Seminar 1 (The Euclidean space $\mathbb{R}^n$)

February 23, 2021

$\boxed{1}$ Let $a=(3,-2,-4)\in\mathbb{R}^3$. Find
$b=(8,6,3)\in\mathbb{R}^3$

$a+b=(11,4,-1)$

$a-b=(-5,-8,-7)$

$-3a+b=(-1,12,15)$

$\langle a,b\rangle = 3\times 8+(-2)\times 6+(-4)\times 3=0 \Rightarrow a\perp b$

$\|a\| = \sqrt{\langle a,a\rangle}=\sqrt{9+4+16}=\sqrt{29}$

$\|b\| = \sqrt{\langle b,b\rangle} = \ldots$

$d(a,b)=\|a-b\|=\sqrt{25+64+49}=\sqrt{\ldots}$

$\boxed{2}$ Prove the Cauchy-Schwarz inequality: $\forall x,y\in\mathbb{R}^n$ one has

(1) $|\langle x,y\rangle|\leq\sqrt{\langle x,x\rangle}\cdot\sqrt{\langle y,y\rangle}$

Solution. Let $f:\mathbb{R}\to\mathbb{R}$, $f(t):=\langle tx-y, tx-y\rangle$

$f(t)=\langle tx-y, tx\rangle+\langle tx-y,-y\rangle=\langle tx,tx-y\rangle+\langle -y,tx-y\rangle$

$= t\langle x,tx-y\rangle-\langle y,tx-y\rangle = t(\langle x,tx\rangle+\langle x,-y\rangle)-(\langle y,tx\rangle+\langle y,-y\rangle)$

$\Rightarrow f(t)=t^2\langle x,x\rangle-2t\langle x,y\rangle+\langle y,y\rangle$

If $x=0_n \Rightarrow$ (1) holds with equality ($0\leq 0$)

If $x\neq 0_n \Rightarrow$ $\left.\begin{array}{l} f \text{ is a second degree polynomial} \\ f(t)\geq 0 \quad \forall t\in\mathbb{R}\end{array}\right\} \Rightarrow \Delta\leq 0$

$\Delta = 4\langle x,y\rangle^2 - 4\langle x,x\rangle\cdot\langle y,y\rangle \le 0 \quad \Rightarrow \quad \langle x,y\rangle^2 \le \langle x,x\rangle\cdot\langle y,y\rangle \quad \Rightarrow \quad (1)$ holds.

<u>Remark</u>. In the above solution we used only the axioms of an inner product $\Rightarrow$

$\Rightarrow$ ineq. (1) holds not only in $\mathbb{R}^n$ but also in every prehilbertian space

For instance $\quad |\langle f,g\rangle| \le \sqrt{\langle f,f\rangle}\cdot\sqrt{\langle g,g\rangle} \quad \forall f,g \in C[a,b]$

$$\Leftrightarrow \quad \left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx}\cdot\sqrt{\int_a^b g^2(x)dx}$$

$$\Leftrightarrow \quad \boxed{\left(\int_a^b f(x)g(x)dx\right)^2 \le \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)}$$

$\boxed{3}$ Prove the parallelogram identity : $\forall x,y\in\mathbb{R}^n$ it holds that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

$\|x\| = \sqrt{\langle x,x\rangle}$

$\Downarrow$

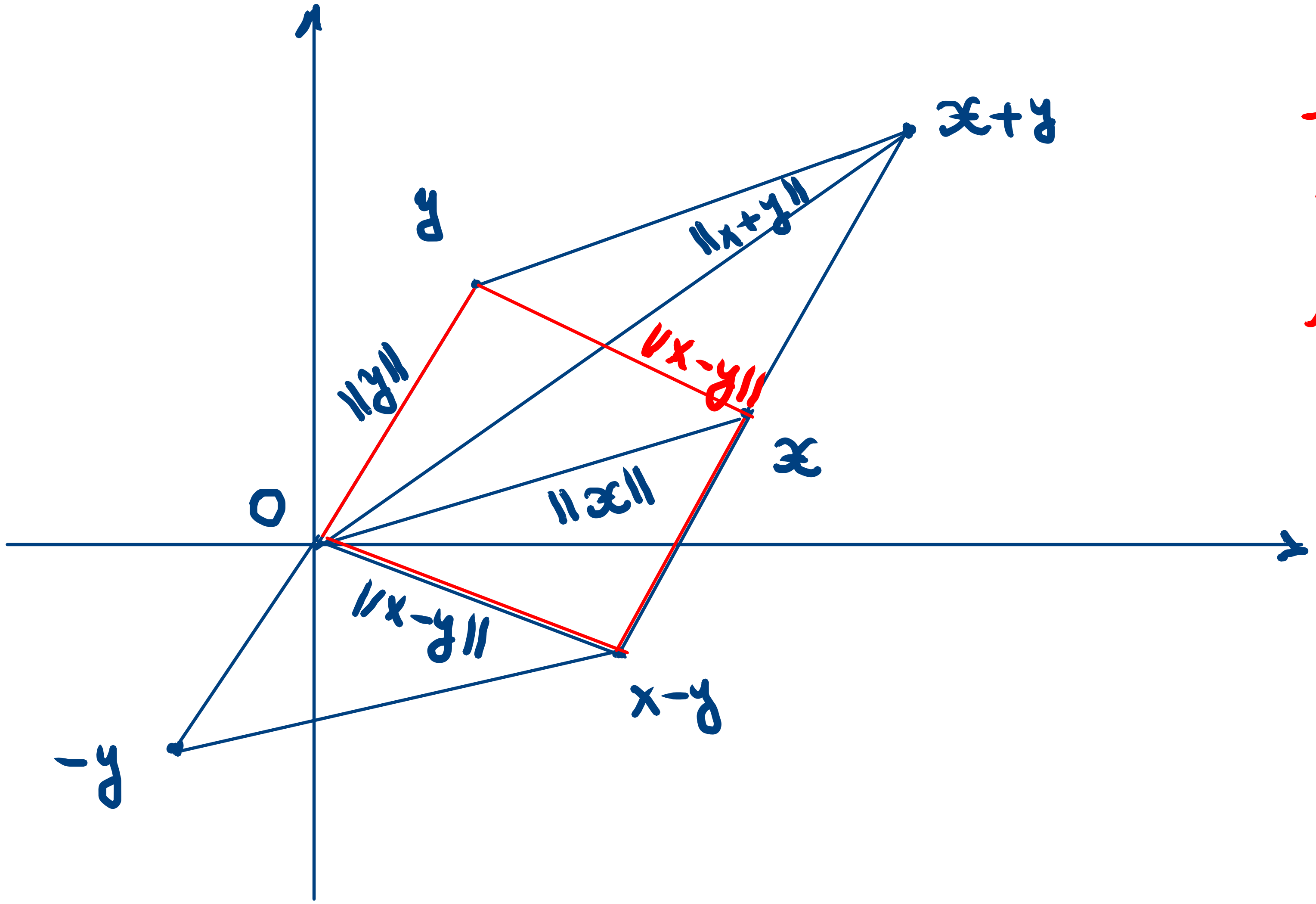
$\|x\|^2 = \langle x,x\rangle$

<u>Solution</u>. $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle =$

$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle y,x\rangle$

Remarks 1° In the above sol. we used only the axioms of an inner product $\Rightarrow$ $\Rightarrow$ the parallelogram identity holds not only in $\mathbb{R}^n$, but also in every prehilbertian space.

2° Geometrical interpretation



$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ expresses the following "known" geometrical property: in each parallelogram the sum of the squares of the two diagonals equals the sum of the squares of the sides

4. Let u and v be two vectors in $\mathbb{R}^n$, show that if $u+v$ and $u-v$ are orthogonal then u and v must have the same length.

Solution. $u+v \perp u-v \Rightarrow \langle u+v, u-v\rangle = 0 \Rightarrow$

$\Rightarrow \langle u,u\rangle + \cancel{\langle v,u\rangle} - \cancel{\langle u,v\rangle} - \langle v,v\rangle = 0$

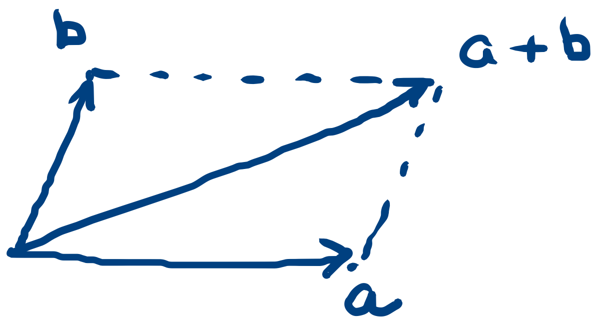
$\Rightarrow \|u\|^2 - \|v\|^2 = 0 \Rightarrow \|u\|^2 = \|v\|^2 \Rightarrow \|u\| = \|v\|$

5. Let a, b and c be three non-zero vectors in $\mathbb{R}^3$, such that $c = |a|b + |b|a$, show that c bisects the angle between a and b.

$c = \|a\|b + \|b\|a$

Solution. Let $\alpha := m(\widehat{a,c})$

$\beta := m(\widehat{b,c})$



X $\alpha = \beta \Leftrightarrow \cos\alpha = \cos\beta \Leftrightarrow \dfrac{\langle a,c\rangle}{\|a\|\cdot\cancel{\|c\|}} = \dfrac{\langle b,c\rangle}{\|b\|\cdot\cancel{\|c\|}}$

$\Leftrightarrow \|b\|\cdot\langle a,c\rangle = \|a\|\langle b,c\rangle$

$\Leftrightarrow \|b\|\cdot\langle a, \|a\|b+\|b\|a\rangle = \|a\|\cdot\langle b, \|a\|b+\|b\|a\rangle$

$\Leftrightarrow \|b\|\left(\|a\|\langle a,b\rangle + \|b\|\langle a,a\rangle\right) = \|a\|\left(\|a\|\langle b,b\rangle + \|b\|\langle b,a\rangle\right)$

$\Leftrightarrow \cancel{\|b\|\cdot\|a\|\langle a,b\rangle} + \cancel{\|b\|^2\cdot\|a\|^2} = \cancel{\|a\|^2\|b\|^2} + \cancel{\|a\|\cdot\|b\|\cdot\langle b,a\rangle}$ ✓

6. Prove that $\|\cdot\|_1 : \mathbb{R}^n \to [0,\infty)$, defined by

$$\|x\|_1 := |x_1| + \cdots + |x_n| \qquad \forall\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

is a norm on $\mathbb{R}^n$.

Solution. We have to verify the norm axioms:

(N_1) $\|x\|_1 = 0 \iff x = 0_n$ ✓

(N_2) $\|\alpha x\|_1 = |\alpha| \cdot \|x\|_1 \quad \forall\ \alpha \in \mathbb{R},\ \forall\ x \in \mathbb{R}^n$

$$\|\alpha x\|_1 = |\alpha x_1| + \cdots + |\alpha x_n| = |\alpha| \underbrace{(|x_1| + \cdots + |x_n|)}_{=\|x\|_1} = |\alpha| \cdot \|x\|_1$$

$\alpha x = (\alpha x_1, \ldots, \alpha x_n)$

(N_3) $\|x+y\|_1 \le \|x\|_1 + \|y\|_1 \quad \forall\ x, y \in \mathbb{R}^n$

$$\|x+y\|_1 = \underbrace{|x_1+y_1|}_{\le |x_1| + |y_1|} + \cdots + |x_n + y_n| \le |x_1| + |y_1| + \cdots + |x_n| + |y_n|$$

$$= \|x\|_1 + \|y\|_1$$

7 (HW) Prove that $\|\cdot\|_\infty : \mathbb{R}^n \to [0,\infty)$, defined by

$$\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\} \qquad \forall\ x = (x_1, \ldots, x_n)$$

is a norm on $\mathbb{R}^n$.

8. Let X be a real vector space, and let $\|\cdot\| : X \to [0,\infty)$ be a norm on X.

One says that $\|\cdot\|$ is generated by an inner product on X if there is an inner product $\langle \cdot , \cdot \rangle$ on X such that

$$\|x\| = \sqrt{\langle x, x\rangle} \qquad \forall x \in X.$$

Prove that if $n \geq 2$, then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not generated by inner products.

Remark? $\Rightarrow$ $\boxed{\text{Pb 3}}$

Solution. Suppose, by contradiction, that $\|\cdot\|_1$ is generated by an inner product in $\mathbb{R}$

$\Rightarrow$ $\|\cdot\|_1$ satisfies the parallelogram identity

$\Rightarrow$ $\|x+y\|_1^2 + \|x-y\|_1^2 = 2\|x\|_1^2 + 2\|y\|_1^2 \quad \forall x, y \in \mathbb{R}^n.$

$x = e_1 = (1,0,0,\dots,0) \qquad \|x\|_1 = 1$

$y = e_2 = (0,1,0,\dots,0) \qquad \|y\|_1 = 1$

$x+y = (1,1,0,\dots,0) \qquad \|x+y\| = 2$

$x-y = (1,-1,0,\dots,0) \qquad \|x-y\| = 2$

$$\Rightarrow \underbrace{2^2 + 2^2}_{8} = \underbrace{2\cdot 1 + 2\cdot 1}_{4} \quad \Rightarrow\Leftarrow$$

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9. Given an arbitrary subset A of $\mathbb{R}^n$, prove that:

a) $\operatorname{cl} A = \mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A)$;

b) $\operatorname{cl} A = (\operatorname{int} A) \cup (\operatorname{bd} A)$;

c) $\operatorname{cl} A = A \cup A'$

$x \in B \setminus C \iff x \in B$ and $x \notin C$

$x \notin B \setminus C \iff x \notin B$ or $x \in C$

<u>Solution</u>. a) We claim that $\operatorname{cl} A \overset{?}{\subseteq} \mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A)$ (1)

Pick an arbitrary $x \in \operatorname{cl} A \Rightarrow \forall V \in \mathcal{V}(x) : V \cap A \neq \emptyset$ (*)

Assume that $x \notin \mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A) \Rightarrow \underbrace{x \notin \mathbb{R}^n}_{\text{false}}$ or $\underline{x \in \operatorname{int}(\mathbb{R}^n \setminus A)} \Rightarrow$

$\Rightarrow \mathbb{R}^n \setminus A \in \mathcal{V}(x)$

$\Downarrow$ (*)

$(\mathbb{R}^n \setminus A) \cap A \neq \emptyset$ - contradiction $\Rightarrow x \in \mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A)$, hence (1) holds

We claim that $\mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A) \overset{?}{\subseteq} \operatorname{cl} A$ (2)

Pick an arbitrary $x \in \mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A) \Rightarrow x \notin \operatorname{int}(\mathbb{R}^n \setminus A)$

Assume that $x \notin \operatorname{cl} A \Rightarrow \exists V \in \mathcal{V}(x)$ s.t. $V \cap A = \emptyset \Rightarrow \left.\begin{array}{l} V \subseteq \mathbb{R}^n \setminus A \\ V \in \mathcal{V}(x) \end{array}\right\} \Rightarrow \mathbb{R}^n \setminus A \in \mathcal{V}(x) \Rightarrow$

$\Rightarrow x \in \operatorname{int}(\mathbb{R}^n \setminus A)$ $\Rightarrow\Leftarrow$

$\Rightarrow x \in \operatorname{cl} A$, hence (2) holds. By (1) and (2) $\Rightarrow$ a) holds.

$x \in \mathrm{cl}A \iff \forall V \in \mathcal{V}(x) : V \cap A \neq \emptyset$

$x \in A' \iff \forall V \in \mathcal{V}(x) : V \cap A \setminus \{x\} \neq \emptyset$

c) $\mathrm{cl}A = A \cup A'$

We know that $\left.\begin{array}{l} A \subseteq \mathrm{cl}A \\ A' \subseteq \mathrm{cl}A \end{array}\right| \Rightarrow A \cup A' \subseteq \mathrm{cl}A \quad (3)$

We claim that $\mathrm{cl}A \overset{?}{\subseteq} A \cup A' \quad (4)$

Let $x \in \mathrm{cl}A$.

Assume that $x \notin A \cup A' \Rightarrow x \notin A$

and

$x \notin A' \Rightarrow \exists V \in \mathcal{V}(x)$ s.t. $V \cap A \setminus \{x\} = \emptyset$

$\Downarrow$

$V \cap A = \emptyset$ or $V \cap A = \{x\}$

$\Downarrow \Uparrow$ $x \in \mathrm{cl}A$

this cannot hold because $x \notin A$

$\Rightarrow x \in A \cup A'$, hence (4) holds

By (3), (4) $\Rightarrow$ c) holds.