ANALYTIC GEOMETRY, PROBLEM SET 3

Warm-up 1. What are the conditions that vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ should satisfy to ensure the existence of a triangle ABC such that $\overrightarrow{AB} \in a$, $\overrightarrow{BC} \in \overline{b}$ and $\overrightarrow{CA} \in \overline{c}$?

1. On the sides of a triangle ABC, one constructs the parallelograms $ABB'A''$, $BCC'B''$, $CAA'C''$. Show that one can construct a triangle MNP such that $\overrightarrow{MN} \in \overline{A'A''}$, $\overrightarrow{NP} \in \overline{B'B''}$ and $\overrightarrow{PM} \in \overline{C'C''}$.

2. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} = \overline{0}$.

3. In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 2\overline{OM}$.

4. Consider, in the 3-dimensional space, the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Prove that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a new parallelogram.

5. Let ABC be a triangle and a, b, c the lengths of its sides, respectively. If A_1 is the intersection point of the internal bisector of the angle $\angle A$ and BC and M is an arbitrary point, show that

$$\overline{MA_1} = \frac{b}{b+c}\overline{MB} + \frac{c}{b+c}\overline{MC}$$

6. If G is the centroid (center of mass) of a triangle ABC in the plane and O is a given point, then

$$\overline{OG} = \frac{\overline{OA} + \overline{OB} + \overline{OC}}{3}$$

7. Let ABC be a triangle, H its orthocenter, O the circumcenter (the center of its circumscribed cercle), G the centroid of the triangle and A' the point on the circumcenter diametrically opposed to A. Then:

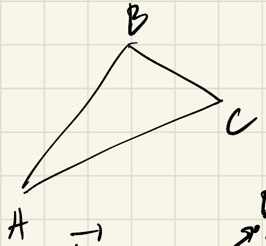
(1) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$; (**Sylvester's formula**)
(2) $\overline{HB} + \overline{HC} = \overline{HA'}$;
(3) $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{HO}$;
(4) $\overline{HA} + \overline{HB} + \overline{HC} = 3\overline{HG}$;
(5) the points H, G, O are collinear and $2GO = HG$. (the **Euler line**)

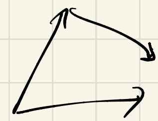
Date: October 8, 2021.

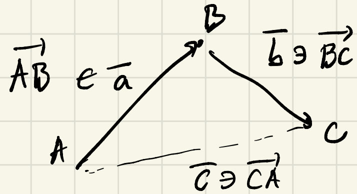
8. Let $ABCD$ be a quadrilateral with $AB \cap CD = \{E\}$, $AD \cap BC = \{F\}$ and the points M, N, P the midpoints of $[BD]$, $[AC]$ and $[EF]$, respectively. Then M, N, P are collinear. (the **Newton-Gauss line**)

Warm-up 1. What are the conditions that vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ should satisfy to ensure the existence of a triangle ABC such that $\overrightarrow{AB} \in \overline{a}$, $\overrightarrow{BC} \in \overline{b}$ and $\overrightarrow{CA} \in \overline{c}$?

the vectors $\overline{a}, \overline{b}, \overline{c}$ are represented on the sides of the triangle ABC



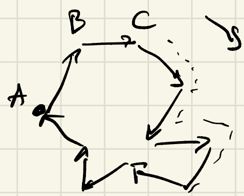




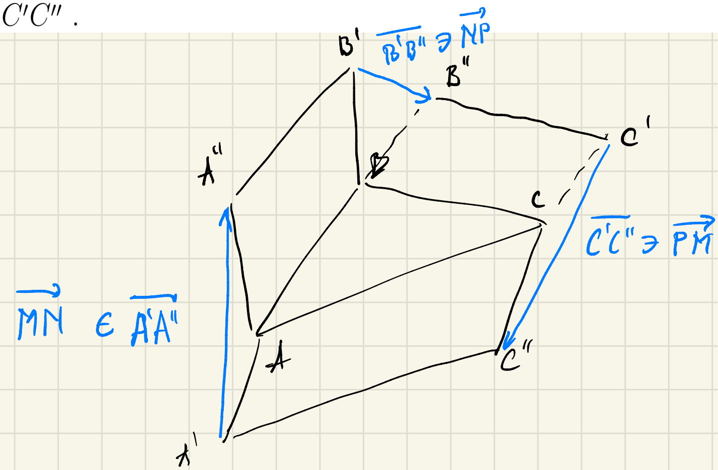
the condition is $\overline{a} + \overline{b} + \overline{c} = \overline{0}$

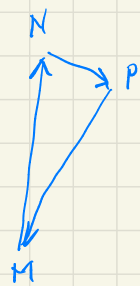
$\overline{a} + \overline{b} = -\overline{c}$

can be generalized to a polygon.



1. On the sides of a triangle ABC, one constructs the parallelograms $ABB'A''$, $BCC'B''$, $CAA'C''$. Show that one can construct a triangle MNP such that $\overrightarrow{MN} \in \overline{A'A''}$, $\overrightarrow{NP} \in \overline{B'B''}$ and $\overrightarrow{PM} \in \overline{C'C''}$.





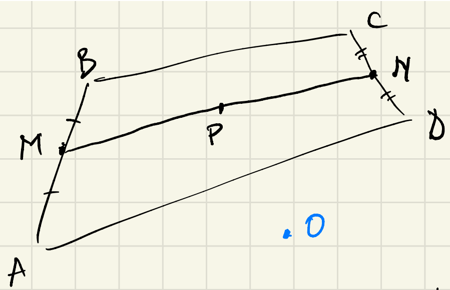
by the warm-up problem we can construct ΔMNP as required iff.

$$\overline{A'A''} + \overline{B'B''} + \overline{C'C''} = \overline{0}$$

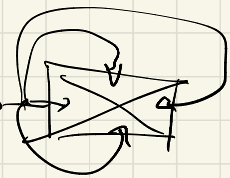
$$\begin{aligned}
\overline{A'A''} &= \overline{A'A} + \overline{AA''}\\
\overline{B'B''} &= \overline{B'B} + \overline{BB''}\\
\overline{C'C''} &= \overline{C'C} + \overline{CC''} \quad \oplus\\
\hline
&\overline{0}
\end{aligned}$$

but $\overline{A'A} = -\overline{CC''}$ and $\overline{AA''} = -\overline{B'B}$ and $\overline{B'B''} = -\overline{C'C}$

2. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} = \overline{0}$. (*)



I $\overline{PA} = \overline{PB} + \overline{BA}$ ... similar for $\overline{PB}$ .... in the triangles



II $\overline{PB} = \overline{PM} + \overline{MB}$

$\overline{PA} = \overline{PM} + \overline{MA}$ $\oplus$

$\overline{PB} + \overline{PA} = 2\overline{PM}$ similar $\overline{PC} + \overline{PD} = 2\overline{PN}$

$$\Rightarrow \overline{PB} + \overline{PA} + \overline{PC} + \overline{PD} = 2\overline{PM} + 2\underbrace{\overline{PN}}_{= -\overline{PM}} = \overline{0}$$

III $\forall$ point O

$$(*) \Leftrightarrow \overline{OA} - \overline{OP} + \overline{OP} - \overline{OB} + \overline{OP} - \overline{OC} + \overline{OP} - \overline{OD} = 0$$

$$\Leftrightarrow \overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 4\overline{OP}$$

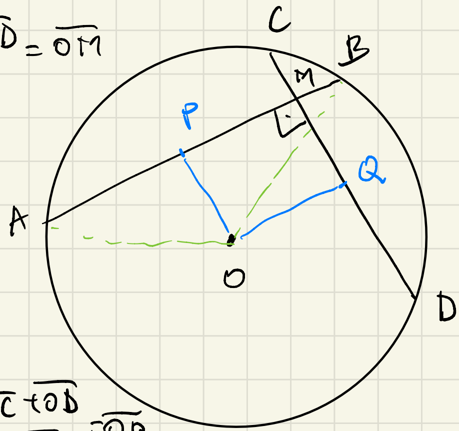
$$\overline{OP} = \frac{1}{2}(\overline{OM} + \overline{ON}), \quad \overline{OM} = \frac{1}{2}(\overline{OA} + \overline{OB}), \quad \overline{ON} = \frac{1}{2}(\overline{OC} + \overline{OD})$$

$$\Leftrightarrow \overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 4 \cdot \frac{1}{2}\left[\frac{\overline{OA} + \overline{OB}}{2} + \frac{\overline{OC} + \overline{OD}}{2}\right]$$

true

3. In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that $\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD}=2\overline{OM}$. $(*)$

$$(*) \iff \frac{\overline{OA}+\overline{OB}}{2}+\frac{\overline{OC}+\overline{OD}}{2}=\overline{OM}$$



P, Q midpoints of $[AB]$ and $[CD]$ respectively

$$\frac{\overline{OA}+\overline{OB}}{2}=\overline{OP} \quad \text{and} \quad \frac{\overline{OC}+\overline{OD}}{2}=\overline{OQ}$$

so

$(*) \iff \overline{OP}+\overline{OQ}=\overline{OM} \iff OPMQ$ is parallelogram.

$\measuredangle PMQ = 90^\circ$

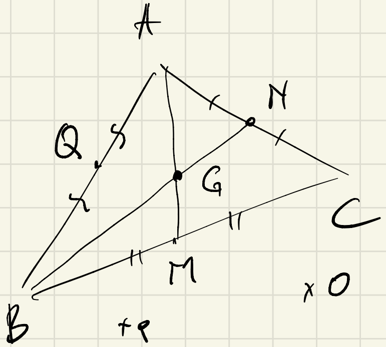
$\stackrel{(**)}{\iff} OPMQ$ is a rectangle since $PM \perp MQ$

for this $(**)$ notice that ΔABO is isosceles $\Rightarrow OP \perp AB$

and similar $OQ \perp CD$

6. If G is the centroid (center of mass) of a triangle ABC in the plane and O is a given point, then

$$\overline{OG} = \frac{\overline{OA} + \overline{OB} + \overline{OC}}{3}$$



Consider $v = \overline{PA} + \overline{PB} + \overline{PC}$ for some point P

we have $v = (\overline{PA} + \overline{PB}) + \overline{PC} = 2\overline{PQ} + \overline{PC}$ (1)

$v = (\overline{PA} + \overline{PC}) + \overline{PB} = 2\overline{PN} + \overline{PB}$ (2)

$v = (\overline{PB} + \overline{PC}) + \overline{PA} = 2\overline{PM} + \overline{PA}$ (3)

$$\left.\begin{array}{l} \text{if } P \in QC \text{ then } v \parallel QC \text{ by (1)} \\ \text{if } P \in BN \text{ then } v \parallel BN \text{ by (2)} \end{array}\right\} \Rightarrow P \in QC \cap BN \Rightarrow v = \overline{0}$$

(in other words $P = G$)

hence by (3) $2\overline{PM} + \overline{PA} = 0$ ie. $2\overline{GM} + \overline{GA} = \overline{0}$ (**)

similar $\left(\begin{array}{l} 2\overline{GN} + \overline{GB} = \overline{0} \\ 2\overline{GQ} + \overline{GC} = \overline{0} \end{array} \right)$

(**) $\Leftrightarrow 2(\overline{OM} - \overline{OG}) + \overline{OA} - \overline{OG} = 0$

$$\frac{2\overline{OM} + \overline{OA}}{3} = \overline{OG} \Leftrightarrow \frac{2\left(\frac{\overline{OB} + \overline{OC}}{2}\right) + \overline{OA}}{3} = \overline{OG}$$

qed.

7. Let ABC be a triangle, H its orthocenter, O the circumcenter (the center of its circumscribed cercle), G the centroid of the triangle and A' the point on the circum~~center~~ circle diametrically opposed to A. Then:

(1) $\overline{OA}+\overline{OB}+\overline{OC}=\overline{OH}$; (**Sylvester's formula**)

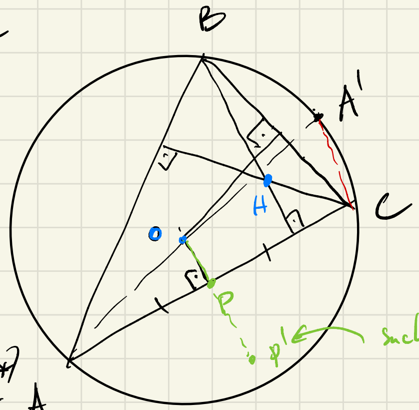
(2) $\overline{HB}+\overline{HC}=\overline{HA'}$;

(3) $\overline{HA}+\overline{HB}+\overline{HC}=2\overline{HO}$;

(4) $\overline{HA}+\overline{HB}+\overline{HC}=3\overline{HG}$; if $2\overline{GO}=\pm\overline{HG}$

(5) the points H, G, O are collinear and $2GO = HG$. (the **Euler line**)

O = intersection of perpendicular bisectors.



such that $2\overline{OP}=\overline{OP'}=\overline{OA}+\overline{OC}$

P is the mid point of AC

notice that

$\overline{OA}+\overline{OC}=2\overline{OP}$

then

(1) $\Leftrightarrow$ $2\overline{OP}+\overline{OB}=\overline{OH}$

$\Leftrightarrow$ $OP'HB$ is a parallelogra. (**)

notice that $OP \perp AC$ and $BH \perp AC \Rightarrow OP \parallel BH$

so in order to show (**) it suffices to show that $2|OP| = |BH|$

AA' is a diameter $\Rightarrow$ $\measuredangle ACA'$ is $90°$

$\Rightarrow \Delta APO \sim \Delta ACA'$ and $2|OP| = |CA'|$

— " — $A'C \perp AC$ and $A'B \perp BA$

$\Rightarrow A'C \parallel BH$ and $A'B \parallel CH$

$\Rightarrow A'CHB$ is a parallelogram $\Rightarrow |A'C| = |BH|$

$\Rightarrow BOP'H$ is a parallelogram. $\Rightarrow$ (**) true $\Rightarrow$ (1) true.