ANALYTIC GEOMETRY, PROBLEM SET 6

Representations of the line in plane

1. Find the equation of the line passing through the intersection point of the lines $d_1 : 2x - 5y - 1 = 0$ and $d_2 : x + 4y - 7 = 0$ and through a point M which divides the segment $[AB]$, given by $A(4, -3)$ and $B(-1, 2)$, into the ratio $k = 2/3$.

2. Find the equation of the line passing through the intersection point of $d_1 : 3x-2y+5 = 0$, $d_2 : 4x + 3y - 1 = 0$ and intersecting the Oy axis at the point A with $OA = 3$.

3. Find the parametric equations of the line through P_1 and P_2, when
(1) $P_1(3, -2)$, $P_2(5, 1)$;
(2) $P_1(4, 1)$, $P_2(4, 3)$.
In each case, find the vector equation of the line passing through these points.

4. Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.

5. Show that the equations $x = 3 - t$, $y = 1 + 2t$ and $x = -1 + 3t$, $y = 9 - 6t$. represent the same line. Write down a director vector for this line.

6. The points $M_1(1, 2)$, $M_2(3, 4)$ and $M_3(5, -1)$ are the midpoints of the sides of a triangle. Write down the equations of the lines determined by the sides of the triangle.

7. Given the line $d : 2x + 3y + 4 = 0$, find the equation of a line d_1 passing through the point $M_0(2, 1)$, in the following situations: a) d_1 is parallel with d; b) d_1 is orthogonal on d; c) the angle determined by d and d_1 is $\pi/4$.

8. The vertices of the triangle $\triangle ABC$ are the intersection points of the lines $d_1 : 4x+3y-5 = 0$, $d_2 : x - 3y + 10 = 0$, $d_3 : x - 2 = 0$. a) Find the coordinates of A, B and C. b) Find the equations of the median lines of the triangle. c) Find the equations of the heights of the triangle.

9. Find the coordinates of the symmetrical of the point $P(-5, 13)$ with respect to the line $d : 2x - 3y - 3 = 0$.

10. Find the coordinates of the point P on the line $d : 2x - y - 5 = 0$, for which the sum $AP + PB$ attains its minimum, when $A(-7, 1)$ and $B(-5, 5)$.

11. Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $4x - y + 2 = 0$, $x - 4y - 8 = 0$ and $x + 4y - 8 = 0$.

Date: October 29, 2021.

1. Find the equation of the line passing through the intersection point of the lines $d_1 : 2x - 5y - 1 = 0$ and $d_2 : x + 4y - 7 = 0$ and through a point M which divides the segment $[AB]$, given by $A(4, -3)$ and $B(-1, 2)$, into the ratio $k = 2/3$.

Let $\{P\} = d_1 \cap d_2$. If $P(x_p, y_p)$.

$$\begin{cases} 2x_p - 5y_p - 1 = 0 \\ x + 4y - 7 = 0 \end{cases} \quad \therefore \quad \boxed{x_p = 3,\ y_p = 1.}$$

$P(3, 1)$.

$$\frac{MA}{MB} = \frac{2}{3} \leftarrow K$$

Let $M(x_M, y_M)$.

Then
$$x_M = \frac{1}{K+1} \cdot x_A + \frac{K}{K+1} \cdot x_B$$
$$= \frac{3}{5} \cdot 4 + \frac{\frac{2}{3}}{\frac{5}{3}} \cdot (-1)$$
$$= \frac{12}{5} - \frac{2}{5} = \underline{\underline{2}}.$$

$$y_M = \frac{1}{k+1} \cdot y_A + \frac{k}{k+1} \cdot y_B$$

$$= \frac{3}{5} \cdot (-3) \qquad \frac{2}{5} \cdot 2$$

$$= \frac{-5}{5} = -1.$$

We found $M(2, -1)$.

$P(3, 1)$.

The vector equation of the line MP is:

$$\overline{MP} = \overline{MO} + \overline{OP} = \overline{OP} - \overline{OM}$$

$$\therefore \overline{MP}(1, 2).$$

A point X in the plane, belongs to the line MP if and only if.

" $\overline{OX} = \overline{OM} + t \cdot \overline{MP}$, where $t \in \mathbb{R}$."

The vector equation of the line.

$X(x, y)$

The parametric equations:

$$\begin{cases} x = 2 + t \cdot 1 \\ y = -1 + t \cdot 2 \end{cases}, \quad t \in \mathbb{R}.$$

2. Find the equation of the line passing through the intersection point of $d_1 : 3x - 2y + 5 = 0$, $d_2 : 4x + 3y - 1 = 0$ and intersecting the Oy axis at the point A with $OA = 3$.

Let $\{P\} = d_1 \cap d_2$; $P(x_P, y_P)$

$$\begin{cases} 3x_P - 2y_P + 5 = 0 \\ 4x_P + 3y_P - 1 = 0 \end{cases}$$

We find $P\left(-\frac{13}{17}, \frac{23}{17}\right)$.

From the hypothesis we find $A(0, \pm 3)$.

I. $A(0,3)$. We want the equation of PA. $\overline{PA}\left(\frac{13}{17}, \frac{28}{17}\right)$.

The symmetric equation of PA.

$$\frac{x-0}{\frac{13}{17}} = \frac{y-3}{\frac{28}{17}}.$$

II. $A(0,-3)$ is solved similarly.

3. Find the parametric equations of the line through P_1 and P_2, when

(1) $P_1(3,-2)$, $P_2(5,1)$;

(2) $P_1(4,1)$, $P_2(4,3)$.

In each case, find the vector equation of the line passing through these points.

(1) A director vector is $\overline{P_1P_2}(2,3)$.

The parametric equations are

$$\begin{cases} x = 3 + 2\cdot t \\ y = -2 + 3\cdot t \end{cases}, \quad t \in \mathbb{R}.$$

Let $P(x, y)$ in the plane and O be the origin. Then $P \in \overline{P_1 P_2}$ iff $\exists t \in \mathbb{R}$ s.t.

$$\overline{OP} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \overline{OP_1} + t \cdot \overline{P_1 P_2}$$

(2) Solved similarly.

4. Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.

$$\begin{cases} x = -5 + 2\cdot t \\ y = 2 + 3\cdot t \end{cases}, \quad t \in \mathbb{R}.$$

d_1 $\qquad$ d_2

5. Show that the equations $x = 3 - t$, $y = 1 + 2t$ and $x = -1 + 3t$, $y = 9 - 6t$, represent the same line. Write down a director vector for this line.

$t = 0 \Rightarrow P_1(3, 1) \in d_1$

$t = 1 \Rightarrow P_2(2, 3) \in d_1$

• Do $P_1, P_2 \in d_2$?

$$\begin{cases} -1 + 3t = 3 \\ 9 - 6t = 1 \end{cases} \Rightarrow t = \frac{4}{3}$$

When $t = \frac{4}{3}$, we see that $P_1 \in d_2$.

$$\begin{cases} -1 + 3t = 2 \\ 9 - 6t = 3 \end{cases} \Rightarrow t = 1.$$

When $t = 1$, we see that $P_2 \in d_2$.

As $P_1 \neq P_2$, d_1 and d_2 are the same line.

Method II. $\quad P_2(2,3) \in d_1 \cap d_2$ (choose $t = 1$ in both param.)

d_1 has d. v. $\quad \overline{v_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

d_2 has d. v. $\quad \overline{v_2} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$

$\overline{v_2} = -3 \cdot \overline{v_1}$, so $\overline{v_1} \parallel \overline{v_2}$.

$\therefore \quad d_1 \parallel d_2$ and $d_1 \cap d_2 \neq \emptyset$

$\therefore \quad d_1 = d_2$.

Method III $\quad d_1 : \begin{cases} x = 3 - t \\ y = 1 + 2t. \end{cases}$

$$t = 3 - x = \frac{y-1}{2} \quad \Leftarrow)$$

$$\frac{x-3}{-1} = \frac{y-1}{2}.$$

$$\Leftrightarrow \quad 2x - 6 = -y + 1 \quad \Leftrightarrow$$

$$\boxed{2x + y - 7 = 0}$$

General eq. of the line.

$$d_2 : \begin{cases} x = -1 + 3t \\ y = 9 - 6t \end{cases}$$

$$\Rightarrow \quad t = \frac{x+1}{3} = \frac{y-9}{-6}$$

$$\Leftrightarrow \quad -6x - 6 = 3y - 27$$

$$6x + 3y - 21 = 0 \quad \Big| : 3$$

$$\boxed{2x + y - 7 = 0}$$ General equation

6. The points $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ are the midpoints of the sides of a triangle. Write down the equations of the lines determined by the sides of the triangle.

Let ΔABC be the triangle.

We want to find $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$.

$$1 = \frac{x_A + x_B}{2}, \quad 3 = \frac{x_B + x_C}{2}, \quad 5 = \frac{x_C + x_A}{2}.$$

$$\therefore \quad 9 = x_A + x_B + x_C.$$

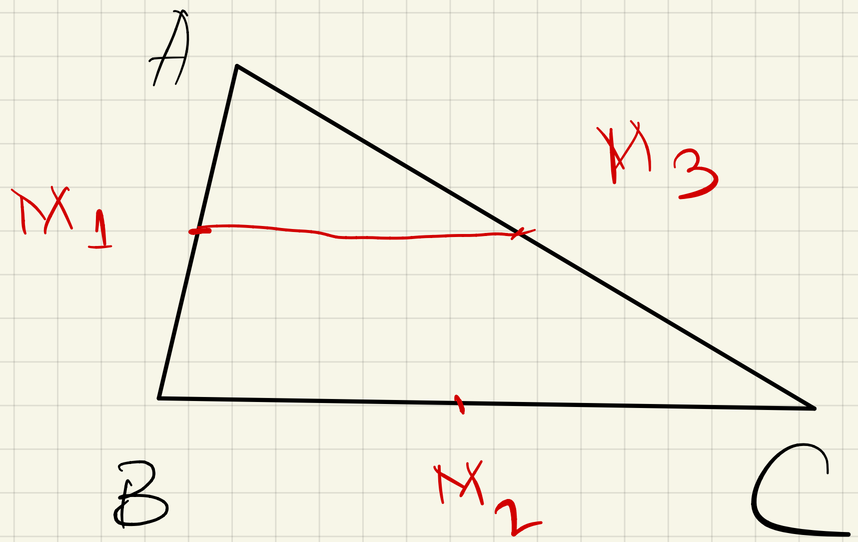
$$x_A = 9 - 2 \cdot 3 = 3, \quad x_B = 9 - 10 = -1.$$

$x_C = 9 - 2 = 7.$

Similarly, find y_A, y_B, y_C.

· Write the equation of lines AB, BC and CA.

<u>Method II</u>.



$M_1M_3 \parallel BC$.

· The equation of $\overline{BC}$:

$\overline{M_1 M_3}$ is a director vector of BC.

The vector eq. of BC is:

$$\overline{OP} = \overline{OM_2} + t \cdot \overline{M_1M_3}, \text{ where } t \in \mathbb{R}.$$

Write down the equations of the lines determined by the sides of the triangle.

7. Given the line $d: 2x+3y+4=0$, find the equation of a line d_1 passing through the point $M_0(2,1)$, in the following situations: a) d_1 is parallel with d; b) d_1 is orthogonal on d; c) the angle determined by d and d_1 is $\pi/4$.

a) Let's go from General eq of d to Parametric equation.

$$d: \; 2x + 3y + 4 = 0.$$

Take $x = t$, then $y = -\frac{4}{3} - \frac{2}{3} \cdot t$.

$$d: \begin{cases} x = t \\ y = -\frac{4}{3} - \frac{2}{3} \cdot t \end{cases}, \; t \in \mathbb{R}.$$

A director vector of d is $\overline{v} = \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$

or, alternatively $\overline{v}' = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

A set of parametric equations for $d_1 \parallel d$

$$d_1 : \begin{cases} x = 2 + 3 \cdot t \\ y = 1 + -2 \cdot t \end{cases}, \quad t \in \mathbb{R}.$$