ANALYTIC GEOMETRY, PROBLEM SET 8

Line bundles, angle between two lines (in 2D) and the equation of the plane (3D)

1. Given the bundle of lines of equations $(1-t)x + (2-t)y + t - 3 = 0, t \in \mathbb{R}$ and $x + y - 1 = 0$, find:

(1) the coordinates of the vertex of the bundle;
(2) the equation of the line in the bundle which cuts Ox and Oy in M, respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$;

2. Let B be the bundle of vertex $M_0(5, 0)$. An arbitrary line from B intersects the lines $d_1 : y - 2 = 0$ and $d_2 : y - 3 = 0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

Try it.

3. Determine the angle between the lines:

(1) $y = 2x + 1$ and $y = -x + 2$;
(2) $y = 3x - 4$ and $x = 3 + t$, $y = -1 - 2t$ for $t \in \mathbb{R}$.
(3) $y = \frac{2}{5}x + 1$ and $4x + 3y - 12 = 0$.

$y - 1 = m \cdot (x - 3)$.

4. Determine the equation of the line which passes through $A(3, 1)$ and makes an angle of $45°$ with the line $2x + 3y - 1 = 0$.

5. Consider the triangle given by the points $A(1, -2)$, $B(5, 4)$ and $C(-2, 0)$. Find the equations of the internal, respectively external bisectors corresponding to the vertex A of this triangle.

6. The points of intersection of the lines $d_1 : x + 2y - 1 = 0$, $d_2 : 5x + 4y - 17 = 0$ and $d_3 : x - 4y + 11 = 0$ determine a triangle. Find the equations of the altitudes of these triangles without determining the coordinates of the vertices of the triangle!

7. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be distinct points in space. Prove that the equation of the plane containing P_1 and P_2 that is parallel to a vector $\overline{a} = (l, m, n)$ is

$\overline{P_1P_2} \parallel \pi$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0.$$

8. Find the equation of the plane passing through $P(7, -5, 1)$ which determines on the positive half-axes three segments of the same length.

9. Find the equation for each of the following planes:

(a) containing $P(2, 1, -1)$ and perpendicular to the vector $\overline{n} = (1, -2, 3)$;

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(b) determined by $O(0,0,0)$, $P_1(3,-1,2)$ and $P_2(4,-2,-1)$;
(c) containing $P(3,4,-5)$ and parallel to both $\overline{a_1}(1,-2,4)$ and $\overline{a_2}(2,1,1)$;
(d) containing the points $P_1(2,-1,-3)$ and $P_2(3,1,2)$ and parallel to the vector $\overline{a}(3,-1,-4)$.

10. Find the equation of the plane containing the perpendicular lines through $P(-2,3,5)$ on the planes $\pi_1 : 4x + y - 3z + 13 = 0$ and $\pi_2 : x - 2y + z - 11 = 0$.

1. Given the bundle of lines of equations $\underbrace{(1-t)x+(2-t)y+t-3=0}_{\ell_t:}, t \in \mathbb{R}$ and ℓ_∞: $x+y-1=0$, find:

(1) the coordinates of the vertex of the bundle;

(2) the equation of the line in the bundle which cuts Ox and Oy in M, respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$;

$P(x_p, y_p)$

Bundle of lines through P in

$$\alpha(x - x_p) + \beta \cdot (y - y_p) = 0, \ \forall (\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Reduced form of the bundle through P

$$\begin{cases} d_t : t \cdot (x - x_p) + y - y_p = 0, & \forall t \in \mathbb{R} \\ d_\infty : x - x_p = 0 \end{cases}$$

Returning to our problem.

Let $P(x_p, y_p)$ be the vertex.

$$\begin{array}{ll} P \in \ell_\infty \ \therefore & \begin{cases} x_p + y_p - 1 = 0 \\ -x_p = 1 \end{cases} \\ P \in \ell_2 \ \therefore & \end{array}$$

Plug $t=2$

So $\boxed{P(-1, 2)}$.

Sanity check: $P \in \ell_t, \ \forall t \in \mathbb{R}$?

$$(1-t)\cdot(-1)+(2-t)\cdot 2+t-3 \overset{?}{=} 0.$$

(1) the coordinates of the vertex of the bundle;
(2) the equation of the line in the bundle which cuts Ox and Oy in M, respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$;

- Check $l_\infty : x+y-1=0.$

$\therefore\ M(1,0)$ and
$N(0,1)$

$$1^2\cdot 1^2 \neq 4\cdot(1^2+1^2)$$

- We search for $\boxed{l_t}$, $t\in\mathbb{R}$.

$$l_t : (1-t)\cdot x + (2-t)\cdot y + t - 3 = 0,\ t\in\mathbb{R}$$

$M \in Ox \cap l_t$

$N \in Oy \cap l_t$

$$\boxed{t \neq 1, 2.}$$

$M(x_M, 0)$ s.t. $(1-t)\cdot x_M = 3-t$

$N(0, y_N)$ s.t $(2-t)\cdot y = 3-t$

$$x_M = \frac{3-t}{1-t}, \quad y_N = \frac{3-t}{2-t}.$$

$$OM^2 = \left(\frac{3-t}{1-t}\right)^2 \text{ and } ON^2 = \left(\frac{3-t}{2-t}\right)^2.$$

Plugging in our equation we get

$$\frac{(3-t)^4}{(1-t)^2\cdot(2-t)^2} = 4\cdot\left[\frac{(3-t)^2}{(1-t)^2} + \frac{(3-t)^2}{(2-t)^2}\right]$$

- If $3-t=0$, then $\boxed{t=3}$

$l_3 : -2x - y = 0. \ni O(0,0)$

$\Rightarrow M(0,0)$ and $N(0,0)$

So l_3 satisfies the required eq^n.

- If $3-t \neq 0$,

$$(3-t)^2 = 4(2-t)^2 + 4(1-t)^2.$$

This is a quadratic equation in t.

Solve it to find $\boxed{t_{1,2}}$.

Answer: $\boxed{l_3,\ l_{t_1},\ l_{t_2}}$.

this triangle.

6. The points of intersection of the lines $d_1 : x + 2y - 1 = 0$, $d_2 : 5x + 4y - 17 = 0$ and $d_3 : x - 4y + 11 = 0$ determine a triangle. Find the equations of the altitudes of these triangles without determining the coordinates of the vertices of the triangle!

Let $\{A\} = d_1 \cap d_2$.

The bundle of lines that pass through A is

$$r\cdot(x+2y-1) + s\cdot(5x+4y-17) = 0,$$

$$(r,s) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

· We look for the altitude h_A in the reduced bundle : (think of $t = \frac{r}{s}$).

$$h_{A,t}: \quad t\cdot(x+2y-1) + 5x+4y-17 = 0.$$

Slope of $h_{A,t}$ is $\boxed{m_t = \frac{-5-t}{4+2t}}$

$$m_t \cdot m_{d_3} = -1.$$

$$m_t \cdot \frac{1}{4} = -1$$

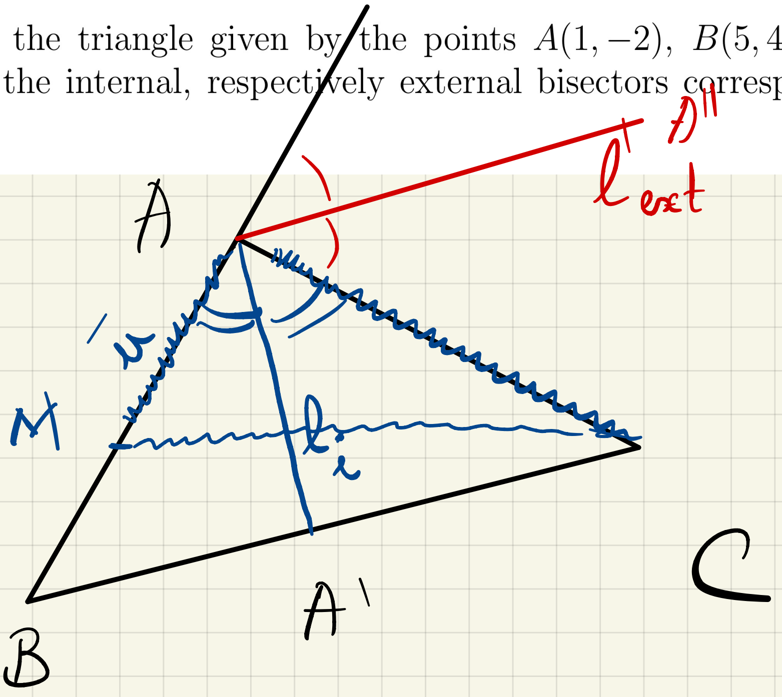
$$\Rightarrow m_t = -4.$$

$$\frac{-5-t}{4+2t} = -4 \iff 5+t = 4(4+2t)$$

$$5+t = 16+8t \Rightarrow -11 = 7t \Rightarrow t = -\frac{11}{7}.$$

$$h_A: -\frac{11}{7}(x+2y-1) + 5x+4y-17 = 0.$$

5. Consider the triangle given by the points $A(1,-2)$, $B(5,4)$ and $C(-2,0)$. Find the equations of the internal, respectively external bisectors corresponding to the vertex A of this triangle.



Method I: Find $m(\sphericalangle BAC) = 90°$.

$\therefore m(\sphericalangle BAA') = 45°$.

Use the formula with the slope.

$\therefore m(\sphericalangle CAA'') = 45°$.

Method II: Use bisector theorem to find.

$$K = \frac{BA'}{CA'} = \frac{AB}{AC}.$$

Then write $\overline{AA'} = \frac{1}{K+1} \cdot \overline{AB} + \frac{K}{K+1} \cdot \overline{AC}$.

Now $\overline{AA'}$ is d.v. of AA'.

$\boxtimes$

Method III

$\overline{AB}(4,6) = 2\cdot \overline{u}$

where $\overline{u}(2,3)$

$\overline{AC}(-3,2)$

$\|\overline{u}\| = \|\overline{AC}\| = \sqrt{9+4} = \sqrt{13}.$

$\boxed{\overline{u}+\overline{AC}}$ is parallel to $\overline{AA'}$ (see the

isosceles ΔAMC)

$(\overline{u}+\overline{AC})(-1,5)$. ← take this as

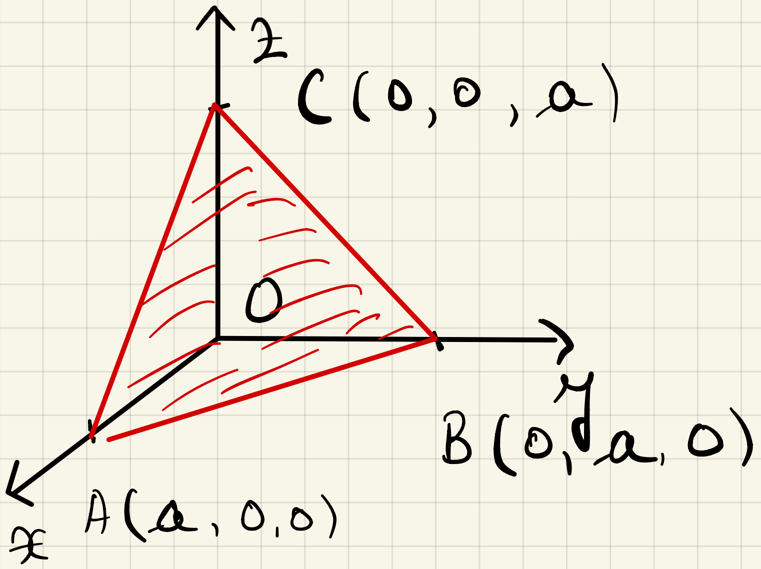
a director vector for AA'.

$$AA' : \begin{cases} x = 1 + (-1)\cdot t \\ y = -2 + 5\cdot t \end{cases}, t\in\mathbb{R}.$$

$$\boxed{AA' : \quad y+2 = -5(x-1).}$$

$$AA'' : \quad y+2 = \frac{1}{5}(x-1) \quad ;$$

8. Find the equation of the plane passing through $P(7,-5,1)$ which determines on the positive half-axes three segments of the same length.



Let π_a be our plane.

$$\pi_a : \begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & a & 0 & 1 \\ 0 & 0 & a & 1 \end{vmatrix} = 0$$

To find a, notice that $P(7,-5,1) \in \pi_a$ and $a > 0$.

$$\begin{vmatrix} 7 & -5 & 1 & 1 \\ a & 0 & 0 & 1 \\ 0 & a & 0 & 1 \\ 0 & 0 & a & 1 \end{vmatrix} = 0.$$