathcal{C} \to \mathbb{R},$$

defined by

$$f(x) = \lim_{n \to \infty} f_n(x), \quad \forall x \in \mathcal{C}.$$

The notation for this **pointwise convergence** is:

$$f_n \xrightarrow{p} f \quad sau \quad f_n \to f.$$

By using the ϵ- characterization for the limit of the sequences of real numbers, at each point of the convergence set, we may deduce the following characterization theorem for the pointwise convergence:

Theorem

$$f_n \xrightarrow{p} f \Longleftrightarrow \forall x \in \mathcal{C}, \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \quad a.i. \quad \forall n \geq n_\varepsilon, \quad |f_n(x) - f(x)| < \varepsilon.$$

In the following we study another convergence notion for sequences of functions, namely, uniform convergence.

Definition: The sequence of functions (f_n) is said to converge uniformly on the set $D_0 \subseteq D$ if

$$\exists f : D \to \mathbb{R}, \quad a.i. \quad \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \quad a.i. \quad \forall n \geq n_\varepsilon, \forall x \in D_0, \; to\, hold |f_n(x) - f(x)| < \varepsilon.$$

The standard notation for uniform convergence is

$$f_n \overset{u}{\rightrightarrows} f \quad sau \quad f_n \rightrightarrows f.$$

Observaţii:

- $\rightrightarrows \Longrightarrow \longrightarrow$ namely, all uniformly convergent sequences of functions are pointwise convergent as well (having as limit, the limit function defined above), but the converse statement does not hold
- the continuity is inherited through uniform convergence
- In practice, whenever we usually determine explicitly the limit function by computin for each $x \in D$

  $$\lim_{n \to \infty} f_n(x).$$

  Afterwards we analyze the uniform convergence, usually by applying the Weirstrass theorem:

Weirstrass' theore, Consider a sequence of functions $(f_n) \subseteq \mathcal{F}(D)$ and a sequence of real numbers $(a_n) \subseteq \mathbb{R}$, such that:

a) $\exists n_0 \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < a_n, \quad \forall n \geq n_\varepsilon, \forall x \in \mathcal{C}$$

b) $\lim_{n \to \infty} a_n = 0$;

Then

$$f_n \rightrightarrows f,$$

The continuity inheritance theorem

If $f_n \rightrightarrows f$, and all the functions f_n, $n \in \mathbb{N}$ are continuous, then so is the limit function f as well.

Sequences and series of functions

$f: \mathbb{N} \to \mathbb{R}$, $f(n) := x_n \in \mathbb{R}$,
is a sequence of real numbers

Definition:

Let $\emptyset \neq A \subseteq \mathbb{R}$.
We denote by $\mathcal{F}(A, \mathbb{R}) = \{ f \mid f: A \to \mathbb{R} \}$
the set of all real (the codomain is $\mathbb{R}$) functions defined on the set A.

Each function $g: \mathbb{N}_k \to \mathcal{F}(A, \mathbb{R})$,
which associates uniquely to each natural number $n \geq k$ a function
is a sequence of functions.
Thus, $g(n) := f_n$, $\forall n \in \mathbb{N}_k$, where $\mathbb{N}_k = \{ n \in \mathbb{N} : n \geq k \}$ for a given $k \in \mathbb{N}$.

Notations:

sequence of functions
$(f_n) \subseteq \mathcal{F}(A, \mathbb{R}) = (f_n)_{n \in \mathbb{N}_k} = (f_n)_{n \geq k}$

Framework:

$(f_n) \subseteq \mathcal{F}(A, \mathbb{R})$ is a sequence of functions defined on $\emptyset \neq A \subseteq \mathbb{R}$

Definition:

A point $x_0 \in A$ is called a (pointwise) convergence point
if the sequence of the real numbers obtained by applying the sequence of functions to that given point x_0, is convergent.
Namely,

$$\exists \lim_{n \to \infty} f_n(x_0) \in \mathbb{R}$$

The set of all convergence points is called the convergence set of the sequence of functions
and is denoted by

$$\mathcal{C} = \{ x \in A : \lim_{n \to \infty} f_n(x) \in \mathbb{R} \}$$

If the convergence set associated to a sequence of functions is non-empty, if $\mathcal{C} \neq \emptyset$,
we may define (associate to it) naturally,
a function called the pointwise limit function to (f_n).
Namely,

$$f: \mathcal{C} \to \mathbb{R}, \quad \forall x \in \mathcal{C}, \quad f(x) = \lim_{n \to \infty} f_n(x) \in \mathbb{R}$$

Notations:

the pointwise convergence function
$f_n \xrightarrow{p} f = f_n \to f$

Examples:

1) Determine the convergence set and the pointwise limit function for the following sequences of functions:

a) $\forall n \in \mathbb{N}$,
$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = x^n \quad \forall x \in \mathbb{R}$

Step 1 → Fix $x \in \mathbb{R}$ random, solve limit and find the convergence set

Choose $x \in \mathbb{R}$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} \infty : x > 1 \\ 1 : x = 1 \\ 0 : |x| < 1 \\ \nexists : x \leq -1 \end{cases} \Rightarrow \exists \lim_{n\to\infty} f_n(x) \in \mathbb{R} \text{ only if } x \in (-1,1] \Rightarrow \mathcal{C} = (-1,1]$$

Step 2 → Find the point wise limit function

$$\mathcal{C} = (-1,1] \neq \emptyset \Rightarrow \exists f : (-1,1] \to \mathbb{R} \text{ such that } f(x) = \begin{cases} 1 : x = 1 \\ 0 : x \in (-1,1) \end{cases}, \quad f_n \to f$$

b) $\forall n \in \mathbb{N}$,
$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = \frac{1}{n} \sin(nx) \quad \forall x \in \mathbb{R}$

1 → Choose $x \in \mathbb{R}$ random

$$\left.\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n} (\sin nx) = 0 \right\} \Rightarrow \forall x \in \mathbb{R}, \exists \lim_{n\to\infty} f_n(x) = 0 \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

$$0 \leq \left|\frac{1}{n} \sin(nx)\right| \leq \frac{1}{n}$$

$$0$$

2 → $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow \exists f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$ such that $f(x) = 0 \quad \forall x \in \mathbb{R}, \quad f_n \to f$

c) $\forall n \in \mathbb{N}$
$f_n : \mathbb{R} \to \mathbb{R}$
$f_n(x) = \frac{nx}{1 + nx} \quad \forall x \in \mathbb{R}$

1 → Choose $x \in \mathbb{R}$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx} = \begin{cases} 1 : x \neq 0 \\ 0 : x = 0 \end{cases} \Rightarrow \exists \lim_{n\to\infty} f_n(x) \in \mathbb{R} \quad \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

2 → $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow \exists f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$ such that $f = \begin{cases} 1 : x \neq 0 \\ 0 : x = 0 \end{cases} \quad \forall x \in \mathbb{R}, \quad f_n \to f$

Theorem: (The ε-characterization theorem for sequences of functions)

Let $(f_n) \subseteq \mathcal{F}(A, \mathbb{R})$ be a sequence of functions and $f : \mathcal{C} \to \mathbb{R}$ be the pointwise limit function.

Then

$$f_n \xrightarrow{\mathcal{C}} f \iff \forall x \in \mathcal{C}, \forall \varepsilon > 0, \underbrace{\exists n_\varepsilon \in \mathbb{N}}_{\text{depends on } n \text{ and } x} \text{ s.t. } \forall n \geq n_\varepsilon \; |f_n(x) - f(x)| < \varepsilon$$

In other words, its a charactirization theorem for the pointwise convergence.

Proof: $f_n \xrightarrow{\mathcal{C}} f \iff \forall x \in \mathcal{C}, \; f(x) = \lim_{n \to \infty} f_n(x)$

$\Updownarrow$ ε-th. for sequences of real numbers

$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon \; |f_n(x) - f(x)| < \varepsilon$

Remark:

1) Looking back at the examples a), b), c) we notice that $\forall n \in \mathbb{N}$, each function $f_n : A \to \mathbb{R}$ is continuous however, the limit function is continuous only for b)

Uniform convergence

Consider $(f_n) \subseteq \mathcal{F}(A, \mathbb{R})$

and $f : \mathcal{C} \to \mathbb{R}$.

The sequence of functions (f_n) is said to converge uniformly on the set $\mathcal{C} \subseteq A$ to f if:

$$\forall \varepsilon > 0, \underbrace{\exists n_\varepsilon \in \mathbb{N}}_{\text{depends only on } n} \text{ s.t } \forall n \geq n_\varepsilon, \; |f_n(x) - f(x)| < \varepsilon \;\; \forall x \in \mathcal{C}$$

Notations:

the uniform convergence

$f_n \overset{\mathcal{C}}{\rightrightarrows} f = f_n \rightrightarrows f$

Remarks:

→ $\rightrightarrows \Rightarrow \to$ namely, all uniformly convergent sequences of functions are pointwise convergent as well (having as limit, the limit function defined above), but the converse statement does not hold

→ the continuity is inherited through uniform convergence

Steps:

① We determine $\mathcal{C}$

② We formulate f, $f_n \to f$

③ We check if $f_n \rightrightarrows f$

Weirstrass theorem

Consider $(f_n) \subseteq \mathcal{F}(A, \mathbb{R})$,

$f: \mathcal{C} \to \mathbb{R}$,

$(a_n) \subseteq \mathbb{R}$ – a sequence of real numbers

such that if

- $\exists\, n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $\forall x \in \mathcal{C}$ $|f_n(x) - f(x)| < a_n \to 0$
- $\lim_{n\to\infty} a_n = 0$

then

$f_n \rightrightarrows f$

The continuity inheritance theorem:

If $f_n \rightrightarrows f$, and all the functions f_n, $n \in \mathbb{N}$, are continuous, then so is the limit function f as well.

Uniform norm:

For a function $f: \mathcal{C} \to \mathbb{R}$, we may define $\|f\|_\infty = \sup_{x \in \mathcal{C}} |f(x)| \in [0, \infty]$

which represents the uniform norm of the function f.

The characterisation of uniform convergence theorem:

$$f_n \rightrightarrows f \iff \lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

Remarks:

1) If $\left.\begin{array}{l}\forall n \in \mathbb{N} \quad f_n \text{ is continuous} \\ f_n \overset{\mathcal{C}}{\rightrightarrows} f\end{array}\right\} \Rightarrow f$ is continuous on $\mathcal{C}$

2) If $\left.\begin{array}{l} f_n \to f \\ \forall n \in \mathbb{N} \quad f_n \text{ is continuous} \\ f \text{ is NOT continuous on } \mathcal{C}\end{array}\right\} \Rightarrow f_n \not\rightrightarrows f$

Steps:

① $\mathcal{C} = \{a \in A : \lim_{n\to\infty} f_n(a) \in \mathbb{R}\}$

② $f: \mathcal{C} \to \mathbb{R}$ $f_n \to f$

③ $\forall n \in \mathbb{N}$, we determine $a_n := \sup |f_n(x) - f(x)|$ $\forall x \in \mathcal{C}$

if it $\exists$ and $\in \mathbb{R}$ $\forall n \in \mathbb{N} \Rightarrow$ we may consider the sequence of real numbers $(a_n)_{n\geq 1}$

- if $\lim_{n\to\infty} a_n = 0 \Rightarrow f_n \rightrightarrows f$
- otherwise $\Rightarrow f_n \not\rightrightarrows f$

Examples:

1) Study $\to$ and $\rightrightarrows$

a) $\forall n \in \mathbb{R}$,

$f_n : \mathbb{R} \to \mathbb{R}$

$f_n(x) = \frac{x^2}{n^2 + x^4}, \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ random

$$\left.\begin{array}{l} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{n^2+x^2} = 0 \end{array}\right\} \Rightarrow \exists \lim_{n\to\infty} f_n(x) \in \mathbb{R} \ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Leftrightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$ $f(x) = 0 \ \forall x \in \mathbb{R}$, $f_n \to f$

3) $\forall n \in \mathbb{N}$, $a_n := \sup |f_n(x) - f(x)| \ \forall x \in \mathcal{C}$

↳ Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{\forall x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \left| \frac{x^2}{n^2+x^4} - 0 \right| = \sup_{x \in \mathbb{R}} \left| \frac{x^2}{n^2+x^4} \right|$$

- We define a helping function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{x^2}{n^2+x^4}$, $\forall x \in \mathbb{R}$

- It is differentiable on $\mathbb{R}$

$$g'(x) = \left(\frac{x^2}{n^2+x^4} \right)' = \frac{2x(n^2+x^4) - 4x^3 \cdot x^2}{(n^2+x^4)^2} \ \forall x \in \mathbb{R}$$

$$= \frac{2n^2x + 2x^5 - 4x^5}{(n^2+x^4)^2} = \frac{2n^2x - 2x^5}{(n^2+x^4)^2}$$

$$2n^2x - 2x^5 = 2x(n^2 - x^4) = 2x(n - x^2)(n + x^2) =$$
$$= 2x(\sqrt{n} - x)(\sqrt{n} + x)\underbrace{(n + x^2)}_{>0}$$

x	$-\infty$		$-\sqrt{n}$		0		$\sqrt{n}$		∞
x		$- - -$	$-$	$- - -$	0	$+ + +$	$+$	$+ + +$	
$\sqrt{n}-x$		$+ + +$	$+$	$+ + +$	$+$	$+ + +$	0	$- - -$	
$\sqrt{n}+x$		$- - -$	0	$+ + +$	$+$	$+ + +$	$+$	$+ + +$	
g'		$+ + +$	0	$- -$	0	$+ +$	0	$- - -$	
g		$\nearrow$	$g(-\sqrt{n})$	$\searrow$	$g(0)$	$\nearrow$	$g(\sqrt{n})$	$\searrow$	

$$g(-\sqrt{n}) = \frac{(-\sqrt{n})^2}{n^2 + (-\sqrt{n})^2} = \frac{1}{2n} \qquad g(\sqrt{n}) = \frac{1}{2n}$$

$$\Rightarrow g(x) \leq \frac{1}{2n} \ \forall x \in \mathbb{R}$$

$$a_n := \frac{1}{2n} \ \forall n \in \mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n} = 0 \xRightarrow[\text{theorem}]{\text{Weierstrass}} f_n \rightrightarrows f$$

b) $\forall n \in \mathbb{N}$,
$f_n : [0,\infty) \to \mathbb{R}$,
$f_n(x) = \frac{1}{1+nx} \quad \forall x \in [0,\infty)$

1) Choose $x \in [0,\infty)$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = \begin{cases} 1 : x=0 \\ 0 : x\neq 0 \end{cases} \quad \left(\tfrac{1}{\infty_+}=0\right)$$

$\Rightarrow \exists \lim_{n\to\infty} f_n(x) \in \mathbb{R} \ \forall x \in [0,\infty) \Rightarrow \mathcal{C} = [0,\infty)$

2) $\mathcal{C} = [0,\infty) \neq \emptyset \Rightarrow$ we introduce $f : [0,\infty) \to \mathbb{R}$, $f(x) = \begin{cases} 1 : x=0 \\ 0 : x\neq 0 \end{cases}$, $f_n \xrightarrow[{[0,\infty)}]{\mathcal{C}} f$

3) $\left.\begin{array}{l} \forall n \in \mathbb{N} \ f_n \text{ is continuous} \\ f \text{ is not continuous at } 0 \end{array}\right\} \Rightarrow f_n \not\rightrightarrows f$

c) $\forall n \in \mathbb{N}$
$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = 2n^2 x e^{-n^2x^2} \quad \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \begin{cases} \frac{\infty}{\infty} \text{ or } \frac{-\infty}{\infty} & : x\neq 0 \\ 0 & : x=0 \end{cases}$$

$$\text{if } x\neq 0 \Rightarrow \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \lim_{n\to\infty} \frac{1}{x}\cdot\frac{2n^2x^2}{e^{n^2x^2}} = \frac{1}{x}\cdot 0 = 0$$

$t = n^2x^2$

$$\lim_{t\to\infty} \frac{2t}{e^t} \underset{\text{l'Hospital}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{t\to\infty} \frac{2}{e^t} = 0$$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0 \ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

2) $C = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0 \ \forall x \in \mathbb{R}$, $f_n \xrightarrow{\mathcal{C}} f$

3) Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \left|\frac{2n^2x}{e^{n^2x^2}}\right| = \sup_{x\in\mathbb{R}} \frac{|2n^2x|}{e^{n^2x^2}} = \sup_{x\in\mathbb{R}} \frac{2n^2|x|}{e^{n^2x^2}}$$

We define the function $g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \frac{2n^2|x|}{e^{n^2x^2}} = \begin{cases} \frac{2n^2x}{e^{n^2x^2}} : x \geq 0 \\ -\frac{2n^2x}{e^{n^2x^2}} : x<0 \end{cases}$$

$\left.\begin{array}{l} \lim_{\substack{x\to 0 \\ x>0}} g(x) = 0 = g(0) = \lim_{\substack{x\to 0 \\ x<0}} g(x) \Rightarrow g \text{ is continuous at } 0 \\ \text{on } \mathbb{R}\setminus\{0\} \end{array}\right\} \Rightarrow$

$\Rightarrow g$ is continuous on $\mathbb{R} \Rightarrow g$ is differentiable on $\mathbb{R}\setminus\{0\}$

$$x>0 : \ g'(x) = 2n^2 \cdot \left(\frac{x}{e^{n^2x^2}}\right)' = 2n^2 \cdot \frac{\cancel{e^{n^2x^2}} - 2n^2x \cdot \cancel{e^{n^2x^2}} \cdot x}{e^{\cancel{2}n^2x^2}} =$$

$$= \frac{2n^2(1-2n^2x^2)}{e^{n^2x^2}}$$

$$x<0: \quad g'(x) = -\frac{2n^2(1-2n^2x^2)}{e^{n^2x^2}}$$

x		$-\frac{1}{n\sqrt{2}}$		0		$\frac{1}{n\sqrt{2}}$	
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$1-2n^2x^2$	$-$	0	$+$	$+$	$+$	0	$-$
g'	$+$	0	$-$	$\vert$	$+$	0	$-$
g	$\nearrow$	0	$\searrow$	$\vert$	$\nearrow$	0	$\searrow$

g is not differentiable at 0

$\Rightarrow -\frac{1}{n\sqrt{2}}$ and $\frac{1}{n\sqrt{2}}$ are the only options for local maximum

$\Rightarrow a_n = g(1) = \frac{1}{\sqrt{2}}$

$\Rightarrow a_n := \frac{1}{\sqrt{2}} \quad \forall n \in \mathbb{N}$

$\lim\limits_{n\to\infty} a_n = \frac{1}{\sqrt{2}} \neq 0 \Rightarrow f_n \not\rightrightarrows f$

Cheat sheet:

$(f_n) \subseteq \mathcal{F}(A, \mathbb{R})$

$\forall n \in \mathbb{N}, \quad f_n : A \to \mathbb{R} \to$ function

$\mathcal{C} = \{x \in A : \exists \lim\limits_{n\to\infty} f_n(x) \in \mathbb{R}\} \to$ the convergence set

$f : \mathcal{C} \to \mathbb{R}, \quad \forall x \in \mathcal{C} \quad f(x) = \lim\limits_{n\to\infty} f_n(x) \to$ the pointwise limit function

$f_n \xrightarrow{\mathcal{C}} f$

$f_n \rightrightarrows f \to$ uniform convergence

$\forall n \in \mathbb{N} \quad a_n := \sup\limits_{x\in\mathcal{C}} |f_n(x) - f(x)|$

$\exists \lim\limits_{n\to\infty} a_n = 0 \Rightarrow f_n \rightrightarrows f$

$\neq 0 \Rightarrow f_n \not\rightrightarrows f$

$$\left.\begin{array}{l} \forall n \in \mathbb{N} \quad f_n \text{ is c.} \\ f_n \to f \\ f \text{ is not continuous} \end{array}\right\} \Rightarrow f_n \not\rightrightarrows f$$

Calculus on $\mathbb{R}$
Homework 12

Lect. dr. Anca Grad
2021

Sequences of functions

Study the pointwise convergence (by specifying the convergence set and the pointwise limit function) and the uniform convergence for the following sequences of functions:

1. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{\cos nx}{n^\alpha}$ unde $\alpha > 0$;
2. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
3. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{x^2}{x^4+n^2}$;
4. $f_n : [0,\infty) \to \mathbb{R},\ f_n(x) = \dfrac{1}{1+nx}$;
5. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$;
6. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{nx}{1+n^2x^2}$;
7. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \sqrt{x^2+\dfrac{1}{n^2}}$;
8. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = n\left(\sqrt{x+\dfrac{1}{n}} - \sqrt{x}\right)$; $f_n \rightrightarrows f$?
9. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{nx}{e^{nx^2}}$;
10. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
11. $f_n : [-1,1] \to \mathbb{R},\ f_n(x) = \dfrac{x}{1+n^2x^2}$;

1. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n^\alpha}$ unde $\alpha > 0$;

$f_n : \mathbb{R} \to \mathbb{R} \quad f_n(x) = \dfrac{\cos nx}{n^\alpha}, \quad \alpha > 0 \ \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ random

$$\left.\begin{aligned} &\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n^\alpha}(\cos nx) = 0 \end{aligned}\right\} \Rightarrow \forall x \in \mathbb{R} \ \exists \lim_{n\to\infty} f_n(x) = 0 \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

$$-1 \leq \cos(nx) \leq 1 \quad | : n^\alpha$$

$$-\frac{1}{n^\alpha} \leq \frac{\cos(nx)}{n^\alpha} \leq \frac{1}{n^\alpha}$$

$$0$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0 \ \forall x \in \mathbb{R}$ $f_n \to f$

3) Choose $n \in \mathbb{N}$ random

$a_n := \sup |f_n(x) - f_n(x)| \quad \forall x \in \mathcal{C} = \mathbb{R}$

$a_n := \sup \left| \dfrac{\cos(nx)}{n^\alpha} - 0 \right| = \sup \left| \dfrac{\cos(nx)}{n^\alpha} \right|$

We define a helping function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \dfrac{\cos(nx)}{n^\alpha}$

g is differentiable on $\mathbb{R}$

$$g'(x) = \left(\frac{\cos(nx)}{n^\alpha}\right)' = \frac{1}{n^\alpha}(\cos(nx))' = \frac{1}{n^\alpha} \cdot n \cdot (-\sin nx) = -\frac{1}{n^{\alpha-1}} \sin nx$$

$$\sin(x) \leq 1 \quad \Rightarrow \quad -1 \leq \sin(nx) \leq 1 \quad \left| \cdot \frac{1}{n^{\alpha-1}} \right.$$

$$-\frac{1}{n^{\alpha-1}} \leq \frac{\sin(nx)}{n^{\alpha-1}} \leq \frac{1}{n^{\alpha-1}} \quad | \cdot (-1)$$

$$\frac{1}{n^{\alpha-1}} \geq \frac{-\sin(nx)}{n^{\alpha-1}} \geq -\frac{1}{n^{\alpha-1}}$$

$$\Rightarrow |g(x)| \leq \frac{1}{n^{\alpha-1}} \to g(x) \leq \frac{1}{n^{\alpha-1}} \quad \begin{matrix} \forall x \in \mathbb{R} \\ \alpha > 0 \end{matrix}$$

$$\Rightarrow a_n := \frac{1}{n^{\alpha-1}}, \quad \alpha > 0$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\alpha-1}} = \begin{cases} \infty & : \alpha \in (0,1) \\ \infty^0 & : \alpha = 1 \\ 0 & : \alpha > 1 \end{cases}$$

for $\alpha = 1$

$$\lim_{n\to\infty} \frac{1}{n^{\alpha-1}} = \lim_{n\to\infty} \frac{1}{n^\alpha \cdot n^{-1}} = \lim_{n\to\infty} \left(\frac{\frac{1}{n^\alpha}}{n}\right) = \lim_{n\to\infty} \left(\frac{n}{n^\alpha}\right) = \lim_{n\to\infty} \left(\frac{n + n^\alpha - n^\alpha}{n^\alpha}\right) = \lim_{n\to\infty} \left(1 + \frac{n - n^\alpha}{n^\alpha}\right) =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{n - n^\alpha}{n^\alpha}\right)^{\frac{n^\alpha}{n - n^\alpha}}\right]^{\frac{n - n^\alpha}{n^\alpha}} = e^{\lim\limits_{n\to\infty} \frac{n - n^\alpha}{n^\alpha}} = e^{-\lim\limits_{n\to\infty} \frac{n^\alpha - n}{n^\alpha}} =$$

$$= e^{-1} = \frac{1}{e}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\alpha-1}} = \begin{cases} \infty & : \alpha \in (0,1) \\ \frac{1}{e} & : \alpha = 1 \\ 0 & : \alpha > 1 \end{cases}$$

$\lim\limits_{n\to\infty} a_n = 0 \iff \alpha > 1 \quad \Rightarrow$ for $\alpha > 1$: $f_n \rightrightarrows f$

$f_n \rightrightarrows f \quad \forall \alpha > 1$

2. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;

1) Choose $x \in [0,1]$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x(1+n^2)}{n^2} = \lim_{n\to\infty} \frac{x + x\cdot n^2}{n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(\overset{0}{\frac{x}{n^2}} + x\right)}{\cancel{n^2}} = x$$

$\Rightarrow \forall x \in [0,1]$ $\lim_{n\to\infty} f_n(x) = x \in [0,1]$

$\Rightarrow \mathcal{C} = [0,1]$

2) $\mathcal{C} = [0,1] \neq \emptyset \Rightarrow$ we introduce $f : [0,1] \to \mathbb{R}$, $f(x) = x$ $\forall x \in [0,1]$, $f_n \xrightarrow{[0,1]} f$

3) Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup \left| \frac{x(1+n^2)}{n^2} - x \right| = \sup \left| \frac{x + xn^2 - xn^2}{n^2} \right| = \sup \left| \frac{x}{n^2} \right| = \sup \frac{x}{n^2}$$

$0 \le x \le 1 \quad |:n^2$

$0 \le \dfrac{x}{n^2} \le \dfrac{1}{n^2} \Rightarrow \underbrace{a_n}_{=\sup \frac{x}{n^2}} := \dfrac{1}{n^2}$

$\Rightarrow a_n := \dfrac{1}{n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{n^2} = 0 \Rightarrow f_n \rightrightarrows f$

3. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{x^2}{x^4 + n^2}$;

$\forall n \in \mathbb{R}$,

$f_n : \mathbb{R} \to \mathbb{R}$

$f_n(x) = \dfrac{x^2}{n^2 + x^4}$, $\forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ random
$\lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{x^2}{n^2 + x^2} = 0$
$\Bigg\} \Rightarrow \exists \lim\limits_{n \to \infty} f_n(x) \in \mathbb{R}\ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Longleftrightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$ $f(x) = 0\ \forall x \in \mathbb{R}$, $f_n \to f$

3) $\underline{\forall n \in \mathbb{N}}$, $a_n := \sup |f_n(x) - f(x)|\ \forall x \in \mathcal{C}$

$\hookrightarrow$ Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{\forall x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \left| \frac{x^2}{n^2 + x^4} - 0 \right| = \sup_{x \in \mathbb{R}} \left| \frac{x^2}{n^2 + x^4} \right|$$

- We define a helping function $g : \mathbb{R} \to \mathbb{R}$
  $g(x) = \dfrac{x^2}{n^2 + x^4}$, $\forall x \in \mathbb{R}$

- It is differentiable on $\mathbb{R}$

$$g'(x) = \left(\frac{x^2}{n^2 + x^4} \right)' = \frac{2x(n^2 + x^4) - 4x^3 \cdot x^2}{(n^2 + x^4)^2}\ \forall x \in \mathbb{R}$$

$$= \frac{2n^2 x + 2x^5 - 4x^5}{(n^2 + x^4)^2} = \frac{2n^2 x - 2x^5}{(n^2 + x^4)^2}$$

$$2n^2 x - 2x^5 = 2x(n^2 - x^4) = 2x(n - x^2)(n + x^2) =$$
$$= 2x(\sqrt{n} - x)(\sqrt{n} + x)\underbrace{(n + x^2)}_{>0}$$

x	$-\infty$		$-\sqrt{n}$		0		$\sqrt{n}$		∞
x		$- - - -$	$-$	$- -$	0	$+ +$	$+$	$+ + +$	
$\sqrt{n} - x$		$+ + + +$	$+$	$+ +$	$+$	$+ +$	0	$- - -$	
$\sqrt{n} + x$		$- - - -$	0	$+ +$	$+$	$+ +$	$+$	$+ + +$	
g'		$+ + + +$	0	$- -$	0	$+ +$	0	$- - -$	
g		$\nearrow$	$g(-\sqrt{n})$	$\searrow$	$g(0)$	$\nearrow$	$g(\sqrt{n})$	$\searrow$	

$$\frac{(-\sqrt{n})^2}{n^2 + (-\sqrt{n})^2} \overset{\|}{=} \frac{1}{2n} \qquad \overset{\|}{\frac{1}{2n}}$$

$\Rightarrow g(x) \leq \dfrac{1}{2n}\ \forall x \in \mathbb{R}$

$a_n := \dfrac{1}{2n}\ \forall n \in \mathbb{N}$

$$\Rightarrow \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2n} = 0 \xRightarrow[\text{theorem}]{\text{Weierstrass}} f_n \rightrightarrows f$$

4. $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{1}{1+nx}$;

$\forall n \in \mathbb{N}$,
$f_n : [0,\infty) \to \mathbb{R}$,
$f_n(x) = \dfrac{1}{1+nx}$ $\forall x \in [0,\infty)$

1) Choose $x \in [0,\infty)$ randomly

$$\left.\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \underset{\left(\frac{1}{\infty_+}=0\right)}{\frac{1}{1+nx}} = \begin{cases} 1 & : x = 0 \\ 0 & : x \neq 0 \end{cases}\right\} \Rightarrow \exists \lim_{n\to\infty} f_n(x) \in \mathbb{R} \ \forall x \in [0,\infty) \Rightarrow \mathcal{C} = [0,\infty)$$

2) $\mathcal{C} = [0,\infty) \neq \emptyset \Rightarrow$ we introduce $f : [0,\infty) \to \mathbb{R}$, $f(x) = \begin{cases} 1 & : x = 0 \\ 0 & : x \neq 0 \end{cases}$, $f_n \xrightarrow[{[0,\infty)}]{\mathcal{C}} f$

3) $\left.\begin{array}{l}\forall n \in \mathbb{N} \ f_n \text{ is continuous} \\ f \text{ is not continuous at } 0\end{array}\right\} \Rightarrow f_n \not\rightrightarrows f$

5. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$;

c) $\forall n \in \mathbb{N}$
$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$ $\quad \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \begin{cases} \frac{\infty}{\infty} \text{ or } \frac{-\infty}{\infty} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$$\text{if } x \neq 0 \Rightarrow \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \lim_{n\to\infty} \frac{1}{x} \cdot \frac{2n^2x^2}{e^{n^2x^2}} = \frac{1}{x} \cdot 0 = 0$$

$t = n^2x^2$

$$\lim_{t\to\infty} \frac{2t}{e^t} \overset{\frac{\infty}{\infty}}{\underset{\text{l'Hospital}}{=}} \lim_{t\to\infty} \frac{2}{e^t} = 0$$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = 0 \ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0 \ \forall x \in \mathbb{R}$, $f_n \to f$

3) Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \left|\frac{2n^2x}{e^{n^2x^2}}\right| = \sup_{x\in\mathbb{R}} \frac{|2n^2x|}{e^{n^2x^2}} = \sup_{x\in\mathbb{R}} \frac{2n^2|x|}{e^{n^2x^2}}$$

We define the function $g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \frac{2n^2|x|}{e^{n^2x^2}} = \begin{cases} \frac{2n^2x}{e^{n^2x^2}} & : x \geq 0 \\ -\frac{2n^2x}{e^{n^2x^2}} & : x < 0 \end{cases}$$

$$\left.\begin{array}{l}\lim\limits_{\substack{x\to 0\\ x>0}} g(x) = 0 = g(0) = \lim\limits_{\substack{x\to 0\\ x<0}} g(x) \Rightarrow g \text{ is continuous at } 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \mathbb{R}\setminus\{0\}\end{array}\right\} \Rightarrow$$

$\Rightarrow g$ is continuous on $\mathbb{R} \Rightarrow g$ is differentiable on $\mathbb{R}\setminus\{0\}$

$$x > 0 : \quad g'(x) = 2n^2 \cdot \left(\frac{x}{e^{n^2x^2}}\right)' = 2n^2 \cdot \frac{\cancel{e^{n^2x^2}} - 2n^2x \cdot \cancel{e^{n^2x^2}} \cdot x}{e^{\cancel{2}n^2x^2}} =$$

$$= \frac{2n^2(1-2n^2x^2)}{e^{n^2x^2}}$$

$$x < 0 : \quad g'(x) = -\frac{2n^2(1-2n^2x^2)}{e^{n^2x^2}}$$

x		$-\frac{1}{n\sqrt{2}}$		0		$\frac{1}{n\sqrt{2}}$	
x	$- - -$	$- - -$	$- - -$	0	$+ + +$	$+ + +$	$+ + +$
$1-2n^2x^2$	$- - -$	0	$+ + +$	$+ + +$	$+ + +$	0	$- - -$
g'	$+ + +$	0	$- -$	$\vert$	$+ + +$	0	$- - -$
g	$\nearrow$	0	$\searrow$	$\vert$	$\nearrow$	0	$\searrow$

g is not differentiable at 0

$\Rightarrow -\frac{1}{n\sqrt{2}}$ and $\frac{1}{n\sqrt{2}}$ are the only options for local maximum

$\Rightarrow a_n = g(1) = \frac{1}{\sqrt{2}}$

$\Rightarrow a_n := \frac{1}{\sqrt{2}} \quad \forall n \in \mathbb{N}$

$\lim\limits_{n\to\infty} a_n = \frac{1}{\sqrt{2}} \neq 0 \Rightarrow f_n \not\rightrightarrows f$

6. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{1+n^2x^2}$;

$\forall n \in \mathbb{N}$,
$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = \frac{nx}{1+n^2x^2} \quad \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ randomly

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$

$\Rightarrow \exists \lim\limits_{n\to\infty} f_n(x) = 0 \in \mathbb{R} \quad \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in \mathbb{R} \quad f_n \xrightarrow{\mathcal{C}} f$

3) Choose $n \in \mathbb{N}$ random

$a_n := \sup |f_n(x) - f(x)| = \sup \left|\frac{nx}{1+n^2x^2}\right| = \sup \frac{|nx|}{1+n^2x^2}$

We define the function $g : \mathbb{R} \to \mathbb{R}$ $g(x) = \frac{|nx|}{1+n^2x^2} = \begin{cases} -\frac{nx}{1+n^2x^2} & : x < 0 \\ \frac{nx}{1+n^2x^2} & : x \geq 0 \end{cases}$

$\lim\limits_{\substack{x\to 0 \\ x>}} g(x) = 0 = g(0) = \lim\limits_{\substack{x\to 0 \\ x<0}} g(x)$

$\Rightarrow g$ is continuous at 0 and on $\mathbb{R}\setminus\{0\}$

$\Rightarrow f$ is continuous on $\mathbb{R}$

$\Rightarrow g$ is differentiable on $\mathbb{R}\setminus\{0\}$

$\underline{x > 0}$

$$g'(x) = \left(\frac{nx}{1+n^2x^2}\right)' = n\left(\frac{x}{1+n^2x^2}\right)' = n\cdot\frac{1+n^2x^2 - x\,n^2 2x}{(1+n^2x^2)^2} =$$

$$= \frac{n(1+n^2x^2-2n^2x^2)}{(1+n^2x^2)^2} =$$

$$= \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = \frac{n(1-nx)(1+nx)}{(1+nx^2)^2}$$

$\underline{x < 0}$

$$g'(x) = \frac{-n(1-nx)(1+nx)}{(1+n^2x^2)^2}$$

g is not differentiable on 0

$1-nx = 0 \Rightarrow 1 = nx \Rightarrow x = \frac{1}{n}$

x	$+\infty$		$-\frac{1}{n}$		0		$\frac{1}{n}$		$-\infty$
$1-nx$		$+$	$+$	$+$	$+$	$+$	0	$-$	
$1+nx$		$-$	0	$+$	$+$	$+$	$+$	$+$	
g'		$-$	0	$+$	$\vert$	$+$	0	$-$	
g		$\searrow$	$g(-\frac{1}{n})$	$\nearrow$	0	$\nearrow$	$g(\frac{1}{n})$	$\searrow$	

$$g\left(-\frac{1}{n}\right) = \frac{-n\cdot -\frac{1}{n}}{1+n^2\left(\frac{1}{n}\right)^2} = \frac{1}{1+n^2\cdot\frac{1}{n^2}} = \frac{1}{2}$$

$g\left(\frac{1}{n}\right) = \frac{1}{2} \Rightarrow$ local maximum

$\Rightarrow a_n = g\left(\frac{1}{n}\right) = \frac{1}{2}$

$\forall n \in \mathbb{N}\ a_n := \frac{1}{2}$

$\lim\limits_{n\to\infty} a_n = \frac{1}{2} \neq 0 \Rightarrow f_n \not\rightrightarrows f$

0

7. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$;

1) Choose $x \in \mathbb{R}$ random

$$\left.\begin{array}{l}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\sqrt{x^2+\frac{1}{n^2}}\right) = \lim_{n\to\infty}\sqrt{x^2+0} = x\end{array}\right\} \Rightarrow \exists \lim_{n\to\infty} f_n(x) = x \in \mathbb{R} \ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = x \quad \forall x \in \mathbb{R} \quad f_n \to f$

3) Choose $n \in \mathbb{N}$ random

$$a_n := \sup|f_n(x) - f(x)| = \sup\left|\sqrt{x^2+\frac{1}{n^2}} - x\right| = \sup\left|\sqrt{\frac{n^2x^2+1}{n^2}} - x\right| = \sup\left|\frac{\sqrt{n^2x^2+1}}{n} - x\right| =$$

$$= \sup\left|\frac{\sqrt{n^2x^2+1} - nx}{nx}\right|$$

We define the function $g : \mathbb{R} \to \mathbb{R}$ $g(x) = \sqrt{x^2+\frac{1}{n^2}} - x$

$$g'(x) = \frac{1}{\cancel{2}\sqrt{x^2+\frac{1}{n^2}}} \cdot \cancel{2}x - 1 = \frac{x - \sqrt{x^2+\frac{1}{n^2}}}{\sqrt{x^2+\frac{1}{n^2}}} < 0$$

$$\Rightarrow g'(x) < 0 \Rightarrow g(0) \geq g(x)$$

$$\sqrt{0^2+\frac{1}{n^2}} - 0 \geq g(x)$$

$$\frac{1}{n} \geq g(x)$$

$$\left.\begin{array}{l} a_n := \frac{1}{n} \ \forall n \in \mathbb{N} \\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \Rightarrow f_n \rightrightarrows f$$

8. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right)$;

$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right)$

amplify with the conjugate

1) Choose $x \in \mathbb{R}$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n\left(\sqrt{x+\tfrac{1}{n}} - \sqrt{x}\right) = \lim_{n\to\infty} \frac{n\cdot(\cancel{x}+\frac{1}{n}-\cancel{x})}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} = \lim_{n\to\infty} \frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{x}+\sqrt{x}} = \lim_{n\to\infty}\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}} \text{ if } x \neq 0$$

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} n\left(\sqrt{0+\tfrac{1}{n}} - \sqrt{0}\right) = \lim_{n\to\infty} n\cdot\frac{1}{\sqrt{n}} = \lim_{n\to\infty}\frac{n}{\sqrt{n}} = \infty$$

$\Rightarrow$

$\Rightarrow C = (0, \infty)$

2) $C = (0,\infty) \neq \emptyset \Rightarrow$ we introduce $f : (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{2\sqrt{x}}$ $\quad f_n \to f$

3) $\forall n \in \mathbb{N}$ f_n is continuous
$f_n \to f$
f is also continuous

So we should study the uniform convergence $f_n \rightrightarrows f$

Choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in(0,\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,\infty)} \left| {}^{2\sqrt{x})}\frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} - {}^{\sqrt{x+\frac{1}{n}}+x)}\frac{1}{2\sqrt{x}}\right| = \sup_{x\in(0,\infty)} \left| {}^{\sqrt{x+\frac{1}{n}}-\sqrt{x})}\frac{2\sqrt{x}-\sqrt{x+\frac{1}{n}}-\sqrt{x}}{\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)2\sqrt{x}}\right| =$$

$$= \sup_{x\in(0,\infty)} \left| \frac{\left(2\sqrt{x}-\sqrt{x+\frac{1}{n}}-\sqrt{x}\right)\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right)}{\left(\cancel{x}+\frac{1}{n}-\cancel{x}\right)2\sqrt{x}}\right| = \sup_{x\in(0,\infty)} \left| \frac{n\cdot\left(\sqrt{x}-\sqrt{x+\frac{1}{n}}\right)\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right)}{2\sqrt{x}}\right| =$$

$$= \sup_{x\in(0,\infty)} \left| \frac{n\cdot\left(\sqrt{x}-\sqrt{x+\frac{1}{n}}\right)^2}{2\sqrt{x}}\right| = \sup_{x\in(0,\infty)} \left| \frac{n\left(\cancel{x}-\cancel{x}+\frac{1}{n}+2\sqrt{x}\sqrt{x+\frac{1}{n}}\right)}{2\sqrt{x}}\right| =$$

$$= \sup_{x\in(0,\infty)} \left| \frac{1+2nx+2\sqrt{(n\cdot x)(n\cdot x+1)}}{2\sqrt{x}}\right| = \sup_{x\in(0,\infty)} \left| \frac{1}{2\sqrt{x}} + n\sqrt{x} + \frac{\sqrt{(n\cdot x)(n\cdot x+1)}}{\sqrt{x}}\right| =$$

We define a function $g : (0,\infty) \to \mathbb{R}$,

$$= \sup_{x\in(0,\infty)} \left| \frac{1}{2\sqrt{x}} + n\sqrt{x} + \sqrt{n(n\cdot x+1)}\right|$$

$$g(x) = \frac{1}{2\sqrt{x}} + n\sqrt{x} + \sqrt{n(n\cdot x+1)}$$

$$g'(x) = \left(\frac{1}{2\sqrt{x}} + n\sqrt{x} + \sqrt{n(n\cdot x+1)}\right)' = -\frac{1}{4x\sqrt{x}} + \frac{n}{2\sqrt{x}} + \frac{n^2}{2\sqrt{n^2x+n}} = \frac{1}{2}\left(\frac{2nx\sqrt{n^2x+n} - \sqrt{n^2x+n} + 2x\sqrt{x}\,n^2}{2x\sqrt{x}\sqrt{n^2x+n}}\right)$$

$$g'(x) = 0 \iff 2nx\sqrt{n^2x+n} - \sqrt{n^2x+n} + 2x\sqrt{x}\,n^2 = 2n^2x\sqrt{x+\frac{1}{n}} - n\sqrt{x+\frac{1}{n}} + 2x\sqrt{x}\,n^2 = \Big| : n$$

$$2nx\sqrt{x+\frac{1}{n}} - \sqrt{x+\frac{1}{n}} + 2xn\sqrt{x} = 0$$

$\left(\ldots - \sqrt{x}\right)$

$$2nx\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)=\sqrt{x+\frac{1}{n}}$$

$$\frac{2\cancel{n}x\cdot\frac{1}{\cancel{n}}}{\sqrt{x+\frac{1}{n}}-\sqrt{x}}=\sqrt{x+\frac{1}{n}}$$

$$\frac{2x}{n}=\left(\sqrt{x+\frac{1}{n}}\right)^2-\sqrt{x}\sqrt{x+\frac{1}{n}}$$

$$2x=x+\frac{1}{n}-\sqrt{x}\sqrt{x+\frac{1}{n}}$$

$$x=\frac{1}{n}-\sqrt{x}\sqrt{x+\frac{1}{n}}$$

$$x+\sqrt{x}\sqrt{x+\frac{1}{n}}=\frac{1}{n}\quad\Rightarrow f_n\rightrightarrows f$$

$$x=\frac{3}{n}$$

$$g\left(\frac{3}{n}\right)=$$

9. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{e^{nx^2}}$;

$f_n : [0,1] \to \mathbb{R}$
$f_n(x) = \dfrac{nx}{e^{nx^2}}$

1) Choose $x \in [0,1]$ random

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{nx}{e^{nx^2}} = \lim\limits_{n\to\infty} \dfrac{1}{x} \cdot \dfrac{nx^2}{e^{nx^2}} = \lim\limits_{n\to\infty} \dfrac{1}{x} \cdot 0 = 0$ $\Big\} \Rightarrow \exists \lim\limits_{n\to\infty} f_n(x) = 0 \in \mathbb{R}\ \forall x \in \mathbb{R} \Rightarrow C = \mathbb{R}$

$t = nx^2$

$\lim\limits_{t\to\infty} \dfrac{t}{e^t} \underset{\text{L'Hospital}}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{n\to\infty} \dfrac{1}{e^t} = 0$

2) $C = \mathbb{R} \neq 0 \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=C} \to \mathbb{R}$, $f(x) = 0$ $\forall x \in \mathbb{R}$ $\quad f_n \xrightarrow{C} f$

3) Choose $n \in \mathbb{N}$ random

$a_n := \sup |f_n(x) - f(x)| = \sup \left| \dfrac{nx}{e^{nx^2}} \right| = \sup \dfrac{n|x|}{e^{nx^2}}$

We introduce the function $g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = \dfrac{n|x|}{e^{nx^2}} = \begin{cases} -\dfrac{nx}{e^{nx^2}} & : x < 0 \\ \dfrac{nx}{e^{nx^2}} & : x \geq 0 \end{cases}$

$\lim\limits_{\substack{x\to 0 \\ x>0}} g(x) = 0 = g(0) = \lim\limits_{\substack{x\to 0 \\ x<0}} g(x)$

$\Rightarrow$ g is continuous at 0 and $\mathbb{R}\setminus\{0\}$

$\Rightarrow$ g is coutinous on $\mathbb{R}$

$\Rightarrow$ g is differentiable on $\mathbb{R}\setminus\{0\}$

$\underline{x > 0}$

$g'(x) = n \cdot \left(\dfrac{x}{e^{nx^2}}\right)' = n \cdot \dfrac{e^{nx^2} - x \cdot e^{nx^2} \cdot n \cdot 2x}{e^{2nx^2}} = n \cdot \dfrac{\cancel{e^{nx^2}}(1 - n2x^2)}{e^{2nx^2}} = \dfrac{n(1-2nx^2)}{e^{nx^2}} =$

$= \dfrac{n(1-\sqrt{2n}\,x)(1+\sqrt{2n}\,x)}{e^{nx^2}}$

$\underline{x < 0}$

$g'(x) = -\dfrac{n(1-\sqrt{2n}\,x)(1+\sqrt{2n}\,x)}{e^{nx^2}}$

g is not differentiable at 0

x	$-\frac{1}{\sqrt{2n}}$ $\quad$ 0 $\quad$ $\frac{1}{\sqrt{2n}}$
$1-\sqrt{2n}\,x$	+ + + + + 0 − − − − − − − −
$1+\sqrt{2n}\,x$	− − − − − − − − − − − 0 + + +
g'	− − − 0 + + + | + + + 0 − − −
g	$\searrow g\left(-\frac{1}{\sqrt{2n}}\right) \nearrow 0 \nearrow g\left(\frac{1}{\sqrt{2n}}\right) \searrow$

$g\left(\dfrac{1}{\sqrt{2n}}\right) = \dfrac{n \cdot \frac{1}{\sqrt{2n}}}{e^{n \cdot \left(\frac{1}{\sqrt{2n}}\right)^2}} = \dfrac{\sqrt{n}}{2} \cdot \dfrac{1}{e^{n \cdot \frac{1}{2n}}} = \dfrac{\sqrt{n}}{2\sqrt{e}} = \sqrt{\dfrac{n}{4e}}$ $\Rightarrow$ local maximum

$\Rightarrow a_n := \sqrt{\dfrac{n}{4e}}$ $\qquad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt{\dfrac{n}{4e}} = \infty \neq 0 \Longrightarrow f_n \not\rightrightarrows f$

10. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x(1+n^2)}{n^2}$;

$f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x(1+n^2)}{n^2}$

1) Choose $x \in [0,1]$ randomly
$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x + x n^2}{n^2} = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$ $\Rightarrow \exists \lim\limits_{n\to\infty} f_n(x) \in \mathbb{R}\ \forall x \in [0,1] \Rightarrow \mathcal{C} = [0,1]$

2) $\mathcal{C} = [0,1] \neq \emptyset \Rightarrow$ we introduce $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$ $\forall x \in [0,1]$ $\quad f_n \xrightarrow{[0,1]} f$

3) $\left.\begin{array}{l} \forall n \in \mathbb{N}\ f_n \text{ is continuous} \\ f \text{ is not continuous at } 0 \end{array}\right\} \Rightarrow f_n \not\rightrightarrows f$

11. $f_n : [-1,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{1+n^2x^2}$;

$f_n : [-1,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{1+n^2x^2}$

1) Choose $x \in [-1,1]$ random
$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x}{1+n^2x^2} = 0$ $\Rightarrow \exists \lim\limits_{n\to\infty} f_n(x) = 0 \in \mathbb{R}\ \forall x \in [-1,1] \Rightarrow \mathcal{C} = [-1,1]$

2) $\mathcal{C} = [-1,1] \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{[-1,1]}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0$ $\forall x \in [-1,1]$ $\quad f_n \xrightarrow{[-1,1]} f$

3) Choose $n \in \mathbb{N}$ random
$a_n := \sup |f_n(x) - f(x)| = \sup |f_n(x)| = \sup \left| \dfrac{x}{1+x^2n^2} \right| = \sup \dfrac{|x|}{1+x^2n^2}$

We introduce the function $g : [-1,1] \to \mathbb{R}$, $g(x) = \begin{cases} \dfrac{-x}{1+x^2n^2} & : x < 0 \\ \dfrac{x}{1+x^2n^2} & : x \geq 0 \end{cases}$

$\lim\limits_{\substack{x\to 0 \\ x>0}} g(x) = 0 = g(0) = \lim\limits_{\substack{x\to 0 \\ x<0}} g(x)$

$\Rightarrow g$ is continuous at 0 and on $[-1,1] \setminus \{0\}$

$\Rightarrow g$ is continuous on $[-1,1]$

$\Rightarrow g$ is differentiable on $[-1,1] \setminus \{0\}$

$$g'(x) = \frac{x'(1+x^2n^2) - x(1+x^2n^2)'}{(1+x^2n^2)^2} = \frac{1+x^2n^2 - x(2xn^2)}{(1+x^2n^2)^2} = \frac{1+x^2n^2-2x^2n^2}{(1+x^2n^2)} = \frac{1-x^2n^2}{(1+x^2n^2)^2} =$$

$$= \frac{(1-xn)(1+xn)}{(1+x^2n^2)^2}$$

x	$-\infty$	$-\frac{1}{n}$	0	$\frac{1}{n}$	$+\infty$
$1-nx$	+ + + + + + + + + + + + +			0 − − − −	
$1+nx$	− − − −	0 + + + + + + + + + + +			
g'	− − − −	0 + +	0 + + +	0 − − − −	
g	↘	$g(-\frac{1}{n})$ ↗	0 ↗	$g(\frac{1}{n})$ ↘	

$$g\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{1 + \frac{1}{\cancel{n^2}}\cancel{n^2}} = \frac{\frac{1}{n}}{1+1} = \frac{1}{2n}$$

$$g(x) \leq g\left(\frac{1}{n}\right)$$

$$\left.\begin{array}{l} a_n := \frac{1}{2n} \\[1em] \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2n} = 0 \end{array}\right\} \Longrightarrow f_n \rightrightarrows f$$