Series of real numbers

A series of real numbers is an ordered pair of two sequences $((x_n),(s_n))_{n\geq k}$, with the property that $\begin{cases} s_k = x_k \\ s_{k+1} = x_k + x_{k+1} \\ \vdots \\ s_n = x_k + \dots + x_n \end{cases}$ $, \forall n > k$

Notation: $\sum_{n\geq k} x_n$

Sum of the series of real numbers

$\sum x_n$ a series of real numbers | If $\exists \lim_{n\to\infty} s_n \in \overline{\mathbb{R}}$, it is called the sum of the series

Notation: $\sum_{n=k}^{\infty} x_n = \lim_{n\to\infty} s_n$

Convergence

$\sum x_n$ a series of real numbers
- → CONVERGENT if $\exists \sum_{n=k}^{\infty} x_n = \lim_{n\to\infty} s_n \in \mathbb{R}$
- → DIVERGENT otherwise
  - → either $\exists \sum_{n=k}^{\infty} x_n \in \{-\infty, \infty\}$
  - → or $\nexists \sum_{n=k}^{\infty} x_n$

Terminology

$\sum_{n\geq k} x_n$ a series of real numbers
- → the sequence (x_n) is called the general (generating) sequence of the series
- → $\forall n \geq k$, the element x_n is called the general term of rank n of the series
- → $\forall n \geq k$, the element s_n is called the partial sum of rank n of the series
- → the sequence (s_n) is called the sequence of partial sums of the series

Recall: Cauchy sequences

(x_n) convergent $\iff$ (x_n) Cauchy sequence (in $\mathbb{R}$)

$\overset{\text{def}}{\iff} \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, \forall p \in \mathbb{N}$, it holds $|x_{n+p} - x_n| < \varepsilon$

The general convergence criteria for series
Cauchy approach

A series $\sum x_n$ is CONVERGENT $\iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, \forall p \in \mathbb{N}$, $|x_{n+1} + \dots + x_{n+p}| < \varepsilon$

Proof:

$\sum x_n$ is convergent $\overset{\text{def}}{\iff} (s_n)$ is convergent

$\underset{\text{CAUCHY SEQ.}}{\overset{T}{\iff}} (s_n)$ is a Cauchy sequence

$\overset{\text{def}}{\iff} \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, \forall p \in \mathbb{N}$ it holds $|s_{n+p} - s_n| < \varepsilon$.

$s_{n+p} = x_1 + x_2 + \dots + x_n + x_{n+1} + \dots + x_{n+p}$

$s_n = x_1 + x_2 + \dots + x_n$

$|s_{n+p} - s_n| = |x_{n+1} + \dots + x_{n+p}|$

$\Rightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, \forall p \in \mathbb{N}$, $|x_{n+1} + \dots + x_{n+p}| < \varepsilon$

Theorem (convergence & limit)

If $\sum x_n$ is CONVERGENT $\Longrightarrow \lim\limits_{n\to\infty} x_n = 0$ $\quad$ (Proof: $(p \Rightarrow q \Leftrightarrow \bar{q} \Rightarrow \bar{p})$)

$\not\Longleftarrow$

but If $\lim\limits_{n\to\infty} x_n \neq 0 \Longrightarrow \sum x_n$ is DIVERGENT

Proof:

Consider $\sum x_n$ convergent $\overset{\text{CON.}}{\underset{\text{CRIT.}}{\Longleftrightarrow}}$ $\forall \varepsilon>0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $\forall p \in \mathbb{N}$ it holds $|x_{n+1}+\ldots+x_{n+p}| < \varepsilon$

for $p=1$ $\Rightarrow \forall \varepsilon>0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$ it holds $|x_{n+1}| < \varepsilon$

$|x_{n+1}-0| < \varepsilon \overset{\varepsilon}{\underset{\text{T}}{\Longrightarrow}} \lim\limits_{n\to\infty} x_{n+1} = 0$

$\Rightarrow \lim\limits_{n\to\infty} x_n = 0$

This theorem is handy in exercises, we compute $\lim\limits_{n\to\infty} x_n$
- $\neq 0$ STOP $\Rightarrow$ DIVERGENT
- $= 0$ FURTHER ANALYSE

Operations with convergent series

$\sum\limits_{n\geq1} x_n$, $\sum\limits_{n\geq1} y_n$ two series of real numbers which have the sums $\sum\limits_{n=1}^{\infty} x_n = x$, $\sum\limits_{n=1}^{\infty} y_n = y$

$a, b \in \mathbb{R}$

If $ax+by$, not a not determination case, then the series $\sum\limits_{n\geq1} ax_n+by_n$ has a sum $\sum\limits_{n=1}^{\infty} ax_n+by_n = ax+by$

If the series $\sum x_n$ has a sum
If $p \in \mathbb{N}$

then $\sum\limits_{n=p}^{\infty} x_n = \sum\limits_{n=1}^{\infty} x_n - (x_1+x_2+\ldots+x_{p-1})$

Exercises

1) Study the nature and existence for the sum of the series:

a) $\sum_{n\geq 1} \frac{1}{n(n+1)}$

$x_n = \frac{1}{n(n+1)} \quad \forall n \geq 1$

$s_1 = x_1 = \frac{1}{1\cdot 2}$

$s_2 = x_1 + x_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$

$s_3 = x_1 + x_2 + x_3 = s_2 + x_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4}$

$\vdots$

$s_n = x_1 + x_2 + \ldots + x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}, \quad \forall n > 1$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}\right) = \lim_{n\to\infty} \left(1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n}} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) =$

$= 1 - 0 =$

$= 1 \quad \exists \in \mathbb{R}$

$\frac{1}{1\cdot 2} = \frac{\overset{2}{/}1}{1} - \frac{1}{2} = \frac{1}{2}$

$\frac{1}{2\cdot 3} = \frac{\overset{3}{/}1}{2} - \frac{\overset{2}{/}1}{3} = \frac{1}{6}$

the series $\sum_{n\geq 1} \frac{1}{n(n+1)}$ has a sum and it is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = 1$

$\lim_{n\to\infty} s_n = 1 \in \mathbb{R} \Rightarrow \sum_{n\geq 1} \frac{1}{n(n+1)}$ convergent

b) $\sum_{n\geq 1} \frac{1}{n}$

Step 1: (collect data) $x_n = \frac{1}{n}, \quad \forall n \geq 1$

$s_1 = x_1 = 1$

$s_2 = 1 + \frac{1}{2}$

$s_3 = 1 + \frac{1}{2} + \frac{1}{3}$

$\vdots$

$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, \quad \forall n > 1$

Step 2: (compute limit) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) = \infty \Rightarrow \exists \notin \mathbb{R}$

Step 3: (conclusion) the series $\sum_{n\geq 1} \frac{1}{n}$ has a sum and it is $\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} s_n = \infty$

$\lim_{n\to\infty} s_n = \infty \notin \mathbb{R} \Rightarrow \sum_{n\geq 1} \frac{1}{n}$ is divergent

c) $\sum_{n\geq 1} (-1)^n$

$x_n = (-1)^n = \begin{cases} 1, & n = 2k \\ -1, & n = 2k-1 \end{cases}, \forall k \in \mathbb{N}$

$s_1 = -1$

$s_2 = -1 + 1 = 0$

$s_3 = -1 + 1 - 1 = -1$

$s_4 = -1 + 1 - 1 + 1 = 0$

$\vdots$

$s_n = -1 + 0 - 1 + \ldots + (-1)^n, \quad \forall n > 1$

$s_n = \begin{cases} 0, & n = 2k \\ -1, & n = 2k-1 \end{cases}$

$$\left.\begin{array}{l}\lim\limits_{n\to\infty} s_{2k} = 0 \\ \qquad\quad \neq \\ \lim\limits_{n\to\infty} s_{2k+1} = -1\end{array}\right\} \Rightarrow \nexists \lim_{n\to\infty} s_n$$

the series $\sum_{n\geq 1}(-1)^n$ doesn't have a sum because $\nexists \lim\limits_{n\to\infty} s_n$

$\nexists \lim\limits_{n\to\infty} s_n \Rightarrow$ the series $\sum_{n\geq 1}(-1)^n$ is divergent

d) $\sum_{n\geq 1}\frac{1}{2^n}$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}\right) = 1$$

the series $\sum_{n\geq 1}\frac{1}{2^n}$ has a sum and it is $\sum_{n=1}^{\infty}\frac{1}{2^n} = \lim\limits_{n\to\infty} s_n = 1$

$\lim\limits_{n\to\infty} s_n = 1 \in \mathbb{R} \Rightarrow \sum_{n\geq 1}\frac{1}{2^n}$ convergent

e) $\sum_{n\geq 1}\frac{1}{n^2}$ is convergent

2) Study the nature of the following series:

a) $\sum_{n\geq 1}(-1)^n$

Step 1: $x_n = (-1)^n$

Step 2: $\nexists \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}(-1)^n$

Step 3: $\Rightarrow \sum_{n\geq 1}(-1)^n$ D

b) $\sum_{n\geq 1} 4^n$

$x_n = 4^n$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} 4^n = \infty$

$\lim\limits_{n\to\infty} x_n = \infty \neq 0 \xRightarrow{T} \sum_{x\geq 1} 4^n$ D

c) $\sum_{n\geq 1}\frac{n+7}{7n+9}$

$x_n = \frac{n+7}{7n+9}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{n+7}{7n+9} = \lim_{n\to\infty}\frac{n(1+\overset{0}{\frac{7}{n}})}{n(7+\underset{0}{\frac{9}{n}})} = \frac{1}{7}$$

$\lim\limits_{n\to\infty} x_n = \frac{1}{7} \neq 0 \xRightarrow{T} \sum_{x\geq 1}\frac{n+7}{7n+9}$ D

a) Determine the sum of the following series:

$$\sum_{n=3}^{\infty} \left(\sqrt{n+1} - \sqrt{n} \right)$$

telescopic series

$x_n = b_{n+1} - b_n \qquad \exists (b_n) \in \mathbb{R}$

If $\lim\limits_{n \to \infty} b_n \in \overline{\mathbb{R}} \Rightarrow \exists \sum\limits_{n=k}^{\infty} x_n = \lim\limits_{n \to \infty} - b_k$

$\lim\limits_{n \to \infty} b_n = \lim\limits_{n \to \infty} \sqrt{n} = \infty^{\frac{1}{2}} = \infty \in \overline{\mathbb{R}} \Rightarrow \exists \sum\limits_{n=3}^{\infty} x_n = \lim\limits_{n \to \infty} - b_k = -\sqrt{3}$