Seminar 2 (Topology in $\mathbb{R}^n$) March 2, 2021

$\boxed{1}$ Let $A, B \subseteq \mathbb{R}^n$. Prove that

$$\operatorname{int}(A \setminus B) \subseteq (\operatorname{int} A) \setminus (\operatorname{int} B) \quad \text{and} \quad (\operatorname{cl} A) \setminus (\operatorname{cl} B) \subseteq \operatorname{cl}(A \setminus B).$$

Give examples of sets A and B for which the above inclusions are strict.

Solution. • Let $x \in \operatorname{int}(A \setminus B) \Rightarrow \exists r > 0$ s.t. $B(x, r) \subseteq A \setminus B = A \cap (\mathbb{R}^n \setminus B)$

$$\left.\begin{array}{l} \Rightarrow B(x,r) \subseteq A \Rightarrow x \in \operatorname{int} A \\ \quad \text{and} \\ B(x,r) \subseteq \mathbb{R}^n \setminus B \Rightarrow x \in \operatorname{ext} B \Rightarrow x \notin \operatorname{int} B \end{array}\right\} \Rightarrow x \in (\operatorname{int} A) \setminus (\operatorname{int} B)$$

$$(\operatorname{int} B) \cap (\operatorname{ext} B) = \emptyset$$

$$C \setminus D = \emptyset \Downarrow C \subseteq D$$

• Let $x \in (\operatorname{cl} A) \setminus (\operatorname{cl} B) \Rightarrow x \in \operatorname{cl} A \Rightarrow \forall V \in \mathcal{V}(x): V \cap A \neq \emptyset$

and

$x \notin \operatorname{cl} B \Rightarrow \exists V_0 \in \mathcal{V}(x): V_0 \cap B = \emptyset$

Suppose that $x \notin \operatorname{cl}(A \setminus B) \Rightarrow \exists V_1 \in \mathcal{V}(x): V_1 \cap (A \setminus B) = \emptyset \Rightarrow (V_1 \cap A) \setminus (V_1 \cap B) = \emptyset$

$$V_1 \cap (A \setminus B) = (V_1 \cap A) \setminus (V_1 \cap B)$$

$V_0 \cap |$

$$\Rightarrow V_1 \cap A \subseteq V_1 \cap B \Rightarrow \underbrace{V_0 \cap V_1 \cap A}_{} \subseteq V_0 \cap V_1 \cap B = V_1 \cap \underbrace{(V_0 \cap B)}_{\emptyset} = \emptyset$$

$$\Rightarrow \underbrace{(V_0 \cap V_1)}_{\in \mathcal{V}(x)} \cap A = \emptyset \quad \Rightarrow\Leftarrow \quad x \in \operatorname{cl} A.$$

- $A = \mathbb{R} \setminus \mathbb{Q}$

  $B = \mathbb{Q}$

  $\mathrm{cl}\, A = \mathrm{cl}\, B = \mathbb{R}$

  $(\mathrm{cl} A) \setminus (\mathrm{cl} B) = \mathbb{R} \setminus \mathbb{R} = \emptyset$

  $\mathrm{cl}(A \setminus B) = \mathrm{cl}\ A = \mathbb{R}$

$A = (\mathbb{R} \setminus \mathbb{Q})^n =$

$= \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j \in \mathbb{R} \setminus \mathbb{Q} \ \forall j = \overline{1,n} \}$

$B = \mathbb{Q}^n = \{ (x_1, \ldots, x_n) \mid x_j \in \mathbb{Q} \ \ \forall j = \overline{1,n} \}$

- $A = (-2, 2) = \mathrm{int}\, A$

  $B = (-1, 1) = \mathrm{int}\, B$

  $A \setminus B = (-2, -1] \cup [1, 2) = (\mathrm{int}\ A) \setminus (\mathrm{int} B)$

  $\mathrm{int}\,(A \setminus B) = (-2, -1) \cup (1, 2)$

$A = B(0_n, 2)$

$B = B\ (0_n, 1)$

$\boxed{2}$ Given two sets $A, B \subseteq \mathbb{R}^n$, prove that:

a) If $A \cup B = \mathbb{R}^n$, then $(\mathrm{cl} A) \cup (\mathrm{int} B) = \mathbb{R}^n$

b) If $A \cap B = \emptyset$, then $(\mathrm{cl}\ A) \cap (\mathrm{int} B) = \emptyset$.

<u>Solution</u> a) Obviously $(\mathrm{cl} A) \cup (\mathrm{int}\, B) \subseteq \mathbb{R}^n$. We claim that $\mathbb{R}^n \subseteq (\mathrm{cl} A) \cup (\mathrm{int} B)$ ✓

$A \cup B = \mathbb{R}^n$

$\downarrow$

Pick $x \in \mathbb{R}^n$. Suppose $x \notin (\mathrm{cl} A) \cup (\mathrm{int}\, B) \Rightarrow$ $x \notin \mathrm{cl} A$ and $x \notin \mathrm{int} B$

$x \notin \mathrm{cl} A \Rightarrow \exists V_0 \in \mathcal{V}(x)$ s.t. $V_0 \cap A = \emptyset \Rightarrow V_0 \subseteq \mathbb{R}^n \setminus A \subseteq B$

$\Rightarrow x \in \mathrm{int}\, B$

$\Rightarrow\!\Leftarrow$

b) Assume, by contradiction, that $\exists\, x \in (\mathrm{cl} A) \cap (\mathrm{int} B) \Rightarrow$

$\Rightarrow$ $x \in \mathrm{cl} A \Rightarrow \forall\, V \in \mathcal{V}(x) : V \cap A \neq \emptyset$

and

$x \in \mathrm{int} B \Rightarrow B \in \mathcal{V}(x)$

$\Rightarrow B \cap A \neq \emptyset$ $\Leftarrow\!\Rightarrow$ hypothesis $A \cap B = \emptyset$

$\boxed{3}$ Prove that $\forall\, A_1, A_2 \subseteq \mathbb{R}^n$ one has

$$\mathrm{cl}(A_1 \cup A_2) = (\mathrm{cl} A_1) \cup (\mathrm{cl} A_2).$$

Is it true that the equality

$$\mathrm{cl}\Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} (\mathrm{cl} A_i)$$

holds for every family $(A_i)_{i \in I}$ of subsets of $\mathbb{R}^n$?

Solution. $A \subseteq B$

$x \in \mathrm{cl} A \Rightarrow \forall\, V \in \mathcal{V}(x) : V \cap A \neq \emptyset \overset{A \subseteq B}{\Longrightarrow} V \cap B \neq \emptyset \Rightarrow x \in \mathrm{cl} B$

$\Rightarrow$ The closure operator has the following monotonicity property

$$A \subseteq B \Rightarrow \mathrm{cl} A \subseteq \mathrm{cl} B$$

$$A_1 \subseteq A_1 \cup A_2 \quad \Rightarrow \quad cl A_1 \subseteq cl(A_1 \cup A_2)$$
$$A_2 \subseteq A_1 \cup A_2 \quad \Rightarrow \quad cl A_2 \subseteq cl(A_1 \cup A_2)$$
$$\Rightarrow (cl A_1) \cup (cl A_2) \subseteq cl(A_1 \cup A_2)$$

We claim that $cl(A_1 \cup A_2) \subseteq (cl A_1) \cup (cl A_2)$ ✓✓

Pick $x \in cl(A_1 \cup A_2) \Rightarrow \forall V \in \mathcal{V}(x) : V \cap (A_1 \cup A_2) \neq \phi$

$\Rightarrow \forall V \in \mathcal{V}(x) : (V \cap A_1) \cup (V \cap A_2) \neq \phi$

$\Rightarrow \forall V \in \mathcal{V}(x) : V \cap A_1 \neq \phi$ or $V \cap A_2 \neq \phi$

? $\Rightarrow [\forall V \in \mathcal{V}(x) : V \cap A_1 \neq \phi]$ or $[\forall V \in \mathcal{V}(x) : V \cap A_2 \neq \phi]$

$\Rightarrow x \in cl A_1$ or $x \in cl A_2$

$\Rightarrow x \in (cl A_1) \cup (cl A_2)$

WRONG

$\underbrace{\forall n \in \mathbb{N} : n \text{ is even or } n \text{ is odd}}_{\text{TRUE}}$

$\underbrace{\underbrace{[\forall n \in \mathbb{N} : n \text{ is even}]}_{\text{FALSE}} \text{ or } \underbrace{[\forall n \in \mathbb{N} : n \text{ is odd}]}_{\text{FALSE}}}_{\text{FALSE}}$

Pick $x \in \mathrm{cl}(A_1 \cup A_2)$. $\Rightarrow \forall V \in \mathcal{V}(x): \quad V \cap (A_1 \cup A_2) \neq \emptyset \quad (*)$

Assume that $x \notin (\mathrm{cl} A_1) \cup (\mathrm{cl} A_2) \Rightarrow$

$\Rightarrow \quad x \notin \mathrm{cl} A_1 \Rightarrow \exists V_1 \in \mathcal{V}(x)$ s.t. $V_1 \cap A_1 = \emptyset$

and

$x \notin \mathrm{cl} A_2 \Rightarrow \exists V_2 \in \mathcal{V}(x)$ s.t. $V_2 \cap A_2 = \emptyset$

Define $\quad V_0 := V_1 \cap V_2 \Rightarrow V_0 \in \mathcal{V}(x)$

$$V_0 \cap (A_1 \cup A_2) = (V_0 \cap A_1) \cup (V_0 \cap A_2)$$

$$= (\underbrace{V_1 \cap V_2 \cap A_1}_{=\emptyset}) \cup (V_1 \cap \underbrace{V_2 \cap A_2}_{=\emptyset}) = \emptyset \quad \Rightarrow \Leftarrow \quad (*)$$

$\Rightarrow x \in (\mathrm{cl} A_1) \cup (\mathrm{cl} A_2)$

? We are looking for a seq. $(A_k)_{k \in \mathbb{N}}$ of subsets of $\mathbb{R}$ s.t.

$$\mathrm{cl}\left(\bigcup_{k=1}^{\infty} A_k\right) \neq \bigcup_{k=1}^{\infty} \mathrm{cl} A_k.$$

$$A_k = \left[-1+\frac{1}{2k}, 1-\frac{1}{2k}\right] = \mathrm{cl}\, A_k \qquad A_k = \overline{B}\left(0_n, 1-\frac{1}{2k}\right)$$

$$\bigcup_{k=1}^{\infty} A_k = (-1, 1) = \bigcup_{k=1}^{\infty} \mathrm{cl}\, A_k$$

$$\neq$$

$$\mathrm{cl}\left(\bigcup_{k=1}^{\infty} A_k\right) = [-1, 1]$$

$$\cancel{A_k = \left[-1-\frac{1}{k}, 1+\frac{1}{k}\right]}$$

$\boxed{4}$ Let (x_k) be a convergent sequence of points in $\mathbb{R}^n$, and let $x := \lim_{k\to\infty} x_k$.
Prove that the set $A := \{x\} \cup \{x_1, x_2, \dots, x_k, \dots\}$ is compact.

Solution. ~~X~~ A is compact $\Leftrightarrow$ every open covering of A has a finite subcovering

Let $(G_i)_{i\in I}$ be an arbitrary open covering of A.

$$x \in A \subseteq \bigcup_{i\in I} G_i \Rightarrow \exists\, i_0 \in I \text{ s.t. } \left.\begin{array}{l} x \in G_{i_0} \\ G_{i_0} \text{ is open} \end{array}\right| \Rightarrow \exists\, \varepsilon > 0 \text{ s.t. } B(x,\varepsilon) \subseteq G_{i_0}$$

Since $(x_k) \to x \Rightarrow \exists\, k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0 : \|x_k - x\| < \varepsilon$

$\Rightarrow \forall k \geq k_0 : x_k \in B(x,\varepsilon) \subseteq G_{i_0}$

$\forall k \in \{1, \dots, k_0 - 1\} \; : \; x_k \in A \subseteq \bigcup_{i \in I} G_i \Rightarrow \exists\, i_k \in I$ s.t. $x_k \in G_{i_k}$

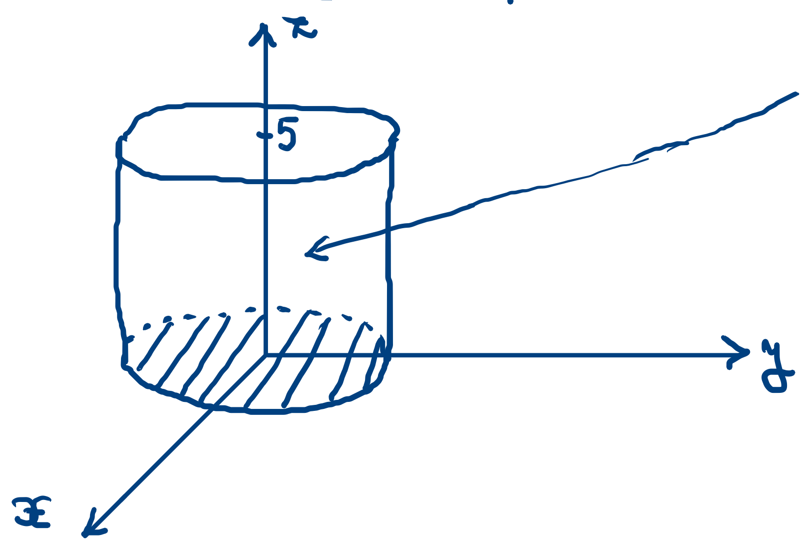
$$\Rightarrow A \subseteq \underbrace{G_{i_0} \cup G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_{k_0-1}}}_{\text{finite subcovering of } A}$$

$\boxed{5}$ Prove that if A is a compact set in $\mathbb{R}^n$, while B is a compact set in $\mathbb{R}^m$, then $A \times B$ is compact in $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

Example

$$\left.\begin{array}{l} A = \{(x,y) \mid x^2+y^2 \le 1\} \text{ compact in } \mathbb{R}^2 \\ B = [0,5] \text{ compact in } \mathbb{R} \end{array}\right\} \Rightarrow A \times B \text{ is compact in } \mathbb{R}^3$$

$A \times B = ?$



$A \times B = \{(x,y,z) \mid (x,y) \in A,\ z \in B\}$

Solution. ~~$\ast$~~ $A\times B$ is compact $\Leftrightarrow$ $A\times B$ is sequentially compact

$\Leftrightarrow$ every sequence in $A\times B$ has a subsequence converging to some point in $A\times B$

Let (x_k) be an arbitrary sequence in $A\times B$

$x_k \in A\times B \Rightarrow \exists a_k \in A \quad \exists b_k \in B$ s.t. $x_k = (a_k, b_k)$

$\Rightarrow$ $\left.\begin{array}{l}(a_k) \text{ is a sequence in } A \\ A \text{ is compact}\end{array}\right\} \Rightarrow \exists (a_{k_j})_{j\geq 1}$ subsequence of (a_k) and $\exists\, a \in A$

s.t. $\lim\limits_{j\to\infty} a_{k_j} = a$

~~(b_k) is a sequence in B, B is compact~~ $\Rightarrow \exists (b_{k_p})_{p\geq 1}$ subsequence of (b_k) and $\exists b\in B$

s.t. $\lim\limits_{p\to\infty} b_{k_p} = b$

$\left.\begin{array}{l}(b_{k_j}) \text{ is a sequence in } B \\ B \text{ is compact}\end{array}\right\} \Rightarrow \exists (b_{k_{j_p}})_{p\geq 1}$ subsequence of $(b_{k_j})_{j\geq 1}$ and

$\exists b \in B$ s.t. $\lim\limits_{p\to\infty} b_{k_{j_p}} = b$

$\Rightarrow (a_{k_{j_p}})_{p\geq 1} \xrightarrow{p\to\infty} a$

$$\lim_{p\to\infty} x_{k_{j_p}} = \lim_{p\to\infty} \left(a_{k_{j_p}}, b_{k_{j_p}} \right) = (a, b) \in A \times B$$