Calculus on $\mathbb{R}$
Week 2

Lect. dr. Anca Grad

Homework 2

Exercise 1

Fill in the following table by using ✓when the set is closed, and ✗ when it is not:

$(-1,2]$	$(-1,1)$	$[-1,1]$	$\mathbb{R}\backslash\{1\}$	$\{1,2,3\}$	$\mathbb{R}\backslash(0,1)$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}\backslash\mathbb{Q}$	$\mathbb{R}$
✗	✗	✓	✗	✓	✗	✓	✓	✗	✗

Sketch the proof for some of them.

Exercise 2

Which of the following sets either open or closed? Try to sketch some proofs.

$$A = \bigcup_{n\in\mathbb{N}\backslash\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right), \quad B = \bigcup_{n\in\mathbb{N}} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$$

$$C = \bigcap_{n\in\mathbb{N}\backslash\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right) \quad D = \bigcap_{n\in\mathbb{N}} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$$

$$E = \bigcup_{n\in\mathbb{N}} \left[-1-\frac{1}{n}, 1+\frac{1}{n}\right] \quad D = \bigcap_{n\in\mathbb{N}} \left(-1-\frac{1}{n}, 1+\frac{1}{n}\right)$$

Excercise 3 (Do not turn this in, we will solve it together at the seminar)Fill in the following table:

Nr.	A	int A	bd A	cl A	ext A	Izo A	A'
1	$(-\infty, -1] \cup (2, +\infty)$						
2	$(-1, 9] \cup [10, 20)$						
3	$\Big((-1, 9] \cup [10, 20)\Big) \cap \mathbb{N}$						
4	$\{1, 2, 3\}$						
5	$\mathbb{N}$						
6	$\mathbb{R} \setminus \{1, 2, 3\}$						
7	$\mathbb{R} \setminus \mathbb{N}$						
8	$\mathbb{Z}$						
9	$\mathbb{R} \setminus \mathbb{Z}$						
10	$\mathbb{Q}$						
11	$\mathbb{R} \setminus \mathbb{Q}$						
12	$\mathbb{R}$						

Exercise 1

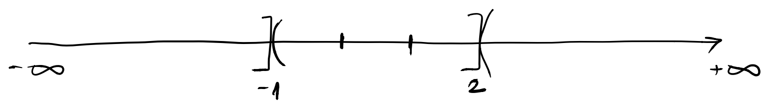
Fill in the following table by using ✓ when the set is closed, and ✗ when it is not:

$(-1,2]$	$(-1,1)$	$[-1,1]$	$\mathbb{R}\backslash\{1\}$	$\{1,2,3\}$	$\mathbb{R}\backslash(0,1)$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}\backslash\mathbb{Q}$	$\mathbb{R}$
✗	✗	✓	✗	✓	✗	✓	✓	✗	✗

Sketch the proof for some of them.

- We shall prove that $(-1,2]$ is not a closed set by CONTRADICTION.

  Assume that $(-1,2]$ is closed. Therefore, $\mathbb{R}\backslash(-1,2]$ is open.

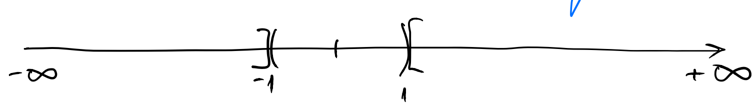
  $\mathbb{R}\backslash(-1,2] = (-\infty, -1] \cup (2, +\infty)$

  $(2, +\infty)$ is open, but $(-\infty, -1]$ is closed, therefore

  $\mathbb{R}\backslash(-1,2]$ is not open $\Rightarrow$ $(-1, 2]$ is not a closed set.

- We shall prove that $(-1,1)$ is not a closed set by CONTRADICTION

  Assume that $(-1,1)$ is closed. Therefore, $\mathbb{R}\backslash(-1,1)$ is open.

  $\mathbb{R}\backslash(-1,1) = (-\infty, -1] \cup [1, +\infty)$

  Both $(-\infty, 1] \cup [1, +\infty)$ are closed and we shall prove this for $[1, +\infty)$.

  $[1, +\infty)$ is closed if $\mathbb{R}\backslash[1,+\infty) = (-\infty, 1)$ is open

  Let $c \in (-\infty, 1)$ be randomly chosen $\Rightarrow \exists r_c > 0$ s.t. $B(c, r_c) \subseteq (-\infty, 1)$

  $c \neq 1 \Rightarrow 1 - c > 0 \Rightarrow \exists r_c := a - c$, then $B(c, a-c) \subseteq (-\infty, a) \Rightarrow$

  $\Rightarrow (-\infty, a) \in \mathcal{V}(c)$

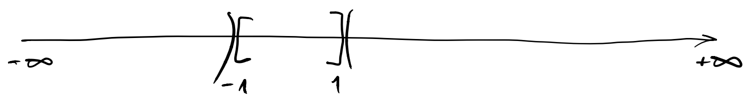
Because c was randomly chosen, then $(-\infty, a)$ is a neighborhood for all its points $\overset{def.}{\iff}$ $(-\infty, 1)$ is an open set $\Rightarrow$ $[1, +\infty)$ is a closed set

Analogically $(-\infty, -1]$ is closed $\Big] \Rightarrow$

$\Rightarrow \mathbb{R} \setminus (-1, 1)$ is closed $\Rightarrow (-1, 1)$ is not closed

- We shall prove that $[-1, 1]$ is a closed set $\iff \mathbb{R} \setminus [-1, 1]$ is open (1)

$\mathbb{R} \setminus [-1, 1] = (-\infty, -1) \cup (1, +\infty)$



We shall prove that $(-\infty, -1)$ is open; consequently $(1, +\infty)$ would be open. (2)

Let $c \in (-\infty, -1)$ be randomly chosen $\Rightarrow \exists r_c > 0$ s.t. $B(c, r_c) \subseteq (-\infty, -1)$

$c \neq -1 \Rightarrow -1 - c > 0 \Rightarrow \exists r_c := -1 - c \Rightarrow$

$\Rightarrow B(c, -1-c) \subseteq (-\infty, -1) \Rightarrow (-\infty, -1) \in \mathcal{V}(c)$

Because c was chosen randomly, $(-\infty, -1)$ is a neighborhood for all its points $\Rightarrow (-\infty, -1)$ is an open set (3)

(1), (2), (3) $\Rightarrow [-1, 1]$ is a closed set

- We shall prove that $\{1, 2, 3\}$ is a closed set $\iff \mathbb{R} \setminus \{1, 2, 3\}$ is open. (1)

$\mathbb{R} \setminus \{1, 2, 3\} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, +\infty)$

We know that $(-\infty, 1)$ & $(3, +\infty)$ are open sets. We shall prove that $(1, 2)$ is open as well and, consequently, $(2, 3)$ will also be open. (2)

Let $c \in (1, 2)$ be arbitrary $\Rightarrow \exists r_c > 0$ s.t. $B(c, r_c) \subseteq (1, 2)$

$c \neq 2 \Rightarrow 2 - c > 0 \Rightarrow \exists r_c := 2 - c \Rightarrow B(c, 2-c) \subseteq (1, 2) \Rightarrow$

$\Rightarrow (1,2) \in \mathcal{V}(c)$

Because c was arbitrary, $(1,2)$ is a neighborhood for all its points

$\Rightarrow (1,2)$ is an open set. (2)

(1), (2) & (3) $\Rightarrow \{1;2;3\}$ is a closed set.

Exercise 2

Which of the following sets either open or closed? Try to sketch some proofs.

$$A = \bigcup_{n\in\mathbb{N}\setminus\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right), \quad B = \bigcup_{n\in\mathbb{N}} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$$

$$C = \bigcap_{n\in\mathbb{N}\setminus\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right) \quad D = \bigcap_{n\in\mathbb{N}} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$$

$$E = \bigcup_{n\in\mathbb{N}} \left[-1-\frac{1}{n}, 1+\frac{1}{n}\right] \quad D = \bigcap_{n\in\mathbb{N}} \left(-1-\frac{1}{n}, 1+\frac{1}{n}\right)$$

1. $A = \bigcup_{n\in\mathbb{N}\setminus\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right)$

For $n=2$: $\left(-1+\frac{1}{2}, 1-\frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

Let $c \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ be chosen randomly $\Rightarrow \exists\, r_c \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ s.t. $B(c, r_c) \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right)$

$c \neq \frac{1}{2} \Rightarrow \frac{1}{2} - c > 0 \Rightarrow \exists\, r_c := \frac{1}{2} - c \Rightarrow$

$\Rightarrow B\left(c, \frac{1}{2} - c\right) \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \left(-\frac{1}{2}, \frac{1}{2}\right) \in \mathcal{V}(c)$

Because c was chosen randomly, $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is a neighborhood for all its points, therefore $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is an open set.

Analogically, every given set $\left(-1+\frac{1}{n}, 1-\frac{1}{2}\right)$, $n \in \mathbb{N}\setminus\{1\}$ is an open set $\Rightarrow A = \bigcup_{n\in\mathbb{N}\setminus\{1\}} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right)$ is an open set.

2. $B = \bigcup_{n\in\mathbb{N}} \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$

For $n=1$: $\left[-1+\frac{1}{1}, 1-\frac{1}{1}\right] = \{0\}$ is a closed set (proven in Exercise 1 for $\{1,2,3\}$)

For $n=2$: $[-\frac{1}{2},\frac{1}{2}]\cup\{0\}=[-\frac{1}{2},\frac{1}{2}]$

To prove that $[-\frac{1}{2},\frac{1}{2}]$ is a closed set, we shall prove that $\mathbb{R}\setminus[-\frac{1}{2},\frac{1}{2}]$ is open. (1)

$\mathbb{R}\setminus[-\frac{1}{2},\frac{1}{2}]=(-\infty,-\frac{1}{2})\cup(\frac{1}{2},+\infty)$

We shall deliver the proof for $(-\infty,-\frac{1}{2})$, as the one for $(\frac{1}{2},+\infty)$ is similar (2)

$$\left.\begin{array}{l}\text{Let } c\in(-\infty,-\frac{1}{2}) \Rightarrow \exists\, r_c>0 \text{ s.t } B(c,r_c)\subseteq(-\infty,-\frac{1}{2}) \\ \qquad c\neq-\frac{1}{2}\Rightarrow -\frac{1}{2}-c>0\Rightarrow \exists\, r_c=-\frac{1}{2}-c\end{array}\right\}\Rightarrow$$

$\Rightarrow B(c,-\frac{1}{2}-c)\subseteq(-\infty,-\frac{1}{2})$

Because c was arbitrary, then $(-\infty,-\frac{1}{2})$ is a neighborhood for all its points $\Rightarrow$ $(-\infty,-\frac{1}{2})$ is an open set. (3)

(1), (2), (3) $\Rightarrow$ $[-\frac{1}{2},\frac{1}{2}]$ is a closed set

Analogically, every given set $[-1+\frac{1}{n},1-\frac{1}{n}]$, $n\in\mathbb{N}$ is a closed set, therefore $B=\bigcup_{n\in\mathbb{N}}[-1+\frac{1}{n},1-\frac{1}{n}]$ is a closed set.

3) $C=\bigcap_{n\in\mathbb{N}\setminus\{1\}}(-1+\frac{1}{n},1-\frac{1}{n})=(-\frac{1}{2},\frac{1}{2})$

According to the demonstration for A, $(-\frac{1}{2},\frac{1}{2})$ is an open set

4) $D=\bigcap_{n\in\mathbb{N}}[-1+\frac{1}{n},1-\frac{1}{n}]=[-\frac{1}{2},\frac{1}{2}]$

According to the demonstration for B, $(-\frac{1}{2},\frac{1}{2})$ is a closed set

5) $E=\bigcup_{n\in\mathbb{N}}[-1-\frac{1}{n},1+\frac{1}{n}]$

For $n=1$: $[-2,2]$

To prove that $[-2,2]$ is a closed set, we have to prove that

$\mathbb{R}\setminus[-2,2]$ is an open set

$\mathbb{R}\setminus[-2,2]=(-\infty,-2)\cup(2,+\infty)$

We shall prove that $(2,+\infty)$ is an open set, $(-\infty,-2)$ having a similar demonstration. (1)

$$\left.\begin{array}{r}\text{Let } c\in(2,+\infty) \text{ be chosen randomly} \Rightarrow \exists\, r_c>0 \text{ s.t. } B(c,r_c)\subseteq(2,+\infty)\\ c\neq 2\Rightarrow c-2>0\Rightarrow \exists\, r_c:=c-2\end{array}\right\}\Rightarrow$$

$\Rightarrow B(c,c-2)\subseteq(2,+\infty)\Rightarrow(2,+\infty)\in\mathcal{V}(c)$

Because c was chosen randomly, $(2,+\infty)$ is a neighborhood for all its points $\Rightarrow (2,+\infty)$ is an open set (2)

(1) & (2) $\Rightarrow (-\infty,-2)\cup(2,+\infty)$ is an open set $\Rightarrow [-2,2]$ is a closed set

6) $F=\bigcap_{n\in\mathbb{N}}\left(-1-\frac{1}{n},\,1+\frac{1}{n}\right)=(-1,1)$

$$\left.\begin{array}{r}\text{Let } c\in(-1,1) \text{ be chosen arbitrary} \Rightarrow \exists\, r_c>0 \text{ s.t. } B(c,r_c)\subseteq(-1,1)\\ c\neq 1\Rightarrow 1-c>0\Rightarrow \exists\, r_c:=1-c\end{array}\right\}\Rightarrow$$

$\Rightarrow B(c,1-c)\subseteq(-1,1)\Rightarrow(-1,1)\in\mathcal{V}(c)$

Because c was chosen randomly, $(-1,1)$ is a neighborhood for all its points $\Rightarrow (-1,1)$ is an open set.