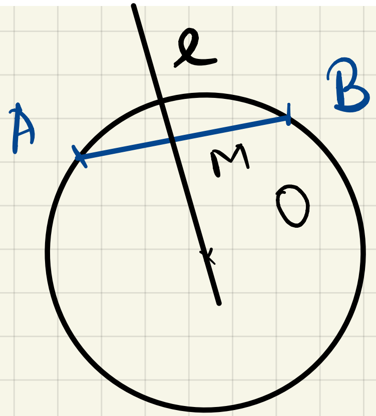
Some notes from the 811 seminar.

1. Find the equation of the circle: a) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $d: 3x - y - 2 = 0$;
b) determined by $A(1,1)$, $B(1,-1)$ and $C(2,0)$; c) tangent to both $d_1 : 2x + y - 5 = 0$ and $d_2 : 2x + y + 15 = 0$, if the tangency point with d_1 is $M(3,1)$.

a)



$OA = OB$, so

$O \in l$, where l is the perp. bisect of $[AB]$

Slope of AB is $m_{AB} = \dfrac{y_B - y_A}{x_B - x_A} = \dfrac{2}{-4} = -\dfrac{1}{2}$.

$l \perp AB$, so $m_l = 2$.

Let M be the midpoint of $[AB]$. $\Rightarrow M(1,2)$.

$\ell: \quad y-2 = 2(x-1).$

$\{O\} = \ell \cap d$, Let $O(x_0, y_0)$. Then

$$\begin{cases} 2x_0 - y_0 = 0 \\ 3x_0 - y_0 - 2 = 0 \end{cases} \Rightarrow \begin{cases} y_0 = 2x_0 \\ 3x_0 - 2x_0 = 2 \end{cases}$$

So $x_0 = 2$ and $y_0 = 4$.

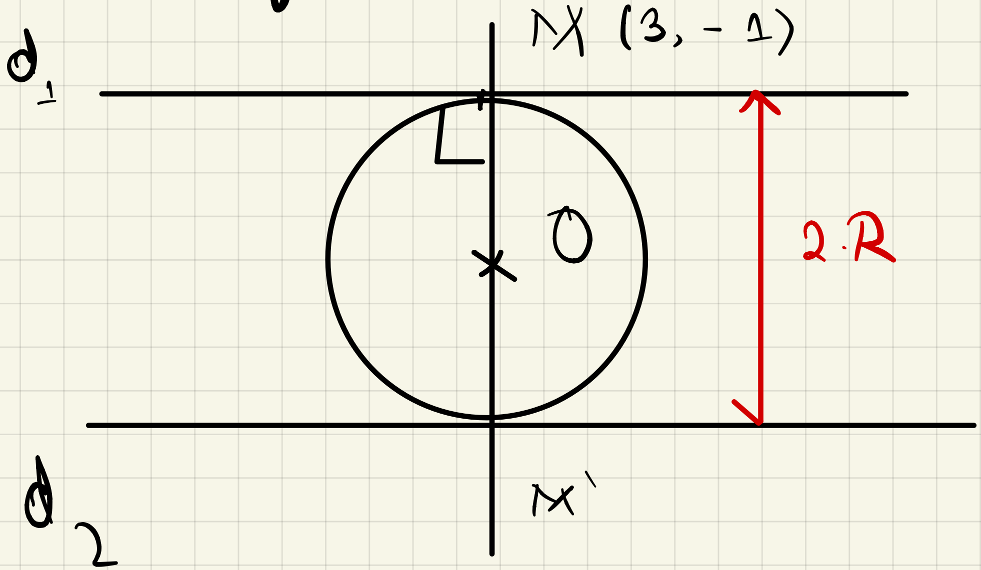
$$R = d(O, A) = \sqrt{(2-3)^2 + (4-1)^2} = \sqrt{10}.$$

So $\mathcal{C}(O, \sqrt{10}) : (x-2)^2 + (y-4)^2 = 10.$

b) See the formula presented on the slides. It involves a 4×4 determinant.

c) $d_1 : 2x + y - 5 = 0$. Observe $d_1 \parallel d_2$.

$d_2 : 2x + y + 15 = 0$



Let $M' = \text{Proj}_{d_2}(M)$.

$m_{d_1} = m_{d_2} = -2 \quad \therefore \quad m_{MM'} = \frac{1}{2}$.

$\therefore \quad MM' : \quad y + 1 = \frac{1}{2}(x - 3)$.

$$\{M'\} = d_2 \cap MM'$$

We solve $\begin{cases} x - 2y - 5 = 0 \\ 2x + y + 15 = 0 \end{cases} \Rightarrow \begin{cases} x = -5 \\ y = -5. \end{cases}$

$\therefore \quad M'(-5, -5)$

$\therefore \quad O(-1, -3) \leftarrow$ midpoint of $[MM']$.

$R = d(M, O) = \sqrt{4^2 + 2^2} = \sqrt{20}.$

$\therefore \quad \boxed{\mathcal{C}(O, \sqrt{20}) : \quad (x+1)^2 + (y+3)^2 = 20.}$

2. a) Determine the position of the point $A(1,-2)$ relative to the circle $C: x^2+y^2-8x-4y-5=0$;
b) Find the intersection between the line $d: 7x-y+12=0$ and the circle $C: (x-2)^2+(y-1)^2-25=0$;
c) Determine the position of the line $d: 2x-y-3=0$ relative to the circle $C: x^2+y^2-3x+2y-3=0$.

Discussed verbally in class.

For b) and c) one has to solve a system of two equations (one linear, one quadratic) in two unknowns.

3. Find the equation of
a) the tangent line to $C: x^2+y^2-5=0$ at the point $A(-1,2)$;
b) the tangent lines to $C: x^2+y^2+10x-2y+6=0$, parallel to $d: 2x+y-7=0$;
c) the tangent lines to $C: x^2+y^2-2x+4y=0$, orthogonal on $d: x-2y+9=0$.

We solve c). Parts a) and b) can

be solved using the same method.

c) the tangent lines to $C: x^2+y^2-2x+4y=0$, orthogonal on $d: x-2y+9=0$.

Let ℓ be the sought-after line.

$$\begin{cases} m_d = \frac{1}{2} \\ d \perp \ell \end{cases} \quad \therefore \quad m_\ell = -2.$$

The equation of ℓ has the form:

ℓ : $y = -2x + n$ for some $n \in \mathbb{R}$ yet to be determined.

ℓ is tangent to $\mathcal{C}$ if and only if $|\ell \cap \mathcal{C}| = 1$.

$$\Leftrightarrow \begin{cases} x^2+y^2-2x+4y=0 \\ y=-2x+m \end{cases}$$ has only one solution $(x,y) \in \mathbb{R}^2$.

$$\Leftrightarrow \quad x^2+(-2x+m)^2-2x-8x+4m=0$$

has only one solution $x \in \mathbb{R}$

$$\Leftrightarrow \quad x^2(1+4)+x(-4m-10)+(m^2+4m)=0$$

has only one solution $x \in \mathbb{R}$.

$$\Leftrightarrow \quad \Delta = 0$$

$$\Leftrightarrow \quad -4(m^2-25)=0 \quad \Leftrightarrow \quad \boxed{m \in \{-5, 5\}}.$$

There are two such lines (which makes sense if we think geometrically).

$$\boxed{\begin{aligned} l_1 &: \quad y = -2x - 5 \\ l_2 &: \quad y = -2x + 5. \end{aligned}}$$

4. Find the foci of the ellipse $\mathcal{E} : 9x^2 + 25y^2 - 225 = 0$.

First, put the ellipse into the canonical eq:

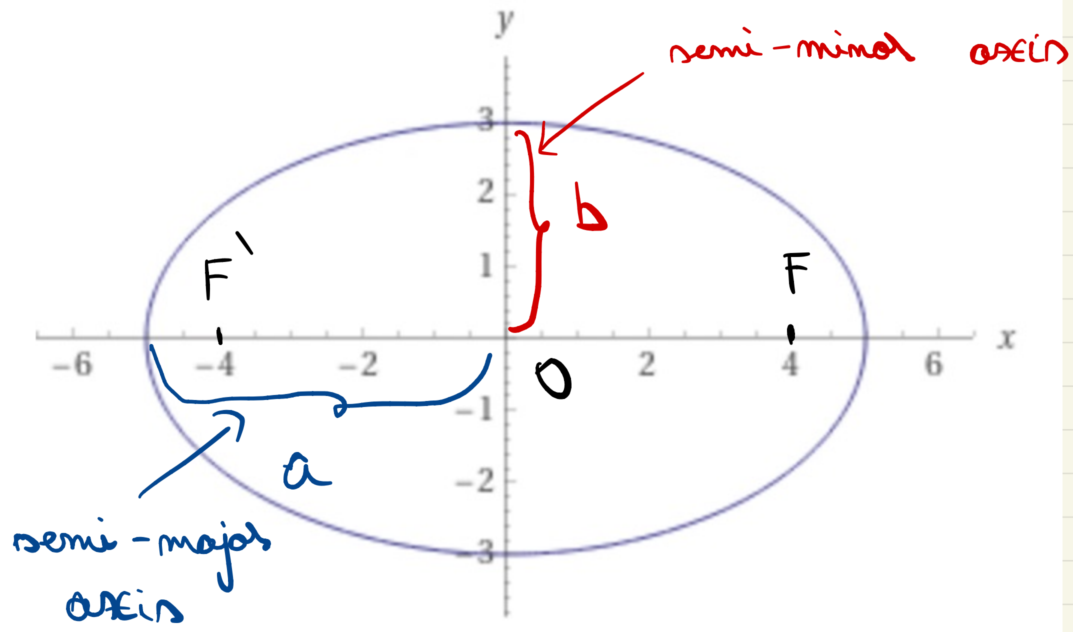
$$\mathcal{E} : \frac{x^2}{\underset{\overset{\shortparallel}{a^2}}{25}} + \frac{y^2}{\underset{\overset{\shortparallel}{b^2}}{9}} = 1.$$

So $a = 5$, $b = 3$. $a > b$ so

$F', F \in Ox$ and $F'(-c, 0)$, $F(c, 0)$

where $c^2 = a^2 - b^2 = 25 - 9 = 16$.

The foci are $F'(-4, 0)$, $F(4, 0)$



5. Find the intersection points between the line $d_1 : x + 2y - 7 = 0$ and the ellipse given by the equation $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

6. Find the position of the line $d : 2x + y - 10 = 0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$.

7. Find the equation of a line which is orthogonal on $d_1 : 2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$.

For 5 and 6 one just has to solve a system of 2 eq. in 2 unknowns (identical method with the analogous problems for circles).

7. Let ℓ be the sought-after line.

$$m_{d_1} = 1.$$

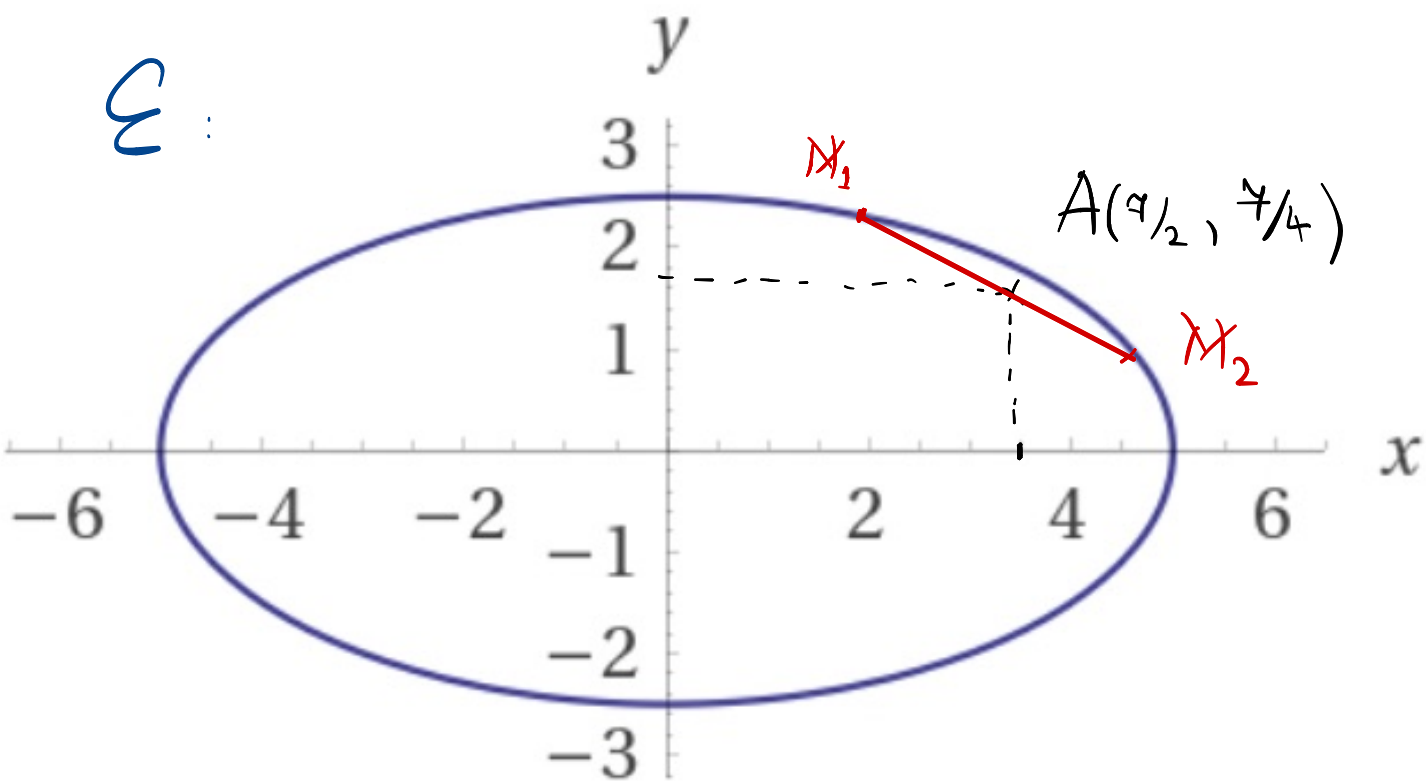
$$\ell \perp m_{d_1} \quad \therefore \quad m_{\ell} = -1.$$

$\ell : y = -x + n$ for some $n \in \mathbb{R}$ that has to be determined.

Determine the possible values for n applying the method in 3. c). (Exercise).

8. Consider the ellipse $x^2 + 4y^2 = 25$. Find the chords on the ellipse which have the point $A(7/2, 7/4)$ as their midpoint.

$\mathcal{E}$:



Let $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ be on $\mathcal{E}$ s.t.

A is the midpoint of $[M_1 M_2]$.

We have :

$$\begin{cases} x_1 + x_2 = 7 \\ y_1 + y_2 = 7/2 \\ x_1^2 + (2y_1)^2 = 25 \\ x_2^2 + (2y_2)^2 = 25 \end{cases}$$

A is mid. of $[M_1 M_2]$

since $M_1, M_2 \in \mathcal{E}$.

Using the first two relations into the last we get :

$$(7 - x_1)^2 + (7 - 2y_1)^2 = 25$$

$$\Leftrightarrow \quad 49 - 14x_1 + 49 - 28y_1 + \underbrace{x_1^2 + 4y_1^2}_{=25} = 25$$

Hence $$14x_1 + 28y_1 = 2\cdot 49 \quad \Big|\, : 14$$

$$x_1 + 2y_1 = 7.$$

$$\Rightarrow \boxed{x_1 = 7 - 2y_1}$$

But we also know

$$x_1^2 + 4y_1^2 = 25,$$ hence.

$$(7-2y_1)^2 + 4y_1^2 = 25 \Rightarrow 49 - 28y_1 + 8y_1^2 = 25$$

$$\Rightarrow 8y_1^2 - 28y_1 + 24 = 0 \quad \Big|\, : 4.$$

$$2y_1^2 - 7y_1 + 6 = 0 \Rightarrow y_1 \in \left\{\frac{3}{2}, 2\right\}.$$

$$x_1 = 7 - 2y_1$$

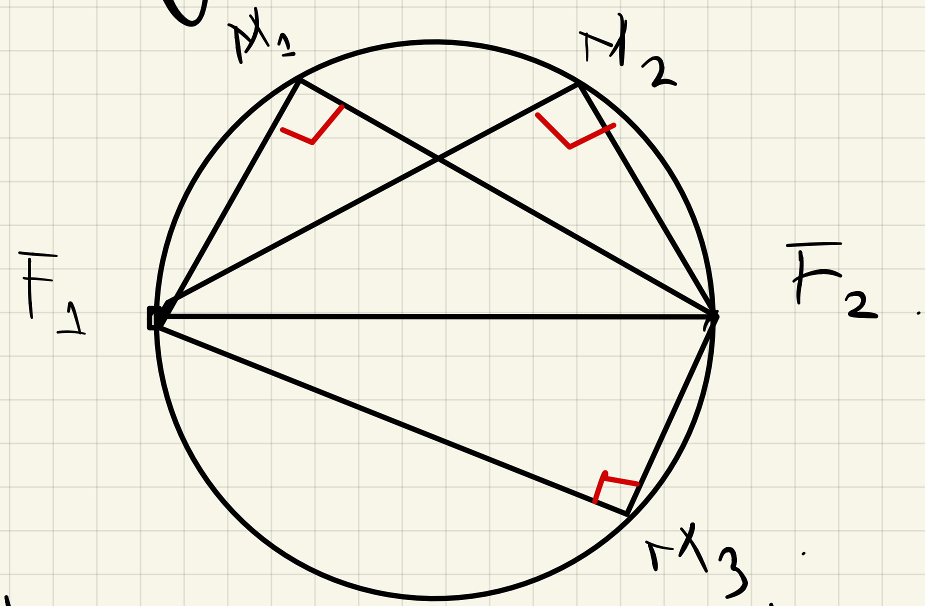
So $M_1 \in \left\{ \left(4, \frac{3}{2}\right), (3, 2) \right\}$.

Since M_2 is such that A is the midpoint of $[M_1 M_2]$ we get that $M_2 \in \left\{ \left(\frac{3}{2}, 2\right), \left(4, \frac{3}{2}\right) \right\}$.

So there is only one such chord, determined by $\left(4, \frac{3}{2}\right)$ and $(3, 2)$.

9. Consider the ellipse $\frac{x^2}{4} + y^2 = 1$ with F_1, F_2 as foci. Find the points M, situated on the ellipse for which the angle $\angle F_1 M F_2$ is right.

Think geometrically:



Given 2 fixed points F_1 and F_2, we have

$\measuredangle F_1 M F_2 = 90^\circ \iff M$ belongs to the circle of diameter $[F_1 F_2]$.

$$\mathcal{E} : \frac{x^2}{2^2} + \frac{y^2}{1^2} = 1 \quad , \text{ hence } \boxed{a = 2, b = 1.}$$

$$c = \sqrt{2^2 - 1^2} = \sqrt{3}.$$

So $F_1(-\sqrt{3}, 0)$, $F_2(\sqrt{3}, 0)$.

The circle with diameter $[F_1 F_2]$ has equation.

$$\mathcal{C}(0, \sqrt{3}) : x^2 + y^2 = 3.$$

We are therefore looking to determine

$$\mathcal{E} \cap \mathcal{C}(0, \sqrt{3}):$$

$M(x,y) \in \mathcal{E} \cap \mathcal{C}(0,\sqrt{3}) \iff$

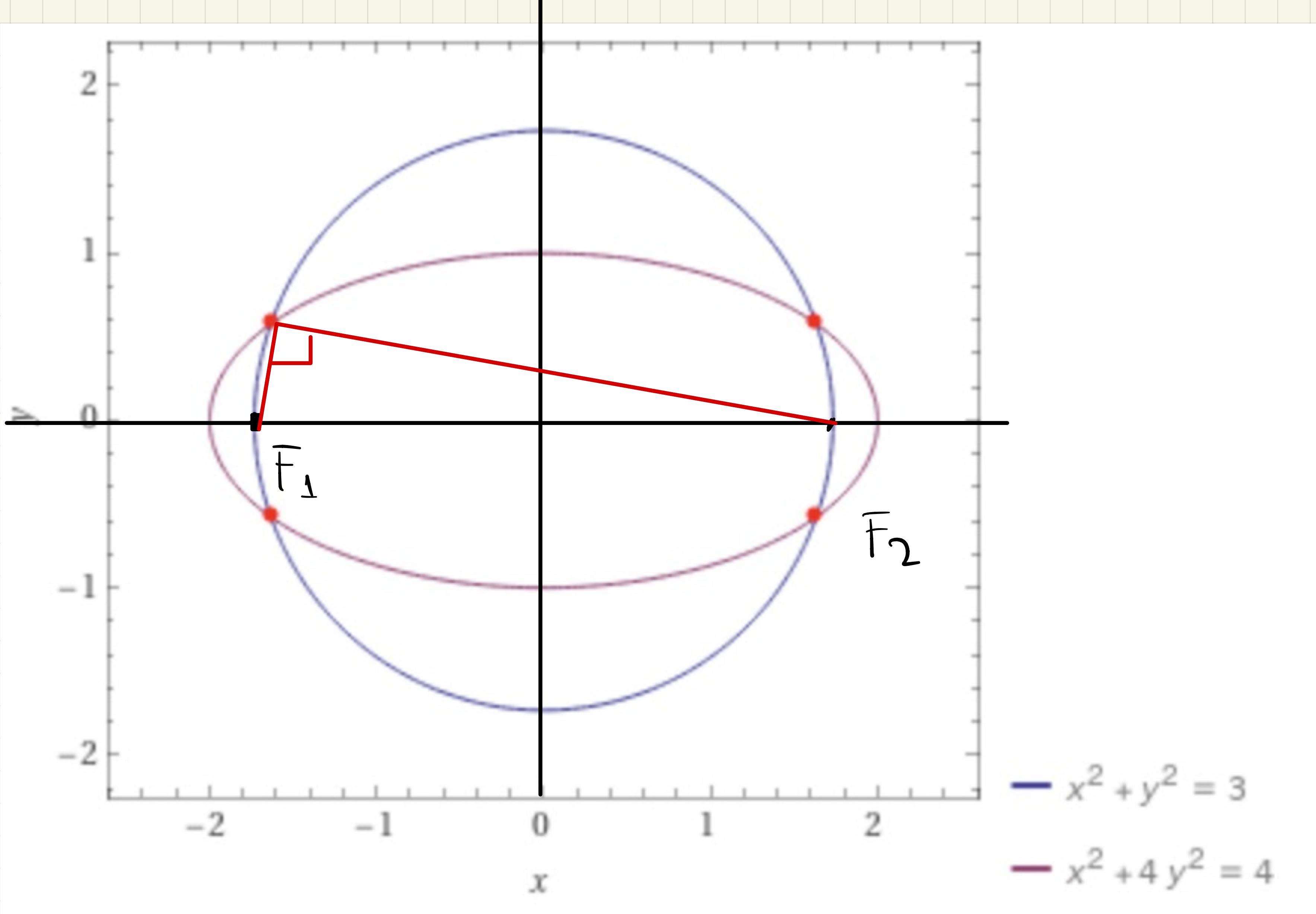
$$\begin{cases} x^2 + 4y^2 = 4 \\ x^2 + y^2 = 3 \end{cases}$$

$\Rightarrow 3y^2 = 1 \Rightarrow y = \pm \frac{1}{\sqrt{3}}$

$\Rightarrow x^2 = \frac{8}{3} \Rightarrow x = \pm 2\sqrt{\frac{2}{3}}$

The required points are

$\left(\frac{1}{\sqrt{3}}, 2\sqrt{\frac{2}{3}}\right), \left(-\frac{1}{\sqrt{3}}, 2\sqrt{\frac{2}{3}}\right), \left(\frac{1}{\sqrt{3}}, -2\sqrt{\frac{2}{3}}\right), \left(-\frac{1}{\sqrt{3}}, -2\sqrt{\frac{2}{3}}\right).$



2
1
0
−1
−2
y
−2
−1
0
1
2
x
F_1
F_2
$x^2 + y^2 = 3$
$x^2 + 4y^2 = 4$

10. Consider the ellipse $\frac{x^2}{4} + y^2 = 1$ with F_1, F_2 as foci. Find the point M on the ellipse for which $\angle F_1MF_2$ is maximal.

Let M be such a point on the ellipse.

Denote by $r := d(M, O)$.

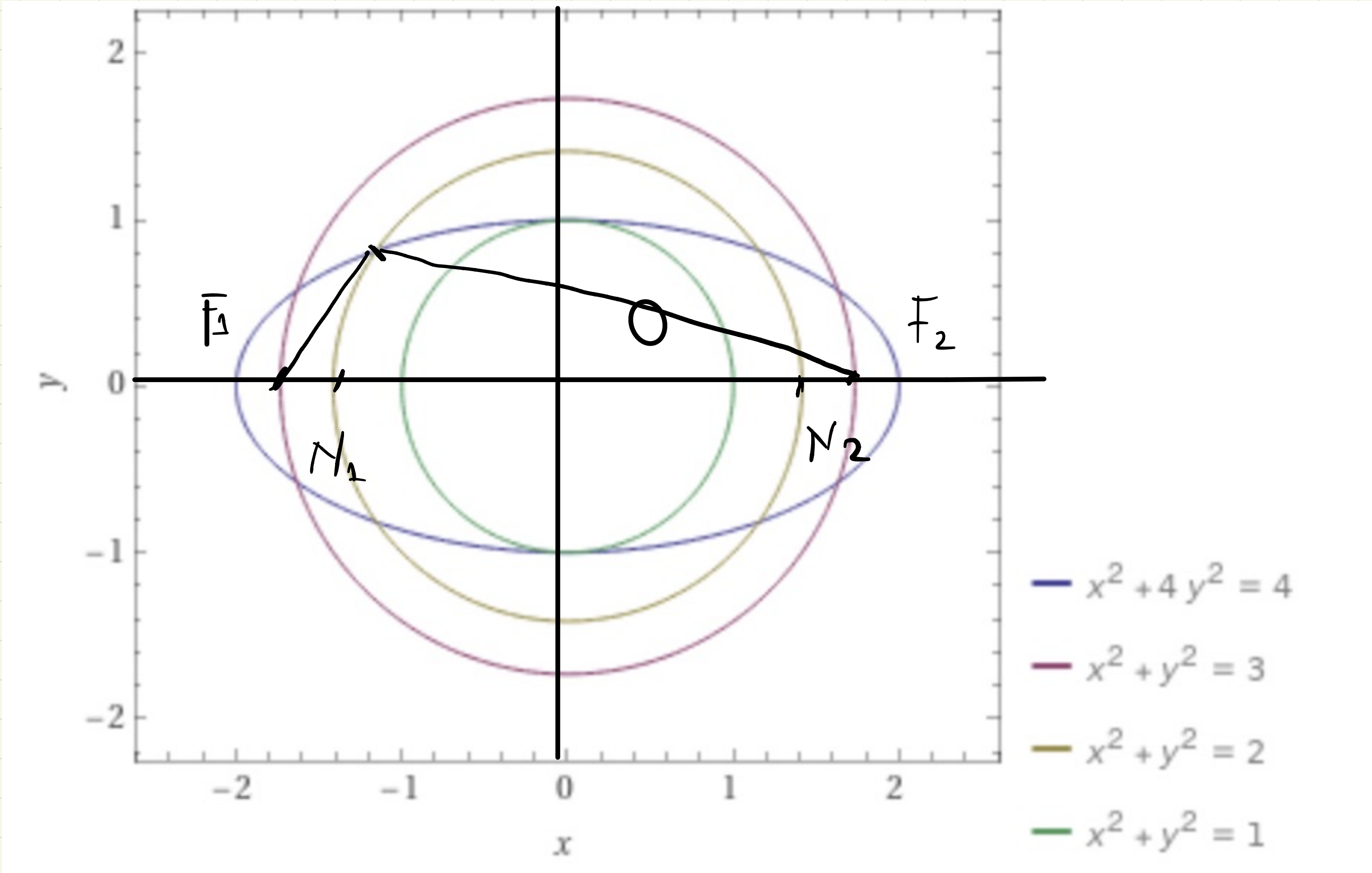
Then M is on the circle $\mathcal{C}(O, r)$.

We saw in problem $\boxed{9.}$ that if $r = \sqrt{3}$ then

$\measuredangle F_1MF_2 = 90^\circ$.

Moreover, $\measuredangle F_1MF_2 < 90^\circ$ if $M \in \mathrm{Ext}(\mathcal{C}(O, \sqrt{3}))$

and $\measuredangle F_1MF_2 > 90^\circ$ if $M \in \mathrm{Int}(\mathcal{C}(O, \sqrt{3}))$.



Exercise: Show that $\measuredangle F_1 M F_2$ is maximised when r is minimal.

So the problem will translate into finding the minimal r s.t. $\mathcal{C}(0, r) \cap \mathcal{E}$ is non-empty.

This is clearly $\boxed{r = 1}$ (the length of the semi-minor axis).

11. Consider the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $d: x + 2y = 1$.

$m_d = -\frac{1}{2}$.

Let $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ be such a chord on the ellipse.

$$m_{M_1M_2} = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{1}{2} \Rightarrow 2(y_1 - y_2) = x_2 - x_1.$$

$$(1) \qquad 9x_1^2 + 25y_1^2 = 225$$

$$(2) \qquad 9x_2^2 + 25y_2^2 = 225.$$

<u>Idea</u>: We want to find a linear relation between $x_1 + x_2$ and $y_1 + y_2$.

Let us subtract (1) and (2).

$$\Rightarrow 9(x_1 - x_2)(x_1 + x_2) + 25(y_1 - y_2)(y_1 + y_2) = 0$$

$$\Rightarrow -18(y_1 - y_2)(x_1 + x_2) + 25(y_1 - y_2)(y_1 + y_2) = 0.$$

Now $y_1 - y_2 \neq 0$, since $m_{M_1 M_2} = -\frac{1}{2} \neq 0$.

Dividing the last equality by $y_1 - y_2$

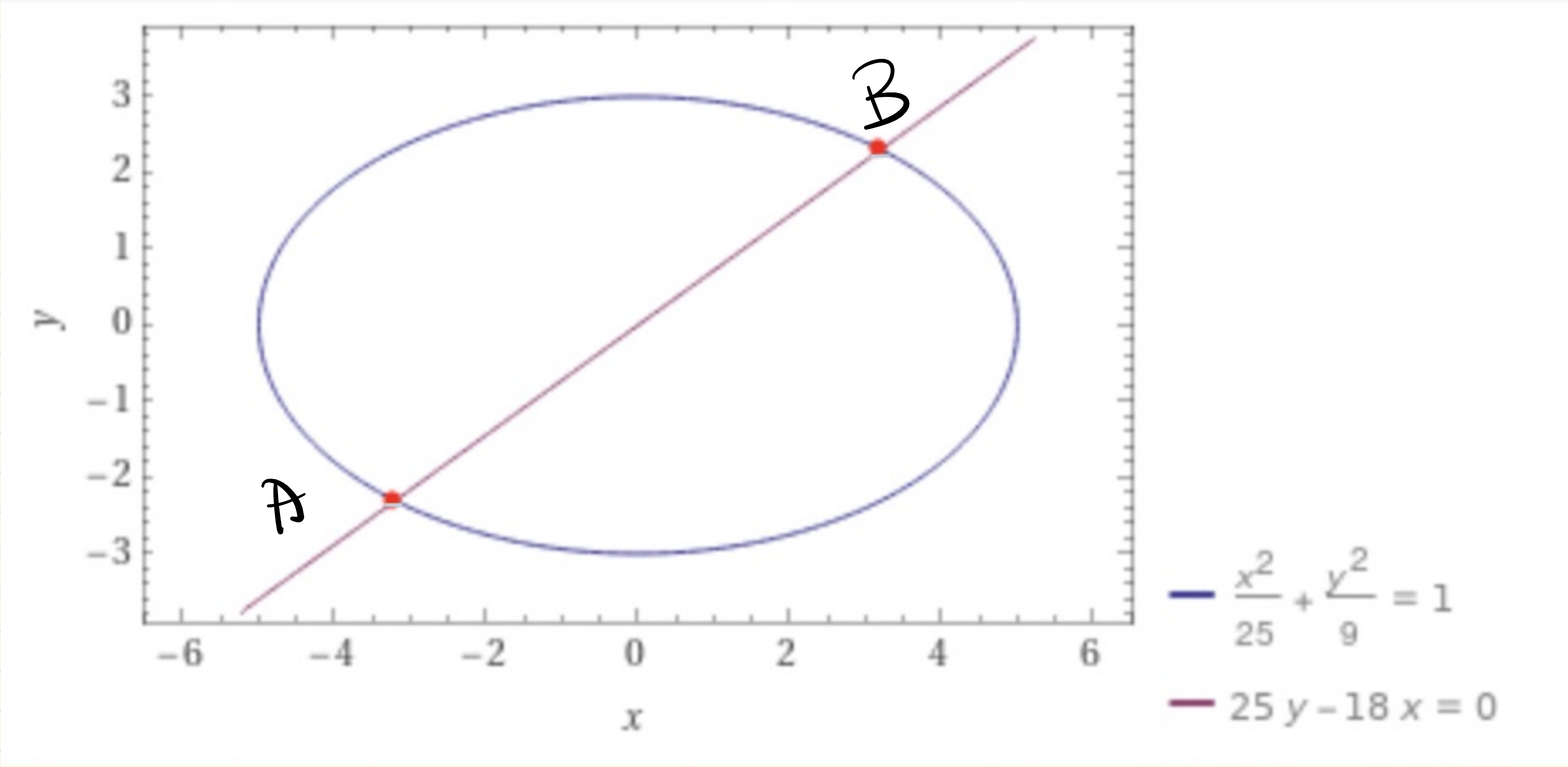
we get

$$-18(x_1 + x_2) + 25(y_1 + y_2) = 0$$

$$\Rightarrow \quad -36\left(\frac{x_1 + x_2}{2}\right) + 50\left(\frac{y_1 + y_2}{2}\right) = 0.$$

The midpoint of $M_1 M_2$ belongs to the line $\boxed{-36x + 50y = 0.}$

$$\Leftrightarrow \boxed{-18x + 25y = 0}.$$



The geometric locus we are looking for is the segment $[AB]$, where the points A and B can be easily determined by solving the system

$$\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1 \\ -18x + 25y = 0 \end{cases}$$