1. Show that the congruency of matrices is an equivalence relation in $\mathrm{Mat}_{n\times n}(\mathbf{K})$.

2. Which of the following are bilinear forms on $\mathbb{R}^n$?

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i |y_i|$
2. $\langle \mathbf{x}, \mathbf{y} \rangle = |\sum_{i=1}^n x_i y_i|$
3. $\langle \mathbf{x}, \mathbf{y} \rangle = (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)$
4. $\langle \mathbf{x}, \mathbf{y} \rangle = \sqrt{\sum_{i=1}^n x_i^2 y_i^2}$
5. $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n (x_i + y_i)^2 - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2$

3. In each of the following, determine the polar bilinear form associated to the given quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$.

1. $q(x,y) = 3x^2 - 8xy - 3y^2$
2. $q(x,y) = 4x^2 - 9xy + 5y^2$
3. $q(x,y) = 6xy$

4. Determine the matrix and the rank of each of the quadratic forms from the previous exercise.

5. In each of the following, determine the polar bilinear form associated to the given quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$.

1. $q(x,y,z) = xz + xy + yz$
2. $q(x,y,z) = 2xy + y^2 - 2xz$
3. $q(x,y,z) = -x^2 - 4xy + 3y^2 + 2z^2$

6. Determine the matrix and the rank of each of the quadratic forms from the previous exercise.

7. Let b be a symmetric bilinear form on a vector space $\mathbf{V}$ and let S be a non-empty subset of $\mathbf{V}$. Show that $S^\perp$ is a vector subspace of $\mathbf{V}$.

8. In each of the following cases find a basis with respect to which the given quadratic form on R^3 is in normal form and calculate the signatures:

1. $4x^2 - 5y^2 + 12z^2$
2. $-x^2 + 9z^2$
3. $-x^2 - y^2 + z^2$
4. $y^2 + 16z^2$

9. Diagonalize each of the quadratic forms of exercise 3, determining the change of coordinates required, and the signatures of the forms.

10. For each of the forms of the preceding exercise, express the matrix B of the diagonalized form as $B = M^t AM$, where A is the matrix of the given form.

11. Diagonalize each of the quadratic forms of exercise 5, determining the change of coordinates required, and the signatures of the forms.

12. For each of the forms of the preceding exercise, express the matrix B of the diagonalized form as $B = M^t AM$, where A is the matrix of the given form.

1. Show that the congruency of matrices is an equivalence relation in $\mathrm{Mat}_{n\times n}(\mathbf{K})$.

we use this symbol to indicate that two matrices are congruent

(last week we used the same symbol for two matrices which are similar)

$$A \sim B \iff \exists M : \quad B = M^t A M$$

- reflexivity : $A = I_n^t A I_n$
- symmetry : $A \sim B \Rightarrow (M^t)^{-1} \big| \; B = M^t A M \; \big| \cdot M^{-1}$

$$(M^{-1})^t B M^{-1} = A \quad \Rightarrow B \sim A$$

- transitivity : $A \sim B$ and $B \sim C \Rightarrow \exists M, N : B = M^t A M \quad C = N^t B N$

$$\Rightarrow C = N^t M^t A M N = (MN)^t A (MN)$$

$$\Rightarrow C \sim A$$

2. Which of the following are bilinear forms on $\mathbb{R}^n$?

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i |y_i|$
2. $\langle \mathbf{x}, \mathbf{y} \rangle = |\sum_{i=1}^n x_i y_i|$
3. $\langle \mathbf{x}, \mathbf{y} \rangle = (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)$
4. $\langle \mathbf{x}, \mathbf{y} \rangle = \sqrt{\sum_{i=1}^n x_i^2 y_i^2}$
5. $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n (x_i + y_i)^2 - \sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2$

1. $\langle x, y \rangle$ is not linear in the second argument since

$$\langle x, -y \rangle = \sum_{i=1}^n x_i |-y_i| = \sum_{i=1}^n x_i |y_i| = \langle x, y \rangle$$

2. $\langle x, y \rangle$ is not linear : take $x = (1, 0 \ldots 0)$ $y = (1, 0 \ldots 0)$ then $\langle x, y \rangle = 1 = \langle -x, y \rangle$

3. $\langle x + cx', y \rangle = \sum_i (x_i + c x_i') \sum_i y_i = \sum x_i \sum y_i + c \sum x_i' \sum y_i = \langle x, y \rangle + c \langle x', y \rangle$

similarly one shows that $\langle \_, \_ \rangle$ is linear in the second argument

4. $\langle x, y \rangle = \langle x, -y \rangle$ so this product is not linear

5. $\langle x, y \rangle = \sum 2 x_i y_i = x^t (2 I_n) y$ which is bilinear

3. In each of the following, determine the polar bilinear form associated to the given quadratic form $q:\mathbb{R}^2 \to \mathbb{R}$.

1. $q(x,y) = 3x^2 - 8xy - 3y^2$
2. $q(x,y) = 4x^2 - 9xy + 5y^2$
3. $q(x,y) = 6xy$

• $b(v,w) = \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right)$

• if $v = v(x_1,y_1)$ and $w = w(x_2,y_2)$ then

$$b((x_1,y_1),(x_2,y_2)) = \frac{1}{2}\left(q(x_1+x_2, y_1+y_2) - q(x_1,y_1) - q(x_2,y_2)\right)$$

1.) • So

$$b((x_1,y_1),(x_2,y_2)) = \frac{1}{2}\Big(3(x_1+x_2)^2 - 8(x_1+x_2)(y_1+y_2) - 3(y_1+y_2)^2$$
$$-3x_1^2 + 8x_1y_1 + 3y_1^2 - 3x_2^2 + 8x_2y_2 + 3y_2^2\Big)$$

$$= \frac{1}{2}\Big(\cancel{3x_1^2} + 6x_1x_2 + \cancel{3x_2^2} - \cancel{8x_1y_1} - 8x_1y_2 - 8x_2y_1 - \cancel{8x_2y_2} - \cancel{3y_1^2} - 6y_1y_2 - \cancel{3y_2^2}$$
$$-\cancel{3x_1^2} + \cancel{8x_1y_1} + \cancel{3y_1^2} - \cancel{3x_2^2} + \cancel{8x_2y_2} + \cancel{3y_2^2}\Big)$$

$$= \frac{1}{2}\left(6x_1x_2 - 8x_2y_1 - 8x_1y_1 - 6y_1y_2\right)$$

$$= 3x_1x_2 - 4x_2y_1 - 4x_1y_1 - 3y_1y_2 = [x_1\ y_1]\underbrace{\begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

• this is the symmetric matrix associated to b

• it is also the symmetric matrix associated to q

2.) $q(x,y) = 4x^2 - 9xy + 5y$ has associated matrix $\begin{bmatrix} 4 & -\frac{9}{2} \\ -\frac{9}{2} & 5 \end{bmatrix}$

$$\Rightarrow b((x_1,y_1),(x_2,y_2)) = [x_1\ y_1]\begin{bmatrix} 4 & -\frac{9}{2} \\ -\frac{9}{2} & 5 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 4x_1x_2 - \frac{9}{2}x_1y_2 - \frac{9}{2}x_2y_1 + 5y_1y_2$$

3) $q(x,y) = 6xy$ has associated matrix $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

$$\Rightarrow b\big((x_1,y_1),(x_2,y_2)\big) = [x_1 \ y_1] \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 3x_1y_2 + 3x_2y_1$$

4. Determine the matrix and the rank of each of the quadratic forms from the previous exercise.

1.) $M = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$ $\det M = -25 \neq 0 \Rightarrow rk(M) = 2$

2.) $M = \begin{bmatrix} 4 & -\frac{9}{2} \\ -\frac{9}{2} & 5 \end{bmatrix}$ $\det M = \frac{161}{4} \neq 0 \Rightarrow rk(M) = 2$

3.) $M = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ $\det M = -9 \neq 0 \Rightarrow rk(M) = 2$

5. In each of the following, determine the polar bilinear form associated to the given quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$.

1. $q(x,y,z) = xz + xy + yz$
2. $q(x,y,z) = 2xy + y^2 - 2xz$
3. $q(x,y,z) = -x^2 - 4xy + 3y^2 + 2z^2$

- $b(v,w) = \frac{1}{2}\big(q(v+w) - q(v) - q(w)\big)$
- if $v = v(x_1,y_1,z_1)$ and $w = w(x_2,y_2,z_2)$ then

$$b\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big) = \frac{1}{2}\big(q(x_1+x_2, y_1+y_2, z_1+z_2) - q(x_1,y_1,z_1) - q(x_2,y_2,z_2)\big)$$

1.) $b\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big) = \frac{1}{2}\big((x_1+x_2)(z_1+z_2) + (x_1+x_2)(z_1+z_2) + (y_1+y_2)(z_1+z_2)$

$$- x_1z_2 - x_1y_1 - y_1z_1 - x_2z_2 - x_2y_2 - y_2z_2$$

$$= \dots = \frac{1}{2}\big(x_1y_2 + x_1z_2 + y_1x_2 + y_1z_2 + z_1x_2 + z_1y_2\big)$$

6. Determine the matrix and the rank of each of the quadratic forms from the previous exercise.

1.) $M = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ $\quad \det M = \frac{1}{4} \quad \Rightarrow \operatorname{rk} M = 3$

2.) $q(x,y,z) = 2xy + y^2 - 2xz$ $\quad M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ $\quad \left.\begin{array}{l} \det M = 0 \\ \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \neq 0 \end{array}\right\} \Rightarrow \operatorname{rk} M = 2$

3) $q(x,y,z) = -4x^2 - 4xy + 3y^2 + 2z^2$ $\quad M = \begin{pmatrix} -4 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \det M = 2 \cdot (-16) \neq 0$

$\Rightarrow \operatorname{rk} M = 3$

7. Let b be a symmetric bilinear form on a vector space $\mathbf{V}$ and let S be a non-empty subset of $\mathbf{V}$. Show that $S^\perp$ is a vector subspace of $\mathbf{V}$.

$S^\perp = \{ w \in V : b(v,w) = 0 \ \forall\, v \in S \}$

$S^\perp$ is a vector subspace of $V \iff \forall\, \alpha_1, \alpha_2 \in K \ \forall\, w_1, w_2 \in S^\perp \ \alpha_1 w_1 + \alpha_2 w_2 \in S^\perp$

So, choose $\alpha_1, \alpha_2 \in K$ and $w_1, w_2 \in S^\perp$

Then $b(v, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 \underbrace{b(v, w_1)}_{=0} + \alpha_2 \underbrace{b(v, w_2)}_{=0} = 0$

since $w_1, w_2 \in S^\perp$

$\Rightarrow \alpha_1 w_1 + \alpha_2 w_2 \in S^\perp$

8. In each of the following cases find a basis with respect to which the given quadratic form on $\mathbb{R}^3$ is in normal form and calculate the signatures:

1. $4x^2 - 5y^2 + 12z^2$
2. $-x^2 + 9z^2$
3. $-x^2 - y^2 + z^2$
4. $y^2 + 16z^2$

Let $e = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$

1.) w.r.t. the basis e the matrix of q is $Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

consider the basis $f = (f_1, f_2, f_3)$ with $f_1 = \frac{e_1}{2}$, $f_2 = \frac{e_3}{2\sqrt{3}}$, $f_3 = \frac{e_2}{\sqrt{5}}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M_{f,e} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} [e_1]_f & [e_2]_f & [e_3]_f \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2\sqrt{3} \\ 0 & \sqrt{5} & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M_{e,f} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{2\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

So $M_f(q) = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{5}} & 0 \end{bmatrix}}_{M_{e,f}^t} \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 12 \end{bmatrix}}_{Q = M_f(q)} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{2\sqrt{2}} & 0 \end{bmatrix}}_{M_{e,f}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -\frac{5}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{3}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\Downarrow$

signature is $(2,1)$

2.) $Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ choose $f = (f_1, f_2, f_3)$ with $f_1 = \frac{e_3}{3}$ $f_2 = e_1$ $f_3 = e_2$

then $M_{e,f} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$

So $M_f(q) = \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{M_{e,f}^t} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{M_f(q)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Downarrow$

signature is $(1,1)$

9. Diagonalize each of the quadratic forms of exercise 3, determining the change of coordinates required, and the signatures of the forms.

1) $q(x,y) = 3x^2 - 8xy - 3y^2$

$= 3\left(x^2 - \frac{8}{3}xy + \left(\frac{4}{3}y\right)^2\right) - 3\left(\frac{4}{3}y\right)^2 - 3y^2$

$= 3\underbrace{\left(x - \frac{4}{3}y\right)}_{x'}^2 - \frac{25}{3}\underbrace{y^2}_{y'=y}$

$= 3x'^2 - \frac{25}{3}y'^2$ has signature $(1,1)$

2) $q(x,y) = 4x^2 - 9xy + 5y^2$

$= 4\left(x^2 - \frac{9}{4}xy + \left(\frac{9}{8}y\right)^2\right) - 4\cdot\left(\frac{9}{8}y\right)^2 + 5y^2$

$= 4\underbrace{\left(x - \frac{9}{8}y\right)}_{x'}^2 - \frac{1}{16}\underbrace{y^2}_{y'=y}$

$= 4x'^2 - \frac{1}{16}y'^2$ has signature $(1,1)$

3) $q(x,y) = 6xy$

$6xy = 6(x'-y')(x'+y') = 6x'^2 - 6y'^2$

choose $\begin{cases} x = x' - y' \\ y = x' + y' \end{cases}$

$\Downarrow$

q has signature $(1,1)$

10. For each of the forms of the preceding exercise, express the matrix B of the diagonalized form as $B = M^t AM$, where A is the matrix of the given form.

1) $A = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}$ $\quad \begin{cases} x' = x - \frac{4}{3}y \\ y' = y \end{cases}$ $\iff \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $M_{e'e}$

$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$ $M_{e,e'} = M_{e'e}^{-1}$

$B = M_{e'}(q) = M_{e,e'}^t \overbrace{M_e(q)}^{A} M_{e,e'} = \begin{bmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -4 & -\frac{25}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{25}{3} \end{bmatrix}$

this is the matrix M that we are looking for

2.) $A = \begin{pmatrix} 4 & -\frac{9}{2} \\ -\frac{9}{2} & 5 \end{pmatrix}$ $\begin{cases} x' = x - \frac{9}{8}y \\ y' = y \end{cases}$ $\Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -\frac{9}{8} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & \frac{9}{8} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 0 \\ \frac{9}{8} & 1 \end{bmatrix} \begin{bmatrix} 4 & -\frac{9}{2} \\ -\frac{9}{2} & 5 \end{bmatrix} \begin{bmatrix} 1 & 9/8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 9/8 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -\frac{9}{2} & \frac{1}{16} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1/16 \end{bmatrix}$$

3.) $A = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ $\begin{cases} x = x' - y' \\ y = x' + y' \end{cases}$ $\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\Rightarrow B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$$

11. Diagonalize each of the quadratic forms of exercise 5, determining the change of coordinates required, and the signatures of the forms.

1. $g(x,y,z) = xz + xy + yz$

$x = x' \quad y = y' - x' \quad z = z' \quad \Leftrightarrow \quad x' = x \quad y' = x+y \quad z' = z$

$$= x'z' + x'(y' - x') + (y' - x')z'$$

$$= x'z' + x'y' - x'^2 + y'z' - x'z'$$

$$= -x'^2 + x'y' + y'z'$$

$$= -\left(x'^2 - x'y' + \left(\frac{y'}{2}\right)^2\right) + \frac{y'^2}{4} + y'z'$$

$$= -\left(x' - \frac{y'}{2}\right)^2 + \frac{1}{4}\left(y'^2 + 4y'z' + (2z')^2\right) - z'^2$$

$$= -\underbrace{\left(x' - \frac{y'}{2}\right)}_{x''}{}^2 + \frac{1}{4}\underbrace{\left(y' + 2z'\right)}_{y''}{}^2 - z'^2 \qquad z'' = z'$$

$$= -x''^2 + \frac{1}{4}y''^2 - z''^2 \quad \rightarrow \text{has signature } (1,2)$$

$x'' = x' - \frac{y'}{2} = x - \frac{x+y}{2} = \frac{x}{2} - \frac{y}{2}$

$y'' = y' + 2z' = x + y + 2z$

$z'' = z$

2.) $q(x,y,z) = 2xy + y^2 - 2xz$

$$= -x^2 + (x^2 + 2xy + y^2) - 2xz$$

$$= (x+y)^2 - (x^2 + 2xz + z^2) + z^2$$

$$= \underbrace{(x+y)}_{x'}{}^2 + \underbrace{z}_{y'}{}^2 - \underbrace{(x+z)}_{z'}{}^2$$

$$= x'^2 + y'^2 - z'^2 \quad \to \text{ signature } (2,1)$$

the actual coordinate change is obtained by expressing x, y, z in terms of x', y', z'

3.) $q(x,y,z) = -x^2 - 4xy + 3y^2 + 2z^2$

$$= -(x^2 + 4xy + (2y)^2) + 4y^2 + 3y^2 + 2z^2$$

$$= -\underbrace{(x+2y)}_{z'}{}^2 + 7\underset{x'}{\underset{\shortparallel}{y}}{}^2 + 2\underset{y'}{\underset{\shortparallel}{z}}{}^2$$

$$= 7x'^2 + 2y'^2 - z'^2 \quad \to \text{ signature } (2,1)$$

12. For each of the forms of the preceding exercise, express the matrix B of the diagonalized form as $B = M^t A M$, where A is the matrix of the given form.

1.) $A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & -1 \\ -1 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

inverse $\quad \frac{1}{\frac{1}{2}+\frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1/2 & 1/2 & 0 \\ -1 & -1 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 1/2 & -1 \\ -1 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$B = M^t A M = \begin{bmatrix} 1 & -1 & 0 \\ 1/2 & 1/2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{bmatrix} 1 & 1/2 & -1 \\ -1 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1/2 & 1/2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/4 & 0 \\ 1/2 & 1/4 & 0 \\ 0 & 1/2 & -1 \end{bmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2.) $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

inverse $\quad \frac{1}{1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$B = M^t A M = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3.) $A = \begin{pmatrix} -1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= x + 2y \end{aligned} \iff \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{M^t} \underbrace{\begin{pmatrix} -1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{B}$$