Course 5 March 23, 2021

$f : A \subseteq \mathbb{R} \to \mathbb{R}^m \qquad f'(a) \in \mathbb{R}^m \qquad df(a) \in L(\mathbb{R}, \mathbb{R}^m)$

3.6. Theorem Let $A \subseteq \mathbb{R}$, let $a \in \operatorname{int} A$, and let $f : A \to \mathbb{R}^m$. Then the following assertions are true:

1° If f is differentiable at $a \Rightarrow f$ is Fréchet diff. at a and

(1) $\quad \forall h \in \mathbb{R} \;:\; df(a)(h) = h \cdot f'(a)$

2° If f is Fréchet diff. at $a \Rightarrow f$ is diff. at a and (1) holds.

Proof. 1° Assume that is diff at $a \Rightarrow \exists \lim\limits_{x \to a} \frac{1}{x-a}\left[f(x) - f(a)\right] = f'(a)$ (*)

Consider $\varphi : \mathbb{R} \to \mathbb{R}^m$, $\varphi(h) := h f'(a) \Rightarrow \varphi \in L(\mathbb{R}, \mathbb{R}^m)$

$$\lim_{x \to a} \frac{1}{|x-a|}\left[f(x) - f(a) - \varphi(x-a)\right] = \lim_{x \to a} \frac{1}{|x-a|}\left[f(x) - f(a) - (x-a) f'(a)\right] =$$

$$= \lim_{x \to a} \underbrace{\frac{x-a}{|x-a|}}_{\pm 1}\left[\frac{1}{x-a}\left(f(x) - f(a)\right) - f'(a)\right] = 0_m$$

$\stackrel{\text{def}}{\Rightarrow} f$ is Fréchet diff at a and $df(a) = \varphi \Rightarrow df(a)(h) = \varphi(h) = h f'(a) \;\forall h \in \mathbb{R}$

2° Assume that f is Fréchet diff at a $\Rightarrow$ $df(a) \in L(\mathbb{R}, \mathbb{R}^m)$ $\Rightarrow$

$\Rightarrow \exists v \in \mathbb{R}^m$ s.t. $df(a)(h) = hv \quad \forall h \in \mathbb{R}$

Def $\Rightarrow 0_m = \lim\limits_{x\to a} \frac{1}{|x-a|}\left[f(x) - f(a) - df(a)(x-a)\right] = \lim\limits_{x\to a} \frac{1}{|x-a|}\left[f(x) - f(a) - (x-a)v\right]$

$$= \lim_{x\to a} \frac{x-a}{|x-a|}\left[\frac{1}{x-a}\left(f(x)-f(a)\right) - v\right]$$

$$\Rightarrow \lim_{x\to a} \left\| \frac{x-a}{|x-a|}\left[\frac{1}{x-a}\left(f(x)-f(a)\right) - v\right]\right\| = 0$$

$$\Rightarrow \lim_{x\to a} \left\| \frac{1}{x-a}\left(f(x)-f(a)\right) - v\right\| = 0$$

$\Rightarrow \lim\limits_{x\to a} \frac{1}{x-a}\left(f(x)-f(a)\right) = v$ $\Rightarrow$ f is diff at a and $f'(a) = v$

$\Rightarrow df(a)(h) = hv = hf'(a)$.

4. Directional derivatives

4.1. Definition Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$, $v \in \mathbb{R}^n$.

One says that f has a directional derivative at a in the direction v if $\exists \lim\limits_{t\to 0} \frac{1}{t}\left[f(a+tv) - f(a)\right] =: b \in \mathbb{R}^m$

4.2. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$. If f is differentiable at a $\Rightarrow$ f has a directional derivative in every direction $v \in \mathbb{R}^n$ and $f'(a;v) = df(a)(v)$.

Proof Let $v \in \mathbb{R}^m$

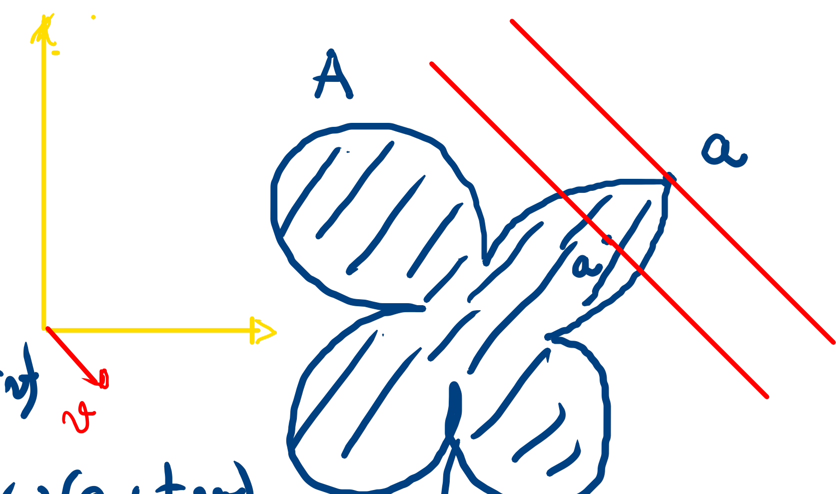
f is diff. at a $\overset{P3.3}{\Longrightarrow}$ $\exists\, \omega : A \to \mathbb{R}^m$ s.t. $\lim\limits_{x \to a} \omega(x) = 0_m$ and

$$\forall x \in A : \quad f(x) = f(a) + df(a)(x-a) + \|x-a\| \cdot \omega(x) \qquad (*)$$

Since $a \in \operatorname{int} A$ $\Rightarrow$ $\exists\, \delta > 0$ s.t. $\forall t \in (-\delta, \delta) :$ $a + tv \in A$

Replacing x by $a+tv$ in $(*)$

$\Rightarrow \forall t \in (-\delta, \delta) :$



$$\frac{1}{t} \mid f(a+tv) = f(a) + df(a)(tv) + \|tv\|\, \omega(a+tv)$$
$$= f(a) + t\, df(a)(v) + |t| \cdot \|v\| \cdot \omega(a+tv)$$

$$\Rightarrow \frac{1}{t}\left[f(a+tv) - f(a)\right] = df(a)(v) + \frac{|t|}{t} \cdot \|v\| \cdot \omega(a+tv)$$

$\forall t \in (-\delta, \delta) \setminus \{0\}$

Letting $t \to 0$ $\Rightarrow$ $\lim\limits_{t \to 0} \frac{1}{t}\left[f(a+tv) - f(a)\right] = df(a)(v)$

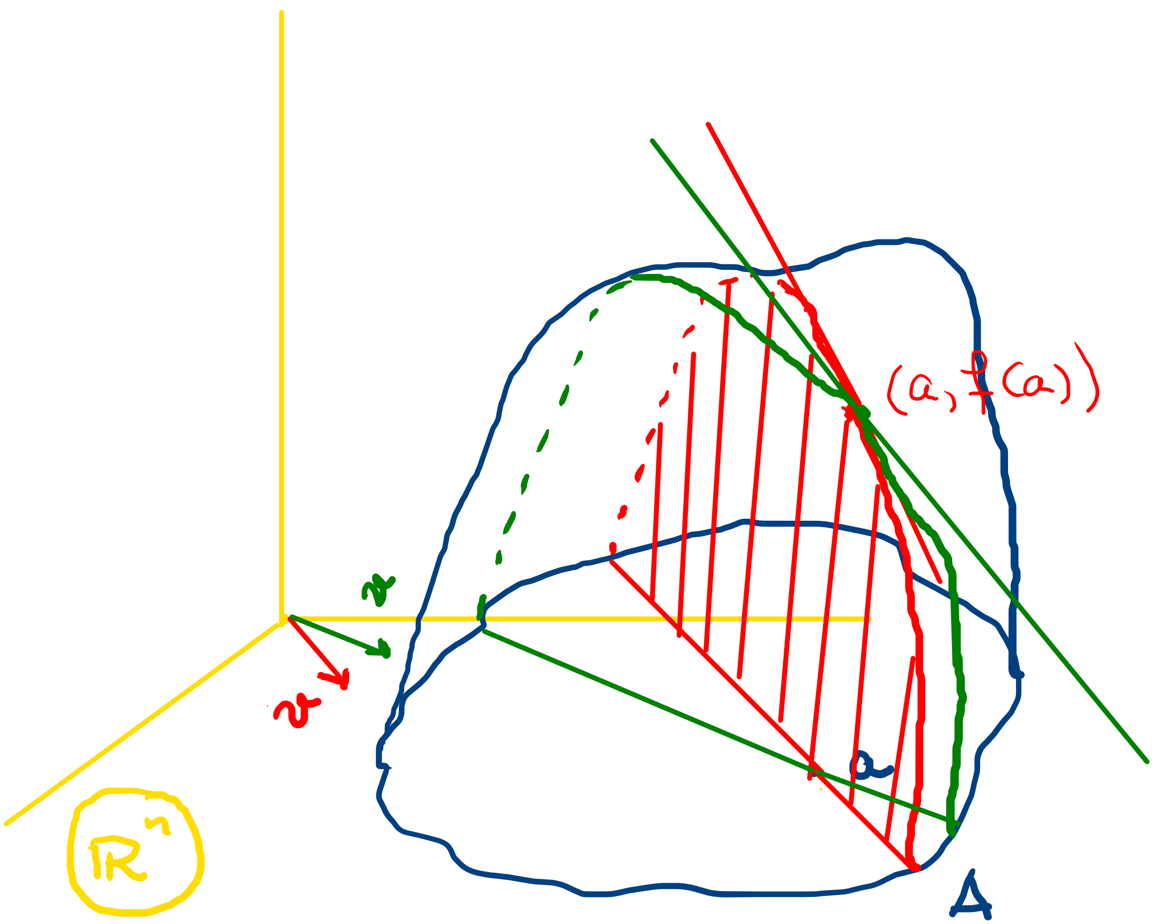
$\Rightarrow \exists\, f'(a;v) = df(a)(v)$.

The vector b is called the directional derivative of f at a in the direction v and it is denoted by $f'(a;v)$ or by $f'_v(a)$

$m=1 \quad \|v\|=1$

$f'(a;v)$ = instantaneous rate of change of f at the point a in the direction v

$(x,y) \in A \mapsto z = f(x,y)$



5. Partial derivatives

5.1. Definition. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$, $j \in \{1, \dots, n\}$.
If f has a directional derivative at a in the direction e_j, then f is said to be partially differentiable at a wrt the variable x_j.
The directional derivative $f'(a; e_j)$ is called the partial derivative of f at a wrt the variable x_j and it is denoted by

$$\frac{\partial f}{\partial x_j}(a) \quad (\text{Jacobi's notation}) \quad \text{or by} \quad f'_{x_j}(a).$$

If f is partially diff. wrt all variables $x_1, \dots, x_n$, then we say that f is partially diff. at a

$$\frac{\partial f}{\partial x_j}(a) = f'(a; e_j) = \lim_{t \to 0} \frac{1}{t}\left[f(a + te_j) - f(a)\right]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[f(a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n)\right]$$

$$x_j := a_j + t$$

$$\frac{\partial f}{\partial x_j}(a) = \lim_{x_j \to a_j} \frac{1}{x_j - a_j}\left[f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n)\right]$$

<u>5.2. Definition</u> (Jacobi matrix) Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$ s.t. $f = (f_1, \dots, f_m)$ f is partially diff. at a. Then we define

$$J(f)(a) = J_f(a) := \begin{array}{c} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \dots & \frac{\partial f_2}{\partial x_n}(a) \\ \text{--} & \text{---} & & \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \dots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \\ \begin{array}{cccc} \frac{\partial f}{\partial x_1}(a) & \frac{\partial f}{\partial x_2}(a) & & \frac{\partial f}{\partial x_n}(a) \end{array} \end{array} \in M_{m \times n}(\mathbb{R})$$

$\hookrightarrow$ the Jacobi matrix of f at a

f is partially diff. wrto x_j at $a \iff f_1, \dots, f_m$ are partially diff wrto x_j at a

$$\frac{\partial f}{\partial x_j}(a) = \left(\frac{\partial f_1}{\partial x_j}(a), \dots, \frac{\partial f_m}{\partial x_j}(a)\right)$$

$m = 1 \implies J(f)(a) = \left(\frac{\partial f}{\partial x_1}(a) \quad \frac{\partial f}{\partial x_2}(a) \quad \dots \quad \frac{\partial f}{\partial x_n}(a) \right) \in \mathcal{M}_{1\times n}(\mathbb{R})$

Consider the vector $\nabla f(a) := \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right) \in \mathbb{R}^n$

∇ = nabla

↳ the gradient of f at a

$$\boxed{J(f)(a) = \nabla f(a)^t}$$

The geometrical meaning of partial derivatives:

$f : A \to \mathbb{R}$ $\quad a \in A \cap A'$, f is differentiable at a

$A \subseteq \mathbb{R}$

The tangent line to G_f at $(a, f(a))$ has the equation

$$y - f(a) = f'(a)(x-a)$$

$$f'(a) \cdot x - y - af'(a) + f(a) = 0$$

$\implies n = (f'(a), -1)$ is a normal vector to G_f

$\alpha x + \beta y + \gamma = 0$

$\implies n = (\alpha, \beta) =$

$= \alpha \vec{i} + \beta \vec{j}$

is a normal vector to the tangent line

$z = f(x,y)$ is the equation of a surface $\qquad f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$

$(x_0, y_0) \in \operatorname{int} A$, $\quad z_0 = f(x_0, y_0)$

$\Rightarrow \quad n = \left(f'_x(x_0,y_0), f'_y(x_0,y_0), -1 \right)$ is a normal vector to the surface (i.e. a normal vector to the tangent plane of the surface at (x_0, y_0, z_0))

Take for instance an elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} = f(x,y)$$

The tangent plane of the paraboloid at (x_0, y_0, z_0)

$$\frac{z+z_0}{2} = \frac{x x_0}{a^2} + \frac{y y_0}{b^2} \quad \Big| \cdot 2$$

$$\frac{2x_0}{a^2} \cdot x + \frac{2y_0}{b^2} \cdot y - z - z_0 = 0$$

$$x \to \frac{x+x_0}{2} \qquad x^2 \to x x_0$$

$$y \to \frac{y+y_0}{2} \qquad y^2 \to y y_0$$

$$z \to \frac{z+z_0}{2} \qquad z^2 \to z z_0$$

$$\Rightarrow \quad n = \Big(\underbrace{\frac{2x_0}{a^2}}_{f'_x(x_0,y_0)}, \underbrace{\frac{2y_0}{b^2}}_{f'_y(x_0,y_0)}, -1 \Big)$$ is the normal to the tangent plane

$\pi: \alpha x + \beta y + \gamma z + \delta = 0$

$n = (\alpha, \beta, \gamma)$ is the normal to π

$F(x,y)=0$ is the implicit equation of a curve $\qquad F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$F(x_0,y_0)=0 \implies (x_0,y_0)\in$ curve

$\nabla F(x_0,y_0)=\left(\frac{\partial F}{\partial x}(x_0,y_0),\ \frac{\partial F}{\partial y}(x_0,y_0)\right)$ is a normal vector the curve at (x_0,y_0) (i.e. a normal vector to the tangent of the curve at (x_0,y_0))

(x_0,y_0) belonging to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The tangent line to the ellipse at (x_0,y_0)

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1 \iff \frac{x_0}{a^2}x+\frac{y_0}{b^2}y-1=0$$

$\Rightarrow \quad n=\left(\frac{x_0}{a^2},\frac{y_0}{b^2}\right)$ is the normal vector of the ellipse at (x_0,y_0)

$$\nabla F(x_0,y_0)=\left(\frac{2x_0}{a^2},\ \frac{2y_0}{b^2}\right)=2n$$

- $F(x,y,z)=0$ the implicit equation of a surface $\qquad F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1$

$\nabla F(x_0,y_0,z_0)$ - is a normal vector of the surface at (x_0,y_0,z_0)

$F(x_0,y_0,z_0)=0$

5.3. Theorem (Fréchet differential vs partial derivatives) Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$ s.t. f is diff. at a. Then:

1° f is partially diff. at a, and $[df(a)] = J(f)(a)$

2° $\forall h = (h_1, \dots, h_n) \in \mathbb{R}^n$: $df(a)(h) = h_1 \dfrac{\partial f}{\partial x_1}(a) + \dots + h_n \dfrac{\partial f}{\partial x_n}(a)$

Proof 1° Since f is differentiable at $a \Rightarrow f$ has directional derivatives in every direction at a, and $f'(a;v) = df(a)(v)$ $\forall v \in \mathbb{R}^n$

$\Rightarrow f$ is partially diff. at a and $\dfrac{\partial f}{\partial x_j}(a) = f'(a;e_j) = df(a)(e_j)$, $j = \overline{1,n}$

But $\dfrac{\partial f}{\partial x_j}(a) = j^{th}$ column in $J(f)(a)$

$df(a)(e_j) = j^{th}$ column in $[df(a)]$

$\Downarrow$

$[df(a)] = J(f)(a)$

2° If $h = (h_1, \dots, h_n) \in \mathbb{R}^n \Rightarrow h = h_1 e_1 + \dots + h_n e_n$

$\Rightarrow df(a)(h) = h_1\, df(a)(e_1) + \dots + h_n\, df(a)(e_n)$

$= h_1 \dfrac{\partial f}{\partial x_1}(a) + \dots + h_n \dfrac{\partial f}{\partial x_n}(a).$

5.4 Corollary Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}$ s.t. f is diff. at a.
Then $\forall h \in \mathbb{R}^n$: $\boxed{df(a)(h) = \langle h, \nabla f(a) \rangle}$

Remark. $f'(a;v) = df(a)(v) = \langle v, \nabla f(a) \rangle$

$\|v\| = 1$ $\quad |f'(a;v)| = |\langle v, \nabla f(a) \rangle| \leq \|v\| \cdot \|\nabla f(a)\| = \|\nabla f(a)\|$

Cauchy-Schwarz

$\Rightarrow -\|\nabla f(a)\| \leq f'(a;v) \leq \|\nabla f(a)\|$

we have equality $\Leftrightarrow$ $v = \dfrac{\nabla f(a)}{\|\nabla f(a)\|}$

we have equality $\Leftrightarrow$ $v = -\dfrac{\nabla f(a)}{\|\nabla f(a)\|}$

Conclusion : $\nabla f(a)$ gives the direction of steepest ascent of f
$-\nabla f(a)$ —— " —— steepest descent of f

Example 1. The elevation of a portion of a hill is given by $z = f(x,y) =$
$= 600 - 2x^2 - 3y$

Consider the point P lying above $(1,2)$ $\Rightarrow P(1,2,f(1,2))$

In which direction will water fall from P?

Answer: $-\nabla f(1,2) = (4,3)$

$$\nabla f(x,y) = (-4x, -3)$$

Example. Captain Gigi is in trouble near the sunny side of Mercury.
The temperature of ship's hull at (x,y,z) is given by

$$T = T(x,y,z) = \frac{T_0}{1+x^2+2y^2+3z^2}$$

In what direction should he proceed in order to decrease the temperature most rapidly?

$(x_0, y_0, z_0) \to$ the direction of steepest descent of temperature is $-\nabla T(x_0,y_0,z_0) = (\ \dots\ )$

Summarizing

f is differentiable at a $\overset{\not\Leftarrow}{\Rightarrow}$ f has directional derivatives in every direction at a $\Rightarrow$ f is partially diff. at a

$\Downarrow$

f is continuous at a

(f has directional derivatives in every direction at a $\not\Rightarrow$ f is continuous at a)

5.4. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$. Suppose that $\exists\, r > 0$ s.t. $B(a,r) \subseteq A$ and

- f is partially diff. at each point of $B(a,r)$
- $\forall j = \overline{1,n}$ the function $\forall x \in B(a,r) \longmapsto \dfrac{\partial f}{\partial x_j}(x) \in \mathbb{R}^m$ is continuous at a.

Then f is differentiable at a.

5.5. Corollary If $A \subseteq \mathbb{R}^n$ is an open set, and $f : A \to \mathbb{R}^m$ is partially differentiable at each point of A, with all partial derivatives $\frac{\partial f}{\partial x_j}$ $(j=\overline{1,n})$ continuous on A, then f is diff. on A.

In order to study the differentiability of a scalar function $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ at some point $a \in \operatorname{int} A$, one can use the following algorithm:

Step I Check the existence of all partial derivatives of f at a

- if at least one partial derivative $\frac{\partial f}{\partial x_j}(a)$ does not exist $\Rightarrow$ f is not diff at a STOP
- otherwise compute $\frac{\partial f}{\partial x_j}(a)$, $j=1,\dots,n$ and pass to Step 2

Step II Check the existence of the limit

$$l = \lim_{x \to a} \frac{f(x) - f(a) - \sum_{j=1}^{n} (x_j - a_j) \frac{\partial f}{\partial x_j}(a)}{\|x-a\|} \qquad h = x-a$$

$$= \lim_{\substack{h_1 \to 0 \\ \dots \\ h_n \to 0}} \frac{f(a_1+h_1, \dots, a_n+h_n) - f(a_1, \dots, a_n) - \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a)}{\sqrt{h_1^2 + \dots + h_n^2}}$$

- if $l = 0 \Rightarrow f$ is differentiable at a
- otherwise $\Rightarrow f$ is not differentiable at a