1. Let $\phi : \mathbf{X} \to \mathbf{Y}$ be an affine morphism. Show that

1. ϕ is injective if and only if $\mathrm{lin}(\phi)$ is injective.
2. ϕ is surjective if and only if $\mathrm{lin}(\phi)$ is surjective.
3. ϕ is bijective if and only if $\mathrm{lin}(\phi)$ is bijective.

2. Consider $\mathbf{v}(2,1,1) \in \mathbb{R}^3$ and $Q(2,2,2) \in \mathbf{A}^3(\mathbb{R})$.

1. Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.
2. Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

3. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbf{A}^3(\mathbf{K})$.

4. In $\mathbf{A}^2(\mathbf{K})$, for the lines/hyperplanes

$$\pi : ax + by + c = 0, \quad \ell : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

5. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

1. $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and
2. $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

6. Give Cartesian equations for the line passing through the point $M(1,0,7)$, parallel to the plane $\pi : 3x - y + 2z - 15 = 0$ and intersecting the line

$$\ell : \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

7. Give Cartesian equations for the projection of the line

$$\ell : 2x - y + z - 1 = 0 \cap x + y - z + 1 = 0$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction of $\overrightarrow{u}(1,1,-2)$. Write down Cartesian equations of the line obtained by reflecting ℓ in the plane π parallel to the direction of $\overrightarrow{u}$.

8. Consider the Euclidean space $\mathbb{E}^3$. Show that the orthogonal reflection $\mathrm{Ref}_{\pi}^{\perp}(x)$ in the plane $\pi : \langle n, x \rangle = p$ is given by

$$\mathrm{Ref}_{\pi}(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2} n$.

9. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4 \quad \text{respectively.}$$

10. Let $\mathbf{X}$ be an affine space. Show that the set T of all translations is a subgroup of $\mathrm{AGL}(\mathbf{X})$. Show that T is a normal subgroup of $\mathrm{AGL}(\mathbf{X})$.

11. Show that $\mathrm{AGL}(\mathbb{R}^n)$ is a subgroup of $\mathrm{GL}(\mathbb{R}^{n+1})$.

1. Let $\phi : \mathbf{X} \to \mathbf{Y}$ be an affine morphism. Show that

1. ϕ is injective if and only if $\mathrm{lin}(\phi)$ is injective.
2. ϕ is surjective if and only if $\mathrm{lin}(\phi)$ is surjective.
3. ϕ is bijective if and only if $\mathrm{lin}(\phi)$ is bijective.

- Recall that ϕ is a map $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b.$ with respect to some coordinate systems of X and Y

where $A = [\mathrm{lin}(\phi)]_{\mathcal{Y},\mathcal{X}}$ and $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

- We can view ϕ as the composition of two maps

$$\phi_1 : \mathbb{R}^n \to \mathbb{R}^m \qquad \phi_1 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\phi_2 : \mathbb{R}^m \to \mathbb{R}^m \qquad \phi_2 \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} + b$$

(this is a translation with the vector b)

So $\phi = \phi_2 \circ \phi_1$

- Since ϕ_2 is bijective

ϕ is injective $\Longleftrightarrow$ ϕ_1 is injective

ϕ is surjective $\Longleftrightarrow$ ϕ_1 is injective

ϕ is bijective $\Longleftrightarrow$ ϕ_1 is bijective

- but $\phi_1 \begin{bmatrix} x \\ \vdots \\ x_n \end{bmatrix} = [\mathrm{lin}\,\phi]_{\mathcal{Y},\mathcal{X}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the map $\mathrm{lin}\,\phi$ with respect to $\mathcal{X}$ and $\mathcal{Y}$ and ϕ_1 is inj., surj. or bij if $\mathrm{lin}\,\phi$ is so.

$\square$

2. Consider $\mathbf{v}(2,1,1) \in \mathbb{R}^3$ and $Q(2,2,2) \in \mathbf{A}^3(\mathbb{R})$.

1. Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.
2. Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

- Let $\ell = Q + \langle v \rangle$

Consider a point $P(x_0, y_0, z_0) \in A^3(\mathbb{R})$

the line containing P and parallel to ℓ is $\ell_P : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 + 2t \\ y_0 + t \\ z_0 + t \end{bmatrix}$

$\ell_P \cap \pi : \quad z_0 + t = 0 \iff t = -z_0$ so $\ell \cap \pi = \begin{bmatrix} x_0 - 2z_0 \\ y_0 - z_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

So, $\quad Pr_{\pi, \ell} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- For the reflection let $P'(x_1, y_1, z_1)$ be the reflection of P in π along ℓ

Then $\quad \frac{1}{2}\left(\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) = \begin{bmatrix} x_0 - 2z_0 \\ y_0 - z_0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 - 4z_0 \\ y_0 - 2z_0 \\ -z_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

Rem clearly, one can also apply the general formulas that we deduced

↳ do this and check that you get the same answer.

3. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbf{A}^3(\mathbf{K})$.

- We show this for projections in hyperplanes (the other cases are similar)
- the vector forms don't change:

$$\mathrm{Pr}_{H,\mathbf{v}}(P) = P - \frac{\varphi(P)}{\operatorname{lin}\varphi(\mathbf{v})}\mathbf{v}.$$

but H is in this case a plane so it has an equation of the form

$H: ax+by+cz+d=0$

and $v = v(v_x, v_y, v_z)$

- For the matrix form we have

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

which in our case becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{av_x+bv_y+cv_z}\begin{pmatrix} v_x a & v_x b & v_x c \\ v_y a & v_y b & v_y c \\ v_z a & v_z b & v_z c \end{pmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x+bv_y+cv_z}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$= \frac{1}{av_x+bv_y+cv_z}\begin{bmatrix} bv_y+cv_z & v_x b & v_x c \\ v_y a & av_x+cv_z & v_y c \\ v_z a & v_z b & av_x+bv_y \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x+bv_y+cv_z}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

4. In $\mathbf{A}^2(\mathbf{K})$, for the lines/hyperplanes

$$\pi : ax + by + c = 0, \quad \ell : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

As in the previous execise we consider $Pr_{\pi,\ell}$ since $Ref_{\pi,\ell}$ is similar

$$Pr_{\pi,\ell}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{av_1+bv_2}\begin{bmatrix} bv_2 & cv_2 \\ bv_1 & cv_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \frac{d}{av_1+bv_1}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

5. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

1. $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and
2. $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

Rem from the definition of these maps it should be clear that the indicated relations are true

1. $$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

So $$\left[\Pr_{H,v}\circ \Pr_{H,v}(P)\right]_K = \left[\Pr_{H,v}\left(\Pr_{H,v}([P]_K)\right)\right]$$

$$= \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\Pr_{H,v}([P]_K) - \frac{a_{n+1}}{v^t\cdot a}v$$

$$= \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\left[\left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)[P]_K - \frac{a_{n+1}}{v^t\cdot a}v\right] - \frac{a_{n+1}}{v^t\cdot a}v$$

$$(*) = \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)^2[P]_K - \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\frac{a_{n+1}}{v^t\cdot a}v - \frac{a_{n+1}}{v^t\cdot a}v$$

$$- \frac{a_{n+1}}{(v^t\cdot a)^2}\left(2v^t\cdot a\cdot \mathrm{Id}_n - v\cdot a^t\right)\cdot v$$

$$\underbrace{\underbrace{2v^t\cdot a\cdot v - \underbrace{v\cdot a^t\cdot v}_{=\, v^t\cdot a\cdot v}}_{v^t\cdot a\cdot v}}_{= -\frac{a_{n+1}}{v^t\cdot a}v}$$

So

$$(*) = \underbrace{\left(Id_n - \frac{v \cdot a^t}{v^t \cdot a} \right)^2}_{\parallel} [p]_k - \frac{a_{n+1}}{v^t \cdot a} v$$

$v^t a \cdot v \cdot a^t$

$$\underbrace{Id_n - 2\,\frac{v \cdot a^t}{v^t \cdot a} + \underbrace{\frac{v \cdot a^t}{v^t \cdot a} \cdot \frac{v \cdot a^t}{v^t \cdot a}}_{\parallel \ \frac{\cancel{(v^t a)}(v \cdot a^t)}{\cancel{(v^t a)} \cdot (v^t \cdot a)}}}$$

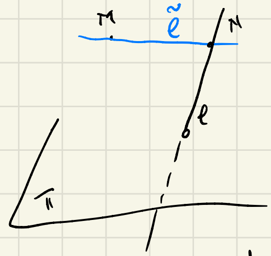
$$= Id_n - \frac{v \cdot a^t}{v^t \cdot a}$$

So $(*) = \left(Id - \frac{v \cdot a^t}{v^t \cdot a} \right) [p]_k - \frac{a_{n+1}}{v^t a} v = [Pr_{H,v}(p)]_k$ $\square$

for 2. the calculation is similar

6. Give Cartesian equations for the line $\tilde{\ell}$ passing through the point $M(1,0,7)$, parallel to the plane $\pi: 3x - y + 2z - 15 = 0$ and intersecting the line

$$\ell: \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$



$\boxed{\text{Method 1}}$ Determine the point $N = \ell \cap \tilde\ell$ using a projection

$N = \mathrm{Pr}_{\ell,\bar\pi}(M)$

$$\mathrm{Pr}_{\ell,\bar\pi}(p) = p' = q - \frac{\varphi(q)}{(\mathrm{lin}\,\varphi)(v)}\, v \quad \text{where}$$

- q is a point on ℓ
- v is a direction vector of ℓ
- $\varphi^{-1}(0)$ is the plane parallel to π and containing p

in our case $\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}_{q} + t \underbrace{\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}}_{v}$ and, if $\tilde\pi \parallel \pi$ and $\tilde\pi \ni p$ then

$\mathrm{lin}\,\varphi(x,y,z) = \mathrm{lin}\,\psi(x,y,z)$

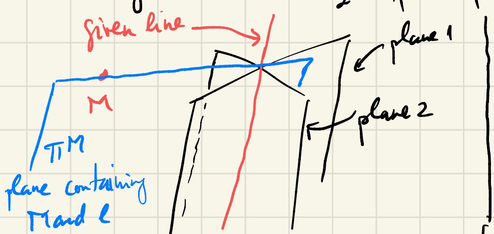
$\tilde\pi: \underbrace{3x - y + 2z + D}_{\varphi(x,y,z)} = 0$

D is determined by the condition $\varphi(p) = 0 \Rightarrow \psi(x,y,z) = (\mathrm{lin}\,\varphi)(x,y,z) - (\mathrm{lin}\,\varphi)(p)$

$\Rightarrow$ We obtain $N = \mathrm{Pr}_{\ell,\bar\pi}(M) = \begin{bmatrix} -\frac{14}{3} \\ \frac{1}{6} \\ -\frac{17}{2} \end{bmatrix}$ $\Rightarrow$ we have two points (M and N) on $\tilde\ell$, so we can write down eq. for $\tilde\ell$

$\boxed{\text{Method 2}}$ Determine $\tilde\ell$ using the pencil of planes passing through the line ℓ

$\ell: \frac{x-1}{4} = \frac{y-3}{2} = z \Leftrightarrow \ell: \begin{cases} \frac{x-1}{4} = z \\ \frac{y-3}{2} = z \end{cases} \Leftrightarrow \begin{cases} x - 4z - 1 = 0 \\ y - 2z - 3 = 0 \end{cases}$ and any other plane containing ℓ has an eq. of the form $\pi_{\alpha,\beta}: \alpha(x - 4z - 1) + \beta(y - 2z - 3) = 0$, $\alpha^2 + \beta^2 > 0$, $\alpha,\beta \in \mathbb{R}$



- if we determine π^M then the line $\tilde\ell$ is $\pi^M \cap \tilde\pi$ (where $\tilde\pi$ is as before the plane parallel to π and containing M)

- $\pi^M = \pi_{\alpha,\beta}$ for some $\alpha,\beta \in \mathbb{R}$

$M \in \pi_{\alpha,\beta} \Rightarrow \alpha(1 - 28 - 1) + \beta(-14 - 3) = 0 \Rightarrow \alpha = -\frac{17}{28}\beta \Rightarrow \pi^M = \pi_{-\frac{17}{28}\beta,\beta}: -17x + 28y + 12z - 67 = 0$

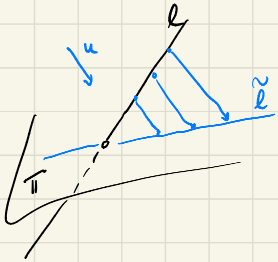
$M \in \tilde\pi \Rightarrow M = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$ satisfies $3x - y + 2z + D = 0 \Rightarrow D = -17$

$$\Rightarrow \tilde\ell: \begin{cases} -17x + 28y + 12z - 67 = 0 \\ 3x - y + 2z - 17 = 0 \end{cases}$$

7. Give Cartesian equations for the projection of the line

$$\ell : 2x - y + z - 1 = 0 \cap x + y - z + 1 = 0$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction of $\vec{u}(1,1,-2)$. Write down Cartesian equations of the line obtained by reflecting ℓ in the plane π parallel to the direction of $\vec{u}$.



$\boxed{\text{Method 1}}$ Using the pencil of planes containing ℓ we can select one plane $\tilde{\pi}$ from this pencil which is parallel to u (there is only one such plane)

$\Rightarrow$ then, we can write $\tilde{\ell}$, the projection of ℓ on π, as the intersection $\pi \cap \tilde{\pi}$

for some $\alpha, \beta \in \mathbb{R}$

Any plane containing ℓ has an equation of the form $\pi_{\alpha,\beta} : \alpha(2x - y + z - 1) + \beta(x + y - z + 1) = 0$

$\Leftrightarrow \pi_{\alpha,\beta} : (2\alpha + \beta)x + (-\alpha + \beta)y + (\alpha - \beta)z - \alpha + \beta = 0$

Further, a plane $\pi_{\alpha\beta}$ is parallel to $u(1,1,-2)$ if and only if

$(2\alpha + \beta)\cdot 1 + (-\alpha + \beta)\cdot 1 + (\alpha - \beta)\cdot(-2) = 0 \Leftrightarrow \alpha = 4\beta$

So, the plane containing ℓ and parallel to u is $\pi_{4\beta,\beta} \Rightarrow \tilde{\ell} : \begin{cases} \pi : x + 2y - z = 0 \\ \pi_{4\beta,\beta} : 3x - y + z - 1 = 0 \end{cases}$

$\boxed{\text{Method 2}}$ Using the projection map $Pr_{\pi,u} : \mathbb{R}^3 \to \pi$ we can project the line ℓ point-wise on π in order to obtain $\tilde{\ell}$

$$Pr_{\pi,u}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{where } \mu = -\frac{\varphi(x,y,z)}{(\text{lin}\,\varphi)(v)}$$

$$\Rightarrow Pr_{\pi,u}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{x + 2y - z}{1 + 2 + 2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5x - x - 2y + z \\ 5x - x - 2y + z \\ 5x + 2x + 4y - 2z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -2 & 1 \\ -1 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The equations that describe individual points on the line ℓ are the parametric equations. So, in order to project individual points from ℓ on π we first transform the given equations for ℓ into parametric equations

$$\ell : \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 2 & -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -3 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right) \Rightarrow \ell : \begin{cases} x = 0 \\ y = z - 1 \\ z = z \end{cases} \Rightarrow \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{p} + z\underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{v} \quad z \in \mathbb{R}$$

$$\Rightarrow Pr_{\pi,u}(\overbrace{p + tv}^{\in \ell}) = \frac{1}{5}\begin{bmatrix} 4 & -2 & 1 \\ -1 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}\left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \underbrace{-\frac{1}{5}\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}}_{\tilde{p}} + \frac{t}{5}\underbrace{\begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}}_{\tilde{v}}$$

✓ parametric equations for $\tilde{\ell}$

8. Consider the Euclidean space $\mathbb{E}^3$. Show that the orthogonal reflection $\mathrm{Ref}_\pi^\perp(x)$ in the plane π : $\langle n, x\rangle = p$ is given by

$$\mathrm{Ref}_\pi(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2}n$.

if $n = n(n_1, n_2, n_3)$ and $x = (x_1, x_2, x_3)$

then $\langle n \cdot x\rangle = p \iff n_1 x_1 + n_2 x_2 + n_3 \cdot x_3 - p = 0$

Use the compact matrix form and notice that

$$\langle n, n\rangle = n^t \cdot n = \|n\|^2$$

9. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4 \quad \text{respectively.}$$

$\boxed{\text{for } \pi_1}$

- $\pi_1 = \varphi^{-1}(0)$ where $\varphi(x,y,z) = 3x - 4z + 1$
- the direction of the reflection is given by the normal vector to π_1 $\quad n = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$

$$\Rightarrow \mathrm{Ref}^{\perp}_{\pi_1}(p) = p + 2\mu n \quad \text{with } \mu = -\frac{\varphi(p)}{(\mathrm{lin}\,\varphi)(n)} = \frac{3x - 4z + 1}{9 + 16} \quad \text{if } p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \mathrm{Ref}^{\perp}_{\pi_1}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{6x - 8z + 2}{25}\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 25x - 18x + 24z - 6 \\ 25y \\ 25z + 24x - 32z + 8 \end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix} 7x & & 24z \\ & 25y & \\ 24x & & -7z \end{pmatrix} + \frac{2}{25}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 7 & 0 & 24 \\ 0 & 25 & 0 \\ 24 & 0 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{2}{25}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$

Check:
- this needs to be the same as replacing in the generic matrix form which we deduced for dimension 3
- the points in π_1 do not change if they are reflected with $\mathrm{Ref}^{\perp}_{\pi}$. Check this on an affine basis of π_1

$\boxed{\text{for } \pi_2}$ the calculations are similar