Proposition 7.3. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$ of dimension n and let $A, B \in \mathrm{M}_n(\mathbf{K})$. Then A and B are similar if and only if there is a linear operator $\phi \in \mathrm{End}(\mathbf{V})$ and bases $\boldsymbol{e}$ and $\boldsymbol{f}$ of $\mathbf{V}$ such that $[\phi]_e = A$ and $[\phi]_f = B$.

- If ϕ, e and f exist then

$$B = [\phi]_f = M_{e,f}^{-1} [\phi]_e M_{e,f} = M^{-1}_{e,f} A M^{-1}_{e,f}$$

where $M_{e,f}$ is the change of basis matrix from f to e.

The above equation says in particular that A and B are similar.

- Conversely, suppose that A and B are similar, i.e there exists a matrix M such that

$$B = M^{-1} A M \qquad (*)$$

- Let $e = (e_1, \dots, e_n)$ be a basis of V
- Let $\phi = \phi_A$ be the linear operator associated to the matrix A
- Let $(f_1, \dots, f_n)$ be vectors whose coordinates (w.r.t e) are the elements in the columns of M, i.e $f_j = m_{1j} e_1 + \dots + m_{nj} e_j$
- M is invertible $\Rightarrow$ rank$(M) = n \Rightarrow f_1, \dots, f_n$ are linearly indep

$$\Rightarrow f = (f_1, \dots, f_n) \text{ is a basis of } V$$

- Moreover, $M = M_{e,f}$ so $(*)$ is

$$B = M_{ef}^{-1} [\phi]_e M_{e,f} \Rightarrow B = [\phi]_f .$$

Proposition 7.6. The eigenvalue associated to an eigenvector is uniquely determined.

. Let v be an eigenvector of ϕ for two eigenvalues λ and $\mu \in \mathbb{K}$

$$\Rightarrow \quad \lambda v = \phi(v) = \mu v$$

$$\Rightarrow \quad (\lambda - v)\, v = 0$$

since $v \neq 0$

$$\Rightarrow \quad \lambda - \mu = 0 \qquad \text{so} \qquad \lambda = \mu$$

Proposition 7.7. If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$ are eigenvectors with the same eigenvalue λ, then for every $c_1, c_2 \in \mathbf{K}$ the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, if it is non-zero, is also an eigenvector with eigenvalue λ.

$$\phi(c_1 v_1 + c_2 v_2) = c_1 \phi(v_1) + c_2 \phi(v_2) \qquad \text{(since } \phi \text{ is linear)}$$

$$= c_1 \lambda v_1 + c_2 \lambda v_2 \qquad \text{(since } v_1, v_2 \text{ are eigenvects. for } \lambda)$$

$$= \lambda (c_1 v_1 + \lambda v_2)$$

Proposition 7.9. If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbf{V}$ are eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, and these λ_i are pairwise distinct, then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

- The proof is by induction on k
- If $k = 1$ the statement is true since $v_1 = 0$
- If $k > 2$ consider a linear combination of the given eigenvectors which equals 0

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0 \qquad (*)$$

$$\Rightarrow \quad \phi(c_1 v_1 + c_2 v_2 + \dots + c_k v_k) = \phi(0)$$

$$\Rightarrow \quad c_1 \phi(v_1) + c_2 \phi(v_2) + \dots + c_k \phi(v_k) = 0$$

$$\Rightarrow \quad c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_k \lambda_k v_k = 0 \qquad (**)$$

by multiplying $(*)$ with λ_1 we also have

$$c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 + \dots + c_k \lambda_1 v_k = 0$$

$$\overset{(**)}{\Rightarrow} \quad c_2 (\lambda_2 - \lambda_1) v_2 + \dots + c_k (\lambda_k - \lambda_1) v_k = 0$$

by induction $v_2, \dots, v_k$ are linearly independent

$$\Rightarrow \quad c_2 (\lambda_2 - \lambda_1) = \dots = c_k (\lambda_k - \lambda_1) = 0$$

Since $\lambda_1 \dots \lambda_k$ are distinct $\lambda_i - \lambda_1 \neq 0 \quad \forall \; 2 \leq i \leq k$

$$\Rightarrow \quad c_2 = \dots = c_k = 0$$

So $(*) \Leftrightarrow c_1 v_1 = 0$

Since v_1 is an eigenvector, $v_1 \neq 0$

$$\Rightarrow c_1 = 0$$

So $(*)$ implies that $c_1 = \dots = c_k = 0$, i.e. $v_1, \dots, v_k$ are linearly independent.

Proposition 7.10. If every $\mathbf{v} \in \mathbf{V} \setminus \{0\}$ is an eigenvector of ϕ then there exists $\lambda \in \mathbf{K}$ such that $\phi = \lambda \, \mathrm{Id}_{\mathbf{V}}$.

- If $\dim V = 1$ the satement is true
- suppose $\dim V > 1$ and let $\{e_1, \dots, e_n\}$ be a basis of V
- hypothesis $\Rightarrow \exists \lambda_1, \dots, \lambda_n \in \mathbb{K} \; : \; \phi(e_i) = \lambda_i e_i$
- for $i \neq j$ consider $v_{ij} = e_i + e_j \Rightarrow \exists \lambda_{ij} \in \mathbb{K} \quad \phi(v_{ij}) = \lambda_{ij} v_{ij}$

$$\Rightarrow \quad \lambda_{ij} e_i + \lambda_{ij} e_j = \lambda_{ij} v_{ij} = \phi(v_{ij}) = \phi(e_i + e_j) = \phi(e_i) + \phi(e_j) = \lambda_i e_i + \lambda_j e_j$$

- e_i, e_j lin. indep $\Rightarrow \lambda_i = \lambda_j = \lambda_{ij} \quad \forall i \neq j \quad \Rightarrow \quad \lambda_1 = \dots = \lambda_k.$

Proposition 7.11. Let $\mathbf{V}$ be a finite dimensional vector space and let $\phi \in \mathrm{End}(\mathbf{V})$. A scalar $\lambda \in \mathbf{K}$ is an eigenvalue of ϕ if and only if the operator

$$\phi - \lambda \mathrm{Id}_{\mathbf{V}} : \mathbf{V} \to \mathbf{V} \quad \text{defined by} \quad (\phi - \lambda \mathrm{Id}_{\mathbf{V}})(\mathbf{v}) = \phi(\mathbf{v}) - \lambda \mathbf{v}$$

is not an isomorphism, that is, if and only if $\det(\phi - \lambda \mathrm{Id}_{\mathbf{V}}) = 0$.

$\phi - \lambda \mathrm{Id}_V$ is not an isomorphism $\iff \ker(\phi - \lambda \mathrm{Id}_V) \neq \langle 0 \rangle$

$\iff \exists\, v \in V,\ v \neq 0,\ (\phi - \lambda \mathrm{Id})(v) = 0$

$\iff \exists\, v \in V,\ v \neq 0 \quad \phi(v) = \lambda v$

$\iff \exists$ an eigenvector v with eigenvalue λ

Proposition 7.13. The definition of P_ϕ is independent of the basis $\boldsymbol{e}$.

- Let $f = \{f_1, \dots, f_n\}$ be another basis of V
- Let $A = [\phi]_e$ and $B = [\phi]_f$

  $\Rightarrow \exists$ matrix M such that $B = M^{-1} A M$
- We need to show that $P_A = P_B$
- We have $B - T \cdot I_n = M^{-1} A M - T \cdot I_n = M^{-1}(A - T \cdot I_n) M$

$\Rightarrow \det(B - T \cdot I_n) = \det(M^{-1}) \cdot \det(A - T \cdot I_n) \cdot \det M = \det(A - T \cdot I_n)$

Corollary 7.14. Let $\mathbf{V}$ be a vector space of dimension n, and let $\phi \in \mathrm{End}(\mathbf{V})$. Then $\lambda \in \mathbf{K}$ is an eigenvalue of ϕ if and only if λ is a root of the polynomial P_ϕ. In particular, ϕ has at most n eigenvalues.

$\Uparrow$ Prop. 7.11

$\Uparrow$ a polynomial of degree n has at most n roots

Proposition 7.15. Let $\mathbf{V}$ be a finite dimensional vector space. An operator $\phi \in \mathrm{End}(\mathbf{V})$ is diagonalizable if and only if there is a basis of $\mathbf{V}$ consisting entirely of eigenvectors of ϕ.

Clear from the definition of diagonalizable operators.

Theorem 7.16. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space of dimension n, and let $\phi \in \mathrm{End}(\mathbf{V})$. If $\{\lambda_1, \dots, \lambda_k\} \subseteq \mathbf{K}$ is the spectrum of ϕ, then

$$\dim(V_{\lambda_1}(\phi)) + \cdots + \dim(V_{\lambda_k}(\phi)) \leq n$$

with equality if and only if ϕ is diagonalizable.

- for $i \in \{1, \dots, k\}$ $\begin{cases} \text{let } d(i) = \dim V_{\lambda_i}(\phi) \\ \text{let } \{e_{i1}, \dots, e_{id(i)}\} \text{ be a basis of } V_{\lambda_i}(\phi) \end{cases}$

- It is enough to show that

$$e_{11}, \dots, e_{1d(1)}, e_{21}, \dots, e_{2d(2)}, \dots, e_{k1}, \dots, e_{kd(k)} \qquad (*)$$

are linearly independent. Indeed, since the maximal number of linearly independent vectors is n, it follows from $(*)$ that $\sum_{i=1}^{k} \underset{\dim V_{\lambda_i}(\phi)}{\underset{\shortparallel}{d(i)}} \leq n$

- Suppose that there exist c_{ij} such that (we need to show that $c_{ij} = 0 \ \forall i,j$)

$$(**) \qquad 0 = \underbrace{c_{11} e_{11} + \dots + c_{1d(1)} e_{1d(1)}}_{v_1} + \underbrace{c_{21} e_{21} + \dots + c_{2d(2)} e_{2d(2)}}_{v_2} + \dots + \underbrace{c_{k1} e_{k1} + \dots + c_{kd(k)} e_{kd(k)}}_{v_k}$$

- Let $v_i = c_{i1} e_{i1} + \dots + c_{id(i)} e_{id(i)} \in V_{\lambda_i}(\phi)$

$$(**) \iff 0 = v_1 + \dots + v_k$$

- Since $e_{i1}, \dots, e_{id(i)}$ is a basis of $V_{\lambda_i}(\phi)$ we have $v_i = 0 \iff c_{i1} = c_{i2} = \dots = c_{id(i)} = 0$

- So $c_{ij} = 0 \ \forall i,j \iff v_i = 0 \ \forall i$ (So, if we show $v_i = 0 \ \forall i$ then we are done)

- Since $v_i \in V_{\lambda_i}$, v_i is an eigenvector for λ_i

- So the non-zero vectors v_i (if there are any such v_i) are eigen vectors for distinct eigen values

Prop 7.9 $\Rightarrow$ the non-zero vectors v_i are linearly independent

$\Rightarrow$ they cannot sum up to 0, but this contradicts $v_1 + \dots + v_k = 0$

So all v_i have to be zero

$\Rightarrow$ $c_{ij} = 0 \ \forall \ ij$

$\Rightarrow$ the vectors in $(*)$ are linearly independent.

Corollary 7.17. If $\dim(\mathbf{V}) = n$ and $\phi \in \operatorname{End}(\mathbf{V})$ has n distinct eigenvalues then it is diagonalizable.

for an eigen value λ $\dim V_\lambda(\phi)$ is at least 1.

$\Rightarrow$ if ϕ has n distinct eigenvalues then

$$n \le \sum_{i=1}^{n} \dim V_{\lambda_i}(\phi) \underset{\text{by Thm 7.16}}{\le} n \quad \Rightarrow \sum_{i=1}^{n} \dim V_{\lambda_i} = n$$

$\Rightarrow$ $\dim V_{\lambda_i} = 1$

$\Rightarrow$ we may choose n distinct eigenvectors corresponding to n distinct eigenvalues

$\Rightarrow$ by Prop 7.9 these vectors will form a basis of V

$\Rightarrow$ since they are eigenvectors, ϕ will be diagonal relative to this basis.

Proposition 7.19. For any operator $\phi \in \mathrm{End}(\mathbf{V})$ and $\lambda \in \mathbf{K}$ one has

$$\dim(\mathbf{V}_\lambda(\phi)) \leq h_\phi(\lambda), \quad (*)$$

that is, the geometric multiplicity is not larger than the algebraic multiplicity.

- The geometric multiplicity for the eigenvalue λ is $\dim(V_\lambda(\phi))$
- The algebraic multiplicity for the eigenvalue λ is the multiplicity of λ in P_ϕ (denoted by $h_\phi(\lambda)$)
- to show $(*)$, let $d = \dim V_\lambda(\phi)$
  let $e_1, \dots, e_n$ be a basis for V such that $\{e_1, \dots, e_d\}$ is a basis for $V_\lambda(\phi)$

then $[\phi]_e = \begin{pmatrix} \lambda I_d & B \\ 0 & C \end{pmatrix}$ where $B \in \mathrm{Mat}_{d,n-d}(\mathbb{K})$, $0 \in \mathrm{Mat}_{n-d,d}(\mathbb{K})$, $C \in \mathrm{Mat}_{n-d,n-d}(\mathbb{K})$

- Expanding the determinant on the first d columns we get

$$P_\phi = \det\left([\phi]_e - T\lambda\right) = \begin{vmatrix} (\lambda - T) I_d & B \\ 0 & C - T \cdot I_{n-d} \end{vmatrix} = (\lambda - T)^d \cdot \underbrace{p(T)}_{= \det(C - T I_{n-d}) = P_C}$$

$\Rightarrow$ the multiplicity of λ in P_ϕ is at least d, i.e

$$d \leq h_\lambda(\phi)$$