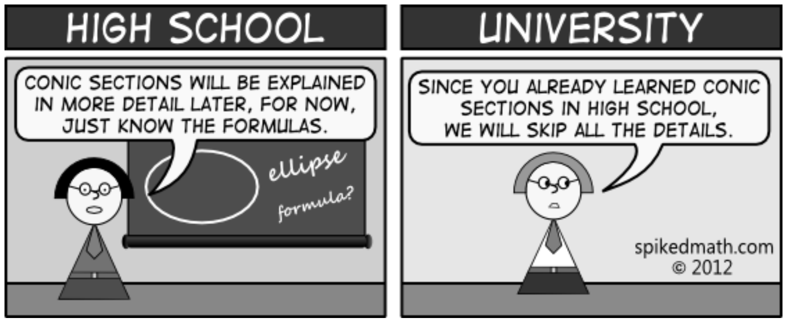
ANALYTIC GEOMETRY, PROBLEM SET 12



1. Find the equation of the circle: a) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $d : 3x - y - 2 = 0$;
b) determined by $A(1,1)$, $B(1,-1)$ and $C(2,0)$; c) tangent to both $d_1 : 2x + y - 5 = 0$ and $d_2 : 2x + y + 15 = 0$, if the tangency point with d_1 is $M(3,-1)$.

2. a) Determine the position of the point $A(1,-2)$ relative to the circle $C : x^2 + y^2 - 8x - 4y - 5 = 0$;
b) Find the intersection between the line $d : 7x - y + 12 = 0$ and the circle $C : (x-2)^2 + (y-1)^2 - 25 = 0$;
c) Determine the position of the line $d : 2x - y - 3 = 0$ relative to the circle $C : x^2 + y^2 - 3x + 2y - 3 = 0$.

3. Find the equation of
a) the tangent line to $C : x^2 + y^2 - 5 = 0$ at the point $A(-1,2)$;
b) the tangent lines to $C : x^2 + y^2 + 10x - 2y + 6 = 0$, parallel to $d : 2x + y - 7 = 0$;
c) the tangent lines to $C : x^2 + y^2 - 2x + 4y = 0$, orthogonal on $d : x - 2y + 9 = 0$.

4. Find the foci of the ellipse $\mathcal{E} : 9x^2 + 25y^2 - 225 = 0$.

5. Find the intersection points between the line $d_1 : x + 2y - 7 = 0$ and the ellipse given by the equation $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

6. Find the position of the line $d : 2x + y - 10 = 0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$.

7. Find the equation of a line which is orthogonal on $d_1 : 2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$.

8. Consider the ellipse $x^2 + 4y^2 = 25$. Find the chords on the ellipse which have the point $A(7/2, 7/4)$ as their midpoint.

9. Consider the ellipse $\frac{x^2}{4} + y^2 = 1$ with F_1, F_2 as foci. Find the points M, situated on the ellipse for which the angle $\angle F_1MF_2$ is right.

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10. Consider the ellipse $\frac{x^2}{4} + y^2 = 1$ with F_1, F_2 as foci. Find the point M on the ellipse for which $\angle F_1 M F_2$ is maximal.

11. Consider the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $x + 2y = 1$.

1. Find the equation of the circle: a) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $d: 3x-y-2=0$;
b) determined by $A(1,1)$, $B(1,-1)$ and $C(2,0)$; c) tangent to both $d_1: 2x+y-5=0$ and $d_2: 2x+y+15=0$, if the tangency point with d_1 is $M(3,1)$.

a) the equation of a circle of radius r centered at the point $P(x_0, y_0)$

is $\mathcal{C}: \quad (x-x_0)^2 + (y-y_0)^2 = r^2$

since $P \in d: 3x-y-2=0 \Rightarrow \boxed{y_0 = 3x_0 - 2}$ so $P = (x_0, 3x_0-2)$ and

$\mathcal{C}: \quad (x-x_0)^2 + (y-3x_0+2)^2 = r^2$

since $A(3,1) \in \mathcal{C}$ and $B(-1,3) \in \mathcal{C}$ we have

$$\begin{cases} (3-x_0)^2 + (1-3x_0+2)^2 = r^2 & \text{and} \\ (-1-x_0)^2 + (3-3x_0+2)^2 = r^2 \end{cases}$$

rearranging the equations we have

$$\begin{cases} 9-6x_0+x_0^2 + 9-18x_0+9x_0^2 = r^2 \\ 1+2x_0+x_0^2 + 25-30x_0+9x_0^2 = r^2 \end{cases} \Leftrightarrow \begin{cases} 10x_0^2 - 24x_0 + 18 = r^2 \\ 10x_0^2 - 28x_0 + 26 = r^2 \end{cases} \ominus$$

$$0 + x_0 - 8 = 0$$

$$\Rightarrow \boxed{x_0 = 2} \Rightarrow \boxed{y_0 = 4}$$

$\Rightarrow$ so, the center of $\mathcal{C}$ is $P(2,4)$

and the radius r is the distance from P to A (and to B)

$$r = d(P,A) = \sqrt{(2-3)^2+(4-1)^2} = \sqrt{1+9} = \sqrt{10}$$

$$\left(\text{check} \quad r = d(P,B) = \sqrt{(2+1)^2+(4-3)^2} = \sqrt{9+1} = \sqrt{10}\right)$$

1. Find the equation of the circle: a) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $d: 3x - y - 2 = 0$;
b) determined by $A(1,1)$, $B(1,-1)$ and $C(2,0)$; c) tangent to both $d_1: 2x + y - 5 = 0$ and $d_2: 2x + y + 15 = 0$, if the tangency point with d_1 is $M(3,-1)$.

b.) from the lecture you know that the circle passing through $A(x_A, y_A)$ $B(x_B, y_B)$ and $C(x_C, y_C)$ has an equation of the form

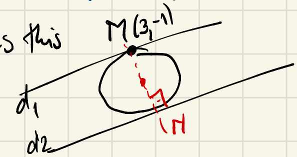
$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_A^2+y_A^2 & x_A & y_A & 1 \\ x_B^2+y_B^2 & x_B & y_B & 1 \\ x_C^2+y_C^2 & x_C & y_C & 1 \end{vmatrix} = 0$$

so, you obtain the equation that is required here if you replace in and expand the determinant

c.) We have a circle which is tangent to two lines $d_1: 2x+y-5$ and $d_2: 2x+y+15=0$

What do you notice by looking at the two lines? they are parallel (why?)

So the picture is this



So the midpoint is on the perpendicular line l passing through M, it has equation

$$l: -1(x-3) + 2(y+1) = 0 \quad \text{(why?)}$$

$$\Leftrightarrow l: -x + 2y + 5 = 0$$

the intersection $N = l \cap d_2$ is the solution to $\begin{cases} 2x + y + 15 = 0 \\ -x + 2y + 5 = 0 \end{cases} \Rightarrow x = 2y+5 \Bigg\} \Rightarrow 5y + 25 = 0$

$$\boxed{\begin{array}{l} \Rightarrow y = -5 \\ \Rightarrow x = -5 \end{array}} \Rightarrow N(-5,-5)$$

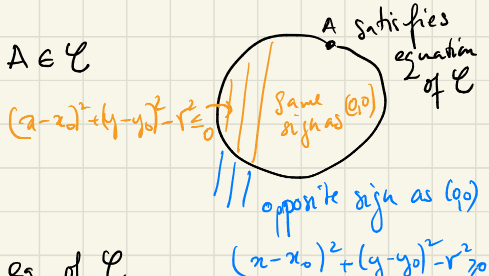
the center of $\mathcal{C}$ is the midpoint of $[M,N]$, it is $P(-1,-3)$

the radius of $\mathcal{C}$ is $d(P,M) = \sqrt{(3+1)^2 + (-1+3)^2} = \sqrt{16+4} = \sqrt{20}$

$$\Rightarrow \mathcal{C}: (x+1)^2 + (y+3)^2 = 20$$

2. a) Determine the position of the point $A(1, -2)$ relative to the circle $C : x^2 + y^2 - 8x - 4y - 5 = 0$;
b) Find the intersection between the line $d : 7x - y + 12 = 0$ and the circle $C : (x-2)^2 + (y-1)^2 - 25 = 0$;
c) Determine the position of the line $d : 2x - y - 3 = 0$ relative to the circle $C : x^2 + y^2 - 3x + 2y - 3 = 0$.

a) $1^2 + (-2)^2 - 8\cdot 1 - 4(-2) - 5 = 0 \Rightarrow A \in \mathcal{C}$



b) $d \cap \mathcal{C}$ = set of points whose coordinates satisfy the eq. of d and the eq. of $\mathcal{C}$

$$d \cap \mathcal{C} : \begin{cases} (x-2)^2 + (y-1)^2 - 25 = 0 \\ 7x - y + 12 = 0 \Rightarrow y = 7x+12 \end{cases} \Rightarrow (x-2)^2 + (7x+11)^2 - 25 = 0$$

$\Rightarrow x^2 - 4x + 4 + 49x^2 + 154x + 121 - 25 = 0$

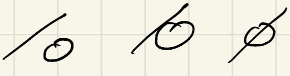
$\Rightarrow 50x^2 + 150x + 100 = 0$

$\Rightarrow 50(x^2 + 3x + 2) = 0$

$\Delta = 9 - 4\cdot 2 = 1 \Rightarrow x_{1,2} = \dfrac{-3 \pm 1}{2} = $ $-2 = x \Rightarrow y = -2$; $-1 = x \Rightarrow y = 5$

$\Rightarrow d \cap \mathcal{C} = \{(-2,-2), (-1,5)\}$

c) for the position of a line and a circle we have three possibilities:



$$d \cap \mathcal{C} : \begin{cases} x^2 + y^2 - 3x + 2y - 3 = 0 \\ 2x - y - 3 = 0 \Rightarrow y = 2x - 3 \end{cases} \Rightarrow x^2 + (2x-3)^2 - 3x + 2(2x-3) - 3 = 0$$

$\Rightarrow x^2 + 4x^2 - 12x + 9 - 3x + 4x - 6 - 3 = 0$

$\Rightarrow 5x^2 - 11x = 0 \Rightarrow x\cdot(5x - 11) = 0$

we have two solutions $\Rightarrow \mathcal{C}$ intersects d in two points.

3. Find the equation of
a) the tangent line to $C : x^2 + y^2 - 5 = 0$ at the point $A(-1, 2)$;
b) the tangent lines to $C : x^2 + y^2 + 10x - 2y + 6 = 0$, parallel to $d : 2x + y - 7 = 0$;
c) the tangent lines to $C : x^2 + y^2 - 2x + 4y = 0$, orthogonal on $d : x - 2y + 9 = 0$.

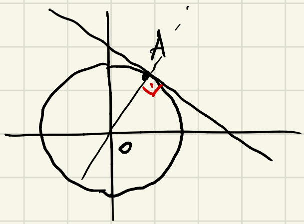
a) For a circle $\mathcal{C}$ with equation $x^2+y^2-r^2=0$

the tangent line at the point $P(x_0,y_0)\in\mathcal{C}$ is

$x_0x+y_0y-r^2=0$ (see lecture)

So, in our case: $-x+2y-5=0$

Method II



for a circle centered at the origin, the tangent line passing through $A\in\mathcal{C}$ is perpendicular to the line OA: $2x+y=0$ (why do we have this eq?)

$\Rightarrow$ the line l has eq: $-x+2y+c=0$

$A\in l \Rightarrow -(-1)+4+c=0 \Rightarrow c=-5$

b) $\underbrace{l \text{ is tangent to } \mathcal{C}}$ and $\underbrace{l \text{ is parallel to } d: 2x+y-7=0}$

$\Downarrow$

l intersects $\mathcal{C}$ in one point $\quad$ $l: 2x+y+c=0$ or $y=-2x-c$

$$\Rightarrow \text{system } (*) \begin{cases} x^2+y^2+10x-2y+6=0 \\ y=-2x-c \end{cases}$$ has a unique solution

$\Updownarrow$

$x^2+(2x+c)^2+10x+4x+2c+6=0$ has a unique solution

$\Updownarrow$ $\quad$ $4x^2+4cx+c^2$

$\Downarrow$

$$5x^2 + (4c+14)\,x + c^2 + 2c + 6 = 0$$

$\hookrightarrow$ has unique solution $\Leftrightarrow \Delta = 0$

$$\Leftrightarrow \cancel{4}(4c^2 + 28c + 49) - \cancel{4}\cdot 5\,(c^2 + 2c + 6) = 0$$

$$\Leftrightarrow -c^2 + 18c + 19 = 0$$

$$\Delta_2 = \underset{\substack{\parallel \\ 4\cdot 9^2}}{18^2} - 4\cdot(-1)\cdot 19 = 4\,(81 + 19) = 4\cdot 100$$

$$\Rightarrow c_{1,2} = \frac{-18 \pm 20}{-2} < \begin{matrix} 19 \\ -1 \end{matrix}$$

so if $c = -1$ or $c = 19$ then system $(*)$ has a unique solution

so we obtained two lines

for $c = -1$ we have $l: \ 2x + y - c = 0$

for $c = 19$ we have $l: \ 2x + y + 19 = 0$

3. Find the equation of
a) the tangent line to $C : x^2 + y^2 - 5 = 0$ at the point $A(-1, 2)$;
b) the tangent lines to $C : x^2 + y^2 + 10x - 2y + 6 = 0$, parallel to $d : 2x + y - 7 = 0$;
c) the tangent lines to $C : x^2 + y^2 - 2x + 4y = 0$, orthogonal on $d : x - 2y + 9 = 0$.

c.) l tangent to $\mathcal{C}$ and l orthogonal to d

$\Updownarrow$ $\Updownarrow$

l intersects $\mathcal{C}$ in one point $\quad$ l parallel to $-x + 2y + c = 0$

$\Updownarrow$ $\Updownarrow$

$l: \ x = 2y + c$ for some constant $c \in \mathbb{R}$

$$l \cap \mathcal{C} : \begin{cases} x^2 + y^2 - 2x + 4y = 0 \\ x = 2y + c \end{cases}$$ has a unique solution

$$\Updownarrow$$

$(2y+c)^2 + y^2 - \cancel{4y} + 2c + \cancel{4y} = 0$ has a unique solution (in y)

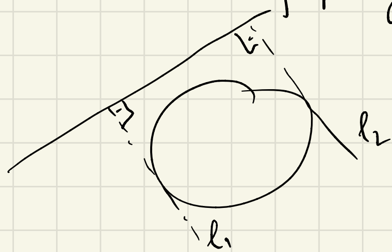
$4y^2 + 4yc + c^2$

$\iff$ $5y^2 + 4cy + c^2 + 2c = 0$ has a unique solution (in y)

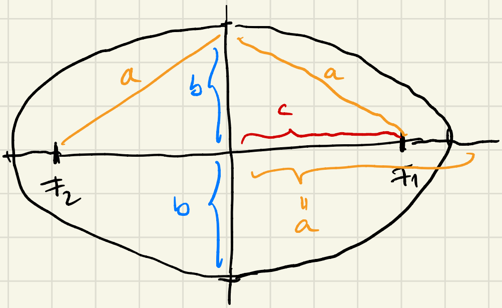
$\iff$ $\Delta = \overset{4}{\cancel{16}}\, c^2 - \cancel{4}\cdot 5\,(c^2+2c) = 0$

$\iff$ $-c^2 - 8c = 0$

$\iff$ $c = 0$ or $c = 8$ (again we obtain two solutions for c which means two lines with the desired property)



4. Find the foci of the ellipse $\mathcal{E} : 9x^2 + 25y^2 - 225 = 0$.



$$\Updownarrow$$

$$\frac{x^2}{25} + \frac{y^2}{9} - 1 = 0$$

$$\Updownarrow$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

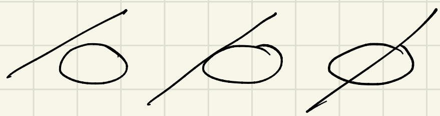
where $2a$ is the sum of the distances from a point on $\mathcal{E}$ to the focal points and if $c^2 = a^2 - b^2$ then $2c$ = focal distance

so $c^2 = 25 - 9 = 16$

$\Rightarrow F_1(4,0)$ and $F_2(-4,0)$

5. Find the intersection points between the line $d_1 : x + 2y - 7 = 0$ and the ellipse given by the equation $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

there are three possibilities :



$d_1 \cap \mathcal{E} = \emptyset$ $\quad d_1 \cap \mathcal{E} = 1\text{pt}$ $\quad d_2 \cap \mathcal{E} =$ two points

$$d_1 \cap \mathcal{E} : \begin{cases} x^2 + 3y^2 - 25 = 0 \\ x + 2y - 7 = 0 \Rightarrow x = 7 - 2y \end{cases} \Rightarrow (7-2y)^2 + 3y^2 - 25 = 0$$

$$49 - 28y + 4y^2 + 3y^2 - 25 = 0$$

$$7y^2 - 28y + 24 = 0$$

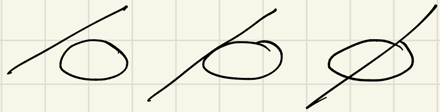
$$\dots \quad y = 2 \pm \frac{2}{\sqrt{2}}$$

$\Rightarrow$ two solutions for y $\Rightarrow$ two intersection points $P_1(\_, y_1)$ and $P_2(\_, y_2)$

...

6. Find the position of the line $d: 2x+y-10=0$ relative to the ellipse $\mathcal{E}: \frac{x^2}{9}+\frac{y^2}{4}-1=0$.

there are three possibilities :



$$d \cap \mathcal{E}: \begin{cases} \frac{x^2}{9}+\frac{y^2}{4}-1=0 \\ 2x+y-10=0 \quad \Rightarrow \quad y = 10-2x \end{cases} \Rightarrow \frac{x^2}{9} + \frac{(10-2x)^2}{4} - 1 = 0$$

$$\frac{(10-2x)^2}{4} = (5-x)^2$$

$$\Rightarrow x^2 + 9\underbrace{(5-x)^2}_{225-90x+9x^2} - 9 = 0 \quad \Rightarrow 10x^2 - 90x + 216 = 0$$

$$\Delta = 90^2 - 4\cdot 10 \cdot 216 = 2^2\cdot 3^3 \cdot 5\underbrace{(3\cdot 5 - 2^4)}_{15-16<0}$$

$$90^2 - 4\cdot 10\cdot 216 = 3^4\cdot 2^2 \cdot 5^2 - 2^6 \cdot 3^3 \cdot 5$$

so $\Delta < 0 \Rightarrow$ the line does not intersect the ellipse

7. Find the equation of a line which is orthogonal on $d_1: 2x-2y-13=0$ and tangent to the ellipse $\mathcal{E}: x^2+4y^2-20=0$.

A line l which is orthogonal to d_1 has an equation of the form

(why?)

$l_c: x+y+c=0$ for some $c \in \mathbb{R}$

l_c is tangent to $\mathcal{E}$ if $l \cap \mathcal{E}$ is a unique point

$$l_c \cap \mathcal{E}: \begin{cases} x^2+4y^2-20=0 \\ x+y+c=0 \quad \Rightarrow \quad x = -y-c \end{cases} \Rightarrow (-y-c)^2+4y^2-20=0$$

this system has a unique solution (x,y) if this equation has a unique solution y

$$y^2+2cy+c^2+4y^2-20=0 \Rightarrow 5y^2+2cy+c^2-20=0$$

$\downarrow$

has a unique solution if

$$0=\Delta_2=\cancel{4}c^2-\cancel{4}\cdot 5\cdot(c^2-20)=0$$

$$-4c^2+100=0 \Rightarrow c^2=25 \Rightarrow c=\pm 5$$

$\Rightarrow$ we have two solutions $c_1=5 \Rightarrow \ell_5: x+y+5=0$ is tangent to $\mathcal{E}$

$c_2=-5 \Rightarrow \ell_{-5}: x+y-5=0$ is tangent to $\mathcal{E}$

$\downarrow$

as expected

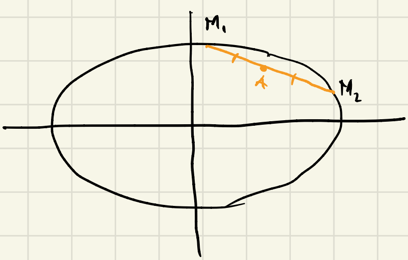
Method II

you know that for a given slope m there are two tangent lines to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

in our case $m=-1$, $a^2=20$, $b^2=5$ and you get the same two lines as above

8. Consider the ellipse $x^2+4y^2=25$. Find the chords on the ellipse which have the point $A(7/2,7/4)$ as their midpoint.



$x^2+4y^2=25 \Rightarrow x=\sqrt{25-4y^2}$ if $x>0$

choose two points M_1 and M_2 as in the picture

$\Rightarrow M_1\left(\sqrt{25-4y_1^2}, y_1\right), M_2\left(\sqrt{25-4y_2^2}, y_2\right)$

the midpoint of the segment $[M_1,M_2]$ is $\left(\frac{\sqrt{25-4y_1^2}+\sqrt{25-4y_2^2}}{2}, \frac{y_1+y_2}{2}\right)=A\left(\frac{7}{2},\frac{7}{4}\right)$

$$\Rightarrow (**)\begin{cases}\sqrt{25-4y_1^2}+\sqrt{25-4y_2^2}=7\\ y_1+y_2=\frac{7}{2} \Rightarrow y_1=\frac{7}{2}-y_2\end{cases} \Rightarrow \sqrt{25-4\left(\frac{7}{2}-y_2\right)^2}+\sqrt{25-4y_2^2}=7$$

$$\frac{49}{4}-7y_2+y_2^2$$

$$\sqrt{25-49+28y_2-4y_2^2} = 7-\sqrt{25-4y_2^2} \qquad |\,()^2$$

$$\cancel{25}-49+28y_2-\cancel{4y_2^2} = 49-14\sqrt{25-4y_2^2}+\cancel{25-4y_2^2}$$

$$-2\cdot 49+4\cdot 7\,y_2 = -14\sqrt{25-4y_2^2} \qquad |:-14$$

$$7-2y_2 = \sqrt{25-4y_2^2} \qquad |\,()^2$$

$$49-28y_2+\cancel{4y_2^2} = 25-\cancel{4y_2^2}$$

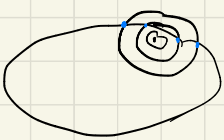
$$24-28y_2=0 \Rightarrow \boxed{y_2=\frac{6}{7}} \Rightarrow \boxed{y_1=\frac{7}{2}-\frac{6}{7}=\frac{49-12}{14}=\frac{37}{14}}$$

So the solutions to the system (**) give the y-coordinates of the two points that we are looking for, with these values we can calculate also the x-coordinates of M_1 and M_2

Notice: in general it could happen that M_1 and M_2 are not on the same side of Oy. So, if we don't find a solution in this form we need to consider the other possibilities:

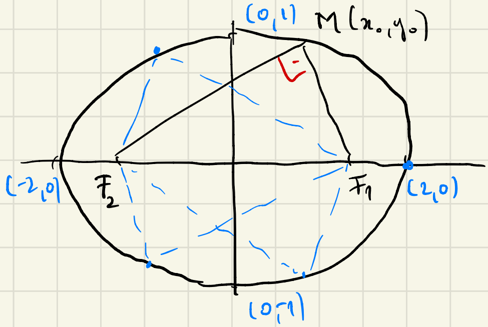
$$M_1\left(\pm\sqrt{25-4y_1^2},\ y_1\right),\ M_2\left(\pm\sqrt{25-4y_2^2},\ y_2\right)$$

different approach:



intersect the ellipse with circles centered at A and discuss when this intersection is a diameter of the circle

9. Consider the ellipse $\frac{x^2}{4}+y^2=1$ with F_1, F_2 as foci. Find the points M, situated on the ellipse for which the angle $\angle F_1 M F_2$ is right.



Remark:
because of the symmetries of the ellipse it is enough to search for such a point in the first quadrant: there are four points with such a property and the other three are obtained by reflecting in the Ox and Oy axes.

$\mathcal{E}: \frac{x^2}{4}+y^2=1 \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a=2 \quad b=1 \quad \Rightarrow c=\sqrt{4-1}=\sqrt{3}$ (the focal distance is $2\sqrt{3}$)

$\Rightarrow$ the focal points are $F_1(\sqrt{3},0)$ and $F_2(-\sqrt{3},0)$

choose $M(x_0,y_0)\in\mathcal{E}$ with $y_0>0$ then $y_0=\sqrt{1-\frac{x_0^2}{4}}$

$$\Rightarrow \overrightarrow{MF_1}\left(\sqrt{3}-x_0, -\sqrt{1-\frac{x_0^2}{4}}\right)$$

and

$$\overrightarrow{MF_2}\left(-\sqrt{3}-x_0, -\sqrt{1-\frac{x_0^2}{4}}\right)$$

the angle $\widehat{F_1MF_2}$ is a right angle $\Leftrightarrow \overrightarrow{MF_1}\perp\overrightarrow{MF_2}$

$$\Leftrightarrow \overrightarrow{MF_1}\cdot\overrightarrow{MF_2}=0$$

$$\Leftrightarrow (\sqrt{3}-x_0)(-\sqrt{3}-x_0)+\left(\sqrt{1-\frac{x_0^2}{4}}\right)^2=0$$

$$\Leftrightarrow x_0^2-3+1-\frac{x_0^2}{4}=0$$

$$\Leftrightarrow \frac{3}{4}x_0^2-2=0$$

$\Leftrightarrow \quad x_0^2 = \frac{8}{3}$

$\Leftrightarrow \quad x_0 = \pm 2\sqrt{\frac{2}{3}}$ so $y_0 = \sqrt{1 - \frac{1}{4} \cdot \frac{8}{3}} = \frac{1}{\sqrt{3}}$

so, for $y_0 > 0$ we found two solutions, two points

$$\left(2\sqrt{\tfrac{2}{3}}, \tfrac{1}{\sqrt{3}}\right) \text{ and } \left(-2\sqrt{\tfrac{2}{3}}, \tfrac{1}{\sqrt{3}}\right)$$

the other two points are obtained by symmetry in the Ox axis:

$$\left(2\sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{3}}\right) \text{ and } \left(-2\sqrt{\tfrac{3}{3}}, -\tfrac{1}{\sqrt{8}}\right)$$

different approach : because the angle is $50°$ these points belong to the circle with diameter $[F_1F_2]$

10. Consider the ellipse $\frac{x^2}{4} + y^2 = 1$ with F_1, F_2 as foci. Find the point M on the ellipse for which $\angle F_1MF_2$ is maximal.

as for problem 11, the focal points of the ellipse $\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a>b)$ are $F_1(c,0)$ and $F_2(-c,0)$

so, for a point $M(x_0,y_0) \in \mathcal{E}$ the angle $\widehat{F_1MF_2}$ is the angle between the vectors

$$\overrightarrow{MF_1}\,(c - x_0, y_0) \text{ and } \overrightarrow{MF_2}\,(-c - x_0, y_0)$$

$$\cos \measuredangle(\overrightarrow{MF_1}, \overrightarrow{MF_2}) = \frac{\overrightarrow{MF_1} \cdot \overrightarrow{MF_2}}{\|\overrightarrow{MF_1}\| \cdot \|\overrightarrow{MF_2}\|} = \frac{x_0^2 - c^2 + y_0^2}{\sqrt{(c-x_0)^2 + y_0^2}\sqrt{(c+x_0)^2 + y_0^2}} \quad (*)$$

$\hookrightarrow$ we need to understand this

$M(x_0,y_0) \in \mathcal{E} \Rightarrow y_0 = b^2\left(1 - \frac{x_0^2}{a^2}\right)$

$\Rightarrow \|\overrightarrow{MF_1}\| = \sqrt{(c-x_0)^2 + y_0^2} = \sqrt{c^2 - 2cx_0 + x_0^2 + b^2 - \frac{b^2}{a^2}x_0^2} = \sqrt{\left(a - \frac{c}{a}x_0\right)^2}$

$= \left|a - \frac{c}{a}x_0\right| = a - \frac{c}{a}x_0$

in a similar way we obtain

$$\|\overrightarrow{MF_2}\| = a + \frac{c}{a} x_0$$

$$c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2$$

so (*) becomes

$$\cos \measuredangle (\overrightarrow{MF_1}, \overrightarrow{MF_2}) = \frac{x_0^2 - c^2 + \overset{\overset{a^2-c^2}{\parallel}}{b^2} - \frac{b^2}{a^2}x_0^2}{\left(a - \frac{c}{a}x_0\right)\left(a + \frac{c}{a}x_0\right)} = \frac{\frac{c^2}{a^2}x_0^2 + a^2 - 2c^2}{a^2 - \frac{c^2}{a^2}x_0^2}$$

$\downarrow$

is a function in $x_0 \in [-a, a]$

if $\mathcal{E}$ is a circle then $c = 0$

$= \frac{a^2}{a^2} = 1$

since the angle is 0

the second derivative in x_0 shows that

x_0	$-a$ . . . 0 . . . a
$\cos \measuredangle (\overrightarrow{MF_1}, \overrightarrow{MF_2})$	$\searrow$ $\boxed{\ }$ $\nearrow$
derivative of —"—	$-$ $-$ 0 $+$ $+$

$$f(z) = \frac{z + a^2 - 2c^2}{-z + a^2}$$

$$f'(z) = z \frac{4(a^2 - c^2)}{(-z + a^2)^2}$$

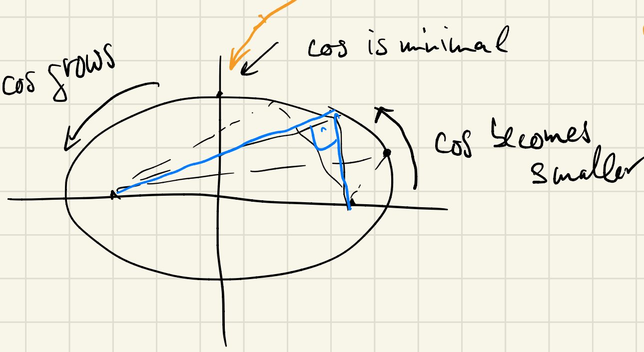
value for $x_0 = 0$

which corresponds to the minimal cos value

which corresponds to the biggest angle

which is attained for $x_0 = 0$

we obtain the point $M(0, b)$

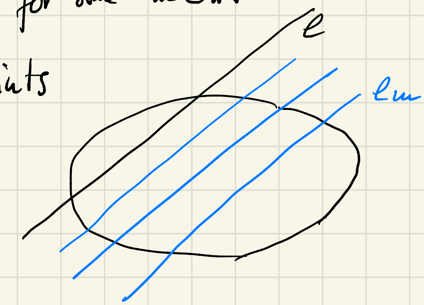


11. Consider the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $x+2y=1$.

- A line which is parallel to the given line has an equation of the form

$$\ell_m: \quad x+2y=m \quad \text{for some } m\in\mathbb{R}$$

- in order to obtain the intersection points with the ellipse we find the solutions to the system



$$\begin{cases} \frac{x^2}{25}+\frac{y^2}{9}=1 \\ x+2y=m \Rightarrow x=-2y-m \end{cases} \Rightarrow \frac{(2y+m)^2}{25}+\frac{y^2}{9}=1$$

$$\frac{4y^2+4my+m^2}{25}+\frac{y^2}{9}=1$$

$$\Rightarrow \quad (9\cdot 4+25)y^2+4\cdot 9\cdot my+9m^2-9\cdot 25=0$$

$$\Delta = 4^2 9^2-4(9\cdot 4+25)(9m^2-9\cdot 25)$$

$$=4\cdot 9\left[4\cdot 9-61(m^2-25)\right]$$

so there are intersection points if $\Delta\geq 0$

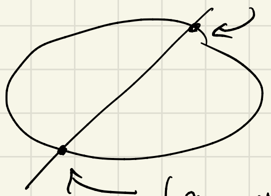
$$\Leftrightarrow \quad 4\cdot 9-61(m^2-25)\geq 0$$

$$\Leftrightarrow \quad \frac{4\cdot 9}{61}+25\geq m^2$$

so $m \in \left[-\sqrt{\frac{4\cdot 9}{61}+25} \, , \, +\sqrt{\frac{4\cdot 9}{61}+25} \right]$

for these values of m the intersection points are

$$(x_1, y_1) = \left(-2y_1 - m \, , \, \frac{-4\cdot 9m + \sqrt{\Delta}}{2\cdot 61} \right)$$



$$= \left(-2\,\frac{-4\cdot 9m + \sqrt{\Delta}}{2\cdot 61} - m \, , \, \frac{-4\cdot 9m + \sqrt{\Delta}}{2\cdot 61} \right)$$

$$(x_2, y_2) = \left(-2\,\frac{-4\cdot 9m - \sqrt{\Delta}}{2\cdot 61} - m \, , \, \frac{-4\cdot 9m - \sqrt{\Delta}}{2\cdot 61} \right)$$

the midpoint of this chord is

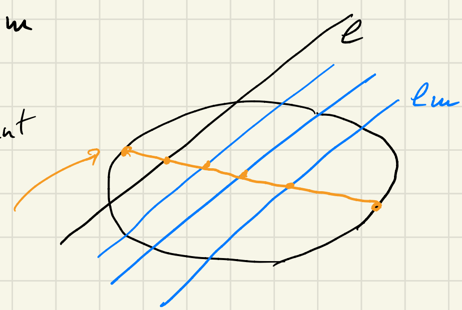
$$\left(\frac{x_1 + x_2}{2} \, , \, \frac{y_1 + y_2}{2} \right) = \left(\frac{-2y_1 - m - 2y_2 - m}{2} \, , \, \frac{y_1 + y_2}{2} \right)$$

$$\frac{-2y_1 - m - 2y_2 - m}{2} = -y_1 - y_2 - m = +\frac{18m}{61} - m = \frac{43}{61}m$$

$$\frac{y_1 + y_2}{2} = \frac{1}{2}\,\frac{-4\cdot 9m + \sqrt{\Delta} - 4\cdot 9m - \sqrt{\Delta}}{2\cdot 61} = \frac{1}{\cancel{2}}\,\frac{-2\cdot 4\cdot 9m}{2\cdot 61} = \frac{-18m}{61}$$

$\Rightarrow$ the geometric locus is the line segment

$$\left\{ \left(\frac{43}{61}m \, , \, \frac{-18}{61}m \right) : m \in [\,\ldots\,] \right\}$$



different approach: · parametrize $M(a\cos\theta, b\sin\theta) \in \mathcal{E}$

· write down line l_M containing M and parallel to the given line

· intersect l_M with $\mathcal{E}$ to find M'

· calculate the midpoint of M and M'