**Theorem 2.3.** Let $\mathbf{A}$ be an affine space with coordinate system $O\mathbf{e}_1\dots\mathbf{e}_n$. Let

$$\begin{aligned} a_{11}x_1+\cdots+a_{1n}x_n &= b_1 \\ &\vdots \\ a_{t1}x_1+\cdots+a_{tn}x_n &= b_t \end{aligned} \tag{2.2}$$

be a system of linear equations in the unknowns $x_1,\dots,x_n$. The set $S$ of points of $\mathbf{A}$ whose coordinates are solutions of (2.2), if there are any, is an affine space of dimension $n-r$ where $r$ is the rank of the matrix of coefficients of the system. The vector subspace associated to $S$ is the vector subspace $\mathbf{W}$ of $\mathbf{V}$ whose equations are given by the associated homogeneous system

$$\begin{aligned} a_{11}x_1+\cdots+a_{1n}x_n &= 0 \\ &\vdots \quad . \\ a_{t1}x_1+\cdots+a_{tn}x_n &= 0 \end{aligned} \tag{2.3}$$

Conversely, for every affine subspace $S$ of $\mathbf{A}$ of dimension $s$ there is a system of $n-s$ linear equations in $n$ unknowns whose solutions correspond precisely to the coordinates of the points in $S$.

- First we show that the set of solutions to (2.2) is an affine subspace of A with the indicated properties.
- If $S=\phi$ there is nothing to show
- If $S\neq\phi$ then there is a point $Q(q_1,\dots,q_n)\in S$

For any $P(p_1,\dots,p_n)\in S$ we have

$$a_{j1}(p_1-q_1)+\dots+a_{jn}(p_n-q_n) = \underbrace{a_{j1}p_1+\dots+a_{jn}p_n}_{=b_j} + \underbrace{a_{j1}q_1+\dots+a_{jn}q_n}_{=b_j} = b_j-b_j=0 \quad \forall j=1,t$$

$\Rightarrow$ $\overrightarrow{QP}$ is a solution to the homogeneous system (2.3)

$\Rightarrow$ $\boxed{S\subseteq T}$, the affine space passing through $Q$ and parallel to $W$

Next we show that $T\subseteq S$ which will imply $S=T$

For any $R(r_1,\dots,r_n)\in T$ we have $\overrightarrow{QR}\in W$

$\Rightarrow$ $(r_1-q_1,\dots,r_n-q_n)$ is a solution to (2.3) $\quad \forall j=1,t$

$$\Rightarrow \quad 0 = a_{j1}(r_1-q_1)+\dots+a_{jn}(r_n-q_n) = a_{j1}r_1+\dots+a_{jn}r_n - \underbrace{(a_{j1}q_1+\dots+a_{jn}q_n)}_{=b_j}$$

$$\Rightarrow \quad a_{j1}r_1+\dots+a_{jn}r_n = b_j \quad \forall j=1,t$$

$\Rightarrow R$ is a solution to (2.2) $\Rightarrow R \in T$

$\Rightarrow S = T$ is an affine subspace of $\mathbb{A}$

- Now, $\dim S = \dim W = \underset{\dim \mathbb{A}}{\underset{\shortparallel}{n}} -$ rank of the matrix of coefficients of (2.2)

- Next we show that an affine subspace $S \subset \mathbb{A}$ has a description by equations as in (2.2)

  - Let $S$ be the affine subspace of $\mathbb{A}$ passing through $Q$ and having associated vector subspace $W$. Let $s = \dim W$

  - Let $$\begin{matrix} a_{11} x_1 + \dots + a_{1n} x_n = 0 \\ \vdots \\ a_{n-s\,1} x_1 + \dots + a_{n-s\,n} x_n = 0 \end{matrix}$$ be equations for $W$.

  - $\forall P \in \mathbb{A}$ : $P \in S \Leftrightarrow \overrightarrow{QP} \in W$

$$\Leftrightarrow a_{j1}(p_1 - q_1) + \dots + a_{jn}(p_n - q_n) = 0 \quad \forall j = 1 \dots n-s$$

$$\Leftrightarrow a_{j1} p_1 + \dots + a_{jn} p_n = b_j \quad \forall j = 1 \dots n-s$$

where we denote by $b_j$ the scalar $a_{j1} q_1 + \dots + a_{jn} q_n$ $\forall j = 1 \dots n-s$

So the points in $S$ are precisely those points in $\mathbb{A}$ for which the coordinates satisfy the system of equations

$$\begin{matrix} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{n-s\,1} x_1 + \dots + a_{n-s\,n} x_n = b_j \end{matrix}$$

**Proposition 2.6.** Let $S$ and $T$ be parallel affine subspaces of $\mathbf{A}$ with $\dim(S) \leq \dim(T)$.

1.) If $S$ and $T$ have a point in common then $S \subseteq T$.

2.) If $\dim(S) = \dim(T)$, and $S$ and $T$ have a point in common then $S = T$.

1) Fix $Q \in S \cap T$

Let $W_S$ be the vector subspace associated to $S$

Let $W_T$ be the vector subspace associated to $T$

Then

$$S = \{ P \in \mathbf{A} : \overrightarrow{QP} \in W_S \} \subseteq \{ P \in \mathbf{A} : \overrightarrow{QP} \subseteq W_T \} = T$$

2.) $\dim W_S = \dim S = \dim T = \dim W_T \Rightarrow W_S = W_T$

by 1) $S \subseteq T$ and $T \subseteq S \Rightarrow S = T$

**Corollary 2.7.** If $S$ is an affine subspace of $\mathbf{A}$ and $P \in \mathbf{A}$, there is a unique affine subspace $T$ of $\mathbf{A}$ which contains $P$, is parallel to $S$ and has the same dimension as $S$.

Let $S, T$ be two affine subspaces of $\mathbf{A}$

$$\left.\begin{array}{l} \cdot \text{ which contain } P \\ \cdot \text{ which are parallel} \end{array}\right\} \overset{P\,2.6.\,2.)}{\Longrightarrow} \quad S = T$$

**Proposition 2.10.** If the intersection $S \cap T$ of two affine subspaces of $\mathbf{A}$ is non-empty it is an affine subspace satisfying

$$\dim(S) + \dim(T) - \dim(\mathbf{A}) \leq \dim(S \cap T) \leq \min\{\dim(S), \dim(T)\}. \tag{2.7}$$

- Let $\dim S = s$ and $\dim T = t$
- by Thm 2-3 we have

$$\left\{\begin{array}{ll} S: \sum_{j=1}^{n} m_{ij} x_j = b_i & i = 1, \ldots, n-s \\ T: \sum_{j=1}^{n} n_{kj} x_j = b_k & k = 1, \ldots, n-t \end{array}\right. \quad (*)$$

- The dimension of $S \cap T$ is $n-r$ where $r$ is the rank of the matrix of coefficients of the system $(*)$
- Notice that

$$r \leq n-s+n-t = 2n-(s+t)$$

- So, if $S \cap T \neq \emptyset$ then

$$\dim(S \cap T) = n-r \geq n - [2n-(s+t)] = s+t-n$$

$$\Rightarrow \dim(S \cap T) \geq \dim S + \dim T - \dim A$$

- Clearly, since $S \cap T \subseteq S, T$ $\dim(S \cap T) \leq \dim S, \dim T$

$$\Rightarrow \dim(S \cap T) \leq \min(\dim S, \dim T)$$

**Proposition 2.11.** Let $S$ and $T$ be two affine subspaces of $\mathbf{A}$ with associated vector subspaces $\mathbf{W}$ and $\mathbf{U}$ respectively. Then $\mathbf{V} = \mathbf{W} + \mathbf{U}$ if and only if $S \cap T \neq \emptyset$ *and*

$$\dim(S \cap T) = \dim(S) + \dim(T) - \dim(\mathbf{A}). \tag{2.8}$$

- Recall (Algebra 1, Lecture 12, Corollary 20)

$$\dim W + \dim U = \dim(W \cap U) + \dim(W + U) \qquad (*)$$

- Let $\dim S = s$ and $\dim T = t$
- by Thm 2.3 we have

$$\left\{ \begin{array}{ll} S: \sum_{j=1}^{n} m_{ij} x_j = b_i & i = 1, \dots, n-s \\ T: \sum_{j=1}^{n} n_{kj} x_j = b_k & k = 1, \dots, n-t \end{array} \right. \qquad (*)$$

- Let $r$ be the rank of the matrix of $(*)$ as in the proof of Proposition 2.10
- Then $(2.8) \Longleftrightarrow$

$$\begin{array}{ccccccc} n - r & = & s & + & t & - & n \\ \parallel & & \parallel & & \parallel & & \\ \dim(S \cap T) & = & \dim S & + & \dim T & - & \dim(A) \\ \parallel & & \parallel & & \parallel & & \\ \dim(W \cap U) & = & \dim(W) & + & \dim(U) & - & \dim(V) \end{array}$$

if $S \cap T \neq \emptyset \longrightarrow$

So $(2.8) \Longleftrightarrow \dim(V) = \underbrace{\dim(W) + \dim(U) - \dim(W \cap U)}_{= \dim(W+U) \text{ by } (*)}$

So $(2.8) \Longleftrightarrow \dim(V) = \dim(W + U)$

$\Longleftrightarrow V = W + U$