1. Find a Cartesian equation of the line ℓ in $\mathbf{A}^2(\mathbb{R})$ containing the points $P = S \cap S'$ and $Q = T \cap T'$ where

$$S : x + 5y - 8 = 0, \quad S' : 3x + 6 = 0, \quad T : 5x - \frac{1}{2}y = 1, \quad T' : x - y = 5.$$

2. Deterimine ~~parametric~~ an equation~~s~~ for the line in $\mathbf{A}^2(\mathbb{C})$ parallel to $\mathbf{v}$ and passing through $S \cap T$ in each of the following cases:

1. $\mathbf{v} = (2,4)$, $S : 3x - 2y - 7 = 0$, $T : 2x + 3y = 0$,
2. $\mathbf{v} = (-5\sqrt{2}, 7)$, $S : x - y = 0$, $T : x + y = 1$.

3. Let ABC be a triangle in some affine space X. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that

$$\overrightarrow{AC'} = \lambda \overrightarrow{BC'} \quad \text{and} \quad \overrightarrow{AB'} = \mu \overrightarrow{CB'}.$$

The lines BB' and CC' meet in the point M. For a fixed but arbitrary point $O \in X$ show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda \overrightarrow{OB} - \mu \overrightarrow{OC}}{1 - \lambda - \mu}.$$

4. Consider the triangle ABC in $\mathbb{E}^n$ with side lengths a, b, c. Let G be its centroid, H the orthocenter and I the incenter. For a fixed but arbitrary point $O \in X$, show that

1. $\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$,
2. $\overrightarrow{OI} = \frac{a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}}{a+b+c}$,
3. $\overrightarrow{OH} = \frac{(\tan \hat{A})\overrightarrow{OA} + (\tan \hat{B})\overrightarrow{OB} + (\tan \hat{C})\overrightarrow{OC}}{\tan \hat{A} + \tan \hat{B} + \tan \hat{C}}$.

5. In some affine space, consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}$$

where $M = AB' \cap A'B$ and $N = AA' \cap BB'$ and where $\overrightarrow{OB} = m\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\overrightarrow{OA'}$.

6. Show that the midpoints of the diagonals of a complete quadrilateral are collinear.

7. Prove the following generalziation of Thales' theorem.

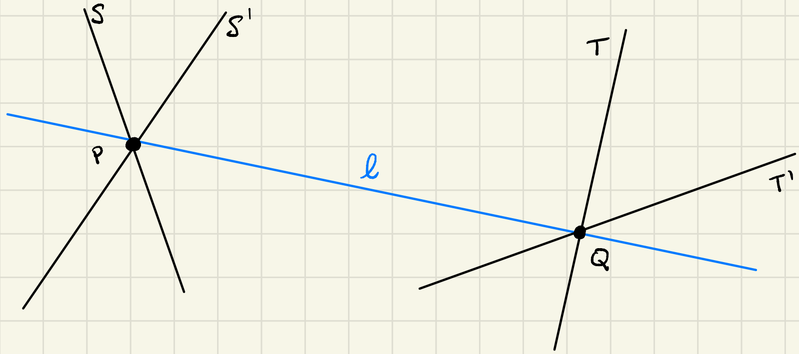
In an affine space $\mathbf{A}$ over $\mathbf{K}$ let H, H', H'' be three distinct parallel hyperplanes, and ℓ_1 and ℓ_2 be lines which are not parallel to H, H', H''. Let $P_i = \ell_i \cap H$, $P_i' = \ell_i \cap H'$, $P_i'' = \ell_i \cap H''$ (for $i = 1, 2$), and let k_1, k_2 be the scalars such that

$$\overrightarrow{P_iP_i''} = k_i \overrightarrow{P_iP_i'} \quad i = 1, 2.$$

Then $k_1 = k_2$.

1. Find a Cartesian equation of the line ℓ in $\mathbf{A}^2(\mathbb{R})$ containing the points $P = S \cap S'$ and $Q = T \cap T'$ where

$$S : x + 5y - 8 = 0, \quad S' : 3x + 6 = 0, \quad T : 5x - \frac{1}{2}y = 1, \quad T' : x - y = 5.$$



Method 1. • Calculate $P = S \cap S'$: $\left\{ \begin{array}{l} x + 5y - 8 = 0 \\ 3x + 6 = 0 \quad \Rightarrow x = -2 \end{array} \right\} \Rightarrow y = 2 \Rightarrow P = P(-2, 2)$

• $Q \in \ell \Rightarrow \ell$ belongs to the pencil of lines determined by T and T':

i.e $\ell \in \mathcal{L}_Q = \{ \alpha (5x - \frac{1}{2}y - 1) + \beta (x - y - 5) = 0 : \alpha, \beta \in \mathbb{R}$ not both zero $\}$

i.e $\ell = \ell_{\alpha, \beta} : \alpha (5x - \frac{1}{2}y - 1) + \beta (x - y - 5) = 0$ for some α, β not both 0

• $P \in \ell \Rightarrow \quad \alpha (-10 - 1 - 1) + \beta (-2 - 2 - 5) = 0$

$\Rightarrow \quad -12\alpha - 9\beta = 0$

$\Rightarrow \quad \alpha = -\frac{3}{4}\beta$

$\Rightarrow \quad \ell = \ell_{-\frac{3}{4}\beta, \beta} : -\frac{3}{4}\cancel{\beta}(5x - \frac{1}{2}y - 1) + \cancel{\beta}(x - y - 5) = 0 \quad | \cdot 4$

$-15x - \frac{3}{2}y + 3 + 4x - 4y - 20 = 0$

$\Rightarrow \ell : -11x - \frac{5}{2}y - 17 = 0$

Method 2: You can use a reduced pencil in the above solution

Method 3: You can calculate $Q = T \cap T'$ first and then use the pencil $\mathcal{L}_P$ to determine ℓ

Method 4: In this example the calculations are easy, so you can calculate both P and Q

2. Deterimine an equation for the line in $\mathbf{A}^2(\mathbb{C})$ parallel to $\mathbf{v}$ and passing through $S \cap T$ in each of the following cases:

1. $\mathbf{v} = (2,4)$, $S : 3x - 2y - 7 = 0$, $T : 2x + 3y = 0$,
2. $\mathbf{v} = (-5\sqrt{2}, 7)$, $S : x - y = 0$, $T : x + y = 1$.

1. The pencil of lines with center $S \cap T$ consists of lines of the form

$$\ell_{\alpha,\beta} : \alpha(3x - 2y - 7) + \beta(2x + 3y) = 0$$

$$\iff \quad (3\alpha + 2\beta)x + (-2\alpha + 3\beta)y - 7\alpha = 0$$

So the direction of $\ell_{\alpha,\beta}$ is given by the homogeneous equation

$$(3\alpha + 2\beta)x + (-2\alpha + 3\beta)y = 0$$

$$\ell_{\alpha,\beta} \parallel v \iff (3\alpha + 2\beta)2 + (-2\alpha + 3\beta)4 = 0 \iff \dots \quad \alpha = 8\beta$$

$\Rightarrow$ the line in $\mathcal{L}_{S\cap T}$ which is parallel to v is

$$\ell_{8\beta,\beta} : \; 24x - 16y - 56 + 2x + 3y = 0$$

$$26x - 13y - 56 = 0$$

2. $\mathcal{L}_{S \cap T}$ consists of $\ell_{\alpha,\beta} : \alpha(x - y) + \beta(x + y - 1) = 0$

$$\iff (\alpha + \beta)x + (-\alpha + \beta)y - \beta = 0$$

$$v \parallel \ell_{\alpha,\beta} \iff (\alpha + \beta)(-5\sqrt{2}) + (-\alpha + \beta)7 = 0 \iff \dots \quad \alpha = \frac{-5\sqrt{2} + 7}{5\sqrt{2} + 7}\beta$$

$\Rightarrow$ the line in $\mathcal{L}_{S\cap T}$ which is parallel to v is

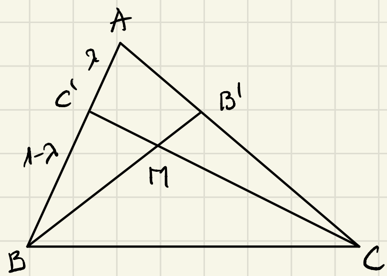
$$\ell_{\frac{-5\sqrt{2}+7}{5\sqrt{2}+7}\beta,\beta} : \underbrace{(\alpha + \beta)}_{= \frac{14}{5\sqrt{2}+7}\cancel{\beta}}x + \underbrace{(-\alpha + \beta)}_{= \frac{10\sqrt{2}}{5\sqrt{2}+7}\cancel{\beta}}y - \cancel{\beta} = 0$$

3. Let ABC be a triangle in some affine space X. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that

$$\overrightarrow{AC'} = \lambda \overrightarrow{BC'} \quad \text{and} \quad \overrightarrow{AB'} = \mu \overrightarrow{CB'}.$$

The lines BB' and CC' meet in the point M. For a fixed but arbitrary point $O \in X$ show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda \overrightarrow{OB} - \mu \overrightarrow{OC}}{1 - \lambda - \mu}.$$



- Fix a point O and let $\{M\} = BB' \cap CC'$
- $M \in BB' \Rightarrow$ (*) $\left\{ \begin{array}{l} \overrightarrow{OM} = (1-t)\overrightarrow{OB} + t\,\overrightarrow{OB'} \text{ for } t \in \mathbb{R} \\ \overrightarrow{OM} = (1-s)\overrightarrow{OC} + s\,\overrightarrow{OC'} \text{ for } s \in \mathbb{R} \end{array} \right.$
- $M \in CC' \Rightarrow$
- since $\overrightarrow{AC'} = \lambda \overrightarrow{BC'}$ we have $\overrightarrow{AB} = \overrightarrow{AC'} + \overrightarrow{C'B} = -\lambda \overrightarrow{C'B} + \overrightarrow{C'B} = (1-\lambda)\overrightarrow{C'B}$

$$\Rightarrow \overrightarrow{BC'} = \frac{1}{\lambda-1}\overrightarrow{AB}$$

- hence $\overrightarrow{OC'} = \overrightarrow{OB} + \overrightarrow{BC'} = \overrightarrow{OB} + \frac{1}{\lambda-1}\overrightarrow{AB} = \overrightarrow{OB} + \frac{1}{\lambda-1}(\overrightarrow{OB} - \overrightarrow{OA})$

so $\overrightarrow{OC'} = -\frac{1}{\lambda-1}\overrightarrow{OA} + \frac{\lambda-1+1}{\lambda-1}\overrightarrow{OB}$

- since $\overrightarrow{AB'} = \mu \overrightarrow{CB'}$ we have $\overrightarrow{AC} = \overrightarrow{AB'} + \overrightarrow{B'C} = (\mu-1)\overrightarrow{CB'}$

$$\Rightarrow \overrightarrow{CB'} = \frac{1}{\mu-1}\overrightarrow{AC}$$

- hence $\overrightarrow{OB'} = \overrightarrow{OC} + \overrightarrow{CB'} = \overrightarrow{OC} + \frac{1}{\mu-1}\overrightarrow{AC} = \overrightarrow{OC} + \frac{1}{\mu-1}(\overrightarrow{OC} - \overrightarrow{OA})$

so $\overrightarrow{OB'} = -\frac{1}{\mu-1}\overrightarrow{OA} + \frac{\mu}{\mu-1}\overrightarrow{OC}$

(*) $\Leftrightarrow \left\{ \begin{array}{l} \overrightarrow{OM} = (1-t)\overrightarrow{OB} - \frac{t}{\mu-1}\overrightarrow{OA} + \frac{t\mu}{\mu-1}\overrightarrow{OC} \\ \overrightarrow{OM} = (1-s)\overrightarrow{OC} - \frac{s}{\lambda-1}\overrightarrow{OA} + \frac{s\lambda}{\lambda-1}\overrightarrow{OB} \end{array} \right\}$ identifying coeffs $\Rightarrow$

$\frac{t}{\mu-1} = \frac{s}{\lambda-1} \Rightarrow t = \frac{\mu-1}{\lambda-1}s$

$1-s = \frac{t\mu}{\mu-1} = \frac{\mu}{\lambda-1}s \Rightarrow s = \frac{\lambda-1}{\lambda+\mu-1}$

$\Rightarrow t = \frac{\mu-1}{\lambda+\mu-1}$

$\Rightarrow \overrightarrow{OM}\cdot(\lambda+\mu-1) = (\lambda+\mu-1-\mu+1)\overrightarrow{OB} - \overrightarrow{OA} + \mu\overrightarrow{OC}$ $\square$

4. Consider the triangle ABC in $\mathbb{E}^n$ with side lengths a, b, c. Let G be its centroid, H the orthocenter and I the incenter. For a fixed but arbitrary point $O \in X$, show that

1. $\overrightarrow{OG} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$,

2. $\overrightarrow{OI} = \frac{a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}}{a+b+c}$,

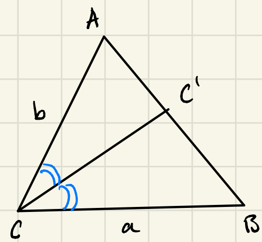
3. $\overrightarrow{OH} = \frac{(\tan\hat{A})\overrightarrow{OA}+(\tan\hat{B})\overrightarrow{OB}+(\tan\hat{C})\overrightarrow{OC}}{\tan\hat{A}+\tan\hat{B}+\tan\hat{C}}$.

In all cases we apply Exercise 3.

1.) let CC' be the median from C. Then $\overrightarrow{AC'} = (-1)\,\overrightarrow{BC'}$ so $\lambda = -1$

let BB' —— " —— B. Then $\overrightarrow{AB'} = (-1)\,\overrightarrow{CB'}$ so $\mu = -1$

$$\Rightarrow \overrightarrow{OG} = \frac{\overrightarrow{OA} - (-1)\overrightarrow{OB} - (-1)\overrightarrow{OC}}{1-(-1)-(-1)}$$



2.) let CC' be the angle bisector of the angle $\hat{C}$.

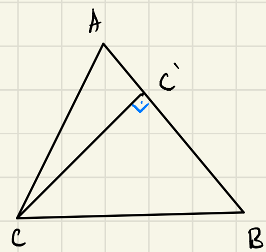
Then, by the angle bisector the

$$\frac{\|\overrightarrow{AC'}\|}{\|\overrightarrow{BC'}\|} = \frac{b}{a}, \text{ so } \overrightarrow{AC'} = -\frac{\|\overrightarrow{AC'}\|}{\|\overrightarrow{BC'}\|}\,\overrightarrow{BC'} = -\frac{b}{a}\,\overrightarrow{BC'} \Rightarrow \lambda = -\frac{b}{a}$$

let BB' be the angle bisector of the angle $\hat{B}$.

Then, similarly $\overrightarrow{AB'} = -\frac{c}{a}\,\overrightarrow{CB'}$ $\Rightarrow \mu = -\frac{c}{a}$

$$\Rightarrow \overrightarrow{OI} = \frac{\overrightarrow{OA} - \left(-\frac{b}{a}\right)\overrightarrow{OB} - \left(-\frac{c}{a}\right)\overrightarrow{OC}}{1-\left(-\frac{b}{a}\right)-\left(-\frac{c}{a}\right)}$$



3.) let CC' be the altitude on the side AB

$$\tan\hat{A} = \frac{|CC'|}{|AC'|} \text{ and } \tan\hat{B} = \frac{|CC'|}{|BC'|} \Rightarrow \frac{\tan\hat{B}}{\tan\hat{A}} = \frac{\|\overrightarrow{AC'}\|}{\|\overrightarrow{C'B}\|}$$

$$\Rightarrow \overrightarrow{AC'} = \frac{\tan\hat{B}}{\tan\hat{A}}\,\overrightarrow{C'B} = -\frac{\tan\hat{B}}{\tan\hat{A}}\,\overrightarrow{BC'} \Rightarrow \lambda = -\frac{\tan\hat{B}}{\tan\hat{A}}$$ and similarly $\mu = -\frac{\tan\hat{C}}{\tan\hat{A}}$

now use the statement in ex 3.

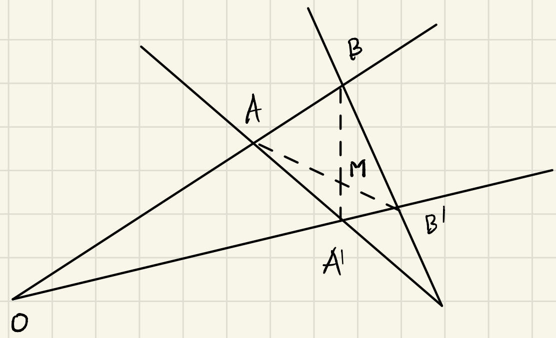
5. In some affine space, consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}$$

where $M = AB' \cap A'B$ and $N = AA' \cap BB'$ and where $\overrightarrow{OB} = m\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\overrightarrow{OA'}$.

Fix a point O



On AB' : $\overrightarrow{OM} = (1-t)\,\overrightarrow{OA} + t\,\overrightarrow{OB'}$

$\overrightarrow{OM} = (1-t)\,\overrightarrow{OA} + t\,n\,\overrightarrow{OA'}$

On $A'B$: $\overrightarrow{OM} = (1-s)\,\overrightarrow{OA'} + s\,\overrightarrow{OB}$

$\overrightarrow{OM} = (1-s)\,m\,\overrightarrow{OA'} + s\,\overrightarrow{OB}$

identifying coefficients we get.

$$\begin{cases} (1-t) = (1-s)\,m \\ tn = s \end{cases} \Rightarrow (1-t) = (1-tn)\,m \Rightarrow 1-m = (1-nm)\,t$$

$\Rightarrow t = \dfrac{1-m}{1-nm}$

$\Rightarrow s = n\dfrac{1-m}{1-nm}$

$\Rightarrow \overrightarrow{OM} = \left(1 - \dfrac{1-m}{1-nm}\right)\overrightarrow{OA} + n\,\dfrac{1-m}{1-nm}\,\overrightarrow{OA'} = \dots$ we obtain the first equality.

For the second one:

On AA' : $\overrightarrow{OM} = (1-t)\,\overrightarrow{OA} + t\,\overrightarrow{OA'}$

On BB' : $\overrightarrow{OM} = (1-s)\,\overrightarrow{OB} + s\,\overrightarrow{OB'} = m(1-s)\,\overrightarrow{OA} + n\,s\,\overrightarrow{OA'}$

identifying coefficients we get

$$\begin{cases} (1-t) = (1-s)\,m \\ t = sn \end{cases} \Rightarrow 1 - sn = m - ms \Rightarrow 1-m = (n-m)\,s$$

$\Rightarrow s = \dfrac{1-m}{n-m}$

$\Rightarrow \overrightarrow{OM} = m\left(1 - \dfrac{1-m}{n-m}\right)\overrightarrow{OA} + n\,\dfrac{1-m}{n-m}\,\overrightarrow{OA'} = \dots$ we obtain the second equality

Exercise 6. Show that the midpoints of the diagonals of a complete quadrilateral are collinear.

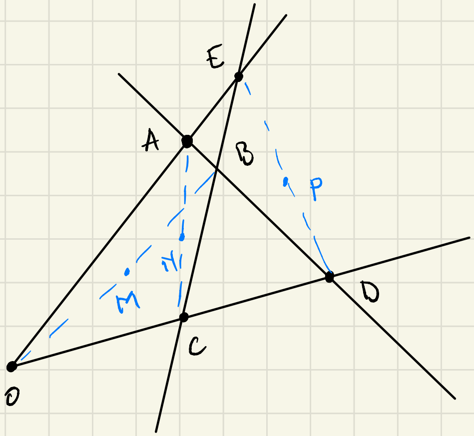


figure determined by four lines, no three of which are concurrent

(the intersection of these lines are the vertices of the complete quadrilateral)

We use the notation in the picture above

- OAEBDC complete quadrilateral
- M midpoint of diagonal OB
- H —— " —— AC
- P —— " —— ED

We have
- $\vec{OH} = \frac{1}{2}\vec{OA} + \frac{1}{2}\vec{OC}$
- $\vec{OP} = \frac{1}{2}\vec{OE} + \frac{1}{2}\vec{OD} = \frac{1}{2} m \vec{OA} + \frac{1}{2} n \vec{OC}$
- $\vec{OM} = \frac{1}{2}\vec{OB}$

$$\vec{OB} = m\frac{1-n}{1-mn}\vec{OA} + n\frac{1-m}{1-mn}\vec{OC} \quad \text{(by ex 4)}$$

$$\Rightarrow \vec{MP} = \vec{OP} - \vec{OM} = \left(\frac{1}{2}m - \frac{1}{2}m\frac{1-n}{1-mn}\right)\vec{OA} + \left(\frac{1}{2}n - \frac{1}{2}n\frac{1-m}{1-mn}\right)\vec{OC}$$

$$= \frac{-mn}{2(1-mn)}\left[(1-m)\vec{OA} + (1-n)\vec{OC}\right]$$

which is parallel to

$$\vec{HP} = \vec{OP} - \vec{OH} = \frac{1}{2}\left[(m-1)\vec{OA} + (n-1)\vec{OC}\right]$$

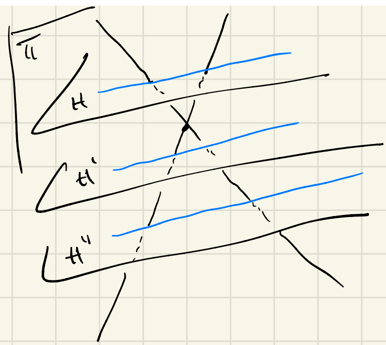
So M, H, P are collinear

Exercise 7. Prove the following generalziation of Thales' theorem.

In an affine space **A** over **K** let H, H', H'' be three distinct parallel hyperplanes, and ℓ_1 and ℓ_2 be lines which are not parallel to H, H', H''. Let $P_i = \ell_i \cap H$, $P_i' = \ell_i \cap H'$, $P_i'' = \ell_i \cap H''$ (for $i = 1, 2$), and let k_1, k_2 be the scalars such that

$$\overrightarrow{P_iP_i''} = k_i \overrightarrow{P_iP_i'} \quad i = 1, 2.$$

Then $k_1 = k_2$.



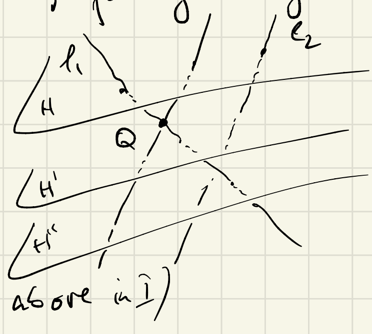
I
- If. l_1 and l_2 intersect in a point then they are coplanar: $l_1, l_2 \subseteq$ plane Π
- Since H, H', H'' are parallel hyperplanes they intersect Π in parallel lines h, h', h''
- So the statement follows from Thales' theorem if we consider h, h', h'', l_1, l_2 in Π

II if. l_1 and l_2 do not intersect (they are skew) choose a point Q on l_1 and a line l_3 passing throug Q and parallel to l_2 and let $P_3 = l_3 \cap H$

$P_3' = l_3 \cap H'$

$P_3'' = l_3 \cap H''$

From the construction $\overrightarrow{P_3P_3''} = k_2 \overrightarrow{P_3P_3'}$



- Using Thales' theorem for l_1 and l_3 (like we did above in I)

we obtain $k_1 = k_2$