Course 8

April 13, 2021

11.3. Theorem. (W. H. Young) Let $A \subseteq \mathbb{R}^n$ be an open set, and let $f: A \to \mathbb{R}$ s.t. f is partially differentiable on A with respect to x_i and x_j ($i,j \in \{1,\dots,n\}$, $i \neq j$).
If $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} : A \to \mathbb{R}$ are differentiable at some point $a \in A$, then

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$$

12. The second order differential

$I \subseteq \mathbb{R}$ open interval

$f: I \to \mathbb{R}$ s.t. f is differentiable on I

$f': I \to \mathbb{R}$

$A \subseteq \mathbb{R}^n$ open set

$f: A \to \mathbb{R}$ s.t. f is differ. on A

$df: A \longrightarrow L(\mathbb{R}^n, \mathbb{R})$

$\forall x \in A \longmapsto df(x) \in L(\mathbb{R}^n, \mathbb{R})$

12.1. Definition Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$. Suppose that $\exists V \in \mathcal{V}(a)$, V open, $V \subseteq A$ s.t.

- f is differentiable at each point of V
- $\forall i = 1, \dots, n$ the function $\forall x \in V \longmapsto \frac{\partial f}{\partial x_i}(x) \in \mathbb{R}$ is differentiable at a

Then f is said to be twice differentiable at a

Remark f differentiable at $a \Rightarrow f$ is partially differ. at a

12.2. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$ s.t. f is twice differentiable at a

Then:

1° $\forall\, i, j \in \{1, \dots, n\}$ the second order partial derivative $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ exists at a

2° $\forall\, i, j \in \{1, \dots, n\}$: $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(a) = \dfrac{\partial^2 f}{\partial x_i \partial x_j}(a)$

<u>12.3. Definition</u> Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}$ a twice differentiable function at a

We define $d^2 f(a) : \mathbb{R}^n \to \mathbb{R}$

$$d^2 f(a)(h) := \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)\, h_i h_j \qquad \forall\, h = (h_1, \dots, h_n) \in \mathbb{R}^n$$

↳ the second order differential of f at a (quadratic form)

Let $C = (c_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix

$$\Phi : \mathbb{R}^n \to \mathbb{R} \qquad \Phi(h) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, h_i h_j \qquad h = (h_1, \dots, h_n) \in \mathbb{R}^n$$

↳ the quadratic form generated by C (associated with C)

$$C = \begin{pmatrix} 2 & -1 \\ -1 & -3 \end{pmatrix} \qquad \Phi(h_1, h_2) = 2h_1^2 - 3h_2^2 - 2h_1h_2$$

$$C = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 0 & 1 \\ 3 & 1 & 4 \end{pmatrix} \qquad \Phi(h_1, h_2, h_3) = h_1^2 + 4h_3^2 - 4h_1h_2 + 6h_1h_3 + 2h_2h_3$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Phi(h_1, h_2) = 2h_1h_2$$

$d^2f(a)$ is the quadratic form associated with the Hessian $H(f)(a) = \nabla^2 f(a)$

Remark $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \operatorname{int} A$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a) - df(a)(x-a)}{\|x-a\|} = 0$$

12.4. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}$ s.t. f is twice differentiable at a

Then

$$\lim_{x \to a} \frac{f(x) - f(a) - df(a)(x-a) - \frac{1}{2} d^2f(a)(x-a)}{\|x-a\|^2} = 0$$

<u>Proof</u> f is twice diff. at a $\Rightarrow$ $\exists V \in \mathcal{V}(a)$, V open, $V \subseteq A$ s.t.

f is diff on V and $\frac{\partial f}{\partial x_i} : V \to \mathbb{R}$ are all diff. at a, $i = \overline{1,n}$

Set $F_i = \frac{\partial f}{\partial x_i}$ $\Rightarrow$ $\lim\limits_{x \to a} \frac{F_i(x) - F_i(a) - dF_i(a)(x-a)}{\|x-a\|} = 0$ (1)

<u>Let $\varepsilon > 0$</u>

By (1) $\Rightarrow$ $\forall i \in \{1, \dots, n\}$ $\exists \delta_i > 0$ s.t. $B(a, \delta_i) \subseteq V$ and

$\forall x \in B(a, \delta_i) \setminus \{a\}$: $\frac{|F_i(x) - F_i(a) - dF_i(a)(x-a)|}{\|x-a\|} < \varepsilon' := \frac{\varepsilon}{n}$

$\forall x \in B(a, \delta_i)$: $|F_i(x) - F_i(a) - dF_i(a)(x-a)| \leq \varepsilon' \|x-a\|$ (1')

Let <u>$\delta := \min\{\delta_1, \dots, \delta_n\} > 0$.</u>

Consider $g : B(a, \delta) \to \mathbb{R}$,

$$g(x) = f(x) - f(a) - df(a)(x-a) - \frac{1}{2} d^2 f(a)(x-a)$$

$$= f(x) - f(a) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)(x_i - a_i)(x_j - a_j)$$

f is diff on V $\Rightarrow$ g is diff on $B(a, \delta)$

$$\frac{\partial g}{\partial x_k}(x) = \frac{\partial f}{\partial x_k}(x) - \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\,(x_i - a_i) - \frac{\partial}{\partial x_k}\,\frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)(x_i - a_i)(x_j - a_j)$$

$$= \frac{\partial f}{\partial x_k}(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\, \underbrace{\frac{\partial}{\partial x_k}(x_i - a_\ell)}_{= \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}} - \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)\, \underbrace{\frac{\partial}{\partial x_k}\Big((x_i - a_i)(x_j - a_j)\Big)}$$

$$= \underbrace{\frac{\partial}{\partial x_k}(x_i - a_i)}_{= \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}} \cdot (x_j - a_j) + (x_i - a_i)\, \underbrace{\frac{\partial}{\partial x_k}(x_j - a_j)}_{= \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}}$$

$$\frac{\partial g}{\partial x_k}(x) = \frac{\partial f}{\partial x_k}(x) - \frac{\partial f}{\partial x_k}(a) - \frac{1}{2}\left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(a)\,(x_j - a_j) + \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_i}(a)\,(x_i - a_i) \right)$$

$$\frac{\partial g}{\partial x_k}(x) = \underbrace{\frac{\partial f}{\partial x_k}(x)}_{F_k(x)} - \underbrace{\frac{\partial f}{\partial x_k}(a)}_{F_k(a)} - \underbrace{\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(a)\,(x_j - a_j)}_{dF_k(a)(x-a)}$$

$$\Rightarrow \quad \forall x \in B(a,\delta) \ \forall k=1,\dots,n) \ : \ \frac{\partial g}{\partial x_k}(x) = F_k(x) - F_k(a) - dF_k(a)(x-a) \qquad (2)$$

Since $B(a,\delta)$ is a convex set $\xRightarrow{MVT}$ $\forall x \in B(a,\delta)$ $\exists \xi_x \in (0,1)$ s.t. $c_x := (1-\xi_x)a + \xi_x x$ we have

$$g(x) - \underbrace{g(a)}_{=0} = dg(c_x)(x-a)$$

$$\Rightarrow \quad |g(x)| = \left| \sum_{k=1}^{n} \frac{\partial g}{\partial x_k}(c_x)(x_k - a_k) \right| \le \sum_{k=1}^{n} \underbrace{\left| \frac{\partial g}{\partial x_k}(c_x) \right|}_{\le \varepsilon' \cdot \|c_x - a\| \le \varepsilon' \|x-a\|} \cdot \overbrace{|x_k - a_k|}^{\le \|x-a\|}$$

By (2) and (1') $\Rightarrow$ $\left| \frac{\partial g}{\partial x_k}(x) \right| = |F_k(x) - F_k(a) - dF_k(a)(x-a)| \le \varepsilon' \|x-a\|$

$$\Rightarrow \quad |g(x)| \le \sum_{k=1}^{n} \varepsilon' \|x-a\|^2 = n\varepsilon' \|x-a\|^2$$

$$\Rightarrow \quad \forall x \in B(a,\delta) \setminus \{a\} \ : \ \frac{|g(x)|}{\|x-a\|^2} = \frac{|f(x) - f(a) - df(a)(x-a) - \frac{1}{2} d^2 f(a)(x-a)|}{\|x-a\|^2} \le n\varepsilon' = \varepsilon$$

13. Necessary and sufficient optimality conditions

$f: I \to \mathbb{R}$ differentiable $\qquad f'(x) = 0 \quad \to \quad a_1, a_2, \dots, a_k$

x		a_1		a_2	a_3	$\dots$	a_k
$f'(x)$	$- -$	0	$+ +$	0	0		0
$f(x)$	$\searrow$		$\nearrow$				

local minimum

$f'(a) = 0 \qquad a$ - critical point

$f''(a) \neq 0 \qquad f''(a) > 0 \Rightarrow a$ is a local minimum

$\qquad\qquad\qquad f''(a) < 0 \Rightarrow a$ is a local maximum

13.1. Definition A quadratic form $\Phi: \mathbb{R}^n \to \mathbb{R}$ is said to be:

- positive definite if $\Phi(h) > 0 \quad \forall h \in \mathbb{R}^n, h \neq 0_n$
- negative definite $\quad \Phi(h) < 0$ ——— " ———
- positive semi-definite if $\Phi(h) \geq 0 \quad \forall h \in \mathbb{R}^n$
- negative semi-definite $\quad \Phi(h) \leq 0 \quad \forall h \in \mathbb{R}^n$
- indefinite if $\exists h, k \in \mathbb{R}^n$ s.t. $\Phi(h) < 0 < \Phi(k)$.

13.2. Theorem (characterization of positive definite quadratic forms).
A quadratic form $\Phi: \mathbb{R}^n \to \mathbb{R}$ is positive definite $\Leftrightarrow \exists \alpha > 0$ s.t. $\forall h \in \mathbb{R}^n$ we have $\Phi(h) \geq \alpha \|h\|^2 \quad (*)$

Proof. $\boxed{\Leftarrow}$ Obvious.

$\boxed{\Rightarrow}$ Φ is continuous on $\mathbb{R}^n$
$S^{n-1} = \{ h \in \mathbb{R}^n \mid \|h\| = 1 \}$ compact

$\Rightarrow$ Φ is bounded and $\exists\, h_0 \in S^{n-1}$ s.t.

$$\Phi(h_0) = \min_{h \in S^{n-1}} \Phi(h) =: \alpha$$

$\Rightarrow$ $h_0 \neq 0_n$ $\Rightarrow$ $\alpha = \Phi(h_0) > 0$

$\uparrow$ Φ is positive definite

Let $h \in \mathbb{R}^n$

- $h = 0_n$ $\Rightarrow$ $(*)$ holds with equality
- $h \neq 0_n$ $\Rightarrow$ $\frac{1}{\|h\|} h \in S^{n-1}$ $\Rightarrow$ $\alpha \leq \Phi\left(\frac{1}{\|h\|} h\right) = \frac{1}{\|h\|^2} \cdot \Phi(h) \quad \Big| \cdot \|h\|^2$

$$\Phi(th) = t^2 \Phi(h)$$

$\Rightarrow$ $\alpha \cdot \|h\|^2 \leq \Phi(h)$

13.3. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}$ s.t. f is twice diff at a.

Then:

1° If a is a local minimum / maximum point for f $\Rightarrow$ $\nabla f(a) = 0_n$ and $d^2 f(a)$ is a positive / negative semi-definite quadratic form.

2° If $\nabla f(a) = 0_n$ and $d^2 f(a)$ is a positive / negative definite quadratic form $\Rightarrow$ a is a local minimum / maximum point for f.

Proof 1° Assume that a is a local minimum for f

Fermat's theorem $\Rightarrow$ $\nabla f(a) = 0_n$

? $d^2 f(a)$ is positive semi-definite $\Leftrightarrow$ $d^2 f(a)(h) \geq 0 \quad \forall h \in \mathbb{R}^n$

Let $h \in \mathbb{R}^n$. Let $\varepsilon > 0$

Since
$$\lim_{x \to a} \frac{f(x) - f(a) - \overbrace{df(a)(x-a)}^{= \langle x-a, \nabla f(a) \rangle = 0} - \frac{1}{2} d^2 f(a)(x-a)}{\|x-a\|^2} = 0 \quad \Rightarrow$$

$\Rightarrow$ $\exists\, \delta > 0$ s.t. $B(a, \delta) \subseteq A$ s.t.

$$\forall\, x \in B(a, \delta) \setminus \{a\} : \quad \frac{\left| f(x) - f(a) - \frac{1}{2} d^2 f(a)(x-a) \right|}{\|x-a\|^2} < \varepsilon \quad \Big| \; \|x-a\|^2$$

$$\forall x \in B(a,\delta) \quad : \quad \left| f(x) - f(a) - \frac{1}{2} d^2 f(a)(x-a) \right| \leq \varepsilon \|x-a\|^2$$

$$\Rightarrow \forall x \in B(a,\delta) \quad : \quad f(x) - f(a) - \frac{1}{2} d^2 f(a)(x-a) \leq \varepsilon \|x-a\|^2$$

$$\left.\begin{array}{l} f(x) - f(a) \leq \frac{1}{2} d^2 f(a)(x-a) + \varepsilon \|x-a\|^2 \\ \text{Since } a \text{ is a local minimum} \Rightarrow \exists V \in \mathcal{V}(a) \text{ s.t. } V \subseteq A \text{ and } f(a) \leq f(x),\ \forall x \in V \end{array}\right\} \Rightarrow$$

$$\Rightarrow \forall x \in B(a,\delta) \cap V \quad : \quad 0 \leq \frac{1}{2} d^2 f(a)(x-a) + \varepsilon \|x-a\|^2$$

Replace x by $a+th$, where $t \neq 0$ is sufficiently small s.t. $a+th \in B(a,\delta) \cap V$

$$\Rightarrow \quad 0 \leq \frac{1}{2} d^2 f(a)(th) + \varepsilon \|th\|^2 = \frac{t^2}{2} d^2 f(a)(h) + \varepsilon t^2 \|h\|^2 \quad \Big| \cdot \frac{1}{t^2}$$

$$0 \leq \frac{1}{2} d^2 f(a)(h) + \varepsilon \|h\|^2 \quad , \quad \forall \varepsilon > 0$$

Letting $\varepsilon \searrow 0 \quad \Rightarrow \quad d^2 f(a)(h) \geq 0.$

2° Assume that $\nabla f(a) = 0_n$ and that $d^2 f(a)$ is a positive definite quadratic form $\Rightarrow$

$\Rightarrow \exists\, \alpha > 0$ s.t. $\forall\, h \in \mathbb{R}^2 : \quad d^2 f(a)(h) \geq \alpha \|h\|^2$

Since $$\lim_{x \to a} \frac{f(x) - f(a) - \overbrace{df(a)(x-a)}^{=0} - \frac{1}{2} d^2 f(a)(x-a)}{\|x-a\|^2} = 0$$

$\Rightarrow \exists\, \delta > 0$ s.t. $B(a,\delta) \subseteq A$ and

$$\forall\, x \in B(a,\delta) \setminus \{a\} : \quad \frac{|f(x) - f(a) - \frac{1}{2} d^2 f(a)(x-a)|}{\|x-a\|^2} < \frac{\alpha}{4}$$

$$\forall\, x \in B(a,\delta) : \quad |f(x) - f(a) - \tfrac{1}{2} d^2 f(a)(x-a)| \leq \frac{\alpha}{4} \|x-a\|^2$$

$$-\frac{\alpha}{4} \|x-a\|^2 \leq f(x) - f(a) - \frac{1}{2} d^2 f(a)(x-a)$$

$$\frac{1}{2} \underbrace{d^2 f(a)(x-a)}_{\geq \alpha \|x-a\|^2} - \frac{\alpha}{4} \|x-a\|^2 \leq f(x) - f(a)$$

$\Rightarrow \forall\, x \in B(a,\delta) : \quad f(x) - f(a) \geq \frac{\alpha}{2} \|x-a\|^2 - \frac{\alpha}{4} \|x-a\|^2 = \frac{\alpha}{4} \|x-a\|^2$

$\Rightarrow \forall\, x \in B(a,\delta) \setminus \{a\} : \quad f(x) > f(a) \quad \Rightarrow$ a is a (strict) local minimum point for f

Sylvester's test: let $C = (c_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix

$$\Phi : \mathbb{R}^n \to \mathbb{R} \qquad \Phi(h) = \sum_{i,j=1}^{n} c_{ij} h_i h_j$$

For every $k = 1, \dots, n$ let $\Delta_k := \det(c_{ij})_{1 \le i,j \le k}$

Then: 1° Φ is positive definite $\iff$ $\Delta_k > 0 \quad \forall k = \overline{1,n}$

2° Φ is negative definite $\iff$ $(-1)^k \Delta_k > 0 \quad \forall k = \overline{1,n}$

3° Φ is positive semi-definite $\implies$ $\Delta_k \ge 0 \quad \forall k = \overline{1,n}$

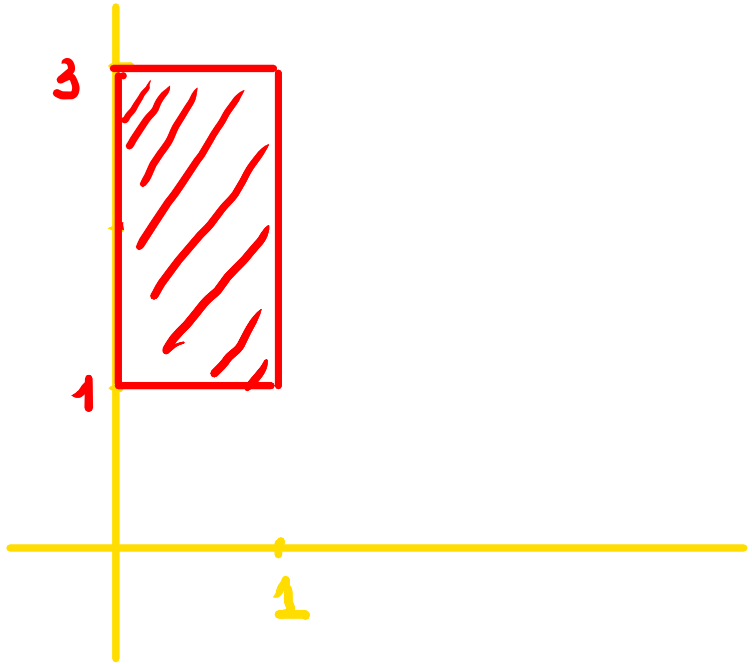
4° Φ is negative semi-definite $\implies$ $(-1)^k \Delta_k \ge 0 \quad \forall k = \overline{1,n}$

Remark If $\nabla f(a) = 0_n$ and $d^2 f(a)$ is an indefinite quadratic form

$\implies$ a is a saddle point.

CHAPTER 3 MULTIPLE INTEGRALS

Evaluate $I=\int_0^1\int_1^3 \frac{1}{x+y}\,dx\,dy \longrightarrow$ double integral



$$I=\int_0^1\int_1^3 \frac{1}{x+y}\,dx\,dy$$

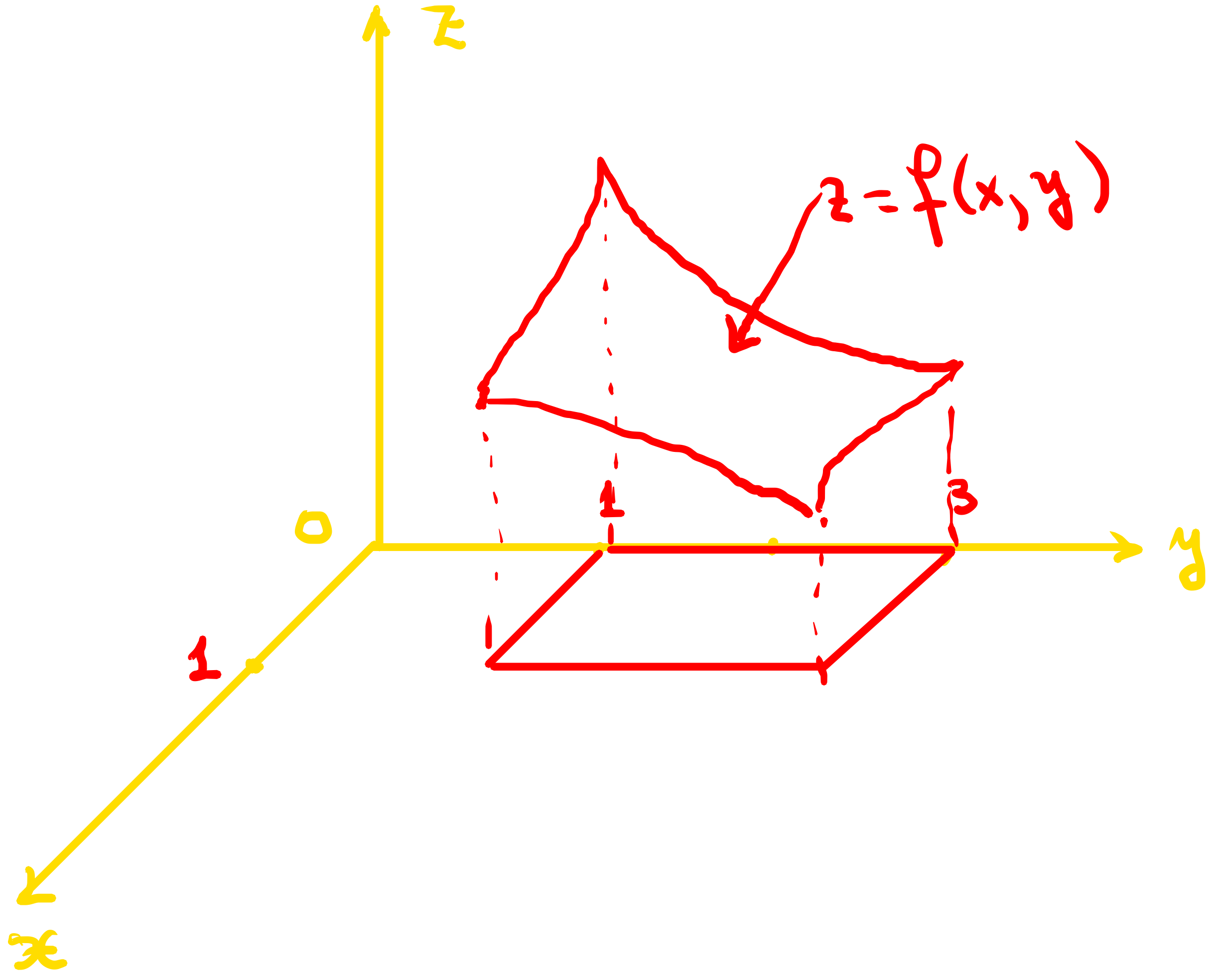
$$I=\int_0^1 \underbrace{\left(\int_1^3 \frac{1}{x+y}\,dy\right)}_{F_1(x)} dx = \int_1^3 \underbrace{\left(\int_0^1 \frac{1}{x+y}\,dx\right)}_{F_2(y)} dy$$

iterated integrals

$$I=\int_0^1 \ln(x+y)\Big|_{y=1}^{y=3} dx = \int_0^1\Big(\ln(x+3)-\ln(x+1)\Big)dx = \int_0^1 (x+3)'\ln(x+3)\,dx - \int_0^1 (x+1)'\ln(x+1)\,dx$$

$$= (x+3)\ln(x+3)\Big|_0^1 - \int_0^1 dx - (x+1)\ln(x+1)\Big|_0^1 + \int_0^1 dx = 4\ln 4 - 3\ln 3 - 2\ln 2 = \ln\frac{4^4}{3^3\cdot 2^2}$$

$$= \boxed{\ln\frac{64}{27}}$$



$f : [0,1] \times [1,3] \longrightarrow \mathbb{R}$, $f(x,y) = \dfrac{1}{x+y}$