ANALYTIC GEOMETRY, PROBLEM SET 6

Representations of the line in plane

1. Find the equation of the line passing through the intersection point of the lines $d_1 : 2x - 5y - 1 = 0$ and $d_2 : x + 4y - 7 = 0$ and through a point M which divides the segment $[AB]$, given by $A(4, -3)$ and $B(-1, 2)$, into the ratio $k = 2/3$.

2. Find the equation of the line passing through the intersection point of $d_1 : 3x-2y+5 = 0$, $d_2 : 4x + 3y - 1 = 0$ and intersecting the Oy axis at the point A with $OA = 3$.

3. Find the parametric equations of the line through P_1 and P_2, when
(1) $P_1(3, -2)$, $P_2(5, 1)$;
(2) $P_1(4, 1)$, $P_2(4, 3)$.
In each case, find the vector equation of the line passing through these points.

4. Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.

$\begin{cases} x = -5 + 2t \\ y = 2 + 3t \end{cases}$

5. Show that the equations $x = 3 - t$, $y = 1 + 2t$ and $x = -1 + 3t$, $y = 9 - 6t$. represent the same line. Write down a director vector for this line.

6. The points $M_1(1, 2)$, $M_2(3, 4)$ and $M_3(5, -1)$ are the midpoints of the sides of a triangle. Write down the equations of the lines determined by the sides of the triangle.

7. Given the line $d : 2x+3y+4 = 0$, find the equation of a line d_1 passing through the point $M_0(2, 1)$, in the following situations: a) d_1 is parallel with d; b) d_1 is orthogonal on d; c) the angle determined by d and d_1 is $\pi/4$.

8. The vertices of the triangle $\triangle ABC$ are the intersection points of the lines $d_1 : 4x+3y-5 = 0$, $d_2 : x - 3y + 10 = 0$, $d_3 : x - 2 = 0$. a) Find the coordinates of A, B and C. b) Find the equations of the median lines of the triangle. c) Find the equations of the heights of the triangle.

9. Find the coordinates of the symmetrical of the point $P(-5, 13)$ with respect to the line $d : 2x - 3y - 3 = 0$.

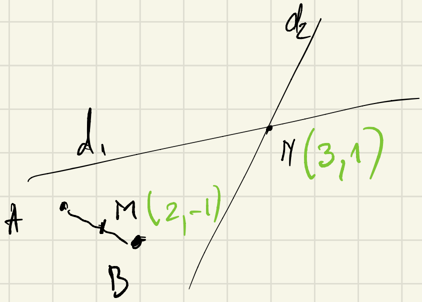
10. Find the coordinates of the point P on the line $d : 2x - y - 5 = 0$, for which the sum $AP + PB$ attains its minimum, when $A(-7, 1)$ and $B(-5, 5)$.

11. Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $4x - y + 2 = 0$, $x - 4y - 8 = 0$ and $x + 4y - 8 = 0$.

Date: October 29, 2021.

1. Find the equation of the line passing through the intersection point of the lines d_1 : $2x - 5y - 1 = 0$ and $d_2 : x + 4y - 7 = 0$ and through a point M which divides the segment $[AB]$, given by $A(4, -3)$ and $B(-1, 2)$, into the ratio $k = 2/3$.

$N(x_N, y_N) \in d_1 \cap d_2$



Method I

$$\begin{cases} N \in d_1 \Rightarrow 2x_N - 5y_N - 1 = 0 \\ N \in d_2 \Rightarrow x_N + 4y_N - 7 = 0 \end{cases}$$

Method II (I')

$$\left.\begin{array}{l} N \in d_1 \Rightarrow y_N = \frac{2}{5}x_N - \frac{1}{5} \\ N \in d_2 \Rightarrow y_N = -\frac{1}{4}x_N + \frac{7}{4} \end{array}\right\} \Rightarrow \frac{2}{5}x_N - \frac{1}{5} = -\frac{1}{4}x_N + \frac{7}{4}$$

$\Rightarrow x_N = \dots$

$y_N = \dots$

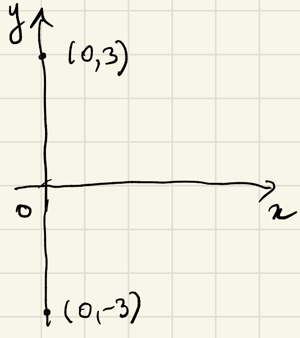
M ?

$$M\left(\frac{x_A + \overset{2/3}{k}\, x_B}{1+k}, \frac{y_A + k\, y_B}{1+k}\right)$$

? $-2x + y + 5 = 0 \Leftarrow \begin{vmatrix} x & y & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 0 \iff \frac{x-2}{3-2} = \frac{y-(-1)}{1-(-1)}$

2. Find the equation of the line passing through the intersection point of $d_1 : 3x - 2y + 5 = 0$, $d_2 : 4x + 3y - 1 = 0$ and intersecting the Oy axis at the point A with $\underbrace{OA = 3}$.

I $A(0,3)$ II $A(0,-3)$



$d_1 \cap d_2 = \left(-\frac{13}{17}, +\frac{23}{17}\right)$

Method 2 using pencils of lines. ?

3. Find the parametric equations of the line through P_1 and P_2, when

(1) $P_1(3,-2)$, $P_2(5,1)$;

(2) $P_1(4,1)$, $P_2(4,3)$.

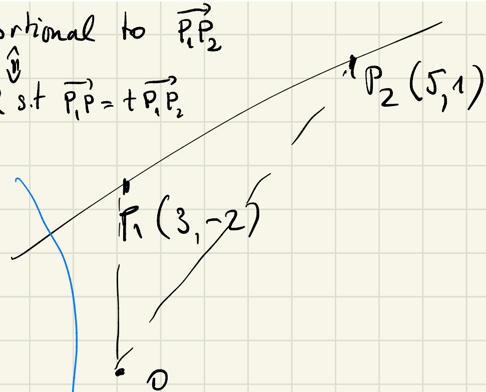
In each case, find the vector equation of the line passing through these points.

$P(x_P, y_P) \in P_1P_2$ iff $\overrightarrow{P_1P}$ is proportional to $\overrightarrow{P_1P_2}$

$\Updownarrow$

$\exists t \in \mathbb{R}$ s.t $\overrightarrow{P_1P} = t\overrightarrow{P_1P_2}$

$\overline{OP} = \overline{OP_1} + t v$, $\quad v = \overline{P_1P_2}$



$\Leftrightarrow \exists t \in \mathbb{R}$ s.t $\overline{P_1O} + \overline{OP} = t\overline{P_1P_2}$

$\Updownarrow$

$$\overline{OP} = \overline{OP_1} + t\overline{P_1P_2}$$

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{cases} x = 3 + 2t \\ y = -2 + 3t \end{cases} \quad t \in \mathbb{R}$$

5. Show that the equations $x = 3 - t$, $y = 1 + 2t$ and $x = -1 + 3t$, $y = 9 - 6t$. represent the same line. Write down a director vector for this line.

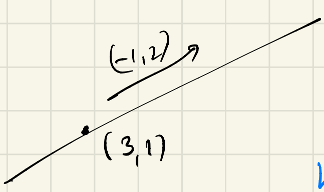
I

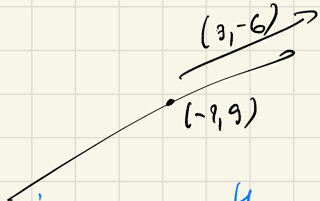
$$\begin{cases} x = 3 - t_1 \\ y = 1 + 2t_1 \end{cases} \qquad \begin{cases} x = -1 + 3t_2 \\ y = 9 - 6t_2 \end{cases}$$

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ director vector $\xleftrightarrow{\text{are proportional}}$ $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$ director vector

$\Downarrow$

parallel.

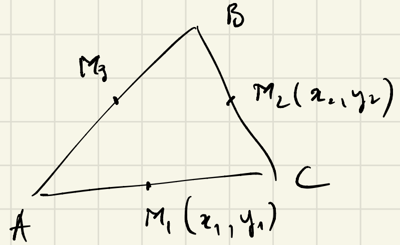




So if they have a point in common they must be equal.

II for each set of param. eqs. determine the general eq. and notice that they are the same. (up to a scalar)

6. The points $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ are the midpoints of the sides of a triangle. Write down the equations of the lines determined by the sides of the triangle.



Method 1. we want to describe AB as

$$y = mx + k$$

Since $AB \parallel M_1M_2$ we obtain m

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2}{2} = 1$$

$\Rightarrow$ $AB: \ y = x + k$

and since $M_3 \in AB$ we have $-1 = 5 + k \Rightarrow k = -6$

$$AB: y = x - 6$$

similar for AC and BC

Method 2. $M_3 = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2} \right)$ since M_3 is mid pt. of AB

$\frac{x_A + x_B}{2} = 5$, $\frac{y_A + y_B}{2} = -1$

$$\left.\begin{array}{l} (1,2) = M_1 = \left(\frac{x_A + x_C}{2}, \frac{y_A + y_C}{2} \right) \\ (3,4) = M_2 = \dots \end{array}\right\} \Rightarrow \text{you get } A, B, C$$

Method 3 $\quad AB: \begin{cases} x = 5 + t \cdot 2 \\ y = -1 + t \cdot 2 \end{cases}$ $\quad \overrightarrow{M_1M_2}$ $\quad$ similar for BC and AC

7. Given the line $d: 2x+3y+4=0$, find the equation of a line d_1 passing through the point $M_0(2,1)$, in the following situations: a) d_1 is parallel with d; b) d_1 is orthogonal on d; c) the angle determined by d and d_1 is $\pi/4$.

a.) $d_1 \parallel d$ $\Downarrow$

$d: 2x+3y+4=0$

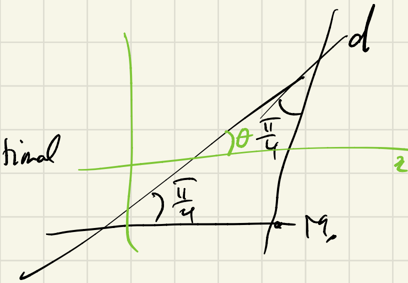
$d: y = \underbrace{-\frac{2}{3}}_{\text{slope}} x - \frac{4}{3}$

$d_1: y = -\frac{2}{3}x + c$
$\ni M_0(2,1)$ $\quad 1 = -\frac{2}{3}\cdot 2 + c \Rightarrow \frac{7}{3} = c$ $\Big\} \Rightarrow y = -\frac{2}{3}x + \frac{7}{3}$

Method 2 $\quad d_1: \dfrac{x-2}{v_x} = \dfrac{y-1}{v_y}$

where (v_x, v_y) is a dir. vector for d_1

Since $d_1 \parallel d$ (v_x, v_y) is proportional to $(-3, 2)$



$d: \dfrac{x - x_M}{v_x} = \dfrac{y - y_M}{v_y} \iff v_y x - v_x y + \dots = 0$

b.) $d_1 \perp d \Rightarrow d_1: \quad y = -\left(-\frac{2}{3}\right)^{-1} x + c$

$y = \frac{3}{2}x + c \ni M_0(2,1) \Rightarrow c = \dots$

c.) $\tan\left(\theta \pm \frac{\pi}{4}\right) = \dfrac{\tan\theta \pm \tan\frac{\pi}{4}}{1 \mp \tan\theta \cdot \tan\frac{\pi}{4}}$ $\quad \tan\theta = -\frac{2}{3}$

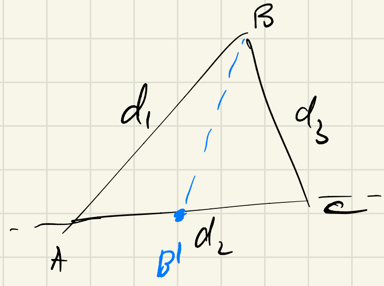
8. The vertices of the triangle $\triangle ABC$ are the intersection points of the lines $d_1 : 4x+3y-5 = 0$, $d_2 : x-3y+10 = 0$, $d_3 : x-2 = 0$. a) Find the coordinates of A, B and C. b) Find the equations of the median lines of the triangle. c) Find the equations of the heights of the triangle.

$A = d_1 \cap d_2$

the coords of $A(x_A, y_A)$ are the sol. to the lin. sys.

$$\begin{cases} 4x+3y-5=0 \\ x-3y+10=0 \end{cases}$$

similar for B and C



$B' = \left(\frac{x_A+x_C}{2}, \frac{y_A+y_B}{2}\right)$