ANALYTIC GEOMETRY, PROBLEM SET 2

More polar coordinates

1. Let $ABCDEF$ be a regular hexagon with size length l. Find the polar coordinates of its vertices in each of the following cases:

a) The center of the hexagon O is chosen as *the pole* and the half-line $[OA$ is set as *the polar axis*.

b) The vertex A is chosen as *the pole* and the half-line $[AB$ is set as *the polar axis*.

2. Find the polar equation corresponding to the given Cartesian equation: a) $y = 5$; b) $x + 1 = 0$; c) $y = 7x$; d) $3x + 8y + 6 = 0$; e) $y^2 = -4x + 4$; f) $x^2 - 12y - 36 = 0$; g) $x^2 + y^2 = 36$; h) $x^2 - y^2 = 25$. Briefly give a geometric interpretation for the solutions to these equations.

3. Find the polar coordinates of the point $P \in \mathcal{E}_2$, whose rectangular (Cartesian) coordinates are $(1 + \cos\alpha, \sin\alpha)$, where $\alpha \in (0, 2\pi)$ is fixed.

Cylindrical and spherical (everything is in 3D here)

Warmp-up 1. In the cylindrical coordinate system, what do the following equations represent in $\mathcal{E}_3$?

a) $r = r_0$, where $r_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed;
c) $z = z_0$, where $z_0 \in \mathbb{R}$ is fixed.

Warm-up 2. In the spherical coordinate system, what do the following equations represent in $\mathcal{E}_3$?

a) $\rho = \rho_0$, where $\rho_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed;
c) $\varphi = \varphi_0$, where $\varphi_0 \in [0, \pi]$ is fixed.

4. Let $P_1(r_1, \theta_1, z_1)$ and $P_2(r_2, \theta_2, z_2)$ be points in $\mathcal{E}_3$ expressed using their cylindrical coordinates. Find the distance P_1P_2, as an expression of r_i, θ_i, z_i, where $i \in \{1, 2\}$.

5. Let $P_1(r_1, \theta_1, \varphi_1)$ and $P_2(r_2, \theta_2, \varphi_2)$ be points in $\mathcal{E}_3$, expressed using their spherical coordinates. Find the distance P_1P_2, as an expression of r_i, θ_i, φ_i, where $i \in \{1, 2\}$.

6. Determine, in cylindrical coordinates, the equation of the surface whose equation in rectangular coordinates is $z = x^2 + y^2 - 2x + y$.

7. Find the equation, in rectangular coordinates, of the surface whose equation in cylindrical coordinates is $r = 4\cos(\theta)$. Explain what the equation describes geometrically.

8. (Non-examinable) Three spheres are pairwise exterior tangent; a plane is tangent to these spheres at points A, B and C. Find the radii of the spheres in terms of a, b, c, representing the lengths of the sides of triangle ABC.

Date: October 2, 2021.

Time permitting...

9. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}=\overline{0}$.

10. In the plane determined by the triangle ABC, let us consider the points M, N, P, Q such that $\overrightarrow{AM}=\frac{2}{3}\overrightarrow{AB}$, $2\overrightarrow{NA}+\overrightarrow{NC}=\overrightarrow{0}$, $\overrightarrow{AP}=\frac{2}{5}\overrightarrow{AB}$ and $3\overrightarrow{QA}+2\overrightarrow{QB}+\overrightarrow{QC}=\overrightarrow{0}$.

(1) Find $\alpha\in\mathbb{R}$ such that $\overrightarrow{QN}=\alpha\cdot\overrightarrow{QM}$.

(2) Find $\beta\in\mathbb{R}$ such that $\overrightarrow{CQ}=\beta\cdot\overrightarrow{QP}$.

(3) Find the value of the ratio $\frac{QA}{QR}$, where $AQ\cap BC=\{R\}$.

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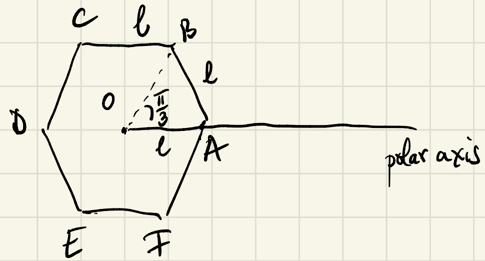
a)



(r, θ)

PS

$A(\ell, 0)$

$B(\ell, \frac{\pi}{3})$

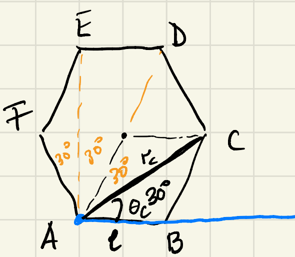
$C(\ell, \frac{2\pi}{3})$

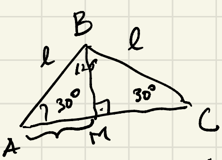
$D(\ell, \pi)$

$E(\ell, \frac{4\pi}{3})$

$F(\ell, \frac{5\pi}{3})$

b)





$AC = ?$

$AM = ?$

$\cos \hat{A} = \frac{AM}{AB}$

$\frac{\sqrt{3}}{2} = \frac{AM}{\ell} \Rightarrow AM = \frac{\sqrt{3}}{2}\ell \Rightarrow AC = \sqrt{3}\ell$

PS

$A(0, 0)$

$B(\ell, 0)$

$C(\sqrt{3}\ell, \frac{\pi}{6})$

$D(2\ell, \frac{\pi}{3})$

$E(\sqrt{3}\ell, \frac{\pi}{2})$

$F(\ell, 4\frac{\pi}{6})$

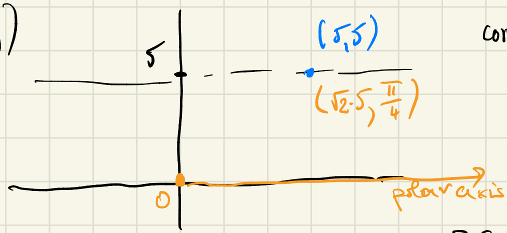
....

2. Find the polar equation corresponding to the given Cartesian equation: a) $y = 5$; b) $x + 1 = 0$; c) $y = 7x$; d) $3x + 8y + 6 = 0$; e) $y^2 = -4x + 4$; f) $x^2 - 12y - 36 = 0$; g) $x^2 + y^2 = 36$; h) $x^2 - y^2 = 25$. Briefly give a geometric interpretation for the solutions to these equations.

polar equations are eq. involving PS

Cartesian — " — — " — — " — RS

a) $y=5$ $\quad P(x,y)$



corresp. eq. in PS

$r\sin\theta = 5$

check. $\sqrt{2}\cdot 5 \cdot \frac{\sqrt{2}}{2} = 5$ ✓

R.S. $\qquad\qquad$ P.S

$P(x,y) \longleftrightarrow P(r,\theta)$

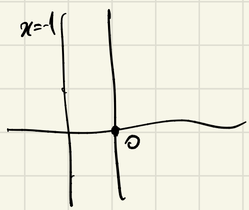
$(*)\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\qquad$ $r = \sqrt{x^2+y^2}$

$\theta = \arctan\frac{y}{x} + k\pi$ where k... (see lect.)

b.) $x+1=0$ in RS has corresponding eq in PS, $\underbrace{r\cos\theta = -1}_{\text{by } (*)}$,

$\Updownarrow$

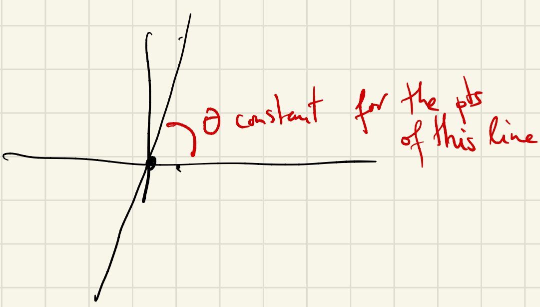
$x=-1$



c.) $y = 7x \rightsquigarrow r\sin\theta = 7r\cos\theta \iff \sin\theta = 7\cos\theta$

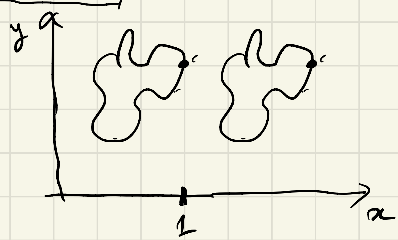
$\iff \frac{\sin\theta}{\cos\theta} = 7$

$\iff \tan\theta = 7$

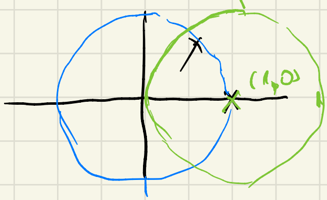


3. Find the polar coordinates of the point $P \in \mathcal{E}_2$, whose rectangular (Cartesian) coordinates are $(1+\cos\alpha, \sin\alpha)$, where $\alpha \in (0, 2\pi)$ is fixed.

$\mathcal{E}_2 \supseteq \{P(1+\cos\alpha, \sin\alpha) : \alpha \in (0, 2\pi)\}$ = " pts of the form $(\cos\alpha, \sin\alpha)$ to which $\underbrace{\text{we add } 1}$ to the first coord"



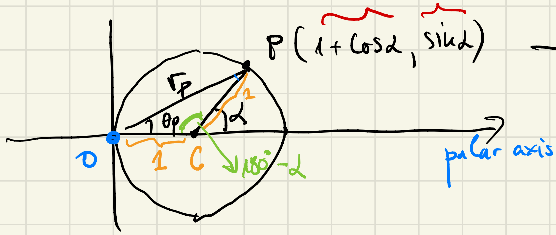
$\mathcal{E}_2 \supseteq \{Q(\cos\alpha, \sin\alpha) : \alpha \in (0, 2\pi)\}$



circle of radius 1

$$OP = \sqrt{(x_P - \underset{0}{\underset{\shortparallel}{x_O}})^2 + (y_P - \underset{0}{\underset{\shortparallel}{y_O}})^2}$$

$P(\underbrace{1+\cos\alpha}_{x_P}, \underbrace{\sin\alpha}_{y_P}) \longrightarrow$



$$r_P = \sqrt{(1+\cos\alpha)^2 + (\sin\alpha)^2}$$

$$r_P = \sqrt{1 + 2\cos\alpha + \underbrace{(\cos\alpha)^2 + (\sin\alpha)^2}_{=1}}$$

$$r_P = \sqrt{2}\sqrt{1+\cos\alpha}$$

$$\arcsin\left(\frac{y_P}{r_P}\right) = \arcsin\left(\frac{\sin\alpha}{\sqrt{2}\sqrt{1+\cos\alpha}}\right) = \ldots$$

$$\arccos\left(\frac{x_P}{r_P}\right) = \arccos\left(\frac{1+\cos\alpha}{\sqrt{2}\sqrt{1+\cos\alpha}}\right) = \ldots$$

ΔOPC isoscel: $180° - 2\theta_P = 180° - \alpha \Rightarrow \theta_P = \dfrac{\alpha}{2}$

$$\Rightarrow P\left(\sqrt{2}\sqrt{1+\cos\alpha}, \frac{\alpha}{2}\right)$$

Warm-up 2. In the spherical coordinate system, what do the following equations represent in $\mathcal{E}_3$?

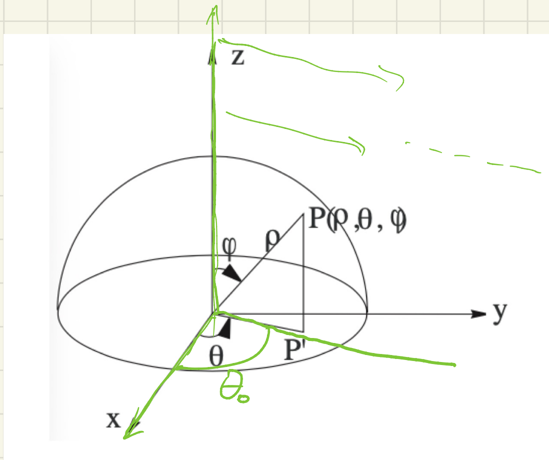
a) $\rho = \rho_0$, where $\rho_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed;
c) $\varphi = \varphi_0$, where $\varphi_0 \in [0, \pi]$ is fixed.

a) ρ is constant equal to $\rho_0 \in \mathbb{R}_{\geq 0}$ $\qquad \rho = \rho_0$

$\rightsquigarrow$ sphere of radius ρ_0

b.) $\theta = \theta_0$ for some $\theta_0 \in [0, 2\pi)$

half plan



5. Let $P_1(r_1, \theta_1, \varphi_1)$ and $P_2(r_2, \theta_2, \varphi_2)$ be points in $\mathcal{E}_3$, expressed using their spherical coordinates. Find the distance P_1P_2, as an expression of r_i, θ_i, φ_i, where $i \in \{1, 2\}$.

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad \text{if } P_i = (x_i, y_i, z_i) \text{ in RS}, \; i = 1,2$$

$$x_2 = \rho_2 \cos\theta_2 \sin\varphi_2, \; y = \rho \sin\theta \sin\varphi, \; z = \rho \cos\varphi$$

$$(P_1P_2)^2 = (\underbrace{\rho_1 \cos\theta_1 \sin\varphi_1}_{x_1} - \underbrace{\rho_2 \cos\theta_2 \sin\varphi_2}_{x_2})^2 + (\rho_1 \sin\theta_1 \sin\varphi_1 - \rho_2 \sin\theta_2 \sin\varphi_2)^2 +$$

$$+ (\rho_1 \cos\varphi_1 + \rho_2 \cos\varphi_2)$$

$$(P_1P_2)^2 = \rho_1^2 \cos\theta_1^2 \sin\varphi_1^2 - 2\rho_1\rho_2 \cos\theta_1 \sin\varphi_1 \cos\theta_2 \sin\varphi_2 + \rho_2^2 \cos\theta_2^2 \sin\varphi_2^2$$

$$+ \rho_1^2 \sin\theta_1^2 \sin\varphi_1^2 - 2\rho_1\rho_2 \sin\theta_1 \sin\varphi_1 \sin\theta_2 \sin\varphi_2 + \rho_2^2 \sin\theta_2^2 \sin\varphi_2^2$$

$$+ (\rho_1 \cos\varphi_1 + \rho_2 \cos\varphi_2)^2$$

$$= \rho_1^2 \sin\varphi_1^2 - 2\rho_1\rho_2 \sin\varphi_1 \sin\varphi_2 \Big(\overbrace{\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2}^{\cos(\theta_1-\theta_2)}\Big)$$

$$+ \rho_2^2 \sin\varphi_2^2$$

$$+ (\ldots)$$

$$P_1P_2 = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \left(\sin\varphi_1 \sin\varphi_2 \cos(\theta_1 - \theta_2) + \cos\varphi_1 \cos\varphi_2 \right)}$$

$$\shortparallel$$

$$P_2P_1$$