1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 1) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

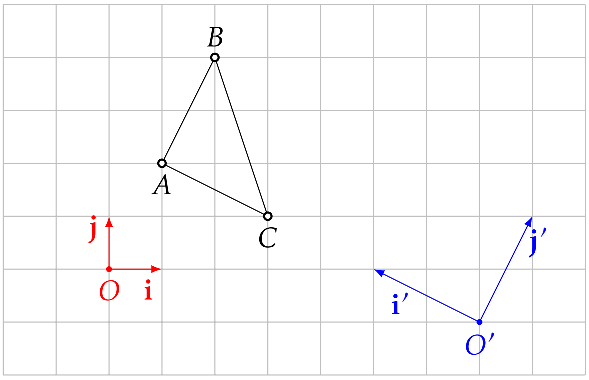


Figure 1: Coordinate systems 2D.

2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

3. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 2) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the point

$$[A]_{\mathcal{K}} = \begin{bmatrix} 3 \\ \sqrt{3} + 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$.

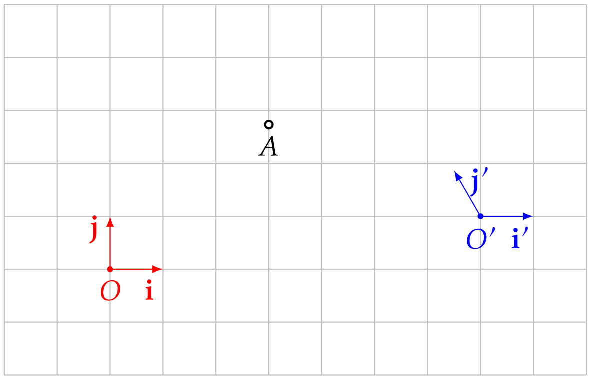


Figure 2: Coordinate systems 2D.

4. We are given the points $A(3,3)$, $B(2,4)$ and $C(5,5)$ with respect to the coordinate system $\mathcal{K}$. Determine the coordinate system $\mathcal{K}'$ such that

$$[A]_{\mathcal{K}'} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}'} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad [C]_{\mathcal{K}'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5. Consider the tetrahedron $ABCD$ (see Fig. 3) and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

1. the coordinates of the vertices of the tetrahedron in the three coordinate systems,
2. the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,
3. the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

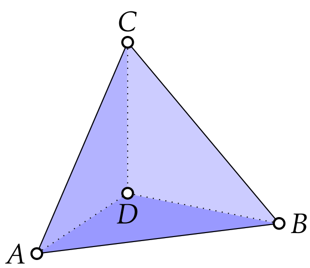


Figure 3: Tetrahedron

6. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ (see Fig. 4) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

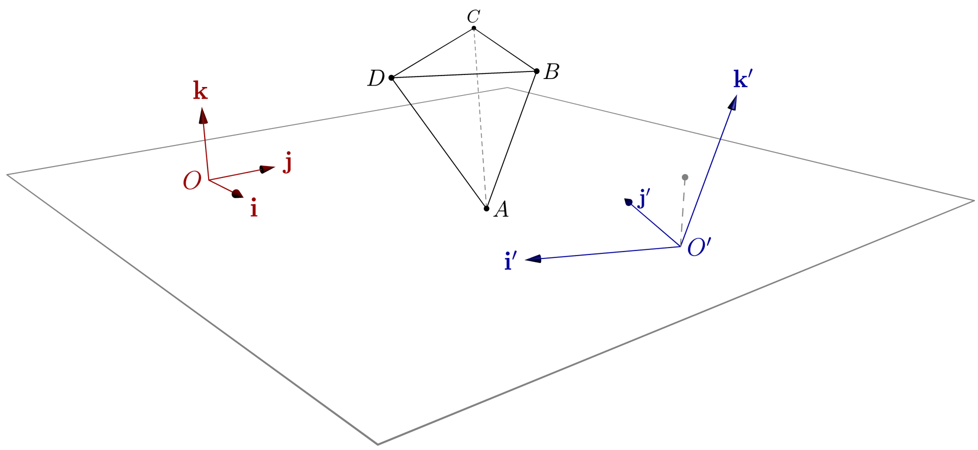


Figure 4: Coordinate systems 3D.

7. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

8. Bob receives from Alice the vertices of the tetrahedron $ABCD$:

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{K}},$$

where $\mathcal{K} = (O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is the coordinate system from which Alice views the tetrahedron (see Fig. 5). Bob wants to view this tetrahedron from his perspective (i.e., from his coordinate system $\mathcal{K}' = (O', \mathbf{e}'_x, \mathbf{e}'_y \mathbf{e}'_z)$). He knows how to describe $\mathcal{K}$ in terms of $\mathcal{K}'$:

$$O = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 0 \end{bmatrix}_{\mathcal{K}'}, \quad \mathbf{e}_x = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}_{\mathcal{K}'}, \quad \mathbf{e}_y = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}_{\mathcal{K}'}, \quad \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{K}'}.$$

What are the coordinates of the vertices of the tetrahedron from Bob's perspective? After he calculates the coordinates, he sends them to Alice. Alice wants to check that the coordinates she obtained from Bob describe the vertices that she sent him. Verify this with the corresponding coordinate transformation.

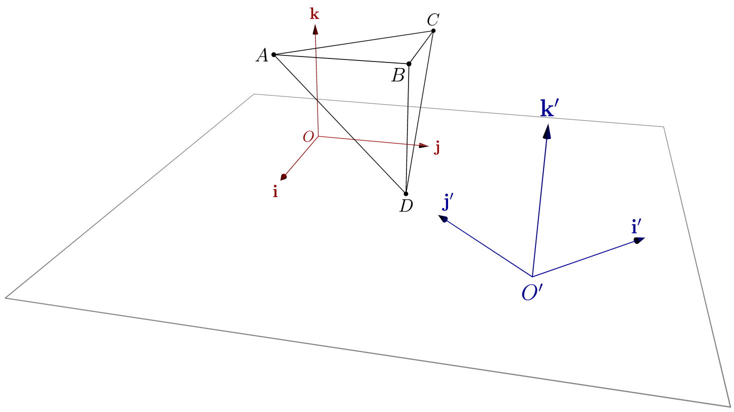


Figure 5: Alice and Bob.

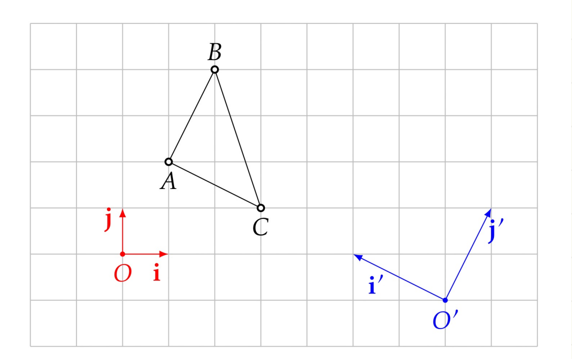
1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 1) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.



- Let $v = (i, j)$ and $w = (i', j')$

- The base change matrix from K' to K is the matrix $M_{v,w}(Id)$

$$M_{K,K'} = M_{v,w} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$[i']_K$ $[j']_K$

are given

- In order to change coordinates, we have $[A]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}([A]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[A]_{\mathcal{K}} + [O]_{\mathcal{K}'}$

- $M_{K',K} = M_{K,K'}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \frac{1}{-5}$

So $$[A]_{K'} = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \frac{1}{-5} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} \frac{1}{-5} \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -15 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$[B]_{K'} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[C]_{K'} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- In order to change coordinates backwards we use $[A]_K = M_{K,K'}([A]_{K'} - [O]_{K'}) = M_{K,K'}[A]_{K'} + [O']_K$

Notice that $[A]_K = M_{K,K'}(M_{K',K}[A]_K + [O]_{K'}) + [O']_K$
$= [A]_K + M_{K,K'}[O]_{K'} + [O']_K = [A]_K$

2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

- In K we have $\vec{AB} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow AB_K : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{= [v]_K}$ $t \in \mathbb{R}$ $(*)$

In K' : $AB_{K'}: \begin{bmatrix} x' \\ y' \end{bmatrix} = M_{K',K} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right)$

$$= \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix} \right)$$

$$= \frac{1}{5} \left(\begin{bmatrix} 0 \\ 5t \end{bmatrix} + \begin{bmatrix} 15 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$AB_{K'} : \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= [v]_{K'}}$$

Notice that if $\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ in K then

in K' we have $\ell : \begin{bmatrix} x' \\ y' \end{bmatrix} = M_{K',K} \left(\begin{bmatrix} x_A \\ y_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix} - [O']_K \right)$

$$= \underbrace{M_{K',K} \left(\begin{bmatrix} x_A \\ y_A \end{bmatrix} - [O']_K \right)}_{= [A]_{K'}} + t \underbrace{M_{K',K} \begin{bmatrix} v_x \\ v_y \end{bmatrix}}_{= [v]_{K'}}$$

- $(*) \Rightarrow \begin{cases} x = 1 + t \\ y = 2 + 2t \end{cases} \Rightarrow t = x - 1 = \frac{y-2}{2} \Rightarrow \ell : 2x - \cancel{2} - y + \cancel{2} = 0$

$\ell : 2x - y = 0$

So, in K we have $\ell : \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\Rightarrow$ in K' we have $\ell : \begin{bmatrix} 2 & -1 \end{bmatrix} \left(M_{K,K'} \begin{bmatrix} x' \\ y' \end{bmatrix} + [O']_K \right)$

$$\Leftrightarrow \begin{bmatrix} 2 & -1 \end{bmatrix} \left(\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = 0$$

$$\Leftrightarrow \begin{bmatrix} -5, & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + 15 = 0$$

$$\Leftrightarrow -5x' + 15 = 0$$

$$\ell_{K'} : x' - 3 = 0$$

AC and BC are treated similarly

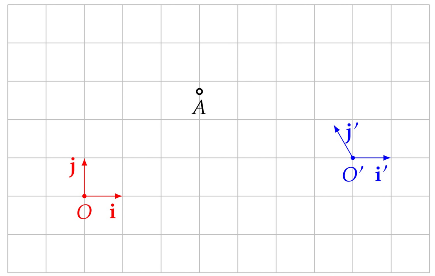
3. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 2) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the point

$$[A]_{\mathcal{K}} = \begin{bmatrix} 3 \\ \sqrt{3}+1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$.



$$M_{K,K'} = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$

$$\Rightarrow M_{K',K} = M_{K,K'}^{-1} = \frac{2}{\sqrt{3}} \begin{bmatrix} \sqrt{3}/2 & 0 \\ 1/2 & 1 \end{bmatrix}^T = \frac{2}{\sqrt{3}} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$$[O]_{K'} = \frac{2}{\sqrt{3}} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix} \right) = \frac{-2}{\sqrt{3}} \begin{bmatrix} \frac{7\sqrt{3}+1}{2} \\ 1 \end{bmatrix}$$

$$[A]_{K'} = M_{K',K} [A]_{K'} + [O]_{K'} = \frac{2}{\sqrt{3}} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ \sqrt{3}+1 \end{bmatrix} - \frac{2}{\sqrt{3}} \begin{bmatrix} \frac{7\sqrt{3}+1}{2} \\ 1 \end{bmatrix}$$

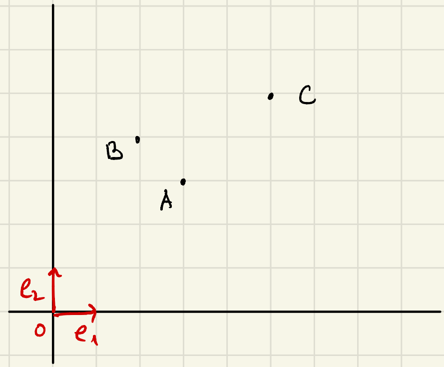
$$= \frac{2}{\sqrt{3}} \begin{bmatrix} \frac{3\sqrt{3} + \sqrt{3} + 1}{2} \\ \sqrt{3}+1 \end{bmatrix} - \frac{2}{\sqrt{3}} \begin{bmatrix} \frac{7\sqrt{3}+1}{2} \\ 1 \end{bmatrix} = \frac{2}{\sqrt{3}} \begin{bmatrix} \frac{-3\sqrt{3}}{2} \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

4. We are given the points $A(3,3)$, $B(2,4)$ and $C(5,5)$ with respect to the coordinate system $\mathcal{K}$. Determine the coordinate system $\mathcal{K}'$ such that

$$[A]_{\mathcal{K}'} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}'} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad [C]_{\mathcal{K}'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We need to find $K' = (O', e_1', e_2')$ expressed w.r.t K so we need

$$M_{K,K'} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \text{ and } [O']_K = \begin{bmatrix} O_1' \\ O_2' \end{bmatrix}$$



We know that

$$\begin{cases} M_{K,K'}[A]_{K'} = [A]_K - [O']_K \\ M_{K,K'}[B]_{K'} = [B]_K - [O']_K \\ M_{K,K'}[C]_{K'} = [C]_K - [O']_K \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} O_1' \\ O_2' \end{bmatrix} \\ \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} O_1' \\ O_2' \end{bmatrix} \\ \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} O_1' \\ O_2' \end{bmatrix} \end{cases} \Leftrightarrow \begin{cases} -m_{11} - m_{12} + O_1' = 3 \\ -m_{21} - m_{22} + O_2' = 3 \\ -2m_{12} + O_1' = 2 \quad \Rightarrow \quad O_1' = 2 + 2m_{12} \\ -2m_{22} + O_2' = 4 \quad \Rightarrow \quad O_2' = 4 + 2m_{22} \\ m_{11} + m_{12} + O_1' = 5 \\ m_{21} + m_{22} + O_2' = 5 \end{cases}$$

$$\Rightarrow \begin{cases} -m_{11} + m_{12} = 1 \\ -m_{21} + m_{22} = -1 \\ m_{11} + 3m_{12} = 3 \\ m_{21} + 3m_{22} = 1 \end{cases} \quad \left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right] \begin{array}{l} \Rightarrow m_{11} = 0 \\ \Rightarrow m_{12} = 1 \\ \Rightarrow m_{21} = 1 \\ \Rightarrow m_{22} = 0 \end{array}$$

$$\Rightarrow O_1' = 4, \; O_2' = 4$$

$$\Rightarrow M_{K,K'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad [O']_K = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

(columns: $[e_1']_K$, $[e_2']_K$)

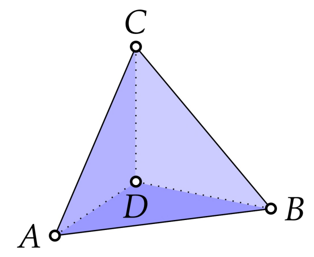
So, $K' = \left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

Exercise 5 Consider the tetrahedron $ABCD$ (see Fig. 3) and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

1. the coordinates of the vertices of the tetrahedron in the three coordinate systems,
2. the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,
3. the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.



a.) $[A]_{K_A} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $[B]_{K_A} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $[C]_{K_A} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $D_{K_A} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$[A]_{K'_A} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $[B]_{K'_A} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $[C]_{K'_A} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $[D]_{K'_A} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$[A]_{K_B} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $[B]_{K_B} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $[C]_{K_B} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $[D]_{K_B} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

b) Let $M = M_{K'_A, K_A}$ $\quad M\,[A]_{K_A} = [A]_{K'_A}$

$M\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $M\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $M\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

basis vectors in K'_A relative to K_A

$M\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \Leftrightarrow M = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

c.) $\overrightarrow{BA} = -\overrightarrow{AB}$ $\quad \left[\overrightarrow{BA}\right]_{K_A} = \begin{bmatrix}-1\\0\\0\end{bmatrix}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$ $\quad [\overrightarrow{BC}]_{K_A} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$ $\quad [\overrightarrow{BD}]_{K_A} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

So $M_{K_A, K_B} = \begin{bmatrix}-1&-1&-1\\0&1&0\\0&0&1\end{bmatrix}$

is the base change matrix from K_B to K_A

6. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ (see Fig. 4) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

$$M_{KK'} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow M_{K'K} = M_{KK'}^{-1} = \begin{bmatrix} 2 & 4 & 0 \\ 4 & -2 & 0 \\ -2 & 1 & -5 \end{bmatrix}^T \cdot \frac{1}{-10} = \frac{1}{10} \begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\det M_{KK'} = -2 - 8$$

$$\Rightarrow [A]_{K'} = \frac{1}{10} \begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} \right) = \frac{1}{10} \begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[B]_{K'} = \frac{1}{10} \begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} \right) = \frac{1}{10} \begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

similarly for $[C]_{K'}$ and $[D]_{K'}$

Notice that $[A]_K = M_{K,K'}\left(M_{K'K}[A]_K + [O]_{K'}\right) + [O']_K = [A]_K + M_{K,K'}[O]_{K'} + [O']_K = [A]_K$

$$M_{KK'}[O]_{K'} = [O']_K$$

7. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

line AB in K
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}}_{A} + t \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\vec{AB}}$$

$[O']_K$

$\Rightarrow$ line AB in K'
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M_{K'K}\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}\right)$$
$$= M_{K'K}\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}\right) + t\, M_{K'K}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{[A]_{K'}} + t\, M_{K'K}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \ldots\ldots$$

If the plane ACD in K is $ax + by + cz + d = 0 \iff [a\ b\ c]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -d$

Then the plane ACD in K' is
$$[a\ b\ c]\left(M_{K,K'}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \underbrace{\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}}_{[O']_K}\right) = -d$$

8. Bob receives from Alice the vertices of the tetrahedron $ABCD$:

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{K}}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{K}},$$

where $\mathcal{K} = (O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is the coordinate system from which Alice views the tetrahedron (see Fig. 5). Bob wants to view this tetrahedron from his perspective (i.e., from his coordinate system $\mathcal{K}' = (O', \mathbf{e}'_x, \mathbf{e}'_y \mathbf{e}'_z)$). He knows how to describe $\mathcal{K}$ in terms of $\mathcal{K}'$:

$$O = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 0 \end{bmatrix}_{\mathcal{K}'}, \quad \mathbf{e}_x = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}_{\mathcal{K}'}, \quad \mathbf{e}_y = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}_{\mathcal{K}'}, \quad \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{K}'}.$$

What are the coordinates of the vertices of the tetrahedron from Bob's perspective? After he calculates the coordinates, he sends them to Alice. Alice wants to check that the coordinates she obtained from Bob describe the vertices that she sent him. Verify this with the corresponding coordinate transformation.

$$M_{K'K} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [P]_{K'} = M_{K'K} \cdot [P]_K + [O]_{K'}$$

so $$[A]_{K'} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 3/\sqrt{2} \\ 1 \end{bmatrix}$$

the calculations of $[B]_{K'}$, $[C]_{K'}$ and $[D]_{K'}$ are similar.