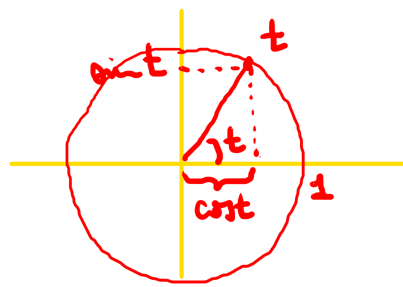
Course 3 March 9, 2021

5. Limits of vector functions

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}^m$ will be called vector function of a real variable.

Examples
- $f : [0, 2\pi] \to \mathbb{R}^2$, $f(t) = (a\cos t, a\sin t)$

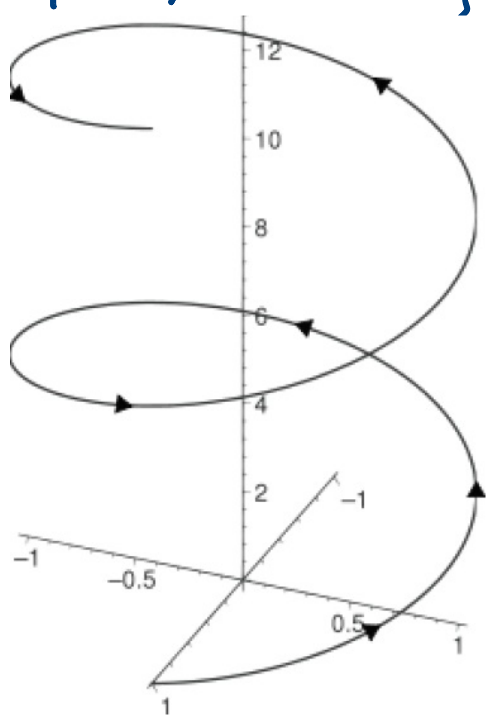


- $f : [0, 2\pi] \to \mathbb{R}^2$

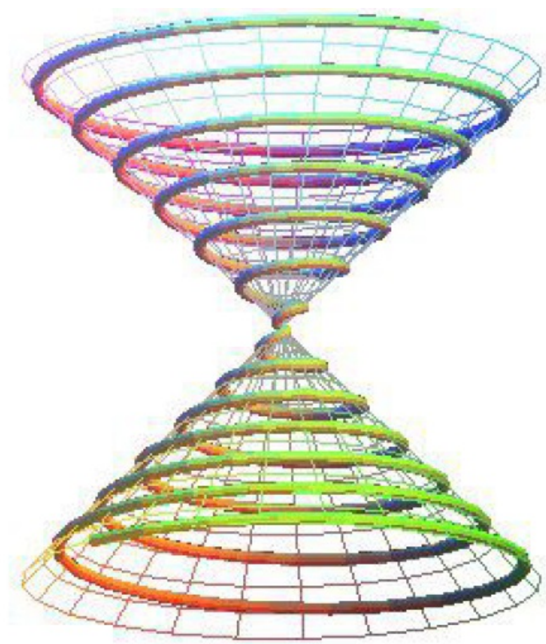
$f(t) = (a\cos t, b\sin t)$ $a, b > 0$

$\operatorname{Im} f = \left\{ (x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} \to$ ellipse

- $f : [0, 4\pi] \to \mathbb{R}^3$, $f(t) = (a\cos t, a\sin t, ct)$.



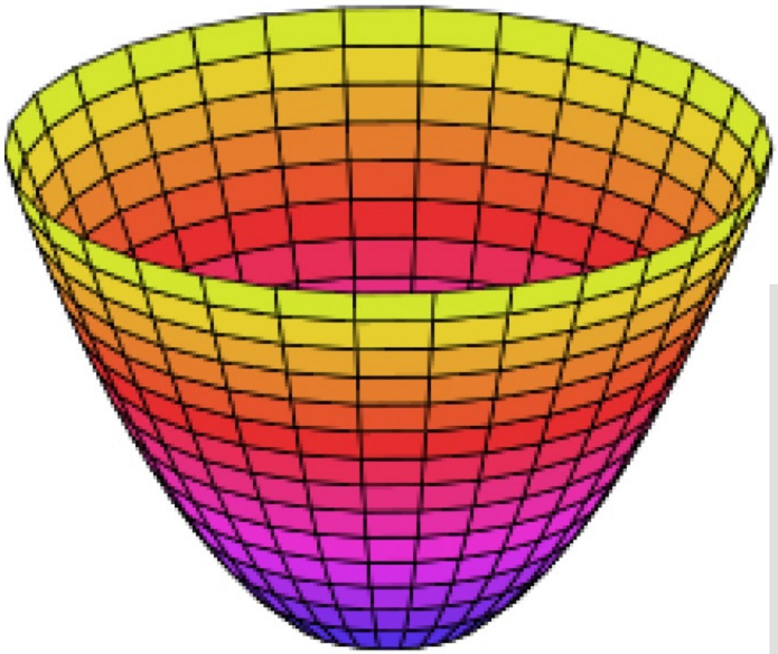
- $\vec{r}(t) = (t\cos t, t\sin t, ct)$ $t \in [0, \infty)$



A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ will be called a scalar function of vector variable (or scalar function of several variables).

<u>Examples</u> • $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 + y^2$

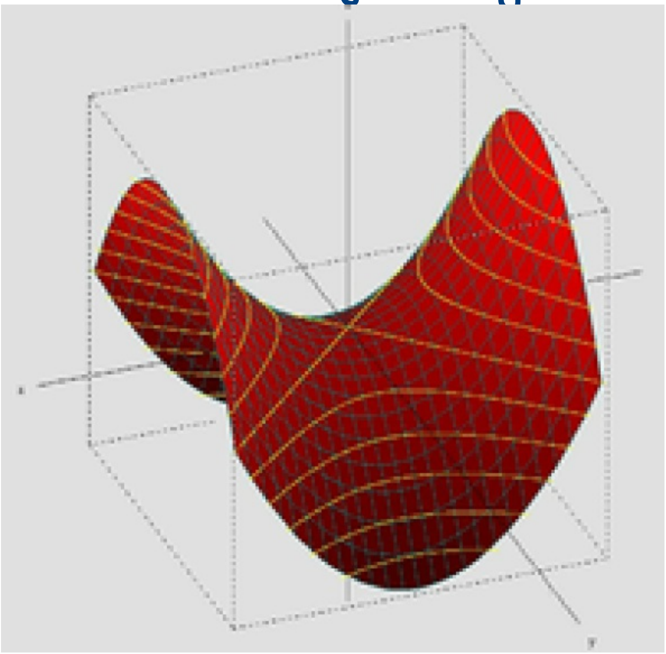
$\Rightarrow z = f(x,y) = x^2 + y^2$ paraboloid of revolution $\left(z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)$



$= \{ (x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2 \}$

• $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 - y^2$

$\Rightarrow z = f(x,y) = x^2 - y^2$ ← hyperbolic paraboloid



Pringles chips

- $F: \mathbb{R}^3 \to \mathbb{R}$ $\quad F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1$

$F(x,y,z) = 0 \quad \leftarrow$ one sheet hyperboloid

A function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is called a vector function of vector variable

- $f: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad f(x,y) = (\sin(x+y), \cos(xy), e^{x-y})$
- $f: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f(x,y,z) = (\underbrace{x - 3y + z^2}_{f_1(x,y,z)}, \underbrace{\ln(1+x^2+y^2+z^2)}_{f_2(x,y,z)})$

Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a given function. We define

$$f_1, \dots, f_m : A \to \mathbb{R}$$

as follows: given any point $x \in A \Rightarrow f(x) \in \mathbb{R}^m$, so $f(x)$ is of the form $f(x) = (y_1, \dots, y_m)$. We set $f_1(x) := y_1, \dots, f_m(x) := y_m$

$f_1, \dots, f_m$ are called the scalar components of f

$$\forall x \in A : \quad f(x) = (f_1(x), \dots, f_m(x))$$

We write $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$ whenever we want to emphasize the scalar components of the vector function f

5.1. Definition Let $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$ a given function, $a \in A'$, $b \in \mathbb{R}^m$.
One says that f has the limit b at the point a if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } \forall x \in A \setminus \{a\} \text{ with } \|x - a\| < \delta \ : \ \|f(x) - b\| < \varepsilon$$

If the limit exists, then it is unique.

Notation $\lim\limits_{x \to a} f(x) = b$.

5.2. Theorem (Characterization of the limit by using sequences). Let $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$ a function, $a \in A'$, $b \in \mathbb{R}^m$. Then

$$\lim_{x \to a} f(x) = b \iff \forall (x_k) \text{ sequence in } A \setminus \{a\} \text{ s.t. } (x_k) \to a \text{ we have } \lim_{k \to \infty} f(x_k) = b$$

Proof $\boxed{\Rightarrow}$ Assume that $\lim\limits_{x \to a} f(x) = b$. Let (x_k) be an arbitrary seq. in $A \setminus \{a\}$ s.t. $(x_k) \to a$.

$\overset{?}{}$ $\lim\limits_{k \to \infty} f(x_k) = b \iff \forall \varepsilon > 0 \ \exists k_0 \in \mathbb{N} \text{ s.t. } k \geq k_0 : \ \|f(x_k) - b\| < \varepsilon$

Let $\varepsilon > 0 \Rightarrow \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with $\|x - a\| < \delta \ : \ \|f(x) - b\| < \varepsilon$

Since $(x_k) \to a \Rightarrow \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0 : \ \|x_k - a\| < \delta$, $x_k \in A \setminus \{a\}$

$\Rightarrow \forall k \geq k_0 : \|f(x_k) - b\| < \varepsilon$

$\boxed{\Leftarrow}$ Suppose that f does not have the limit b at the point a

$\Rightarrow \exists \varepsilon > 0$ s.t. $\forall \delta > 0 \ \exists x \in A \setminus \{a\}$ with $\|x-a\| < \delta$ s.t. $\|f(x)-b\| \geq \varepsilon$

In particular for $\delta = \frac{1}{k} \Rightarrow \forall k \geq 1 \ \exists x_k \in A \setminus \{a\}$ with $\|x_k - a\| < \frac{1}{k}$ s.t.

$$\|f(x_k)-b\| \geq \varepsilon$$

$\Rightarrow (x_k)$ is a sequence in $A \setminus \{a\}$ with $\lim_{k\to\infty} x_k = a$ $\overset{\text{our hypothesis}}{\Rightarrow}$

$\Rightarrow \lim_{k\to\infty} f(x_k) = b$ $\Rightarrow\Leftarrow$ $\|f(x_k)-b\| \geq \varepsilon \ \forall k \geq 1$.

The obtained contradiction shows that $\lim_{x\to a} f(x) = b$.

<u>Remark</u> $\lim_{x\to a} f(x) = b \iff \lim_{x\to a} \|f(x)-b\| = 0$

<u>5.3. Theorem</u> Let $A \subseteq \mathbb{R}^n$, $a \in A'$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$, and let $b = (b_1, \dots, b_m) \in \mathbb{R}^m$. Then

$$\lim_{x\to a} f(x) = b \iff \lim_{x\to a} f_i(x) = b_i \quad \forall i \in \{1, \dots, m\}$$

$$f(x) = (f_1(x), \dots, f_m(x))$$

$x \to a$ $\downarrow$ b $\quad\downarrow b_1 \quad \dots \quad \downarrow b_m$

Example $b = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$ $f(x,y)=\frac{xy}{x^2+y^2}$

We are going to prove that the limit b does not exist

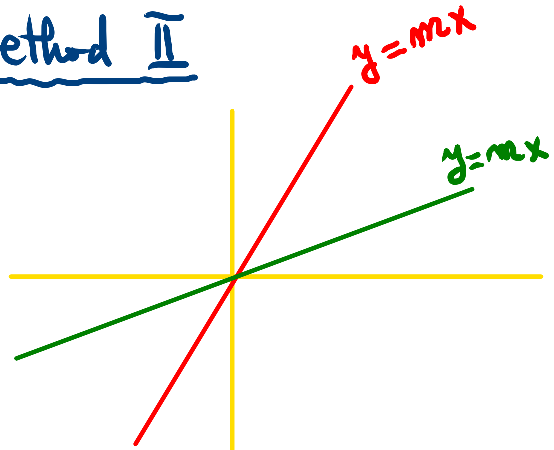
Method I

$\left(\frac{1}{k},\frac{1}{k}\right) \xrightarrow{k\to\infty} (0,0)$ $f\left(\frac{1}{k},\frac{1}{k}\right) = \frac{1}{2} \xrightarrow{k\to\infty} \frac{1}{2}$

$\left(\frac{2}{k},\frac{1}{k}\right) \xrightarrow{k\to\infty} (0,0)$ $f\left(\frac{2}{k},\frac{1}{k}\right) = \frac{2/k^2}{5/k^2} = \frac{2}{5} \xrightarrow{k\to\infty} \frac{2}{5}$

TS2 $\Rightarrow \nexists \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$

Method II



$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{x\cdot mx}{x^2+m^2x^2} = \frac{m}{1+m^2}$

If $\lim_{x\to 0} f(x,mx)$ depends on m $\Rightarrow$

$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist

! If $\lim_{x\to 0} f(x,mx)$ does not depend on m $\Rightarrow$?

6. Continuity of vector functions

6.1. Definition Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ a given function, and $a \in A$.
One says that f is continuous at a if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in A \text{ with } \|x - a\| < \delta \ : \ \|f(x) - f(a)\| < \varepsilon$$

If a is an isolated point of A $\Rightarrow$ f is continuous at a

6.2. Theorem (characterization of continuity by using sequences).
Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ a given function, and $a \in A$. Then

f is continuous at a $\Longleftrightarrow$ $\forall (x_k)$ sequence in A s.t $(x_k) \to a$ we have $\lim\limits_{k \to \infty} f(x_k) = f(a)$

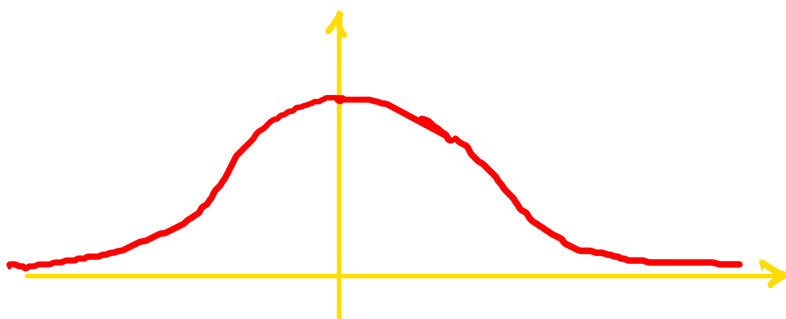
6.3. Theorem Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ a function, and $a \in A \cap A'$.
Then f is continuous at a $\Longleftrightarrow$ $\exists \lim\limits_{x \to a} f(x) = f(a)$.

6.4. Theorem. Let $A \subseteq \mathbb{R}^n$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$ a function, and $a \in A$.
Then f is continuous at a $\Longleftrightarrow$ $f_i : A \to \mathbb{R}$ is continuous at a $\forall i = \overline{1, m}$.

6.5 Theorem If A is a compact subset of $\mathbb{R}^n$ and $f: A \to \mathbb{R}^m$ is a continuous function on A, then $f(A)$ is a compact subset of $\mathbb{R}^m$.

Remark $\left.\begin{array}{l} A \text{ open/closed} \\ f \text{ continuous on } A \end{array}\right| \not\Rightarrow f(A)$ is open/closed

Counterexample $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \frac{1}{x^2+1}$ continuous



$\mathbb{R}$ is both open and closed

but $f(\mathbb{R}) = (0,1]$ which is not open nor closed

Proof. $f(A)$ is a compact set in $\mathbb{R}^m \Leftrightarrow f(A)$ is sequentially compact in $\mathbb{R}^m$

$\Leftrightarrow \forall (y_k)$ seq. in $f(A)$ $\exists (y_{k_j})_{j\geq 1}$ subseq

and $\exists\, b \in f(A)$ s.t. $\lim\limits_{j\to\infty} y_{k_j} = b$

Let (y_k) be an arbitrary seq. in $f(A)$

$\Rightarrow \forall k \geq 1 \quad \exists x_k \in A$ s.t. $y_k = f(x_k)$

$\Rightarrow \left.\begin{array}{l} (x_k) \text{ is a seq. in } A \\ A \text{ is compact} \end{array}\right\} \Rightarrow \exists (x_{k_j})_{j\geq 1}$ subseq. of (x_k) and $\exists\, a \in A$ s.t. $\lim\limits_{j\to\infty} x_{k_j} = a$

Since f is continuous at a $\Rightarrow$ $\lim_{j\to\infty} \underbrace{f(x_{k_j})}_{=y_{k_j}} = f(a) =: b \in f(A)$

$\Rightarrow$ $\lim_{j\to\infty} y_{k_j} = b \in f(A)$

<u>6.6. Theorem</u> (K. Weierstrass). If A is a compact subset of $\mathbb{R}^n$, and $f: A \to \mathbb{R}$ is continuous on A, then f is bounded and attains its bounds on A, i.e., $\exists\, a, b \in A$ s.t. $f(a) = \inf f(A)$ and $f(b) = \sup f(A)$.

<u>Proof</u>. $\left.\begin{array}{l} A \text{ compact} \\ f \text{ continuous on } A \end{array}\right| \overset{T6.5}{\Longrightarrow} f(A)$ is compact in $\mathbb{R}$ $\Rightarrow$

$\Rightarrow$ $f(A)$ is bounded $\Rightarrow$ f is bounded

and

$f(A)$ is closed $\Rightarrow$ $\inf f(A) \in f(A)$ and $\sup f(A) \in f(A)$

$\Downarrow$ $\exists a \in A : f(a) = \inf f(A)$ $\qquad$ $\Downarrow$ $\exists b \in A : f(b) = \sup f(A)$

Chapter 2. DIFFERENTIAL CALCULUS IN $\mathbb{R}^n$

1. The normed space of linear mappings

<u>1.1. Definition</u>. A function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is called a <u>linear mapping</u> if

$$\forall \alpha, \beta \in \mathbb{R} \quad \forall x, y \in \mathbb{R}^n \quad : \quad \varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y)$$

Denote

$$L(\mathbb{R}^n, \mathbb{R}^m) := \{ \varphi : \mathbb{R}^n \to \mathbb{R}^m \mid \varphi \text{ is a linear mapping} \}$$

If $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \implies$
- $\varphi(0_n) = 0_m$
- $\varphi(-x) = -\varphi(x) \quad \forall x \in \mathbb{R}^n$
- $\forall \alpha_1, \dots, \alpha_k \in \mathbb{R}$, $\forall x_1, \dots, x_k \in \mathbb{R}^n$ we have
  $$\varphi(\alpha_1 x_1 + \dots + \alpha_k x_k) = \alpha_1 \varphi(x_1) + \dots + \alpha_k \varphi(x_k)$$

<u>1.2. Theorem</u> Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$\varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \iff \exists v_1, \dots, v_n \in \mathbb{R}^m \text{ s.t. } \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n :$$

$$\varphi(x) = x_1 v_1 + \dots + x_n v_n$$

$v_1 = \varphi(e_1), \dots, v_n = \varphi(e_n)$

1.3. Theorem. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$. Then

$$\varphi \in L(\mathbb{R}^n, \mathbb{R}) \iff \exists\, v \in \mathbb{R}^m \text{ s.t. } \varphi(x) = \langle x, v \rangle \quad \forall x \in \mathbb{R}^n$$

1.4. Definion (the matrix of a linear mapping) Let $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let $\{e_1, \dots, e_n\}$ be the canonical basis in $\mathbb{R}^n$.

Set

$v_1 := \varphi(e_1) \in \mathbb{R}^m \Rightarrow v_1$ has the form $v_1 = (v_{11}, v_{12}, \dots, v_{1m})$

$v_2 := \varphi(e_2) \in \mathbb{R}^m \Rightarrow v_2$ — " — $v_2 = (v_{21}, v_{22}, \dots, v_{2m})$

$\vdots$

$v_n := \varphi(e_n) \in \mathbb{R}^m \Rightarrow v_n$ — " — $v_n = (v_{n1}, v_{n2}, \dots, v_{nm})$

$$[\varphi] := \begin{pmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ \underbrace{v_{1m}}_{\varphi(e_1)} & \underbrace{v_{2m}}_{\varphi(e_2)} & \cdots & \underbrace{v_{nm}}_{\varphi(e_n)} \end{pmatrix} \in M_{m \times n}(\mathbb{R})$$

→ the matrix of the linear mapping φ

<u>Convention</u>: a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ in a matrix expression (equality) will be identified with the column matrix $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n\times 1}(\mathbb{R})$

For instance $\langle x, y \rangle = x^t \cdot y$

$$x^t \cdot y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^t \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \underbrace{(x_1 \; \dots \; x_n)}_{1\times n} \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{n\times 1} = \underbrace{x_1 y_1 + \dots + x_n y_n}_{1\times 1}$$

$$\boxed{\varphi(x) = [\varphi] x = [\varphi] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$$