Integrals

Basic Properties/Formulas/Rules

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant. $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$ where $F(x) = \int f(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^a f(x)\,dx = 0$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $\int_a^b c\,dx = c(b-a)$

If $f(x) \geq 0$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$

If $f(x) \geq g(x)$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Common Integrals

Polynomials

$\int dx = x + c$ $\int k\,dx = k\,x + c$ $\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\int \frac{1}{x}dx = \ln|x| + c$ $\int x^{-1}\,dx = \ln|x| + c$ $\int x^{-n}dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$

$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$ $\int x^{\frac{p}{q}}dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$

Trig Functions

$\int \cos u\,du = \sin u + c$ $\int \sin u\,du = -\cos u + c$ $\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$ $\int \csc u \cot u\,du = -\csc u + c$ $\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$ $\int \cot u\,du = \ln|\sin u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$ $\int \sec^3 u\,du = \frac{1}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + c$

$\int \csc u\,du = \ln|\csc u - \cot u| + c$ $\int \csc^3 u\,du = \frac{1}{2}\left(-\csc u \cot u + \ln|\csc u - \cot u|\right) + c$

Exponential/Logarithm Functions

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$ $\int a^u\,du = \frac{a^u}{\ln a} + c$ $\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\sin(bu) - b\cos(bu)\right) + c$ $\int u\mathbf{e}^u du = (u-1)\mathbf{e}^u + c$

$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c$ $\int \frac{1}{u\ln u}du = \ln|\ln u| + c$

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Tabel de integrale nedefinite

1.	$\int dx = x + C$	21.	$\int \frac{dx}{\sin^2 x} = -ctg\,x + C$
2.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	22.	$\int \frac{dx}{\cos^2 x} = tg\,x + C$
3.	$\int x\,dx = \frac{x^2}{2} + C$	23.	$\int (1 + tg^2 x)\,dx = tg\,x + C$
4.	$\int \sqrt{x}\,dx = \frac{2}{3}x\sqrt{x} + C$	24.	$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$
5.	$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$	25.	$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \ln\left\lvert\frac{x-1}{x+1}\right\rvert + C$
6.	$\int \frac{dx}{x} = \ln\lvert x\rvert + C$	26.	$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} arctg\,\frac{x}{a} + C$
7.	$\int \frac{dx}{ax+b} = \frac{1}{a}\ln\lvert ax+b\rvert + C$	27.	$\int \frac{dx}{x^2+1} = arctg\,x + C$
8.	$\int \frac{dx}{x^2} = -\frac{1}{x} + C$	28.	$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + C$
9.	$\int \ln x dx = x\ln x - x + C$	29.	$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left\lvert x + \sqrt{x^2 - a^2}\right\rvert + C$
10.	$\int e^x dx = e^x + C$	30.	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$
11.	$\int e^{-x} dx = -e^{-x} + C$	31.	$\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$
12.	$\int a^x\,dx = \frac{a^x}{\ln a} + C$	32.	$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + C$
13.	$\int \sin x\,dx = -\cos x + C$	33.	$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} + C$
14.	$\int \cos x\,dx = \sin x + C$	34.	$\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 + a^2}) + C$
15.	$\int tg\,x\,dx = -\ln\lvert\cos x\rvert + C$	35.	$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left\lvert x + \sqrt{x^2 - a^2}\right\rvert + C$
16.	$\int ctg\,x\,dx = \ln\lvert\sin x\rvert + C$	36.	$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$
17.	$\int f'(x)\,dx = f(x) + C$	37.	$\int f(x)\,dx = F(x) + C$
18.	$\int f(x)f'(x)\,dx = \frac{f^2(x)}{2} + C$	38.	$\int f(x)F(x)\,dx = \frac{F^2(x)}{2} + C$
19.	$\int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x)\rvert + C$	39.	$\int \frac{f(x)}{F(x)} dx = \ln\lvert F(x)\rvert + C$
20.	$\int f^n(x)f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + C$	40.	$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

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Tabel de integrale nedefinite

21.	$\int \frac{1}{\sin^2 x} dx = -\,ctg\,x + C$	1.	$\int dx = x + C$
22.	$\int \frac{1}{\cos^2 x} dx = tg\,x + C$	2.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
23.	$\int (1 + tg^2 x)\, dx = tg\,x + C$	3.	$\int x\, dx = \frac{x^2}{2} + C$
24.	$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$	4.	$\int \sqrt{x}\, dx = \frac{2}{3} x\sqrt{x} + C$
25.	$\int \frac{1}{x^2 - 1} dx = \frac{1}{2} \ln\left\lvert\frac{x-1}{x+1}\right\rvert + C$	5.	$\int \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x} + C$
26.	$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} arctg\, \frac{x}{a} + C$	6.	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
27.	$\int \frac{1}{x^2 + 1} dx = arctg\, x + C$	7.	$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln\lvert ax+b\rvert + C$
28.	$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}) + C$	8.	$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$
29.	$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left\lvert x + \sqrt{x^2 - a^2}\right\rvert + C$	9.	$\int \ln x\, dx = x\ln x - x + C$
30.	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$	10.	$\int e^x dx = e^x + C$
31.	$\int \frac{1}{\sqrt{1 - x^2}} dx = \arcsin x + C$	11.	$\int e^{-x} dx = -e^{-x} + C$
32.	$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + C$	12.	$\int a^x\, dx = \frac{a^x}{\ln a} + C$
33.	$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} + C$	13.	$\int \sin x\, dx = -\cos x + C$
34.	$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + C$	14.	$\int \cos x\, dx = \sin x + C$
35.	$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln\left\lvert x + \sqrt{x^2 - a^2}\right\rvert + C$	15.	$\int tg\, x\, dx = -\ln\lvert\cos x\rvert + C$
36.	$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$	16.	$\int ctg\, x\, dx = \ln\lvert\sin x\rvert + C$
37.	$\int f(x)\, dx = F(x) + C$ F primitiva funcției f	17.	$\int f'(x)\, dx = f(x) + C$
38.	$\int F(x) f(x)\, dx = \frac{F^2(x)}{2} + C$	18.	$\int f(x) f'(x)\, dx = \frac{f^2(x)}{2} + C$
39.	$\int \frac{f(x)}{F(x)} dx = \ln\lvert F(x)\rvert + C$	19.	$\int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x)\rvert + C$
40.	$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$	20.	$\int f^n(x) f'(x)\, dx = \frac{f^{n+1}(x)}{n+1} + C$

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Derivate

1. $c' = 0$
2. $x' = 1$
3. $(x^n)' = nx^{n-1}$
4. $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$
5. $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$
6. $(lnx)' = \frac{1}{x}$
7. $(e^x)' = e^x$
8. $(a^x)' = a^x\, lna$
9. $(sin\, x)' = cos\, x$
10. $(cos\, x)' = -sin\, x$
11. $(tg\, x)' = \frac{1}{cos^2 x} = 1 + tg^2 x$
12. $(ctg\, x)' = -\frac{1}{sin^2 x}$
13. $(arctg\, x)' = \frac{1}{x^2+1}$
14. $(arcsin\, x)' = \frac{1}{\sqrt{1-x^2}}$
15. $\left(\sqrt{x^2 \pm a^2}\right)' = \frac{x}{\sqrt{x^2 \pm a^2}}$
16. $\left(\sqrt{a^2 - x^2}\right)' = -\frac{x}{\sqrt{a^2 - x^2}}$
17. $(arcctg\, x)' = -\frac{1}{x^2+1}$
18. $(arccos\, x)' = -\frac{1}{\sqrt{1-x^2}}$
19. $\left(\sqrt[n]{x}\right)' = \frac{1}{n\sqrt[n]{x^{n-1}}}$
20. $(log_a\, x)' = \frac{1}{x\, lna}$

Integrale

1. $\int dx = x + C$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
3. $\int \sqrt{x}\, dx = \frac{2}{3}x\sqrt{x} + C$
4. $\int \frac{1}{x^2} dx = -\frac{1}{x} + C$
5. $\int \frac{1}{x} dx = ln|x| + C$
6. $\int e^x dx = e^x + C$
7. $\int a^x\, dx = \frac{a^x}{ln\, a} + C$
8. $\int cos\, x\, dx = sin\, x + C$
9. $\int sin\, x\, dx = -cos\, x + C$
10. $\int \frac{1}{cos^2 x} dx = tg\, x + C$
11. $\int \frac{1}{sin^2 x} dx = -ctg\, x + C$
12. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} arctg\, \frac{x}{a} + C$
13. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = arcsin\, \frac{x}{a} + C$
14. $\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} + C$
15. $\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} + C$
16. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\, ln\left|\frac{x-a}{x+a}\right| + C$
17. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = ln(x + \sqrt{x^2 + a^2}) + C$
18. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = ln\left|x + \sqrt{x^2 - a^2}\right| + C$
19. $\int tg\, x\, dx = -ln\, |cos\, x| + C$
20. $\int ctg\, x\, dx = ln\, |sin\, x| + C$
21. $\int \frac{1}{ax+b} dx = \frac{1}{a} ln|ax+b| + C$
22. $\int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} ln|x^2 \pm a^2| + C$

Derivatives

Basic Properties/Formulas/Rules

$\frac{d}{dx}\left(cf(x)\right) = cf'(x)$, c is any constant. $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, n is any number. $\frac{d}{dx}(c) = 0$, c is any constant.

$(f\,g)' = f'\,g + f\,g'$ – **(Product Rule)** $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ – **(Quotient Rule)**

$\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)g'(x)$ **(Chain Rule)**

$\frac{d}{dx}\left(\mathbf{e}^{g(x)}\right) = g'(x)\,\mathbf{e}^{g(x)}$ $\frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)}$

Common Derivatives

Polynomials

$\frac{d}{dx}(c) = 0$ $\frac{d}{dx}(x) = 1$ $\frac{d}{dx}(cx) = c$ $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\frac{d}{dx}\left(cx^n\right) = ncx^{n-1}$

Trig Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trig Functions

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$

Exponential/Logarithm Functions

$\frac{d}{dx}\left(a^x\right) = a^x \ln(a)$ $\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$

$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x},\quad x > 0$ $\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x},\quad x \neq 0$ $\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x \ln a},\quad x > 0$

Hyperbolic Trig Functions

$\frac{d}{dx}(\sinh x) = \cosh x$ $\frac{d}{dx}(\cosh x) = \sinh x$ $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$ $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Trig Substitutions

If the integral contains the following root use the given substitution and formula.

$$\sqrt{a^2-b^2x^2} \quad \Rightarrow \quad x=\frac{a}{b}\sin\theta \quad \text{and} \quad \cos^2\theta=1-\sin^2\theta$$

$$\sqrt{b^2x^2-a^2} \quad \Rightarrow \quad x=\frac{a}{b}\sec\theta \quad \text{and} \quad \tan^2\theta=\sec^2\theta-1$$

$$\sqrt{a^2+b^2x^2} \quad \Rightarrow \quad x=\frac{a}{b}\tan\theta \quad \text{and} \quad \sec^2\theta=1+\tan^2\theta$$

Partial Fractions

If integrating $\int \frac{P(x)}{Q(x)}\,dx$ where the degree (largest exponent) of $P(x)$ is smaller than the degree of $Q(x)$ then factor the denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Products and (some) Quotients of Trig Functions

$\int \sin^n x\cos^m x\,dx$

1. **If n is odd.** Strip one sine out and convert the remaining sines to cosines using $\sin^2 x=1-\cos^2 x$, then use the substitution $u=\cos x$
2. **If m is odd.** Strip one cosine out and convert the remaining cosines to sines using $\cos^2 x=1-\sin^2 x$, then use the substitution $u=\sin x$
3. **If n and m are both odd.** Use either 1. or 2.
4. **If n and m are both even.** Use double angle formula for sine and/or half angle formulas to reduce the integral into a form that can be integrated.

$\int \tan^n x\sec^m x\,dx$

1. **If n is odd.** Strip one tangent and one secant out and convert the remaining tangents to secants using $\tan^2 x=\sec^2 x-1$, then use the substitution $u=\sec x$
2. **If m is even.** Strip two secants out and convert the remaining secants to tangents using $\sec^2 x=1+\tan^2 x$, then use the substitution $u=\tan x$
3. **If n is odd and m is even.** Use either 1. or 2.
4. **If n is even and m is odd.** Each integral will be dealt with differently.

Convert Example : $\cos^6 x=\left(\cos^2 x\right)^3=\left(1-\sin^2 x\right)^3$

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SUBSTITUTION IN INTEGRALS

a) $$\int \frac{1}{1+\frac{1}{\sin x}}\,dx = \int \frac{\sin x}{\sin x+1}\,dx =$$

$$= \int \frac{\sin x+1}{\sin x+1}\,dx - \int \frac{1}{\sin x+1}\,dx =$$

$$= 1 - \int \frac{1}{\sin x+1}\,dx = 1 - \int \frac{1}{\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}+1} \quad 1+tg^2\frac{x}{2} =$$

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$$= 1 - \int \frac{1+tg^2\frac{x}{2}}{tg^2\frac{x}{2}+1+2tg\frac{x}{2}} = \cancel{1} = 1 + \int \frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+2tg\frac{x}{2}+1}\,dx.$$

$$tg\frac{x}{2} = u \Rightarrow$$

$$\Rightarrow \frac{x}{2} = arctg\, u \Rightarrow x = 2arctg\, u \;|\, d \Rightarrow$$

$$\Rightarrow dx = 2\cdot\frac{1}{u^2+1}\,du$$

$$I = \int \frac{2u}{u^2+2u+1}\cdot\frac{1}{u^2+1}\,du = \int \frac{2u}{(u+1)^2}\cdot\frac{1}{u^2+1}\,du =$$

$$= \int \frac{(u+1)^2-u^2-1}{(u+1)^2}\cdot\frac{1}{u^2+1}\,du = \int\left(\frac{(u+1)^2}{(u+1)^2}-\frac{u^2+1}{(u+1)^2}\right)\cdot\frac{1}{u^2+1}\,du =$$

$$= \int\left(1-\frac{u^2+1}{(u+1)^2}\right)\cdot\frac{1}{u^2+1}\,du =$$

$$= \int \frac{1}{u^2+1}\,du - \int \frac{\cancel{u^2+1}}{(u+1)^2}\cdot\frac{1}{\cancel{(u^2+1)}}\,du =$$

$$= arctg\, u - \frac{(u+1)^{-1}}{-1} = arctg\left(tg\frac{x}{2}\right) - \frac{\left(tg\frac{x}{2}+1\right)^{-1}}{-1} =$$

$$= \frac{x}{2} + \frac{1}{tg\frac{x}{2}+1}$$

b) $\int \frac{1}{3\sin x + 4\cos x} dx = \int \frac{1}{3 \cdot \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} + 4 \cdot \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}} dx =$

$= \int \frac{1+tg^2\frac{x}{2}}{6tg\frac{x}{2}+4-4tg^2\frac{x}{2}} dx = \int \frac{\cancel{1+u^2}}{\cancel{2}(-2u^2+3u+2)} \cdot \cancel{2} \cdot \frac{1}{\cancel{u^2+1}} du$

$u = tg\frac{x}{2} \Rightarrow arctg\, u = \frac{x}{2} \Rightarrow 2arctg\, u = x$

$dx = 2 \frac{1}{u^2+1}$

$= -\int \frac{du}{2u^2-3u-2} = \int \frac{1}{(u-2)(2u+1)} du$

$= \int \left(\frac{1}{5(u-2)} - \frac{2}{5(2u+1)} \right) du =$

$= \frac{1}{5} \int \frac{1}{u-2} du - \frac{2}{5} \int \frac{1}{2u+1} du =$

$= \frac{1}{5} \ln(u-2) - \frac{2}{5} \int \frac{1}{2u+1} du = \frac{1}{5} \ln(u-2) - \frac{1}{5} \int \frac{(2u+1)'}{2u+1} du$

$= \frac{1}{5} \ln(u-2) - \frac{1}{5} \ln(2u+1) =$

$= \frac{1}{5} \ln \frac{u-2}{2u+1} = \frac{1}{5} \ln \frac{tg\frac{x}{2}-2}{2tg\frac{x}{2}+1}$

c) $\int \frac{\sqrt{9-x^2}}{x^2} dx = \int \frac{\sqrt{9-(3\sin u)^2}}{9\sin^2 u} \cdot \cos u\, du =$

$x = 3\sin u$

$dx = \cos u\, du$

$= \int \frac{\sqrt{9-9\sin^2 u}}{9\sin^2 u} \cdot \cos u\, du = \int \frac{\cancel{3}\sqrt{1-\sin^2 u}}{\cancel{9}_3 \sin^2 u} \cdot \cos u\, du =$

$= \int \frac{1}{3} \frac{\sqrt{\cos^2 u}}{\sin^2 u} \cdot \cos u\, du = \int \frac{1}{3} \frac{\cos^2 u}{\sin^2 u} du = \int \frac{1}{3} ctg^2 u\, du$

$$= \frac{1}{3}\int ctg^2 u\, du = \frac{1}{3}\int \frac{\cos^2 u}{\sin^2 u}\, du = \frac{1}{3}\int \frac{\frac{1}{1+tg^2 u}}{\frac{tg^2 u}{1+tg^2 u}}\, du$$

$$= \frac{1}{3}\int \frac{1}{tg^2 u}\, du = \frac{1}{3}\int \frac{1}{\left(\frac{\sin a}{\cos a}\right)^2} =$$

$$= \frac{1}{3}\int \frac{1}{\left(\frac{\frac{2tg\frac{u}{2}}{1+tg^2\frac{u}{2}}}{\frac{1-tg^2\frac{u}{2}}{1+tg^2\frac{u}{2}}}\right)^2} = \frac{1}{3}\int \frac{1}{\left(\frac{2tg\frac{u}{2}}{1-tg^2\frac{u}{2}}\right)^2}\, du =$$

$$= \frac{1}{3}\int \left(\frac{1-tg^2\frac{u}{2}}{2tg\frac{u}{2}}\right)^2 du = \frac{1}{3}\int \frac{1-2tg^2\frac{u}{2}+tg^4\frac{u}{2}}{2tg^2\frac{u}{2}}\, du =$$

$$= \frac{1}{3}\int \cancel{\frac{-2tg^2\frac{u}{2}}{2tg^2\frac{u}{2}}}\, du + \frac{1}{3}\int \frac{1+tg^4\frac{u}{2}}{2tg^2\frac{u}{2}}\, du =$$

$$= -\frac{1}{3}u + \frac{1}{3}\int \frac{1+v^4}{\cancel{2}v^2}\cdot \cancel{2}\cdot \frac{1}{v^2+1}\, dv =$$

$$v = tg^2\frac{u}{2} \Rightarrow u = 2\, arctg\, v$$

$$du = 2\cdot\frac{1}{v^2+1}$$

$$= -\frac{1}{3}u + \frac{1}{3}\int \frac{1+v^4}{v^2(v^2+1)}\, du$$

$$1+v^4 = t \Rightarrow ?$$

$$\int \frac{\sqrt{9-x^2}}{x^2}\, dx = -\int \sqrt{9-x^2}\cdot\left(\frac{1}{x}\right)' dx$$

$$= -\frac{\sqrt{9-x^2}}{x} + \int \frac{1}{\sqrt{9-x^2}}\, dx = -\frac{\sqrt{9-x^2}}{2} + \arcsin\frac{x}{3}$$

d) $\int \frac{1}{\sqrt{(x^2+1)}^3}dx = \int \frac{1}{(x^2+1)\sqrt{x^2+1}}dx =$

$x = \tan(u)$

$\operatorname{arctg} x = u \Rightarrow \frac{1}{x^2+1} \cdot dx = du$

$= \int \frac{1}{\sqrt{tg^2 u+1}}du = \int \frac{1}{\sqrt{\frac{\sin^2 u}{\cos^2 u}+1}}du =$

$= \int \frac{\cos u}{\sqrt{\sin^2 u+\cos^2 u}}du = \int \frac{\cos u}{\sqrt{1}}du = \sin x + C \Rightarrow$

$\Rightarrow I = \sin(\operatorname{arctg} x) + C =$

$= \frac{x}{\sqrt{x^2+1}} + C$

e) $\int \frac{1}{\sqrt{(x^2-8)}^3}dx = \int \frac{1}{(x^2-8)^{\frac{3}{2}}}dx =$

$x = \frac{2^{\frac{3}{2}}}{\cos t} \Rightarrow \frac{x}{2^{\frac{3}{2}}} = \frac{1}{\cos t} \Rightarrow$

$\Rightarrow \frac{2^{\frac{3}{2}}}{x} = \cos t \Rightarrow \arccos\left(\frac{2^{\frac{3}{2}}}{x}\right) = t$

$dx = 2^{\frac{3}{2}} \cdot -\frac{1}{\cos^2 t} \cdot \sin t\, dt$

$= \int \frac{1}{\left(\left(\frac{2^{\frac{3}{2}}}{\cos t}\right)^2 - 8\right)^{\frac{3}{2}}} \cdot 2^{\frac{3}{2}} \cdot -\frac{1}{\cos^2 t} \cdot \sin t\, dt =$

$= \int \frac{\cos^{2\cdot\frac{3}{2}} t}{\left(2^{\frac{3}{2}\cdot 2} - 8\cos^2 t\right)^{\frac{3}{2}}} \cdot 2^{\frac{3}{2}} \cdot -\frac{1}{\cos^2 t} \cdot \sin t\, dt =$

$= \int \frac{\cos^3 t}{(8-8\cos^2 t)^{\frac{3}{2}}} \cdot 2^{\frac{3}{2}} \cdot -\frac{1}{\cos^2 t} \cdot \sin t\, dt$

$$=\int \frac{\cos t\cdot \sin t}{8^{\frac{3}{2}}\cdot(1-\cos^2 t)^{\frac{3}{2}}}\cdot 2^{\frac{3}{2}}\,dt$$

$$=-\int \frac{-\cos t\sin t\cdot 2^{\frac{3}{2}}}{2^{3\cdot\frac{3}{2}}(1-\cos^2 t)^{\frac{3}{2}}}dt=$$

$$=-\int \frac{-\cos t\sin t\, 2^{\frac{3}{2}-3\cdot\frac{3}{2}}}{(1-\cos^2 t)^{\frac{3}{2}}}dt=-\int\frac{\cos t\sin t\cdot 2^{\frac{3-9}{2}}}{(1-\cos^2 t)^{\frac{3}{2}}}dt=$$

$$=-\int \frac{\cos t\sin t\, 2^{\frac{-6}{2}}}{(1-\cos^2 t)^{\frac{3}{2}}}dt=-\int \frac{2^{-3}\cos t\sin t}{(1-\cos^2 t)^{\frac{3}{2}}}dt=$$

$$\cos^2 t'=-2\cos t\cdot\sin t$$

$$=-2^{-4}\int\frac{(1-\cos^2 t)'}{(1-\cos^2 t)^{\frac{3}{2}}}dt=-2^{-4}\cdot\frac{(1-\cos^2 t)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}dt=$$

$$=-2^{-4}\cdot\frac{(\sin^2 t)^{-\frac{1}{2}}}{-\frac{1}{2}}=2^{-4}\cdot 2\cdot(\sin^2 t)^{-\frac{1}{2}}=$$

$$=2^{-3}\cdot\frac{1}{\sin t}=2^{-3}\cdot\frac{1}{\sin\left(\arccos\frac{2^{\frac{3}{2}}}{x}\right)}=$$

$$=2^{-3}\cdot\frac{1}{\sqrt{1-\left(\frac{2^{\frac{3}{2}}}{x}\right)^2}}=2^{-3}\cdot\frac{x}{\sqrt{x^2-8}}=\frac{x}{8\sqrt{x^2-8}}+C$$

f) $\int\sqrt{2x-x^2}\,dx=\int\sqrt{1-(1-x)^2}\,dx=$

$1-x=1\cos t \Rightarrow x=1-\cos t \Rightarrow \arccos(1-x)=t$

$dx=1+\sin t\,dt$

$$=\int\sqrt{1-(\cos t)^2}(1+\sin t)\,dt$$

$$=\int \sqrt{\sin^2 t}\,(1+\sin t)\,dt=\int \sin t\,(1+\sin t)\,dt=$$

$$=-\int \cancel{(\cos t)'}\,(1+\sin t)\,dt=\int (\sin t+\sin^2 t)\,dt=$$

$$=\int \sin t\,dt+\int \sin^2 t\,dt=$$

$$=-\cos t+\int \frac{1-\cos(2t)}{2}\,dt=$$

$$=-\cos t+\frac{1}{2}t-\frac{1}{2}\int \cos(2t)\,dt=$$

$$=-\cos t+\frac{1}{2}t-\frac{1}{2}\,\frac{\sin(2t)}{2}=$$

$$=-\cos t+\frac{1}{2}t-\frac{1}{4}\sin(2t)=$$

$$=-\cos(\arccos(1-x))+\frac{1}{2}\arccos(1-x)-\frac{1}{4}\sin(2\arccos(1-x))$$

$$==x-1+\frac{1}{2}\arccos(1-x)-\frac{1}{4}\sqrt{1-(1-x)^2}$$

g) $\int \sqrt{4-x^2}\,dx=$

$x=2\sin t \Rightarrow \frac{x}{2}=\sin t \Rightarrow \arcsin\frac{x}{2}=t$

$dx=2\cos t$

$$\int \sqrt{4-(2\sin t)^2}\cdot 2\cos t\,dt=\int \sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt=$$

$$=\int 2\sqrt{1-\sin^2 t}\cdot 2\cos t\,dt=\int 2\sqrt{\cos^2 t}\cdot 2\cos t\,dt=$$

$$=4\int \cos^2 t\,dt=4\int \frac{\cos(2t)-1}{2}\,dt=$$

$$=2\int \cos(2t)\,dt-2\int dt=\cancel{2}\cdot\frac{\sin(2t)}{\cancel{2}}-2t=$$

$$=2\sin t\cos t-2t=2\sin\left(\arcsin\frac{x}{2}\right)\cos\left(\arcsin\frac{x}{2}\right)-2\arcsin\frac{x}{2}$$

$$=\cancel{2}\cdot\frac{x}{\cancel{2}}\cdot\sqrt{1-\left(\frac{x}{2}\right)^2}-2\arcsin\frac{x}{2}=\frac{x}{2}\sqrt{4-x^2}-2\arcsin\frac{x}{2}$$

h) $\int x\sqrt{1+x^2}\,dx = \int tg\,t \cdot \sqrt{1+tg^2 t} \cdot \frac{1}{\cos^2 t}\,dt =$

$x = 1\,tg\,t \Rightarrow arctg\,x = t$

$dx = \frac{1}{\cos^2 t}\,dt \quad \frac{1}{x^2+1}\,dx = dt$

$= \int tg\,t \cdot \sqrt{\frac{\cos^2 t + \sin^2 t}{\cos^2 t}} \cdot \frac{1}{\cos^2 t}\,dt =$

$= \int tg\,t \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos^2 t}\,dt = \int \frac{\sin t}{\cos^4 t}\,dt =$

$= -\int \frac{-\sin t\,dt}{\cos^4 t} = -\int \frac{(\cos t)'}{\cos^4 t}\,dt =$

$= -\frac{\cos^{-3} t}{-3} = \frac{1}{3} \cdot \frac{1}{\cos^3 t} = \frac{1}{3} \cdot \frac{1}{\cos^3(arctg\,x)} =$

$= \frac{1}{3} \cdot \frac{1}{(\cos(arctg\,x))^3} = \frac{1}{3} \cdot \frac{1}{\left(\sqrt{1+x^2}\right)^3} + C$

EXERCISE 2

a) $\int \frac{2x-1}{x^2-3x+2}\,dx = \int \frac{2x-2}{x^2-3x+2}\,dx + \int \frac{1}{x^2-3x+2}\,dx =$

$x^2-3x+2 = x^2-x-2x+2 = x(x-1)-2(x-1) = (x-1)(x-2)$

$= \int \frac{2\cancel{(x-1)}}{\cancel{(x-1)}(x-2)}\,dx + \int \frac{1}{\left(x-\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{2}{4}}\,dx =$

$= \int \frac{2}{x-2}\,dx + \int \frac{1}{\left(x-\frac{3}{2}\right)^2 + \frac{-9+8}{4}}\,dx =$

$= 2\int \frac{(x-2)'}{x-2}\,dx + \int \frac{\left(x-\frac{3}{2}\right)'}{\left(x-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\,dx =$

$= 2\ln(x-2) + \frac{1}{2\cdot\frac{1}{2}} \ln\left|\frac{x-\frac{3}{2}-\frac{1}{2}}{x-\frac{3}{2}+\frac{1}{2}}\right| =$

$$= 2\ln(x-2) + \ln\left|\frac{x-2}{x-1}\right| =$$

$$= 2\ln(x-2) + \ln(x-2) - \ln(x-1) =$$

$$= 3\ln(x-2) - \ln(x-1) + C$$

b) $$\int \frac{4}{(x-1)(x+1)^2} dx = \int 4\left(\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}\right) dx$$

$$= \int \frac{4}{4(x-1)} dx - \int \frac{4}{4(x+1)} dx - \int \frac{\cancel{4}^{2}}{\cancel{2}(x+1)^2} dx =$$

$$= \int \frac{1}{x-1} dx - \int \frac{1}{x+1} dx - 2\int \frac{1}{(x+1)^2} dx =$$

$$= \ln(x-1) - \ln(x+1) - 2\frac{(x+1)^{-2+1}}{-2+1} =$$

$$= \ln(x-1) - \ln(x+1) + 2(x+1)^{-1} =$$

$$= \ln(x-1) - \ln(x+1) + 2 \cdot \frac{1}{x+1} + C$$

c) $$\int \frac{1}{x^3 - x^4} dx = \int \frac{1}{x^3(1-x)} dx$$

$$\frac{?}{x^3} + \frac{?}{1-x} = \frac{?}{x} + \frac{?}{x^2} + \frac{?}{x^3} + \frac{?}{1-x} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{1-x}$$

$$\frac{1}{x^3(1-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{1-x} \Leftrightarrow$$

$$\Leftrightarrow 1 = Ax^2(1-x) + Bx(1-x) + C(1-x) + Dx^3$$

$$1 = Ax^2 - Ax^3 + Bx - Bx^2 + C - Cx + Dx^3$$

$$1 = x^3(D-A) + x^2(A-B) + x(B-C) + C$$

$$\left.\begin{array}{l} D-A=0 \\ A-B=0 \\ B-C=0 \\ C=1 \end{array}\right\} \Rightarrow A=B=C=D=1$$

$$\int \frac{1}{x^3(1-x)} dx = \int \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{1-x}\right) dx =$$

$$= \int \frac{1}{x}dx + \int \frac{1}{x^2}dx + \int \frac{1}{x^3}dx + \int \frac{1}{1-x}dx$$

$$= \cancel{\ln|x|} + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + \frac{x^{-3}}{-3} \cancel{- \ln|x|}$$

$$= -\frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3} + C$$

d) $$\int \frac{2x+5}{x^2+5x+10}dx = \int \frac{(x^2+5x+10)'}{x^2+5x+10}dx =$$

$$= \ln(x^2+5x+10) + C$$

c) $$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+\frac{1}{2})^2 - \frac{1}{4} + 1}dx =$$

$$= \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}dx = \frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) =$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

EXERCISE 3

a) $$\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx = \int \frac{(\sqrt{x+1}-\sqrt{x})dx}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})} =$$

$$= \int \frac{\sqrt{x+1}-\sqrt{x}}{\cancel{x}+1\cancel{-x}}dx = \int(\sqrt{x+1}-\sqrt{x})dx =$$

$$= \int \sqrt{x+1}\,dx - \int \sqrt{x}\,dx = \int (x+1)^{\frac{1}{2}}dx - \int x^{\frac{1}{2}}dx =$$

$$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2(x+1)^{\frac{3}{2}} - 2x^{\frac{3}{2}}}{3} + C$$

b) $$I = \int \frac{1}{x+\sqrt{x-1}}dx$$

$$u = \sqrt{x-1} \Rightarrow u^2+1 = x \Rightarrow dx = 2u\,du$$

$$=\int \frac{2u\,du}{u^2+1+\sqrt{u^2+1-1}}=\int \frac{2u\,du}{u^2+1+u}=$$

$$=\int \frac{(2u+1)du}{u^2+u+1}-\int \frac{1}{u^2+u+1}du=$$

$$=\int \frac{(u^2+u+1)'}{u^2+u+1}du-\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2u+1}{\sqrt{3}}\right)+C=$$

$$=\ln(u^2+u+1)-\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2u+1}{\sqrt{3}}\right)+C=$$

$$=\ln(x-1+\sqrt{x-1}+1)-\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2\sqrt{x-1}+1}{\sqrt{3}}\right)+C$$

$$=\ln(x+\sqrt{x-1})-\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2\sqrt{x-1}+1}{\sqrt{3}}\right)+C$$

EXERCISE 4

a) $I=\int \frac{1}{1+\sqrt{x^2+2x-2}}dx=$

$$=\int \frac{1-\sqrt{x^2+2x-2}}{1-x^2-2x+2}dx=\int \frac{1-\sqrt{x^2+2x-2}}{-(x^2+2x-3)}dx=$$

$$=\int \frac{\sqrt{x^2+2x-2}-1}{(x^2+2x-3)}dx=\int \frac{\sqrt{x^2+2x-2}}{x^2+2x-3}dx-\int \frac{1}{x^2+2x-3}dx$$

$$\int \frac{\sqrt{x^2+2x-2}}{x^2+2x-3}dx=\int \frac{\sqrt{(x+1)^2-3}}{(x+1)^2-4}dx=\int \frac{\sqrt{u^2-3}}{u^2-4}$$

$u=x+1 \Rightarrow u-1=x \Rightarrow dx=du$

$$I=\int \frac{1}{1+\sqrt{x^2+2x-2}}dx$$

$\sqrt{x^2+2x-2}=t-x \Rightarrow \cancel{x^2}+2x-2=t^2-2xt+\cancel{x^2} \Rightarrow$

$\Rightarrow 2x=t^2-2xt+2 \Rightarrow 2x+2xt=t^2+2 \Rightarrow$

$\Rightarrow 2x(t+1)=t^2+2 \Rightarrow x=\frac{t^2+2}{2(t+1)}$

$$=\int \frac{1}{(t+2)^2}\cdot\left(\frac{1}{t+1}-\frac{3}{(t+1)^2}\right)dt$$

$$=\int \frac{1}{(t+2)^2}\cdot\frac{1}{t+1}\,dt-\int\frac{3}{(t+2)^2(t+1)^2}\,dt$$

$$=\int -\frac{1}{t+2}\,dt-\int\frac{1}{(t+2)^2}\,dt+\int\frac{1}{(t+1)}\,dt+\int\frac{6}{t+2}\,dt+$$

$$+\int\frac{3\,dt}{(t+2)^2}-\int\frac{6\,dt}{(t+1)}+\int\frac{3}{(t+1)^2}\,dt$$

$$=5\int\frac{1}{t+2}\,dt+2\int\frac{1}{(t+2)^2}\,dt-5\int\frac{1}{t+1}\,dt+3\int\frac{1}{(t+1)^2}\,dt$$

$$=5\ln(t+2)+2\cdot\frac{(t+2)^{-1}}{-1}-5\ln(t+1)+3\,\frac{(t+1)^{-1}}{-1}=$$

$$=5\ln(t+2)-2\cdot\frac{1}{t+2}-5\ln(t+1)-3\cdot\frac{1}{t+1}$$

$$=5\ln\left(\frac{t+2}{t+1}\right)-2\cdot\frac{1}{t+2}-3\,\frac{1}{t+1}$$

$$=5\ln\left(\frac{\sqrt{x^2+2x-2}+2}{\sqrt{x^2+2x-2}-1}\right)-3\,\frac{1}{\sqrt{x^2+2x-2}+1}-2\,\frac{1}{\sqrt{x^2+2x-2}+2}+C$$

Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+c \qquad \int \sin^{-1}u\,du = u\sin^{-1}u+\sqrt{1-u^2}+c$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c \qquad \int \tan^{-1}u\,du = u\tan^{-1}u-\frac{1}{2}\ln\left(1+u^2\right)+c$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)+c \qquad \int \cos^{-1}u\,du = u\cos^{-1}u-\sqrt{1-u^2}+c$$

Hyperbolic Trig Functions

$$\int \sinh u\,du = \cosh u+c \qquad \int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u+c \qquad \int \operatorname{sech}^2 u\,du = \tanh u+c$$

$$\int \cosh u\,du = \sinh u+c \qquad \int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u+c \qquad \int \operatorname{csch}^2 u\,du = -\coth u+c$$

$$\int \tanh u\,du = \ln\left(\cosh u\right)+c \qquad \int \operatorname{sech} u\,du = \tan^{-1}\left|\sinh u\right|+c$$

Miscellaneous

$$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right|+c \qquad \int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+c$$

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right|+c$$

$$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2}-\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+c$$

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+c$$

$$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2}+\frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right)+c$$

Standard Integration Techniques

Note that all but the first one of these tend to be taught in a Calculus II class.

u Substitution

Given $\int_a^b f\left(g\left(x\right)\right)g'\left(x\right)dx$ then the substitution $u=g\left(x\right)$ will convert this into the integral, $\int_a^b f\left(g\left(x\right)\right)g'\left(x\right)dx = \int_{g(a)}^{g(b)} f\left(u\right)\,du$.

Integration by Parts

The standard formulas for integration by parts are,

$$\int u\,dv = uv-\int v\,du \qquad \int_a^b u\,dv = uv\Big|_a^b-\int_a^b v\,du$$

Choose u and dv and then compute du by differentiating u and compute v by using the fact that $v=\int dv$.