Series with positive terms (SPT)

Let $\sum x_n$ be a series of real numbers.
It is called a SPT if $x_n > 0, \ \forall n \in \mathbb{N}$
Each SPT has a sum.
Each divergent SPT has the sum infinity.
i.e. if $\sum x_n$ is D $\Rightarrow \exists \sum_{n=1}^{\infty} x_n = \infty$

P1

$\sum x_n$ a SPT $\Rightarrow$ (s_n) is INCREASING

Proof:

for a random $n \in \mathbb{N}$ s.t. $s_{n+1} - s_n = (x_1 + x_2 + \dots + x_n + x_{n+1}) - (x_1 + x_2 + \dots + x_n) = x_{n+1} > 0$
therefore (s_n) increasing

P2

$\sum x_n$ a SPT $\Rightarrow \exists \sum_{n=1}^{\infty} x_n \in \overline{\mathbb{R}}$ $[\in (0, \infty]]$ (each SPT has a sum)

Proof:

$$\sum x_n \text{ a SPT} \xRightarrow{P_1} (s_n) \text{ increasing} \xRightarrow{\text{Weierstrass}} \exists \underbrace{\lim_{n\to\infty} s_n}_{\overset{\text{def}}{=} \sum_{n=1}^{\infty} x_n} \in \overline{\mathbb{R}}$$

Theorem

$\sum x_n$ a SPT
$\sum x_n$ is convergent $\Longleftrightarrow$ (s_n) is bounded

Proof:

"$\Rightarrow$"

$$\sum x_n \text{ is convergent} \overset{\text{def}}{\Longleftrightarrow} \exists \lim_{n\to\infty} s_n \in \mathbb{R} \Rightarrow (s_n) \text{ bounded}$$

"$\Leftarrow$"

$\sum x_n$ is SPT

$$(s_n) \text{ bounded} \Longrightarrow (\text{Weierstrass})\ (s_n) \text{ is convergent} \Rightarrow \exists \lim_{n\to\infty} s_n \in \mathbb{R} \Longleftrightarrow \sum x_n \text{ is convergent}$$

(s_n) increasing

The Cauchy condensation criteria

$\sum x_n$ SPT s.t. (x_n) decreasing

Then $\sum x_n \sim \sum 2^n x_{2^n}$ (same nature)

e. g. the proof of nature of the generalized harmonic series

Study the nature of this series:

① $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n^\alpha} = \begin{cases} \infty & : \alpha < 0 \\ 1 & : \alpha = 0 \\ 0 & : \alpha > 0 \end{cases}$ $\left(\frac{1}{n^\alpha} = \frac{1}{n^0} = 1 \quad \forall n \in \mathbb{N} \Rightarrow x_n = 1 \to \text{a constant sequence} \right)$

If $\alpha \le 0 \Rightarrow \lim_{n\to\infty} x_n \neq 0 \Rightarrow \sum x_n$ is D. $\left.\begin{matrix} \\ x_n > 0 \end{matrix}\right\} \Rightarrow \sum_{n=1}^{\infty} x_n = \infty$

If $\alpha > 0 \Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow ?$

② SPT ✓ $\boxed{\alpha > 0}$

We apply the Cauchy condensation criteria:

$\frac{x_{n+1}}{x_n} = \left(\frac{n}{n+1}\right)^\alpha < 1 \Rightarrow (x_n)$ is decreasing $\xrightarrow{\text{condensation}} \boxed{\sum x_n \sim \sum 2^n \cdot x_{2^n}} = \sum y_n$

$$y_n = 2^n \cdot x_{2^n} = 2^n \cdot \frac{1}{(2^n)^\alpha} = \frac{1}{2^{n\alpha - n}} = \frac{1}{2^{n(\alpha-1)}} = \left(\frac{1}{2^{\alpha-1}}\right)^n$$

$\sum y_n = \sum \left(\frac{1}{2^{\alpha-1}}\right)^n$ which is a geometric series of ratio $q = \frac{1}{2^{\alpha-1}}$

(because α is a constant) $\alpha > 0$

it is $\begin{cases} C & \text{if } 0 < |q| < 1 \\ D & \text{otherwise} \end{cases}$

in this case $0 < q$ • $\Rightarrow \sum y_n$ is C $\Leftrightarrow \boxed{q < 1} \Leftrightarrow \frac{1}{2^{\alpha-1}} < 1 \iff$

$\iff 1 < 2^{\alpha-1} \iff \alpha - 1 > 0 \iff \boxed{\alpha > 1}$.

• $\sum y_n$ is D otherwise $\left(\alpha \le 1 \text{ and } \boxed{\alpha > 0} \right)$

from ②

In conclusion ①&② $\sum \frac{1}{n^\alpha}$ is $\begin{cases} C & \text{if } \alpha > 1 \\ D & \text{if } \alpha \le 1. \end{cases}$

Comparison criteria for SPT

C1C

$\sum x_n, \sum y_n$ SPT s.t. $\exists a>0$ and $\forall n \geq k$, $x_n \leq a y_n$ $\Longrightarrow$ $\begin{cases} \sum y_n \text{ is convergent} \Rightarrow \sum x_n \text{ is convergent} \\ \sum x_n \text{ is divergent} \Rightarrow \sum y_n \text{ is divergent} \end{cases}$
$\exists k>0$

C1C has two cases in which it can't be applied: $\sum y_n$ is divergent; $\sum x_n$ is convergent

C2C

Let $\sum x_n, \sum y_n$ SPT s.t $\exists \lim\limits_{n\to\infty} \frac{x_n}{y_n} = l \in \overline{\mathbb{R}}$

Then

If $l \in (0,\infty)$ $\Rightarrow$ $\sum x_n \sim \sum y_n$ (same nature)

If $l=0$ $\Rightarrow$ C1C

If $l=\infty$ $\Rightarrow$ C1C

$\sum x_n \leftrightarrow \sum y_n$ $\begin{cases} \sum y_n \ D \Rightarrow \sum x_n \ D \\ \sum x_n \ C \Rightarrow \sum y_n \ C \end{cases}$

C3C

Let $\sum x_n, \sum y_n$ SPT s.t $\exists a>0$ and $\forall n \geqslant k$ $\frac{x_{n+1}}{x_n} \leq a \cdot \frac{y_{n+1}}{y_n}$ $\Longrightarrow$ C1C
$\exists k \in \mathbb{N}$

C1C $\Longleftrightarrow$ C2C $\Longleftrightarrow$ C3C

e.g.

1) Study the nature of the series:

a) $\sum\limits_{n\geq 1} \frac{3^n}{4^n+2^n}$ $\qquad x_n = \frac{3^n}{4^n+2^n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3^n}{4^n\left(1+\frac{2^n}{4^n}\right)} = \lim_{n\to\infty} \frac{\left(\frac{3}{4}\right)^n}{\left(1+\left(\frac{2}{4}\right)^n\right)} = \frac{0}{1+0} = 0$$

$$\frac{3^n}{2\cdot 2^n} \leq x_n \leq \left(\frac{3}{4}\right)^n$$

$y_n = \left(\frac{3}{4}\right)^n$, $\forall n \in \mathbb{N}$

$\sum y_n$ is a geometric series of ratio $q = \frac{3}{4} \in (0,1) \Rightarrow \sum y$ is convergent

$\left.\begin{array}{l} x_n \leq y_n, \ \forall n \in \mathbb{N} \\ \sum y_n \text{ is convergent} \end{array}\right\} \overset{C1C}{\Longrightarrow} \sum x_n$ is convergent

b) $\sum_{n\geq 1} \frac{n^3+3n}{\sqrt{n^5+7n^2+3}}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^3+3n}{\sqrt{n^5+7n^2+3}} = \infty \implies \sum x_n \ \ D. \ \ (C1C)$$

c) $\sum_{n\geq 1} \frac{n^3+3n}{\sqrt{n^7+7n^2+3}}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^3+3n}{\sqrt{n^7+7n^2+3}} = 0$$

C2C

$$x_n = \frac{n^3+3n}{\sqrt{n^7+5n^2+3}} \qquad y_n = \frac{1}{n^\alpha}$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{n^3+3n}{\sqrt{n^7+5n^2+3}} \cdot n^\alpha = \lim_{n\to\infty} \frac{n^{\alpha+3}\left(1+\frac{3}{n^2}\right)}{n^{\frac{7}{2}}\left(1+\frac{5}{n^2}+\frac{3}{n^7}\right)} \overset{\alpha+3=\frac{7}{2}}{=} 1 \in (0,\infty)$$

here we ~~choose~~ choose α

$$\exists\, \underset{\substack{\parallel \\ 1}}{l} \in (0,\infty) \overset{C1C}{\Rightarrow} \sum x_n \sim \sum \frac{1}{n^{\frac{1}{2}}}$$

$$\left.\begin{array}{l} \exists\, l \in (0,\infty) \overset{C1C}{\Rightarrow} \sum x_n \sim \sum \frac{1}{n^{\frac{1}{2}}} \\ \sum \frac{1}{n^{\frac{1}{2}}} \ D. \ \frac{1}{2} < 1 \end{array}\right\} \overset{C2C}{\Longrightarrow} \sum x_n \text{ is } D$$

Consequence of the D'Alambert criterion for SPT

$\sum x_n$ a SPT

$\exists\, l = \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ if $\begin{cases} l<1 \to \sum x_n \text{ is convergent} \\ l=1 \\ l>1 \to \sum x_n \text{ is divergent} \end{cases}$

$\Vert$ Cesaro-Stolz

if $\exists \lim\limits_{n\to\infty} \sqrt[n]{x_n}$!

The Raabe-Duhamel criterion

If $\exists\, T = \lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right)$ $\begin{cases} l>1 \to \sum x_n \text{ is convergent} \\ l=1 \\ l<1 \to \sum x_n \text{ is divergent} \end{cases}$

Algorithm for analysing sequences with reductible consecutive terms

① $l = \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ $\begin{array}{ll} <1 & C \\ >1 & D \\ =1 & \text{back to hypothesis} \\ & \text{or} \\ & \text{go to 2} \end{array}$

② $T = \lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right)$ $\begin{array}{ll} >1 & C \\ <1 & D \\ =1 & \text{go to hyp.} \\ & \text{or} \\ & \text{Kummer criterion} \end{array}$

$\boxed{\text{Example}}$ Discuss depending on the values of $\boxed{a>0}$, the nature of the series

$$\sum_{n\geq 1} \frac{n!}{a\cdot(a+1)\cdot\ldots\cdot(a+n-1)}$$

$\boxed{\text{SPT}}$ ✓

at this stage $\lim\limits_{n\to\infty} x_n$?

① $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{\cancel{a\cdot(a+1)}\cdot\ldots\cdot(\cancel{a+n-1})\cdot(a+n)} \cdot \frac{\cancel{a\cdot(a+1)\cdot\ldots\cdot(a+n)}}{\cancel{n!}} = \lim\limits_{n\to\infty} \frac{n+1}{a+n} = 1$?

② $\lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{a+n}{n+1} - 1\right) = \lim\limits_{n\to\infty} \frac{n(a+n-n-1)}{n+1} = \lim\limits_{n\to\infty} \frac{n}{n+1}\cdot(a-1) = 1\cdot(a-1) = \boxed{a-1}$

$\frac{n}{n+1} \to 1$

$T = a-1$ $\quad \sum x_n$ C if $a-1>1 \iff a>2$ ✓

D if $a-1<1 \iff a<2$ ✓

? if $\boxed{a-1=1} \iff \boxed{a=2}$

I can go back to the hypothesis with $\boxed{a=2}$ $\quad x_n = \frac{n!}{2\cdot 3\cdot\ldots\cdot(2+n-1)} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$\sum \frac{1}{n+1}$ D $\sim \sum \frac{1}{n}$ (C2C)

conclusion $\sum x_n : \begin{cases} C & \text{for } a>2 \\ D & \text{for } a\leq 2 \end{cases}$ ✓

SPT

T_1 (the Cauchy condensation criteria)

Consider $\sum x_n$ to be an SPT st. (x_n) is decreasing. Then

$\sum 2^n x_{2^n}$

s have the same nature)

! e of the generalized harmonic series

a constant)

$n^{\alpha} = \frac{1}{n^0} = 1 \quad \forall n \in \mathbb{N} \Rightarrow x_n = 1 \to$ a constant sequence)

$\Delta.\}\ \Rightarrow \sum_{n=1}^{\infty} x_n = \infty$

)

Application 2: Study the nature of the series

$$\sum_{n\geq 3} \frac{1}{\ln(\ln n)}$$

SPT ✓

$$\frac{x_{n+1}}{x_n} = \frac{\frac{1}{\ln(\ln(n+1))}}{\frac{1}{\ln(\ln n)}} = \frac{\ln(\ln n)}{\ln(\ln(n+1))} < 1 \Rightarrow (x_n) \text{ is decreasing}$$

SPT

$$\sum x_n \sim \sum 2^n x_{2^n}$$

$$y_n = 2^n \cdot x_{2^n} = 2^n \cdot \frac{1}{\ln(\ln 2^n)} = 2^n \cdot \frac{1}{\ln(n \cdot \ln 2)} = \frac{2^n}{\ln n + \ln 2}$$

? what is the nature of $\sum y_n$? $\quad \sum_{n\geq 3} \frac{2^n}{\ln n + \ln 2}$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{2^n}{\ln n + \ln 2} = \infty$$

proof

Method 1: Define two function

$f: (3,\infty) \to \mathbb{R} \qquad f(x) = 2^x$

$g: (3,\infty) \to \mathbb{R} \qquad g(x) = \ln x + \ln 2$

f, g are diff. on $(3,\infty)$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{2^x \ln 2}{\frac{1}{x} + 0} = \lim_{x\to\infty} x \cdot 2^x \ln 2 = \infty \ ✓$$

Method 2: C-S = L'H for sequences.
(does not involve functions)

$$y_n = \frac{2^n}{\ln n + \ln 2} \quad \forall n \geq 3 \quad \begin{cases} a_n := 2^n \\ b_n := \ln n + \ln 2 \end{cases}$$

⊙ b_n is increasing

⊙ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} (\ln n + \ln 2) = \infty \Rightarrow (b_n)$ is unbounded

⊙ $$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{2^{n+1} - 2^n}{\ln(n+1) + \cancel{\ln 2} - \ln n - \cancel{\ln 2}} = \lim_{n\to\infty} \frac{2^n(2-1)}{\ln\left(\frac{n+1}{n}\right)} = \frac{\infty \cdot 1}{0_+} = \infty$$

C-S

$\Rightarrow \lim_{n\to\infty} y_n = \infty \neq 0$

$\Rightarrow$ $\sum y_n$ is D $\overset{\text{condensation}}{\Longleftrightarrow}$ $\sum x_n$ is D $= \sum \frac{1}{\ln(\ln n)}$