ANALYTIC GEOMETRY, PROBLEM SET 4

Projections. Dot product. Cross product.

1. Find the orthogonal projection $pr_{\overline{u}}(\overline{v})$, where $\overline{v} = 10\overline{a} + 2\overline{b}$, $\overline{u} = 5\overline{a} - 12\overline{b}$, if $\overline{a} \perp \overline{b}$ and $||\overline{a}|| = ||\overline{b}|| \neq 0$.

2. Using the dot product, prove the **Cauchy-Buniakowski-Schwarz** inequality, i.e. show that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

3. For a tetrahedron $ABCD$, show that $\cos(\widehat{\overline{AB}, \overline{CD}}) = \frac{AD^2+BC^2-AC^2-BD^2}{2AB \cdot CD}$. (the 3D version of the **cosine theorem**)

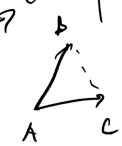
4. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds: $9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$.

5. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ and $A'B'$, respectively, are concurrent. Then the perpendicular lines through A', B', C' on BC, CA and AB, respectively are also concurrent. (Steiner's theorem on **orthologic triangles**)

$\frac{1}{2} ||\overline{AB} \times \overline{AC}||$

cross prod.

6. Find the area of the plane triangle having the vertices $A(1, 0, 1)$, $B(0, 2, 3)$, $C(2, 1, 0)$.



7. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \overline{a}$, $\overline{CA} = \overline{b}$ and $\overline{AB} = \overline{c}$ if and only if $\overline{a} \times \overline{b} = \overline{b} \times \overline{c} = \overline{c} \times \overline{a}$.

8. Find a vector orthogonal on both $\overline{u}$ and $\overline{v}$, if:
a) $\overline{u} = -7\overline{i} + 3\overline{j} + \overline{k}$ and $\overline{v} = 2\overline{i} + 4\overline{k}$ b) $\overline{u} = (-1, -1, -1)$ and $\overline{v} = (2, 0, 2)$.

9. Let a, b, and c denote the lengths of the sides of $\triangle ABC$. We write O for its circumcenter, R for the length of its circumradius, H for its orthocenter and G for the centroid. Show that
a) $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$; b) $OG^2 = R^2 - 1/9(a^2 + b^2 + c^2)$.

$w(x, y, z)$ I) $w \perp u \Leftrightarrow w \cdot u = 0$ → lin eq

$w \perp v \Leftrightarrow w \cdot v = 0$ → lin eq.

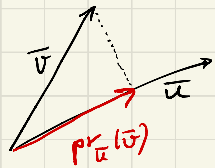
II) $w \parallel u \times v$

Date: October 14, 2021.

1. Find the orthogonal projection $pr_{\overline{u}}(\overline{v})$, where $\overline{v} = 10\overline{a} + 2\overline{b}$, $\overline{u} = 5\overline{a} - 12\overline{b}$, if $\overline{a} \perp \overline{b}$ and $||\overline{a}|| = ||\overline{b}|| \neq 0$.

$$\Updownarrow \quad \bar a \cdot \bar b = 0$$

$$pr_{\bar u}(\bar v) = \frac{\bar v \cdot \bar u}{\|\bar u\|^2}\,\bar u \qquad\qquad pr_{\overline{b}}\overline{u} = \frac{\overline{u}\cdot\overline{b}}{|\overline{b}|^2}\cdot\overline{b};$$



$$\bar v \cdot \bar u = (10\bar a + 2\bar b)\cdot(5\bar a - 12\bar b)$$

$$= 50\bar a^2 + \cancel{10\bar a\bar b}^{\,0} - 24\bar b^2 - \cancel{120\bar b\bar a}^{\,0}$$

$$\bar b^2 = \|\bar a\|^2$$

$$= 26\bar a^2$$

$$\|\bar u\|^2 = u\cdot u = (5\bar a - 12\bar b)\cdot(5\bar a - 12\bar b) = \dots\dots \quad \dots\dots \quad c\cdot\|\bar a\|^2 \ ?$$

$$pr_{\bar u}(v) = \frac{26\cancel{\bar a^2}}{c\,\cancel{\bar a^2}}(5\bar a - 12\bar b)$$

2. Using the dot product, prove the **Cauchy-Buniakowski-Schwarz** inequality, i.e. show that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

$v(a_1, a_2, a_3)$

$w(b_1, b_2, b_3)$

$$(v\cdot w)^2 \ \leq\ \|v\|^2 \cdot \|w\|^2$$

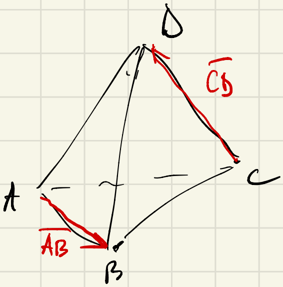
(Method I) Lagrange's identity

(Method II) $v\cdot w^2 = \|v\|^2\cdot\|w\|^2\cdot\cos(\widehat{v,w}) \leq \|v\|^2\cdot\|w\|^2$

3. For a tetrahedron $ABCD$, show that $\cos(\widehat{\overline{AB},\overline{CD}}) \overset{(*)}{=} \frac{AD^2+BC^2-AC^2-BD^2}{2AB\cdot CD}$. (the 3D version of the **cosine theorem**)

$$\| $$

$$\frac{\overline{AB}\cdot\overline{CD}}{AB\cdot CD}$$



$$(*) \iff 2\,\overline{AB}\cdot\overline{CD} = \underset{\overline{AD}^2}{\underset{\|}{AD^2}} + \underset{\overline{BC}^2}{\underset{\|}{BC^2}} - \underset{\overline{AC}^2}{\underset{\|}{AC^2}} - \underset{\overline{BD}^2}{\underset{\|}{BD^2}}$$

$$\overline{AB}\cdot\overline{CD} = (\overline{AD}+\overline{DB})\cdot(\overline{CA}+\overline{AD})$$

$$= \overline{AD}^2 + \underbrace{\overline{DB}\cdot\overline{CA} + \overline{AD}\cdot\overline{CA}}_{\overline{AB}\cdot\overline{CA}} + \overline{DB}\cdot\overline{AD}$$

$$= \overline{AD}^2 + \underbrace{(\overline{AC}+\overline{CB})\cdot(\overline{CB}+\overline{BA})}_{\overline{AC}\cdot\overline{CB} + \overline{CB}\cdot\overline{CB} + \overline{AC}\cdot\overline{BA} + \overline{CB}\cdot\overline{BA}} + \overline{DB}\cdot\overline{AD}$$

$$\overline{CB}\cdot\overline{CB} = -\overline{BC}\cdot -\overline{BC} = \overline{BC}^2$$

$$\overline{AC}\cdot(\overline{CB}+\overline{BA}) = \overline{AC}\cdot\underbrace{\overline{CA}}_{-\overline{AC}} = -\overline{AC}^2$$

$$= \overline{AD}^2 + \overline{BC}^2 - \overline{AC}^2 + \overline{CB}\cdot\overline{BA} + \overline{DB}\cdot\overline{AD}$$

$$\overline{DB}\cdot\overline{AD} = \overline{DB}\cdot(\overline{AB}+\overline{BD}) = \overline{DB}\cdot\overline{AB} - \overline{BD}^2$$

$$= \overline{AD}^2 + \overline{BC}^2 - \overline{BD}^2 - \overline{AC}^2 + \underset{-\overline{BC}\cdot(-\overline{AB})}{\underset{\|}{\overline{CB}}\cdot\underset{\|}{\overline{BA}}} + \overline{DB}\cdot\overline{AB}$$

$$\underbrace{(\overline{BC}+\overline{DB})\cdot\overline{AB}}_{\overline{DC}\cdot\overline{AB}}$$

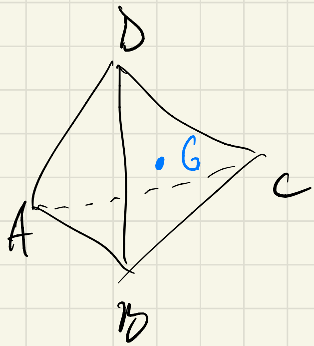
$$-\overline{CD}\cdot\overline{AB}$$

$$\Rightarrow \quad 2\,\overline{CD}\cdot\overline{AB} = \overline{AD}^2 + \overline{BC}^2 - \overline{BD}^2 - \overline{AC}^2 \qquad \square$$

4. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds: $9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$.

Hpto:

$$\overline{OG} = \frac{\overline{OB}+\overline{OC}+\overline{OD}}{3}$$



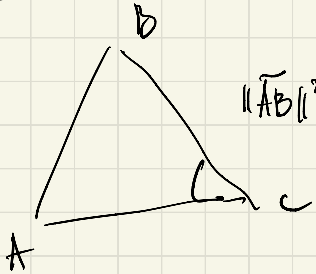
$$\overline{AG} = \frac{\overline{AB}+\overline{AC}+\overline{AD}}{3}$$

$$9\cdot\overline{AG}^2 = \left(\frac{\overline{AB}+\overline{AC}+\overline{AD}}{\cancel{3}}\right)^2\cdot\cancel{9}$$

$$\Rightarrow \quad 9\,\overline{AG}^2 = (\overline{AB}+\overline{AC}+\overline{AD})^2$$

$$= \overline{AB}^2 + \overline{AC}^2 + \overline{AD}^2 + \underbrace{2\overline{AB}\cdot\overline{AC}}_{\overset{?}{=}\frac{1}{2}(\|\overline{AB}\|^2+\|\overline{AC}\|^2-\|\overline{BC}\|^2} + 2\overline{AB}\cdot\overline{AD} + 2\overline{AC}\cdot\overline{AD}$$

Cosine law

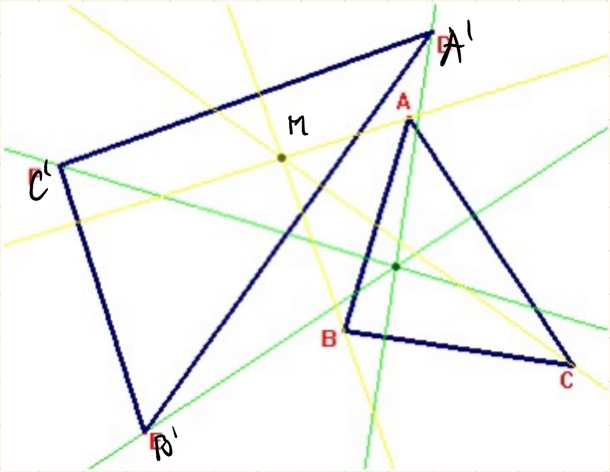


$$\|\overline{AB}\|^2 = \|\overline{AC}\|^2 + \|\overline{CB}\|^2 - 2\,\underbrace{\|\overline{AC}\|\cdot\|\overline{CB}\|\cdot\cos\hat{C}}$$

$$\cos\hat{C} = \frac{\overline{CA}\cdot\overline{CB}}{\|\overline{CA}\|\cdot\|\overline{CB}\|}$$

$$\overline{CA}\cdot\overline{CB} = \frac{1}{2}\left(\|\overline{AC}\|^2 + \|\overline{CB}\|^2 - \|\overline{AB}\|^2\right)$$

5. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ and $A'B'$, respectively, are concurrent. Then the perpendicular lines through A', B', C' on BC, CA and AB, respectively are also concurrent.
(Steiner's theorem on **orthologic triangles**)



We know :

(1) $\overline{AM}\cdot\overline{B'C'}=0$

(2) $\overline{BM}\cdot\overline{C'A'}=0$

(3) $\overline{CM}\cdot\overline{A'B'}=0$

Let N = intersection of the perperp. line from A' on BC with the perp line from B' on AC

we have $\overline{A'N}\cdot\overline{BC}=0$

$\overline{B'N}\cdot\overline{AC}=0$

we need to show that

$N \in$ perp. line from C' on AB $\iff \overline{C'N}\cdot\overline{AB}=0$

$\iff (\overline{C'A'}+\overline{A'N})\cdot(\overline{AC}+\overline{CB})=0$

$\iff \overline{C'A'}\cdot\overline{AB}+\underbrace{\overline{A'N}\cdot\overline{AC}}_{(\overline{A'B'}+\overline{B'N})\cdot\overline{AC}}+0=0$

$\iff \boxed{\overline{C'A'}\cdot\overline{AB}+\overline{A'B'}\cdot\overline{AC}=0}$

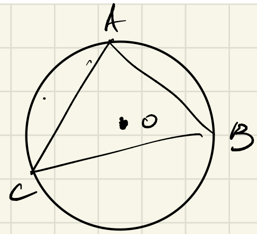
↳ show that this follows from (1) (2) (3)

9. Let a, b, and c denote the lengths of the sides of $\triangle ABC$. We write O for its circumcenter, R for the length of its circumradius, H for its orthocenter and G for the centroid. Show that
a) $OH^2 = 9R^2 - (a^2+b^2+c^2)$; b) $OG^2 = R^2 - 1/9(a^2+b^2+c^2)$.

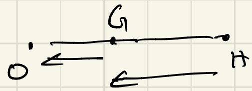
$\vec{OH}\cdot\vec{OH}=(\overline{OH})^2=(\overline{OA}+\overline{OB}+\overline{OC})^2$

$=\underbrace{\overline{OA}^2}_{R^2}+\underbrace{\overline{OB}^2}_{R^2}+\underbrace{\overline{OC}^2}_{R^2}+\underbrace{2\overline{OA}\cdot\overline{OB}}_{2\cdot\frac{1}{2}(\overline{OA}^2+\overline{OB}^2-\overline{AB}^2)}+\dots$

$R^2 \quad R^2 \quad c^2$



b.) $2\,\overrightarrow{GO} = \overrightarrow{HG}$



$$\Rightarrow 3\,\overrightarrow{GO} = \overrightarrow{HO}$$

$$\overrightarrow{GO}^2 = \frac{\overrightarrow{HO}^2}{9}$$