Serii de puteri

Fie $(a_n)_{n\geq 0} \subseteq \mathbb{R}$ un şir de numere reale. Se numeşte **serie de puteri** o serie de funcţii de forma

$$\sum_{n\geq 0} a_n x^n,$$

cu observaţia că prima funcţie din această serie de funcţii este funcţia constantă a_0. Astfel, pentru un $x_0 \in \mathbb{R}$, se obţine o serie de numere reale,

$$\sum_{n\geq 0} a_n {x_0}^n = a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_n x_0^n + ...$$

Un punct $x_0 \in \mathbb{R}$ se numeşte **punct de convergenţă** dacă seria de numere reale $\sum\limits_{n\geq 0} a_n x_0^n$, este convergentă, adică

$$\sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R}\,.$$

Mulţimea tuturor punctelor de convergenţă formează **mulţimea de convergenţă a seriei de puteri**, notată prin

$$\mathcal{C} = \left\{ x_0 \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R} \right\}.$$

Se constată că în cazul seriilor de puteri întotdeauna

$$0 \in \mathcal{C},$$

deoarece

$$\sum_{n=0}^{\infty} a_n 0 = a_0 \in \mathbb{R}\,.$$

Raza de convergenţă a seriei de puteri este

$$R = \frac{1}{\lambda} \quad unde \quad \lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Conform teoremei lui Cauchy-Hadamard,

$$(-R, R) \subseteq \mathcal{C} \subseteq [-R, R].$$

Cazurile particulare în care

$$x = -R \quad si \quad x = R$$

trebuie analizate separat, pentru a se stabili cu exactitate $\mathcal{C}$.

Toate exerciţiile au acelaşi enunţ: stabiliţi raza de convergenţă şi mulţimea de convergenţă a următoarelor serii de puteri:

Exemeplul 1

$$\sum_{n\geq 1} n^n x^n.$$

Rezolvare: Şirul care generează seria de puteri este $(a_n)_{n\geq 1}$, are termenul general

$$a_n = n^n, \quad \forall n \in \mathbb{N}.$$

Calculăm

$$\lambda = \lim_{n\to\infty} \sqrt[n]{n^n} = \lim_{n\to\infty} n = \infty \Longrightarrow R = 0.$$

De aceea

$$\mathcal{C} = \{0\}.$$

Observaţie: Seriile de puteri pot fi scrise ca fiind dezoltate în jururl unor puncte arbitrare în $\mathbb{R}$, caz în care au formularea;

$$\sum_{n\geq 0} a_n (x - x_0)^n.$$

Pentru aceste cazuri raza de convergenţă se calculează exact dupa modelul de mai sus. Singura diferenţă apare la formularea mulţimii de oconvergenţă, astfel;

$$(x_0 - R, x_0 + R) \subseteq \mathcal{C} \subseteq [x_0 - R, x_0 + R].$$

Cazurile în care $x = x_0 - R$ şi $x = x_0 + R$ trebuie analizate separat pentru a preciza cu exactitate multţimea de convergenţă.

Exemplul 2 :

$$\sum_{n\geq 1} \frac{(-1)^n}{n(2n+1)} (x+2)^n.$$

Rezolvare: Seria de puteri este dezolvtată în jurul punctului $x_0 = -2$, iar şirul care o generează este $(a_n)_{n\geq 1}$, având termenul general

$$a_n = \frac{(-1)^n}{n(2n+1)}, \quad \forall n \in \mathbb{N}.$$

Calculăm

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(n+1)(2n+3)}{n(2n+1)} \right| = 1 \Longrightarrow R = \frac{1}{1} = 1.$$

Deci

$$(-2-1, -2+1) = (-3, -1) \subseteq \mathcal{C}.$$

Verificăm pe rând capetele intervalului de convergenţă.

Pentru $x = -3$, seria de numere reale

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} \quad \sim \sum_{n\geq 1} \frac{1}{n^2},$$

care este convergentă, deci $-3 \in \mathcal{C}$.

Pentru $x = -1$, seria de numere reale $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)}$ este o serie alternată. Deoarece şirul de numere reale $\left(\frac{1}{n(2n+1)}\right)_{n\geq 1}$ este descrescă tor, cu limita 0, din criteriul lui Leibniz, rezultă că avem convergenţă, astfel, $-1 \in \mathcal{C}$.

Power series

Definition:

Consider a certain sequence $(a_n) \subseteq \mathbb{R}$

The structure $\sum_{n \geq 0} a_n \cdot x^n \ (\forall x \in \mathbb{R})$ is called a power series

with the convention that the first function in this series, is the constant function a_0.

Thus, for a given $x_0 \in \mathbb{R}$, we get the following series of real numbers

$$\sum_{n \geq 0} a_n x_0^n = a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 + \dots + a_n x_0^n + \dots$$

A point $x_0 \in \mathbb{R}$ is called a convergence point of the series of real numbers $\sum_{n \geq 0} a_n x_0^n$ if the series is convergent

thus

$$\sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R}.$$

The set of all convergence points is said to be the convergence set of the power series and is denoted by

$$\mathcal{C} = \left\{ x_0 \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R} \right\}$$

For each power series $0 \in \mathcal{C} \Rightarrow \mathcal{C} \neq \emptyset$

due to the fact that $\sum_{n=0}^{\infty} a_n 0 = a_0 \in \mathbb{R}$

The convergence radius of the power series is $\begin{cases} R = \sup \mathcal{C} \\ R = \dfrac{1}{\lambda} \\ R \geq 0 \end{cases}$ where $\lambda = \lim\limits_{n \to \infty} \dfrac{|a_{n+1}|}{|a_n|} \overset{\text{C.S.}}{=} \lim\limits_{n \to \infty} \sqrt[n]{|a_n|}$

According to Cauchy-Hadamard's theorem,

$$(-R, R) \subseteq \mathcal{C} \subseteq [-R, R].$$

The particular cases $x = -R$ and $x = R$ must be analysed separately in order to determine exactly $\mathcal{C}$.

Remark:

The power series may be written as developed around another point in $\mathbb{R}$, case in which they are stated as

$$\sum_{n \geq 0} a_n (x - x_0)^n.$$

In such case the convergence radius is computed accordingly to the algorithm below.

The convergence radius is computed exactly like in the case below,

the only difference relies in the formulation of the convergence set, namely:

$$(x_0 - R, \ x_0 + R) \subseteq \mathcal{C} \subseteq [x_0 - R, x_0 + R]$$

The cases when $x = x_0 - R$ and $x = x_0 + R$ must be analysed separately in order to specify completely the convergence case.

The Abel theorem for power series

Let $\sum a_n \cdot x^n$ be a power series.

Then:

1° If $0 \neq t \in \mathscr{C} \Rightarrow$ the series $\sum a_n \cdot u^n$ is A.C. $\forall u \in \mathbb{R}$ s.t. $|u| < |t|$

2° If $t \notin \mathscr{C} \Rightarrow$ the series $\sum a_n \cdot v^n$ is D. $\forall v \in \mathbb{R}$ s.t. $|t| < |v|$

Proof:

1°. Consider a random $0 \neq t \in \mathscr{C} \Longleftrightarrow$ the series $\sum\limits_{n \geq 0} a_n \cdot t^n$ is C

$$\Downarrow$$

$$\lim_{n \to \infty} a_n \cdot t^n = 0$$

$$\Downarrow$$

the sequence $(a_n \cdot t^n)$ is bounded

$\Rightarrow \exists M > 0$ s.t. $|a_n \cdot t^n| < M \quad (a_n \cdot t^n \in B(0, M))$ (*)

Consider a $u \in \mathbb{R}$ s.t. $|u| < |t|$, random

$$\underbrace{|a_n \cdot u^n|}_{\alpha_n} \underset{t \neq 0}{=} \left|a_n \cdot u^n \cdot \frac{t^n}{t^n}\right| = \left|a_n \cdot t^n \cdot \left(\frac{u}{t}\right)^n\right| = |a_n \cdot t^n| \cdot \left|\frac{u}{t}\right|^n \overset{(*)}{<} \underbrace{M \cdot \left|\frac{u}{t}\right|^n}_{\beta_n}$$

$|\alpha_n| \leq \beta_n$

$\left.\begin{array}{l} 0 \leq |u| \leq |t| \\ t \neq 0 \end{array}\right\} \Rightarrow 0 \leq \left|\frac{u}{t}\right| < 1$ ↳ constant $= q$

$\sum \beta_n = \sum M \cdot \left|\frac{u}{t}\right|^n = M \cdot \sum \left|\frac{u}{t}\right|^n$

↳ $M \cdot \sum q^n$ – geometric series

$0 \leq q < 1 \rightarrow \sum \beta_n$ is C.

$\Rightarrow \left.\begin{array}{l} |\alpha_n| \leq \beta_n \\ \sum \beta_n \text{ is C.} \end{array}\right\} \overset{CIC}{\Rightarrow} \sum |\alpha_n|$ is C.

$$\Updownarrow$$

$\sum \alpha_n$ is A.C. $\Longleftrightarrow \sum\limits_{n \geq 0} a_n \cdot u^n$ is A.C. (u random)

2°. Choose a $v \in \mathbb{R}$ s.t. $|t| < |v|$. We want $v \notin \mathscr{C}$.

Assume by contradiction that $v \in \mathscr{C} \Rightarrow \left.\begin{array}{l} \sum a_n \cdot v^n \text{ is C.} \\ |t| < |v| \end{array}\right\} \overset{1°}{\Longrightarrow} \sum a_n \cdot t^n$ is A.C.

$$\Downarrow$$

$\sum a_n \cdot t^n$ is C

$$\Downarrow$$

$t \in \mathscr{C}$

Remark:

If $t \in \mathscr{C} \Rightarrow -t \in \mathscr{C}$ or $\notin \mathscr{C}$

Steps:

① $R \Rightarrow (-R; R) \subseteq \mathscr{C} \subseteq [-R; R]$

② analyze particulary the series $\sum a_n \cdot R^n$ and $\sum a_n \cdot (-R)^n$ each individually

③ deliver $\mathscr{C}$ (with/without $-R, R \in \mathscr{C}$, according to step II)

Theorem (Connection between R and $\mathcal{C}$)

Consider $\sum a_n \cdot x^n$ a power series.
Then the following are equivalent:

1° $R = \sup \mathcal{C} \in \overline{\mathbb{R}}$

$\Updownarrow$

2° $\begin{cases} a) \ \sum a_n \cdot x^n \text{ is C } \ \forall x \in \mathbb{R} \text{ with } |x| < R \\ b) \ \sum a_n \cdot x^n \text{ is D } \ \forall x \in \mathbb{R} \text{ with } |x| > R \end{cases}$

$\Updownarrow$

3° $(-R, R) \subseteq \mathcal{C} \subseteq [-R, R]$

Theorem (Cauchy–Hadamard)

Consider $\sum a_n \cdot x^n$ a power series and $\lambda = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|} \overset{C.S.}{=} \lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

Then $R = \begin{cases} \infty & : \lambda = 0 \\ \frac{1}{\lambda} & : \lambda \in (0, \infty) \\ 0 & : \lambda = \infty \end{cases}$

Proof: $R = \sup \mathcal{C}$

Case 1 : $\lambda = 0$

Consider a random $x \in \mathbb{R}^*$, $\sum a_n x^n$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \cdot |x|^{\frac{n}{n}} = |x| \cdot \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x| \cdot 0 = 0$$

$\underset{\text{criterion}}{\overset{\text{ROOT}}{\Longrightarrow}}$ the series $\sum |a_n x^n|$ is C. $\Leftrightarrow \sum a_n \cdot x^n$ is A.C.

$\Downarrow$

$\sum a_n \cdot x^n$ is C.

$\left.\begin{array}{l} \Rightarrow x \in \mathcal{C} \\ x \text{ random in } \mathbb{R}^* \end{array}\right\} \Rightarrow \left.\begin{array}{l} \mathbb{R}^* \subseteq \mathcal{C} \\ \{0\} \in \mathcal{C} \end{array}\right\} \Rightarrow \mathbb{R} \subseteq \mathcal{C} \subseteq \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

$\mathcal{C} = \mathbb{R} \Rightarrow \sup \mathcal{C} = R = \infty$ ✓

Case 2 : $\alpha \in (0, \infty)$

Consider a random $x \in \mathbb{R}^*$, $\sum a_n x^n$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \cdot |x| = |x| \cdot \lambda \ \begin{array}{ll} < 1 & \text{C.} \\ > 1 & \text{D.} \\ = 1 & ? \end{array}$$

- if $|x| \cdot \lambda < 1 \Rightarrow x \in \mathcal{C}$
  $\Updownarrow$
  $|x| < \frac{1}{\lambda} \Longleftrightarrow x \in \left(-\frac{1}{\lambda}, \frac{1}{\lambda}\right)$
  $\Bigg\} \Rightarrow \left(-\frac{1}{\lambda}, \frac{1}{\lambda}\right) \subseteq \mathcal{C}$ (1)

- if $|x| \cdot \lambda > 1 \Rightarrow x \notin \mathcal{C}$
  $\Updownarrow$
  $|x| > \frac{1}{\lambda} \Longleftrightarrow x \in \left(-\infty, -\frac{1}{\lambda}\right) \cup \left(\frac{1}{\lambda}, \infty\right)$
  $\Bigg\}$ $\left(-\infty, -\frac{1}{\lambda}\right) \cup \left(\frac{1}{\lambda}, \infty\right) \cap \mathcal{C} = \emptyset$ $\Rightarrow \mathcal{C} \subseteq \left[-\frac{1}{\lambda}, \frac{1}{\lambda}\right]$ (2)

Thus (1 and 2) $\Rightarrow \left(-\frac{1}{\lambda}, \frac{1}{\lambda}\right) \subseteq \mathcal{C} \subseteq \left[-\frac{1}{\lambda}, \frac{1}{\lambda}\right] \underset{\text{theorem}}{\overset{\text{Abel}}{\Longleftrightarrow}} R = \frac{1}{\lambda}$ ✓

Case 3: $\alpha = \infty$

Consider a random $x \in \mathbb{R}^*$, $\sum a_n x^n$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n \cdot x^n|} = |x| \cdot \lim_{n\to\infty} \sqrt[n]{|a_n|} = \underset{>0}{|x|} \cdot \infty = \infty > 1 \Rightarrow \sum a_n \cdot x^n \text{ is } D \Bigg\} \Rightarrow$$

$$\Rightarrow \forall x \in \mathbb{R}^*, \ x \notin \mathcal{C} \Rightarrow \left.\begin{matrix} \mathcal{C} \subseteq \{0\} \\ \{0\} \subseteq \mathcal{C} \end{matrix}\right\} \Rightarrow \begin{matrix} \mathcal{C} = \{0\} \\ \sup \mathcal{C} = 0 = R \end{matrix}$$

Examples:

a) Determine the convergence radius and the convergence set of the following power series:

1) $\sum_{n \geq 0} x^n \quad a_n = 1$

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{1} = 1 \Rightarrow R = \frac{1}{\lambda} = \frac{1}{1} = 1$$

For $x = 1$, the series of real numbers $\sum_{n=0}^{\infty} 1 = \infty$ is D, thus $1 \notin \mathcal{C}$.

For $x = -1$, the series of real numbers $\sum_{n=0}^{\infty} (-1)^n$ does not have a sum, due to the fact that the sequence of partial sums is constantly oscilating between 1 and 0, therefore $\nexists \lim_{n\to\infty} (-1)^n$.
Thus $-1 \notin \mathcal{C}$.

We conclude that $\mathcal{C} = (-1, 1)$

2) $\sum_{n \geq 1} \frac{(-1)^n}{n(2n+1)} (x+2)^n$

$$a_n = \frac{(-1)^n}{n(2n-1)} \quad \forall n \in \mathbb{N}$$

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \left| \frac{(n+1)(2n+3)}{n(2n+1)} \right| = 1 \Rightarrow R = \frac{1}{1} = 1$$

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Exemplul 2 :

$$\sum_{n\geq 1} \frac{(-1)^n}{n(2n+1)}(x+2)^n.$$

Rezolvare: Seria de puteri este dezolvtată în jurul punctului $x_0 = -2$, iar şirul care o generează este $(a_n)_{n\geq 1}$, având termenul general

$$a_n = \frac{(-1)^n}{n(2n+1)}, \quad \forall n \in \mathbb{N}.$$

Calculăm

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{(n+1)(2n+3)}{n(2n+1)}\right| = 1 \Longrightarrow R = \frac{1}{1} = 1.$$

Deci

$$(-2-1, -2+1) = (-3, -1) \subseteq \mathcal{C}.$$

Verificăm pe rând capetele intervalului de convergenţă.

Pentru $x = -3$, seria de numere reale

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} \quad \sim \sum_{n\geq 1} \frac{1}{n^2},$$

care este convergentă, deci $-3 \in \mathcal{C}$.

Pentru $x = -1$, seria de numere reale $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)}$ este o serie alternată. Deoarece şirul de numere reale $\left(\frac{1}{n(2n+1)}\right)_{n\geq 1}$ este descrescă tor, cu limita 0, din criteriul lui Leibniz, rezultă că avem convergenţă, astfel, $-1 \in \mathcal{C}$.

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În concluzie

$$\mathcal{C} = [-3, -1].$$

Exercises: Determine both the convergence radius and the convergence set for the following power series:

$$a)\sum_{n\geq 0}(n+1)^n x^n \quad b)\sum_{n\geq 0}\frac{1}{n}x^n \quad c)\sum_{n\geq 0}\frac{(-1)^n}{n}x^n \quad d)\sum_{n\geq 0}\frac{1}{n(n+1)}x^n \quad e)\sum_{n\geq 0}\frac{1}{n!}x^n \quad f)\sum_{n\geq 0}n!x^n.$$

1. $\sum x^n$
2. $\sum_{n\geq 1}\frac{1}{n}x^n$
3. $\sum_{n\geq 1}\frac{1}{n(n+1)}x^n$
4. $\sum_{n\geq 0}\frac{1}{n!}x^n$
5. $\sum_{n\geq 1}n!x^n$
6. $\sum_{n\geq 0}\left(\sqrt[3]{n^2+n+1}-\sqrt[3]{n^2-n-1}\right)^n x^n.$

1. $\sum x^n$

$a_n = 1, \ \forall n \in \mathbb{N}$

$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{1} = 1 \Rightarrow R = \frac{1}{1} = 1$

Cauchy $\longrightarrow$ Hadamard $\quad (-1,1) \subseteq \mathcal{C} \subseteq [-1,1]$

$\sum_{n=0}^{\infty} 1^n = \lim_{n\to\infty} (\overbrace{1+\dots+1}^{n \text{ times}}) = \infty \Rightarrow \Delta \Rightarrow 1 \notin \mathcal{C}$

$\sum_{n=0}^{\infty} (-1)^n, \quad \exists \lim_{n\to\infty} (-1)^n \Rightarrow -1 \notin \mathcal{C}$

$\mathcal{C} = (-1,1)$

2. $\sum_{n\geq 1} \frac{1}{n} x^n$

$a_n = \frac{1}{n}, \ \forall n \in \mathbb{N}$

$\lambda = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow R = \frac{1}{1} = 1$

Cauchy $\longrightarrow$ Hadamard $\quad (-1,1) \subseteq \mathcal{C} \subseteq [-1,1]$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is Δ. $\Rightarrow 1 \notin \mathcal{C}$

$\sum_{n=1}^{\infty} \frac{1}{n} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ alternating

$\left.\begin{array}{l} \frac{1}{n} \text{ is decreasing} \\ \lim_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \xRightarrow{\text{Leibniz}} \mathrm{C} \Rightarrow -1 \in \mathcal{C}$

$\Rightarrow \mathcal{C} = [-1,1)$

geometric s. $\quad \nexists \lim_{n\to\infty}$

$x=-1 \Rightarrow \sum (-1)^n$ divergent and $\nexists \sum_{n=1}^{\infty} (-1)^n$

$\lim_{n\to\infty} s_n = \infty \quad \exists \lim_{n\to\infty} \neq 0$

$x = 1 \Rightarrow \sum_{n\geq 1} 1^n = \sum_{n\geq 1} 1$ divergent and $\sum_{n=1}^{\infty} (1)^n = \infty$

1. $\sum x^n$

$u_n = x^n$

$a_n = 1$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{1} = 1 \Rightarrow R = 1 \Rightarrow \cancel{\mathcal{C} = 1} \quad (-1,1) \subseteq \mathcal{C}$$

$(-R,R)$

$$\Rightarrow \left.\begin{matrix} -1 \notin \mathcal{C} \\ 1 \notin \mathcal{C} \end{matrix}\right\} \Rightarrow \mathcal{C} = (-1,1)$$

$\sum (\frac{1}{2})^n$ C

$\sum (2)^n$ D

power series

2. $\sum_{n\geq 1} \frac{1}{n} x^n$

$u_n = \frac{1}{n} \cdot x^n$

$a_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow R = \frac{1}{1} = 1 \Rightarrow (-1;1) \subseteq \mathcal{C} \subseteq [-1,1] \quad (1)$$

$\rightarrow$ if $x=-1 \Rightarrow u_n = \frac{(-1)^n}{n} \Rightarrow \sum \frac{(-1)^n}{n}$ is convergent $\left(\text{Leibniz Theorem} \left\{ \begin{matrix} \frac{(-1)^n}{n} \text{ is decreasing} \\ \lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \end{matrix} \right. \right)$

$\Rightarrow -1 \in \mathcal{C} \quad (2)$

$\rightarrow$ if $x = 1 \Rightarrow u_n = \frac{1}{n} \Rightarrow \sum \frac{1}{n}$ is divergent $\Rightarrow 1 \notin \mathcal{C} \quad (3)$

From (1), (2), (3) $\Rightarrow \mathcal{C} = [-1,1)$

harmonic series with $\alpha = 1$

$\sum \frac{2^n}{n}$ is D

3. $\sum_{n\geq 1} \frac{1}{n(n+1)} x^n$

$u_n = \frac{1}{n(n+1)} x^n$

$a_n = \frac{1}{n(n+1)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n\cancel{(n+1)}}{\cancel{(n+1)}(n+2)} = \lim_{n\to\infty} \frac{n}{n+2} = 1 \Rightarrow R = 1 \Rightarrow (-1;1) \subseteq \mathcal{C} \subseteq [-1,1] \quad (1)$$

- if $x = -1 \Rightarrow u_n = \frac{(-1)^n}{n(n+1)} \Rightarrow \sum u_n$ is convergent $\Rightarrow -1 \in \mathcal{C}$ (2)
  (Leibniz)
- if $x = 1 \Rightarrow u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

telescopic series

series $\sim \sum \frac{1}{n^2} \Rightarrow C$

$2 > 1$

$$s_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

$$\left.\begin{matrix} s_{n+1} - s_n = 1 - \frac{1}{n+2} - 1 + \frac{1}{n+1} = \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0, \ \forall n \in \mathbb{N} \Rightarrow (s_n)_n \text{ increasing} \\ 0 < s_n < 1 \nearrow (s_n)_n \text{ bounded} \end{matrix}\right\} \Rightarrow$$

$\Rightarrow (s_n)_n$ convergent $\Rightarrow \sum u_n$ is convergent $\Rightarrow 1 \in \mathcal{C} \quad (3)$

From (1), (2), (3) $\Rightarrow \mathcal{C} = [-1,1] \longrightarrow \sum \frac{1}{2^n(n+1)}$ C. $\quad \sum \frac{2^n}{n(n+1)}$ D.

4. $\sum_{n\geq 0}\frac{1}{n!}x^n$

$$u_n = \frac{1}{n!}\cdot x^n$$

$$a_n = \frac{1}{n!}$$

ratio ↓ $\left(\frac{1}{0_+}\right)$ convergence ↓ set

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{n!}{n!\cdot(n+1)} = \lim_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow R = +\infty \Rightarrow \boxed{\mathcal{C}=\mathbb{R}}$$

$(-\infty,\infty)\subseteq\mathcal{C}$

$\mathbb{R}\subseteq\mathcal{C}$

$\sum \frac{(-\frac{1}{2})^n}{n!}$ c.

$\sum \frac{2^n}{n!}$ also c.

5. $\sum_{n\geq 1} n!x^n$

$$u_n = n!\cdot x^n$$

$$a_n = n!$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)\cancel{!}}{\cancel{n!}} = \lim_{n\to\infty}(n+1) = +\infty \Rightarrow R = 0 \Rightarrow \mathcal{C} = \{0\}$$

6. $\sum_{n\geq 0}\left(\sqrt[3]{n^2+n+1}-\sqrt[3]{n^2-n-1}\right)^n x^n.$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

$$\lambda = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\overset{\infty}{\underbrace{\sqrt[3]{n^2+n+1}}_{a}} - \overset{\infty}{\underbrace{\sqrt[3]{n^2-n-1}}_{b}} = \lim_{n\to\infty}\frac{a^3-b^3}{a^2+ab+b^2} = \lim_{n\to\infty}\frac{(n^2+n+1)-(n^2-n-1)}{\sqrt[3]{(n^2+n+1)^2}+\sqrt[3]{(n^2+n+1)(n^2-n-1)}+\sqrt[3]{(n^2-n-1)^2}} =$$

$$= \lim_{n\to\infty}\frac{2n+2}{n^{4/3}\underbrace{\left[\underbrace{\sqrt[3]{\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^2}}_{=1}+\underbrace{\sqrt[3]{\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^2\left(1-\frac{1}{n}-\frac{1}{n^2}\right)}}_{=1\cdot 1=1}+\underbrace{\sqrt[3]{\left(1-\frac{1}{n}-\frac{1}{n^2}\right)^2}}_{=1}\right]}_{3}} \sim \frac{2n}{3n^{\frac{4}{3}}} = 0_+ \Rightarrow R = \frac{1}{0_+} = \infty$$

$\Rightarrow (-\infty,\infty)\subseteq\mathcal{C}$

$\Leftrightarrow \mathbb{R}\subseteq\mathcal{C}$, $\mathcal{C}\subseteq\mathbb{R}$ $\Big\}\Leftrightarrow \mathcal{C}\subseteq\mathbb{R}$

$\Rightarrow \sum\left(-\frac{1}{2}\right)^n\cdot\left[\sqrt[3]{\ }-\sqrt[3]{\ }\right]^n$ is c.