Continuous functions

$\emptyset \neq \Delta \subseteq \mathbb{R}$, $f : \Delta \to \mathbb{R}$, $x_0 \in \Delta$ | The function f is continuous at x_0 if $\forall (x_n) \subseteq \Delta$ with $\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$.

Remark:

By using the definition, it can be easily proved that all elementary functions are continuous on their greatest definition domain.
Therefore, unless you are explicitly required to prove the continuity of such elementary functions, you may assume by default that they are continuous.

Characterization theorem with neighbourhoods:

f is continuous at x_0 if and only if $\forall V \in \mathcal{V}(f(x_0))$, $\exists U \in \mathcal{V}(x_0)$ such that $f(x) \in V$

Characterization theorem with ε and δ:

f is continuous at x_0 if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$,
such that $\forall x \in \Delta$ with $|x - x_0| < \delta$,

it holds $|f(x) - f(x_0)| < \varepsilon$.

Theorem characterising the connection between limits and continuity:

$x_0 \in D \cap D' = \Delta \setminus \operatorname{Iso}\Delta$ | The following statements are true:

1. f is continuous at $x_0 \Rightarrow \exists \lim_{x \to x_0} f(x) = f(x_0 - 0) = f(x_0 + 0) = f(x_0)$

2. if $\begin{cases} \exists f(x_0 - 0) \\ \exists f(x_0 + 0) \\ f(x_0 - 0) = f(x_0 + 0) = f(x_0) \end{cases} \Rightarrow f$ is continuous at x_0

Prove a function is continuous at a point

$\emptyset \neq A \subseteq \mathbb{R}$
$f: A \to \mathbb{R}$
$a \in A$

f is continuous at a if one of the following:

I $a \in \mathcal{J}_{zo}A \to f$ not continuous at a

II $a \in A \setminus \mathcal{J}_{zo}A \subseteq A'$ $\xrightarrow{def}$ $\forall (a_n) \subseteq A$ s.t. $\lim_{n\to\infty} a_n = a$, it holds $\lim_{n\to\infty} f(a_n) = f(a)$

the set of the accumulation points

$\xrightarrow[th.]{neigh.}$ $\forall V \in \mathcal{V}(f(a))$, $\exists U \in \mathcal{V}(a)$ s.t. $\forall x \in U \cap A$ it holds $f(x) \in V$

$\xrightarrow[th.]{\varepsilon,\delta}$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in A$ with $|x-a| < \delta$ it holds $|f(x) - f(a)| < \varepsilon$

$\xrightarrow[way]{highschool}$ $\exists \lim_{x\to a} f(x) = f(a)$

Remark:

Whenever we are not specifically asked to prove that f is continuous at a certain point, we write that it is a composition of elementary functions, thus it is continuous.

Prove a function is not continuous at a point

$\emptyset \neq A \subseteq \mathbb{R}$
$f: A \to \mathbb{R}$
$a \in A$

f is NOT continuous at a if one of the following:

I $a \in \mathcal{J}_{zo}A \to f$ not continuous at a

II $a \in A \setminus \mathcal{J}_{zo}A \subseteq A'$ $\xrightarrow{def}$

- We emphasize just a sequence $(b_n) \subseteq A$ with $\lim_{n\to\infty} b_n = a$, but for which $\lim_{n\to\infty} f(b_n) \neq f(a)$

  (in this case, $\lim_{x\to a} f(x)$ might exist, but its $\neq f(a)$)

- We emphasize two sequences (b_n) and $(t_n) \subseteq A$ s.t.
  $\lim_{n\to\infty} b_n = \lim_{n\to\infty} t_n = a$
  but
  $\lim_{n\to\infty} f(b_n) \neq \lim_{n\to\infty} f(t_n)$ (in this case $\nexists \lim_{x\to a} f(x)$)

$\xrightarrow[way]{highschool}$

- $\exists \lim_{\substack{x\to a\\x<a}} f(x) \neq \lim_{\substack{x\to a\\x>a}} f(x)$ (in this case $\nexists \lim_{x\to a} f(x)$)

- $\exists \lim_{x\to a} f(x) \neq f(a)$

$\xrightarrow[th.]{neigh.}$
$\xrightarrow[th.]{\varepsilon,\delta}$ the hard way

Continuity for Dirichlet type functions

Dirichlet functions

$f : \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \begin{cases} g(x) & : x \in \mathbb{Q} \\ h(x) & : x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ $\quad \to$ unclear how to compute side limits

f is continuous on the set $\mathcal{C} = \{x \in \mathbb{R} : g(x) = h(x)\}$

We use ε and δ characterisation theorem

f is discontinuous on $\mathbb{R} \setminus \mathbb{Q}$

(to prove, we just deny the definition and emphasize for a $t \in \mathbb{R} \setminus \mathbb{Q}$)

$$\left.\begin{cases} \exists (b_n) \subseteq \mathbb{Q} \text{ with } \lim\limits_{n \to \infty} b_n = t \\ \exists (t_n) \subseteq \mathbb{R} \setminus \mathbb{Q} \text{ with } \lim\limits_{n \to \infty} t_n = t \end{cases} \text{ s.t. } \begin{array}{c} \lim\limits_{n \to \infty} f(b_n) = \lim\limits_{n \to \infty} g(b_n) = g(t) \\ \neq \\ \lim\limits_{n \to \infty} f(t_n) = \lim\limits_{n \to \infty} h(t_n) = h(t) \end{array} \right\} \Rightarrow t \notin \mathcal{C}$$

$$\left.\begin{array}{l} t \in \mathcal{C} \overset{\text{def.}}{\Rightarrow} f \text{ is not continuous at } t \\ \qquad\qquad\qquad\qquad\qquad t\text{-random} \end{array}\right\} \Rightarrow f \text{ is not continuous on } \mathbb{R} \setminus \mathbb{Q}$$

Continuous Functions

Exercise 1: By using the defintion, verify if the following function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

is continuous.

Exercise 2:
a) By using the ε and δ characterization theorem, prove that the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x & : x \in \mathbb{Q} \\ -x & : x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases}$$

is continuous at 0.

b) By using the definition, prove that the function from **a)** has not other continuity points, except for 0, thus $\forall x \in \mathbb{R}\backslash\{0\}$, f is discontinuous at x.

Exercise 3: Study the continuity of the functions:

a) $f : (-\infty, 0] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in (-\infty, 0) \\ 7 & : x = 0, \end{cases}$$

b)$f : [-1, 2] \cup \{4\} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x + 3 & : x \in [-1, 2] \\ 0 & : x = 4. \end{cases}$$

Exercise 4: Study the continuity of the functions:

a)$f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{\sin x^2}{|x|} & : x \neq 0 \\ 0 & : x = 0. \end{cases} .$$

b)$f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{x^{-1}} & : x \in (0, \infty) \\ 0 & : x = 0, \\ x^2 + 2x + \sin x & : x \in (-\infty, 0). \end{cases} .$$

c)$f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in \mathbb{Q} \\ \cos x & : x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases} .$$

d) $f : [-2,1] \cup \{3\} \to \mathbb{R}$,

$$f(x) = \begin{cases} \cos(\pi x) & : x \in [-2,0] \\ 1+\sin x & : x = (0,1] \\ 2 & : x = 3. \end{cases}$$

Exercise 5: Study, by discussing the parameter $a \in \mathbb{R}$, the continuity of the following functions:

a) $f : [1,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sqrt{a^2 - 2ax + x^2} & : x \in [1,2] \\ \\ 3a + 2x & : x \in (2,3]. \end{cases}$$

b) $f : (0,\pi) \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{3x} & : x \in (0,1] \\ \\ a\dfrac{\sin(x-1)}{x^2 - 5x + 4} & : x \in (1,\pi). \end{cases}$$

Exercise 6: Let $0 < a < b \in \mathbb{R}$ and $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$, defined by:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\backslash\{0,1\}.$$

- a) Prove that f is a continuous function.
- b) Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = f(x), \forall x \in \mathbb{R}\backslash\{0,1\}$.
- c) Compute $\lim\limits_{x\to-\infty} F(x)$ and $\lim\limits_{x\to\infty} F(x)$.

Exercise 1: By using the defintion, verify if the following function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

is continuous.

$f: \mathbb{R} \to \mathbb{R}, \ f(x) = x^2$

f continuous at $x_0 \in \mathbb{R} \overset{def.}{\Longleftrightarrow} \forall (x_n) \subseteq \mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = x_0 \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(x_0)$

Step 1: (necessities) Choose x_0 randomly

Let $(x_n) \subseteq \mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = x_0$

Step 2: (compute) $f(x_n) = x_n^2$

$\left.\begin{array}{l} \lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} x_n^2 = x_0^2 = f(x_0) \\ x_0 \text{ random} \end{array}\right\} \Rightarrow \forall (x_n) \subseteq \mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = x_0 \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(x_0)$

$\Downarrow$

f continuous for any $x_0 \in \mathbb{R}$

$\Downarrow$

f continuous on $\mathbb{R}$

Exercise 2:

a) By using the ε and δ characterization theorem, prove that the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x & : x \in \mathbb{Q} \\ -x & : x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

is continuous at 0.

b) By using the definition, prove that the function from **a)** has not other continuity points, except for 0, thus $\forall x \in \mathbb{R} \setminus \{0\}$, f is discontinuous at x.

$f: \mathbb{R} \to \mathbb{R}, \ f(x) = \begin{cases} x & : x \in \mathbb{Q} \\ -x & : x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

a) f continuous at $0 \underset{TH.}{\overset{\varepsilon, \delta}{\Longleftrightarrow}} \forall \varepsilon > 0, \ \exists \delta > 0$ s.t. $\forall x \in \left\langle \begin{array}{l} \mathbb{Q} \\ \mathbb{R} \setminus \mathbb{Q} \end{array}\right.$ with $|x - 0| < \delta$, it holds $|f(x) - f(0)| < \varepsilon$

$|x| < \delta$

$x < \delta$

$\to x_0 = 0 \in \mathbb{Q}$

Choose $\varepsilon > 0$ randomly (1)

$\exists \delta > 0$ with $|x| < \delta \underset{(2)}{\Rightarrow} |f(x)| < \delta \Rightarrow |f(x) - 0| < \delta \Rightarrow |f(x) - f(0)| < \delta$

jump to the end $\Big|$ $|f(x) - f(0)| < \varepsilon$

$\left.\begin{array}{l} |f(x) - f(0)| < \delta \\ \text{Choose } \varepsilon := \delta > 0 \end{array}\right] \Rightarrow |f(x) - f(0)| < \varepsilon, \ \forall x \in \mathbb{Q} \ (3)$

From (1), (2), (3) $\Rightarrow \forall \varepsilon > 0, \ \exists \delta > 0$ with $|x| < \delta$, it holds $|f(x) - f(0)| < \varepsilon \Leftrightarrow f$ continuous at 0

$\to x_0 = 0 \in \mathbb{R}$

Choose $\varepsilon > 0$ randomly (1)

$\exists\, \delta > 0$ with $|x| < \delta \underset{(2)}{\Rightarrow} |f(x)| < \delta \Rightarrow |f(x) - 0| < \delta \Rightarrow |f(x) - f(0)| < \delta$

jump to the end $\Big|$ $|f(x) - f(0)| < \varepsilon$

$$\left.\begin{array}{l} |f(x) - f(0)| < \delta \\ \text{Choose } \varepsilon := \delta > 0 \end{array}\right\} \Rightarrow |f(x) - f(0)| < \varepsilon, \ \forall x \in \mathbb{R} \setminus \mathbb{Q} \ (3)$$

From (1), (2), (3) $\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ with $|x| < \delta$, it holds $|f(x) - f(0)| < \varepsilon \Leftrightarrow f$ continuous at 0

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x & : x \in \mathbb{Q} \\ -x & : x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

f continuous at $x_0 \in \mathbb{R} \setminus \{0\} \overset{\text{def.}}{\Longleftrightarrow} \forall (x_n) \subseteq \mathbb{R} \setminus \{0\}$ with $\lim\limits_{n \to \infty} x_n = x_0 \Rightarrow \lim\limits_{n \to \infty} f(x_n) = f(x_0)$

$x_0 \in \mathbb{Q} \setminus \{0\}$

Let $(x_n) \subseteq \mathbb{Q}$ with $\lim\limits_{n \to \infty} x_n = x_0$. Then $f(x_n) = x_n$

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = x_0 \quad (1)$$

$x_0 \in \mathbb{R} \setminus \mathbb{Q} \setminus \{0\}$

Let $(y_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ with $\lim\limits_{n \to \infty} y_n = x_0$. Then $f(y_n) = -x_0$

$$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} (-x_0) = -x_0 \quad (2)$$

if f has a limit at $x_0 \in \mathbb{R} \setminus \{0\}$, it is unique

therefore

f continuous at $x_0 \in \mathbb{R} \setminus \{0\} \Longleftrightarrow$

$$x_0 = -x_0$$
$$2x_0 = 0$$
$$x_0 = 0 \notin \mathbb{R} \setminus \{0\}$$

$\nexists$ another point at which f could be continuous

$\Rightarrow f$ is discontinuous on $\mathbb{R} \setminus \{0\}$

Exercise 3: Study the continuity of the functions:

a) $f : (-\infty, 0] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in (-\infty, 0) \\ 7 & : x = 0, \end{cases}$$

b) $f : [-1, 2] \cup \{4\} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x + 3 & : x \in [-1, 2] \\ 0 & : x = 4. \end{cases}$$

a) $f : (-\infty, 0] \to \mathbb{R}$ $\quad f(x) = \begin{cases} \sin x : & x \in (-\infty, 0) \\ 7 : & x = 0 \end{cases}$

f continuous on $(-\infty, 0)$ (elementary function)

the problem of continuity is for $x_0 = 0$

f continuous at $x_0 = 0 \overset{\text{side}}{\underset{\text{limits}}{\Longleftrightarrow}} \lim\limits_{\substack{x \to 0 \\ x > 0}} f(x) = \lim\limits_{\substack{x \to 0 \\ x < 0}} f(x) = f(x_0)$

$$\left.\begin{aligned} &\lim_{\substack{x \to 0 \\ x > 0}} f(x) = 7 \\ &\lim_{\substack{x \to 0 \\ x < 0}} f(x) = \lim_{\substack{x \to 0 \\ x < 0}} \sin x = \sin 0 = 0 \\ &f(0) = 7 \end{aligned}\right\} \Rightarrow f \text{ is discontinuous at } 0 \Rightarrow f \text{ continuous on } (-\infty, 0)$$

b) $f : [-1, 2] \cup \{4\} \to \mathbb{R}$, $\quad f(x) = \begin{cases} 2x + 3 & : x \in [-1, 2] \\ 0 & : x = 4 \end{cases}$

f continuous on $[-1, 2]$ (elementary functions)
the problem of continuity is on the point $x_0 = 4$
but $x_0 = 4$ is an isolated point $\Rightarrow$ f is definetly discontinuous at 4

Exercise 4: Study the continuity of the functions:

a) $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases}\dfrac{\sin x^2}{|x|} & : x\neq 0\\ 0 & : x=0.\end{cases}.$$

a) $f:\mathbb{R}\to\mathbb{R}$, $\quad f(x)=\begin{cases}\dfrac{\sin x^2}{|x|} & : x\neq 0\\ 0 & : x=0\end{cases} = \begin{cases}\dfrac{\sin x^2}{x} & , x>0\\ 0 & , x=0\\ -\dfrac{\sin x^2}{x} & , x<0\end{cases}$

f is continuous on $\mathbb{R}\setminus\{0\}$ (elementary functions)
the problem of continuity is on the point $x_0=0$

f is continuous at $x_0=0 \overset{\text{def}}{\Longleftrightarrow} \forall (x_n)\subseteq\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n=0 \Rightarrow \lim\limits_{n\to\infty} f(x_n)=f(0)$

Consider $(x_n)\subseteq\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n=0$

$f(x_n)=0$

$\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{n\to\infty} 0 = 0 = \lim\limits_{n\to\infty} f(0)$

$\Rightarrow f$ continuous on $0\in\mathbb{R}$

$\Rightarrow f$ continuous on $\mathbb{R}$

b) $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases}e^{x^{-1}} & : x\in(0,\infty)\\ 0 & : x=0,\\ x^2+2x+\sin x & : x\in(-\infty,0).\end{cases}.$$

$f:\mathbb{R}\to\mathbb{R}$, $\quad f(x)=\begin{cases}e^{x^{-1}} & : x\in(0,\infty)\\ 0 & : x=0\\ x^2+2x+\sin x & : x\in(-\infty,0)\end{cases}$

f is continuous on $(-\infty,0)\cup(0,\infty)=\mathbb{R}\setminus\{0\}$ (operations with elementary functions)
the problem of continuity is in the point $x_0=0$

f is continuous at $x_0=0 \overset{\text{side}}{\underset{\text{limit}}{\Longleftrightarrow}} \lim\limits_{\substack{x\to 0\\ x>0}} f(x)=\lim\limits_{\substack{x\to 0\\ x<0}} f(x)=f(0)$

$\lim\limits_{\substack{x\to 0\\ x>0}} f(x)=\lim\limits_{\substack{x\to 0\\ x>0}} e^{x^{-1}}=\lim\limits_{\substack{x\to 0\\ x>0}} e^{\frac{1}{x}}=e^{\frac{1}{0_+}}=e^{\infty}=\infty$

$\lim\limits_{\substack{x\to 0\\ x<0}} f(x)=\lim\limits_{\substack{x\to 0\\ x<0}} x^2+2x+\sin x=0$

$f(0)=0$

$\neq \Rightarrow f$ is not continuous at $x_0=0$

c) $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in \mathbb{Q} \\ \cos x & : x \in \mathbb{R}\setminus\mathbb{Q}, \end{cases}.$$

$$f:\mathbb{R}\to\mathbb{R}, \quad f(x) = \begin{cases} \sin x & : x \in \mathbb{Q} \\ \cos x & : x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Let x_0 be a continuity point.

$\xRightarrow{\text{def.}}$ Consider $(x_n) \subseteq \mathbb{Q}$ with $\lim\limits_{n\to\infty} x_n = x_0$

$$f(x_n) = \sin x_n$$
$$\lim_{n\to\infty} f(x_n) = f(x_0) = \sin x_0 \quad (1)$$

Consider $(y_n) \subseteq \mathbb{Q}\setminus\mathbb{R}$ with $\lim\limits_{n\to\infty} y_n = x_0$

$$f(y_n) = f(x_0) = \cos x_0 \quad (2)$$

f continuous at x_0 $\Rightarrow$ the limit is unique $\Rightarrow$ $\sin x_0 = \cos x_0 \Longleftrightarrow x_0 \in \left\{\frac{\pi}{4} + \pi n,\ n\in\mathbb{Z}\right\}$

f continuous on $\mathscr{C} = \left\{x \mid x = \frac{\pi}{4} + \pi n,\ n\in\mathbb{Z}\right\}$

f discontinuous on $\mathbb{R}\setminus\mathscr{C}$

d) $f : [-2, 1] \cup \{3\} \to \mathbb{R}$,

$$f(x) = \begin{cases} \cos(\pi x) & : x \in [-2, 0] \\ 1 + \sin x & : x = (0, 1] \\ 2 & : x = 3. \end{cases}$$

$$f:[-2,1]\cup\{3\}\to\mathbb{R}, \quad f(x) = \begin{cases} \cos(\pi x) & : x \in [-2,0] \\ 1+\sin x & : x = (0,1] \\ 2 & : x = 3 \end{cases}$$

f continuous on $[-2,1]\setminus\{0,3\}$ (elementary function)
f discontinuous at $x_0 = 3$ because $x_0 = 3$ is an isolated point

f continuous at $x_0 = 0 \underset{\text{limits}}{\overset{\text{side}}{\Longleftrightarrow}} \lim\limits_{\substack{x\to 0\\ x>0}} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} f(x) = f(0)$

$$\left.\begin{array}{l} \lim\limits_{\substack{x\to 0\\ x>0}} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} (1+\sin x) = 1+0 = 1 \\[2ex] \lim\limits_{\substack{x\to 0\\ x<0}} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} (\cos \pi x) = \cos 0 = 1 \\[2ex] f(0) = \cos(\pi\cdot 0) = \cos 0 = 1 \end{array}\right\} = \ \Rightarrow f \text{ continuous at } 0$$

$\Rightarrow f$ continuous on $[-2,1]$

Exercise 5: Study, by discussing the parameter $a \in \mathbb{R}$, the continuity of the following functions:

a) $f : [1,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sqrt{a^2 - 2ax + x^2} & : x \in [1,2] \\ 3a + 2x & : x \in (2,3]. \end{cases}$$

b) $f : (0,\pi) \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{3x} & : x \in (0,1] \\ a\dfrac{\sin(x-1)}{x^2 - 5x + 4} & : x \in (1,\pi). \end{cases}$$

Exercise 6: Let $0 < a < b \in \mathbb{R}$ and $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$, defined by:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\backslash\{0,1\}.$$

- a) Prove that f is a continuous function.
- b) Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = f(x), \forall x \in \mathbb{R}\backslash\{0,1\}$.
- c) Compute $\lim\limits_{x\to-\infty} F(x)$ and $\lim\limits_{x\to\infty} F(x)$.

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$a \in \mathbb{R}$

a) $f : [1,3] \to \mathbb{R}$

$$f(x) = \begin{cases} \sqrt{a^2-2ax+x^2}, & x \in [1,2] \\ 3a+2x, & x \in (2,3] \end{cases}$$

f continuous on $[1,3] \setminus \{2\}$ (operations with elementary functions).
The problem of continuity is for the point $x_0 = 2$.
We shall compute the side limits at $x_0 = 2$,
as well as the value of the function in the said point.

$$\exists\, l_l(2) = \lim_{\substack{x\to 2\\ x<2}} f(x) = \lim_{\substack{x\to 2\\ x<2}} \sqrt{a^2-2ax+x^2} = \sqrt{a^2-4a+4} = \sqrt{(a-2)^2} = |a-2|$$

$$\exists\, l_r(2) = \lim_{\substack{x\to 2\\ x>2}} f(x) = \lim_{\substack{x\to 2\\ x>2}} (3a+2x) = 3a+4$$

$$f(2) = \sqrt{a^2-4a+4} = |a-2|$$

$\to$ if $l_l(2) = l_r(2) \Rightarrow |a-2| = 3a+4$
- $a-2 = -3a-4 \Rightarrow 4a = -2 \Rightarrow a = -\frac{1}{2}$
- $a-2 = 3a+4 \Rightarrow 2a = -6 \Rightarrow a = -3$

$\Rightarrow a \in \{-\frac{1}{2}, -3\} \Rightarrow f$ is continuous at $x_0 = 2 \Rightarrow f$ is continuous on $[1,3]$

$\to$ if $a \notin \{-\frac{1}{2}, -3\} \Rightarrow l_l(2) \neq l_r(2) \Rightarrow f$ is discontinous at $x_0 = 2 \Rightarrow$ the domain of continuity is $D_2 = [1,3] \setminus \{2\}$

b) $f : (0,\pi) \to \mathbb{R}$

$$f(x) = \begin{cases} e^{3x}, & x \in (0,1] \\ a \cdot \dfrac{\sin(x-1)}{x^2-5x+4}, & x \in (1,\pi) \end{cases}$$

$x^2 - 5x + 4 = 0$
$\Delta = 25 - 16 = 9$
$x_{1,2} = \frac{5 \pm 3}{2}$; $x_1 = 1$, $x_2 = 4 \notin (1,\pi)$

f continuous on $(0,\pi) \setminus \{1\}$. (operations with elementary functions).
The problem of continuity is for the point $x_0 = 1$.
We shall compute the side limits at $x_0 = 1$,
as well as the value of the function in the said point.

$$\exists\, l_l(1) = \lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} e^{3x} = e^3$$

$$\exists\, l_r(1) = \lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} a \cdot \frac{\sin(x-1)}{x^2-5x+4} = a \lim_{\substack{x\to 1\\ x>1}} \frac{\sin(x-1)}{(x-1)(x-4)} = a \lim_{\substack{x\to 1\\ x>1}} \frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{(x-4)} = a \cdot \frac{1}{1-4} = -\frac{a}{3}$$

$$f(1) = e^3$$

$\to$ if $-\frac{a}{3} = e^3 \Rightarrow a = -3e^3 \Rightarrow l_l(1) = l_r(1) = f(1) \Rightarrow f$ continuous at $x_0 = 1 \Rightarrow f$ continuous on $(0,\pi)$

$\to$ if $a \in \mathbb{R} \setminus \{-3e^3\} \Rightarrow l_l(1) \neq l_r(1) \Rightarrow f$ discontinuous at $x_0 = 1 \Rightarrow$ the domain of continuity for f

is $D_c = (0,\pi)\setminus\{1\}$

Exercise 6: Let $0 < a < b \in \mathbb{R}$ and $f : \mathbb{R}\setminus\{0,1\} \to \mathbb{R}$, defined by:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\setminus\{0,1\}.$$

- a) Prove that f is a continuous function.
- b) Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = f(x), \forall x \in \mathbb{R}\setminus\{0,1\}$.
- c) Compute $\lim_{x\to-\infty} F(x)$ and $\lim_{x\to\infty} F(x)$.

$0 < a < b \in \mathbb{R}$

$f : \mathbb{R}\setminus\{0,1\} \to \mathbb{R}$

$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}$

a) Let x_0 be a continuity point.

Let $(x_n) \subseteq \mathbb{R}$ with $\lim_{n\to\infty} x_n = x_0$

$$f(x_n) = \left(\frac{b^{x_n} - a^{x_n}}{x_n(b-a)}\right)^{\frac{1}{x_n-1}}$$

$$\lim_{n\to\infty} f(x_n) = \left(\frac{b^{x_0} - a^{x_0}}{x_0(b-a)}\right)^{\frac{1}{x_0-1}}, \ \forall x_0 \in \mathbb{R}\setminus[0,1] \Rightarrow f \text{ continuous on } \mathbb{R}\setminus[0,1]$$

b) f continuous on $\mathbb{R}\setminus\{0,1\}$

We shall compute the limits at $x=0$ and $x=1$

$$\lim_{x\to 0}\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{x\to 0}\left(\frac{1}{b-a}\right)^{\frac{1}{x-1}}\cdot\left(\frac{b^x-1}{x}-\frac{a^x-1}{x}\right)^{\frac{1}{x-1}} = (b-a)\cdot(\ln b - \ln a)^{-1} = \frac{b-a}{\ln\frac{b}{a}} \Rightarrow \text{the limit } \exists$$

$\forall a, b \in \mathbb{R}, 0<a<b$

$$\lim_{x\to 1}\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{x\to 1}\left(1+\frac{b^x-a^x-x(b-a)}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{x\to 1}\left(\left(1+\frac{b^x-a^x-x(b-a)}{x(b-a)}\right)^{\frac{x(b-a)}{b^x-a^x-x(b-a)}}\right)^{\frac{b^x-a^x-x(b-a)}{x(x-1)(b-a)}} =$$

$$= e^{\lim_{x\to 1}\frac{b^x-a^x-xb+x\cdot a}{x(x-1)(b-a)}} = e^{\frac{1}{b-a}\lim_{x\to 1}\frac{b(b^{x-1}-x)-a(a^{x-1}-x)}{x-1}} = e^{\frac{1}{b-a}\lim_{x\to 1}\left(b\cdot\frac{b^{x-1}-x}{x-1}-a\cdot\frac{a^{x-1}-x}{x-1}\right)} =$$

$$= e^{\frac{1}{b-a}(b\ln b - a\ln a)}$$

Hence, let $F:\mathbb{R}\to\mathbb{R}$, $F(x) = \begin{cases} f(x), & x \in \mathbb{R}\setminus\{0,1\} \\ \frac{b-a}{\ln b-\ln a}, & x=0 \\ e^{\frac{\ln b^b-\ln a^a}{b-a}}, & x=1 \end{cases}$ $\quad \forall\ 0<a<b\in\mathbb{R}$

c) $$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty}\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{t\to\infty}\left(\frac{\left(\frac{1}{b}\right)^t-\left(\frac{1}{a}\right)^t}{-t(b-a)}\right)^{\frac{-1}{t+1}} = \lim_{t\to\infty}\left(\frac{\left(\frac{1}{a}\right)^t-\left(\frac{1}{b}\right)^t}{t(b-a)}\right)^{\frac{-1}{t+1}} =$$

$x=-t \xrightarrow{x\to-\infty} +\infty$

$$= \lim_{t\to\infty}\left(\frac{\left(\frac{1}{a}\right)^t\left(1-\left(\frac{a}{b}\right)^t\right)}{t(b-a)}\right)^{\frac{-1}{t+1}} = \lim_{t\to\infty}\frac{\left(\frac{1}{a}\right)^{\frac{-t}{t+1}}\left(1-\left(\frac{a}{b}\right)^t\right)^{\frac{-1}{t+1}}}{t^{\frac{-1}{t+1}}(b-a)^{\frac{-1}{t+1}}} =$$

$$= a\cdot\lim_{t\to\infty}\frac{1}{t^{\frac{-1}{t+1}}} = a\lim_{t\to\infty} t^{t+1} = a\cdot e^{\lim_{n\to\infty}(t+1)\ln t} = \infty$$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty}\left(\frac{b^x\left(1-\left(\frac{a}{b}\right)^x\right)}{x(b-a)}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty}\frac{b^{\frac{x}{x-1}}\cdot\left(1-\left(\frac{a}{b}\right)^x\right)^{\frac{1}{x-1}}}{x^{\frac{1}{x-1}}\cdot(b-a)^{\frac{1}{x-1}}} = b\cdot\lim_{x\to\infty}\frac{1}{x^{\frac{1}{x-1}}} =$$

$$= b\cdot e^{\lim_{x\to\infty}\frac{-\ln x}{x-1}} = b\cdot e^0 = b, \quad b\in\mathbb{R},\ 0>a>b$$

$$\lim_{x \to \infty} \frac{-\ln x}{x-1} = \left(\frac{-\infty}{\infty}\right) \text{L'Hospital} = -\lim_{x \to \infty} \frac{\frac{1}{x}}{1} = -\lim_{x \to \infty} \frac{1}{x} = 0$$

Exercițiul 6: Fie $0 < a < b \in \mathbb{R}$ şi $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$, definită prin:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\backslash\{0,1\}.$$

- a) Arătați că f este o funcție continuă.
- b) Arătacți că există o funcție coninuă $F : \mathbb{R} \to \mathbb{R}$ asfel încât $F(x)$ $\mathbb{R}\backslash\{0,1\}$.
- c) Calculați $\lim_{x\to-\infty} F(x)$ şi $\lim_{x\to\infty} F(x)$.

c) $L = \lim_{x\to\infty}\left(\frac{b^x-a^x}{x(b-a)}\right)^{\frac{1}{x-1}}$

$x \to \infty \Rightarrow \boxed{\frac{1}{x-1} \to 0}$

$0<a<b \Rightarrow b-a>0$ constant

$\Rightarrow \lim_{x\to\infty}\left(\frac{1}{b-a}\right)^{\frac{1}{x-1}} = c^0 = 1$

$L = 1 \cdot \lim_{x\to\infty}\left(\frac{b^x-a^x}{x}\right)^{\frac{1}{x-1}}$

$\lim_{x\to\infty}\frac{b^x-a^x}{x} = \lim_{x\to\infty}\frac{b^x}{x}\left[1-\left(\frac{a}{b}\right)^x\right]$, $\left(\frac{a}{b}\right)^x \to 0$, $1-0=1$

$0<a<b \Rightarrow \boxed{0<\frac{a}{b}<1}$

$\Rightarrow \lim_{x\to\infty}\left(\frac{a}{b}\right)^x = 0$

$= \lim_{x\to\infty}\frac{b^x}{x} = \begin{cases}\infty : & b>1\\ 0 : & b\le 1\end{cases}$ because $\lim_{x\to\infty} b^x = \begin{cases}\infty : & b>1\\ 1 : & b=1\\ 0 : & |b|<1\\ \nexists : & b\le -1\end{cases}$

$\Rightarrow \lim_{x\to\infty}\left(\frac{b^x-a^x}{x}\right)^{\frac{1}{x-1}} = \infty^0$ or 0^0

$= \lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} \cdot \underbrace{\left(1-\left(\frac{a}{b}\right)^x\right)^{\frac{1}{x-1}}}_{1^0=1} = \lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} \underline{\underline{\infty^0}}\ 0^0$

$\lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty} e^{\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}}} = e^{\lim_{x\to\infty}\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}}}$

$\infty^0, 0^0 \to$ with e $\quad \boxed{x = e^{\ln x}}$ $\quad \boxed{\ln a^b = b\ln a}$

$\cdot \lim_{x\to\infty}\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty}\frac{1}{x-1}\ln\left(\frac{b^x}{x}\right) \underline{\underline{\frac{\infty}{\infty}}}$

if $\begin{cases} b>1 \Rightarrow \frac{b^x}{x}\to\infty \Rightarrow \ln\frac{b^x}{x}\to\infty\\ b\le 1 \Rightarrow \frac{b^x}{x}\to 0 \Rightarrow \ln\frac{b^x}{x}\to -\infty\end{cases}$

$= \lim_{x\to\infty}\frac{\ln(b^x)-\ln x}{x-1} = \lim_{x\to\infty}\frac{x\ln b-\ln x}{x-1} =$

$\stackrel{L'H}{=} \lim_{x\to\infty}\frac{\ln b - \frac{1}{x}}{1} = \lim_{x\to\infty}\left[\ln b - \frac{1}{x}\right] =$

$= \ln b$

$\lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = e^{\lim_{x\to\infty}\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}}} = e^{\ln b} = b$

$\boxed{L = b}$

$\boxed{\lim_{x\to-\infty} F(x) = a}$ Hw

Dirichlet functions

$f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\begin{cases} g(x) & : x\in\mathbb{Q}\\ h(x) & : x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

- for this function, it is unclear how to compute side limits
- they are c. on the set $\mathcal{C}=\{x\in\mathbb{R} : g(x)=h(x)\}$

in order to prove c. it is easy to use ε and δ theorem

- they are discontinuous on $\mathbb{R}\setminus\mathcal{C}$

in order to prove that, just deny the definition, and emphasize: for a $t\in\mathbb{R}\setminus\mathcal{C}$

Hw3 $\begin{cases} \cdot\ \exists (b_m)\subseteq\mathbb{Q} \text{ with } \lim_{m\to\infty} b_m = t & \text{s.t. } \lim_{m\to\infty} f(b_m)=\lim_{m\to\infty} g(b_m)=g(t)\\ \cdot\ \exists (t_m)\subseteq\mathbb{R}\setminus\mathbb{Q} \text{ with } \lim_{m\to\infty} t_m = t & \lim_{m\to\infty} f(t_m)=\lim_{m\to\infty} h(t_m)=h(t)\end{cases}$ $\neq)\ t\notin\mathcal{C}$

$\overset{def}{\Rightarrow}$ f is not c. at t, t-random $\Big\}\Rightarrow$ f is not c. on $\mathbb{R}\setminus\mathcal{C}$.

Example: The classical Dirichlet function $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\begin{cases}1 & : x\in\mathbb{Q}\\ 0 & : x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

Study the continuity of f.

f is nowhere continuous on $\mathbb{R}$

Choose $a\in\mathbb{R}$ randomly (2 sequences $\underset{(b_m)}{\mathbb{Q}}$ and $\underset{(t_m)}{\mathbb{R}\setminus\mathbb{Q}}$)

$f(b_m)=1 \quad \forall m\in\mathbb{N} \Rightarrow (f(b_m))$ is the constant sequence 1

$\Rightarrow \lim_{m\to\infty} f(b_m)=1$

$f(t_m)=0 \quad \forall m\in\mathbb{N} \Rightarrow (f(t_m))$ → the constant sequence 0

$\Rightarrow \lim_{m\to\infty} f(t_m)=0$ $\neq$

$\Rightarrow f$ is not c. at a, a-random $\Big\} \Rightarrow \boxed{f \text{ is disc. on } \mathbb{R}}$.

Ex2 (H8) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\begin{cases} x & : x\in\mathbb{Q}\\ -x & : x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

b) f is $\boxed{\text{not c.}}$ on $\mathbb{R}\setminus\{0\}$. Choose $a\in\mathbb{R}\setminus\{0\}$

$(b_m)\subseteq\mathbb{Q}$ with $\lim_{m\to\infty} b_m = a$, $\quad f(b_m)=b_m \quad \forall m\in\mathbb{N}$, $\lim_{m\to\infty} f(b_m)=\lim_{m\to\infty} b_m = a$

$(t_m)\subseteq\mathbb{R}\setminus\mathbb{Q}$ — " — $\quad f(t_m)=-t_m \quad \forall m\in\mathbb{N}$, $\lim_{m\to\infty} f(t_m)=\lim_{m\to\infty} -t_m = -a$

$a=-a \Leftrightarrow \boxed{a=0}$ ↯ $\Rightarrow f$ is NOT c. at a, a random $\Big\}$ f is not c. on $\mathbb{R}\setminus\{0\}$

Ex1 $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=x^2$.

Prove that f is c. on $\mathbb{R}$. (with the definition)

We prove that:

$\forall (a_m)\subseteq A$ with $\lim_{m\to\infty} a_m = a$, $\exists \lim_{m\to\infty} f(a_m)=f(a)$.

Choose $\boxed{(a_m)\subseteq A \text{ with } \lim_{m\to\infty} a_m = a}$ randomly

$f(a_m)=a_m^2 \quad \forall m\in\mathbb{N}$, $\Rightarrow \lim_{m\to\infty} f(a_m)=\lim_{m\to\infty} a_m^2 = \boxed{a^2}=f(a)$ ($a\in\mathbb{R}$)

(a_m) → random $\overset{def}{\Longrightarrow}$ f is c. at a, a random $\Big\} \Rightarrow f$ is c. on $\mathbb{R}$.

Ex2 Hw8a) $f(x)=\begin{cases} x & : x\in\mathbb{Q}\\ -x & : x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

a) prove that it is c. at 0.

Assume we do it like before.

choose $(a_m)\subseteq\mathbb{R}$ with $\boxed{\lim_{m\to\infty} a_m = 0}$ $\quad f(a_m)=\begin{cases} -a_m & : a_m\in\mathbb{Q}\\ a_m & : a_m\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

?$\exists$ $\lim_{m\to\infty} f(a_m)=\begin{cases} ?\\ ?\end{cases}$

(x_{2m}) $\hookrightarrow \mathbb{N}$ $\quad (x_{2m+1})$ $\mathbb{N}$

$a_m = ?$ $\quad a_m\in\mathbb{Q}$? $\quad$ We cannot name proper subsequences

- with ε and δ is easier

We prove that: $\forall\varepsilon>0, \exists\delta>0$ s.t. $\forall x\in\mathbb{R}$ with $|x-0|<\delta$, it holds $|f(x)-f(0)|<\varepsilon$

① Choose $\boxed{\varepsilon>0}$ randomly

② sup end $\quad |f(x)-\cancel{f(0)}|<\varepsilon$ ($f(0)=0$) $\Leftrightarrow |f(x)|<\varepsilon$ $\quad$? $|x|<\varepsilon$

$|f(x)|=\begin{cases} |x| & : x\in\mathbb{Q}\\ |-x| & : x\in\mathbb{R}\setminus\mathbb{Q}\end{cases} = |x|$

③ middle $\quad$ assign $\boxed{\delta := \varepsilon}$

$\forall x\in\mathbb{R}$ with $|x-0|<\delta$, it holds $|f(x)-f(0)|=|x|<\delta=\varepsilon$

for $\varepsilon>0$ random $\exists\ \delta:=\varepsilon>0$ s.t. $\forall x\in\mathbb{R}$ with $|x-0|<\delta$, it holds $|f(x)-f(0)|<\varepsilon$, ε-random $\Big\}\overset{T\varepsilon\delta}{\Rightarrow}$ f is c. at 0.

a) Prove that f is c. at 0 with the help of the <u>ε,δ theorem</u>

f is c. at $a \Leftrightarrow \forall \varepsilon>0, \exists \delta>0$ s.t. $\forall x\in A$ with $|x-a|<\delta$, it holds $|f(x)-f(a)| < \varepsilon$

$\boxed{a=0}$

$\forall \varepsilon>0, \exists \delta>0$ s.t. $\forall x \in A$ with $\boxed{|x|<\delta}$ it holds $\underbrace{|f(x)-f(0)|}<\varepsilon$

free to choose

$|f(x)|<\varepsilon$

$$f(x)=\begin{cases} x & : x\in\mathbb{Q} \\ -x & : x\in\mathbb{R}\setminus\mathbb{Q}\end{cases} \Rightarrow \underset{0\in\mathbb{Q}}{f(0)=0}$$

① Choose $\varepsilon>0$ randomly

② imp. to the end: $|f(x)|<\varepsilon$?

$$|f(x)|=\begin{cases} |x| & : x\in\mathbb{Q} \\ |-x| & : x\in\mathbb{R}\setminus\mathbb{Q}\end{cases} = |x|$$

$|f(x)|=|x|<\varepsilon$

③ middle: $\boxed{\exists\, \delta:=\varepsilon}$ $\quad \forall x\in A$ with $|x|<\delta$, it holds $|f(x)|=|x|<\delta=\varepsilon$

for $\varepsilon>0$ we $\exists\, \delta:=\varepsilon$ s.t. $\forall x\in A$ with $|x|<\delta$, it hold $|f(x)|<\varepsilon \Big\} \Rightarrow \forall\varepsilon>0$

ε - random

$\Downarrow T\varepsilon,\delta$

$\boxed{f \text{ is c. at } 0}$

<u>Conclusion</u>: f is c. only at 0 and disc. on $\mathbb{R}\setminus\{0\}$

it is a function with just a single continuity point.

Exercițiul 3: Studiați continuitatea funcției

a) $f : (-\infty, 0] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in (-\infty, 0) \\ 7 & : x = 0, \end{cases}$$

is is c. on $(-\infty, 0)$
not at 0 $\lim_{\substack{x \to 0 \\ x<0}} f(x) = \sin 0 = 0 \neq f(0) = 7$

b) $f : [-1, 2] \cup \{4\} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x + 3 & : x \in [-1, 2] \\ 0 & : x = 4. \end{cases}$$

f is c. on $[-1,2]$
$\boxed{\text{c. at } 4}$ as $\boxed{4 \in \mathrm{Iz}\, A}$

Exercițiul 4: Studiați continuitatea funcțiilor:

a) $f : \mathbb{R} \to \mathbb{R}$,

c.

$$f(x) = \begin{cases} \dfrac{\sin x^2}{|x|} & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

c. on $\mathbb{R} \setminus \{0\}$ as an elementary function
c. at 0 $\lim_{x \to 0} \underbrace{\frac{\sin x^2}{x^2}}_{\to 1} \left(\frac{\cancel{x}x^2}{|x|} \right) = 0$

b) $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{x^{-1}} & : x \in (0, \infty) \\ 0 & : x = 0, \\ x^2 + 2x + \sin x & : x \in (-\infty, 0). \end{cases}$$

c. on $\mathbb{R} \setminus \{0\}$ as an elementary functions
$\lim_{\substack{x \to 0 \\ x<0}} e^{x-1} = \frac{1}{e} \neq 0 = f(0) \Rightarrow$ not c. at 0

c) $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in \mathbb{Q} \\ \cos x & : x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

$\to$ f is c. on $\mathcal{C}$ [proof ε and δ $\to$ Ex 2.]

$\to$ f. is not c. on $\mathbb{R} \setminus \mathcal{C}$
$\to$ with the def.

$\boxed{\sin x = \cos x}$

$\mathcal{C} = \{ x \in \mathbb{R} : \sin x = \cos x \} = \left\{ \frac{\pi}{4} + k\pi : k \in \mathbb{Z} \right\}$

HW

$(b_n) \subseteq \mathbb{Q}$ $(t_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$

Exercițiul 3: Studiați continuitatea funcției

a) $f : (-\infty, 0] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in (-\infty, 0) \\ 7 & : x = 0, \end{cases}$$

is is c. on $(-\infty, 0)$
not at 0 $\lim_{\substack{x \to 0 \\ x<0}} f(x) = \sin 0 = 0 \neq f(0) = 7$

b) $f : [-1, 2] \cup \{4\} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x + 3 & : x \in [-1, 2] \\ 0 & : x = 4. \end{cases}$$

f is c. on $[-1,2]$
$\boxed{\text{c. at } 4}$ as $\boxed{4 \in \mathrm{Iso}\, A}$

Exercițiul 4: Studiați continuitatea funcțiilor:

a) $f : \mathbb{R} \to \mathbb{R}$,

c.

$$f(x) = \begin{cases} \dfrac{\sin x^2}{|x|} & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

c. on $\mathbb{R} \setminus \{0\}$ as an elementary function
c. at 0 $\lim_{x \to 0} \underbrace{\left(\frac{\sin x^2}{x^2}\right)}_{\to 1} \cdot \underbrace{\left(\frac{x^2}{|x|}\right)}_{0} = 0$

b) $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{x^{-1}} & : x \in (0, \infty) \\ 0 & : x = 0, \\ x^2 + 2x + \sin x & : x \in (-\infty, 0). \end{cases}$$

c. on $\mathbb{R} \setminus \{0\}$ as an elementary functions
$\lim_{\substack{x \to 0 \\ x<0}} e^{x-1} = \frac{1}{e} \neq 0 = f(0) \Rightarrow$ not c. at 0

c) $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in \mathbb{Q} \\ \cos x & : x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

→ f is c. on $\mathcal{C}$ [proof ε and δ → Ex 2.]

f is not c. on $\mathbb{R} \setminus \mathcal{C}$
→ with the def.

$\boxed{\sin x = \cos x}$

$\mathcal{C} = \{x \in \mathbb{R} : \sin x = \cos x\} = \left\{\frac{\pi}{4} + k\pi : k \in \mathbb{Z}\right\}$

HW

$(b_n) \subseteq \mathbb{Q}$ $(t_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$

Exercițiul 6: Fie $0 < a < b \in \mathbb{R}$ și $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$, definită prin:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\backslash\{0,1\}.$$

- a) Arătați că f este o funcție continuă.
- b) Arătacți că există o funcție coninuă $F : \mathbb{R} \to \mathbb{R}$ asfel încât $F(x)$ $\mathbb{R}\backslash\{0,1\}$.
- c) Calculați $\lim_{x\to-\infty} F(x)$ și $\lim_{x\to\infty} F(x)$.

c) $L = \lim_{x\to\infty}\left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}$

$x \to \infty \Rightarrow \boxed{\frac{1}{x-1} \to 0}$

$0 < a < b$
$\Rightarrow b - a > 0$ constant

$\Rightarrow \lim_{x\to\infty}\left(\frac{1}{b-a}\right)^{\frac{1}{x-1}} = C^0 = 1$

$L = 1 \cdot \lim_{x\to\infty}\left(\frac{b^x - a^x}{\cancel{x}}\right)^{\frac{1}{x-1}}$

$\lim_{x\to\infty}\frac{b^x - a^x}{x} = \lim_{x\to\infty}\frac{b^x}{x}\left[1 - \left(\frac{a}{b}\right)^x\right]$, $\left(\frac{a}{b}\right)^x \to 0$, $\underbrace{1 - 0 = 1}$

$0 < a < b \Rightarrow \boxed{0 < \frac{a}{b} < 1}$

$\Rightarrow \lim_{x\to\infty}\left(\frac{a}{b}\right)^x = 0$

$= \lim_{x\to\infty}\frac{b^x}{x} = \begin{cases} \infty : & b > 1 \\ 0 : & b \le 1 \end{cases}$

because $\lim_{x\to\infty} b^x = \begin{cases} \infty : & b > 1 \\ 1 : & b = 1 \\ 0 : & |b| < 1 \\ \nexists : & b \le -1 \end{cases}$

$\Rightarrow \lim_{x\to\infty}\left(\frac{b^x - a^x}{x}\right)^{\frac{1}{x-1}} = \infty^0$ or 0^0

$= \lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} \cdot \underbrace{\left(1 - \left(\frac{a}{b}\right)^x\right)^{\frac{1}{x-1}}}_{1^0 = 1} = \lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} \underline{\underline{\overset{\infty^0}{=}}}_{0^0}$

$\lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty} e^{\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}}} = e^{\lim_{x\to\infty}\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}}}$

$\infty^0, 0^0 \to$ with e

$\boxed{x = e^{\ln x}}$

$\boxed{\ln a^b = b \ln a}$

- $\lim_{x\to\infty}\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = \lim_{x\to\infty}\frac{1}{x-1}\ln\left(\frac{b^x}{x}\right) \overset{\frac{\infty}{\infty}}{=}$

if $\begin{cases} b > 1 \Rightarrow \frac{b^x}{x} \to \infty \Rightarrow \ln\frac{b^x}{x} \to \infty \\ b \le 1 \Rightarrow \frac{b^x}{x} \to 0 \Rightarrow \ln\frac{b^x}{x} \to -\infty \end{cases}$

$= \lim_{x\to\infty}\frac{\ln(b^x) - \ln x}{x-1} = \lim_{x\to\infty}\frac{x\ln b - \ln x}{x-1} =$

$\overset{L'H}{=} \lim_{x\to\infty}\frac{\ln b - \frac{1}{x}}{1} = \lim_{x\to\infty}\left[\ln b - \frac{1}{x}\right] =$

$= \ln b$

$\lim_{x\to\infty}\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}} = e^{\lim_{x\to\infty}\ln\left(\frac{b^x}{x}\right)^{\frac{1}{x-1}}} = e^{\ln b} = b$

$\boxed{L = b}$

(*) $\boxed{\lim_{x\to-\infty} F(x) = a}$

Hw: