Neighbourhoods

$a \in \mathbb{R}$, $V \subseteq \mathbb{R}$ | set V neighbourhood of the point a $\iff \exists r_V > 0$ s.t. $B(a, r_V) \subseteq V$

$= \{x \in \mathbb{R} : |x-a| < r_V\} = (a - r_V, a + r_V)$

(the ball of some radius $r_V > 0$, centred at a is included in the set)

Properties:

$\boxed{P_1}$

If $\left.\begin{array}{l} V \in \mathcal{V}(a) \\ V \subseteq W \end{array}\right\} \Rightarrow W \in \mathcal{V}(a)$

$\boxed{P_2}$

If $\left.\begin{array}{l} V \in \mathcal{V}(a) \\ W \in \mathcal{V}(a) \end{array}\right\} \Rightarrow V \cap W \in \mathcal{V}(a)$

$\boxed{P_3}$

$\mathbb{R} \in \mathcal{V}(a)$

$\boxed{P_4}$

$\forall V \in \mathcal{V}(a),\ \exists T \subseteq V$ s.t. $\forall t \in T,\ V \in \mathcal{V}(t)$

Proof:

Proof: (V_1) $V \in \mathcal{V}(a) \overset{def.}{\Longrightarrow} \left.\begin{array}{l} \boxed{\exists r_V > 0} \text{ s.t. } B(a, r_V) \subseteq V \\ V \subseteq W \end{array}\right\} \Rightarrow \boxed{B(a, r_V) \subseteq W}$

$\overset{def.}{\Longrightarrow} W \in \mathcal{V}(a)$ ✓

(V_2) $V \in \mathcal{V}(a) \overset{def.}{\Longrightarrow} \exists r_V > 0$ s.t. $B(a, r_V) \subseteq V$

$W \in \mathcal{V}(a) \overset{def.}{\Longrightarrow} \exists r_W > 0$ s.t. $B(a, r_W) \subseteq W$

$a - r_V$, a, $a + r_V$, $a + r_W$, $a - r_W$

$\exists r = \min\{r_V, r_W\} > 0$ s.t. $B(a, r) \subseteq V \cap W$

$\overset{def}{\Longrightarrow} V \cap W \in \mathcal{V}(a)$ ✓

(V_3) $\mathbb{R} \overset{?}{\in} \mathcal{V}(a)$

$a-2$, a, $a+2$

$\exists r = 2 > 0$ s.t. $B(a, 2) \subseteq \mathbb{R} \overset{def}{\Longrightarrow} \mathbb{R} \in \mathcal{V}(a)$ ✓

(V_4) We know: $V \in \mathcal{V}(a)$

We want: $\exists T \subseteq V$ s.t. $\forall t \in T,\ V \in \mathcal{V}(t)$

$V \in \mathcal{V}(a) \overset{def}{\Longrightarrow} \exists r_V > 0$ s.t. $B(a, r_V) \subseteq V$ (1)

$a - r_V$, a, $a + r_V$

$T = ?$

We prove that $\forall t \in B(a, r_V),\ V \in \mathcal{V}(t)$

Choose a $t \in B(a, r_V)$, randomly

$t - (a - r_V) > 0$, t

$a - r_V$, a, $a + r_V$

$a + r_V - t > 0$

$\boxed{\exists r} = \min\{t - (a - r_V),\ a + r_V - t\}\ \boxed{> 0}$

s.t. $\boxed{B(t, r)} \subseteq B(a, r_V)\ \boxed{\subseteq V}$ (1)

$\left.\begin{array}{l} \Rightarrow V \in \mathcal{V}(t) \\ t \to \text{was randomly chosen} \end{array}\right\} \Rightarrow \forall t \in B(a, r_V),\ V \in \mathcal{V}(t)$

$T =$

Remarks: (1) $\mathbb{R}$ is a neighbourhood for all of its points

Proof: apply $\boxed{V_3}$

(2) Not all the neighbourhoods are intervals

Example: $(-1, 2) \in \mathcal{V}(0) \overset{\boxed{V_1}}{\Longrightarrow} (-1, 2) \cup \{3\} \in \mathcal{V}(0)$

↪ it is not an interval

-1, 0, 1, 2

$\exists r = \frac{1}{3} > 0$ s.t. $B(0, \frac{1}{3}) \subseteq (-1, 2) \Rightarrow \in \mathcal{V}(0)$

(3) A set is not necesserly a neighbourhood for all of its points

Example: • $A = (-1, 2] \cup \{3\}$

-1, 0, 1, 2, 3, t

• $A \notin \mathcal{V}(3)$ $[\forall r_V > 0,\ B(3, r_V) \not\subseteq V]$

$t = 3 + \frac{r_V}{2} \in B(3, r_V)$ ✓

$\notin V$

• $A \notin \mathcal{V}(2)$ $[\forall r_V > 0,\ B(2, r_V) \not\subseteq V]$

$t = 2 + \frac{\min\{r_V, 1\}}{2}$ ✓

$t \notin V$

• $\forall a \in (-1, 2),\ A \in \mathcal{V}(a)$

(4) $\exists$ sets which are not neighbourhoods for any of their points

Examples: $\{7\}$, $\{1, 2, \dots, 10\}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ (Seminar)

Examples :

neighbourhood of $+\infty$ $\rightarrow$ a subset of $\mathbb{R}$

$(a, +\infty)$, $a \in \mathbb{R}$

a subset which contains all sufficiently large real numbers

neighbourhood of $-\infty$ $\rightarrow$ a subset of $\mathbb{R}$

$(-\infty, a)$, $a \in \mathbb{R}$

a subset which contains all sufficiently large real numbers

Open and closed sets

$A \subseteq \mathbb{R}$ | A open $\Longleftrightarrow \forall a \in A,\ A \in \mathcal{V}(a)$
(is a neighbourhood for all its points)

A closed $\Longleftrightarrow \mathbb{R}\setminus A$ is open $\Longleftrightarrow \forall a \in \mathbb{R}\setminus A,\ \mathbb{R}\setminus A \in \mathcal{V}(a)$

Theorem

$\mathbb{R}$ and $\mathbb{Q}$ both open and closed sets

$\forall (G_i)_{i\in I}$ family of open sets $\Rightarrow \bigcup_{i\in I} G_i$ is open

$\forall (G_j)_{j\in J}$ a finite family of open sets $\Rightarrow \bigcap_{j\in J} G_j$ is open

$\forall (G_i)_{i\in I}$ family of closed sets $\Rightarrow \bigcup_{i\in I} G_i$ is closed

$\forall (G_j)_{j\in J}$ a finite family of closed sets $\Rightarrow \bigcap_{j\in J} G_j$ is closed

Examples of open sets:

$\forall a<b \in \mathbb{R}$, (a,b) open set

$\forall a \in \mathbb{R}$, $(-\infty, a)$ open sets
(a, ∞)

Examples of closed sets:

$\forall a<b \in \mathbb{R}$ $[a,b]$ closed

$\forall a \in \mathbb{R}$ $(-\infty, a]$ and $[a, \infty)$ closed

{isolated point} closed

$\mathbb{N}, \mathbb{Z}$ closed

- The interval (-3, 3) is open, because if x is any number in (-3, 3), then $-3 < x < 3$. or equivalently, $-3 - x < 0 < 3 - x$. Now let $\varepsilon = \min(3 + x, 3 - x)$. Then $\varepsilon > 0$, and the interval $(x - \varepsilon / 2, x + \varepsilon / 2)$ is contained in (-3, 3). In fact, the same argument works for any interval of the form (a, b) with a, b real numbers.
- The interval [4, 7] is closed, because its complement consists of the two open sets $(-\infty, 4)$ and $(7, \infty)$.
- The interval (-4, 5] is neither open nor closed. It is not open, because the point $x = 5$ is contained in the set, but every neighborhood of that point is not contained inside the set. It is not closed, because its complement $(-\infty, -4]$ and $(5, \infty)$ is not open at $x = -4$.
- The interval $(0, \infty)$ and also the interval $(-\infty, 0)$ are both open, because every point in the set contains a neighborhood contained inside the set.
- The interval $[0, \infty)$ and also the interval $(-\infty, 0]$ are both closed, because their complements are the open sets mentioned above.

REMARK

If A NOT OPEN $\not\Rightarrow$ A CLOSED

If A NOT CLOSED $\not\Rightarrow$ A OPEN

e.g.

$(-1, 2]$ → not open $\quad (-1,2] \notin v(2)$

→ assume that $(-1, 2]$ is closed $\Leftrightarrow$ $\mathbb{R} \setminus (-1, 2] = (-\infty, -1] \cup (2, \infty)$ is open

false $\notin v(-1)$

⇒ $(-1, 2]$ is not closed

neither open, nor closed

Separation axioms

in $\mathbb{R}$: $\forall x<y\in\mathbb{R}$, $\exists z\in\mathbb{R}$ s.t. $x<z<y$

in $\mathbb{Q}$: $\forall x<y\in\mathbb{R}$, $\exists t\in\mathbb{Q}$ s.t. $x<t<y$

in $\mathbb{R}\setminus\mathbb{Q}$: $\forall x<y\in\mathbb{R}$, $\exists s\in\mathbb{Q}$ s.t. $x<s<y$

$\exists\, x,y\in\mathbb{R}$ s.t $\forall n\in\mathbb{N}$ $x\nless n\nless y$

Archimedes axiom

$\forall x>0\in\mathbb{R}$, $\exists n_x\in\mathbb{N}$ s.t. $x<n_x$

$\forall y>0\in\mathbb{R}$, $\exists n_y\in\mathbb{N}$ s.t. $0<\frac{1}{n_y}<n_y$

BOUNDS

$\emptyset \neq A \subseteq \mathbb{R}$, an element $x \in \mathbb{R}$ is

- a LOWER BOUND of the set A if $\forall a \in A,\ x \leq a$
  $LB(A) = \{x \in \mathbb{R} : x \leq a,\ \forall a \in A\}$ the set of all l.b. of A
- a UPPER BOUND of the set A if $\forall a \in A,\ x \geq a$
  $UB(A) = \{x \in \mathbb{R} : x \geq a,\ \forall a \in A\}$ the set of all u.b. of A
- $\emptyset \neq A \subseteq \mathbb{R}$ is a BOUNDED set if $LB(A) \neq \emptyset$ and $UB(A) \neq \emptyset$

INFIMUM AXIOM $\inf A$

- $\forall\ \emptyset \neq A \subseteq \mathbb{R}$, lower bounded $(LB(A) \neq \emptyset)$
  $\exists$ the MOST LOWER BOUNDED (the infimum) $\boxed{\text{in } \mathbb{R}}$
- If $LB(A) = \emptyset$ by convention $\inf A = -\infty$

SUPREMUM AXIOM $\sup A$

- $\forall\ \emptyset \neq A \subseteq \mathbb{R}$, upper bounded $(UB(A) \neq \emptyset)$
  $\exists$ the LEAST UPPER BOUND (the supremum) $\boxed{\text{in } \mathbb{R}}$
- If $UB(A) = \emptyset$ by convention $\sup A = \infty$

- Both the inf and sup $\exists$ for all non-empty subsets of $\mathbb{R}$ (but are not always real)
- inf and sup are independent

MINIMUM ($\min A$)

- An element $x \in \mathbb{R}$ is said to be
  the minimum of A if $x = A \cap LB(A)$
- If $\inf A \in A \Rightarrow \exists \min A = \inf A$
- $\min A$ does not always exist,
  even if $\inf A \in \mathbb{R}$

MAXIMUM ($\max A$)

- An element $x \in \mathbb{R}$ is said to be
  the maximum of A if $x = A \cap UB(A)$
- If $\sup A \in A \Rightarrow \exists \max A = \sup A$
- $\max A$ does not always exist,
  even if $\sup A \in \mathbb{R}$

- min and max are independent

EXAMPLE:

$A = (-1, 2] \cup \{3\}$

$LB(A) = (-\infty, -1] \xRightarrow{AS} \exists \inf A \in \mathbb{R} = -1$ $\quad\nexists \min A$ because $-1 \notin A$

$UB(A) = [3, \infty) \xRightarrow{AS} \exists \sup A \in \mathbb{R} = 3$ $\quad\exists \max A = \sup A = 3$ because $3 \in A$

Consider $A \subseteq \mathbb{R}$

INTERIOR OF A

$int\, A = \{x \in \mathbb{R} : A \in \mathcal{V}(x)\}$

→ $int\, A \subseteq A$

→ the greatest open set, included in A

→ A is open $\Leftrightarrow A = int\, A$

CLOSURE OF A

$cl\, A = \{x \in \mathbb{R} : \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

→ $A \subseteq cl\, A$

→ the least (smallest) closed set, which includes A

→ A is closed $\Leftrightarrow A = cl\, A$

$int\, A \subseteq A \subseteq cl\, A$

BOUNDARY (BORDER) of A

$bd\, A = \{x \in \mathbb{R} : \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \text{ and } V \cap (\mathbb{R} \setminus A) = \emptyset\}$

→ $bd\, A \subseteq cl\, A$

→ $bd\, A = cl\, A \setminus int\, A \Leftrightarrow int\, A \cup bd\, A = cl\, A$

→ $int\, A \cap bd\, A = \emptyset$

→ always closed

THE SET OF ISOLATED POINTS

$\mathcal{J}_{zo} A = \{x \in A : \exists V \in \mathcal{V}(x) \text{ s.t. } V \cap A = \{x\}\}$

→ $\mathcal{J}_{zo} A \subseteq A$

THE SET OF THE ACCUMULATION POINTS

$A' = cl\, A \setminus \mathcal{J}_{zo} A = \{x \in A : \forall V \in \mathcal{V}(x),\ V \cap A \setminus \{x\} \neq \emptyset\}$

Example: a) $A = (-1, 2] \cup \{3\}$

$int\, A = (-1, 2)$

$cl\, A = \underbrace{[-1, 2]}_{cl} \cup \underbrace{\{3\}}_{cl}$ cl.

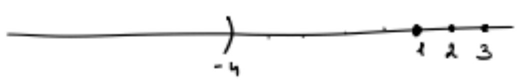
$bd\, A = cl\, A \setminus int\, A = \{-1, 2, 3\}$

$\boxed{\mathcal{J}_{zo}\, A = \{3\}}$

↳ $\exists r = \frac{1}{2}$ s.t. $B(3, \frac{1}{2}) \cap A = \{3\}$

$A' = [-1, 2]$

b) $B = (-\infty, -4) \cup \mathbb{N}$



$int\, B = (-\infty, -4)$

$cl\, B = \underbrace{(-\infty, -4]}_{cl} \cup \underbrace{\mathbb{N}}_{cl}$ $\cup$ closed $\overset{?}{\Rightarrow}$ closed

$\mathbb{R} \setminus (\quad) = \underline{(-4, 1) \cup (1, 2) \cup \ldots}$ Open

closed

$bd\, B = \{-4\} \cup \mathbb{N}$

$\mathcal{J}_{zo}\, B = \mathbb{N} \Rightarrow A' = (-\infty, -4]$