Calculus on $\mathbb{R}$ Lect. dr. ANCA GRAD
Homework 12 **2021**

Sequences of functions

Study the pointwise convergence (by specifying the convergence set and the pointwise limit function) and the uniform convergence for the following sequences of functions:

1. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{\cos nx}{n^\alpha}$ unde $\alpha > 0$;
2. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
3. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{x^2}{x^4+n^2}$;
4. $f_n : [0,\infty) \to \mathbb{R},\ f_n(x) = \dfrac{1}{1+nx}$;
5. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$;
6. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{nx}{1+n^2x^2}$;
7. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$;
8. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = n\left(\sqrt{x+\dfrac{1}{n}} - \sqrt{x}\right)$; $f_n \rightrightarrows f$?
9. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{nx}{e^{nx^2}}$;
10. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;
11. $f_n : [-1,1] \to \mathbb{R},\ f_n(x) = \dfrac{x}{1+n^2x^2}$;

Theory

Fie $\emptyset \neq D \subseteq \mathbb{R}$. We denote by

$$\mathcal{F}(D) = \{f|\quad f : D \to \mathbb{R}\}$$

the set of all the functions defined on the set D. A **seq**
function $x : \mathbb{N}_k \to \mathcal{F}(D)$, which associates uniquely to ea
function. Thus

$$x(n) := f_n, \quad \forall n \in \mathbb{N}_k.$$

1. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n^\alpha}$ unde $\alpha > 0$;

$f_n : \mathbb{R} \to \mathbb{R} \quad f_n(x) = \dfrac{\cos nx}{n^\alpha} \quad , \ \alpha > 0 \ \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n^\alpha}(\cos nx) = 0$$

$\Rightarrow \forall x \in \mathbb{R} \ \exists \lim_{n\to\infty} f_n(x) = 0 \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

$$-1 \leq \cos(nx) \leq 1 \quad | : n^\alpha$$

$$-\frac{1}{n^\alpha} \leq \frac{\cos(nx)}{n^\alpha} \leq \frac{1}{n^\alpha}$$

$$\to 0$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0 \ \forall x \in \mathbb{R}$ $f_n \to f$

3) Choose $n \in \mathbb{N}$ random

$a_n := \sup |f_n(x) - f_n(x)| \quad \forall x \in \mathcal{C} = \mathbb{R}$

$a_n := \sup \left| \dfrac{\cos(nx)}{n^\alpha} - 0 \right| = \sup \left| \dfrac{\cos(nx)}{n^\alpha} \right|$

We define a helping function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \dfrac{\cos(nx)}{n^\alpha}$

g is differentiable on $\mathbb{R}$

$$g'(x) = \left(\frac{\cos(nx)}{n^\alpha}\right)' = \frac{1}{n^\alpha}(\cos(nx))' = \frac{1}{n^\alpha} \cdot n \cdot (-\sin nx) = -\frac{1}{n^{\alpha-1}} \sin nx$$

$\sin(x) \leq 1 \quad \Rightarrow -1 \leq \sin(nx) \leq 1 \quad \left| \cdot \dfrac{1}{n^{\alpha-1}} \right.$

$$-\frac{1}{n^{\alpha-1}} \leq \frac{\sin(nx)}{n^{\alpha-1}} \leq \frac{1}{n^{\alpha-1}} \quad | \cdot (-1)$$

$$\frac{1}{n^{\alpha-1}} \geq \frac{-\sin(nx)}{n^{\alpha-1}} \geq -\frac{1}{n^{\alpha-1}}$$

$$\Rightarrow |g(x)| \leq \frac{1}{n^{\alpha-1}} \to g(x) \leq \frac{1}{n^{\alpha-1}} \quad \forall x \in \mathbb{R},\ \alpha > 0$$

$\Rightarrow a_n := \dfrac{1}{n^{\alpha-1}} \quad , \ \alpha > 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\alpha-1}} = \begin{cases} \infty & : \alpha \in (0,1) \\ \infty^0 & : \alpha = 1 \\ 0 & : \alpha > 1 \end{cases}$$

for $\alpha = 1$

$$\lim_{n\to\infty} \frac{1}{n^{\alpha-1}} = \lim_{n\to\infty} \frac{1}{n^\alpha \cdot n^{-1}} = \lim_{n\to\infty} \left(\frac{1}{\frac{n^\alpha}{n}}\right) = \lim_{n\to\infty} \left(\frac{n}{n^\alpha}\right) = \lim_{n\to\infty} \left(\frac{n + n^\alpha - n^\alpha}{n^\alpha}\right) = \lim_{n\to\infty} \left(1 + \frac{n - n^\alpha}{n^\alpha}\right) =$$

$$= \lim_{n\to\infty} \left\{\left(1 + \frac{n-n^\alpha}{n^\alpha}\right)^{\frac{n^\alpha}{n-n^\alpha}}\right\}^{\frac{n-n^\alpha}{n^\alpha}} = e^{\lim_{n\to\infty} \frac{n-n^\alpha}{n^\alpha}} = e^{-\lim_{n\to\infty} \frac{n^\alpha - n}{n^\alpha}} =$$

$$= e^{-1} = \frac{1}{e}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\alpha-1}} = \begin{cases} \infty & : \alpha \in (0,1) \\ \frac{1}{e} & : \alpha = 1 \\ 0 & : \alpha > 1 \end{cases}$$

$\lim_{n\to\infty} a_n = 0 \iff \alpha > 1 \quad \Rightarrow$ for $\alpha > 1 : \ f_n \rightrightarrows f$

$f_n \rightrightarrows f \quad \forall \alpha > 1$

2. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$;

1) Choose $x \in [0,1]$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x(1+n^2)}{n^2} = \lim_{n\to\infty} \frac{x + x\cdot n^2}{n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}(\overset{0}{\frac{x}{n^2}} + x)}{\cancel{n^2}} = x$$

$\Rightarrow \forall x \in [0,1]$ $\lim_{n\to\infty} f_n(x) = x \in [0,1]$

$\Rightarrow \mathscr{C} = [0,1]$

2) $\mathscr{C} = [0,1] \neq \emptyset \Rightarrow$ we introduce $f : [0,1] \to \mathbb{R}$, $f(x) = x$ $\forall x \in [0,1]$, $f_n \xrightarrow{[0,1]} f$

3) Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup \left| \frac{x(1+n^2)}{n^2} - x \right| = \sup \left| \frac{x + xn^2 - xn^2}{n^2} \right| = \sup \left| \frac{x}{n^2} \right| = \sup \frac{x}{n^2}$$

$0 \le x \le 1 \quad | : n^2$

$0 \le \frac{x}{n^2} \le \frac{1}{n^2} \Rightarrow a_n := \frac{1}{n^2}$

$= \sup \frac{x}{n^2}$

$\Rightarrow a_n := \frac{1}{n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0 \Rightarrow f_n \rightrightarrows f$

3. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{x^2}{x^4+n^2}$;

$\forall n \in \mathbb{R}$,

$f_n : \mathbb{R} \to \mathbb{R}$

$f_n(x) = \dfrac{x^2}{n^2+x^4}, \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ random
$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x^2}{n^2+x^2} = 0$ $\Bigg\} \Rightarrow \exists \lim\limits_{n\to\infty} f_n(x) \in \mathbb{R} \ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Longleftrightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$ $f(x) = 0$ $\forall x \in \mathbb{R}$, $f_n \to f$

3) $\forall n \in \mathbb{N}$, $a_n := \sup |f_n(x) - f(x)|$ $\forall x \in \mathcal{C}$

$\hookrightarrow$ Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{\forall x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| \frac{x^2}{n^2+x^4} - 0 \right| = \sup_{x\in\mathbb{R}} \left| \frac{x^2}{n^2+x^4} \right|$$

- We define a helping function $g : \mathbb{R} \to \mathbb{R}$
  $g(x) = \dfrac{x^2}{n^2+x^4}, \ \forall x \in \mathbb{R}$

- It is differentiable on $\mathbb{R}$

$$g'(x) = \left(\frac{x^2}{n^2+x^4}\right)' = \frac{2x(n^2+x^4) - 4x^3 \cdot x^2}{(n^2+x^4)^2} \quad \forall x \in \mathbb{R}$$

$$= \frac{2n^2x + 2x^5 - 4x^5}{(n^2+x^4)^2} = \frac{2n^2x - 2x^5}{(n^2+x^4)^2}$$

$$2n^2x - 2x^5 = 2x(n^2 - x^4) = 2x(n-x^2)(n+x^2) = 2x(\sqrt{n}-x)(\sqrt{n}+x)\underbrace{(n+x^2)}_{>0}$$

x	$-\infty$		$-\sqrt{n}$		0		$\sqrt{n}$		∞
x		$-\ -\ -$	$-$	$-\ -$	0	$+\ +$	$+$	$+\ +\ +$	
$\sqrt{n}-x$		$+\ +\ +$	$+$	$+\ +$	$+$	$+\ +$	0	$-\ -\ -$	
$\sqrt{n}+x$		$-\ -\ -$	0	$+\ +$	$+$	$+\ +$	$+$	$+\ +\ +$	
g'		$+\ +\ +$	0	$-\ -$	0	$+\ +$	0	$-\ -\ -$	
g		$\nearrow$	$g(-\sqrt{n})$	$\searrow$	$g(0)$	$\nearrow$	$g(\sqrt{n})$	$\searrow$	

$$g(-\sqrt{n}) = \frac{(-\sqrt{n})^2}{n^2+(-\sqrt{n})^2} = \frac{1}{2n} \qquad g(\sqrt{n}) = \frac{1}{2n}$$

$$\Rightarrow g(x) \leq \frac{1}{2n} \quad \forall x \in \mathbb{R}$$

$$a_n := \frac{1}{2n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n} = 0 \xRightarrow[\text{theorem}]{\text{Weierstrass}} f_n \rightrightarrows f$$

4. $f_n : [0,\infty) \to \mathbb{R},\ f_n(x) = \dfrac{1}{1+nx}$;

$\forall n \in \mathbb{N}$,
$f_n : [0,\infty) \to \mathbb{R}$,
$f_n(x) = \dfrac{1}{1+nx}\ \forall x \in [0,\infty)$

1) Choose $x \in [0,\infty)$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \underset{\left(\frac{1}{\infty_+}=0\right)}{\frac{1}{1+nx}} = \begin{cases} 1 : x = 0 \\ 0 : x \neq 0 \end{cases} \Rightarrow \exists \lim_{n\to\infty} f_n(x) \in \mathbb{R}\ \forall x \in [0,\infty) \Rightarrow \mathcal{C} = [0,\infty)$$

2) $\mathcal{C} = [0,\infty) \neq \emptyset \Rightarrow$ we introduce $f : [0,\infty) \to \mathbb{R}$, $f(x) = \begin{cases} 1 : x = 0 \\ 0 : x \neq 0 \end{cases}$, $f_n \xrightarrow[{[0,\infty)}]{\mathcal{C}} f$

3) $\left.\begin{array}{l} \forall n \in \mathbb{N}\ f_n \text{ is continuous} \\ f \text{ is not continuous at } 0 \end{array}\right\} \Rightarrow f_n \not\rightrightarrows f$

5. $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$;

c) $\forall n \in \mathbb{N}$
$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}} \quad \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ randomly

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \begin{cases} \frac{\infty}{\infty} \text{ or } \frac{-\infty}{\infty} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

if $x \neq 0 \Rightarrow \lim\limits_{n\to\infty} \dfrac{2n^2x}{e^{n^2x^2}} = \lim\limits_{n\to\infty} \dfrac{1}{x} \cdot \dfrac{2n^2x^2}{e^{n^2x^2}} = \dfrac{1}{x} \cdot 0 = 0$

$t = n^2x^2$

$$\lim_{t\to\infty} \frac{2t}{e^t} \overset{\frac{\infty}{\infty}}{\underset{\text{l'Hospital}}{=}} \lim_{t\to\infty} \frac{2}{e^t} = 0$$

$\Rightarrow \lim\limits_{n\to\infty} f_n(x) = 0 \quad \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

2) $C = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0\ \forall x \in \mathbb{R}$, $f_n \to f$

3) Choose $n \in \mathbb{N}$ randomly

$$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \left| \frac{2n^2x}{e^{n^2x^2}} \right| = \sup_{x\in\mathbb{R}} \frac{|2n^2x|}{e^{n^2x^2}} = \sup_{x\in\mathbb{R}} \frac{2n^2|x|}{e^{n^2x^2}}$$

We define the function $g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \frac{2n^2|x|}{e^{n^2x^2}} = \begin{cases} \frac{2n^2x}{e^{n^2x^2}} : x \geq 0 \\ -\frac{2n^2x}{e^{n^2x^2}} : x < 0 \end{cases}$$

$$\left.\begin{array}{l} \lim\limits_{\substack{x\to 0 \\ x>0}} g(x) = 0 = g(0) = \lim\limits_{\substack{x\to 0 \\ x<0}} g(x) \Rightarrow g \text{ is continuous at } 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \mathbb{R}\setminus\{0\} \end{array}\right\} \Rightarrow$$

$\Rightarrow g$ is continuous on $\mathbb{R} \Rightarrow g$ is differentiable on $\mathbb{R}\setminus\{0\}$

$x>0: \quad g'(x) = 2n^2 \cdot \left(\frac{x}{e^{n^2x^2}}\right)' = 2n^2 \cdot \frac{\cancel{e^{n^2x^2}} - 2n^2x \cdot \cancel{e^{n^2x^2}} \cdot x}{e^{\cancel{2}n^2x^2}} =$

$$= \frac{2n^2(1-2n^2x^2)}{e^{n^2x^2}}$$

$x<0: \quad g'(x) = -\frac{2n^2(1-2n^2x^2)}{e^{n^2x^2}}$

x		$-\frac{1}{n\sqrt{2}}$		0		$\frac{1}{n\sqrt{2}}$	
x	$-\,-\,-$	$-$	$-\,-$	0	$+\,+\,+$	$+$	$+\,+\,+$
$1-2n^2x^2$	$-\,-\,-$	0	$+\,+\,+$	$+$	$+\,+$	0	$-\,-\,-$
g'	$+\,+\,+$	0	$-\,-$	$\vert$	$+\,+\,+$	0	$-\,-\,-$
g	$\nearrow$	0	$\searrow$	$\vert$	$\nearrow$	0	$\searrow$

g is not differentiable at 0

$\Rightarrow -\frac{1}{n\sqrt{2}}$ and $\frac{1}{n\sqrt{2}}$ are the only options for local maximum

$\Rightarrow a_n = g(1) = \frac{1}{\sqrt{2}}$

$\Rightarrow a_n := \frac{1}{\sqrt{2}} \quad \forall n \in \mathbb{N}$

$\lim\limits_{n\to\infty} a_n = \frac{1}{\sqrt{2}} \neq 0 \Rightarrow f_n \not\rightrightarrows f$

6. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{1+n^2x^2}$;

$\forall n \in \mathbb{N}$,
$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = \frac{nx}{1+n^2x^2} \quad \forall x \in \mathbb{R}$

1) Choose $x \in \mathbb{R}$ randomly

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$

$\Rightarrow \exists \lim\limits_{n\to\infty} f_n(x) = 0 \in \mathbb{R} \quad \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f: \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in \mathbb{R} \quad f_n \xrightarrow{\mathcal{C}} f$

3) Choose $n \in \mathbb{N}$ random

$a_n := \sup |f_n(x) - f(x)| = \sup \left|\frac{nx}{1+n^2x^2}\right| = \sup \frac{|nx|}{1+n^2x^2}$

We define the function $g: \mathbb{R} \to \mathbb{R} \quad g(x) = \frac{|nx|}{1+n^2x^2} = \begin{cases} -\frac{nx}{1+n^2x^2} & : x<0 \\ \frac{nx}{1+n^2x^2} & : x \geq 0 \end{cases}$

$\lim\limits_{\substack{x\to 0\\ x>}} g(x) = 0 = g(0) = \lim\limits_{\substack{x\to 0\\ x<0}} g(x)$

$\Rightarrow g$ is continuous at 0 and on $\mathbb{R}\setminus\{0\}$

$\Rightarrow f$ is continuous on $\mathbb{R}$

$\Rightarrow g$ is differentiable on $\mathbb{R}\setminus\{0\}$

$\underline{x>0}$

$$g'(x) = \left(\frac{nx}{1+n^2x^2}\right)' = n\left(\frac{x}{1+n^2x^2}\right)' = n\cdot\frac{1+n^2x^2 - x\,n^2 2x}{(1+n^2x^2)^2} =$$

$$= \frac{n(1+n^2x^2-2n^2x^2)}{(1+n^2x^2)^2} =$$

$$= \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = \frac{n(1-nx)(1+nx)}{(1+nx^2)^2}$$

$\underline{x<0}$

$$g'(x) = \frac{-n(1-nx)(1+nx)}{(1+n^2x^2)^2}$$

g is not differentiable on 0

$1-nx=0 \Rightarrow 1=nx \Rightarrow x=\frac{1}{n}$

x	$+\infty$		$-\frac{1}{n}$		0		$\frac{1}{n}$		$-\infty$
$1-nx$		$+$	$+$	$+$	$+$	$+$	0	$-$	
$1+nx$		$-$	0	$+$	$+$	$+$	$+$	$+$	
g'		$-$	0	$+$	|	$+$	0	$-$	
g		$\searrow$	$g(-\frac{1}{n})$	$\nearrow$	0	$\nearrow$	$g(\frac{1}{n})$	$\searrow$	

$$g\left(-\frac{1}{n}\right) = \frac{-n\cdot -\frac{1}{n}}{1+n^2\left(\frac{1}{n}\right)^2} = \frac{1}{1+n^2\cdot\frac{1}{n^2}} = \frac{1}{2}$$

$g\left(\frac{1}{n}\right) = \frac{1}{2} \Rightarrow$ local maximum

$\Rightarrow a_n = g\left(\frac{1}{n}\right) = \frac{1}{2}$

$\forall n\in\mathbb{N}\ a_n := \frac{1}{2}$

$\lim\limits_{n\to\infty} a_n = \frac{1}{2} \neq 0 \Rightarrow f_n \not\rightrightarrows f$

0

7. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$;

$$\left.\begin{array}{l} \text{1) Choose } x \in \mathbb{R} \text{ random} \\ \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \left(\sqrt{x^2+\frac{1}{n^2}}\right) = \lim\limits_{n\to\infty} \sqrt{x^2+0} = x \end{array}\right\} \Rightarrow \exists \lim_{n\to\infty} f_n(x) = x \in \mathbb{R} \ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = x \quad \forall x \in \mathbb{R} \quad f_n \to f$

3) Choose $n \in \mathbb{N}$ random

$$a_n := \sup |f_n(x) - f(x)| = \sup \left| \sqrt{x^2+\frac{1}{n^2}} - x \right| = \sup \left| \sqrt{\frac{n^2x^2+1}{n^2}} - x \right| = \sup \left| \frac{\sqrt{n^2x^2+1}}{n} - x \right| =$$

$$= \sup \left| \frac{\sqrt{n^2x^2+1} - nx}{nx} \right|$$

We define the function $g : \mathbb{R} \to \mathbb{R}$ $g(x) = \sqrt{x^2+\frac{1}{n^2}} - x$

$$g'(x) = \frac{1}{\cancel{2}\sqrt{x^2+\frac{1}{n^2}}} \cdot \cancel{2}x - 1 = \frac{x - \sqrt{x^2+\frac{1}{n^2}}}{\sqrt{x^2+\frac{1}{n^2}}} < 0$$

$$\Rightarrow g'(x) < 0 \Rightarrow g(0) \geq g(x)$$

$$\sqrt{0^2+\frac{1}{n^2}} - 0 \geq g(x)$$

$$\frac{1}{n} \geq g(x)$$

$$\left.\begin{array}{l} a_n := \frac{1}{n} \quad \forall n \in \mathbb{N} \\ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \Rightarrow f_n \rightrightarrows f$$

8. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right)$:

$f_n : \mathbb{R} \to \mathbb{R}$,
$f_n(x) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right)$

1) Choose $x \in \mathbb{R}$ random

$$\left.\begin{aligned} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right) = \lim_{n\to\infty} \frac{n\cdot\left(\cancel{x}+\frac{1}{\cancel{n}}-\cancel{x}\right)}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} = \lim_{n\to\infty} \frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}} = \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{x}+\sqrt{x}} = \lim_{n\to\infty} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \exists \lim_{n\to\infty} f_n(x) = \frac{1}{2\sqrt{x}} \in \mathbb{R} \quad \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = \frac{1}{2\sqrt{x}}$ $\quad f_n \to f$

3) Choose $n \in \mathbb{N}$ random

$$a_n := \sup |f_n(x) - f(x)| = \sup\left| n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right) - \frac{1}{2\sqrt{x}}\right| = \sup\left| \overset{2\sqrt{x}}{\frac{1}{\sqrt{x+\frac{1}{n}}+\sqrt{x}}} - \overset{2\sqrt{x+\frac{1}{n}}+\sqrt{x}}{\frac{1}{2\sqrt{x}}}\right| =$$

$$= \sup\left| \frac{2\sqrt{x} - 2\sqrt{x+\frac{1}{n}} - \sqrt{x}}{2\sqrt{x}\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)}\right| = \sup\left| \frac{\sqrt{x} - 2\sqrt{x+\frac{1}{n}}}{2\sqrt{x}\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)}\right|$$

9. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{e^{nx^2}}$;

$f_n : [0,1] \to \mathbb{R}$

$f_n(x) = \dfrac{nx}{e^{nx^2}}$

1) Choose $x \in [0,1]$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{1}{x}\cdot\frac{nx^2}{e^{nx^2}} = \lim_{n\to\infty} \frac{1}{x}\cdot 0 = 0 \quad \Bigg\} \Rightarrow \exists \lim_{n\to\infty} f_n(x) = 0 \in \mathbb{R} \ \forall x \in \mathbb{R} \Rightarrow \mathcal{C} = \mathbb{R}$$

$t = nx^2$

$$\lim_{t\to\infty} \frac{t}{e^t} \overset{\frac{\infty}{\infty}}{\underset{\text{L'Hospital}}{=}} \lim_{n\to\infty} \frac{1}{e^t} = 0$$

2) $\mathcal{C} = \mathbb{R} \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{\mathbb{R}}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0 \ \forall x \in \mathbb{R}$ $\quad f_n \xrightarrow{\mathcal{C}} f$

3) Choose $n \in \mathbb{N}$ random

$$a_n := \sup |f_n(x) - f(x)| = \sup \left|\frac{nx}{e^{nx^2}}\right| = \sup \frac{n|x|}{e^{nx^2}}$$

We introduce the function $g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = \dfrac{n|x|}{e^{nx^2}} = \begin{cases} -\dfrac{nx}{e^{nx^2}} & : x < 0 \\ \dfrac{nx}{e^{nx^2}} & : x \geq 0 \end{cases}$

$$\lim_{\substack{x\to 0\\ x>0}} g(x) = 0 = g(0) = \lim_{\substack{x\to 0\\ x<0}} g(x)$$

$\Rightarrow$ g is continuous at 0 and $\mathbb{R}\setminus\{0\}$

$\Rightarrow$ g is continous on $\mathbb{R}$

$\Rightarrow$ g is differentiable on $\mathbb{R}\setminus\{0\}$

$x > 0$

$$g'(x) = n\cdot\left(\frac{x}{e^{nx^2}}\right)' = n\cdot\frac{e^{nx^2} - x\cdot e^{nx^2}\cdot n\cdot 2x}{e^{2nx^2}} = n\cdot\frac{\cancel{e^{nx^2}}(1-n2x^2)}{e^{2\cancel{nx^2}}} = \frac{n(1-2nx^2)}{e^{nx^2}} =$$

$$= \frac{n(1-\sqrt{2n}\,x)(1+\sqrt{2n}\,x)}{e^{nx^2}}$$

$x < 0$

$$g'(x) = -\frac{n(1-\sqrt{2n}\,x)(1+\sqrt{2n}\,x)}{e^{nx^2}}$$

g is not differentiable at 0

x		$-\frac{1}{\sqrt{2n}}$		0		$\frac{1}{\sqrt{2n}}$	
$1-\sqrt{2n}x$	$+ + + + +$	0	$- - -$	$- -$	$- - -$	$-$	$- -$
$1+\sqrt{2n}x$	$- - - -$	$-$	$- - -$	$- -$	$- - -$	0	$+ + +$
g'	$- - -$	0	$+ + +$	$\vert$	$+ + +$	0	$- - -$
g	$\searrow$	$g\left(-\frac{1}{\sqrt{2n}}\right)$	$\nearrow$	0	$\nearrow$	$g\left(\frac{1}{\sqrt{2n}}\right)$	$\searrow$

$$g\left(\frac{1}{\sqrt{2n}}\right) = \frac{n\cdot\frac{1}{\sqrt{2n}}}{e^{n\cdot\left(\frac{1}{\sqrt{2n}}\right)^2}} = \frac{\sqrt{n}}{2}\cdot\frac{1}{e^{n\cdot\frac{1}{2n}}} = \frac{\sqrt{n}}{2\sqrt{e}} = \sqrt{\frac{n}{4e}} \Rightarrow \text{local maximum}$$

$$\Rightarrow a_n := \sqrt{\frac{n}{4e}} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\sqrt{\frac{n}{4e}} = \infty \neq 0 \Longrightarrow f_n \not\rightrightarrows f$$

10. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x(1+n^2)}{n^2}$;

$f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x(1+n^2)}{n^2}$

1) Choose $x \in [0,1]$ randomly
$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{x + x n^2}{n^2} = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$ $\Rightarrow \exists \lim_{n\to\infty} f_n(x) \in \mathbb{R}\ \forall x \in [0,1] \Rightarrow \mathcal{C} = [0,1]$

2) $\mathcal{C} = [0,1] \neq \emptyset \Rightarrow$ we introduce $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x=0 \\ 1, & x \neq 0 \end{cases}$ $\forall x \in [0,1]$ $\quad f_n \xrightarrow{[0,1]} f$

3) $\left.\begin{array}{l} \forall n \in \mathbb{N}\ f_n \text{ is continuous} \\ f \text{ is not continuous at } 0 \end{array}\right\} \Rightarrow f_n \not\rightrightarrows f$

11. $f_n : [-1,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{1+n^2x^2}$;

$f_n : [-1,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{1+n^2x^2}$

1) Choose $x \in [-1,1]$ random
$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{x}{1+n^2x^2} = 0$ $\Bigg\} \Rightarrow \exists \lim_{n\to\infty} f_n(x) = 0 \in \mathbb{R}\ \forall x \in [-1,1] \Rightarrow \mathcal{C} = [-1,1]$

2) $\mathcal{C} = [-1,1] \neq \emptyset \Rightarrow$ we introduce $f : \underbrace{[-1,1]}_{=\mathcal{C}} \to \mathbb{R}$, $f(x) = 0$ $\forall x \in [-1,1]$ $\quad f_n \xrightarrow{[-1,1]} f$

3) Choose $n \in \mathbb{N}$ random
$a_n := \sup |f_n(x) - f(x)| = \sup |f_n(x)| = \sup \left| \dfrac{x}{1+x^2n^2} \right| = \sup \dfrac{|x|}{1+x^2n^2}$

We introduce the function $g : [-1,1] \to \mathbb{R}$, $g(x) = \begin{cases} \dfrac{-x}{1+x^2n^2} & : x < 0 \\ \dfrac{x}{1+x^2n^2} & : x \geq 0 \end{cases}$

$\lim_{\substack{x\to 0 \\ x>0}} g(x) = 0 = g(0) = \lim_{\substack{x\to 0 \\ x<0}} g(x)$

$\Rightarrow g$ is continuous at 0 and on $[-1,1] \setminus \{0\}$

$\Rightarrow g$ is continuous on $[-1,1]$

$\Rightarrow g$ is differentiable on $[-1,1] \setminus \{0\}$

$$g'(x) = \frac{x'(1+x^2n^2) - x(1+x^2n^2)'}{(1+x^2n^2)^2} = \frac{1+x^2n^2 - x(2xn^2)}{(1+x^2n^2)^2} = \frac{1+x^2n^2-2x^2n^2}{(1+x^2n^2)} = \frac{1-x^2n^2}{(1+x^2n^2)^2} =$$
$$= \frac{(1-xn)(1+xn)}{(1+x^2n^2)^2}$$

x	$-\infty$		$-\frac{1}{n}$		0		$\frac{1}{n}$		$+\infty$
$1-nx$		+ + + +	+	+ + +	+	+ + +	0	− − −	
$1+nx$		− − −	0	+ + +	+	+ + +	+	+ + +	
g'		− − −	0	+ +	0	+ + +	0	− − −	
g		$\searrow$	$g(-\frac{1}{n})$	$\nearrow$	0	$\nearrow$	$g(\frac{1}{n})$	$\searrow$	

$$g\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{1 + \frac{1}{\cancel{n^2}}\cancel{n^2}} = \frac{\frac{1}{n}}{1+1} = \frac{1}{2n}$$

$$g(x) \leq g\left(\frac{1}{n}\right)$$

$$\left.\begin{array}{l} a_n := \frac{1}{2n} \\ \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{2n} = 0 \end{array}\right\} \Longrightarrow f_n \rightrightarrows f$$