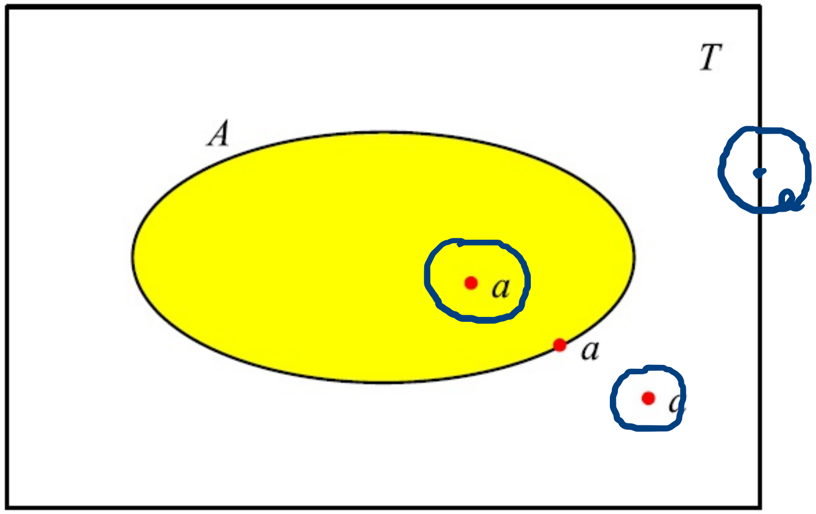
Course 10 April 27, 2021

3.1. Theorem (characterization of Jordan measurable sets).

A bounded set $A \subseteq \mathbb{R}^n$ is Jordan measurable $\iff$ bd$A$ has Lebesgue measure zero

Proof. Choose a generalized rectangle $T \subseteq \mathbb{R}^n$ s.t. cl$A \subseteq$ int$T$



A is Jordan measurable $\overset{\text{def}}{\iff}$

$\iff \chi_A$ is Riemann integrable over T

$\iff$ disc$(\chi_A|_T)$ has Lebesgue measure zero

Lebesgue's criterion

• if $a \in \text{int}A \Rightarrow \exists r > 0$ s.t. $B(a,r) \subseteq A$

$\Rightarrow \chi_A(x) = 1 \quad \forall x \in B(a,r)$

$\Rightarrow \chi_A$ is continuous at a

• if $a \in \text{ext}A \Rightarrow \exists r > 0$ s.t. $B(a,r) \subseteq \mathbb{R}^n \setminus A \Rightarrow \chi_A(x) = 0 \quad \forall x \in B(a,r) \cap T$

$\Rightarrow \chi_A|_T$ is continuous at a

- if $a \in \text{bd}\, A = (\text{cl}\, A) \cap \text{cl}(\mathbb{R}^n \setminus A) \Rightarrow$

$\Rightarrow$ $a \in \text{cl}\, A$ $\Rightarrow$ $\exists (x_k)$ seq. in A s.t. $(x_k) \to a$; $\chi_A(x_k) = 1 \quad \forall k \geq 1$

and

$a \in \text{cl}(\mathbb{R}^n \setminus A) \Rightarrow \exists (y_k)$ seq in $T \setminus A$ s.t. $(y_k) \to a$; $\chi_A(y_k) = 0 \quad \forall k \geq 1$

$$\lim_{k \to \infty} \chi_A(x_k) = 1 \neq 0 = \lim_{k \to \infty} \chi_A(y_k)$$

$\Rightarrow$ χ_A is discontinuous at a

$\Rightarrow$ $\text{disc}\left(\chi_A\big|_T\right) = \text{bd}\, A$

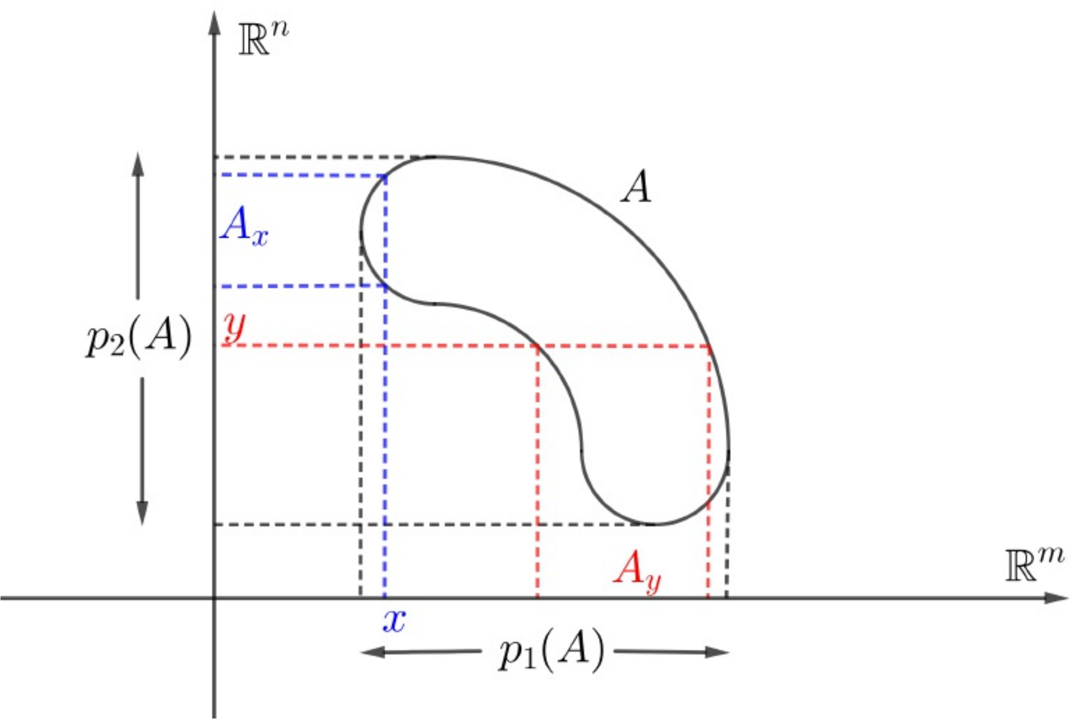
$\Rightarrow$ A is Jordan measurable $\Leftrightarrow$ bd A has Lebesgue measure zero

3.2. Theorem (Lebesgue's criterion on Jordan measurable sets).

Let $A \subseteq \mathbb{R}^n$ be a Jordan measurable set. A function $f : A \to \mathbb{R}$ is R-integrable over A $\Leftrightarrow$ $\begin{cases} \cdot\ f \text{ is bounded} \\ \cdot\ \text{disc}(f) \text{ has Lebesgue measure zero} \end{cases}$

3.3. Corollary $\left.\begin{array}{l} \text{If } A \subseteq \mathbb{R}^n \text{ is Jordan measurable} \\ f : A \to \mathbb{R} \text{ is continuous and bounded} \end{array}\right\} \Rightarrow f$ is R-integrable over A.

4. Computation of multiple integrals over bounded sets by means of iterated integrals



Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$ be a bounded set

- $\forall\, x \in \mathbb{R}^m$ we consider the set

$$A_x := \{ y \in \mathbb{R}^n \mid (x,y) \in A \}$$

↳ the section of A through x

We define

$$p_1(A) := \{ x \in \mathbb{R}^m \mid A_x \neq \phi \}$$

↳ projection of A onto $\mathbb{R}^m$

- $\forall\, y \in \mathbb{R}^n$ we consider the set $A_y := \{ x \in \mathbb{R}^m \mid (x,y) \in A \}$

↳ the section of A through y

We define

$$p_2(A) := \{ y \in \mathbb{R}^n \mid A_y \neq \phi \}$$

↳ projection of A onto $\mathbb{R}^n$

Let $f: A \to \mathbb{R}$ be an arbitrary function

f is said to be partially integrable over A if

- $\forall x \in p_1(A)$ the section $f_x : A_x \to \mathbb{R}$, $f_x = f(x, \cdot)$ is Riemann integrable over A_x
- $\forall y \in p_2(A)$ the section $f_y : A_y \to \mathbb{R}$, $f_y = f(\cdot, y)$ is Riemann integrable over A_y

Under the above assumptions we may define

$$F_1 : p_1(A) \to \mathbb{R} \qquad F_1(x) := \int_{A_x} f_x(y)\,dy = \int_{A_x} f(x,y)\,dy$$

$$F_2 : p_2(A) \to \mathbb{R} \qquad F_2(y) := \int_{A_y} f_y(x)\,dx = \int_{A_y} f(x,y)\,dx$$

If F_1 is R-integrable over $p_1(A)$, then we may consider its integral

$$\int_{p_1(A)} F_1(x)\,dx = \int_{p_1(A)} \left(\int_{A_x} f(x,y)\,dy \right) dx$$

If F_2 is R-integrable over $p_2(A)$, then we may consider its integral

$$\int_{p_2(A)} F_2(y)\,dy = \int_{p_2(A)} \left(\int_{A_y} f(x,y)\,dx \right) dy$$

iterated integrals of f

Note that in the special case when $A = S \times T$, where $S \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^n$ are generalized rectangles, then the above definitions coincide with those given in section 2.

<u>4.1. Theorem</u> (G. Fubini) Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be a bounded set, and let $f : A \to \mathbb{R}$ s.t. f is Riemann integrable and partially integrable over A.

Then F_1 is R-integrable over $p_1(A)$

F_2 is R-integrable over $p_2(A)$ and one has

$$\int_A f(x,y)\,dxdy = \int_{p_1(A)} F_1(x)\,dx = \int_{p_2(A)} F_2(y)\,dy$$

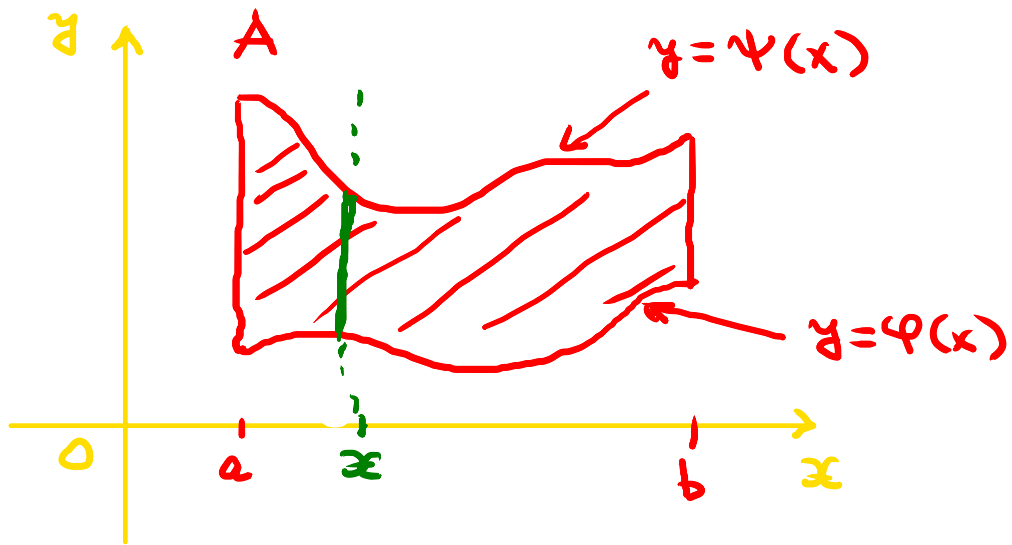
$$\Updownarrow$$

$$\underbrace{\int_A f(x,y)\,dxdy}_{m+n \text{ vars}} = \underbrace{\int_{p_1(A)} \Big(\underbrace{\int_{A_x} f(x,y)\,dy}_{n \text{ vars}} \Big) dx}_{m \text{ variables}} = \underbrace{\int_{p_2(A)} \Big(\underbrace{\int_{A_y} f(x,y)\,dx}_{m \text{ vars}} \Big) dy}_{n \text{ variables}}$$

4.2. Definition (normal domains in $\mathbb{R}^2$). A set $A \subseteq \mathbb{R}^2$ is said to be a normal domain w.r.t. the x-axis if $\exists\, a, b \in \mathbb{R}$, $a < b$

$\exists\, \varphi, \psi : [a, b] \to \mathbb{R}$ continuous functions with $\varphi \leq \psi$

s.t. $A = \{ (x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ \varphi(x) \leq y \leq \psi(x) \}$

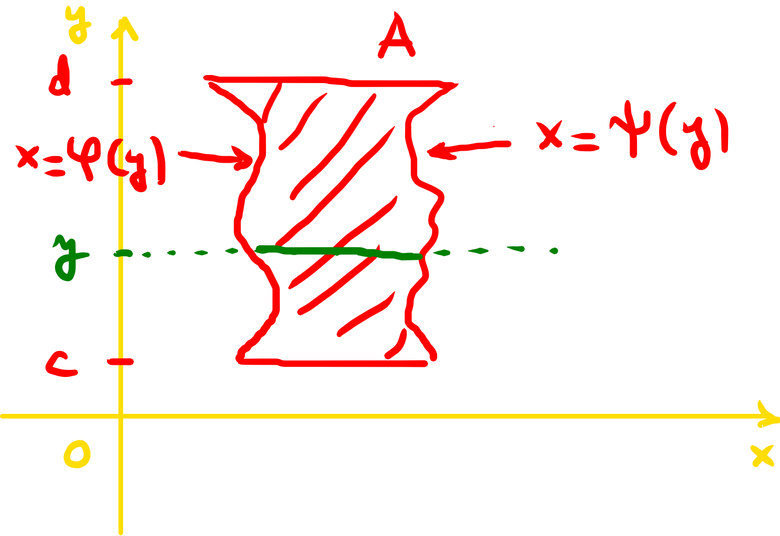


$p_1(A) = [a, b]$

$\forall x \in [a, b] : \quad A_x = [\varphi(x), \psi(x)]$

$A \subseteq \mathbb{R}^2$ is called a normal domain w.r.t. the y-axis if $\exists\, c < d$ and $\exists\, \varphi, \psi : [c, d] \to \mathbb{R}$ continuous functions, with $\varphi \leq \psi$ s.t.

$$A = \{ (x, y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ \varphi(y) \leq x \leq \psi(y) \}$$



$$p_2(A)=[c,d]$$

$$\forall\, y\in[c,d] \; : \quad A_y=[\varphi(y),\psi(y)]$$

It can be proved that if $A\subseteq\mathbb{R}^2$ is a normal domain, then A is compact and Jordan measurable.

<u>4.3. Corollary</u>. If $A\subseteq\mathbb{R}^2$ is a normal domain w.r.t. the x-axis and $f: A\to\mathbb{R}$ is continuous, then f is R-integrable over A and

$$\iint_A f(x,y)\,dx\,dy=\int_a^b\left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right)dx$$

$f=1$ $\qquad m(A)=\mathcal{A}(A)=\int_a^b\big(\psi(x)-\varphi(x)\big)\,dx$

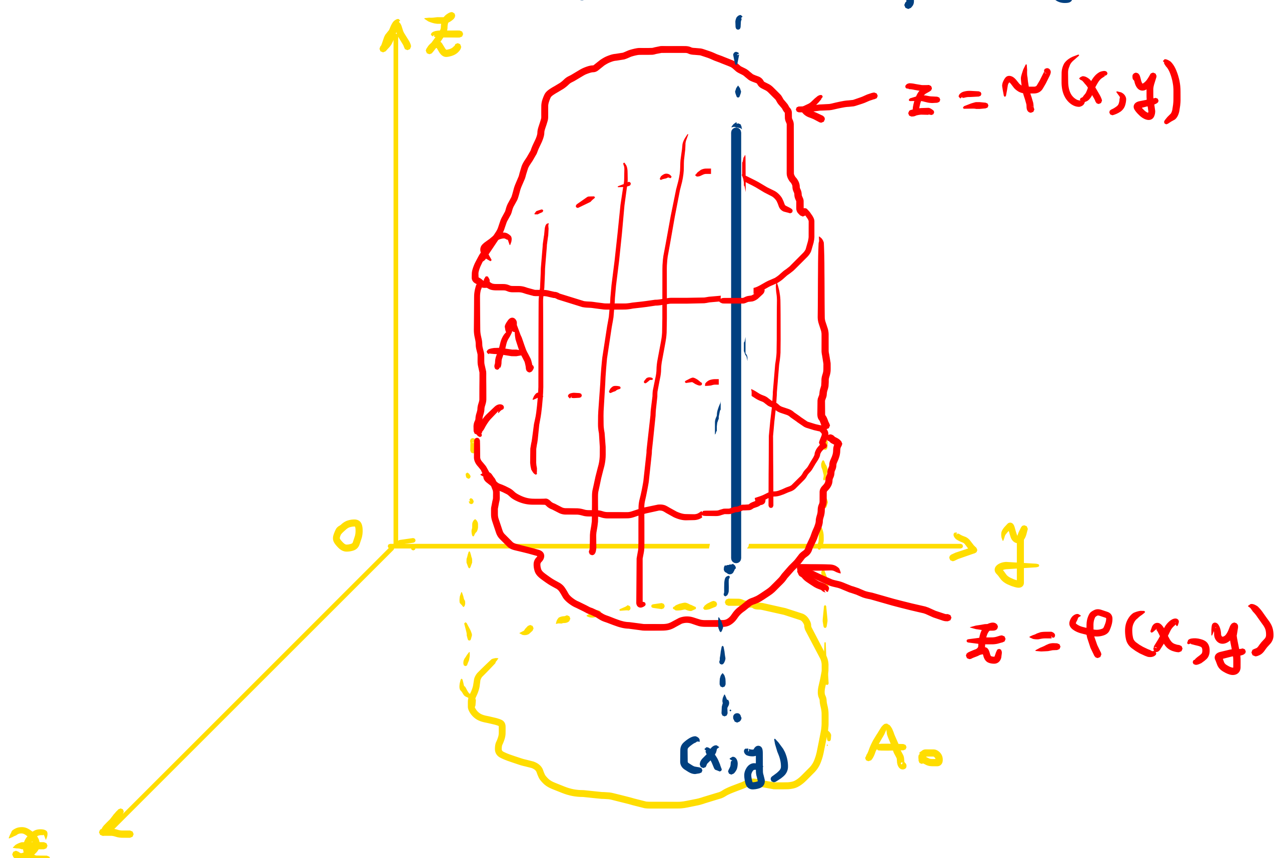
If $A\subseteq\mathbb{R}^2$ is a normal domain w.r.t. the y-axis and $f: A\to\mathbb{R}$ is continuous then f is R-integrable over A and

$$\iint_A f(x,y)\,dx\,dy=\int_c^d\left(\int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\right)dy$$

$f=1$ $\qquad m(A)=\mathcal{A}(A)=\int_c^d\big(\psi(y)-\varphi(y)\big)\,dy$

4.4. Definition (normal domains in $\mathbb{R}^3$) A set $A \subseteq \mathbb{R}^3$ is said to be a normal domain w.r.t. the xOy-plane if $\exists A_0 \subseteq \mathbb{R}^2$ Jordan measurable $\exists \varphi, \psi : A_0 \to \mathbb{R}$ continuous and bounded

s.t. $A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in A_0,\ \varphi(x,y) \le z \le \psi(x,y)\}$



$p_1(A) = A_0$

$\forall (x,y) \in A_0$

$A_{(x,y)} = [\varphi(x,y), \psi(x,y)]$

Similarly one can define normal domains in $\mathbb{R}^3$ w.r.t. the xOz or w.r.t. the yOz plane.

4.5. Corollary If $A \subseteq \mathbb{R}^3$ is a normal domain w.r.t. xOy, and $f: A \to \mathbb{R}$ is continuous and bounded $\Rightarrow$ f is R-integrable over A and

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{A_0} \left(\int_{z=\varphi(x,y)}^{z=\psi(x,y)} f(x,y,z)\,dz \right) dx\,dy$$

$f=1$ $\qquad m(A) = Vol(A) = \iint_{A_0} \Big(\psi(x,y) - \varphi(x,y) \Big)\,dx\,dy$

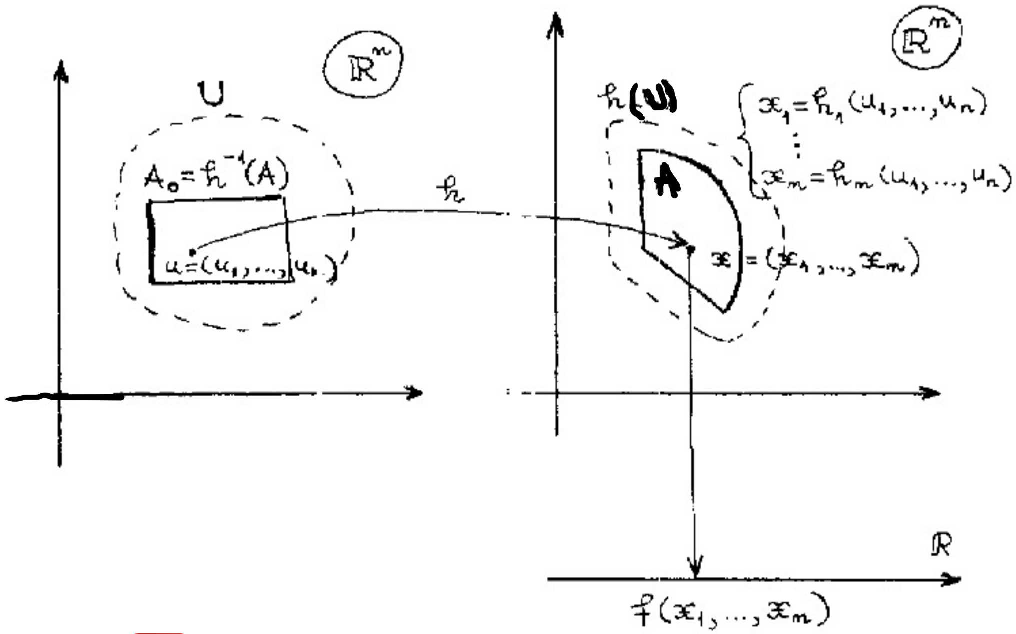
5. Change of variables in multiple integrals

5.1. Theorem Let
- $U \subseteq \mathbb{R}^n$ be an open set
- $h: U \to \mathbb{R}^n$ be an injective function, of class C^1 on U and with the property that $\det J(h)(u) \neq 0$ for all $u \in U$
- $A \subseteq \mathbb{R}^n$ Jordan measurable s.t. $\mathrm{cl}\,A \subseteq h(U)$

Then: 1° The set $A_0 = h^{-1}(A)$ is Jordan measurable

2° If $f: A \to \mathbb{R}$ is R-integrable over A, then $(f \circ h) \cdot |\det J(h)| : A_0 \to \mathbb{R}$ is R-integrable over A_0 and

$$\int_A f(x)\,dx = \int_{A_0} (f \circ h)(u) \cdot |\det J(h)(u)|\,du$$

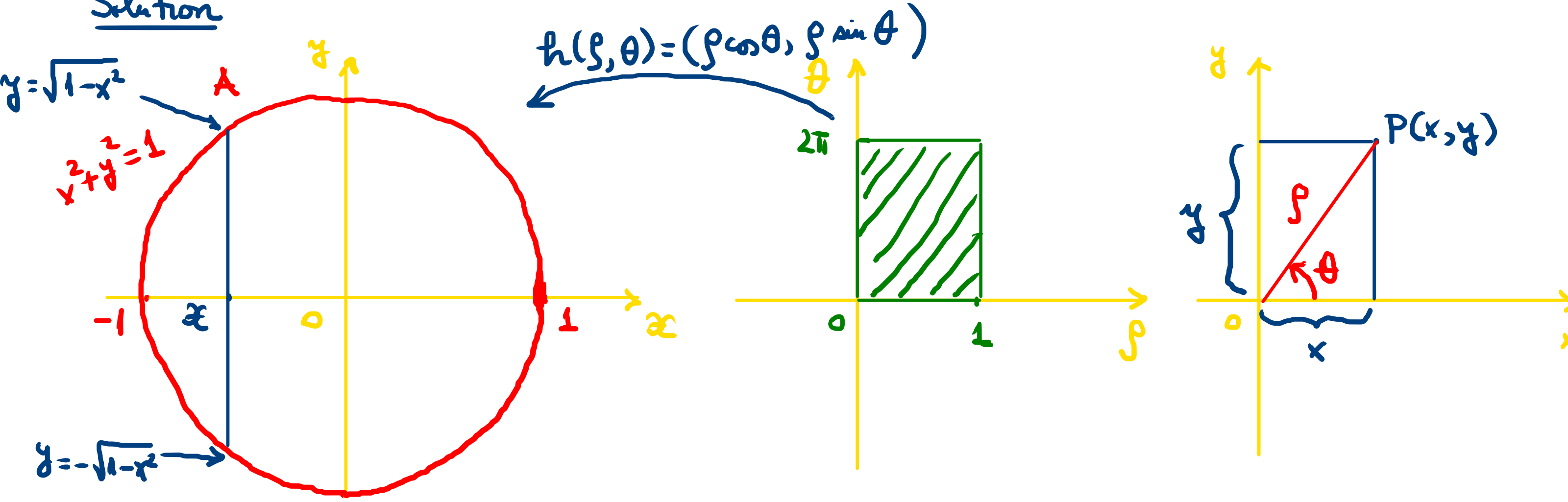


$$\int \cdots \int_A f(x_1, \dots, x_n)\, dx_1 \cdots dx_n = \int \cdots \int_{A_0} f(h_1(u_1, \dots, u_n), \dots, h_n(u_1, \dots, u_n)) \times |\det J(h)(u_1, \dots, u_n)|\, du_1 \cdots du_n$$

$$\begin{cases} x_1 = h_1(u_1, \dots, u_n) \\ \vdots \\ x_n = h_n(u_1, \dots, u_n) \end{cases}$$

Example Evaluate $I = \iint_A e^{x^2+y^2}\,dx\,dy$, where $A : x^2+y^2 \le 1$

Solution



x, y = Cartesian coordinates of P

ρ, θ = Polar coordinates of P

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

If we try to compute I by using Fubini's theorem, then we get

$$I = \int_{x=-1}^{x=1} \left(\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} e^{x^2+y^2}\,dy \right) dx = \int_{x=-1}^{x=1} \left(e^{x^2} \underbrace{\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} e^{y^2}\,dy}_{\text{cannot be evaluated}} \right) dx$$

We evaluate I by passing to polar coordinates

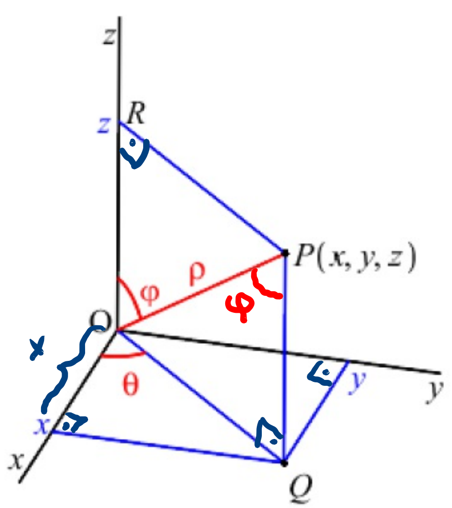
$$I = \iint_{[0,1]\times[0,2\pi]} e^{(\rho\cos\theta)^2+(\rho\sin\theta)^2}\cdot\rho\, d\rho\, d\theta$$

$$\det J(h)(\rho,\theta) = \frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \boxed{\rho}$$

$$I = \int_{\rho=0}^{\rho=1}\int_{\theta=0}^{\theta=2\pi} \rho e^{\rho^2}\, d\rho\, d\theta = \left(\int_0^1 \rho e^{\rho^2} d\rho\right)\left(\int_0^{2\pi} d\theta\right) = \cancel{2}\pi \cdot \frac{1}{\cancel{2}} e^{\rho^2}\Big|_{\rho=0}^{\rho=1}$$

$$\boxed{I = \pi(e-1)}$$

<u>Spherical coordinates</u>



x, y, z = Cartesian coordinates of P

ρ, φ, θ = spherical coordinates of P

$$\left.\begin{array}{l} \cos\theta = \dfrac{x}{OQ} \Rightarrow x = OQ\cos\theta \\ \sin\varphi = \dfrac{OQ}{\rho} \Rightarrow OQ = \rho\sin\varphi \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{array}}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} = \rho^2\sin\varphi$$

6. Applications of multiple integrals

6.1. Center of mass of a plane plate. Consider a plane plate whose shape is a set $A \subseteq \mathbb{R}^2$. If the plate is non-homogeneous, then we consider known its surface density $\rho(x,y)$ for $\forall (x,y) \in A$
(g/cm^2)

Then

$$m = \iint_A \rho(x,y)\,dx\,dy$$

$\hookrightarrow$ the mass of the plate

If the plate is homogeneous $\Rightarrow \rho(x,y) = \rho \Rightarrow m = \rho \cdot \mathcal{A}(A)$

The center of mass of the plate is the point G having the coordinates

$$x_G = \frac{\iint_A x\rho(x,y)\,dx\,dy}{\iint_A \rho(x,y)\,dx\,dy} \qquad y_G = \frac{\iint_A y\,\rho(x,y)\,dx\,dy}{\iint_A \rho(x,y)\,dx\,dy}$$

If $\rho(x,y)=\rho$

$$x_G = \frac{\iint_A x\,dx\,dy}{\mathcal{A}(A)} \qquad y_G = \frac{\iint_A y\,dx\,dy}{\mathcal{A}(A)}$$

6.2. <u>Center of mass of a solid body</u> Consider a solid body whose shape is given by $A \subseteq \mathbb{R}^3$

$\rho(x,y,z)$ = density of the body at $(x,y,z) \in A$

$$m = \iiint_A \rho(x,y,z)\,dx\,dy\,dz$$

↳ the mass of the body

$$\rho(x,y,z)=\rho \quad \Rightarrow \quad m = \rho \cdot \mathrm{Vol}(A)$$

The center of mass G has the coordinates

$$x_G = \frac{\iiint_A x\,\rho(x,y,z)\,dx\,dy\,dz}{\iiint_A \rho(x,y,z)\,dx\,dy\,dz} \qquad y_G = \dots \qquad z_G = \dots$$

$$\rho(x,y,z) = \rho$$

$$x_G = \frac{\iiint_A x\,dx\,dy\,dz}{Vol(A)} \qquad y_G = \frac{\iiint_A y\,dx\,dy\,dz}{Vol(A)} \qquad z_G = \dots$$