1. Which of the following are affine subspace?

1.	a line in $\mathbf{E}^2$	7.	a circle in $\mathbf{E}^2$ centered at $(-1,1)$
2.	a circle in $\mathbf{E}^2$ centered at $(0,0)$	8.	a halfline in $\mathbf{E}^2$ or $\mathbf{E}^3$
3.	$(0,1)$ in $\mathbf{E}^2$	9.	a parabola in $\mathbf{E}^2$
4.	the half plane $x > y$ in $\mathbf{E}^2$	10.	a plane in $\mathbf{E}^3$
5.	a line in $\mathbf{E}^3$	11.	a disk in $\mathbf{E}^3$
6.	a triangle in $\mathbf{E}^2$ or $\mathbf{E}^3$	12.	a sphere in $\mathbf{E}^3$

2. Have a look at the third problem set of the geometry cours from last semester. Which problems on that list are purely affine, i.e. involve only vectors and vector relations in a vector space?

3. Let $\mathbf{A}$ be an affine space over the $\mathbf{K}$-vector space $\mathbf{V}$. Show that $\overrightarrow{PP} = 0$ for every $P \in \mathbf{A}$ and $\overrightarrow{PQ} = -\overrightarrow{QP}$ for every $P,Q \in \mathbf{A}$.

4. Give two distinct affine structures for the Euclidean plane $\mathbf{E}^2$. (*hint:* $\mathbb{R}$ and $\mathbb{C}$.)

5. Let S be an affine subspace of the affine space $\mathbf{A}$. Show that if $\dim(S) = \dim(\mathbf{A})$ then $S = \mathbf{A}$.

6. Which of the following admits the structure of an affine space? (explain why)

1. $\mathcal{C}(\mathbb{R})$ = the set of continuous functions $\mathbb{R} \to \mathbb{R}$.
2. $\{P \in \mathbf{K}[x] : \deg(P) \leq n\}$ = the set of polynomials of degree at most n.
3. $\{P \in \mathbf{K}[x] : \deg(P) = n\}$ = the set of polynomials of degree n.

7. Let p be a prime number.

1. Show that $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is a field.
2. How many vectors does an $\mathbb{F}_p$-vector space have?
3. How many points does an affine space over an $\mathbb{F}_p$-vector space have?

8. Let $\mathbf{A}$ be an affine space and consider four points $A,B,C,D \in \mathbf{A}$. Show that if $\overrightarrow{AB} = \overrightarrow{CD}$ then $\overrightarrow{AC} = \overrightarrow{BD}$.

9. In the affine space $\mathbf{A}^2(\mathbb{C})$ consider the line passing through the point $A(4,-2\mathbf{i})$ and having direction vector $\mathbf{a}(7+\mathbf{i},1)$. Give several parametric equations for this line.

10. In the affine space $\mathbf{A}^3(\mathbb{C})$ consider the plane passing through the point $A(2+\mathbf{i},5,-\mathbf{i})$ and parallel to the vectors $\mathbf{a}(2,3,1)$ and $\mathbf{b}(-1,-11,3)$. Give several parametric equations for this plane.

11. In the affine space $\mathbf{A}^4(\mathbb{R})$ consider

$$\text{the plane} \quad \alpha = \left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\4\\1\\2\end{bmatrix} \quad \text{and the line} \quad \beta = \left\langle \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\3\\-1\\1\end{bmatrix}.$$

Determine $\alpha \cap \beta$.

12. Let $\mathbf{V}$ be a vector space of dimension at least 5 and let $\mathbf{A}$ be an affine space with associated vector space $\mathbf{V}$. Consider three distinct points $a, b, c \in \mathbf{A}$ and a plane $\pi = \langle v_1, v_2 \rangle + a$. Determine an affine subspace in $\mathbf{A}$ of dimension at most 4 which contains a, b, c and π.

13. Show that the definition of the affine subspace $\langle P_0, \ldots, P_n \rangle$ generated by $P_0, \ldots, P_n$ does not depend on the point P_0.

14. Let $\mathbf{A}$ be an affine space over the vector subspace $\mathbf{V}$ and let $C \in \mathbf{A}$. For each $P \in \mathbf{A}$, the *reflection* of P in C is the point $\mathrm{Ref}_C(P)$ satisfying the vector identity

$$\overrightarrow{C\,\mathrm{Ref}_C(P)} = -\overrightarrow{CP}.$$

This defines a map $\mathrm{Ref}_C : \mathbf{A} \to \mathbf{A}$. Show that $\mathrm{Ref}_C(P)$ maps affine subspaces to affine subspaces.

15. Show that the definition of a k-simplex with vertices $P_0, \ldots, P_k$ doesn't depend on the choice of P_0.

16. Show that a k-simplex is convex.

1. Which of the following are affine subspace?

1. a line in $\mathbf{E}^2$
2. a circle in $\mathbf{E}^2$ centered at $(0,0)$
3. $(0,1)$ in $\mathbf{E}^2$
4. the half plane $x > y$ in $\mathbf{E}^2$
5. a line in $\mathbf{E}^3$
6. a triangle in $\mathbf{E}^2$ or $\mathbf{E}^3$
7. a circle in $\mathbf{E}^2$ centered at $(-1,1)$
8. a halfline in $\mathbf{E}^2$ or $\mathbf{E}^3$
9. a parabola in $\mathbf{E}^2$
10. a plane in $\mathbf{E}^3$
11. a disk in $\mathbf{E}^3$
12. a sphere in $\mathbf{E}^3$

- The affine structure of $\mathbb{E}^2$ and $\mathbb{E}^3$ is given by the map

$$(A,B) \to \vec{AB} \quad \text{where } \vec{AB} \text{ is the equipollence class from last semester}$$

- A set S is an affine subspace of $\mathbb{E}^n$ $(n=2,3)$ if and only if there exists a subspace W of vectors such that for a fixed (but arbitrary) point $Q \in S$ we have

$$W = \{ \vec{QP} : P \in S \}$$

  - this means that the set of all vectors which can be represented by points in S is a vector subspace of the vector space associated to $\mathbb{E}^2$ (or $\mathbb{E}^3$)

$\Rightarrow$ S is an affine subspace $\Leftrightarrow$ $\{ \vec{AB} : A, B \in S \}$ is a vector subspace

1) a line ℓ in $\mathbb{E}^2$:

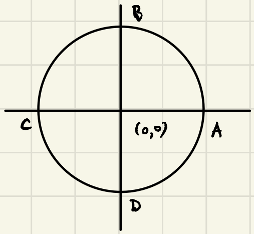
- we know that for any two distinct points $A, B \in \ell$ the vector $\vec{AB}$ is a direction vector for ℓ
- we know that any two direction vectors are proportional, i.e linearly dependent

- we know that any non-empty set contains at least one point and so, $\vec{AA} = 0$ can be represented in such a set

$\Rightarrow$ $\{ \vec{AB} : A, B \in \ell \} = \{0\} \cup \{ v : v \text{ is a direction vector for } \ell \}$

this is a 1-dimensional vector subspace generated by any direction vector of ℓ

2. A circle $\mathcal{C}$ in $\mathbb{E}^2$ centered at $(0,0)$



- we consider the case where $\mathcal{C}$ is of radius 1. The other cases are similar.

- consider the points $A(1,0)$, $B(0,1)$, $C(-1,0)$, $D(0,-1)$

  then $\vec{CA}(2,0)$ and $\vec{DB}(0,2)$ are vectors which can be represented by points in $\mathcal{C}$

- If $\{ \vec{PQ} : P, Q \in \mathcal{C} \}$ is a vector subspace then

$$\vec{CA} + \vec{DB} = \vec{MN} \text{ for some } M, N \in \mathcal{C}$$

- However $\vec{CA} + \vec{DB} = (2,2) \Rightarrow$ it has length $2\sqrt{2}$

  But all vectors in $\{ \vec{PQ} : P, Q \in \mathcal{C} \}$ have length at most the diameter of $\mathcal{C}$ which is 2

$\Rightarrow$ the vector $\vec{CA} + \vec{DB}$ cannot be represented by points in $\mathcal{C}$

$\Rightarrow$ $\{ \vec{PQ} : P, Q \in \mathcal{C} \}$ is not a vector space

$\Rightarrow$ $\{\vec{PQ} : P, Q \in \mathcal{C}\}$ is not a vector subspace

$\Rightarrow$ $\mathcal{C}$ is not an affine subspace of $\mathbb{E}^2$

- the other cases can be treated similarly
- In fact :
  - the only affine subspaces of $\mathbb{E}^2$ are
    - points
    - lines
    - $\mathbb{E}^2$ itself.
  - the only affine subspaces of $\mathbb{E}^3$ are
    - points
    - lines
    - planes
    - $\mathbb{E}^3$ itself.

2. Have a look at the third problem set of the geometry cours from last semester. Which problems on that list are purely affine, i.e. involve only vectors and vector relations in a vector space?

are affine

Warm-up 1.
1, 2, 4, 5, 6, 8

are not affine

3, 7

- in affine geometry we have
  - line segments, so we have polygons
    so we have triangles, parallelograms and quadrilaterals
  - midpoints
  - angle bisectors can also be described purely vectorial

- in affine geometry we don't have
  - distances so we don't have circles
  - orthogonality so we don't have heights in triangles
    so, no orthocenter

all these notions can be added to an affine space. We will do this later in the course

3. Let **A** be an affine space over the **K**-vector space **V**. Show that $\overrightarrow{PP} = 0$ for every $P \in \mathbf{A}$ and $\overrightarrow{PQ} = -\overrightarrow{QP}$ for every $P, Q \in \mathbf{A}$.

- The second axiom of an affine space says that $\forall\, P, Q, R$ we have

$$\text{(AS2)} \qquad \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

- So, by choosing $Q = P$ we have

$$\overrightarrow{PP} + \overrightarrow{PR} = \overrightarrow{PR}$$

Hence, $\overrightarrow{PP}$ is the neutral element of the vector space associated to $\mathbf{A}$

- If in (AS2) we choose $R = P$ we have

$$\overrightarrow{PQ} + \overrightarrow{QP} = \overrightarrow{PP} = 0$$

Hence, $\overrightarrow{PQ}$ is the inverse element of $\overrightarrow{QP}$, i.e. $\overrightarrow{PQ} = -\overrightarrow{QP}$

4. Give two distinct affine structures for the Euclidean plane $\mathbf{E}^2$. (*hint:* $\mathbb{R}$ and $\mathbb{C}$.)

- Giving an affine structure to a set S means
  - choosing a vector space V and
  - describing a map $S \times S \to V$ which satisfies the axioms
- If $S = \mathbb{E}^2$ we may choose $W = \mathbb{R}^2$
  - fix a coordinate sytem Oxy in $\mathbb{E}^2$ and define

    $\mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{R}^2$ by $\left(P(x_P, y_P), Q(x_Q, y_Q)\right) \mapsto (x_Q - x_P, y_Q - y_P)$
  - this is the usual map that associates to two points P and Q the coordinates of the vector $\overrightarrow{PQ}$
- If $S = \mathbb{E}$ we may also choose $W = \mathbb{C}$ (viewed as a 1-dimensional complex vector space)
  - Relative to a coordinate system Oxy in $\mathbb{E}^2$ we can define

    $\mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{C}^2$ by $\left(P(x_P, y_P), Q(x_Q, y_Q)\right) \mapsto (x_Q - x_P) + i(y_Q - y_P)$
- in both cases it is easy to see that (AS1) and (AS2) are satisfied.
- Notice that the real affine space $\mathbb{E}^2$ has dimension 2

  the complex affine space $\mathbb{E}^2$ has dimension 1

5. Let S be an affine subspace of the affine space $\mathbf{A}$. Show that if $\dim(S) = \dim(\mathbf{A})$ then $S = \mathbf{A}$.

- Since S is an affine subspace of A we have $S \subseteq \mathbb{A}$
- We need to show that $\mathbb{A} \subseteq S$

  or, equivalently, $\forall P \in A : P \in S$

  - Let V be the vector space associated to A
  - Let W be the vector subspace associated to S
  - notice that $\dim W = \dim S = \dim \mathbb{A} = \dim V$

    $\Rightarrow$ so $W = V$.

  - Choose a point $P \in A$
  - $P \in S$ if and only if for $Q \in S$ we have $\overrightarrow{QP} \in W$
    - Since $W = V$, $\forall Q \in S$ $\overrightarrow{QP} \in W = V$

    $\Rightarrow$ So, $P \in S$

6. Which of the following admits the structure of an affine space? (explain why)

1. $\mathcal{C}(\mathbb{R})$ = the set of continuous functions $\mathbb{R} \to \mathbb{R}$.
2. $\{P \in \mathbf{K}[x] : \deg(P) \le n\}$ = the set of polynomials of degree at most n.
3. $\{P \in \mathbf{K}[x] : \deg(P) = n\}$ = the set of polynomials of degree n.

1.) the set $C(\mathbb{R})$ of continuous functions with values in $\mathbb{R}$ is a real vector space with the following operations

$\forall f, g \in \mathcal{C}(\mathbb{R}) \quad ((f+g)(x) = f(x) + g(x))$

$\forall f \in \mathcal{C}(\mathbb{R}) \; \forall c \in \mathbb{R} \quad (cf)(x) = c \cdot f(x)$

Any vector space has the structure of an affine space

in our case $\mathcal{C}(\mathbb{R})_a$

2.) Notice first that

- $\mathbb{K}[x]$ is a $\mathbb{K}$-vector space
  - the sum of two polynomials is a polynomial
  - a scalar times a polynomial is a polynomial

- The set of polynomials in one indeterminate x can be identified with the set of polynomial functions on $\mathbb{R}$ with values in $\mathbb{R}$ so $\mathbb{R}[x]$ is a vector subspace of $\mathcal{C}(\mathbb{R})$ it is therefore also an affine subspace of $\mathcal{C}(\mathbb{R})_a$

Consider $S = \{ P \in \mathbb{K}[x] : \deg P \leq n \}$

- since $\deg(P+Q) \leq \deg(P) + \deg(Q)$

  it follows that S is closed under addition

- since $\forall c \in \mathbb{K} \ \forall P \in \mathbb{K}[x] \ \deg(c \cdot P) \leq \deg(P)$

  it follows that S is closed under multiplication by scalars

$\Rightarrow$ S is a vector subspace of $\mathbb{K}[x]$

- in particular it is a $\mathbb{K}$-vector space

$\Rightarrow$ S is an affine space

3.) Consider $S = \{ P \in \mathbb{K}[x] : \deg P = n \}$ as a subset of $\mathbb{K}[x]$

- We show that S is not an affine subspace of $\mathbb{K}[x]_a$ unless $n = -\infty$,

  i.e unless $S = \{0\}$

- Consider $W = \{ \overrightarrow{QP} = P - Q : P, Q \in S \}$ (this is the set of vectors which can be represereted by points in S)

- if S is an affine subspace of $\mathbb{K}[x]_a$ then W is a vector subspace of $\mathbb{K}[x]$, the associated vector subspace of S

- notice that if $P = 2x^n$ and $Q = x^n$ then

$$\overrightarrow{QP} = 2x^n - x^n = x^n \in W$$

- However, notice also that for $R=0$

$$\overrightarrow{QR} = \overrightarrow{QP} = x^n \in W$$

so by the definition of affine subspaces R has to be in S

- this is a contradiction with $\deg 0 = -\infty$

$\Rightarrow$ S is not an affine subspace of $\mathbb{K}[x]_a$

- notice that we proved here that S is not an <u>affine subspace</u> of $\mathbb{K}[x]_a$
- this doesn't mean that there is no affine space structure on S which is unrelated to that of $\mathbb{K}[x]_a$

- if you find one, let me know

7. Let p be a prime number.

1. Show that $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is a field.
2. How many vectors does an $\mathbb{F}_p$-vector space have?
3. How many points does an affine space over an $\mathbb{F}_p$-vector space have?

1.) $\forall$ integer n $\mathbb{Z}/n\mathbb{Z}$ is a ring

- the elements are the sets $[a] = \{\dots, a-n, a, a+n, a+2n \dots\}$ for $a \in \{0, \dots, n-1\}$
- addition is given by

$$[a]+[b]=[a+b]$$

- multiplication is given by

$$[a]\cdot[b] = [a\cdot b]$$

- notice that $[a]+[b]=[c] \iff a+b=c \mod n$

$$[a]\cdot[b]=[c] \iff a\cdot b = c \mod n$$

- for a prime number p we denote by $\mathbb{F}_p$ the ring $\mathbb{Z}/p\mathbb{Z}$.

and show that each element $[a]$ is invertible:

- since $a<p$ and p is a prime $\Rightarrow \gcd(a,p)=1$

(Bézout's identity) $\Rightarrow \exists\, m,n \in \mathbb{Z}$ such that $ma+np=1$

$$\Rightarrow \quad ma = 1 \mod p$$

$\Rightarrow [m]\cdot[a] = 1$ i.e $[m]$ is the inverse of $[a]$

2) If V is an n-dimensional $\mathbb{F}_p$-vector space

then V is isomorphic to $\mathbb{F}_p^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{F}_p\}$

$\Rightarrow |V| = p^n$

3) • If A is an affine space over the $\mathbb{F}_p$-vector space V

the $\dim A = \dim V$

• by axiom (AS 1) there is a bijection between points in A

and vectors in V. Fix $Q \in A$ then the bijection is

$$A \ni P \longleftrightarrow \overrightarrow{QP} \in V$$

Hence $|A| = |V| = p^n$

8. Let $\mathbf{A}$ be an affine space and consider four points $A, B, C, D \in \mathbf{A}$. Show that if $\overrightarrow{AB} = \overrightarrow{CD}$ then $\overrightarrow{AC} = \overrightarrow{BD}$.

$$\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{CD} \qquad \text{(by axiom AS2)}$$

$$\Rightarrow \overrightarrow{AC} = -\overrightarrow{CB} + \overrightarrow{CD}$$

$$\Rightarrow \overrightarrow{AC} = \overrightarrow{BC} + \overrightarrow{CD} \qquad \text{(by exercise 3)}$$

$$\Rightarrow \overrightarrow{AC} = \overrightarrow{BD} \qquad \text{(by axiom AS2)}$$

9. In the affine space $\mathbf{A}^2(\mathbb{C})$ consider the line passing through the point $A(4, -2\mathbf{i})$ and having direction vector $\mathbf{a}(7 + \mathbf{i}, 1)$. Give several parametric equations for this line.

Denote by ℓ the line

$$\ell = \{ P \in \mathbb{A}^2(\mathbb{C}) : \overrightarrow{AP} = t \cdot a \quad t \in \mathbb{C} \}$$

$$\Updownarrow$$

$$\underset{\substack{\| \\ P}}{\overrightarrow{OP}} = \underset{\substack{\| \\ A}}{\overrightarrow{OA}} + t \cdot a \qquad \text{where } O = (0,0)$$

- one set of parametric equations is

$$\ell : \begin{cases} x = 4 + t(7+i) \\ y = -2i + t \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2i \end{bmatrix} + t \begin{bmatrix} 7+i \\ 1 \end{bmatrix} \quad t \in \mathbb{C}$$

- since $A + a = \begin{bmatrix} 13+i \\ 1-2i \end{bmatrix}$ is also a point on ℓ, and $i \cdot v$ is also a direction vector, another set of parametric equations is

$$\ell : \begin{cases} x = 13 + i + t(-1 + 7i) \\ y = 1 - 2i + ti \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13+i \\ 1-2i \end{bmatrix} + t \begin{bmatrix} -1+7i \\ i \end{bmatrix}$$

etc ...

10. In the affine space $\mathbf{A}^3(\mathbb{C})$ consider the plane passing through the point $A(2+\mathbf{i},5,-\mathbf{i})$ and parallel to the vectors $\mathbf{a}(2,3,1)$ and $\mathbf{b}(-1,-11,3)$. Give several parametric equations for this plane.

- Let π be this plane

$$\pi : \begin{cases} x = 2+i + 2t - s \\ y = 5 + 3t - 11s \\ z = -i + t + 3s \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+i \\ 5 \\ -i \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ -11 \\ 3 \end{bmatrix} \quad t,s \in \mathbb{C}$$

- a and $a+ib$ are also direction vectors so

$$\pi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+i \\ 5 \\ -i \end{bmatrix} + t \begin{bmatrix} 2-i \\ 3-11i \\ 1+3i \end{bmatrix} + s \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

11. In the affine space $\mathbf{A}^4(\mathbb{R})$ consider

$$\text{the plane} \quad \alpha = \left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\4\\1\\2\end{bmatrix} \quad \text{and the line} \quad \beta = \left\langle \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\3\\-1\\1\end{bmatrix}.$$

Determine $\alpha \cap \beta$.

The coordinates of a an arbitrary point in α are described by the parametric equations

$$\alpha: \quad \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}2\\4\\1\\2\end{bmatrix} + t\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \qquad \text{with } t, s \in \mathbb{R}$$

Similarly, an arbitrary point on the line β is described by

$$\beta: \quad \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}2\\3\\-1\\1\end{bmatrix} + u\begin{bmatrix}2\\3\\-1\\1\end{bmatrix} \qquad \text{with } u \in \mathbb{R}$$

A point P lies both in α and in β if there are $t, s, u \in \mathbb{R}$ such that

$$\begin{bmatrix}2\\4\\1\\2\end{bmatrix} + t\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}x_P\\y_P\\z_P\\w_P\end{bmatrix} = \begin{bmatrix}2\\3\\-1\\1\end{bmatrix} + u\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$$

So, the points $P \in \alpha \cap \beta$ correspond to the solutions of the system

$$\begin{bmatrix}2\\4\\1\\2\end{bmatrix} + t\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\3\\-1\\1\end{bmatrix} + u\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$$

$$\iff \quad t\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + u\begin{bmatrix}-1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\-1\\-2\\-1\end{bmatrix} \iff \begin{matrix} t - u = 0 \\ t + s - u = -1 \\ t + u = -2 \\ t + s - u = -1 \end{matrix}$$

the extended matrix of this system is
(augmented)

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & -2 \\ 1 & 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \iff \begin{cases} t = -1 \\ s = -1 \\ u = -1 \end{cases}$$

the system has a unique solution $(-1, -1, -1)$

$\Rightarrow$ $\alpha \cap \beta$ has a unique point which corresponds to the parameter $u = -1$ in our parametrization of β, so

$$\alpha \cap \beta = \left\{ \begin{bmatrix}1\\2\\0\\0\end{bmatrix} \right\}$$