1. Show that translations are isometries.

2. Show that an isometry is bijective.

3. Determine the matrix form of a rotation with angle 45° having the same center of rotation as the rotation

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

4. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

5. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

6. Find the eigenvectors for each of the following symmetric matrices:

$$A = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}, \quad B = \begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 128 & 240 \\ 240 & 450 \end{bmatrix}.$$

7. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

8. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

9. Show that $O(n)$ is a subgroup of $\mathrm{AGL}(\mathbb{R}^n)$. Show that $SO(n)$ is a normal subgroup of $O(n)$.

10. Show that the Gram-Schmidt process produces an orthonormal basis.

11. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,0)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

12. Prove that in a Euclidean vector space $(\mathbf{V}, \langle \_, \_ \rangle)$ the following identities hold, for any $\mathbf{v}, \mathbf{w} \in \mathbf{V}$.

1. $||\mathbf{v}+\mathbf{w}||^2 + ||\mathbf{v}-\mathbf{w}||^2 = 2||\mathbf{v}||^2 + 2||\mathbf{w}||^2$,
2. $||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 = 4\langle \mathbf{v}, \mathbf{w} \rangle$.

13. Consider two points $P(a,b)$ and $Q(c,d)$. Show that a rotation around P with angle θ followed by a rotation around Q with angle $-\theta$ is a translation and determine the corresponding translation vector.

14. Show that in $\mathbb{E}^2$ orthogonal reflections in lines are isometries. Show that in $\mathbb{E}^n$ orthogonal reflections in hyperplanes are isometies.

2. Show that an isometry is bijective.

$\downarrow$

map which preserves distances $f: \mathbb{E}^n \to \mathbb{E}^n$

$d(P,Q) = d(f(P), f(Q))$

injectivity : $f(P) = f(Q) \overset{?}{\Rightarrow} P = Q$

if $f(P) = f(Q) \Rightarrow d(f(P), f(Q)) = 0$

$\Rightarrow d(P,Q) = 0 \Rightarrow P = 0$

surjectivity : isometries are affine maps $f \in AGL(\mathbb{E}^n)$

$\Rightarrow f(x) = Ax + b$ w.r.t some coordinate system

- it is the composition of two maps $f_1(x) = Ax$ and $f_2(x) = x + b$
- f_2 is bijective $\Rightarrow$ $\begin{cases} f \text{ is surjective} \Leftrightarrow f_1(x) \text{ is surj.} \\ f \text{ is injective} \Leftrightarrow f_1(x) \text{ is injective} \end{cases}$
- by the first part f_1 is injective, it is an injective linear map $\mathbb{R}^n \to \mathbb{R}^n$ $\Rightarrow$ it is also surj.

3. Determine the matrix form of a rotation with angle 45° having the same center of rotation as the rotation

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

- that f is a rotation, follows from the classification of isometrie in dim 2
- the center of the rotation f is the point satisfying the equation

$$f(x) = x$$

$$\Leftrightarrow \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\iff \begin{cases} \left(\frac{2}{\sqrt{13}}-1\right)x_1 - \frac{3}{\sqrt{13}}x_2 = -1 \\ \frac{3}{\sqrt{13}}x_1 + \left(\frac{2}{\sqrt{13}}-1\right)x_2 = 2 \end{cases}$$

$$\begin{vmatrix} -1 & -\frac{3}{\sqrt{13}} \\ 2 & \frac{2}{\sqrt{13}}-1 \end{vmatrix} = -\frac{2}{\sqrt{13}} + 1 + \frac{6}{\sqrt{13}} = \frac{4}{\sqrt{13}} + 1$$
$$= \frac{4+\sqrt{13}}{\sqrt{13}}$$

$$\begin{vmatrix} \frac{2}{\sqrt{13}}-1 & -1 \\ \frac{3}{\sqrt{13}} & 2 \end{vmatrix} = \frac{4}{\sqrt{13}} - 2 + \frac{3}{\sqrt{13}} = \frac{7-2\sqrt{13}}{\sqrt{13}}$$

$$\begin{vmatrix} \frac{2}{\sqrt{13}}-1 & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}}-1 \end{vmatrix} = \frac{1}{13}\begin{vmatrix} 2-\sqrt{13} & -3 \\ 3 & 2-\sqrt{13} \end{vmatrix} = \frac{1}{13}\left[4-4\sqrt{13}+13+9\right] = \frac{1}{13}\left(26-4\sqrt{13}\right)$$
$$= \frac{2\sqrt{13}-4}{\sqrt{13}} = 2\left(\frac{\sqrt{13}-2}{\sqrt{13}}\right)$$

$$\Rightarrow x_1 = \frac{4+\sqrt{13}}{2(\sqrt{13}-2)} = \frac{1}{2}\frac{(4+\sqrt{13})(\sqrt{13}+2)}{(13-4)} = \frac{1}{18}\left(4\sqrt{13}+8+13+2\sqrt{13}\right) = \frac{1}{18}\left(21+6\sqrt{13}\right)$$

$$\Rightarrow x_1 = \frac{1}{6}\left(7+2\sqrt{13}\right)$$

and $$x_2 = \frac{7-2\sqrt{13}}{2(\sqrt{13}-2)} = \frac{1}{18}(7-2\sqrt{13})(\sqrt{13}+2) = \frac{1}{18}\left(7\sqrt{13}+14-26-4\sqrt{13}\right) = \frac{1}{18}\left(-12+3\sqrt{13}\right)$$

$$\Rightarrow x_2 = \frac{1}{6}\left(-4+\sqrt{13}\right)$$

$\Rightarrow$ the center of the rotation f is $C_f = \begin{bmatrix} \frac{7+2\sqrt{13}}{6} \\ \frac{-4+\sqrt{13}}{6} \end{bmatrix}$

A rotation of angle $45°$ around C_f is obtained by

1. translating C_f in the origin
2. rotating with angle $45°$ around the origin
3. translating back.

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$$
$$\shortparallel$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \underbrace{\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\Bigg(\underbrace{\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{7+2\sqrt{13}}{6} \\ \frac{-4+\sqrt{13}}{6} \end{bmatrix}}_{1}\Bigg)}_{2} + \begin{bmatrix} \frac{7+2\sqrt{13}}{6} \\ \frac{-4+\sqrt{13}}{6} \end{bmatrix}}_{3}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \frac{1}{6\sqrt{2}}\begin{bmatrix} 11+\sqrt{13} \\ 3+3\sqrt{13} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 7+2\sqrt{13} \\ -4+\sqrt{13} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{6\sqrt{2}} \begin{bmatrix} -11 - \sqrt{13} + 7\sqrt{2} + 2\sqrt{26} \\ -3 - 3\sqrt{13} - 4\sqrt{2} + \sqrt{26} \end{bmatrix}$$

4. Determine the cosine of the $\underbrace{\text{angle of the rotation}}$ f given in the previous exercise and find the inverse rotation, f^{-1}.

denote it by θ

then $\cos\theta = \frac{2}{\sqrt{13}}$

$f(x) = Ax + b$ if $f^{-1}(x) = \tilde{A}x + \tilde{b}$ then $x = f^{-1} \circ f(x) = \tilde{A}(Ax+b) + \tilde{b}$

$$= \tilde{A}Ax + \tilde{A}b + \tilde{b}$$

$\Rightarrow \tilde{A} = A^{-1}$ and $\tilde{b} = -\tilde{A}b$

$$\left(\frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \right)^{-1} = \frac{1}{\frac{4+9}{13}} \begin{bmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix}^T = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

we expected this since $A \in O(2)$ so $\det A = 1$

we expected this since $A \in O(2) \Rightarrow A^{-1} = A^T$

$$\tilde{b} = -\frac{1}{\sqrt{13}} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{13}} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\Rightarrow f^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{\sqrt{13}} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

<u>Rem</u> $\operatorname{tr} A = \operatorname{tr} A^T = \operatorname{tr} A^{-1}$ for $A \in O(n)$

$\Rightarrow \cos\theta_A = \cos\theta_{A^{-1}}$

obviously $\theta_{A^{-1}} = -\theta_A$ but with the cosine-trace formula we don't see this

5. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

$$T(x) = Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}(x)$$

$$\Rightarrow T^{-1}(x) = \left(Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}\right)^{-1}(x) = T^{-1}_{(-2,5)} \circ Rot^{-1}_{-\frac{\pi}{3}}(x)$$

$$= T_{(2,-5)} \circ Rot_{\frac{\pi}{3}}(x)$$

$$= \begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

6. Find the eigenvectors for each of the following symmetric matrices:

$$A = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}, \quad B = \begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 128 & 240 \\ 240 & 450 \end{bmatrix}.$$

A:

$$\begin{vmatrix} 73-T & 36 \\ 36 & 52-T \end{vmatrix} = (73-T)(52-T) - 6^4 = 4 \cdot 13 \cdot 73 - 125T + T^2 - 6^4$$

$$= T^2 - 125T + 4\underbrace{\left(949 - \underbrace{2^2 \cdot 3^4}_{324}\right)}_{625}$$

$$= T^2 - 5^3 T + 4 \cdot 5^4$$

$$\Delta = 5^6 - 2^4 \cdot 5^4$$
$$= 5^4(25-16)$$

$$\Rightarrow T_{1,2} = \frac{5^3 \pm 5^2 3}{2}$$

$$= \frac{25(5 \pm 3)}{2} \quad \begin{matrix} 100 \\ 25 \end{matrix}$$

B: 338, −169

C: 578, 0

7. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cdots \\ \sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_1 & \cdots \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & \cdots \\ \sin(\theta_1+\theta_2) & \cdots \end{bmatrix}$$

8. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

- $A \in SO(3) \iff AA^t = I_3$ & $\det A = 1$

$$A^T = \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

On checks that $\det A = 1$ so, yeah, $A \in SO(3)$

- the axis of rotation is the line passing through the origin in the direction of the eigenvectors for the eigenvalue 1

this is obtained by solving $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\iff \begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y = 0 \Big\} \Rightarrow 2x + z = 0 \Rightarrow z = -2x$$

$\Rightarrow$ eigenspace for $\lambda = 1$ is $V_1 = \{(t, 0, -2t) : t \in \mathbb{R}\}$ ← this is a line passing through the origin, it is the rotation axis

the eigenvectors are the non-zero vectors in V_1