Calculus on $\mathbb{R}$ Lect. dr. ANCA GRAD
Homework 11 **2021**

Improper Integrals

Exercise 1: Study (with the help of the definiton and by using the Leibniz-Newton formula) the imporper integrability of the following functions, and, in case of convergence, determine the value of the improper integral.

For the examples within this exercise, the following steps should be followed:

Step 1: Compute the nondeterminate integral of f.

Pasul 2: Choose an antiderivative of f (usually the function having the constant $c = 0$ is chosen).

Pasul 3: Compute the limit towards a from the antiderivative. If the limit exists and is finite, than we are in a convergence case.

a)

$$f : (-1, 1) \to \mathbb{R} \quad f(x) = \frac{1}{\sqrt{1 - x^2}}$$

b)

$$f : [1, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{x(x + 1)}.$$

c)

$$f : (0, 1] \to \mathbb{R} \quad f(x) = \ln x.$$

d)

$$f : [0, 1) \to \mathbb{R} \quad f(x) = \frac{\arcsin x}{\sqrt{1 - x^2}}.$$

e)

$$f : (0, 1] \to \mathbb{R} \quad f(x) = \frac{\ln x}{\sqrt{x}}.$$

f)

$$f : [e, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{x \cdot (\ln x)^3}.$$

g)

$$f : \left(\frac{1 + \sqrt{3}}{2}, 2\right] \to \mathbb{R} \quad f(x) = \frac{1}{x\sqrt{2x^2 - 2x - 1}}.$$

h)

$$f : [0, \infty) \to \mathbb{R}, f(x) = \frac{\pi}{2} - \operatorname{arctg} x.$$

i)

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{1+x^2},$$

j)

$$f\colon \left(\frac{1}{3}, 3\right] \to \mathbb{R} \quad f(x) = \frac{1}{\sqrt[3]{3x-1}}$$

k)

$$f\colon [1,\infty) \to \mathbb{R} \quad f(x) = \frac{x}{(1+x^2)^2}.$$

Exercise 2: Study the improper integrability (with the help of the comparison criteria) for the following functions. (Notice that in case of convergence, this criteria does not provide us the directly with the value of the improper integral, since the criteria mainly assures the nature).

For the examples within this exercise, the following steps should be followed:

Step 1: Determine the problematic boundary points of the domain,

Step 2: Compute

$$\lim_{x \uparrow b} (b-x)^p f(x)$$

and set p such that the value of the limit to belong to $\in (0,\infty)$, for $f : [a,b) \to [0,\infty)$.

If the domain is open at în a, then we compute

$$L = \lim_{x \downarrow a} (x-a)^p f(x)$$

and if the domain is upper unbounde and we compute (deci $[a,\infty)$)

$$L = \lim_{x \to \infty} x^p f(x).$$

Step 3: For the first two cases, if $p < 1$ we have convergent improper integrability, while in the third one, improper integrability is convergent if $p > 1$.

a)

$$f : [1,\infty) \to \mathbb{R} \quad f(x) = \frac{1}{s\sqrt{1+x^2}}$$

b)

$$f : [0, \frac{\pi}{2}) \to \mathbb{R} \quad f(x) = \frac{1}{\cos x}$$

c)

$$f : (0,\infty) \to \mathbb{R} \quad f(x) = \left(\frac{arctgx}{x}\right)^2$$

d)

$$f : (1,\infty) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$$

e)

$$f : (0,1) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{1-x^2}} \right)^2$$

a)

$$f:(-1,1)\to\mathbb{R} \quad f(x)=\frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Let $F:(-1;1)\to\mathbb{R}$, $F(x)=\arcsin x$ be an antiderivative of f

$$\lim_{\substack{x\to -1\\ x>-1}} F(x) = \lim_{\substack{x\to -1\\ x>-1}} \arcsin x = \arcsin(-1) = -\frac{\pi}{2}\in\mathbb{R} \to \text{we are in a convergence case}$$

$$\lim_{\substack{x\to -1\\ x<-1}} F(x) = \arcsin 1 = \frac{\pi}{2}\in\mathbb{R} \to \text{we are in a convergence case}$$

b)

$$f:[1,\infty)\to\mathbb{R} \quad f(x)=\frac{1}{x(x+1)}.$$

$$\int \frac{1}{x(x+1)}\,dx = \int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \int\frac{1}{x}\,dx - \int\frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C =$$

$$= \ln\frac{x}{x+1} + C$$

Let $F:[1,+\infty)\to\mathbb{R}$, $F(x)=\ln\frac{x}{x+1}$ be an antiderivative of f

$$\lim_{a\to\infty}\int_1^a f(x)\,dx = \lim_{a\to\infty}\left(F(a)-F(1)\right) = \lim_{a\to\infty}\left(\ln\frac{a}{a+1} - \ln\frac{1}{2}\right) = \ln 2 + \lim_{a\to\infty}\ln\frac{a}{a+1} =$$

$$= \ln 2 + \ln 1 = \ln 2 \in\mathbb{R} \to \text{convergence case}$$

c)

$$f:(0,1]\to\mathbb{R} \quad f(x)=\ln x.$$

$$\int \ln x\,dx = x\cdot\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx = x\ln x - x + C$$

Let $F:(0,1]\to\mathbb{R}$, $F(x)=x\ln x - x$ be an antiderivative of f

$$\lim_{\substack{a\to 0\\ a>0}}\int_a^1 f(x)\,dx = \lim_{\substack{a\to 0\\ a>0}}\left(F(1)-F(a)\right) = \lim_{\substack{a\to 0\\ a>0}}\left(-1-(a\ln a - a)\right) = -1-\lim_{\substack{a\to 0\\ a>0}}(a\ln a - a) = -1-\lim_{\substack{a\to 0\\ a>0}} a\ln a = -1-\lim_{\substack{a\to 0\\ a>0}}\frac{\ln a}{\frac{1}{a}} =$$

$$= -1-\left(\frac{\infty}{\infty}\right) l'H,\ -1-\lim_{\substack{a\to 0\\ a>0}}\frac{\frac{1}{a}}{-\frac{1}{a}} = -1+\lim_{\substack{a\to 0\\ a>0}} a = -1\in\mathbb{R}$$

d)

$$f:[0,1)\to\mathbb{R} \quad f(x)=\frac{\arcsin x}{\sqrt{1-x^2}}.$$

$$\int\frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{\arcsin^2 x}{2} + C$$

Let $F:[0,1)\to\mathbb{R}$, $F(x)=\frac{\arcsin^2 x}{2}$ be an antiderivative of f

$$\lim_{\substack{a\to 1\\ a<1}}\int_0^a f(x)\,dx = \lim_{\substack{a\to 1\\ a<1}}\left(F(a)-F(0)\right) = \lim_{\substack{a\to 1\\ a<1}}\frac{\arcsin^2 a}{2} = \frac{\arcsin^2 1}{2} = \frac{\pi^2}{8}\in\mathbb{R} \Rightarrow \text{convergence case}$$

e)

$$f:(0,1]\to\mathbb{R}\quad f(x)=\frac{\ln x}{\sqrt{x}}.$$

$$\int\frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 2\int\frac{\sqrt{x}}{x}\,dx = 2\sqrt{x}\ln x - 2\int\frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

Let $F:(0,1)\to\mathbb{R}$ be an antiderivative of f, $F(x)=2\sqrt{x}(\ln x-2)$

$$\lim_{\substack{a\to0\\a>0}}\int_a^1 f(x)\,dx = \lim_{\substack{a\to0\\a>0}}(F(1)-F(a)) = \lim_{\substack{a\to0\\a>0}}\left(-4-2\sqrt{a}(\ln a-2)\right) = -4-2\lim_{\substack{a\to0\\a>0}}(\sqrt{a}\cdot\ln a-2) =$$

$$= -4-2\lim_{\substack{a\to0\\a>0}}\sqrt{a}\cdot\ln a = -4-2\lim_{\substack{a\to0\\a>0}}\frac{\ln a}{\frac{1}{\sqrt{a}}}\ \text{l'H.} = -4-2\lim_{\substack{a\to0\\a>0}}\frac{\cancel{\frac{1}{a}}}{-\frac{1}{\cancel{a}}\cdot\frac{1}{2\sqrt{a}}} = -4+2\lim_{\substack{a\to0\\a>0}}2\sqrt{a} = -4+0\in\mathbb{R}$$

$\Rightarrow$ convergence case

f)

$$f:[e,\infty)\to\mathbb{R}\quad f(x)=\frac{1}{x\cdot(\ln x)^3}.$$

$$\int\frac{1}{(\ln x)^3}\cdot\frac{1}{x} = \frac{(\ln x)^{-2}}{-2}+c = \frac{-1}{2\ln^2 x}+c$$

Let $F:[e,+\infty)\to\mathbb{R}$, $F(x)=\frac{1}{2\ln^2 x}$ be an antiderivative of f

$$\lim_{a\to+\infty}\int_e^a f(x)\,dx = \lim_{a\to+\infty}(F(a)-F(e)) = \lim_{a\to+\infty}\left(-\frac{1}{2\ln^2 a}+\frac{1}{2}\right) = \frac{1}{2}-\lim_{a\to+\infty}\frac{1}{2\ln^2 a} = \frac{1}{2}-0=\frac{1}{2}\in\mathbb{R}\Rightarrow \text{convergent case}$$

g)

$$f:\left(\frac{1+\sqrt{3}}{2},2\right]\to\mathbb{R}\quad f(x)=\frac{1}{x\sqrt{2x^2-2x-1}}.$$

$$\mathcal{I}=\int f(x)\,dx = \int\frac{1}{x\sqrt{2x^2-2x-1}}\,dx = \int\frac{1}{x^2\sqrt{2-\frac{2}{x}-\frac{1}{x^2}}}\,dx$$

$$\left.\begin{aligned}\frac{1}{x}&=t\\ -\frac{1}{x^2}\,dx&=dt\end{aligned}\right\}\Rightarrow \mathcal{I}'=-\int\frac{dt}{\sqrt{2-2t-t^2}} = -\int\frac{dt}{\sqrt{2-(t^2+2t+1-1)}} = -\int\frac{dt}{\sqrt{3-(t+1)^2}} =$$

$$= \arcsin\frac{t+1}{\sqrt{3}} \Rightarrow \mathcal{I}=-\arcsin\frac{\frac{1}{x}+1}{\sqrt{3}}$$

$$\mathcal{I}=-\arcsin\frac{1+x}{x\sqrt{3}}+C$$

Let $F:\left(\frac{1+\sqrt{3}}{2},2\right]\to\mathbb{R}$, $F(x)=-\arcsin\frac{1+x}{x\sqrt{3}}$ be an antiderivative of f

$$\lim_{\substack{a\to\frac{1+\sqrt{3}}{2}\\a>\frac{1+\sqrt{3}}{2}}}\int_a^2 f(x)\,dx = \lim_{\substack{a\to\frac{1+\sqrt{3}}{2}\\a>\frac{1+\sqrt{3}}{2}}}(F(2)-F(a)) = \lim_{\substack{a\to\frac{1+\sqrt{3}}{2}\\a>\frac{1+\sqrt{3}}{2}}}\left(-\arcsin\frac{\sqrt{3}}{2}+\arcsin\frac{1+a}{a\sqrt{3}}\right) = -\frac{\pi}{2}+\arcsin\frac{1+\frac{1+\sqrt{3}}{2}}{\sqrt{3}+3} =$$

$$= -\frac{\pi}{2}+\arcsin\frac{\cancel{3+\sqrt{3}}}{\cancel{\sqrt{3}+3}} = -\frac{\pi}{2}+\frac{\pi}{2} = 0$$

h)

$$f : [0, \infty) \to \mathbb{R}, f(x) = \frac{\pi}{2} - \text{arctg}x.$$

$$\int \left(\frac{\pi}{2} - \text{arctg}x\right) dx = \frac{\pi}{2}\cdot x - \int \text{arctg}x \, dx = \frac{\pi}{2}\cdot x - x\,\text{arctg}x + \int \frac{x}{1+x^2} dx = \frac{\pi}{2}\cdot x - x\,\text{arctg}\,x + \frac{1}{2}\ln(x^2+1) + C$$

Let $F: [0, +\infty) \to \mathbb{R}$, $F(x) = \frac{\pi x}{2} - x\,\text{arctg}x + \frac{1}{2}\ln(x^2+1)$ be an antiderivative of f

$$\lim_{a\to+\infty} \int_0^a f(x) = \lim_{a\to\infty} (F(a) - F(0)) = \lim_{a\to\infty} \left(\frac{\pi a}{2} - \text{arctg}\,a + \frac{1}{2}\ln(a^2+1)\right) =$$

$$= \lim_{a\to\infty} \left(a \cdot \frac{\frac{\pi}{2} - \text{arctg}\,a}{\text{tg}\left(\frac{\pi}{2} - \text{arctg}\,a\right)} \cdot \text{tg}\left(\frac{\pi}{2} - \text{arctg}\,a\right) + \frac{1}{2}\ln(a^2+1)\right) =$$

$$= \lim_{a\to+\infty} \left(\frac{\frac{\pi}{2} - \text{arctg}\,a}{\text{tg}\left(\frac{\pi}{2} - \text{arctg}\,a\right)} \cdot \frac{\cancel{a}}{\cancel{a}} + \frac{1}{2}\ln(a^2+1)\right) = 1 + \infty = +\infty \Rightarrow \text{divergence case}$$

i)

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{1+x^2},$$

$$\int \frac{1}{1+x^2} dx = \text{arctg}x + C$$

Let $F: \mathbb{R} \to \mathbb{R}$, $F(x) = \text{arctg}x$ be an antiderivative of f

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \text{arctg}x = -\frac{\pi}{2}$$

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \text{arctg}x = \frac{\pi}{2}$$

j)

$$f: \left(\frac{1}{3}, 3\right] \to \mathbb{R} \quad f(x) = \frac{1}{\sqrt[3]{3x-1}}$$

$$I = \int \frac{1}{\sqrt[3]{3x-1}} dx$$

$$\left.\begin{array}{l} 3x-1 = t^3 \\ 3\,dx = 3t^2\,dt \end{array}\right] \Rightarrow I' = \int \frac{t^2 dt}{t} = \frac{t^2}{2} \Rightarrow I = \frac{\sqrt[3]{(3x-1)^2}}{2} + C$$

Let $F: \left(\frac{1}{3}, 3\right] \to \mathbb{R}$, $F(x) = \frac{\sqrt[3]{(3x-1)^2}}{2}$ be an antiderivative of f

$$\lim_{\substack{a\to\frac{1}{3}\\ a>\frac{1}{3}}} \int_a^3 f(x) = \lim_{\substack{a\to\frac{1}{3}\\ a>\frac{1}{3}}} (F(3) - F(a)) = \lim_{\substack{a\to\frac{1}{3}\\ a>\frac{1}{3}}} \left(2 - \frac{\sqrt[3]{(3a-1)^2}}{2}\right) = 2 - \frac{\sqrt[3]{(3\cdot\frac{1}{3}-1)^2}}{2} = 2 - 0 = 2 \in \mathbb{R} \Rightarrow \text{convergence case}$$

k)

$$f: [1, \infty) \to \mathbb{R} \quad f(x) = \frac{x}{(1+x^2)^2}.$$

$$\int \frac{x}{(1+x^2)^2} dx = \frac{1}{2}\int \frac{2x}{(1+x^2)^2} dx = \frac{1}{2} \cdot \frac{-1}{1+x^2} + C$$

Let $F: [1, +\infty) \to \mathbb{R}$, $F(x) = \frac{-1}{2(1+x^2)}$ be an antiderivative of f

$$\lim_{a\to+\infty} \int_1^a f(x)\,dx = \lim_{a\to+\infty} (F(a) - F(1)) = \lim_{a\to+\infty} \left(\frac{-1}{2(1+a^2)} + \frac{1}{4}\right) = \frac{1}{4} - 0 = \frac{1}{4} \in \mathbb{R} \Rightarrow \text{convergence case}$$

a)

$$f:[1,\infty)\to\mathbb{R} \quad f(x)=\frac{1}{x\sqrt{1+x^2}}$$

$$\lim_{x\to+\infty} x^p\cdot\frac{1}{x\sqrt{1+x^2}} = \lim_{x\to+\infty}\frac{x^p}{x^2\sqrt{1+\frac{1}{x^2}}} \overset{\text{for } p=2}{=\!=} 1\in(0,+\infty) \Rightarrow \text{convergent improper integrability}$$

b)

$$f:[0,\tfrac{\pi}{2})\to\mathbb{R} \quad f(x)=\frac{1}{\cos x}$$

Problematic boundary points: $\frac{\pi}{2}$

$$\left.\begin{array}{l}\lim\limits_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} \left(\frac{\pi}{2}-x\right)^p\cdot\frac{1}{\cos x} = \lim\limits_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}\frac{\frac{\pi}{2}-x}{\sin\left(\frac{\pi}{2}-x\right)} = 1 \\ \qquad\qquad (\text{for } p=1)\end{array}\right\} \Rightarrow \text{divergent improper integrability}$$

c)

$$f:(0,\infty)\to\mathbb{R} \quad f(x)=\left(\frac{arctgx}{x}\right)^2$$

$$\lim_{\substack{x\to 0\\ x>0}}(x-0)^p\cdot\frac{\operatorname{arctg}^2 x}{x^2} = \lim_{\substack{x\to 0\\ x>0}} x^p\cdot\frac{\operatorname{arctg}^2 x}{x^2} \overset{\text{for } p=0}{=\!=} 1\in(0,+\infty) \Rightarrow \text{convergent improper integrability}$$

$$\lim_{x\to+\infty} x^p\cdot\frac{\operatorname{arctg}^2 x}{x^2} \overset{\text{for } p=2}{=\!=} \frac{\pi}{2}^2\in(0,+\infty) \rightarrow \text{convergent improper integrability}$$

(not sure...)

d)

$$f:(1,\infty)\to\mathbb{R} \quad f(x)=\left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$$

$$\lim_{\substack{x\to 1\\ x>1}}(x-1)^p\cdot\frac{\ln^2 x}{x^2(x^2-1)} = \lim_{\substack{x\to 1\\ x>1}}(x-1)^p\cdot\left(\frac{\ln(1+x-1)}{x-1}\right)^2\cdot(x-1)^2\cdot\frac{1}{x^2(x-1)(x+1)} =$$

$$= \lim_{\substack{x\to 1\\ x>1}}(x-1)^{p-1}\cdot\left(\frac{\ln(1+x-1)}{x-1}\right)^2\cdot\frac{1}{x^2(x+1)} =$$

$$= \lim_{\substack{x\to 1\\ x>1}}(x-1)^{p-1}\cdot\left(\frac{\ln(1+x-1)}{x-1}\right)^2\cdot\frac{1}{x^2(x+1)} \overset{\text{for } p=1}{=\!=} \frac{1}{2}\in(0,+\infty) \Rightarrow \text{divergent improper integrability}$$

$$\lim_{x\to+\infty} x^p\cdot\frac{\ln^2 x}{x^2(x^2-1)} = \lim_{x\to+\infty} x^{p-2}\cdot\frac{\ln^2 x}{x^2-1} \overset{\text{for } p=4}{=\!=} \frac{1}{2}\in(0,+\infty) \Rightarrow \text{convergent improper integrability}$$

e)

$$f:(0,1)\to\mathbb{R} \quad f(x)=\left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2$$

$$\lim_{\substack{x\to 1\\ x<1}}(1-x)^p\cdot\frac{\ln^2 x}{x^2(x-1)(x+1)} = \frac{1}{2}\lim_{\substack{x\to 1\\ x<1}}(1-x)^p\cdot\frac{\ln^2(1+x-1)}{(x-1)^2}\cdot\frac{(x-1)^{\cancel{2}}}{x^2\cancel{(x-1)}} = \frac{1}{2}\lim_{\substack{x\to 1\\ x<1}}(1-x)^p\cdot(x-1) = \frac{1}{2}\lim_{\substack{x\to 1\\ x<1}}(-1)^p\cdot(x-1)^{p+1}\rightarrow$$