ANALYTIC GEOMETRY, PROBLEM SET 10

Mostly angles in 3D.

1. Show that the line $d = \begin{cases} x = 0 \\ y = t \\ z = t \end{cases}$ is contained inside the plane $6x + 4y - 4z = 0$.

2. Determine whether the line given by $x = 3 + 8t$, $y = 4 + 5t$, and $z = -3 - t$, $t \in \mathbb{R}$ is parallel to the plane $x - 3y + 5z - 12 = 0$.

3. Prove that the lines $d_1 : \begin{cases} x = 1 + 4t \\ y = 5 - 4t \\ z = -1 + 5t \end{cases}, t \in \mathbb{R}$ and $d_2 : \begin{cases} x = 2 + 8t \\ y = 4 - 3t \\ z = 5 + t \end{cases}, t \in \mathbb{R}$ are skew.

4. Find the parametric equations of the line passing through $(5, 0, -2)$ and parallel to the planes $x - 4y + 2z = 0$ and $2x + 3y - z + 1 = 0$.

5. Find the equation of the plane containing the point $P(2, 0, 3)$ and the line $d : \begin{cases} x = -1 + t \\ y = t \\ z = -4 + 2t \end{cases}$.

6. Let $M_1(2, 1, -1)$ and $M_2(-3, 0, 2)$ be two points. Find:
a) the equation of the bundle of planes passing through M_1 and M_2;
b) the plane π from the bundle, which is orthogonal on xOy;
c) the plane ρ from the bundle, which is orthogonal on π.

7. Find the angle determined by d_1 and d_2, when: a) $d_1 : x = 4 - t, y = 3 + 2t, z = -2t, t \in \mathbb{R}$ and $d_2 : x = 5 + 2s, y = 1 + 3s, z = 5 - 6s, s \in \mathbb{R}$.
b) $d_1 : \frac{x-1}{2} = \frac{y+5}{7} = \frac{z-1}{-1}$ and $d_2 : \frac{x+3}{-2} = \frac{y-9}{1} = \frac{z}{4}$.

8. Find the angle determined by the planes $\pi_1 : x - \sqrt{2}y + z - 1 = 0$ and $\pi_2 : x + \sqrt{2}y - z + 3 = 0$.

9. Find the coordinates of the orthogonal projection of the point P(2, 1, 1) on the plane $\pi : x + y + 3z + 5 = 0$.

10. Determine the orthogonal projection of the point $A(1, 3, 5)$ on the line which is given as the intersection of the planes $2x + y + z - 1 = 0$ and $3x + y + 2z - 3 = 0$.

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11. Determine the equations of the planes which pass through the points $P(0, 2, 0)$ and $Q(-1, 0, 0)$ and which form an angle of $60°$ with the Oz axis.

12. Find the equations of the projection of the line $d : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$ on the plane $\pi : x + 2y - z = 0$.

13. Find the angle determined by the lines $d_1 : \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases}$ and $d_2 : \begin{cases} x - y - z - 1 = 0 \\ x - y + 2x + 1 = 0 \end{cases}$.

14. Find the angle determined by the planes $\pi_1 : x+3y+2z+1 = 0$ and $\pi_2 : 3x+2y-z-6 = 0$.

15. Find the angle determined by the plane xOy and the line M_1M_2, where $M_1(1, 2, 3)$ and $M_2(-2, 1, 4)$.

1. Show that the line $d = \begin{cases} x = 0 \\ y = t \\ z = t \end{cases}$ is contained inside the plane $\Pi: 6x + 4y - 4z = 0$.

Method I

the line d is contaiint in the plane Π if all points of d lie in Π

$\Leftrightarrow$ all $\underbrace{\text{points of } d}$ satisfy the equation of Π

$\downarrow$

from the equation of d we have that any point P of d has coordinates $P(0,t,t)$ for some $t \in \mathbb{R}$

and $6 \cdot 0 + 4t - 4t = 0$ so any such P satisfies the eq. of Π

$\Rightarrow d = \{P(0,t,t) : t \in \mathbb{R}\} \subseteq \Pi$

Method II

the line d is containd in the plane Π if $\begin{cases} \text{① } d \text{ is parallel to } \Pi \\ \quad \text{and} \\ \text{② } d \text{ and } \Pi \text{ have one point in common} \end{cases}$

we can check ① with a direction vector of d, eg $v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

and a normal vector for Π, e.g. $n = \begin{pmatrix} 6 \\ 4 \\ -4 \end{pmatrix}$

$d \parallel \Pi \Leftrightarrow v \cdot n = 0 \Leftrightarrow 6 \cdot 0 + 4 \cdot 1 - 4 \cdot 1 = 0$ true

we can check ② with any point on d, eg $P(0,0,0)$ $(t=0)$

$P \in \Pi \Leftrightarrow 6 \cdot 0 + 4 \cdot 0 - 4 \cdot 0 = 0$ true

Method III

the line d is containd in Π if two distinct points in d are contained in Π

we can check this with $P(0,0,0)$ $(t=0)$ and $Q(0,1,1)$ for $t=1$

2. Determine whether the line given by $x = 3 + 8t$, $y = 4 + 5t$, and $z = -3 - t$, $t \in \mathbb{R}$ is parallel to the plane $x - 3y + 5z - 12 = 0$.

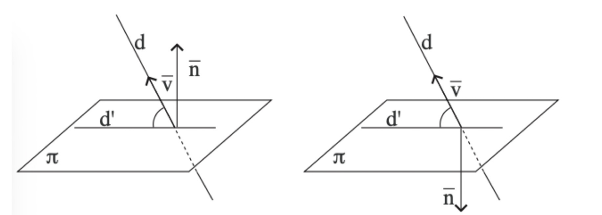
Method I $\quad d: \begin{cases} x = 3 + 8t \\ y = 4 + 5t \\ z = -3 - t \end{cases} \quad t \in \mathbb{R}$ $\qquad \pi: x - 3y + 5z - 12 = 0$

$v = \begin{pmatrix} 8 \\ 5 \\ -1 \end{pmatrix}$ is a direction vector for d $\qquad n = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ is a normal vector for π

$d \parallel \pi \iff$ the angle between d and π is zero

from the lecture we know that:

$$m(\widehat{d,\pi}) = \begin{cases} \frac{\pi}{2} - m(\widehat{\overline{v},\overline{n}}), & \text{if } \overline{v}\cdot\overline{n} \geq 0 \\ m(\widehat{\overline{v},\overline{n}}) - \frac{\pi}{2}, & \text{if } \overline{v}\cdot\overline{n} < 0 \end{cases} \tag{7}$$



$$m(\widehat{d,\pi}) = \begin{cases} \frac{\pi}{2} - \arccos\frac{\overline{v}\cdot\overline{n}}{|\overline{v}||\overline{n}|}, & \text{if } \overline{v}\cdot\overline{n} \geq 0 \\ \arccos\frac{\overline{v}\cdot\overline{n}}{|\overline{v}||\overline{n}|} - \frac{\pi}{2}, & \text{if } \overline{v}\cdot\overline{n} < 0 \end{cases} \tag{8}$$

for us $v \cdot n = 8 - 15 - 5 = -12$

so $m(\widehat{d,\pi}) = m(\widehat{\overline{v},\overline{n}}) - \frac{\pi}{2}$

which is zero if and only if

$m(\widehat{\overline{v},\overline{n}}) = \frac{\pi}{2} = \arccos\frac{\overline{v}\cdot\overline{n}}{|\overline{v}||\overline{n}|}$

$\iff \cos\frac{\pi}{2} = \overline{v}\cdot\overline{n}$

$\iff 0 = \overline{v}\cdot\overline{n}$ a contradiction since $v \cdot n = -12$

Short story:

$-12 = v \cdot n \neq 0 \Rightarrow v \not\perp n \Rightarrow d \not\parallel \pi$

Method II check if there is an intersection point $d \cap \pi$

3. Prove that the lines $d_1 : \begin{cases} x = 1 + 4t \\ y = 5 - 4t \\ z = -1 + 5t \end{cases}, t \in \mathbb{R}$ and $d_2 : \begin{cases} x = 2 + 8t \\ y = 4 - 3t \\ z = 5 + t \end{cases}, t \in \mathbb{R}$ are skew.

$P_1 = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix}$ direction vector for d_1 $\quad P_2 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}$ direction vector for d_2

d_1 and d_2 are skew $\Leftrightarrow \begin{cases} ① \ d_1 \nparallel d_2 \text{ (not parallel)} \\ ② \ d_1 \cap d_2 = \emptyset \text{ (they don't intersect)} \end{cases}$

- we can check ① using the direction vectors v_1 and v_2

$d_1 \nparallel d_2 \Leftrightarrow v_1 \nparallel v_2 \Leftrightarrow$ the components of these vectors are not proportional (i.e. they are linearly independent)

since $\frac{4}{8} \neq \frac{-4}{-3} \neq \frac{5}{1}$ $\quad v_1 \nparallel v_2 \Rightarrow d_1 \nparallel d_2$

- we can check ② in several ways:

t = parameter for d_1 $\quad s$ = parameter for d_2

Method I the system $\begin{cases} 1 + 4t = 2 + 8s & (=x) \\ 5 - 4t = 4 - 3s & (=y) \\ -1 + 5t = 5 + s & (=z) \end{cases}$ does not have solutions (t,s)

equation 1 $\Rightarrow t = \frac{1+8s}{4} \overset{\text{eq. 3}}{\Rightarrow} \frac{5}{4}(1+8s) = 6 + s$

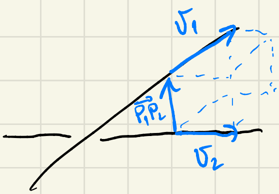
$5 + 40s = 24 + 4s$

$36s = 19 \Rightarrow s = \frac{19}{36}$ ↯

$\overset{\text{eq. 2}}{\Rightarrow} -1 - 8s = -1 - 3s \Rightarrow s = 0$

$\Rightarrow$ the system is incompatible $\Rightarrow$ no solution (t,s)

$\Rightarrow d_1 \cap d_2 = \emptyset$



Method II since $d_1 \nparallel d_2$ (by ①), d_1 intersects d_2 if and only if they are coplanar

d_1 and d_2 are coplanar $\iff$ $\overrightarrow{P_1P_2}$, v_1, v_2 are linearly dependent

$$\iff \begin{vmatrix} 1 & -1 & 6 \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 0$$

$$= -4 - 72 - 40 + 192 + 4 + 15 \neq 0$$

$\Downarrow$

$\overrightarrow{P_1P_2}$, v_1, v_2 are linearly independent

$\Downarrow$

d_1 and d_2 are <u>not</u> coplanar.

4. Find the parametric equations of the line passing through $\underbrace{(5, 0, -2)}_{P}$ and parallel to the planes $\underbrace{x - 4y + 2z = 0}_{\pi_1}$ and $\underbrace{2x + 3y - z + 1 = 0}_{\pi_2}$.

$n_1(1,-4,2)$ is a normal vector for π_1

$n_2(2,3,-1)$ is a normal vector for π_2

let ℓ be the line passing through P and parallel to π_1 and π_2

$$\Rightarrow \ell : \begin{cases} x = 5 + v_x \lambda \\ y = 0 + v_y \lambda \\ z = -2 + v_z \lambda \end{cases}$$

$\lambda \in \mathbb{R}$ where $v(v_x, v_y, v_z)$ is a direction vector for ℓ

$$\left.\begin{array}{l} \text{since } \ell \parallel \pi_1 \Rightarrow v \perp n_1 \\ \text{since } \ell \parallel \pi_2 \Rightarrow v \perp n_2 \end{array}\right\} \Rightarrow v \parallel n_1 \times n_2$$

in particular $n_1 \times n_2$ is a direction vector for ℓ

$$\Rightarrow \text{we can choose } v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & -4 & 2 \\ 2 & 3 & -1 \end{vmatrix} = i(4-6) - j(-1-4) + k(3+8)$$

$$\Rightarrow v(-2, 5, 11)$$

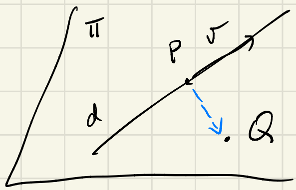
$$\Rightarrow \ell : \begin{cases} x = 5 - 2\lambda \\ y = 0 + 5\lambda \\ z = -2 + 11\lambda \end{cases}$$

5. Find the equation of the plane containing the point $P(2,0,3)$ and the line $d: \begin{cases} x = -1 + t \\ y = t \\ z = -4 + 2t \end{cases}$.

$t=0$

a point on d is $Q = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is a direction vector for d

Let π be the plane that we are looking for



if Q is not on d then $\begin{cases} \pi \parallel v \\ \pi \parallel \overrightarrow{PQ} \\ \pi \ni Q \end{cases}$ so an equation for π is

$$\pi : \begin{vmatrix} x-(-1) & y-0 & z-(-4) \\ 1 & 1 & 2 \\ -3 & 0 & -7 \end{vmatrix} = 0$$

$$\Leftrightarrow \pi : (x+1)\cdot(-7) - y\cdot(-7+6) + (z+4)\cdot 3 = 0$$

$$-7x - 7 + y + 3z + 12 = 0$$

$$\Leftrightarrow \pi : -7x + y + 3z + 5 = 0$$

test $\pi \ni P$ $\quad -14 + 9 + 5 = 0$ ✓

$\pi \parallel v$ $\quad -7 + 1 + 6 = 0$ ✓

$\pi \parallel \overrightarrow{PQ}$ $\quad 21 - 21 = 0$ ✓

6. Let $M_1(2,1,-1)$ and $M_2(-3,0,2)$ be two points. Find:

a) the equation of the bundle of planes passing through M_1 and M_2;
b) the plane π from the bundle, which is orthogonal on xOy;
c) the plane ρ from the bundle, which is orthogonal on π.

a) the line M_1M_2 has equations $\frac{x-2}{-3-2} = \frac{y-1}{0-1} = \frac{z-(-1)}{2-(-1)}$

$$\Leftrightarrow \frac{x-2}{-5} = \frac{y-1}{-1} = \frac{z+1}{3}$$

so M_1M_2 is the set of points whose coordinats satisfy the system

$$\begin{cases} \frac{x-2}{-5} = 1-y \\ 1-y = \frac{z+1}{3} \end{cases} \Leftrightarrow \begin{cases} x-2+5-5y=0 \\ 3-3y-z-1=0 \end{cases} \Leftrightarrow \begin{cases} x-5y+3=0 & \leftarrow \text{plane } \pi_1 \\ 3y+z-2=0 & \leftarrow \text{plane } \pi_2 \end{cases}$$

a plane containing the points M_1 and M_2 contains the line M_1M_2

so the bundle of planes containing M_1 and M_2 is

"$\lambda_1\pi_1 + \lambda_2\pi_2 = 0$" λ_1, λ_2 real values not both zero

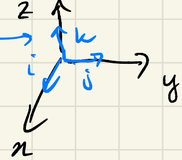
i.e the bundle is

$$\pi_{\lambda_1,\lambda_2} : \lambda_1(x-5y+3) + \lambda_2(3y+z-2) = 0 \quad \text{—||—}$$

b) • a plane in $\pi_{\lambda_1,\lambda_2}$ is orthogonal to the plane xOy if the normal vectors of $\pi_{\lambda_1,\lambda_2}$ are orthogonal to the normal vectors of xOy

• it is enough to check that one normal vector of $\pi_{\lambda_1,\lambda_2}$ is orthogonal to one normal vector of xOy

- a normal vector of xOy is $k(0,0,1)$



- a normal vector for $\Pi_{\lambda_1,\lambda_2}$ is

$$n_{\lambda_1,\lambda_2} = (\lambda_1, -5\lambda_1 + 3\lambda_2, \lambda_2)$$

- so $\Pi_{\lambda_1,\lambda_2} \perp xOy \iff n_{\lambda_1,\lambda_2} \perp k \iff n_{\lambda_1,\lambda_2} \cdot k = 0$

$$\iff \lambda_2 = 0 \quad \Big\} (*)$$

for $\lambda_2 = 0$ we obtain

$\Pi_{\lambda_1,0}$: $\lambda_1(x - 5y + 3) = 0 \iff x - 5y + 3 = 0$ which is the plane Π_1

$\Rightarrow \Pi = \Pi_1$

Rem in $(*)$ we obtained an equation which was easy to solve even though we worked with the full bundle of planes if the equations obtained in this way are more difficult to solve, one can use the reduced bundle

$$\Pi_1 + \lambda \Pi_2 = 0$$

and check separately the plane $\Pi_1 = 0$

c) • we are looking for a plane ρ in $\{\Pi_{\lambda_1,\lambda_2}\}$ which is orthogonal on the plane $\Pi = \Pi_1$

- a normal vector for Π_1 is $n(1,-5,0)$

- let us use the reduced bundle

$$\pi_\lambda : \quad (x-5y+3) + \lambda(3y+z-2) = 0$$

- a normal vector for π_λ is $n_\lambda(1, -5+3\lambda, \lambda)$
- if n denotes a normal vector for ρ then

$$\rho \perp \pi_\lambda \iff n \cdot n_\lambda = 0 \iff 1 + 25 - 15\lambda = 0 \iff \lambda = \frac{26}{15}$$

$$\Rightarrow \rho : \quad (x-5y+3) + \frac{26}{15}(3y+z-2) = 0$$

$\vdots$

$\rho : \quad \ldots$

7. Find the angle determined by d_1 and d_2, when: a) $d_1 : x = 4-t, y = 3+2t, z = -2t, t \in \mathbb{R}$ and $d_2 : x = 5+2s, y = 1+3s, z = 5-6s, s \in \mathbb{R}$.
b) $d_1 : \frac{x-1}{2} = \frac{y+5}{7} = \frac{z-1}{-1}$ and $d_2 : \frac{x+3}{-2} = \frac{y-9}{1} = \frac{z}{4}$.

a) $d_1 : \begin{cases} x = 4-t \\ y = 3+2t \\ z = -2t \end{cases}$ $\qquad d_2 : \begin{cases} x = 5+2s \\ y = 1+3s \\ z = 5-6s \end{cases}$

$v_1 \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ is a direction vector for d_1 $\qquad v_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ is a direction vector for d_2

$$\frac{v_1 \cdot v_2}{\|v_1\| \cdot \|v_2\|} = \frac{-2+6-12}{\underbrace{\sqrt{1+4+4}}_{9}\ \underbrace{\sqrt{4+9+36}}_{49}} = \frac{-8}{3 \cdot 7}$$

$v_1 \cdot v_2 \leq 0$ so the acute angle is $\pi - \arccos \frac{-8}{21}$

b) $d_1 : \frac{x-1}{2} = \frac{y+5}{7} = \frac{z-1}{-1}$ $\qquad d_2 : \frac{x+3}{-2} = \frac{y-9}{1} = \frac{z}{4}$

$\hookrightarrow v_1(2, 7, -1)$ is a direction vector for d_1 $\qquad \hookrightarrow v_2(-2, 1, 4)$ is a direction vector for d_2

$$\frac{v_1 \cdot v_2}{\|v_1\| \cdot \|v_2\|} = \frac{-4+7-4}{\sqrt{4+49+1}\ \underbrace{\sqrt{4+1+16}}_{21}} = \frac{-1}{9\sqrt{14}}$$

$$\begin{array}{r|l} 54 & 2 \\ 27 & 3^3 \end{array}$$

$v_1 \cdot v_2 \leq 0$ so the acute angle is $\pi - \arccos \frac{-1}{9\sqrt{14}}$

8. Find the angle determined by the planes $\pi_1 : x-\sqrt{2}y+z-1=0$ and $\pi_2 : x+\sqrt{2}y-z+3=0$.

$\hookrightarrow n_1(1, -\sqrt{2}, 1)$ is a normal vector for π_1

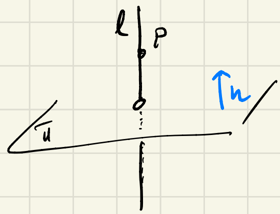
$\hookrightarrow (1, \sqrt{2}, -1)$ is a normal vector for π_2

$$\frac{n_1 \cdot n_2}{\|n_1\| \cdot \|n_2\|} = \frac{1-2-1}{\sqrt{1+2+1}\sqrt{1+2+1}} = \frac{-2}{4} = -\frac{1}{2}$$

$n_1 \cdot n_2 < 0$ so the acute angle is $\pi - \arccos\left(-\frac{1}{2}\right) = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

9. Find the coordinates of the orthogonal projection of the point P(2, 1, 1) on the plane $\pi : x + y + 3z + 5 = 0$.

$\hookrightarrow n(1,1,3)$ is a normal vector for π



• the orthogonal projection of P on π is the intersection of the line ℓ which contains P and is perpendicular on π

$\Downarrow$

n is a direction vector for ℓ

$$\Rightarrow \ell : \begin{cases} x = 2 + t \\ y = 1 + t \\ z = 1 + 3t \end{cases}$$

$$\ell \cap \pi : \quad \overset{x}{(2+t)} + \overset{y}{(1+t)} + 3\overset{z}{(1+3t)} + 5 = 0$$

$$11 + 11t = 0 \Rightarrow t = -1$$

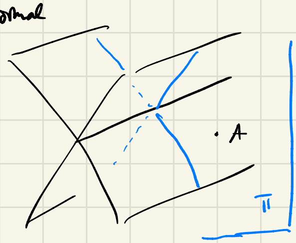
$\Rightarrow \ell \cap \pi$ is the point $Q(2-1, 1-1, 1+3(-1))$

$Q(1, 0, -2)$

10. Determine the orthogonal projection of the point $A(1,3,5)$ on the line which is given as the intersection of the planes $2x+y+z-1=0$ and $3x+y+2z-3=0$.

Π_1 with normal vector $n_1(2,1,1)$

Π_2 with normal vector $n_2(3,1,2)$



Let $\ell = \Pi_1 \cap \Pi_2$ (the intersection line of the two planes)

the orthogonal projection of the point A on the line ℓ is the intersection point of ℓ with the plane Π which contains A and is orthogonal on the line ℓ

$\Downarrow$

a direction vector of ℓ is a normal vector for Π (1)

$\ell = \Pi_1 \cap \Pi_2 \Rightarrow$ a direction vector for ℓ is $n_1 \times n_2$ (2)

From (1) & (2) it follows that $n_1 \times n_2$ is a normal vector for Π

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = i(2-1) - j(4-3) + k(2-3)$$

so $n_1 \times n_2 (1,-1,-1)$

$$\Rightarrow \Pi: \quad 1(x-1) - 1(y-3) - 1(z-5) = 0$$

$$\Leftrightarrow \Pi: \quad x - y - z + 7 = 0$$

- so the proj of A on ℓ is $\Pi \cap \ell$
- in order to calculate the intersection we need ℓ in parametric form

$$\ell: \begin{cases} 2x+y+z-1=0 & \Rightarrow y = -2x - z + 1 \\ 3x+y+2z-3=0 & \Rightarrow y = -3x - 2z + 3 \end{cases} \Rightarrow -2x - z + 1 = -3x - 2z + 3$$

$\Downarrow$

$$\begin{pmatrix} 2 & 1 & 1 & -1 \\ 3 & 1 & 2 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

$x + z - 2 = 0$

$\Rightarrow x = 2 - z$

$\Rightarrow y = -4 + 2z - z + 1$

$\Rightarrow y = -3 + z$

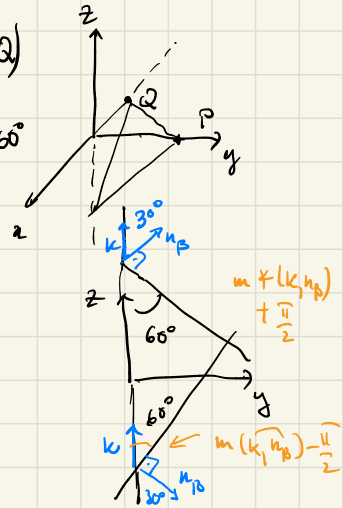
so $\ell : \begin{cases} x = 2 - z \\ y = -3 + z \\ z = z \end{cases}$

so $\pi \cap \ell$: $(2-z) - (-3+z) - z + 7$

$12 - 3z = 0 \Rightarrow z = 4 \Rightarrow \begin{array}{l} y = 1 \\ x = -2 \end{array}$

the point $\pi \cap \ell$ is $(-2, 1, 4)$ = projection of A on ℓ

11. Determine the equations of the planes which pass through the points $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the Oz axis.



- from all the planes containing P and Q (the line PQ) we want to select those which form an angle of 60° with Oz.

- PQ : $\dfrac{x-0}{-1-0} = \dfrac{y-2}{0-2} = \dfrac{z-0}{0}$

$$-2x = -y+2 \qquad z=0$$

$$\underbrace{2x-y+2=0}_{\pi_1} \qquad \underbrace{z=0}_{\pi_2}$$

$$PQ = \pi_1 \cap \pi_2$$

- the bundle of planes passing through the line PQ is

$$\pi_{\alpha\beta} : \alpha(x-y+2)+\beta z=0$$

a reduced bundle is $\pi_\beta : x-y+2+\beta z=0$

↳ has normal vector $n_\beta(1,-1,\beta)$

- a direction vector for Oz is $k(0,0,1)$

$$60° = m(\angle(Oz,\pi_\beta)) = m(\angle(k,n_\beta)) \pm 90°$$

so $\quad m(\angle(k,n_\beta)) = 150°$ or $30°$

$$\underbrace{\frac{\pi}{2}+\frac{\pi}{3}}_{\frac{5\pi}{6}} \qquad \frac{\pi}{6}$$

so $\cos \measuredangle(k, n_\beta) = -\frac{\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{2}$

$$\| \quad \frac{k \cdot n_\beta}{\|k\| \cdot \|n_\beta\|} = \frac{\beta}{1 \cdot \sqrt{1+1+\beta^2}} = \pm \frac{\sqrt{3}}{2}$$

So $\frac{\beta^2}{2+\beta^2} = \frac{3}{4}$ $\iff 4\beta^2 = 6 + 3\beta^2$

$\iff \beta^2 = 6$

$\iff \beta = \sqrt{6}$ or $-\sqrt{6}$

we found two planes

$\Pi_{-\sqrt{6}}$: $x - y - \sqrt{6}\, z + 2 = 0$

and

$\Pi_{\sqrt{6}}$: $x - y + \sqrt{6}\, z + 2 = 0$

12. Find the equations of the projection of the line $d : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$ on the plane $\pi : x + 2y - z = 0$.

$\hookrightarrow$ has normal vector $n(1,2,-1)$

- the projection of d on π is the intersection of a plane π_2 which contains d with π

- we look for π_2 in the bundle of planes containing d.

$$\pi_\lambda : \quad 2x - y + z - 1 + \lambda(x + y - z + 1) = 0$$

$\hookrightarrow$ has normal vector $n_\lambda(2+\lambda, -1+\lambda, 1-\lambda)$

$$\pi_\lambda \perp \pi \iff n \cdot n_\lambda = 0 \iff 2 + \lambda + 2(-1+\lambda) - (1-\lambda) = 0$$

$$4\lambda - 1 = 0 \Rightarrow \lambda = \frac{1}{4}$$

so $\pi_2 = \pi_{1/4} : \; 2x - y + z - 1 + \frac{1}{4}(x + y - z + 1) = 0$

$\iff \; ---- = 0$

and the projection of d on π is $\pi \cap \pi_{1/4} : \begin{cases} x + 2y - z = 0 \\ --- \end{cases}$

13. Find the angle determined by the lines $d_1 : \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases}$ and $d_2 : \begin{cases} x - y - z - 1 = 0 \\ x - y + 2x + 1 = 0 \end{cases}$

one plane

another plane

- we calculate the angle with the direction vectors of the two lines (see lecture)

- a direction vector for d_1 can be calculated either

  - by translating the two equations into parametric equations and reading off a direction vector

  - by calculating the cross product of two normal vectors corresponding to the planes which intersect in d_1

14. Find the angle determined by the planes $\pi_1 : x+3y+2z+1 = 0$ and $\pi_2 : 3x+2y-z-6 = 0$.

$\hookrightarrow n_1(1, 3, 2)$ is a normal vector for π_1

$\hookrightarrow n_2(3, 2, -1)$ is a normal vector for π_2

from the lecture

$$m(\widehat{\pi_1, \pi_2}) = \begin{cases} \arccos \dfrac{\overline{n}_1 \cdot \overline{n}_2}{|\overline{n}_1||\overline{n}_2|}, & \text{if } \overline{n}_1 \cdot \overline{n}_2 \geq o \\ \pi - \arccos \dfrac{\overline{n}_1 \cdot \overline{n}_2}{|\overline{n}_1||\overline{n}_2|}, & \text{if } \overline{n}_1 \cdot \overline{n}_2 < o \end{cases}$$

$$\frac{n_1 \cdot n_2}{\|n_1\| \cdot \|n_2\|} = \frac{3+6-2}{\sqrt{1+9+4}\sqrt{9+4+1}} = \frac{7}{14} = \frac{1}{2}$$

$n_1 \cdot n_2 \geq 0$ so $m(\widehat{\pi_1, \pi_2}) = \arccos \frac{1}{2} = 60^\circ$

15. Find the angle determined by the plane xOy and the line M_1M_2, where $M_1(1, 2, 3)$ and $M_2(-2, 1, 4)$.

from the lecture:

$$m(\widehat{d,\pi}) = \begin{cases} \frac{\pi}{2} - \arccos \frac{\overline{v} \cdot \overline{n}}{|\overline{v}||\overline{n}|}, & \text{if } \overline{v} \cdot \overline{n} \geq 0 \\ \arccos \frac{\overline{v} \cdot \overline{n}}{|\overline{v}||\overline{n}|} - \frac{\pi}{2}, & \text{if } \overline{v} \cdot \overline{n} < 0 \end{cases}$$

in our case $n = k\,(0,0,1)$ and $v = \overrightarrow{M_1M_2}\,(-3,-1,1)$

$$\frac{v \cdot n}{\|v\| \cdot \|n\|} = \frac{1}{1 \cdot \sqrt{9+1+1}} = \frac{1}{\sqrt{11}}$$

so $m(\widehat{M_1M_2, xOy}) = \frac{\pi}{2} - \arccos \frac{1}{\sqrt{11}}$