ANALYTIC GEOMETRY, PROBLEM SET 5

Various problems with vectors

1. Given the vectors $\overline{a}(3,-1,-2)$ and $\overline{b}(1,2,-1)$. Compute
$$\overline{a}\times\overline{b},\ (2\overline{a}+\overline{b})\times\overline{b} \text{ and } (2\overline{a}+\overline{b})\times(2\overline{a}-\overline{b}).$$

2. Find the distances between the opposite sides of the parallelogram constructed on $\overrightarrow{AB}(6,0,2)$ şi $\overrightarrow{AC}(1.5,2,1)$.

3. Find the vector $\overline{p}$, knowing that $\overline{p}$ is perpendicular on $\overline{a}(2,3,-1)$ and $\overline{b}(1,-1,3)$ and its dot product with $\overline{c}(2,-3,4)$ is equal to 51.

4. Given the points $A(1,-1,2)$, $B(5,-6,2)$ and $C(1,3,-1)$, find the length of the altitude from the vertex B in the triangle $\triangle ABC$.

5. Given the vectors $\overline{a}(2,-3,1)$, $\overline{b}(-3,1,2)$ and $\overline{c}(1,2,3)$, compute $(\overline{a}\times\overline{b})\times\overline{c}$ and $\overline{a}\times(\overline{b}\times\overline{c})$.

6. Let $ABCD$ be a convex quadrilateral. Show that if the diagonal AC passes through the midpoint of the diagonal BD, then the triangles ACB and ACD have equal areas.

7. Prove that the points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are situated in the same plane.

8. Find the volume of the tetrahedron which has $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,1)$ and $D(4,1,3)$ as vertices.

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be coplanar representatives of vectors with modulus 1 and such that A, B, C are on the same side of a line that passes through O. Show that $||\overline{OA}+\overline{OB}+\overline{OC}|| \geq 1$.

10. Let $ABCD$ be a quadrilateral and E, F the midpoints of $[AB]$ and $[CD]$. Denote by K, L, M and N the midpoints of the segments $[AF], [CE], [BF]$ and $[DE]$, respectively. Prove that $KLMN$ is a parallelogram.

Date: October 23, 2021.

I expect you are able to prove equalities as the ones below. Have a go at them!

12. Let $\overline{a}, \overline{b}, \overline{c}$ be vectors in $\mathcal{V}_3$. Prove the following formulae:

1. $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{a} \cdot \overline{b}) \cdot \overline{c} = \begin{vmatrix} \overline{b} & \overline{c} \\ \overline{a} \cdot \overline{b} & \overline{a} \cdot \overline{c} \end{vmatrix}$; (Compare components)
2. $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{b} \cdot \overline{c}) \cdot \overline{a} = \begin{vmatrix} \overline{b} & \overline{a} \\ \overline{b} \cdot \overline{c} & \overline{a} \cdot \overline{c} \end{vmatrix}$.
3. $(\overline{a} \times \overline{b}) \cdot (\overline{c} \times \overline{d}) = \begin{vmatrix} \overline{a} \cdot \overline{c} & \overline{a} \cdot \overline{d} \\ \overline{b} \cdot \overline{c} & \overline{b} \cdot \overline{d} \end{vmatrix}$;
4. $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = (\overline{a}, \overline{c}, \overline{d}) \cdot \overline{b} - (\overline{b}, \overline{c}, \overline{d}) \cdot \overline{a} = (\overline{a}, \overline{b}, \overline{d}) \cdot \overline{c} - (\overline{a}, \overline{b}, \overline{c}) \cdot \overline{d}$;
5. $(\overline{a} \times \overline{b}, \overline{b} \times \overline{c}, \overline{c} \times \overline{a}) = (\overline{a}, \overline{b}, \overline{c})^2$

1. Given the vectors $\overline{a}(3,-1,-2)$ and $\overline{b}(1,2,-1)$. Compute

$$\overline{a}\times\overline{b},\ (2\overline{a}+\overline{b})\times\overline{b} \text{ and } (2\overline{a}+\overline{b})\times(2\overline{a}-\overline{b}).$$

(Cosmin D.)

$$\overline{a}\times\overline{b} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 5\overline{i} + \overline{j} + 7\overline{k}$$

$$(2\overline{a}+\overline{b})\times\overline{b} = (2\overline{a})\times\overline{b} + \overline{b}\times\overline{b}$$

$$= 2(\overline{a}\times\overline{b}) + \overline{0}$$

$$= 10\overline{i} + 2\overline{j} + 14\overline{k}.$$

$$(2\overline{a}+\overline{b})\times(2\overline{a}-\overline{b}) = \underbrace{2\overline{a}\times 2\overline{a}}_{\overline{0}} + 2\overline{a}\times(-\overline{b})$$

$$+ \overline{b}\times 2\overline{a} + \underbrace{\overline{b}\times(-\overline{b})}_{\overline{0}}$$

$$= 2\overline{a}\times(-\overline{b}) + \overline{b}\times 2\overline{a}$$

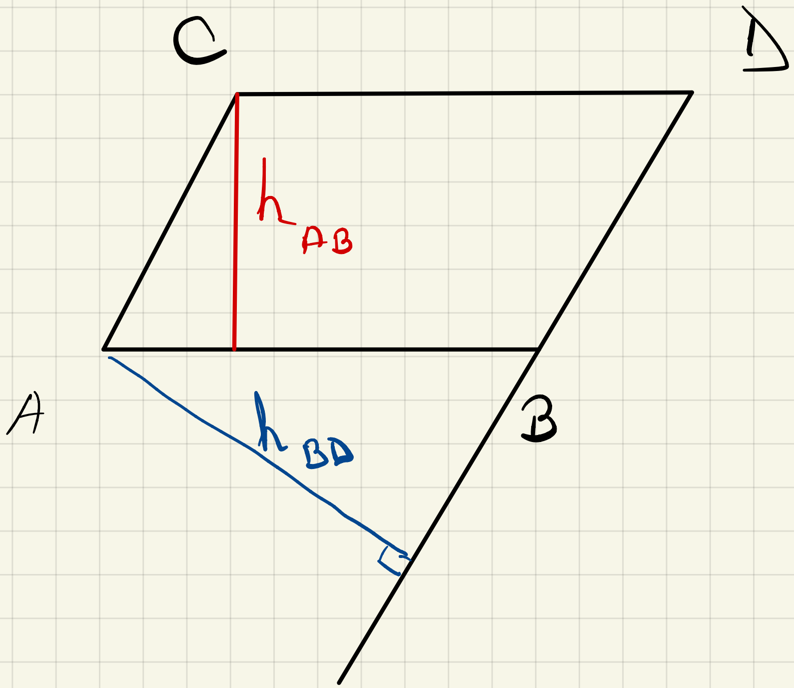
$$= -2(\bar{a} \times \bar{b}) + 2(\underbrace{\bar{b} \times \bar{a}}_{-\bar{a} \times \bar{b}})$$

$$= -4(\bar{a} \times \bar{b})$$

$$= -20\bar{i} - 4\bar{j} - 28\bar{k}.$$

$\boxtimes$

2. Find the distances between the opposite sides of the parallelogram constructed on $\overrightarrow{AB}(6, 0, 2)$ şi $\overrightarrow{AC}(1.5, 2, 1)$.



$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 6 & 0 & 2 \\ \frac{3}{2} & 2 & 1 \end{vmatrix} = -4\bar{i} - 3\bar{j} + 12\bar{k}.$$

$$\text{Area}[ABCD] = \|\overline{AB} \times \overline{AC}\| = \sqrt{16 + 9 + 144} = \sqrt{169}$$

$$= 13.$$

$$\text{Area}[ABCD] = h_{AB} \cdot \|\overline{AB}\| = h_{BD} \cdot \|\overline{AC}\|.$$

$$\|\overline{AB}\| = \sqrt{40} = 2\sqrt{10}$$

$$\|\overline{AC}\| = \frac{\sqrt{29}}{2}.$$

$$h_{AB} = \frac{13}{2\sqrt{10}} = \frac{13\sqrt{10}}{20}.$$

$$h_{AC} = \frac{2 \cdot 13}{\sqrt{29}} = \frac{26\sqrt{29}}{29}.$$

$\boxtimes$

3. Find the vector $\overline{p}$, knowing that $\overline{p}$ is perpendicular on $\overline{a}(2,3,-1)$ and $\overline{b}(1,-1,3)$ and its dot product with $\overline{c}(2,-3,4)$ is equal to 51.

Let $\overline{p}(p_1, p_2, p_3) \in V_3$.

$$\overline{p}\cdot\overline{a} = 0 \iff 2p_1 + 3p_2 - p_3 = 0$$

$$\overline{p}\cdot\overline{b} = 0 \iff p_1 - p_2 + 3p_3 = 0$$

$$\overline{p}\cdot\overline{c} = 51 \iff 2p_1 - 3p_2 + 4p_3 = 51.$$

Solve the system to find $p_1, p_2, p_3 \in \mathbb{R}$.

4. Given the points $A(1,-1,2)$, $B(5,-6,2)$ and $C(1,3,-1)$, find the length of the altitude from the vertex B in the triangle $\triangle ABC$.

$\overline{BC}(-4, 9, -3)$

$\overline{BA}(-4, 5, 0)$

$$\overline{BC} \times \overline{BA} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ -4 & 9 & -3 \\ -4 & 5 & 0 \end{vmatrix} = 15\overline{i} - 12\overline{j} - 16\overline{k}.$$

$$\|\overline{BC} \times \overline{BA}\| = \sqrt{225 + 144 + 256}$$

$$= \sqrt{625}$$

$h_B \cdot \|\overline{AC}\| = 25.$

$\overline{AC}(0, 4, -3) \Rightarrow \|\overline{AC}\| = 5.$

$\therefore \boxed{h_B = 5.}$

from the vertex B in the triangle $\triangle ABC$.

5. Given the vectors $\overline{a}(2, -3, 1)$, $\overline{b}(-3, 1, 2)$ and $\overline{c}(1, 2, 3)$, compute $(\overline{a} \times \overline{b}) \times \overline{c}$ and $\overline{a} \times (\overline{b} \times \overline{c})$.

$$\overline{a} \times \overline{b} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} \overset{?}{=} -7\overline{i} + \underline{8\overline{j}} + 11\overline{k}$$

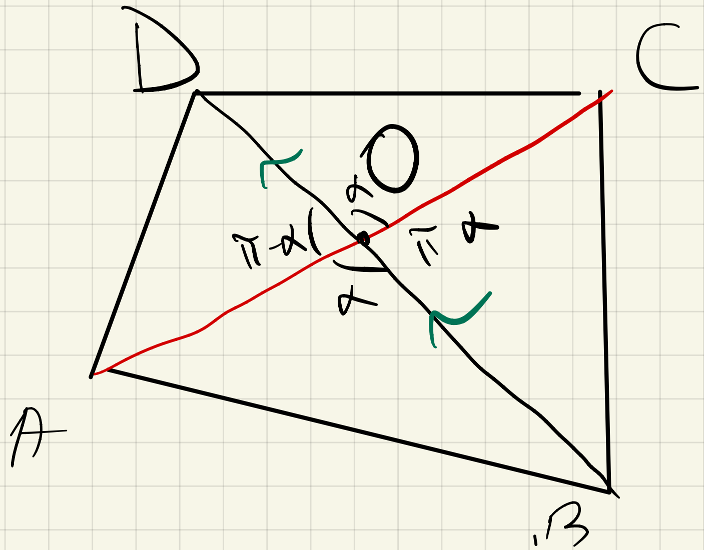
Maybe $0 \cdot \overline{j}$

$$(\overline{a} \times \overline{b}) \times \overline{c} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ -7 & 8 & 11 \\ 1 & 2 & 3 \end{vmatrix} = \underline{2\overline{i} + 32\overline{j} - 22\overline{k}}$$

$$\bar b \times \bar c = \begin{vmatrix} \bar i & \bar j & \bar k \\ -3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\bar i + 11\bar j - 7\bar k$$

$$\bar a \times (\bar b \times \bar c) = \begin{vmatrix} \bar i & \bar j & \bar k \\ 2 & -3 & 1 \\ -1 & 11 & -7 \end{vmatrix} = 10\bar i + 13\bar j + 19\bar k$$

6. Let $ABCD$ be a convex quadrilateral. Show that if the diagonal AC passes through the midpoint of the diagonal BD, then the triangles ACB and ACD have equal areas.



$$[ABO] = \frac{AO \cdot OB \cdot \sin\alpha}{2}$$

$\vdots$

Proof using "$\times$".

- It's enough to show that

$$\| \overline{AB} \times \overline{AC} \| = \| \overline{AC} \times \overline{AD} \|.$$

- O - midpoint $[BD]$, $\overline{AO} = \frac{1}{2}(\overline{AB} + \overline{AD})$.

$\overline{AO} = \alpha \cdot \overline{AC}$ for some $\alpha \in \mathbb{R}$.

$\Rightarrow \overline{AB} + \overline{AD} = 2\alpha \cdot \overline{AC} \quad \Big| \times \overline{AC}$ (on the right)

$$\overline{AB} \times \overline{AC} + \overline{AD} \times \overline{AC} = \overline{0}$$

$$\overline{AB} \times \overline{AC} = - \overline{AD} \times \overline{AC}$$

$$\therefore \overline{AB} \times \overline{AC} = \overline{AC} \times \overline{AD}$$

$$\therefore \| \overline{AB} \times \overline{AC} \| = \| \overline{AC} \times \overline{AD} \|$$

$\blacksquare$

8. Find the volume of the tetrahedron which has $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,1)$ and $D(4,1,3)$ as vertices.

$$\text{Volume}(ABCD) = \frac{1}{6} \cdot \left| (\overline{AB}, \overline{AC}, \overline{AD}) \right|$$

$$\overline{AB} = (3, 6, 3)$$

$$\overline{AC} = (1, 3, 0)$$

$$\overline{AD} = (2, 2, 2)$$

$$(\overline{AB}, \overline{AC}, \overline{AD}) = \begin{vmatrix} 3 & 6 & 3 \\ 1 & 3 & 0 \\ 2 & 2 & 2 \end{vmatrix} = -6.$$

$$\text{Volume}(ABCD) = 1.$$

7. Prove that the points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are situated in the same plane.

A, B, C, D are coplanar if and only if

$\overline{AB}, \overline{AC}, \overline{AD}$ are linearly dependent.

$\Leftrightarrow (\overline{AB}, \overline{AC}, \overline{AD}) = 0$.

We can check that

$$(\overline{AB}, \overline{AC}, \overline{AD}) = \begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 0.$$

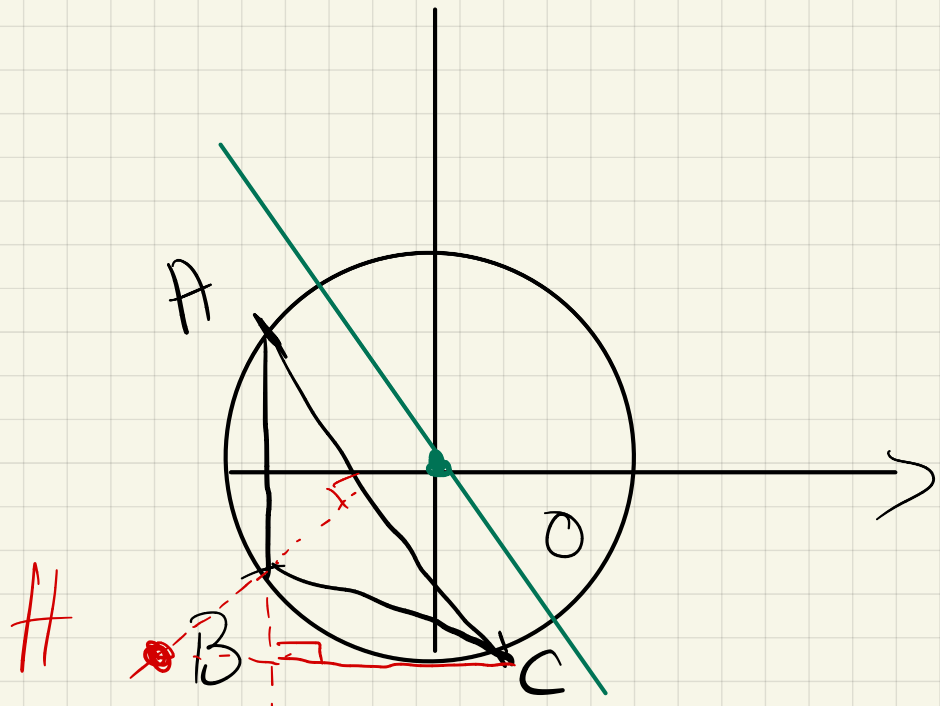
$\boxtimes$

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be coplanar representatives of vectors with modulus 1 and such that A, B, C are on the same side of a line that passes through O. Show that $||\overline{OA}+\overline{OB}+\overline{OC}|| \geq 1$.

Consider a triangle ABC.

$OA = OB = OC \Rightarrow O$ - circumcenter.

Choose a system of coordinates with origin in O.



ΔABC is an obtuse triangle.

$\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$, where H - orthocenter.

(see previous seminars).

Since ΔABC is obtuse, H lies outside $\mathscr{C}(0, 1)$.

$\Rightarrow \quad \|\overline{OH}\| \geq 1.$