1. Show that similarity of matrices is an equivalence relation in $\mathrm{Mat}_n(\mathbf{K})$.

2. Show that the eigenvalue associated to an eigenvector is uniquely determined.

3. Show that if $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$ are eigenvectors with the same eigenvalue λ, then for every $c_1, c_2 \mathbf{K}$ the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, if it is non-zero, is also an eigenvector with eigenvalue λ.

4. Give the characteristic polynomial, the eigenvectors and eigenvalues for the identity matrix Id_n and the zero matrix 0_n.

5. Give the eigenvalues of $\mathrm{lin}(\mathrm{Pr}_{H,\mathbf{v}})$, $\mathrm{lin}(\mathrm{Pr}_{\ell,\mathbf{W}})$, $\mathrm{lin}(\mathrm{Ref}_{H,\mathbf{v}})$ and $\mathrm{lin}(\mathrm{Ref}_{\ell,\mathbf{W}})$. What can you say about the eigenvectors?

6. Give the characteristic polynomial, the eigenvectors and eigenvalues for the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Are these matrices diagonalizable?

7. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

8. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

9. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator

$$\phi(x, y, z) = (x + y - z, y + z, 2x).$$

Find the matrix $[\phi]_{\boldsymbol{b}}$ where

$$\boldsymbol{b} = \{(1,1,0), (-1,0,1), (1,1,1)\}.$$

10. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:

$$\begin{bmatrix} -6 & 2 & -5 \\ -4 & 4 & -2 \\ 10 & -3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}$$

11. Show that if $A = (a_{ij}) \in \mathrm{Mat}_{2\times 2}(\mathbf{K})$ then

$$P_A(T) = T^2 - \mathrm{tr}(A)T + \det(A)$$

where $\mathrm{tr}(A) = a_{11} + a_{22}$ is the *trace* of A.

1. Show that similarity of matrices is an equivalence relation in $\mathrm{Mat}_n(\mathbf{K})$.

$A \sim B \iff \exists M : \quad B = M^{-1} A M$

- reflexivity: $\quad A = I_n^{-1} A I_n$
- symmetry: $\quad A \sim B \Rightarrow M \mid B = M^{-1} A M \mid \cdot M^{-1}$

$$\Rightarrow (M^{-1})^{-1} B M^{-1} = A \quad \Rightarrow B \sim A$$

- transitivity: $\quad A \sim B$ and $B \sim C \Rightarrow \exists M, N : B = M^{-1} A M \quad C = N^{-1} B N$

$$\Rightarrow C = N^{-1} M^{-1} A M N = (MN)^{-1} A (MN)$$

$$\Rightarrow C \sim A$$

2. Show that the eigenvalue associated to an eigenvector is uniquely determined.

Let v be an eigenvector of $\phi \in \mathrm{End}(V)$

if $\quad \phi(v) = \lambda v \quad$ and $\phi(v) = \mu v$

$\Rightarrow \lambda v = \mu v$

$\Rightarrow (\lambda - \mu) v = 0$

$\Rightarrow$ since $v \neq 0$, $\quad \lambda - \mu = 0$

$\Rightarrow \lambda = \mu$

3. Show that if $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$ are eigenvectors with the same eigenvalue λ, then for every ~~$c_1, c_2 \mathbf{K}$~~ $c_1, c_2 \in \mathbb{K}$ the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, if it is non-zero, is also an eigenvector with eigenvalue λ.

$$
\begin{aligned}
\phi(c_1 v_1 + c_2 v_2) &= c_1 \phi(v_1) + c_2 \phi(v_2) && \text{(since } \phi \text{ is linear)} \\
&= c_1 \lambda v_1 + c_2 \lambda v_2 && \text{(since } v_1, v_2 \in V_\lambda(\phi)) \\
&= \lambda \underbrace{(c_1 v_1 + c_2 v_2)} && \text{(since } V \text{ is a vector space)}
\end{aligned}
$$

if this vector is non-zero, it is an eigenvector for the eigenvalue λ

4. Give the characteristic polynomial, the eigenvectors and eigenvalues for the identity matrix Id_n and the zero matrix 0_n.

- $$P_{I_n} = \det(I_n - T \cdot I_n) = \begin{vmatrix} 1-T & 0 & \cdots & 0 \\ 0 & 1-T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-T \end{vmatrix} = (1-T)^n$$

$\Rightarrow$ I_n has a unique eigenvalue $\lambda = 1$

and any vector of V is an eigenvector of I_n

$$\phi_{I_n}(v) = 1 \cdot v = v$$

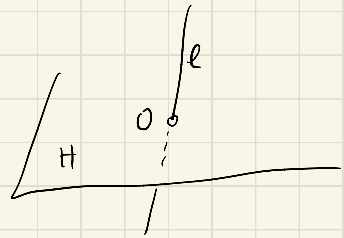
- $$P_{O_n} = \det(O_n - T \cdot T_n) = \begin{vmatrix} -T & 0 & \cdots & 0 \\ 0 & -T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -T \end{vmatrix} = (-1)^n T$$

$\Rightarrow$ O_n has a unique eigenvalue $\lambda = 0$

and any vector of V is an eigenvector of O_n

$$\phi_{O_n}(v) = 0 \cdot v = 0$$

5. Give the eigenvalues of lin($\mathrm{Pr}_{H,\mathbf{v}}$), lin($\mathrm{Pr}_{\ell,\mathbf{W}}$), lin($\mathrm{Ref}_{H,\mathbf{v}}$) and lin($\mathrm{Ref}_{\ell,\mathbf{W}}$). What can you say about the eigenvectors?



- let $\phi = \mathrm{lin}(Pr_{H,v})$ then $\phi(\overrightarrow{AB}) = \overrightarrow{Pr_{H,v}(A)\,Pr_{H,v}(B)}$

- let W be the vector subspace associated to H

  let $\mathcal{W} = \{w_2, \dots, w_n\}$ be a basis of W

- let ℓ be a line with direction vector v

- let $\{O\} = \ell \cap H$ then $v = \overrightarrow{OP_1}$ $\quad w_i = \overrightarrow{OP_i}$ $\quad i = 2, \dots, n$

$$\Rightarrow \quad \phi(\overrightarrow{OP_1}) = \overrightarrow{Pr_{H,v}(O)\,Pr_{H,v}(P_1)} = \overrightarrow{OO} = 0$$

$$\phi(\overrightarrow{OP_i}) = \overrightarrow{Pr_{H,v}(O)\,Pr_{H,v}(P_i)} = \overrightarrow{OP_i} = \overrightarrow{OP_i} \quad \forall i = 2, \dots, n$$

- since $v \nparallel H$, $b = \{v, w_2, \dots, w_n\}$ is a basis of V and

$$[\phi]_b = \begin{bmatrix} 0 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & I_{n-1} \end{bmatrix}$$

- ϕ has two eigenvalues: 0 and 1

  $V_0(\phi) = \langle v \rangle \qquad V_1(\phi) = W$

- let $\phi = \mathrm{lin}\, Ref_{H,v}$ with W, w_i, P_i, O as above $\phi(\overrightarrow{OP_i}) = \overrightarrow{OP_i}$ $i=2,n$

  and $\phi(\overrightarrow{OP_1}) = \overrightarrow{Ref_{H,v}(O)\,Ref_{H,v}(P_1)} = \overrightarrow{OP'} = -\overrightarrow{OP}$ since O is midpoint of $[PP']$

  $\Rightarrow [\phi]_b = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots\ 1 \end{bmatrix}$, spectrum of ϕ is $\{1, -1\}$, $V_{-1}(\phi) = \langle v \rangle$ $\quad V_1(\phi) = W$

similarly one shows that

$$[\text{lin } Pr_{\ell,w}]_b = \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ 0 & & & 1 \end{bmatrix}, \quad \text{sectrum of } \text{lin } Pr_{\ell,w} \text{ is } \{0,1\}$$

$$\dim V_0(\text{lin } Pr_{\ell,w}) = n-1$$

$$\dim V_1(\text{lin } Pr_{\ell,w}) = 1$$

$$[\text{lin } Ref_{\ell,w}]_b = \begin{bmatrix} -1 & & & 0 \\ & \ddots & & \\ & & -1 & \\ 0 & & & 1 \end{bmatrix}, \quad \text{spectrum of } \text{lin } Ref_{\ell,w} \text{ is } \{-1,1\}$$

$$\dim V_{-1}(\text{lin } Ref_{\ell,w}) = n-1$$

$$\dim V_1(\text{lin } Ref_{\ell,w}) = 1$$

6. Give the characteristic polynomial, the eigenvectors and eigenvalues for the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Are these matrices diagonalizable?

$P_A = \det(A - T \cdot I_n) = (-1)^n T^n \Rightarrow A$ has only one eigenvalue $\lambda = 0$

if A would be diagonalizable then $A = O_n$ which is not the case

$\Rightarrow A$ is not diagonalizable

$P_B = \det(B - T \cdot I_n) = (1-T)^n \Rightarrow B$ has only one eigenvalue $\lambda = 1$

if B would be diagonalizable then $B = I_n$ which is not the case

$\Rightarrow B$ is not diagonalizable

7. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

- $P_A = \det(A - T\cdot I_n) = \det\begin{vmatrix} -T & 1 \\ -1 & -T \end{vmatrix} = T^2 + 1$
- P_A does not have roots in $\mathbb{R} \Rightarrow A$ is not diagonalizable
- P_A has roots in $\mathbb{C}$ so A viewed in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is diagonalizable

$$T^2+1=0 \qquad \lambda_1 = i \qquad \lambda_2 = -i$$

By Proposition 7.9 P_A is diagonalizable (over $\mathbb{C}$)

- to find the eigenvectors we solve the system

$$A\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \begin{cases} v_2 = i v_1 \\ -v_1 = i v_2 \end{cases} \quad \text{the solutions are } \begin{bmatrix} v_1 \\ i v_1 \end{bmatrix}$$

$$\Leftrightarrow V_i(A) = \left\langle \begin{bmatrix} 1 \\ i \end{bmatrix} \right\rangle$$

$$B\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -i\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \begin{cases} v_2 = -i v_1 \\ -v_1 = -i v_2 \end{cases} \quad \text{the solutions are } \begin{bmatrix} v_1 \\ -i v_1 \end{bmatrix}$$

$$\Leftrightarrow V_{-i}(B) = \left\langle \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\rangle$$

8. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ the matrix is already diagonal

- so the eigenvalues are 1 and -1
- $V_1(A) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \qquad V_{-1}(A) = \langle \begin{bmatrix} 0 \\ -1 \end{bmatrix} \rangle$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad P_A = \det(A - T\cdot I_2) = \det\begin{bmatrix} 1-T & 1 \\ 0 & 1-T \end{bmatrix} = (1-T)^2$

- so A has only one eigenvalue $\lambda = 1$

$$A\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = v_1 \\ v_2 = v_2 \end{cases} \text{ the solutions are } \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$

$\Rightarrow V_1(A) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$

notice that $\underbrace{\dim V_1(A)}_{\text{geometric multiplicity}} \lneq \underbrace{h_A(1)}_{\text{algebraic multiplicity}}$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad P_A = \det(A - T\cdot I_2) = \det\begin{bmatrix} 1-T & 0 \\ 1 & 1-T \end{bmatrix} = (1-T)^2$

- so A has only one eigenvalue $\lambda = 1$

$$A\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 = v_1 \\ v_1 + v_2 = v_2 \end{cases} \text{ the solutions are } \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$

$\Rightarrow V_1(A) = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$

notice that $\underbrace{\dim V_1(A)}_{\text{geometric multiplicity}} \lneq \underbrace{h_A(1)}_{\text{algebraic multiplicity}}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad P_A = \det(A - T \cdot I_2) = \det \begin{bmatrix} 1-T & 1 \\ 1 & 1-T \end{bmatrix} = (1-T)^2 - 1$$

$$= T^2 - 2T = T(T-2)$$

· so A has two eigenvalues $\lambda_1 = 0$ and $\lambda = 2$

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = 0 \\ v_1 + v_2 = 0 \end{cases} \text{ the solutions are } \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix}$$

$$\Rightarrow V_0(A) = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$$

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = 2v_1 \\ v_1 + v_2 = 2v_2 \end{cases} \text{ the solutions are } \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$$

$$\Rightarrow V_2(A) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$$

9. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator

$$\phi(x, y, z) = (x + y - z, y + z, 2x).$$

Find the matrix $[\phi]_{\boldsymbol{b}}$ where

$$\boldsymbol{b} = \{(1,1,0), (-1,0,1), (1,1,1)\}.$$

Let e be the canonical basis of $\mathbb{R}^3$ $\quad e = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

then

$$[\phi_e(x,y,z)]_e = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[\phi]_b = M_{e,b}^{-1} [\phi]_e M_{e,b}$$

$$M_{e,b} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow M_{e,b}^{-1} = \frac{1}{\underset{\substack{\parallel \\ 0+1+0 \\ -0-(-1)-1}}{\det M_{e,b}}} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow [\phi]_b = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \cdot \underbrace{\left[\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right]}$$

$$= \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & 2 \\ 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 1 \\ -1 & 3 & 1 \\ 3 & -5 & 1 \end{bmatrix}$$

10. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:

$$A = \begin{bmatrix} -6 & 2 & -5 \\ -4 & 4 & -2 \\ 10 & -3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix} = B$$

$$P_A = \det(A - T \cdot I_3) = \det \begin{bmatrix} -6-T & 2 & -5 \\ -4 & 4-T & -2 \\ 10 & -3 & 8-T \end{bmatrix} =$$

$$= (-6-T) \begin{vmatrix} 4-T & -2 \\ -3 & 8-T \end{vmatrix} - 2 \begin{vmatrix} -4 & -2 \\ 10 & 8-T \end{vmatrix} - 5 \begin{vmatrix} -4 & 4-T \\ 10 & -3 \end{vmatrix}$$

$$= -(6+T)(32 - 12T + T^2 - 6) - 2(4T - 32 + 20) - 5(12 - 40 + 10T)$$

$$= \dots = -T^3 + 6T^2 - 12T + 8$$

$$T^3 - 3 \cdot 2 T^2 + 3 \cdot 2^2 T + 2^3 = 0$$

$$\Leftrightarrow (T-2)^3 = 0 \Rightarrow \text{ 2 is the only eigenvalue of } A$$

To find the eigenvectors we solve

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 10 & -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 10 & -3 & 6 \end{bmatrix} \underset{L_3 + L_1 + \frac{1}{2} L_2}{\sim} \begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 2 $\Rightarrow$ the solution space has dimension 1

$$\Rightarrow \dim V_2(A) = 1 \lneq h_2(A) = 3$$

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}$ We see that 1 is an eigenvalue and that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for 1.

$$P_B = \det \begin{bmatrix} 1-T & 0 & 0 \\ 0 & -3-T & -15 \\ 0 & 2 & 8-T \end{bmatrix} = (1-T)\Big(\underbrace{(3+T)(T-8)+30}_{T^2+5T-24+30}\Big) = (1-T)(T+2)(T+3)$$

$\Rightarrow$ the spectrum of B is $\{1, -2, -3\}$

B is diagonalizable by prop. 7.9

11. Show that if $A = (a_{ij}) \in \mathrm{Mat}_{2\times 2}(\mathbf{K})$ then

$$P_A(T) = T^2 - \mathrm{tr}(A)T + \det(A)$$

where $\mathrm{tr}(A) = a_{11} + a_{22}$ is the *trace* of A.

$$P_A = \det \begin{vmatrix} a_{11} - T & a_{12} \\ a_{21} & a_{22} - T \end{vmatrix} = (a_{11} - T)(a_{22} - T) - a_{21}a_{12}$$

$$= T^2 - \underbrace{(a_{11} + a_{22})}_{= \mathrm{tr} A} T + \underbrace{a_{11}a_{22} - a_{21}a_{12}}_{= \det A}$$