ANALYTIC GEOMETRY, PROBLEM SET 8

Line bundles, angle between two lines (in 2D) and the equation of the plane (3D)

1. Given the bundle of lines of equations $(1-t)x + (2-t)y + t - 3 = 0, t \in \mathbb{R}$ and $x + y - 1 = 0$, find:

(1) the coordinates of the vertex of the bundle;
(2) the equation of the line in the bundle which cuts Ox and Oy in M, respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$;

2. Let B be the bundle of vertex $M_0(5,0)$. An arbitrary line from B intersects the lines $d_1 : y - 2 = 0$ and $d_2 : y - 3 = 0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

3. Determine the angle between the lines:

(1) $y = 2x + 1$ and $y = -x + 2$;
(2) $y = 3x - 4$ and $x = 3 + t$, $y = -1 - 2t$ for $t \in \mathbb{R}$.
(3) $y = \frac{2}{5}x + 1$ and $4x + 3y - 12 = 0$.

4. Determine the equation of the line which passes through $A(3,1)$ and makes an angle of $45°$ with the line $2x + 3y - 1 = 0$.

5. Consider the triangle given by the points $A(1,-2)$, $B(5,4)$ and $C(-2,0)$. Find the equations of the internal, respectively external bisectors corresponding to the vertex A of this triangle.

6. The points of intersection of the lines $d_1 : x + 2y - 1 = 0$, $d_2 : 5x + 4y - 17 = 0$ and $d_3 : x - 4y + 11 = 0$ determine a triangle. Find the equations of the altitudes of these triangles without determining the coordinates of the vertices of the triangle!

7. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be distinct points in space. Prove that the equation of the plane containing P_1 and P_2 that is parallel to a vector $\overline{a} = (l, m, n)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0.$$

8. Find the equation of the plane passing through $P(7,-5,1)$ which determines on the positive half-axes three segments of the same length.

9. Find the equation for each of the following planes:

(a) containing $P(2,1,-1)$ and perpendicular to the vector $\overline{n} = (1,-2,3)$;

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(b) determined by $O(0,0,0)$, $P_1(3,-1,2)$ and $P_2(4,-2,-1)$;
(c) containing $P(3,4,-5)$ and parallel to both $\overline{a_1}(1,-2,4)$ and $\overline{a_2}(2,1,1)$;
(d) containing the points $P_1(2,-1,-3)$ and $P_2(3,1,2)$ and parallel to the vector $\overline{a}(3,-1,-4)$.

10. Find the equation of the plane containing the perpendicular lines through $P(-2,3,5)$ on the planes $\pi_1 : 4x + y - 3z + 13 = 0$ and $\pi_2 : x - 2y + z - 11 = 0$.

1. Given the bundle of lines of equations $(1-t)x+(2-t)y+t-3=0, t \in \mathbb{R}$ and $x+y-1=0$, find:

(1) the coordinates of the vertex of the bundle;

(2) the equation of the line in the bundle which cuts Ox and Oy in M, respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2+ON^2)$;

Remark:

We have $l_t : (1-t)x + (2-t)y + t - 3 = 0$ for $t \in \mathbb{R}$

$$\Leftrightarrow \quad (x+2y-3) + t(-x-y+1) = 0$$

$$\overset{s=-t}{\Leftrightarrow} \quad (x+2y-3) + s(x+y-1) = 0$$

which is a reduced bundle. Adding the line with equation $x+y-1=0$ we obtain the line bundle

$$l_{\alpha,\beta} : \alpha(x+2y-3) + \beta(x+y-1) = 0$$

which this execise is considering

(1) • In order to find the vertex of this line bundle one can intersect any two distinct lines of the line bundle

• starting from what is given, we may choose the line corresponding to the parameter $t=1$ which is

$$l_{t=1} : \quad y = 2$$

and intersect it with the line $l_\infty : x+y-1=0$ which is singled out in the description

- the intersection point $l_{t=1} \cap l_\infty$ is the vertex and is obtaint with the system $\begin{cases} y=2 \\ x+y-1=0 \end{cases}$

so, the intersection point is $P(-1,2)$

- since all lines of a line bundle intersect in a common point, the vertex, you can calculate the vertex by intersecting <u>any</u> two distinct lines

(2) As noted in the starting remark the line bundle is the set of lines

$$\{ l_t : (1-t)x + (2-t)y + t - 3 = 0 \}_{t\in\mathbb{R}} \cup \{ l_\infty : x+y-1=0 \}$$

- we need to check all these lines and see which of them satisfies the requirement

- we start with l_t and calculate the intersection point with the axes

- $M = l_t \cap O_x (y=0) \Rightarrow M\left(\frac{3-t}{1-t}, 0\right) \Rightarrow \vec{OM}\left(\frac{3-t}{1-t}, 0\right)$ where $t \neq 1$

  $N = l_t \cap O_y (x=0) \Rightarrow N\left(0, \frac{3-t}{2-t}\right) \Rightarrow \vec{ON}\left(0, \frac{3-t}{2-t}\right)$ where $t \neq 2$ (**)

- we check the condition on $\vec{OM}$ and $\vec{ON}$ in coordinates

$$\vec{OM}^2 \cdot \vec{ON}^2 = 4\left(\vec{OM}^2 + \vec{ON}^2\right)$$

$$\Leftrightarrow \left(\frac{3-t}{1-t}\right)^2 \cdot \left(\frac{3-t}{2-t}\right)^2 = 4\left[\left(\frac{3-t}{1-t}\right)^2 + \left(\frac{3-t}{2-t}\right)^2\right] \quad (*)$$

$\boxed{\text{if } t=3}$ then $M = N = O(0,0)$ the line d_3 passes through the origin and satisfies the condition.

$\boxed{\text{if } t \neq 3}$ then we can divide (*) by $(3-t)^2$ to obtain

$$\frac{(3-t)^2}{(1-t)^2(2-t)^2} = 4\left(\frac{1}{(1-t)^2} + \frac{1}{(2-t)^2}\right)$$

$$\Leftrightarrow \quad (3-t)^2 = 4\left[(2-t)^2 + (1-t)^2\right]$$

$$\Leftrightarrow \quad 7t^2 - 18t + 11 = 0$$

$$\Leftrightarrow \quad (t-1)(7t-11) = 0$$

So we found two possible values for t: 1 and $\frac{11}{7}$

But $t=1$ was excludet above (see (**))

$\Rightarrow$ we found one extra line $l_{\frac{11}{7}}: -4x+3y-10=0$

- we also need to check $l_\infty : x+y-1=0$
- $M = l_\infty \cap Ox \Rightarrow M=(1,0)$

  $N = l_\infty \cap Oy \Rightarrow N=(0,1)$

  $\Rightarrow OM^2 = ON^2 = 1$. So the condition is not satisfied in this case

Remark: Such calulations depend on the choice of the generic equation which describes the line bundle

$\hookrightarrow$ you could also work with the equation

$$t(x+1) + y - 2 = 0 \quad \text{(of the same linebundle)}$$

why?

2. Let B be the bundle of vertex $M_0(5,0)$. An arbitrary line from B intersects the lines $d_1 : y-2=0$ and $d_2 : y-3=0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

Remark:

there are two obvious lines passing through $M_0(5,0)$:

$$\ell_1 : x=5 \quad \text{and} \quad \ell_2 : y=0$$

So any line passing through M_0 has an equation of the form

$$\ell_{\alpha,\beta} : \alpha(x-5)+\beta y=0$$

One corresponding reduced line bundle is

$\ell_t : t(x-5)+y=0$ where we omit the line $\ell_\infty : x=5$

- we prove the statement for the lines $\{d_t\}_{t\in\mathbb{R}}$

$$M_1 = d_t \cap d_1 \, (y=2) \Rightarrow M_1\left(\frac{5t-2}{t}, 2\right)$$ and $t\neq 0$

(the line d_0 has equation $y=0$ and does not intersect d_1)

$$M_2 = d_t \cap d_2 \, (y=3) \Rightarrow M_2\left(\frac{5t-3}{t}, 3\right)$$

$$\Rightarrow \overrightarrow{OM_2}\left(\frac{5t-3}{t}, 3\right)$$

$\Rightarrow$ the line passing through M_1 and parallel to $\overrightarrow{OM_2}$ is

$$\tilde{\ell}_t : \frac{x-\frac{5t-2}{t}}{\frac{5t-3}{t}} = \frac{y-2}{3}$$

$$3tx - \cancel{15t} - 6$$

$$5ty - \cancel{10t} - 3y + 6$$

$$\Leftrightarrow \tilde{\ell}_t : 3(tx-5t-2)=(5t-3)(y-2)$$

$$\Leftrightarrow \tilde{\ell}_t : 3tx-5ty-5t+3y-12=0$$

$\Leftrightarrow \tilde{\ell}_t : \ t(3x-5y-5) + 3y-12 = 0$

which is a reduced line bundle $\Rightarrow$ all these lines intersect in one point

- it remains to check that the line $\ell_\infty : x=5$ also gives such a line

$$M_1 = \ell_\infty \cap d_1 \ (y=2) \Rightarrow M_1(5,2)$$

$$M_2 = \ell_\infty \cap d_2 \ (y=3) \Rightarrow M_2(5,3)$$

$\Rightarrow$ the line passing through M_1 and parallel to $\overrightarrow{OM_2}$ is

$$\tilde{\ell}_\infty : \ \frac{x-5}{5} = \frac{y-2}{3} \Leftrightarrow 3x-5y-5=0$$

which is exactly the line missing from the reduced line bundle $\tilde{\ell}_t$

3. Determine the angle between the lines:

(1) $y = 2x+1$ and $y = -x+2$;

(2) $y = 3x-4$ and $x = 3+t,\ y = -1-2t$ for $t \in \mathbb{R}$.

(3) $y = \frac{2}{5}x+1$ and $4x+3y-12=0$.

Method I (direction vectors)

- find direction vectors for each line

(1) $\begin{cases} x = x \\ y = 2x+1 \end{cases} \Rightarrow v_1(1,2)$ is a direction vector

$\begin{cases} x = x \\ y = -x+2 \end{cases} \Rightarrow v_2(1,-1)$ is a direction vector

- find the angle with the scalar product.

$$\cos(\measuredangle v_1, v_2) = \frac{v_1 \cdot v_2}{\|v_1\| \cdot \|v_2\|} = \frac{-1}{\sqrt{5} \cdot \sqrt{2}}$$

$$\Rightarrow \measuredangle(v_1, v_2) = \arccos\left(\frac{-1}{\sqrt{10}}\right)$$

Method $\underline{\text{II}}$ (slopes)

if α is the angle between the two lines then

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = 3$$

$$\Rightarrow \measuredangle(v_1, v_3) = \arctan(3)$$

Method $\underline{\text{III}}$?

4. Determine the equation of the line which passes through $A(3, 1)$ and makes an angle of $45°$ with the line $2x + 3y - 1 = 0$.

Method I

- the line is $l_1: \begin{cases} x = -\frac{3}{2}y + \frac{1}{2} \\ y = y \end{cases}$ $\Rightarrow$ a direction vector is $(-\frac{3}{2}, 1)$

another direction vector is

$2 \cdot (-3/2, 1) = (-3, 2) = v_1$

- let l_2 be the line passing through $A(3,1)$ and making an angle of $45°$ with l_1
- let $v_2(a,b)$ be a direction vector for l_2 of norm $\sqrt{26}$

then $\|v_2\| = \sqrt{26} = \sqrt{a^2+b^2}$ $(\ast)$

$$\frac{\sqrt{2}}{2} = \cos(45°) = \frac{v_1 \cdot v_2}{\|v_1\| \cdot \|v_2\|} = \frac{-3a + 2b}{\sqrt{9+4}\ \sqrt{26}}$$

$$\Leftrightarrow 13 = -3a + 2b \Leftrightarrow b = \frac{13 + 3a}{2}$$

$$\overset{(\ast)}{\Rightarrow} 26 = a^2 + \frac{(13+3a)^2}{2} \Rightarrow a = -5 \text{ or } a = -1$$

Case 1 $\quad a = -5 \Rightarrow b = -1 \Rightarrow v_2 = v_2(-5, -1) \Rightarrow l_2: \begin{cases} x = 3 + t(-5) \\ y = 1 + t(-1) \end{cases}$

Case 2 $\quad a = -1 \Rightarrow b = 5 \Rightarrow v_2 = v_2(-1, 5) \Rightarrow l_2: \begin{cases} x = 3 + t(-1) \\ y = 1 + t(5) \end{cases}$

Method II

- look for l_2 in the form $y - 1 = m(x-3)$

$$1 = \tan(45)° = \left| \frac{m - \left(\frac{-2}{3}\right)}{1 + \left(-\frac{2}{3}\right)m} \right| = \left| \frac{3m+2}{3-2m} \right|$$

Case 1 $\quad 3m + 2 = 3 - 2m \Rightarrow m = \frac{1}{5}$

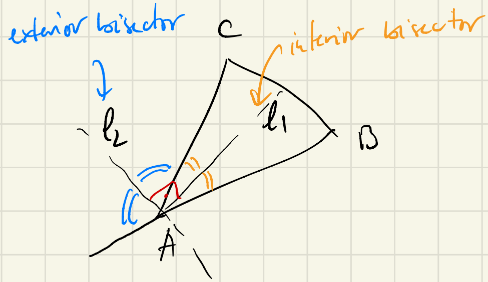
$$\Rightarrow l_1 : y - 1 = \frac{1}{5}(x-3)$$

Case 2 $\quad 3m + 2 = 2m - 3 \Rightarrow m = -5$

$$\Rightarrow l_2 : y - 1 = -5(x-3)$$

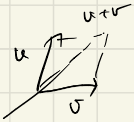
5. Consider the triangle given by the points $A(1,-2)$, $B(5,4)$ and $C(-2,0)$. Find the equations of the internal, respectively external bisectors corresponding to the vertex A of this triangle.

let l_1 be the interior bisector
and l_2 be the exterior bisector



notice that $l_1 \perp l_2$

- l_1 is the bisector of the angle $\sphericalangle(\vec{AB}, \vec{AC})$
- in general if u and v are two vectors of the same length then the bisector of the angle $\sphericalangle(u,v)$ has $u+v$ as direction vector



- since $\sphericalangle(\vec{AB}, \vec{AC}) = \sphericalangle\left(\frac{\vec{AB}}{\|\vec{AB}\|}, \frac{\vec{AC}}{\|\vec{AC}\|}\right)$

a direction vector for l_1 is $v = \frac{\vec{AB}}{\|\vec{AB}\|} + \frac{\vec{AC}}{\|\vec{AC}\|}$

$$\vec{AB} = (4,6) \text{ and } \vec{AC} = (-3,2)$$

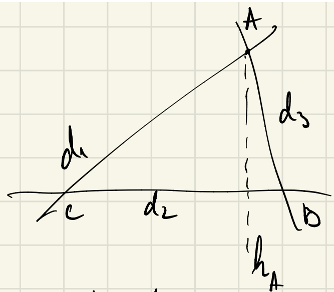
$$\Rightarrow v = \frac{(4,6)}{\sqrt{16+36}} + \frac{(-3,2)}{\sqrt{9+4}} = \frac{1}{\sqrt{13}}\left[\frac{(4,6)}{2} + (-3,2)\right]$$

$\Rightarrow \sqrt{13}\cdot v = (-1,5)$ is also a direction vector for l_1

$$\Rightarrow l_1 = \begin{cases} x = 1 + t(-1) \\ y = -2 + t \cdot 5 \end{cases}$$

- for l_2 notice that $(5,1)\cdot(-1,5) = 0 \Leftrightarrow (5,1) \perp (-1,5)$
$\Rightarrow (5,1)$ is a direction vector for l_2

6. The points of intersection of the lines $d_1 : x + 2y - 1 = 0$, $d_2 : 5x + 4y - 17 = 0$ and $d_3 : x - 4y + 11 = 0$ determine a triangle. Find the equations of the altitudes of these triangles without determining the coordinates of the vertices of the triangle!



- let h_A be the height from A
- it is a line passing through the intersection point $d_1 \cap d_3$
- in other words, it belongs to the line bundle with vertex $d_1 \cap d_3$
- in other words, it has an equation of the form

$$\alpha(x+2y-1) + \beta(x-4y+11) = 0 \qquad (*)$$

- on the other hand $h_A \perp d_2$ i.e. the slope of h_A is $-\left(\frac{-5}{4}\right)^{-1} = \frac{4}{5}$
- so h_A is a line in the bundle $(*)$ with slope $\frac{4}{5}$

let us look for h_A in the reduced bundle

$$d_t : t(x+2y-1) + x - 4y + 11 = 0$$

$$\Leftrightarrow (t+1)x + (2t-4)y - t + 11 = 0$$

d_t has slope $\frac{t+1}{4-2t}$ and $t \neq 2$ else $d_t \parallel Ox \not\perp d_2$

so, we get h_A by imposing $\frac{t+1}{4-2t} = \frac{4}{5} \Leftrightarrow t = -\frac{11}{3}$

$\Rightarrow h_A = d_{-\frac{11}{3}} : \ldots\ldots$

7. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be distinct points in space. Prove that the equation of the plane containing P_1 and P_2 that is parallel to a vector $\overline{a} = (l, m, n)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0.$$

The vector $\overrightarrow{P_1P_2}$ has components $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$
and is parallel to the plane
if $\overrightarrow{P_1P_2}$ is not parallel to a then you can use from the course

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \qquad (13)$$

which is the analytic equation of the plane determined by a point and two nonparallel directions.

9. Find the equation for each of the following planes:
(a) containing $P(2, 1, -1)$ and perpendicular to the vector $\overline{n} = (1, -2, 3)$;

$$1(x-2) - 2(y-1) + 3(z-(-1)) = 0$$

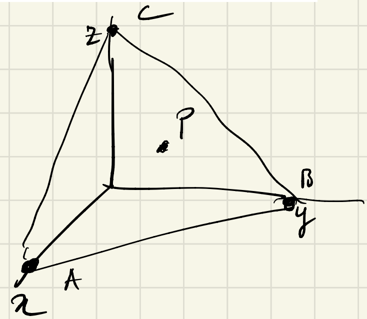
(b) determined by $O(0, 0, 0)$, $P_1(3, -1, 2)$ and $P_2(4, -2, -1)$;
(c) containing $P(3, 4, -5)$ and parallel to both $\overline{a_1}(1, -2, 4)$ and $\overline{a_2}(2, 1, 1)$;
(d) containing the points $P_1(2, -1, -3)$ and $P_2(3, 1, 2)$ and parallel to the vector $\overline{a}(3, -1, -4)$.

b.) $\begin{vmatrix} x & y & z \\ 3 & -1 & 2 \\ 4 & -2 & -1 \end{vmatrix} = 0$
$\quad = \ldots$

c.) $\begin{vmatrix} x-3 & y-4 & z-(-5) \\ 1 & -2 & 4 \\ 2 & 1 & 1 \end{vmatrix} = 0$
$\quad = \ldots$

d.) use exercise 7.

8. Find the equation of the plane passing through $P(7, -5, 1)$ which determines on the positive half-axes three segments of the same length.



let π be this plane

$$\pi: ax + by + cz + d = 0$$

$A = \pi \cap Ox \ (y=0, z=0) \Rightarrow A\left(\frac{-d}{a}, 0, 0\right)$

$B = \pi \cap Oy \Rightarrow B\left(0, \frac{-d}{b}, 0\right)$

$C = \pi \cap Oz \Rightarrow C\left(0, 0, \frac{-d}{c}\right)$

"which detrmines ... segments of the same length"

$\Leftrightarrow \quad -\frac{d}{a}, -\frac{d}{b}, -\frac{d}{c} > 0 \quad$ and $\quad -\frac{d}{a} = -\frac{d}{b} = -\frac{d}{c}$

$\Leftrightarrow \quad a = b = c \quad$ and $\quad -\frac{d}{a} > 0$

so $\quad \pi: \ ax + ay + az + d = 0 \quad | : a$

$\qquad x + y + z = -k \quad$ and $\quad k > 0$

now use the fact that $P \in \pi$ to obtain k.

10. Find the equation of the plane containing the perpendicular lines through $P(-2, 3, 5)$ on the planes $\pi_1 : 4x + y - 3z + 13 = 0$ and $\pi_2 : x - 2y + z - 11 = 0$.

- if the plane π contains the perpendicular lines through $P(-2,3,5)$ on π_1 and π_2 then π is parallel to the normal vectors of π_1 and the normal vectors of π_2

- $(4,1,-3)$ is a normal vector for π_1
  $(1,-2,1)$ — " — π_2

$$\Rightarrow \quad \pi : \begin{vmatrix} x-(-2) & y-3 & z-5 \\ 4 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

$$\| $$

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