# Continuous Functions

**Exercise 1:** By using the defintion, verify if the following function

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=x^2$$

is continuous.

**Exercise 2:**
**a)** By using the $\varepsilon$ and $\delta$ characterization theorem, prove that the function

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=\begin{cases} x & : x\in\mathbb{Q}\\ -x & : x\in\mathbb{R}\backslash\mathbb{Q},\end{cases}$$

is continuous at 0.

**b)** By using the definition, prove that the function from **a)** has not other continuity points, except for 0, thus $\forall x\in\mathbb{R}\backslash\{0\}$, $f$ is discontinuous at $x$.

**Exercise 3:** Study the continuity of the functions:

**a)** $f:(-\infty,0]\to\mathbb{R}$,

$$f(x)=\begin{cases}\sin x & : x\in(-\infty,0)\\ 7 & : x=0,\end{cases}$$

**b)**$f:[-1,2]\cup\{4\}\to\mathbb{R}$,

$$f(x)=\begin{cases}2x+3 & : x\in[-1,2]\\ 0 & : x=4.\end{cases}$$

**Exercise 4:** Study the continuity of the functions:

**a)**$f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases}\dfrac{\sin x^2}{|x|} & : x\neq 0\\ 0 & : x=0.\end{cases}\quad .$$

**b)**$f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases}e^{x^{-1}} & : x\in(0,\infty)\\ 0 & : x=0,\\ x^2+2x+\sin x & : x\in(-\infty,0).\end{cases}\quad .$$

**c)**$f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases}\sin x & : x\in\mathbb{Q}\\ \cos x & : x\in\mathbb{R}\backslash\mathbb{Q},\end{cases}\quad .$$

**d)** $f : [-2, 1] \cup \{3\} \to \mathbb{R}$,

$$f(x) = \begin{cases} \cos(\pi x) & : x \in [-2, 0] \\ 1 + \sin x & : x = (0, 1] \\ 2 & : x = 3. \end{cases}$$

**Exercise 5:** Study, by discussing the parameter $a \in \mathbb{R}$, the continuity of the following functions:

**a)** $f : [1, 3] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sqrt{a^2 - 2ax + x^2} & : x \in [1, 2] \\ \\ 3a + 2x & : x \in (2, 3]. \end{cases}$$

**b)** $f : (0, \pi) \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{3x} & : x \in (0, 1] \\ \\ a\dfrac{\sin(x-1)}{x^2 - 5x + 4} & : x \in (1, \pi). \end{cases}$$

**Exercise 6:** Let $0 < a < b \in \mathbb{R}$ and $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}$, defined by:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\backslash\{0, 1\}.$$

- a) Prove that $f$ is a continuous function.
- b) Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = f(x), \forall x \in \mathbb{R}\backslash\{0, 1\}$.
- c) Compute $\lim\limits_{x \to -\infty} F(x)$ and $\lim\limits_{x \to \infty} F(x)$.

# Theory briefing

**General Hypotheses**

$$\begin{cases} \emptyset \neq D \subseteq \mathbb{R} \\ f : D \to \mathbb{R} \\ x_0 \in D. \end{cases}$$

**Definition**:

The function $f$ is said to be **continuous at** $x_0$ if

$$\forall (x_n) \subseteq D \quad \text{with} \quad \lim_{n \to \infty} x_n = x_0 \Longrightarrow \lim_{n \to \infty} f(x_n) = f(x_0).$$

**Characterization theorem with neighborhoods**:

f is continuous at $x_0$ if and only if

$$\forall V \in \mathcal{V}(f(x_0)), \quad \exists U \in \mathcal{V}(x_0) \quad \text{such that} \quad f(x) \in V,$$

**Characterization theorem with $\varepsilon$ and $\delta$**:

$f$ is continuous at $x_0$ if and only if

$\forall \varepsilon > 0, \quad \exists \delta > 0,$ such that $\quad \forall x \in D \quad$ with $\quad |x - x_0| < \delta, \quad$ it holds $\quad |f(x) - f(x_0)| < \varepsilon.$

**Theorem characterizing the connection between limits and continuity**:

Let $x_0 \in D \cap D' = D \setminus IzoD$. The following statements are true:

1. $f$ is continuous at în $x_0 \Longrightarrow \exists \lim\limits_{x \to x_0} f(x) = f(x_0 - 0) = f(x_0 + 0) = f(x_0)$.
2. If $\begin{cases} \exists f(x_0 - 0) \\ \exists f(x_0 + 0) \\ f(x_0 - 0) = f(x_0 + 0) = f(x_0) \end{cases} \Longrightarrow f$ is continuous at $x_0$.

**Remark:** By using the definition, it can be easily proved that all elementary functions are continuous on their greatest definition domain. Therefore, unless you are explicitly required to prove the continuity of such elementary functions, you may assume by default that they are continuous.

**Exercise 1:** By using the defintion, verify if the following function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

is continuous.

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$f$ is continuous at $x_0 \in \mathbb{R} \iff \forall (x_n) \subseteq \mathbb{R}$ with $\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$

Choose $x_0$ randomly

Let $(x_n) \subseteq \mathbb{R}$ with $\lim_{n\to\infty} x_n = x_0$

$f(x_n) = x_n^2 \Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = x_0^2 = f(x_0) \quad (1)$

$x_0$ was arbitrary $\Rightarrow$ (1) holds for any $x_0 \in \mathbb{R} \Rightarrow$ (1) holds in general $\Rightarrow f$ is continuous

**Exercise 2:**

**a)** By using the $\varepsilon$ and $\delta$ characterization theorem, prove that the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x & : x \in \mathbb{Q} \\ -x & : x \in \mathbb{R}\setminus\mathbb{Q}, \end{cases}$$

is continuous at 0.

**b)** By using the definition, prove that the function from **a)** has not other continuity points, except for 0, thus $\forall x \in \mathbb{R}\setminus\{0\}$, $f$ is discontinuous at $x$.

a) $f : \mathbb{R} \to \mathbb{R}$

$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}, \quad x_0 = 0$

$f$ is continuous at $x_0 = 0 \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \Delta$ with $|x - x_0| < \delta$, the following holds: $|f(x) - f(x_0)| < \varepsilon$

→ if $x_0 = 0 \in \mathbb{Q}$:

Choose $\varepsilon > 0$ randomly (1)

$\exists \delta > 0$ s.t. $|x - x_0| < \delta$ (2) $\Rightarrow |x| < \delta \Rightarrow |f(x)| < \delta \Rightarrow$

$|f(x) - f(x_0)| < \varepsilon \Rightarrow |x| < \varepsilon$

$\Rightarrow |f(x) - 0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta \Big\} \Rightarrow |f(x) - f(x_0)| < \varepsilon, \ \forall x \in \mathbb{Q} \quad (3)$

Choose $\varepsilon := \delta > 0$

(1), (2), (3) $\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon, \ \forall x \in \mathbb{Q}$

→ if $x_0 = 0 \in \mathbb{R}\setminus\mathbb{Q}$:

Choose $\varepsilon > 0$ randomly (1')

$\exists \delta > 0$ s.t. $|x - x_0| < \delta$ (2') $\Rightarrow |x| < \delta \Rightarrow |-f(x)| < \delta \Rightarrow$

$\Rightarrow |0 - f(x)| < \delta \Rightarrow |-(f(x) - f(x_0))| < \delta \Rightarrow |f(x) - f(x_0)| < \delta, \ \forall x \in \mathbb{R} \Big\} \Rightarrow |f(x) - f(x_0)| < \varepsilon, \ \forall x \in \mathbb{R}\setminus\mathbb{Q}$ (3')

Choose $\varepsilon := \delta > 0$

(1'), (2'), (3') $\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon, \ \forall x \in \mathbb{R}\setminus\mathbb{Q}$

In conclusion, $f$ is continuous at $x_0 = 0$

b) Let $x_0$ be a continuity point.

Let $(x_n)_n \subseteq \mathbb{Q}$ a sequence with $\lim_{n\to\infty} x_n = x_0$. Then $f(x_n) = x_n$.

According to a), $f$ is continuous if $\forall x \in \mathbb{Q} \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0) = x_0$ (1)

Let $(y_n)_n \subseteq \mathbb{R}\setminus\mathbb{Q}$ a sequence with $\lim_{n\to\infty} y_n = x_0$. Then $f(y_n) = -y_n$

According to a), $f$ is continuous if $\forall y \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow \lim\limits_{n\to\infty} f(y_n) = f(x_0) = -x_0$ (2)

If $f$ is continuous in $x_0 \Rightarrow$ the limit is unique $\Rightarrow x_0 = -x_0$

$2x_0 = 0$

$\Rightarrow x_0 = 0$ is the unique solution.

Therefore, $\nexists$ another point at which $f$ could be continuous

$\Rightarrow f$ is discontinuous at $\forall x \in \mathbb{R}\setminus\{0\}$

**Exercise 3:** Study the continuity of the functions:

**a)** $f : (-\infty, 0] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in (-\infty, 0) \\ 7 & : x = 0, \end{cases}$$

**b)** $f : [-1, 2] \cup \{4\} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x+3 & : x \in [-1, 2] \\ 0 & : x = 4. \end{cases}$$

a) $f : (-\infty, 0] \to \mathbb{R}$

$$f(x) = \begin{cases} \sin x & : x \in (-\infty, 0) \\ 7 & : x = 0 \end{cases}$$

$f$ continuous on $(-\infty, 0)$ (elementary function).

The problem of the continuity is for the point $x_0 = 0$.

We shall compute the side limits at $x_0 = 0$.

$$\left.\begin{array}{l} \exists\, l_l(0) = \lim\limits_{\substack{x\to 0\\ x<0}} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} \sin x = 0 \quad (1) \\ \nexists\, l_r(0) \\ f(0) = 7 \quad (2) \end{array}\right\} \text{from (1) and (2)} \Rightarrow 0 \neq 7 \Rightarrow f \text{ is discontinuous at } x_0 = 0$$

$\Rightarrow f$ is continuous on $(-\infty; 0)$

b) $f : [-1, 2] \cup \{4\} \to \mathbb{R}$

$$f(x) = \begin{cases} 2x+3, & x \in [-1, 2] \\ 0, & x = 4 \end{cases}$$

$f$ is continuous on $[-1, 2]$ (operations with elementary functions)

$x = 4$ is an isolated point $\Rightarrow$ the study of continuity is out of question

**Exercise 4:** Study the continuity of the functions:

**a)** $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{\sin x^2}{|x|} & : x \neq 0 \\ 0 & : x = 0. \end{cases}.$$

**b)** $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{x^{-1}} & : x \in (0, \infty) \\ 0 & : x = 0, \\ x^2 + 2x + \sin x & : x \in (-\infty, 0). \end{cases}.$$

**c)** $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in \mathbb{Q} \\ \cos x & : x \in \mathbb{R}\setminus\mathbb{Q}, \end{cases}.$$

**d)** $f : [-2, 1] \cup \{3\} \to \mathbb{R}$,

$$f(x) = \begin{cases} \cos(\pi x) & : x \in [-2, 0] \\ 1 + \sin x & : x = (0, 1] \\ 2 & : x = 3. \end{cases}$$

a) $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{\sin x^2}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -\dfrac{\sin x^2}{x}, & x < 0 \\ 0, & x = 0 \\ \dfrac{\sin x^2}{x}, & x > 0 \end{cases}$$

$f$ continuous on $\mathbb{R}\setminus\{0\}$ (operations with elementary functions).

The problem of continuity is for the point $x_0 = 0$.

We shall compute the side limits at $x_0 = 0$,

as well as the value of the function in the said point.

$$\exists\, l_l(0) = \lim_{\substack{x\to 0\\ x<0}} f(x) = \lim_{\substack{x\to 0\\ x<0}} -\frac{\sin x^2}{x} = -\lim_{\substack{x\to 0\\ x<0}} \frac{\sin x^2}{x^2}\cdot\frac{x^{\cancel{2}}}{\cancel{x}} = -\lim_{\substack{x\to 0\\ x<0}} \frac{\sin x^2}{x^2}\cdot x = -1\cdot 0 = 0 \quad (1)$$

$$\exists\, l_r(0) = \lim_{\substack{x\to 0\\ x>0}} f(x) = \lim_{\substack{x\to 0\\ x>0}} \frac{\sin x^2}{x} = \lim_{\substack{x\to 0\\ x>0}} \frac{\sin x^2}{x^2}\cdot\frac{x^{\cancel{2}}}{\cancel{x}} = \lim_{\substack{x\to 0\\ x>0}} \frac{\sin x^2}{x^2}\cdot x = 1\cdot 0 = 0 \quad (2)$$

$f(0) = 0 \quad (3)$

$(1),(2),(3) \Rightarrow l_l(0) = l_r(0) = f(0) = 0 \Rightarrow f$ is continuous at $x_0 = 0 \Rightarrow f$ is continuous on $\mathbb{R}$

b) $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \begin{cases} e^{x^{-1}}, & x\in(0,+\infty)\\ 0, & x = 0\\ x^2+2x+\sin x, & x\in(-\infty,0)\end{cases}$$

$f$ continuous on $\mathbb{R}\setminus\{0\}$ (operations with elementary functions).
The problem of continuity is for the point $x_0 = 0$.
We shall compute the side limits at $x_0 = 0$,
as well as the value of the function in the said point.

$$\exists\, l_l(0) = \lim_{\substack{x\to 0\\ x<0}} (x^2+2x+\sin x) = 0+0+0 = 0 \quad (1)$$

$$\exists\, l_r(0) = \lim_{\substack{x\to 0\\ x>0}} e^{x^{-1}} = \lim_{\substack{x\to 0\\ x>0}} e^{\frac{1}{x}} = e^{\frac{1}{0_+}} = e^{+\infty} = \infty \quad (2)$$

$f(0) = 0 \quad (3)$

$(1),(2),(3) \Rightarrow \begin{matrix} l_l(0)\neq l_r(0)\\ f(0)\neq l_r(0)\end{matrix} \Rightarrow f$ is discontinuous at $x_0 = 0 \Rightarrow f$ is continuous on $\mathbb{R}\setminus\{0\}$
or
(the domain of continuity is $D_c = \mathbb{R}\setminus\{0\}$)

c) $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \begin{cases}\sin x, & x\in\mathbb{Q}\\ \cos x, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

Let $x_0$ be a continuity point.
Let $(x_n)_n \subseteq \mathbb{Q}$, with $\lim\limits_{n\to\infty} x_n = x_0$

Then $f(x_n) = \sin x_n \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(x_0) = \sin x_0 \quad (1)$

Let $(y_n)_n \subseteq \mathbb{R}\setminus\mathbb{Q}$, with $\lim\limits_{n\to\infty} y_n = x_0$

Then $f(y_n) = \cos y_n \Rightarrow \lim\limits_{n\to\infty} f(y_n) = f(x_0) = \cos x_0 \quad (2)$

$f$ continuous at $x_0 \Leftrightarrow \sin x_0 = \cos x_0$
$\Rightarrow x_0 \in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$ are the only points at which $f$ is continuous $\Rightarrow$
$\Rightarrow f$ discontinuous on $\mathbb{R}\setminus\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$

d) $f:[-2,1]\cup\{3\}\to\mathbb{R}$

$$f(x) = \begin{cases}\cos(\pi x), & x\in[-2,0]\\ 1+\sin x, & x\in(0,1]\\ 2, & x = 3\end{cases}$$

$f$ continuous on $[-2,1]\setminus\{0,3\}$ (operations with elementary functions).
$x=3$ is an isolated point $\Rightarrow$ the study of continuity is out of question
The problem of continuity is for the point $x_0=0$.
We shall compute the side limits at $x_0=0$,
as well as the value of the function in the said point.

$$\exists\ l_l(0) = \lim_{\substack{x\to\infty \\ x<0}} f(x) = \lim_{\substack{x\to\infty \\ x<0}} \cos(\pi x) = \cos 0 = 1 \quad (1)$$

$$\exists\ l_r(0) = \lim_{\substack{x\to\infty \\ x>0}} f(x) = \lim_{\substack{x\to 0 \\ x>0}} (1+\sin x) = 1+0=1 \quad (2)$$

$$f(0) = \cos 0 = 1 \quad (3)$$

$(1),(2),(3) \Rightarrow l_l(0)=l_r(0)=f(0) \Rightarrow f$ is continuous at $x_0=0 \Rightarrow$ the domain of continuity for $f$ is $\Delta_c=[-2,1]$

**Exercise 5:** Study, by discussing the parameter $a\in\mathbb{R}$, the continuity of the following functions:

**a)** $f:[1,3]\to\mathbb{R}$,

$$f(x)=\begin{cases}\sqrt{a^2-2ax+x^2} & : x\in[1,2]\\ 3a+2x & : x\in(2,3].\end{cases}$$

**b)** $f:(0,\pi)\to\mathbb{R}$,

$$f(x)=\begin{cases}e^{3x} & : x\in(0,1]\\ a\dfrac{\sin(x-1)}{x^2-5x+4} & : x\in(1,\pi).\end{cases}$$

$a\in\mathbb{R}$

a) $f:[1,3]\to\mathbb{R}$

$$f(x)=\begin{cases}\sqrt{a^2-2ax+x^2}, & x\in[1,2]\\ 3a+2x, & x\in(2,3]\end{cases}$$

$f$ continuous on $[1,3]\setminus\{2\}$ (operations with elementary functions).
The problem of continuity is for the point $x_0=2$.
We shall compute the side limits at $x_0=2$,
as well as the value of the function in the said point.

$$\exists\ l_l(2)=\lim_{\substack{x\to 2 \\ x<2}} f(x) = \lim_{\substack{x\to 2 \\ x<2}} \sqrt{a^2-2ax+x^2} = \sqrt{a^2-4a+4} = \sqrt{(a-2)^2} = |a-2|$$

$$\exists\ l_r(2) = \lim_{\substack{x\to 2 \\ x>2}} f(x) = \lim_{\substack{x\to 2 \\ x>2}} (3a+2x) = 3a+4$$

$$f(2)=\sqrt{a^2-4a+4}=|a-2|$$

$\rightarrow$ if $l_l(2)=l_r(2) \Rightarrow |a-2|=3a+4$
- $a-2=-3a-4 \Rightarrow 4a=-2 \Rightarrow a=-\frac{1}{2}$
- $a-2=3a+4 \Rightarrow 2a=-6 \Rightarrow a=-3$

$\Rightarrow a\in\{-\frac{1}{2},-3\} \Rightarrow f$ is continuous at $x_0=2 \Rightarrow f$ is continuous on $[1,3]$

$\rightarrow$ if $a\notin\{-\frac{1}{2},-3\} \Rightarrow l_l(2)\neq l_r(2) \Rightarrow f$ is discontinous at $x_0=2 \Rightarrow$ the domain of continuity is $\Delta_2=[1,3]\setminus\{2\}$

b) $f:(0,\pi)\to\mathbb{R}$

$$f(x)=\begin{cases} e^{3x}, & x\in(0,1] \\ a\cdot\frac{\sin(x-1)}{x^2-5x+4}, & x\in(1,\pi)\end{cases}$$

$x^2-5x+4=0$

$\Delta=25-16=9$

$x_{1,2}=\frac{5\pm3}{2}$ $\begin{cases} x_1=1 \\ x_2=4\notin(1,\pi)\end{cases}$

$f$ continuous on $(0,\pi)\setminus\{1\}$. (operations with elementary functions).

The problem of continuity is for the point $x_0=1$.

We shall compute the side limits at $x_0=1$,

as well as the value of the function in the said point.

$\exists\, l_l(1)=\lim\limits_{\substack{x\to1\\x<1}} f(x)=\lim\limits_{\substack{x\to1\\x<1}} e^{3x}=e^3$

$\exists\, l_r(1)=\lim\limits_{\substack{x\to1\\x<1}} f(x)=\lim\limits_{\substack{x\to1\\x>1}} a\cdot\frac{\sin(x-1)}{x^2-5x+4}=a\lim\limits_{\substack{x\to1\\x>1}}\frac{\sin(x-1)}{(x-1)(x-4)}=a\lim\limits_{\substack{x\to1\\x>1}}\frac{\sin(x-1)}{(x-1)}\cdot\frac{1}{(x-4)}=a\cdot\frac{1}{1-4}=-\frac{a}{3}$

$f(1)=e^3$

$\rightarrow$ if $-\frac{a}{3}=e^3\Rightarrow a=-3e^3\Rightarrow l_l(1)=l_r(1)=f(1)\Rightarrow f$ continuous at $x_0=1\Rightarrow f$ continuous on $(0,\pi)$

$\rightarrow$ if $a\in\mathbb{R}\setminus\{-3e^3\}\Rightarrow l_l(1)\neq l_r(1)\Rightarrow f$ discontinuous at $x_0=1\Rightarrow$ the domain of continuity for $f$ is $D_c=(0,\pi)\setminus\{1\}$

**Exercise 6:** Let $0<a<b\in\mathbb{R}$ and $f:\mathbb{R}\setminus\{0,1\}\to\mathbb{R}$, defined by:

$$f(x)=\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}},\forall x\in\mathbb{R}\setminus\{0,1\}.$$

- a) Prove that $f$ is a continuous function.
- b) Prove that there exists a continuous function $F:\mathbb{R}\to\mathbb{R}$ such that $F(x)=f(x),\forall x\in\mathbb{R}\setminus\{0,1\}$.
- c) Compute $\lim\limits_{x\to-\infty}F(x)$ and $\lim\limits_{x\to\infty}F(x)$.

$0<a<b\in\mathbb{R}$

$f:\mathbb{R}\setminus\{0,1\}\to\mathbb{R}$

$f(x)=\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}}$

a) Let $x_0$ be a continuity point.

Let $(x_n)\subseteq\mathbb{R}$ with $\lim\limits_{n\to\infty}x_n=x_0$

$f(x_n)=\left(\frac{b^{x_n}-a^{x_n}}{x_n(b-a)}\right)^{\frac{1}{x_n-1}}$

$\lim\limits_{n\to\infty}f(x_n)=\left(\frac{b^{x_0}-a^{x_0}}{x_0(b-a)}\right)^{\frac{1}{x_0-1}}$, $\forall x_0\in\mathbb{R}\setminus\{0,1\}\Rightarrow f$ continuous on $\mathbb{R}\setminus\{0,1\}$

b) $f$ continuous on $\mathbb{R}\setminus\{0,1\}$

We shall compute the limits at $x=0$ and $x=1$

$\lim\limits_{x\to0}\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}}=\lim\limits_{x\to0}\left(\frac{1}{b-a}\right)^{\frac{1}{x-1}}\cdot\left(\frac{b^x-1}{x}-\frac{a^x-1}{x}\right)^{\frac{1}{x-1}}=(b-a)\cdot(\ln b-\ln a)^{-1}=\frac{b-a}{\ln\frac{b}{a}}\Rightarrow$ the limit $\exists$ $\forall a,b\in\mathbb{R},\ 0<a<b$

$\lim\limits_{x\to1}\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}}=\lim\limits_{x\to1}\left(1+\frac{b^x-a^x-x(b-a)}{x(b-a)}\right)^{\frac{1}{x-1}}=\lim\limits_{x\to1}\left(\left(1+\frac{b^x-a^x-x(b-a)}{x(b-a)}\right)^{\frac{x(b-a)}{b^x-a^x-x(b-a)}}\right)^{\frac{b^x-a^x-x(b-a)}{x(x-1)(b-a)}}=$

$=e^{\lim\limits_{x\to1}\frac{b^x-a^x-xb+x\cdot a}{x(x-1)(b-a)}}=e^{\frac{1}{b-a}\lim\limits_{x\to1}\frac{b(b^{x-1}-x)-a(a^{x-1}-x)}{x-1}}=e^{\frac{1}{b-a}\lim\limits_{x\to1}\left(b\cdot\frac{b^{x-1}-x}{x-1}-a\cdot\frac{a^{x-1}-x}{x-1}\right)}=$

$$= e^{\frac{1}{b-a}(b\ln b - a\ln a)}$$

Hence, let $F: \mathbb{R} \to \mathbb{R}$, $F(x) = \begin{cases} f(x), & x \in \mathbb{R}\setminus\{0,1\} \\ \frac{b-a}{\ln b - \ln a}, & x = 0 \\ e^{\frac{\ln b^b - \ln a^a}{b-a}}, & x = 1 \end{cases}$ $\forall\ 0 < a < b \in \mathbb{R}$

c) $\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty}\left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{t\to\infty}\left(\frac{\left(\frac{1}{b}\right)^t - \left(\frac{1}{a}\right)^t}{-t(b-a)}\right)^{\frac{-1}{t+1}} = \lim_{t\to\infty}\left(\frac{\left(\frac{1}{a}\right)^t - \left(\frac{1}{b}\right)^t}{t(b-a)}\right)^{\frac{-1}{t+1}} =$

$x = -t \xrightarrow{x\to-\infty} +\infty$

$$= \lim_{t\to\infty}\left(\frac{\left(\frac{1}{a}\right)^t\left(1-\left(\frac{a}{b}\right)^t\right)}{t(b-a)}\right)^{\frac{-1}{t+1}} = \lim_{t\to\infty}\frac{\left(\frac{1}{a}\right)^{\frac{-t}{t+1}}\left(1-\left(\frac{a}{b}\right)^t\right)^{\frac{-1}{t+1}}}{t^{\frac{-1}{t+1}}(b-a)^{\frac{-1}{t+1}}} =$$

$$= a\cdot\lim_{t\to\infty}\frac{1}{t^{\frac{-1}{t+1}}} = a\lim_{t\to\infty} t^{t+1} = a\cdot e^{\lim_{u\to\infty}(t+1)\ln t} = \infty$$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty}\left(\frac{b^x\left(1-\left(\frac{a}{b}\right)^x\right)}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty}\frac{b^{\frac{x}{x-1}}\cdot\left(1-\left(\frac{a}{b}\right)^x\right)^{\frac{1}{x-1}}}{x^{\frac{1}{x-1}}\cdot(b-a)^{\frac{1}{x-1}}} = b\cdot\lim_{x\to\infty}\frac{1}{x^{\frac{1}{x-1}}} =$$

$$= b\cdot e^{\lim_{x\to\infty}\frac{-\ln x}{x-1}} = b\cdot e^0 = b, \quad b\in\mathbb{R},\ 0 > a > b$$

$\lim_{x\to\infty}\frac{-\ln x}{x-1} = \left(\frac{-\infty}{\infty}\right)$ L'Hospital $= -\lim_{x\to\infty}\frac{\frac{1}{x}}{1} = -\lim_{x\to\infty}\frac{1}{x} = 0$