ANALYTIC GEOMETRY, PROBLEM SET 3

Warm-up 1. What are the conditions that vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ should satisfy to ensure the existence of a triangle ABC such that $\overrightarrow{AB} \in a$, $\overrightarrow{BC} \in \overline{b}$ and $\overrightarrow{CA} \in \overline{c}$?

1. On the sides of a triangle ABC, one constructs the parallelograms $ABB'A''$, $BCC'B''$, $CAA'C''$. Show that one can construct a triangle MNP such that $\overrightarrow{MN} \in \overline{A'A''}$, $\overrightarrow{NP} \in \overline{B'B''}$ and $\overrightarrow{PM} \in \overline{C'C''}$.

2. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} = \overline{0}$.

3. In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 2\overline{OM}$.

4. Consider, in the 3-dimensional space, the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Prove that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a new parallelogram.

5. Let ABC be a triangle and a, b, c the lengths of its sides, respectively. If A_1 is the intersection point of the internal bisector of the angle $\angle A$ and BC and M is an arbitrary point, show that

$$\overline{MA_1} = \frac{b}{b+c}\overline{MB} + \frac{c}{b+c}\overline{MC}$$

6. If G is the centroid (center of mass) of a triangle ABC in the plane and O is a given point, then

$$\overline{OG} = \frac{\overline{OA} + \overline{OB} + \overline{OC}}{3}$$

7. Let ABC be a triangle, H its orthocenter, O the circumcenter (the center of its circumscribed cercle), G the centroid of the triangle and A' the point on the circumcenter diametrically opposed to A. Then:

(1) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$; (**Sylvester's formula**)
(2) $\overline{HB} + \overline{HC} = \overline{HA'}$;
(3) $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{HO}$;
(4) $\overline{HA} + \overline{HB} + \overline{HC} = 3\overline{HG}$;
(5) the points H, G, O are collinear and $2GO = HG$. (the **Euler line**)

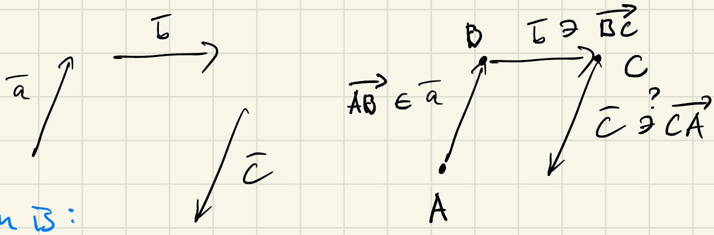
Date: October 8, 2021.

8. Let $ABCD$ be a quadrilateral with $AB \cap CD = \{E\}$, $AD \cap BC = \{F\}$ and the points M, N, P the midpoints of $[BD]$, $[AC]$ and $[EF]$, respectively. Then M, N, P are collinear. (the **Newton-Gauss line**)

Warm-up 1. What are the conditions that vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ should satisfy to ensure the existence of a triangle ABC such that $\overrightarrow{AB} \in \bar{a}$, $\overrightarrow{BC} \in \bar{b}$ and $\overrightarrow{CA} \in \bar{c}$?

rem: if $\bar{a} \cap \bar{b} \ni \overrightarrow{AB}$

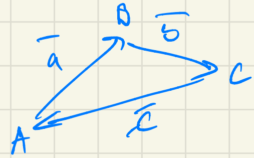
$\Rightarrow \bar{a} = \bar{b}$



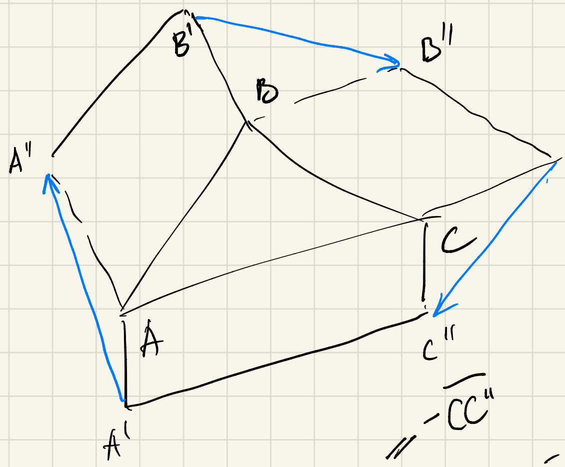
the condition is:

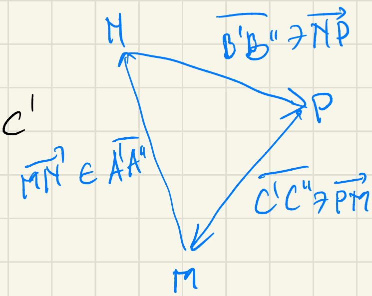
$$\bar{a}+\bar{b}+\bar{c}=0$$

$$\bar{a}+\bar{b}=-\bar{c}$$



1. On the sides of a triangle ABC, one constructs the parallelograms $ABB'A''$, $BCC'B''$, $CAA'C''$. Show that one can construct a triangle MNP such that $\overrightarrow{MN} \in \overline{A'A''}$, $\overrightarrow{NP} \in \overline{B'B''}$ and $\overrightarrow{PM} \in \overline{C'C''}$.





? $\overrightarrow{MN}+\overrightarrow{NP}=\overrightarrow{MP}$

$= -\overrightarrow{PM}$

$\overset{?}{\Leftarrow}$ $\overline{A'A''}+\overline{B'B''}+\overline{C'C''}=\bar{0}$

$$\overline{A'A''}=\overline{A'A}+\overline{AA''}=-\overline{B'B}$$

$$// \ -\overline{CC''}$$

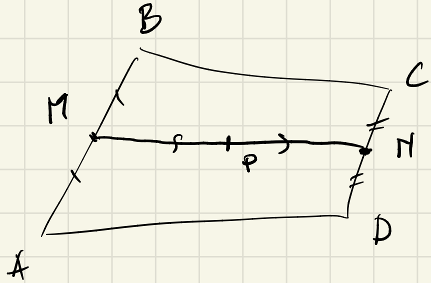
$$\overline{B'B''}=\overline{B'B}+\overline{BB''}=-\overline{C'C}$$

$$\overline{C'C''}=\overline{C'C}+\overline{CC''} \quad (+)$$

$$\overline{A'A''}+\overline{B'B''}+\overline{C'C''}=\bar{0}$$

$\Rightarrow$ One can construct the desired triangle MNP (by warmup 1)

2. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}=\overline{0}$.



I)

$$\overline{PA} = \overline{PM} + \overline{MA}$$

$$\overline{PB} = \overline{PM} + \overline{MB}$$

$$\overline{PC} = \overline{PN} + \overline{NC}$$

$$\overline{PD} = \overline{PN} + \overline{ND} \quad (+)$$

$$\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD} = \underbrace{2\,\overline{PM} + 2\,\overline{PN}}_{=\overline{0}} + \underbrace{\overline{MA}+\overline{MB}}_{=\overline{0}} + \underbrace{\overline{NC}+\overline{ND}}_{=\overline{0}}$$

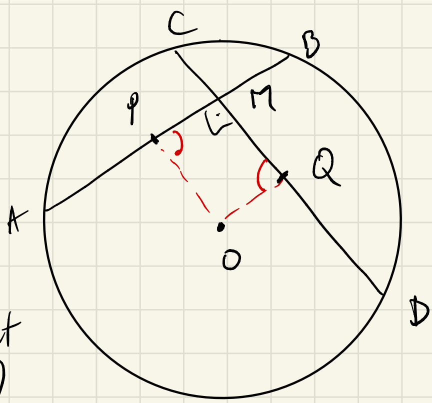
II) ...

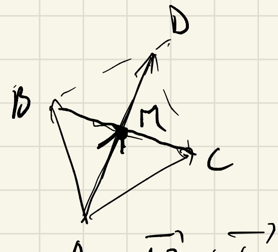
III) ...

3. In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that $\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD}=2\overline{OM}$. $(*)$

$\overrightarrow{OA}+\overrightarrow{OB}=2\overrightarrow{OP}$

if P is the midpoint of $[AB]$





$\overrightarrow{AB}+\overrightarrow{AC}$
$=2\overline{AM}$

$\overline{OC}+\overline{OD}=2\overline{OQ}$

if Q is the midpoint of $[CD]$

So

$(*) \iff \cancel{2}\overline{OM} = \cancel{2}\overline{OP} + \cancel{2}\overline{OQ}$

$\iff \overline{OM} = \overline{OP} + \overline{OQ}$ (is this true?)

$\iff OQMP$ is a parallelogram.

$\iff$ (since $\angle PMQ$ is $90°$) $OQMP$ is a rectangle

$\iff \angle P = 90°$ $\angle Q = 90°$ $(**)$

notice $\Delta\, OCD$ is isoscel and Q is mid-pt. of $[CD]$

$\Rightarrow OQ \perp CD$

similar in $\Delta\, OAB$: $OP \perp AB$

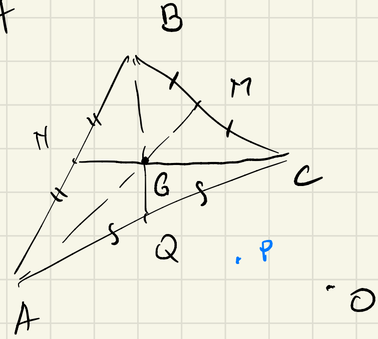
$\Rightarrow (**)$ true

$\Rightarrow (*)$ true.

6. If G is the centroid (center of mass) of a triangle ABC in the plane and O is a given point, then

$$\overline{OG} = \frac{\overline{OA}+\overline{OB}+\overline{OC}}{3}$$

intersection of medians



for some point P consider the vector $v = \overline{PA}+\overline{PB}+\overline{PC}$

$$v = (\overline{PA}+\overline{PB}) + \overline{PC} = 2\overline{PN}+\overline{PC} \qquad (1)$$

$$v = (\overline{PA}+\overline{PC}) + \overline{PB} = 2\overline{PQ}+\overline{PB} \qquad (2)$$

$$v = (\overline{PB}+\overline{PC}) + \overline{PA} = 2\overline{PM}+\overline{PA} \qquad (3)$$

in other words $P=G$

$$\left.\begin{array}{l} \text{if } P\in NC \text{ it follows from (1) that } v \parallel NC \\ \text{if } P\in QB \text{ — '' — (2) that } v \parallel QB \end{array}\right\} \Rightarrow \text{if } P\in NC\cap QB \Rightarrow v=\overline{0}$$

hence, by (3), $2\overline{PM}+\overline{PA}=\overline{0}$ (if $P=G$)

So $2\overline{GM}+\overline{GA}=0$

So $\forall$ point O $\quad 2(\overline{GO}+\overline{OM}) + \overline{GO}+\overline{OA} = 0$

$$\Leftrightarrow 3\underbrace{\overline{GO}}_{-\overline{OG}} + 2\overline{OM}+\overline{OA} = 0$$

$$\Leftrightarrow 2\overline{OM}+\overline{OA} = 3\overline{OG}$$

$$\Leftrightarrow \quad \overline{OG} = \frac{2\overline{OM} + \overline{OA}}{3}$$

rem: since M is the midpoint of $[BC]$ $\overline{OM} = \frac{1}{2}(\overline{OC} + \overline{OB})$

$$\Leftrightarrow \quad \overline{OG} = \frac{2\left(\frac{\overline{OB}+\overline{OC}}{2} + \overline{OA}\right)}{3}$$

$$\Leftrightarrow \quad \overline{OG} = \frac{\overline{OB} + \overline{OC} + \overline{OA}}{3}$$

$\square$

7. Let ABC be a triangle, H its orthocenter, O the circumcenter (the center of its circumscribed cercle), G the centroid of the triangle and A' the point on the circum~~center~~ circle diametrically opposed to A. Then:

(1) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$; (**Sylvester's formula**)
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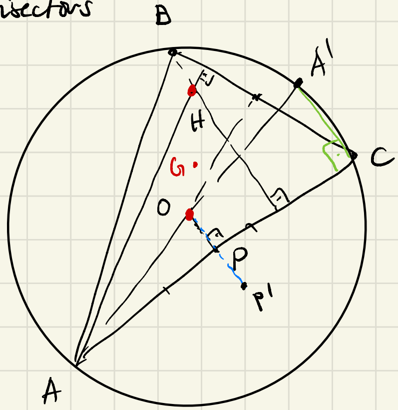
(4) $\overline{HA} + \overline{HB} + \overline{HC} = 3\overline{HG}$;
(5) the points H, G, O are collinear and $2GO = HG$. (the **Euler line**) $\Leftrightarrow 2\overrightarrow{GO} = \pm\overrightarrow{HG}$

O = intersection of perpendicular bisectors

H = — " — heights



- let P be the midpoint of [AC]

then $\overline{OA} + \overline{OC} = 2\overline{OP} = \overline{OP'}$

- AA' is a diameter

$\Rightarrow AA'C$ is a right triangle with $\angle C = 90°$ $\Big\} \Rightarrow |OP| = 2|A'C|$ (**)

- $\Delta\, OAC$ is isosceles $\Rightarrow OP \perp AC$

- AA' is a diameter $\angle ABA' = 90° = \angle A'CA$, i.e. $BA' \perp BA$ and $CA' \perp CA$

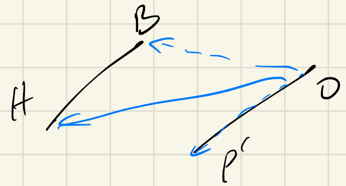
$\Downarrow$ $\qquad$ $\Downarrow$

$BA' \parallel CH$ $\qquad$ $CA' \parallel BH$

$\Rightarrow CA'BH$ is a parallelogram

$$\left.\begin{array}{c}\Rightarrow \quad \overline{HB} = \overline{CA'} \\ (**)\end{array}\right\} \Rightarrow \overline{HB} = 2\,\overline{PO} = \overline{P'O} \Rightarrow$$

HBOP' is a parallelogram



see

$$\Rightarrow \quad \underbrace{\overline{OB} + \underbrace{\overline{OP'}}_{\overline{OA} + \overline{OC}}}_{\overline{OA} + \overline{OB} + \overline{OC}} = \overline{OH}$$