Sequences of real numbers

A sequence (x_n) of real numbers is an ordered list of numbers $x_n \in \mathbb{R}$, indexed by the natural numbers $n \in \mathbb{N}$.

terms of the sequence

Unlike sets, the terms in a sequence may be repeated.

A sequence (x_n) of real numbers is a function $f : \mathbb{N}_k \to \mathbb{R}$,

domain: $\mathbb{N}_k = \{ n \in \mathbb{N} \cup \{0\} \mid n \geq k \}$, when $k \in \mathbb{N} \cup \{0\}$ fixed

which assigns to each $n \in \mathbb{N}_k$, a unique value $f(n) \in \mathbb{R}$.

Notations:

$(x_n)_{n \geq k} = (x_n)_{n \in \mathbb{N}_k} = (x_n) \subseteq \mathbb{R} \rightarrow$ sequence of real numbers

$\forall n \in \mathbb{N}_k \quad f(n) := x_n \rightarrow$ general term of rank n of the sequence

$f(\mathbb{N}_k) = \{ x_k, x_{k+1}, \dots, x_n, \dots \} \rightarrow$ image of the sequence

$f(\mathbb{N}_k) \rightarrow$ a subset of $\mathbb{R}$

Remark:

→ sequences may be introduced → EXPLICITY

when $\forall n \in \mathbb{N}$, x_n has an expression that depends on n

e.g. $x_n = n, \ \forall n \in \mathbb{N}$

$a_n := \frac{n^2+1}{n^3}, \ \forall n \in \mathbb{N}$

→ IMPLICITLY

for recurrent sequences

e.g. $\begin{cases} a > 0, \quad 0 < x_0 < \frac{1}{a} \\ x_{n+1} = 2a x_n - x_n^2, \ \forall n \in \mathbb{N} \end{cases}$

Monotonicity

A sequence of real numbers $(x_n) \subseteq \mathbb{R}$ is said to be:

INCREASING (strict crescător) if $x_n < x_{n+1}, \forall n \in \mathbb{N}_k$	NONDECREASING (crescător) if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}_k$
DECREASING (strict descrescător) if $x_n > x_{n+1}, \forall n \in \mathbb{N}_k$	NONINCREASING (descrescător) if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}_k$
MONOTONIC if it is one of the above	

Boundedness

A sequence of real numbers $(x_n) \subseteq \mathbb{R}$ is said to be:

UPPER BOUNDED if $UB f(\mathbb{N}_k) \neq \emptyset$	LOWER BOUNDED if $LB f(\mathbb{N}_k) \neq \emptyset$
BOUNDED if both upper and lower bounded	

Convergence

A sequence of real numbers $(x_n) \subseteq \mathbb{R}$ is said to be:

CONVERGENT if $\exists \lim\limits_{n\to\infty} a_n \in \mathbb{R}$	DIVERGENT in all the other cases $\nexists \lim\limits_{n\to\infty} a_n$ or $\exists \lim\limits_{n\to\infty} \in \{-\infty, \infty\}$

Limit

An element $\ell \in \overline{\mathbb{R}}$ is said to be a limit of a sequence of real numbers $(x_n) \subseteq \mathbb{R}$

if $\forall V \in v(\ell)$, $\exists n_V \in \mathbb{N}_k$ s.t. $\forall n \geq n_V$, $x_n \in V$.

In practice, we work with balls in these 3 cases:

1°| $\ell \in \mathbb{R}$ $\quad V \leftrightarrow B(\ell, \varepsilon) \Longleftrightarrow (\ell-\varepsilon, \ell+\varepsilon)$

2°| $\ell = \infty$ $\quad V \leftrightarrow B(\infty, \varepsilon) \Longleftrightarrow (\varepsilon, \infty]$

3°| $\ell = -\infty$ $\quad V \leftrightarrow B(-\infty, \varepsilon) \Longleftrightarrow [-\infty, -\varepsilon)$

Theorem (limit uniques)

Each sequence $(a_n) \subseteq \mathbb{R}$ has at most one limit.

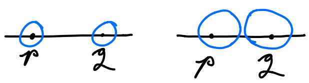
Proof:

Assume that $\exists (a_n) \subseteq \mathbb{R}$ with two distinct limits $p \neq q$.

$p = \lim\limits_{n\to\infty} a_n \Longleftrightarrow \exists V \in v(p)$, $\exists n_V \in \mathbb{N}$ s.t. $\forall n \geq n_V$, $a_n \in V$

$q = \lim\limits_{n\to\infty} a_n \Longleftrightarrow \exists W \in v(q)$, $\exists n_W \in \mathbb{N}$ s.t. $\forall n \geq n_W$, $a_n \in W$

$p \neq q$



Lemma: If two real numbers are different, they are different through neighbourhoods as well.

If $p \neq q$, $\exists U \in v(p)$, $\exists T \in v(q)$ s.t. $U \cap T = \emptyset$

Proof:

$r := |p-q| > 0$ $\qquad U = B(p, \frac{r}{2})$, $T = B(q, \frac{r}{2})$ $\quad U \cap T = \emptyset$

Lemma $p \neq q$ $\longrightarrow$ $\exists U \in v(p)$, $\exists T \in v(q)$ s.t. $U \cap T = \emptyset$

from $U \in v(p) \Rightarrow \exists n_V \in \mathbb{N}$ s.t. $\forall n \geq n_V$, $a_n \in U$

from $T \in v(q) \Rightarrow \exists n_T \in \mathbb{N}$ s.t. $\forall n \geq n_T$, $a_n \in T$

$\Rightarrow n' = \max\{n_V, n_T\} \in \mathbb{N} \Rightarrow$ $n' \geq n_V \Rightarrow a_{n'} \in U$, $n' \geq n_T \Rightarrow a_{n'} \in T$ $\Rightarrow a_{n'} \in U \cap T = \emptyset$ contrad. ↯ it exists

$\Rightarrow (a_n)$ has at most one limit.

Remark:

→ for a random sequence $(a_n) \subseteq \mathbb{R}$ just one of the following holds:
- it doesn't have a limit
- it has an unique limit

Characterization theorem with ε of the limit

Let $(a_n) \subseteq \mathbb{R}$ be a sequence of real numbers and let $l \in \overline{\mathbb{R}}$.

L1 $l \in \mathbb{R}$

$$l = \lim_{n\to\infty} a_n \iff \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon,\ |x_n - l| < \varepsilon$$

L2 $l = \infty$

$$l = \lim_{n\to\infty} a_n \iff \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon,\ x_n > \varepsilon$$

L3 $l = -\infty$

$$l = \lim_{n\to\infty} a_n \iff \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon,\ x_n < -\varepsilon$$

The Weierstrass theorem:

Let $(a_n) \subseteq \mathbb{R}$ be a sequence of real numbers.

→ if (a_n) INCREASING $\Rightarrow \exists \lim\limits_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$

→ if (a_n) DECREASING $\Rightarrow \exists \lim\limits_{n\to\infty} a_n := \inf\{a_n : n \in \mathbb{N}\}$

Remark:

monotonicity $\Rightarrow \exists$ of a limit (not necessarily finite)

$\not\Leftarrow$ $\exists$ sequences which do have a limit and are not monotonic $\quad a_n := \frac{(-1)^n}{n}$

monotonicity + boundedness $\Rightarrow$ $\exists$ convergence ($\exists$ a finite limit)

$\not\Leftarrow$

convergence $\Rightarrow$ boundedness

In practice, we use the squeeze criterion

S1: $\lim\limits_{n\to\infty} x_n = \ell \in \mathbb{R}$ If $\exists$ a sequence $(a_n) \subseteq \mathbb{R}$ s.t. $|x_n - \ell| \le a_n$

$\lim\limits_{n\to\infty} a_n = 0$

then $\lim\limits_{n\to\infty} x_n = \ell$

S2: $\lim\limits_{n\to\infty} x_n = \infty$ If $\exists$ a sequence $(a_n) \subseteq \mathbb{R}$ s.t. $x_n \ge a_n$

$\lim\limits_{n\to\infty} a_n = \infty$

then $\lim\limits_{n\to\infty} x_n = \infty$

S3: $\lim\limits_{n\to\infty} x_n = -\infty$ If $\exists$ a sequence $(a_n) \subseteq \mathbb{R}$ s.t. $x_n \le a_n$

$\lim\limits_{n\to\infty} a_n = -\infty$

then $\lim\limits_{n\to\infty} x_n = -\infty$

In highschool: $b_n \le x_n \le a_n$

ℓ

If $\exists \lim\limits_{n\to\infty} x_n = \ell \Rightarrow \ell$ is the limit of all subsequences of (x_n)

If $\exists$ at least two subsequences of (x_n) with a different limit $\Rightarrow \nexists \lim\limits_{n\to\infty} x_n$

Cauchy sequence

Let $(x_n) \subseteq \mathbb{R}$ be a sequence of real numbers.

It is a Cauchy sequence if $\Leftrightarrow (x_n)$ is convergent (in $\mathbb{R}$!!) (in other spaces $\text{conv} \underset{\not\Leftarrow}{\Rightarrow} \text{Cauchy}$)

$\Leftrightarrow \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $|x_{n+p} - x_n| < n_\varepsilon$
$\forall p \in \mathbb{N}$

e.g.

1) Study wheter the sequences are Cauchy sequences:

a) $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$

We try to prove that it is convergent $\Longleftrightarrow$ Cauchy sequence

$\forall \varepsilon > 0,\ \exists v_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq v_\varepsilon$, $|x_{n+p} - x_n| < \varepsilon$
$\forall p \in \mathbb{N}$

Choose $\varepsilon > 0$ random

jump to the end $\left\{ |x_{n+p} - x_n| < \varepsilon \right.$

Choose $n, p \in \mathbb{N}$ random

$|x_{n+p} - x_n| = \left| \cancel{1} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{3^2}} + \ldots + \cancel{\frac{1}{n^2}} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+p)^2} - \cancel{1} - \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}} - \ldots - \cancel{\frac{1}{n^2}} \right| =$

$= \left| \frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+p)^2} \right|$

$\left| \frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+p)^2} \right| < \varepsilon$

$\frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+p)^2} < \varepsilon$

$\frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+p)^2} \leq \frac{1}{n^2}$

$\frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+p)^2} \leq \frac{p}{n^2} < \varepsilon$

$\downarrow$

it depends on p

try to get rid of p — use a diff. aproach

$\frac{1}{(n+1)^2} \leq \frac{1}{(n+1)^2}, \ \ldots, \ \frac{1}{(n+p)^2} \leq \frac{1}{(n+p)(n+p+1)}$

$\Rightarrow |x_{n-p} - x_n| \leq \frac{1}{n(n+1)} + \ldots + \frac{1}{(n+p)(n+p+1)}$

$|x_{n-p} - x_n| \geq \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} + \ldots + \cancel{\frac{1}{n+p}} - \frac{1}{n+p-1}$

$\geq \frac{1}{n} - \frac{1}{n+p-1} \leq \frac{1}{n}$

$\Rightarrow |x_{n-p} - x_n| \leq \frac{1}{n}$

$x_n \to 0 \Rightarrow$ for $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $n \geq n_v$, $|x_v - 0| < \varepsilon$

So for, $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall v \geq v_\varepsilon$

The Cesaro – Stolz theorem:

Let $(a_n), (b_n) \subseteq \mathbb{R}$ two sequences of real numbers
s.t. $\to b_n \neq 0$, $\forall n \in \mathbb{N}$
$\to (b_n)$ strictly monotonic
unbounded $(\to \pm\infty)$
$\to \exists \lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in \overline{\mathbb{R}}$

then $\exists \lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$

$(x_n) \subseteq \mathbb{R}_+$ $\exists \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$

$\exists \lim\limits_{n\to\infty} \sqrt[n]{x_n} = \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$

$\lim\limits_{n\to\infty} \sqrt[v]{v} \underset{C.S.}{\overset{C_4}{\Longrightarrow}} \lim\limits_{n\to\infty} \frac{v+1}{v} = 1$

$\lim\limits_{n\to\infty} \sqrt[v]{v!} \underset{C.S.}{\overset{C_4}{\Longrightarrow}} \lim\limits_{n\to\infty} \frac{(v+1)!}{v!} =$

$= \lim\limits_{v\to\infty} (v+1) = \infty$

C_1:

Consider x_n s.t. $\exists \lim\limits_{n\to\infty} x_n$.

Then there $\exists \lim\limits_{n\to\infty} \frac{x_1+x_2+\ldots+x_n}{n} = \lim\limits_{n\to\infty} x_n$

Proof:

$a_n = x_1 + x_2 + \ldots + x_n$
$b_n = n$

a) $b_n \neq 0 \ \forall n \in \mathbb{N}$

b) $b_{n+1} - b_n = n+1-n = 1 > 0 \ \to$ increasing

c) $\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim\limits_{n\to\infty} \frac{x_{n+1}}{n+1-n} = \lim\limits_{n\to\infty} x_{n+1}$

$a, b, c \overset{C.S.}{\Longrightarrow} \exists \lim\limits_{n\to\infty} \frac{x_1+\ldots+x_n}{n} = \lim\limits_{n\to\infty} x_n$

C_2:

Consider $(x_n) \subseteq \mathbb{R}_+^*$ s.t. $\exists \lim\limits_{n\to\infty} x_n$

$\exists \lim\limits_{n\to\infty} \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}} = \lim\limits_{n\to\infty} x_n$

Proof: C_1, $y_n = \frac{1}{x_n}$

C_3:

Consider $(x_n) \subseteq \mathbb{R}_+^*$ s.t. $\exists \lim\limits_{n\to\infty} x_n$

$\Rightarrow \exists \lim\limits_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = \lim\limits_{n\to\infty} x_n$

Proof: The standard inequality

Sequences of Real Numbers part 1

Exercise 1: Study the monotonicity, boundedness and convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, having the general term:

$$a)\quad x_n=\frac{3^n+5^n}{7^n},\quad b)\quad x_n=\frac{(-1)^n}{n},\quad c)\quad x_n=\frac{4^n}{n!},\quad d)\quad x_n=\frac{n}{n^2+1}.$$

Exercise2: Using the characterising theorem with ε prove that

$$a)\lim_{n\to\infty}\frac{2n}{n^2+1}=0\qquad b)\lim_{n\to\infty}\frac{2n^2}{-2n+4}=-\infty.$$

Exercise 3: Compute the limit of the sequences of real numbers having the following general terms:

$$a)\quad \frac{3^n+1}{5^n+1},\qquad b)\quad \frac{9^n+(-3)^n}{9^{n-1}+3},\qquad c)\quad \left(\sin\frac{\pi}{10}\right)^n,\qquad d)\sqrt{4n^2+2n+1}-2n,$$

$$e)\quad \left(7+\frac{1-2n^3}{3n^4+2}\right)^2,\qquad f)\quad \sqrt[3]{n^3+n+3}-\sqrt[3]{n^3+1},\qquad g)\quad \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}},$$

$$h)\quad \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n}\right).$$

Exercise 4: Let $t\in\mathbb{R}$.

a) Prove that there exists an decreasing sequence of rational numbers converging to t.

b) Prove that there exists a increasing sequence of irrational numbers converging to t.

Exercise 5: Let $a > 0$ and let $x_0 \in \mathbb{R}$ be such that $0 < x_0 < \frac{1}{a}$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, defined recursively by:

$$x_{n+1} = 2x_n - ax_n^2, \forall n \in \mathbb{N}.$$

Study the convergence of the sequence by following the next steps:

a) Prove by induction that $x_n < \frac{1}{a}, \forall n \in \mathbb{N}$.

b) Prove by induction that $0 < x_n, \forall n \in \mathbb{N}$.

c) By using a) and b) prove that $(x_n)_{n\in\mathbb{N}}$ is increasing.

d) Compute the limit of the sequence.

Sequences of real numbers   2nd part

Exercise 1: Study the nature (covergence or divergence) of the following sequence of real numbers.

$$x_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\dots\left(1 - \frac{1}{n^2}\right),\ n \geq 2.$$

In case it is convergent, compute its limit.

Exercise 2: Determine the limits of the following sequences of real numbers, having as general term:

$$a)\ \frac{3^n}{4^n}, \qquad b)\ \frac{2^n + (-2)^n}{3^n}, \qquad c)\ \frac{5 - n^3}{n^2 + 1}, \qquad d)\ \left(2 + \frac{4^n + (-5)^n}{7^n + 1}\right)^{2n^3 - n^2},$$

$$e)\ \frac{1 + 2 + \ldots + n}{n^2}, \qquad f)\ \left(\frac{n^3 + 4n + 1}{2n^3 + 5}\right)^{\frac{-2n^4+1}{n^4+3n+1}}, \qquad g)\ (\cos(-2013))^n,$$

$$h)\ \left(\frac{n^5 + 3n + 1}{2n^5 - n^4 + 3}\right)^{\frac{3n - n^4}{n^3+1}}.$$

Exercise 3: Determine the limits of the following sequences of real numbers, having as general term:

$$a)\ \left(1 + \frac{1}{-n^3 + 3n}\right)^{n^2 - n^3}, \quad b)\ (3n^2 + 5)ln\left(1 + \frac{1}{n^2}\right),$$

$$c)\frac{n^n}{1^1 + 2^2 + ... + n^n}$$

$$d)\frac{x_1 + 2x_2 + ... + nx_n}{n^2},$$

when $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, with the limit $x \in \mathbb{R}$.

Exercise 4: Determine the limits of the following sequences of real numbers, having as general term:

$$a)x_n = \frac{a^n - a^{-n}}{a^n + a^{-n}}, \quad a \neq 0$$

$$b)y_n = \frac{a^n + b^n}{a^{n+1} + b^{n+1}}, \quad a \neq -b$$

$$c)z_n = \frac{1 + a + ... + a^n}{1 + b + ... + b^n}, \quad a, b > 0.$$

Exercises:

1) With the ε-theorem prove that:

a) $\lim_{n\to\infty} n = \infty$

$\lim_{n\to\infty} n = \infty \iff \forall \varepsilon>0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon,\ n>\varepsilon$. $(*)$

Choose $\varepsilon>0$ random

jump to the end $\Big|$ $n>\varepsilon$ $\qquad \exists n' = [\varepsilon]+1 \in \mathbb{N}$

or

$\xRightarrow[TH.]{ARCHIMEDES} \exists n'' \in \mathbb{N}$ s.t. $n''>\varepsilon$

$\left.\begin{array}{ll}\forall n \geq \begin{matrix} n' \\ n''\end{matrix} & n>\varepsilon \\ & \varepsilon\text{-random}\end{array}\right\} \Rightarrow (*)$ holds $\Rightarrow \lim_{n\to\infty} n = \infty$

b) $\lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} \frac{1}{n} = 0 \in \mathbb{R} \iff \forall \varepsilon>0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon,\ \left|\frac{1}{n}-0\right|<\varepsilon$. $(*)$

Choose $\varepsilon>0$ random

jump to the end $\Big|$ $\left|\frac{1}{n}\right|<\varepsilon \iff \frac{1}{n}<\varepsilon \xRightarrow[TH.]{ARCHIMEDES} \exists n_\varepsilon \in \mathbb{N}$ s.t. $0<\frac{1}{n_\varepsilon}<\varepsilon$

For $n \geq n_\varepsilon \Rightarrow \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon$

$\left.\begin{array}{ll}\forall n \geq n_\varepsilon & \left|\frac{1}{n}-0\right|<\varepsilon \\ & \varepsilon\text{-random}\end{array}\right\} \Rightarrow (*)$ holds $\Rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$

c) $\lim_{n\to\infty} a^n = \begin{cases} \infty, & a>1 \\ 1, & a=1 \\ 0, & |a|<1 \\ \nexists, & a\leq -1 \end{cases}$

Consider $a \in \mathbb{R}$.

Case 1 | $a>1$

$\lim_{n\to\infty} a^n = \infty \iff \forall \varepsilon>0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon,\ a^n>\varepsilon$ $(*)$

Choose $\varepsilon>0$ random.

jump to the end $\Big|$ $a^n>\varepsilon$ $\qquad a^n>\varepsilon \ |\cdot \log_a,\ a>1$

$n>\log_a \varepsilon \xRightarrow[TH.]{ARCHIMEDES} \exists n_\varepsilon \in \mathbb{N}$ s.t. $n_\varepsilon > \log_a \varepsilon$

if $\left.\begin{array}{l} n\geq n_\varepsilon \\ n_\varepsilon > \log_a \varepsilon\end{array}\right\} \Rightarrow n>\log_a \varepsilon$

$a^n > a^{\log_a \varepsilon}$

$a^n > \varepsilon$

$\left.\begin{array}{l}\text{for } \varepsilon>0,\ \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon,\ a^n>\varepsilon \\ \varepsilon\text{-random}\end{array}\right\} \Rightarrow (*)$ holds $\iff \lim_{n\to\infty} a^n = \infty$ for $a>1$

Case 2 | $a=1$

$\lim_{n\to\infty} a = 1 \in \mathbb{R} \Rightarrow$

$\forall n \in \mathbb{N},\ a^n = 1 \Rightarrow (a^n)_{n\in\mathbb{N}}$ is, in this case, the constant sequence 1, with $\lim\limits_{n\to\infty} 1 = 1$.

Exercise 1: Study the monotonicity, boundedness and convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, having the general term:

$$a)\quad x_n = \frac{3^n+5^n}{7^n},\qquad b)\quad x_n = \frac{(-1)^n}{n},\qquad c)\quad x_n = \frac{4^n}{n!},\qquad d)\quad x_n = \frac{n}{n^2+1}.$$

$a)\quad x_n = \frac{3^n+5^n}{7^n},$

MONOTONICITY:

$$x_n = \frac{3^n+5^n}{7^n}$$

$$x_{n+1} = \frac{3^{n+1}+5^{n+1}}{7^{n+1}} = \frac{3\cdot 3^n + 5\cdot 5^n}{7\cdot 7^n} = \frac{3\cdot 3^n}{7\cdot 7^n} + \frac{5\cdot 5^n}{7\cdot 7^n} = \frac{3}{7}\cdot\frac{3^n}{7^n} + \frac{5}{7}\cdot\frac{5^n}{7^n} = \frac{3\cdot 5}{3\cdot 5\cdot 7}\left(\frac{3^n+5^n}{7^n}\right) = \frac{15}{115}\left(\frac{3^n+5^n}{7^n}\right) = \frac{\cancel{15}}{\cancel{115}}\cdot x_n = \frac{3}{23}\cdot x_n < x_n$$

$x_{n+1} < x_n \ \forall n\in\mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}}$ is decreasing $\Rightarrow (x_n)$ is monotonic

BOUNDEDNESS:

$3^n > 0 \quad \forall n\in\mathbb{N}$

$5^n > 0 \quad \forall n\in\mathbb{N}$

$3^n+5^n > 0,\ \forall n\in\mathbb{N}$

$7^n > 0,\ \forall n\in\mathbb{N}$

$x_n = \frac{3^n+5^n}{7^n} > 0,\ \forall n\in\mathbb{N} \quad\Rightarrow\quad x_n > 0\ \forall n\in\mathbb{N}\ (1)$

$\frac{3^n+5^n}{7^n} = \left(\frac{3}{7}\right)^n + \left(\frac{5}{7}\right)^n,\ \forall n\in\mathbb{N}$

$\frac{3}{7} < 1,\ \forall n\in\mathbb{N}$

$\frac{5}{7} < 1,\ \forall n\in\mathbb{N}$

--- $()^n$

$\left(\frac{3}{7}\right)^n < 1,\ \forall n\in\mathbb{N}$

$\left(\frac{5}{7}\right)^n < 1,\ \forall n\in\mathbb{N}$

--- $(+)$

$x_n = \left(\frac{3}{7}\right)^n + \left(\frac{5}{7}\right)^n < 2,\ \forall n\in\mathbb{N} \quad\Rightarrow\quad x_n < 2,\ \forall n\in\mathbb{N}\ (2)$

From (1) and (2) $\Rightarrow 0 < x_n < 2,\ \forall n\in\mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}}$ upper and lower bounded $\Rightarrow (x_n)_{n\in\mathbb{N}}$ bounded

CONVERGENCE:

$(x_n)_{n\in\mathbb{N}}$ monotonic and bounded $\Rightarrow (x_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow \exists \lim\limits_{n\to\infty} x_n \in\mathbb{R}$

LIMIT:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{3}{7}\right)^n + \left(\frac{5}{7}\right)^n = 0+0 = 0$$

$b)\quad x_n = \frac{(-1)^n}{n},\qquad -1,\ \frac{1}{2},\ \frac{-1}{3},\ \frac{1}{4},\ \frac{-1}{5},\ \frac{1}{6}$

MONOTONICITY:

$$x_n = \frac{(-1)^n}{n}$$

$$x_n = \begin{cases} \frac{1}{n}, & n = 2k \\ \frac{-1}{n}, & n = 2k+1 \end{cases},\ k\in\mathbb{N}$$

$$\left.\begin{array}{l} x_{2k} = \frac{1}{2k} > 0\ \forall k\in\mathbb{N} \\ x_{2k+1} = \frac{-1}{2k+1} < 0\ \forall k\in\mathbb{N} \end{array}\right\} \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ not monotonic}$$

BOUNDEDNESS:

$$x_n = \frac{(-1)^n}{n} \qquad \left.\begin{array}{l} x_{2k} = \frac{1}{2k} > 0 \ \forall k \in \mathbb{N} \\ x_{2k+1} = \frac{-1}{2k+1} < 0 \ \forall k \in \mathbb{N} \end{array}\right\} \Rightarrow -1 \leq x_n < 1, \ \forall n \in \mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ bounded}$$

$$x_1 = \frac{(-1)}{1} = -1$$

$$x_2 = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$x_3 = \frac{-1}{3}$$

$$x_4 = \frac{1}{4}$$

$\vdots$

it approaches 0

CONVERGENCE:

$$\left.\begin{array}{l} \lim\limits_{k\to\infty} x_{2k} = \lim\limits_{k\to\infty} \frac{1}{2k} = 0 \\ \lim\limits_{k\to\infty} x_{2k+1} = \lim\limits_{k\to\infty} \frac{1}{2k+1} = 0 \end{array}\right\} \Rightarrow \exists \lim_{n\to\infty} x_n = 0 \in \mathbb{R} \Rightarrow (x_n)_n \text{ convergent}$$

LIMIT:

$$\lim_{n\to\infty} x_n = 0$$

c) $x_n = \dfrac{4^n}{n!}$,

MONOTONICITY:

$$x_n = \frac{4^n}{n!}$$

$$x_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4\cdot 4^n}{(n+1)\cdot n!} = \frac{4}{n+1}\cdot\frac{4^n}{n!} = \frac{4}{n+1}\cdot x_n$$

$$\left.\begin{array}{l} \text{for } n \leq 3 \Rightarrow x_{n+1} \geq x_n \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ nondecreasing} \\ \text{for } n > 3 \Rightarrow x_{n+1} < x_n \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ decreasing} \end{array}\right\} \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ nonmonotonic}$$

BOUNDEDNESS:

$$x_n = \frac{4^n}{n!}$$

$4^n > 0, \ \forall n \in \mathbb{N}, \ n \leq 3$

$n! > 0, \ \forall n \in \mathbb{N}, \ n \leq 3$

$$\left.\begin{array}{l} \frac{4^n}{n!} > 0, \ \forall n \in \mathbb{N}, \ n \leq 3 \\ \frac{4^n}{n!} < \frac{4^3}{3!} = \frac{64}{6} = \frac{32}{3}, \ \forall n \in \mathbb{N}, \ n > 3 \end{array}\right\} \Rightarrow 0 < x_n < \frac{32}{3} \ \forall n \in \mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ lower and upper bounded}$$

$$\Rightarrow (x_n)_{n\in\mathbb{N}} \text{ bounded}$$

CONVERGENCE:

$$\lim_{n\to\infty} \frac{4^n}{n!} = 0 \Rightarrow \exists \lim_{n\to\infty} x_n \in \mathbb{R} \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ convergent}$$

LIMIT:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{4^n}{n!} = 0$$

d) $x_n = \dfrac{n}{n^2+1}$.

MONOTONICITY:

$$x_n = \frac{n}{n^2+1}$$

$$x_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$$

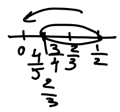
$$\frac{x_n}{x_{n+1}} = \frac{n}{n^2+1} \cdot \frac{(n+1)^2+1}{n+1} = \frac{n}{n^2+1}\,\frac{(n+1)^2}{n+1} + \frac{n}{n^2+1}\cdot\frac{1}{n+1} =$$

$$= \frac{n(n+1)^2+n}{(n^2+1)(n+1)} =$$

$$= \frac{n((n+1)^2+1)}{(n^2+1)(n+1)} = \frac{n(n^2+2n+2)}{(n^3+n^2+n+1)} = \frac{n^3+2n^2+2n}{n^3+n^2+n+1} > 1$$

$\frac{x_n}{x_{n+1}} > 1 \quad \forall n \in \mathbb{N} \Rightarrow x_n > x_{n+1} \ \forall n \in \mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}}$ decreasing $\forall n \in \mathbb{N}$

BOUNDEDNESS:



$x_n = \frac{n}{n+1}$ $\qquad x_n = \frac{n}{n+1} > 0 \quad \forall n \in \mathbb{N}$

$x_1 = \frac{1}{2}$

$x_2 = \frac{2}{3}$ $\qquad x_n = \frac{n}{n+1} < \frac{1}{2} \ \forall n \in \mathbb{N}$

$x_3 = \frac{3}{4}$

$x_4 = \frac{4}{5}$

$\vdots$ $\qquad \Rightarrow \quad 0 < x_n < \frac{1}{2}, \ \forall n \in \mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}}$ lower and upper bounded $\Rightarrow (x_n)_{n\in\mathbb{N}}$ bounded

CONVERGENCE:

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = \lim\limits_{n\to\infty} \frac{n}{n(1+\underbrace{\frac{1}{n}}_{0})} = 1 \in \mathbb{R} \Rightarrow \exists \lim\limits_{n\to\infty} x_n \in \mathbb{R} \Rightarrow (x_n)_{n\in\mathbb{N}}$ convergent

(x_n) monotonic and bounded $\Rightarrow (x_n)_{n\in\mathbb{N}}$ convergent

LIMIT:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n(1+\underbrace{\frac{1}{n}}_{0})} = 1$$

Exercise2: Using the characterising theorem with ε prove that

$$a)\ \lim_{n\to\infty} \frac{2n}{n^2+1} = 0 \qquad b)\ \lim_{n\to\infty} \frac{2n^2}{-2n+4} = -\infty.$$

$$a)\ \lim_{n\to\infty} \frac{2n}{n^2+1} = 0$$

$x_n = \dfrac{2n}{n^2+1}$, $(x_n)_{n\in\mathbb{N}}$

$\lim\limits_{n\to\infty} x_n = 0 \overset{\varepsilon}{\underset{TH.}{\Longleftrightarrow}} \forall \varepsilon>0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $|x_n - 0| < \varepsilon$ $(*)$

$\| \quad |x_n| < \varepsilon$

$\left|\dfrac{2n}{n+1}\right| < \varepsilon$

Choose $\varepsilon > 0$ random

jump to the end

$\left|\dfrac{2n}{n^2+1}\right| < \varepsilon \iff \dfrac{2n}{n^2+1} < \varepsilon \iff 2n < (n^2+1)\varepsilon$

$0 < n^2\varepsilon + \varepsilon - 2n$

$n^2\varepsilon - 2n + \varepsilon > 0$

$\Delta = 4 - 4\varepsilon^2 = 4(1-\varepsilon^2)$

$n_{1,2} = \dfrac{2 \pm \sqrt{4(1-\varepsilon^2)}}{2\varepsilon} = \dfrac{2\pm 2\sqrt{1-\varepsilon^2}}{2\varepsilon} \overset{ARCHIMEDES}{\underset{TH.}{\Longrightarrow}} \exists n_\varepsilon \in \mathbb{N}$ s.t. $n_\varepsilon > \dfrac{2+2\sqrt{1-\varepsilon^2}}{2\varepsilon}$ $\Rightarrow$

$n \geq n_\varepsilon \Rightarrow \dfrac{n}{n^2+1} \leq \dfrac{n_\varepsilon}{n^2+1}$

$\Rightarrow \dfrac{n}{n^2+1} < \varepsilon \iff \left|\dfrac{n}{n^2+1}\right| < \varepsilon \iff |x_n - 0| < \varepsilon$, ε random $\Big\} \Rightarrow (*)$ holds

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{2n}{n^2+1}$

$$b)\ \lim_{n\to\infty} \frac{2n^2}{-2n+4} = -\infty.$$

$x_n = \dfrac{2n^2}{-2n+4} = -\infty$, $(x_n)_{n\in\mathbb{N}}$

$\lim\limits_{n\to\infty} x_n = -\infty \overset{\varepsilon}{\underset{TH.}{\Longleftrightarrow}} \forall \varepsilon>0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $x_n < -\varepsilon$ $(*)$

Choose $\varepsilon > 0$ random

jump to the end

$x_n < -\varepsilon$

$x_n < -\varepsilon$

$\dfrac{2n}{4-2n} < -\varepsilon$

$\dfrac{2n}{4-2n} + \varepsilon < 0$

Exercise 3: Determine the limits of the following sequences of real numbers, having as general term:

$$a)\quad \left(1+\frac{1}{-n^3+3n}\right)^{n^2-n^3}, \quad b)\quad (3n^2+5)ln\left(1+\frac{1}{n^2}\right),$$

$$c)\frac{n^n}{1^1+2^2+...+n^n}$$

$$d)\frac{x_1+2x_2+...+nx_n}{n^2},$$

when $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, with the limit $x \in \mathbb{R}$.

Exercise 4: Let $t \in \mathbb{R}$.

a) Prove that there exists an decreasing sequence of rational numbers converging to t.

b) Prove that there exists a increasing sequence of irrational numbers converging to t.

Exercise 5: Let $a > 0$ and let $x_0 \in \mathbb{R}$ be such that $0 < x_0 < \frac{1}{a}$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, defined recursively by:

$$x_{n+1} = 2x_n - ax_n^2, \forall n \in \mathbb{N}.$$

Study the convergence of the sequence by following the next steps:

a) Prove by induction that $x_n < \frac{1}{a}, \forall n \in \mathbb{N}$.

b) Prove by induction that $0 < x_n, \forall n \in \mathbb{N}$.

c) By using a) and b) prove that $(x_n)_{n\in\mathbb{N}}$ is increasing.

d) Compute the limit of the sequence.

Exercise 2: Determine the limits of the following sequences of real numbers, having as general term:

$$a)\ \frac{3^n}{4^n}, \quad b)\ \frac{2^n+(-2)^n}{3^n}, \quad c)\ \frac{5-n^3}{n^2+1}, \quad d)\ \left(2+\frac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2},$$

$$e)\ \frac{1+2+\ldots+n}{n^2}, \quad f)\ \left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}}, \quad g)\ (\cos(-2013))^n,$$

$$h)\ \left(\frac{n^5+3n+1}{2n^5-n^4+3}\right)^{\frac{3n-n^4}{n^3+1}}.$$

$a)$ $\frac{3^n}{4^n}$,

$$\lim_{n\to\infty}\frac{3^n}{4^n}=\lim_{n\to\infty}\underbrace{\left(\frac{3}{4}\right)^n}_{0<\frac{3}{4}<1}=0$$

$b)$ $\frac{2^n+(-2)^n}{3^n}$,

$$\lim_{n\to\infty}\frac{2^n+(-2)^n}{3^n}=\lim_{n\to\infty}\frac{(2-2)^n}{3^n}=0$$

$c)$ $\frac{5-n^3}{n^2+1}$,

$$\lim_{n\to\infty}\frac{5-n^3}{n^2+1}=-\lim_{n\to\infty}\frac{n^3-5}{-n^2-1}=-\infty$$

$d)$ $\left(2+\frac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2}$,

$$\lim_{n\to\infty}\left(2+\frac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2}=\lim_{n\to\infty}\left(2+\frac{(4-5)^n}{7^n+1}\right)^{2n^3-n^2}=\lim_{n\to\infty}\left(2+\frac{(-1)^n}{7^n+1}\right)^{2n^2-n^2}$$