Course 4 March 16, 2021

1.5. Theorem If $\varphi, \psi \in L(\mathbb{R}^n, \mathbb{R}^m)$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha\varphi + \beta\psi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $[\alpha\varphi + \beta\psi] = \alpha[\varphi] + \beta[\psi]$

In fact $\forall \varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \longmapsto [\varphi] \in \mathcal{M}_{m\times n}(\mathbb{R})$

is an isomorphism of vector spaces

1.6. Theorem If $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$, and $\psi \in L(\mathbb{R}^m, \mathbb{R}^p)$, thend $\psi\circ\varphi \in L(\mathbb{R}^n, \mathbb{R}^p)$ and $[\psi\circ\varphi] = [\psi]\cdot[\varphi]$

1.7. Theorem. Let $\varphi = (\varphi_1, \dots, \varphi_m) : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$\varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \iff \forall i = 1, \dots, m : \varphi_i : \mathbb{R}^n \to \mathbb{R} \text{ is a linear mapping}$$

1.8. Theorem. Given $\varphi \in L(\mathbb{R}^n, \mathbb{R}^n)$, the following statements are equivalent:

1° φ is bijective

2° φ is injective

3° φ is surjective

4° $\det[\varphi] \neq 0$.

$$n = \dim \mathbb{R}^n = \dim \operatorname{Ker}(\varphi) + \dim \operatorname{Im}(\varphi)$$

Goal: to endow the vector space $L(\mathbb{R}^n, \mathbb{R}^m)$ with a norm.

<u>Def</u>. A function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is called a Lipschitz function if $\exists\, \alpha \geq 0$ s.t.

$$\forall\, x, y \in A: \quad \|f(x) - f(y)\| \leq \alpha \|x - y\|$$

<u>1.9. Theorem</u> Every linear mapping $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ is a Lipschitz function

<u>Proof</u>. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$\Rightarrow \quad \varphi(x) = \varphi(x_1 e_1 + \dots + x_n e_n) = x_1 \varphi(e_1) + \dots + x_n \varphi(e_n)$

$\Rightarrow \quad \|\varphi(x)\| \leq \|x_1 \varphi(e_1)\| + \dots + \|x_n \varphi(e_n)\|$

$$= \underbrace{|x_1|}_{\leq \|x\| = \sqrt{x_1^2 + \dots + x_n^2}} \cdot \|\varphi(e_1)\| + \dots + \underbrace{|x_n|}_{\leq \|x\|} \cdot \|\varphi(e_n)\|$$

$\Rightarrow \quad \|\varphi(x)\| \leq \|x\| \big(\underbrace{\|\varphi(e_1)\| + \dots + \|\varphi(e_n)\|}_{= \alpha} \big)$

$\Rightarrow \quad \|\varphi(x)\| \leq \alpha \|x\| \quad \forall\, x \in \mathbb{R}^n$

$\Rightarrow \quad \forall\, x, y \in \mathbb{R}^n: \quad \|\varphi(x) - \varphi(y)\| = \|\varphi(x - y)\| \leq \alpha \cdot \|x - y\|$

$\Rightarrow \quad \varphi$ is Lipschitz $\quad \Rightarrow \quad \varphi$ is continuous on $\mathbb{R}^n$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cancel{\Leftarrow}$

$f'(x) = \dfrac{1}{2\sqrt{x}} \quad x \in (0, 1]$

$f: [0, 1] \to \mathbb{R}, \quad f(x) = \sqrt{x}$

f is continuous on $[0, 1]$ but it is not Lipschitz.

Let $S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1 \}$

↳ unit sphere in $\mathbb{R}^n$

S^{n-1} is closed and bound $\Rightarrow$ S^{n-1} is compact

If $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ $\Rightarrow$ φ is continuous

$\Big\}$ $\xRightarrow{\text{Weierstrass Theorem}}$ we may introduce

$$\|\varphi\| := \max_{x \in S^{n-1}} \|\varphi(x)\| = \max_{x_1^2 + \dots + x_n^2 = 1} \|\varphi(x_1, \dots, x_n)\|$$

↳ the norm of the linear mapping φ

<u>1. 10. Theorem</u>. Given $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\psi \in L(\mathbb{R}^m, \mathbb{R}^p)$, the following assertions are true:

1° $\forall x \in \mathbb{R}^n$: $\|\varphi(x)\| \le \|\varphi\| \cdot \|x\|$ (1)

2° $\|\psi \circ \varphi\| \le \|\psi\| \cdot \|\varphi\|$

<u>Proof</u> 1° If $x = 0_n \Rightarrow \varphi(x) = 0_m \Rightarrow$ (1) holds with equality

If $x \neq 0_n \Rightarrow \frac{1}{\|x\|} \cdot x \in S^{n-1} \underset{\text{def } \|\varphi\|}{\Rightarrow} \left\| \varphi\left(\frac{1}{\|x\|} \cdot x \right) \right\| \le \|\varphi\| \Rightarrow \frac{1}{\|x\|} \|\varphi(x)\| \le \|\varphi\|$

$\Rightarrow \|\varphi(x)\| \le \|\varphi\| \cdot \|x\|$

2° Since $\psi \circ \varphi \in L(\mathbb{R}^n, \mathbb{R}^p) \Rightarrow \exists x_0 \in S^{n-1}$ s.t. $\|\psi \circ \varphi\| = \|(\psi \circ \varphi)(x_0)\|$

$\Rightarrow \|\psi \circ \varphi\| = \|\psi(\varphi(x_0))\| \underset{1°}{\le} \|\psi\| \cdot \|\varphi(x_0)\| \underset{1°}{\le} \|\psi\| \cdot \|\varphi\| \cdot \underbrace{\|x_0\|}_{=1} = \|\psi\| \cdot \|\varphi\|$

1.11. Theorem The function $\|\cdot\| : L(\mathbb{R}^n, \mathbb{R}^m) \to [0, \infty)$, defined by

$$\|\varphi\| := \max_{x \in S^{n-1}} \|\varphi(x)\| \qquad \varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$$

is a norm on the vector space $L(\mathbb{R}^n, \mathbb{R}^m)$

Proof We have to check the axioms of a norm.

(N_1) $\quad \|\varphi\| = 0 \iff \varphi = \theta \qquad \theta : \mathbb{R}^n \to \mathbb{R}^m,\ \theta(x) = 0_m\ \forall x \in \mathbb{R}^n$

$\boxed{\Leftarrow}$ Obvious $\qquad \boxed{\Rightarrow}$ $\|\varphi(x)\| \le \underbrace{\|\varphi\|}_{=0} \cdot \|x\| = 0 \implies \|\varphi(x)\| = 0\ \forall x \in \mathbb{R}^n$

$$\Downarrow$$

$$\varphi(x) = 0_m\ \forall x \in \mathbb{R}^n$$

$$\Downarrow$$

$$\varphi = \theta$$

(N_2) $\quad \|\alpha\varphi\| = |\alpha| \cdot \|\varphi\| \qquad \forall \alpha \in \mathbb{R},\ \forall \varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$

$$\max_{x \in S^{n-1}} \|(\alpha\varphi)(x)\| = |\alpha| \cdot \|\varphi\| \iff \begin{cases} \cdot\ \|(\alpha\varphi)(x)\| \le |\alpha| \cdot \|\varphi\| \quad \forall x \in S^{n-1} \\ \cdot\ \exists x_0 \in S^{n-1} : \|(\alpha\varphi)(x_0)\| = |\alpha| \cdot \|\varphi\| \end{cases}$$

- $\forall x \in S^{n-1}$: $\quad \|(\alpha\varphi)(x)\| = \|\alpha\,\varphi(x)\| = |\alpha| \cdot \underbrace{\|\varphi(x)\|}_{\le \|\varphi\|} \le |\alpha| \cdot \|\varphi\|$

- Weierstrass Thm $\implies \exists x_0 \in S^{n-1}$ s.t. $\|\varphi\| = \|\varphi(x_0)\| \implies \|(\alpha\varphi)(x_0)\| = \|\alpha\varphi(x_0)\|$

$$= |\alpha| \cdot \|\varphi(x_0)\|$$
$$= |\alpha| \cdot \|\varphi\|$$

(N_3) $\quad \|\varphi+\psi\| \leq \|\varphi\| + \|\psi\| \quad \forall\, \varphi, \psi \in L(\mathbb{R}^n, \mathbb{R}^m)$

Since $\varphi+\psi \in L(\mathbb{R}^n, \mathbb{R}^m) \Rightarrow \exists\, x_0 \in S^{n-1}$ s.t. $\|\varphi+\psi\| = \|(\varphi+\psi)(x_0)\| \Rightarrow$

$$\Rightarrow \|\varphi+\psi\| = \|\varphi(x_0)+\psi(x_0)\| \leq \underbrace{\|\varphi(x_0)\|}_{\leq \|\varphi\|} + \underbrace{\|\psi(x_0)\|}_{\leq \|\psi\|} \leq \|\varphi\| + \|\psi\|.$$

Remark. Let $A \subseteq \mathbb{R}$, let $a \in A \cap A'$, and let $f: A \to \mathbb{R}$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

2. Differentiation of vector functions of a real variable

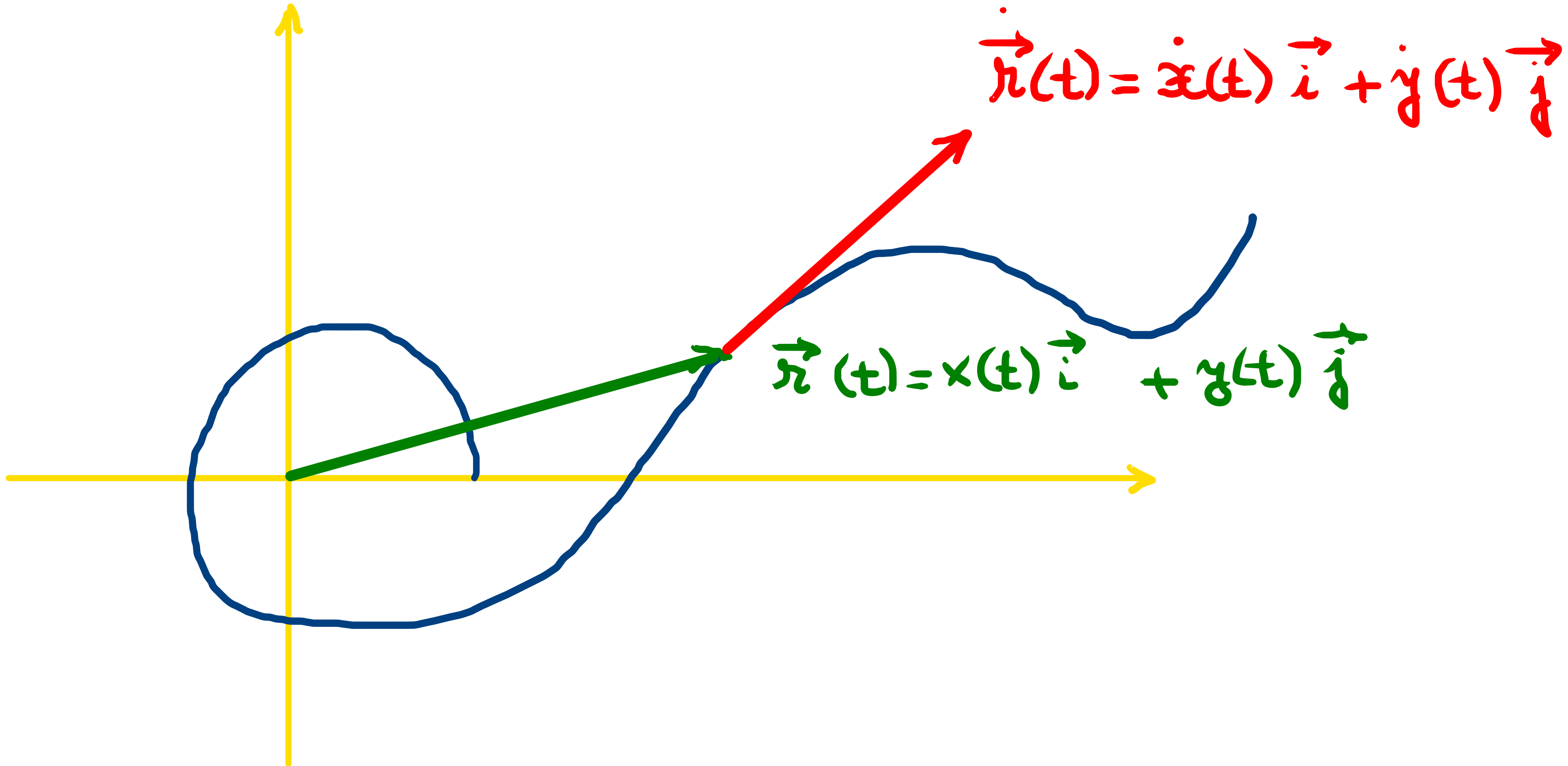
2.1. Definition Let $A \subseteq \mathbb{R}$, let $a \in A \cap A'$, and let $f: A \to \mathbb{R}^m$. If there exists the limit

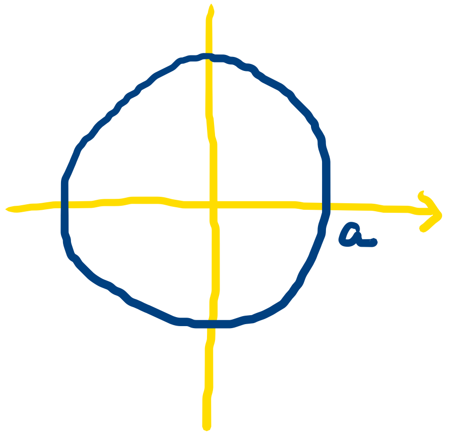
$$\lim_{x\to a} \frac{1}{x-a}\left[f(x)-f(a)\right] = b \in \mathbb{R}^m$$

then one says that f is differentiable at a. The vector $b \in \mathbb{R}^m$ is called the derivative of f at a and it is denoted by $f'(a)$ (Lagrange notation) or by $\frac{df}{dx}(a)$ (Leibniz notation)

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} \qquad \vec{r} : A \subseteq \mathbb{R} \to \mathbb{R}^2$$

$$\dot{\vec{r}}(t) = \underset{\substack{\parallel \\ x'(t)}}{\dot{x}(t)}\,\vec{i} + \dot{y}(t)\,\vec{j} \qquad \text{(Newton's notation)}$$





$x(t) = a\cos t$

$y(t) = a\sin t$

2.2. Theorem. Let $A \subseteq \mathbb{R}$, $a \in A \cap A'$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$. Then the following assertions are true:

1° If f is differentiable at a $\Rightarrow$ $f_1, \dots, f_m$ are diff at a and

(*) $\qquad f'(a) = (f_1'(a), \dots, f_m'(a))$

2° If $f_1, \dots, f_m$ are diff. at a $\Rightarrow$ f is diff. at a and (*) holds.

Remark. The MVT does not remain true for vector functions

$f:[0,2\pi]\to\mathbb{R}^2$, $f(x)=(\cos x,\sin x)$

Suppose that $\exists\, c\in(0,2\pi)$ s.t. $\left.\begin{array}{l} f(2\pi)-f(0)=2\pi\cdot f'(c) \\ f(2\pi)=(1,0)=f(0) \\ f'(x)=(-\sin x,\cos x) \end{array}\right\}\Rightarrow$

$\Rightarrow\ (0,0)=2\pi(-\sin c,\cos c)\ \Rightarrow\ \sin c=0=\cos c$ $\Rightarrow\Leftarrow$

- $f=(f_1,\dots,f_m):[a,b]\longrightarrow\mathbb{R}^m$

$\not\exists\, c\in(a,b):\ f(b)-f(a)=(b-a)f'(c) \iff \begin{cases} f_1(b)-f_1(a)=(b-a)f_1'(c) \\ \vdots \\ f_m(b)-f_m(a)=(b-a)f_m'(c) \end{cases}$

2.3. Theorem (MVT for vector functions of real variable). If $f:[a,b]\to\mathbb{R}^m$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c\in(a,b)$ s.t.

$$\|f(b)-f(a)\|\le(b-a)\cdot\|f'(c)\|$$

<u>Proof</u> If $f(b)=f(a) \Rightarrow c$ can be chosen arbitrarily in (a,b).

Assume next that $f(b)\neq f(a)$. Consider the vector

$$v = \frac{1}{\|f(b)-f(a)\|}\left[f(b)-f(a)\right] \quad \Rightarrow \quad v\in\mathbb{R}^m, \quad v=(v_1,\dots,v_m)$$

$$\|v\|=1$$

Let $g:[a,b]\to\mathbb{R}$, $\quad g(x) := \langle v, f(x)\rangle$

$$= v_1 f_1(x)+\dots+v_m f_m(x) \quad \text{where } f=(f_1,\dots,f_m)$$

Since f is continuous on $[a,b] \Rightarrow f_1,\dots,f_m$ are continuous on $[a,b]$ $\Big\}$
is differentiable on $(a,b) \xRightarrow{T2.2} f_1,\dots,f_m$ are diff. on (a,b) $\Big\} \Rightarrow$

$\Rightarrow g$ is continuous on $[a,b]$ and diff. on (a,b)

By applying the MVT for real-valued functions of a real variable to $g \Rightarrow$

$\Rightarrow \exists\, c\in(a,b)$ s.t. $\quad g(b)-g(a) = (b-a)\,g'(c). \qquad (1)$

We have $\quad g(b)-g(a) = \langle v, f(b)\rangle - \langle v, f(a)\rangle = \langle v, f(b)-f(a)\rangle =$

$$= \left\langle \frac{1}{\|f(b)-f(a)\|}\left[f(b)-f(a)\right], f(b)-f(a)\right\rangle =$$

$$= \frac{1}{\|f(b)-f(a)\|}\langle f(b)-f(a), f(b)-f(a)\rangle = \frac{1}{\cancel{\|f(b)-f(a)\|}}\cdot\|f(b)-f(a)\|^2$$

$$\Rightarrow \quad g(b)-g(a)=\|f(b)-f(a)\| \qquad (2)$$

$$g'(x)=v_1 f_1'(x)+\dots+v_m f_m'(x)=\langle v, f'(x)\rangle \Rightarrow g'(c)=\langle v, f'(c)\rangle \qquad (3)$$

$$\begin{matrix}(1)\\(2)\\(3)\end{matrix} \Rightarrow \|f(b)-f(a)\| = (b-a)\langle v, f'(c)\rangle \le (b-a)\,|\langle v, f'(c)\rangle| \le \leftarrow \text{Cauchy-Schwarz}$$

$$\le (b-a)\cdot \underbrace{\|v\|}_{=1}\cdot \|f'(c)\| = (b-a)\cdot\|f'(c)\|$$

Remark. $A\subseteq\mathbb{R}$, $f: A\to\mathbb{R}$, $a\in A\cap A'$

$\varphi\in L(\mathbb{R},\mathbb{R}) \Leftrightarrow$
$\Leftrightarrow \exists c\in\mathbb{R}$ s.t. $\varphi(x)=cx \ \forall x\in\mathbb{R}$

f is differentiable at a $\Leftrightarrow \exists \lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\in\mathbb{R}$

$\Leftrightarrow \exists c\in\mathbb{R}$ s.t. $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}=c$

$\Leftrightarrow \exists c\in\mathbb{R}$ s.t. $\lim\limits_{x\to a}\dfrac{f(x)-f(a)-c(x-a)}{x-a}=0$

$\Leftrightarrow \exists \varphi\in L(\mathbb{R},\mathbb{R})$ s.t. $\lim\limits_{x\to a}\dfrac{f(x)-f(a)-\varphi(x-a)}{x-a}=0$

! Maurice Fréchet

3. Differentiation of vector functions of vector variable

3.1. Lemma. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}^m$, $\varphi_1, \varphi_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$. If

$$\lim_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - \varphi_1(x-a)\right] = 0_m = \lim_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - \varphi_2(x-a)\right]$$

then $\varphi_1 = \varphi_2$.

3.2. Definition. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}^m$. One says that f is (Fréchet) differentiable at a if $\exists\, \varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t.

(1) $$\lim_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - \varphi(x-a)\right] = 0_m.$$

If f is differentiable at a $\overset{L3.1}{\Longrightarrow}$ $\exists!$ linear mapping φ satisfying (1).

This unique linear mapping is called the (Fréchet) differential of f at a and it will be denoted by $df(a)$.

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ $\qquad\qquad$ $df(a): \mathbb{R}^n \to \mathbb{R}^m$

f is diff. at $a \in \operatorname{int} A$ $\qquad\qquad$ $\forall\, h \in \mathbb{R}^n: \; df(a)(h) \in \mathbb{R}^m$

We have

(2) $$\lim_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - df(a)(x-a)\right] = 0_m$$

3.3. Proposition Given $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$, the following assertions are true:

1° If f is diff. at a $\Rightarrow$ $\exists\, \omega : A \to \mathbb{R}^m$ s.t.

(3) $\lim\limits_{x \to a} \omega(x) = 0_m$

and

(4) $\forall x \in A :$ $f(x) = f(a) + df(a)(x-a) + \|x-a\| \cdot \omega(x)$

2° If $\exists\, \varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\exists\, \omega : A \to \mathbb{R}^m$ s.t.

(3) $\lim\limits_{x \to a} \omega(x) = 0_m$

and

(5) $\forall x \in A :$ $f(x) = f(a) + \varphi(x-a) + \|x-a\| \omega(x)$

then f is diff. at a and $df(a) = \varphi$.

Proof. 1° f diff at a $\Rightarrow$ $\lim\limits_{x \to a} \underbrace{\frac{1}{\|x-a\|}\left[f(x) - f(a) - df(a)(x-a)\right]}_{:= \omega(x) \text{ for } x \in A \setminus \{a\}} = 0_m$

$\omega(a) = 0_m$

$\Rightarrow$ ω satisfies (3) and (4)

2° By (5) $\Rightarrow$ $\lim\limits_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - \varphi(x-a)\right] = \lim\limits_{x \to a} \omega(x) = 0_m$

$\Rightarrow$ f is diff at a and $df(a) = \varphi$

3.4. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$. If f is diff. at a $\Rightarrow$

$\Rightarrow$ f is continuous at a

Proof f is diff. at a $\overset{3.3}{\Longrightarrow}$ $\exists\, \omega : A \to \mathbb{R}^m$ s.t. $\lim\limits_{x \to a} \omega(x) = 0_m$ and

$$\forall x \in A: \quad f(x) = f(a) + df(a)(x-a) + \|x-a\|\,\omega(x)$$

$$0 \leq \|f(x) - f(a)\| = \|df(a)(x-a) + \|x-a\|\,\omega(x)\|$$

$$\leq \|df(a)(x-a)\| + \|x-a\| \cdot \|\omega(x)\| \leq$$

$$\leq \|df(a)\| \cdot \|x-a\| + \|x-a\| \cdot \|\omega(x)\| \qquad \|\omega(x)\| \xrightarrow{x \to a} 0$$

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$$0 \xleftarrow{x \to a}$$

$\Rightarrow$ $\lim\limits_{x \to a} \|f(x) - f(a)\| = 0$ $\Rightarrow$ $\lim\limits_{x \to a} f(x) = f(a)$ $\Rightarrow$ f is continuous at a

3.5. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$. Then:

1° If f is diff at a $\Rightarrow$ $f_1, \dots, f_m$ are diff at a and

$$(6) \qquad df(a) = \big(df_1(a), \dots, df_m(a)\big)$$

2° If $f_1, \dots, f_m$ are diff. at a $\Rightarrow$ f is diff. at a and (6) holds

$n=1 \qquad f: A \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m$

$a \in A \cap A' \qquad f'(a) \in \mathbb{R}^m$

?

$a \in \operatorname{int} A \qquad df(a) \in L(\mathbb{R}, \mathbb{R}^m)$