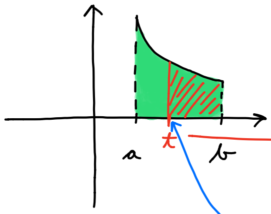
Improper integrals



$x = a$ is a vertical asymptote to the graph of f

$t \in (a,b) \Rightarrow \exists \int_t^b f(x)\,dx$

$? \lim\limits_{\substack{t \to a \\ t > a}} \int_t^b f(x)\,dx$

We classify the improper integrals into 3 types:

Type 1 → with the domain $[a,b)$ or $[a,\infty)$ where $a < b$

Type 2 → with the domain $(a,b]$ or $(-\infty,b]$ where $a < b$

Type 3 → with the domain (a,b) or $(-\infty,a)$ or (b,∞) where $a < b$

Definition:

1) Let $-\infty < a < b \leq \infty$ (type 1)

If f is LRi (locally Riemann integrable $\Longleftrightarrow$ Ri on each $[c,d] \subseteq [a,b)$)

and if $\exists \lim\limits_{\substack{t \to b \\ t < b}} \int_a^t f(x)\,dx \in \overline{\mathbb{R}}$

it is called the improper integral of the function f on $[a,b)$ and is denoted by $\int_a^b f(x)\,dx$.

2) Let $-\infty \leq a < b < \infty$

If $f:(a,b]$ is LRi

then $\int_{a_+}^b f(x)\,dx = \lim\limits_{\substack{t \to a \\ t > a}} \int_t^b f(x)\,dx$

3) Let $-\infty \leq a < b \leq \infty$

If $f:(a,b)$ is LRi on (a,b)

then if $\exists\, c \in (a,b)$ s.t. $\begin{cases} \exists \int_{a_+}^c f(x)\,dx \in \overline{\mathbb{R}} \\ \exists \int_c^{b_-} f(x)\,dx \in \overline{\mathbb{R}} \\ \text{the sum } \int_{a_+}^c f(x)\,dx + \int_c^{b_-} f(x)\,dx \ \exists \in \overline{\mathbb{R}} \\ \qquad = \int_{a_+}^{b_-} f(x)\,dx \end{cases}$

it is called the improper integral of the function f on (a,b) and is denoted by $\int_{a_+}^{b_-} f(x)\,dx$.

The choice for $c \in (a,b)$ is up to the reader.

Convergence criteria for improper integrals

Consider $f:[a,b)\to[0,\infty)$ LRI.
Then

1) If $\exists c\in(a,b)$ s.t. $\forall x\in[c,b)$ $f(x)\le g(x)$ then $\begin{cases}\int_a^{b-} g \text{ C.} \Rightarrow \int_a^{b-} f \text{ C.}\\ \int_a^{b-} f \text{ D.} \Rightarrow \int_a^{b-} g \text{ D.}\end{cases}$ C1C

2) If $\exists c\in(a,b)$ s.t. $\forall x\in[c,b)$ $\alpha\le\underbrace{\frac{f(x)}{g(x)}}_{g(x)\neq 0}\le\beta$ then $\int_a^{b-} f\sim\int_a^{b-} g$ C2C
$\exists\, 0<\alpha<\beta<\infty$

(with limits)
3) If $\exists c\in(a,b)$ then $\int_a^{b-} f\sim\int_a^{b-} g$

$g(x)>0$ $\forall x\in[a,b)$

$\exists L=\lim\limits_{\substack{t\to b\\ t<b}}\frac{f(t)}{g(t)}\in(0,\infty)$

In order to apply comparison for the ii of f we find:

① $[a,b)$, $b\in\mathbb{R}$ $\quad g(x)=\underbrace{\frac{1}{(b-x)^p}}_{0_+}$ $\quad$ C for $p<1$, D for $p\ge1$

② $(a,b]$, $a\in\mathbb{R}$ $\quad g(x)=\underbrace{\frac{1}{(x-a)^p}}_{0+}$ $\quad$ C for $p<1$, D for $p\ge 1$

(① and ② $\sim$)

③ $[a,\infty)$, $a\in\mathbb{R}$ $\quad g(x)=\frac{1}{x^p}$ $\quad$ C for $p>1$, D for $p\le1$

$$L=\lim_{\substack{t\to\text{problem}\\ t\neq\text{problem}}}\frac{f(x)}{g(x)}=\begin{cases}(b-x)^p\cdot f(x) & : [a,b)\\ (x-a)^p\cdot f(x) & : (a,b]\\ x^p\cdot f(x) & : [a,\infty)\end{cases}$$

Summarizing table for the comparison criteria
$L\in(0,\infty)$

Domain	L	p	nature
$[a,b)$ $(a,b]$	$<\infty$	<1	C.
	>0	≥1	D.
$[a,\infty)$	$<\infty$	>1	C.
	>0	<1	D.

Steps to study the improper integrability
and, in case of convergence, the value of the improper integral

Idea → we use the definition and the Leibniz-Newton formula

Step 1 → Compute the nondeterminate integral of f

Step 2 → Choose an antiderivative of f (usually the function having the constant $c=0$)

Step 3 → Compute the limit towards a from the antiderivative.
If the limit exists and is finite $\Rightarrow$ convergence case.

Idea → we use the comparison criteria
(in case of convergence, you have to calculate separatly the value of the ii)

Step 1 → Determine the problematic boundary points of the domain

Step 2 → for $f:[a,b) \to [0,\infty)$
compute $L = \lim\limits_{\substack{x \to b \\ x < b \\ (x \uparrow b)}} (b-x)^p \cdot f(x)$ and set p such that the value of the limit $\in (0,\infty)$

→ for $f:(a,b] \to [0,\infty)$
compute $L = \lim\limits_{\substack{x \to a \\ x > a \\ (x \downarrow a)}} (x-a)^p \cdot f(x)$ and set p such that the value of the limit $\in (0,\infty)$

→ for $f:[a,\infty) \to [0,\infty)$
compute $L = \lim\limits_{x \to \infty} x^p \cdot f(x)$ and set p such that the value of the limit $\in (0,\infty)$

Step 3 → For the first two cases
- if $p<1 \Rightarrow$ convergent improper integral.
- if $p>1 \Rightarrow$ divergent —"—

For the third case
- if $p>1 \Rightarrow$ divergent improper integral.
- if $p<1 \Rightarrow$ convergent —"—

e.g.

1) Study the existence and the nature of the ii for the function

a) $f:[a,b)\to\mathbb{R}$, $f(x)=\frac{1}{(b-x)^p}$ $\forall x\in[a,b)$

$a<b$

$p\in\mathbb{R}$ constant

I. $\int\frac{1}{(b-x)^p}\,dx$ (antiderivative of f)

$\rightarrow p=1$ $\int\frac{1}{(b-x)}\,dx=\int\frac{(b-x)'}{(b-x)}\cdot(-1)\,dx=-\int\frac{(b-x)'}{b-x}\,dx=$

$=-\ln|\underbrace{b-x}_{>0}|+\mathcal{C}=-\ln(b-x)+\mathcal{C}$

$\rightarrow p\neq 1$ $\int\frac{1}{(b-x)^p}\,dx=\int(b-x)^{-p}\,dx=\int-(b-x)'\cdot(b-x)^{-p}\,dx=$

$=-\frac{(b-x)^{-p+1}}{-p+1}+\mathcal{C}=$

$=\frac{(b-x)^{1-p}}{p-1}+\mathcal{C}$

II

($\lim F(t)$, $t\to$ problem, $t\neq$ value)

in our case the problem point is b

$p=1$ $\rightarrow$ We choose $F(x)=-\ln(b-x)$, $\forall x\in[a,b)$ to be an antiderivative of f

Compute $\lim\limits_{\substack{t\to b\\ t<b}}F(t)=\lim\limits_{\substack{t\to b\\ t<b}}-\ln(b-t)=-\ln(0_+)=-(-\infty)=\infty$

$\begin{matrix}t\to b\\ t<b\end{matrix}\Rightarrow b-t>0$

$\rightarrow\exists\Rightarrow$ the ii of f on $[a,b)$ $\exists$ and it is $=\infty\Rightarrow D$

$p\neq1$ $\rightarrow$ We choose $F(x)=\frac{(b-x)^{1-p}}{p-1}$, $\forall x\in[a,b)$, an antiderivative of f

$\rightarrow$ Analyse $\lim\limits_{\substack{t\to b\\ t<b}}F(t)=\lim\limits_{\substack{t\to b\\ t<b}}\frac{(b-t)^{1-p}}{p-1}=\frac{1}{p-1}\cdot\lim\limits_{\substack{t\to b\\ t<b}}(\underbrace{b-p}_{0_+})^{1-p}=$

$=\frac{1}{p-1}\cdot\begin{cases}0 & :1-p>0\\ \infty & :1-p<0\end{cases}=\begin{cases}0 & :p<1\\ \frac{\infty}{p-1} & :1<p\end{cases}=\begin{cases}0 & :p<1\\ \infty & :1<p\end{cases}$

$\rightarrow\exists\Rightarrow\exists$ ii of f on (a,b) $\overset{not}{=}\int_a^{b-}f(x)\,dx=\lim\limits_{\substack{t\to b\\ t<b}}F(t)-F(a)=\begin{cases}0-F(a) & :p<1\\ \infty-F(a) & ;p>1\end{cases}$

$=\begin{cases}-F(a) & :p<1 & C\\ \infty & :p>1 & D\end{cases}$

Same for $f(x)=\frac{1}{(x-a)^p}$

b) $f:[0,1) \to \mathbb{R}$

$$f(x) = \frac{1}{\sqrt[4]{1-x^4}}$$

$f(x) > 0$ we can compare it to $g(x) = \dfrac{1}{(1-x)^p}$

$[a,b)$ (b)

problem 1

$$L = \lim_{\substack{x\to 1\\ x<1}} \underbrace{(1-x)^p}_{\substack{1-x>0\\ 0_+}} \cdot f(x) = \lim_{\substack{x\to 1\\ x<1}} (1-x)^p \cdot \frac{1}{\sqrt[4]{1-x^4}} \overset{\frac{0}{0}}{=} \lim_{\substack{x\to 1\\ x<1}} (1-x)^p \cdot \frac{1}{(1-x)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}\cdot(1+x^2)^{\frac{1}{4}}} =$$

$$= \frac{1}{2^{\frac{1}{4}+\frac{1}{4}}} \cdot \lim_{\substack{x\to 1\\ x<1}} \frac{(1-x)^p}{(1-x)^{\frac{1}{4}}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow L = \dfrac{1}{\sqrt{2}} \in (0,\infty) \to$ apply c.t.

$L \in (0,\infty)$

$p = \frac{1}{4} < 1$

$\Rightarrow \int_0^{1-} f(x)\,dx$ is C

Substitutions in Integrals

1 Euler's substitions

Sometime when in the formulation of the function to be integrated we encounter

$$\sqrt{ax^2+bx+c},$$

where $a, b, c \in \mathbb{R}$, we consider a new variable t, in one of the following cases:

$$\sqrt{ax^2+bx+c} = \begin{cases} \pm\sqrt{a}x \pm t & \text{if } a > 0 \\ \pm x \cdot t \pm \sqrt{c} & \text{if } c > 0 \\ t(x-x_0) & \text{if } x_0 \text{ is a solution of the equation } ax^2+bx+c=0. \end{cases}$$

2 Weirstras' (trigonometric) substitutions

For functions in whose formulations are involved trigonometric functions, there is a usual substitution, namely:

$$tg\frac{x}{2} = t.$$

If we denote by $R(\sin x, \cos x)$ the expression to be integrated, sometimes we may consider other substitutions, which might lead us faster to the expected solution. Hence:P

- If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, then choose $\cos x = t$.
- If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, then choose $\sin x = t$.
- If $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$, then choose $tg\ x = t$.

Recall the following trigonometric identities:

$$\cos^2 x = \frac{1}{1+tg^2 x} \qquad \sin^2 x = \frac{tg^2 x}{1+th^2 x}.$$

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} \qquad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

3 Other tigonometric substitution

Sometimes, when the integrating function contains square roots of second degree polynomials (alternatively to using Euler's substitutions) we may pass to trigonometric functions, in the following situations:

- When$\int R(x, \sqrt{r^2 - x^2}dx$ choose $x = r\sin$ or $x = r\cos t$.
- When $\int R(x, \sqrt{r^2 + x^2}dx$ choose $x = rtgt$ or $x = rctgt$.
- When $\int R(x, \sqrt{x^2 - r^2}dx$ choose $x = \frac{r}{\cos x}$ or $x = \frac{r}{\sin x}$.

Exercise 1:

a) $\int \frac{1}{1+\frac{1}{sinx}}dx, \quad x \in (\pi, \pi);$

b) $\int \frac{1}{3\sin x + 4\cos x}dx \quad x \in (\pi, \pi);$

c) $\int \frac{\sqrt{9-x^2}}{x^2}dx, \quad x \in (-3, 3);$

d) $\int \frac{1}{\sqrt{(x^2+1)^3}}dx, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);$

e) $\int \frac{1}{\sqrt{(x^2-8)^3}}dx \quad x \in (-\sqrt{8}, \sqrt{8});$

f) $\int \sqrt{2x - x^2}dx \quad x \in (0, 2);$

g) $\int \sqrt{4 - x^2}dx \quad x \in (-2, 2);$

h) $\int x\sqrt{1 + x^2}dx.$

Exercise 2:

Determine

$$a) \quad \int \frac{2x-1}{x^2-3x+2}\mathrm{d}x, \;\; x \in]2, +\infty[;$$

Calculus on $\mathbb{R}$ Lect. dr. ANCA GRAD
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Improper Integrals

Exercise 1: Study (with the help of the definiton and by using the Leibniz-Newton formula) the imporper integrability of the following functions, and, in case of convergence, determine the value of the improper integral.

For the examples within this exercise, the following steps should be followed:

Step 1: Compute the nondeterminate integral of f.

Pasul 2: Choose an antiderivative of f (usually the function having the constant $c = 0$ is chosen).

Pasul 3: Compute the limit towards a from the antiderivative. If the limit exists and is finite, than we are in a convergence case.

a)

$$f : (-1, 1) \to \mathbb{R} \quad f(x) = \frac{1}{\sqrt{1-x^2}}$$

b)

$$f : [1, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{x(x+1)}.$$

c)

$$f : (0, 1] \to \mathbb{R} \quad f(x) = \ln x.$$

d)

$$f : [0, 1) \to \mathbb{R} \quad f(x) = \frac{\arcsin x}{\sqrt{1-x^2}}.$$

e)

$$f : (0, 1] \to \mathbb{R} \quad f(x) = \frac{\ln x}{\sqrt{x}}.$$

f)

$$f : [e, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{x \cdot (\ln x)^3}.$$

g)

$$f : \left(\frac{1+\sqrt{3}}{2}, 2 \right] \to \mathbb{R} \quad f(x) = \frac{1}{x\sqrt{2x^2 - 2x - 1}}.$$

h)

$$f : [0, \infty) \to \mathbb{R}, f(x) = \frac{\pi}{2} - \text{arctg} x.$$

i)

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{1+x^2},$$

j)

$$f\colon \left(\frac{1}{3}, 3\right] \to \mathbb{R} \quad f(x) = \frac{1}{\sqrt[3]{3x-1}}$$

k)

$$f\colon [1, \infty) \to \mathbb{R} \quad f(x) = \frac{x}{(1+x^2)^2}.$$

Exercise 2: Study the improper integrability (with the help of the comparison criteria) for the following functions. (Notice that in case of convergence, this criteria does not provide us the directly with the value of the improper integral, since the criteria mainly assures the nature).

For the examples within this exercise, the following steps should be followed:

Step 1: Determine the problematic boundary points of the domain,

Step 2: Compute

$$\lim_{x \uparrow b} (b-x)^p f(x)$$

and set p such that the value of the limit to belong to $\in (0, \infty)$, for $f : [a, b) \to [0, \infty)$.

If the domain is open at în a, then we compute

$$L = \lim_{x \downarrow a} (x-a)^p f(x)$$

and if the domain is upper unbounde and we compute (deci $[a, \infty)$)

$$L = \lim_{x \to \infty} x^p f(x).$$

Step 3: For the first two cases, if $p < 1$ we have convergent improper integrability, while in the third one, improper integrability is convergent if $p > 1$.

a)

$$f : [1, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{s\sqrt{1+x^2}}$$

b)

$$f : [0, \frac{\pi}{2}) \to \mathbb{R} \quad f(x) = \frac{1}{\cos x}$$

c)

$$f : (0, \infty) \to \mathbb{R} \quad f(x) = \left(\frac{arctgx}{x}\right)^2$$

d)

$$f : (1, \infty) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$$

e)

$$f : (0, 1) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{1 - x^2}} \right)^2$$

a)

$$f:(-1,1)\to\mathbb{R}\quad f(x)=\frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Let $F:(-1;1)\to\mathbb{R}$, $F(x)=\arcsin x$ be an antiderivative of f

$$\lim_{\substack{x\to -1\\ x>-1}} F(x) = \lim_{\substack{x\to -1\\ x>-1}} \arcsin x = \arcsin(-1) = -\frac{\pi}{2}\in\mathbb{R} \to \text{we are in a convergence case}$$

$$\lim_{\substack{x\to -1\\ x<-1}} F(x) = \arcsin 1 = \frac{\pi}{2}\in\mathbb{R} \to \text{we are in a convergence case}$$

b)

$$f:[1,\infty)\to\mathbb{R}\quad f(x)=\frac{1}{x(x+1)}.$$

$$\int \frac{1}{x(x+1)}\,dx = \int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \int\frac{1}{x}\,dx - \int\frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C =$$

$$= \ln\frac{x}{x+1} + C$$

Let $F:[1,+\infty)\to\mathbb{R}$, $F(x)=\ln\frac{x}{x+1}$ be an antiderivative of f

$$\lim_{a\to\infty}\int_1^a f(x)\,dx = \lim_{a\to\infty}\left(F(a)-F(1)\right) = \lim_{a\to\infty}\left(\ln\frac{a}{a+1}-\ln\frac{1}{2}\right) = \ln 2 + \lim_{a\to\infty}\ln\frac{a}{a+1} =$$

$$= \ln 2 + \ln 1 = \ln 2 \in\mathbb{R} \to \text{convergence case}$$

c)

$$f:(0,1]\to\mathbb{R}\quad f(x)=\ln x.$$

$$\int \ln x\,dx = x\cdot\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx = x\ln x - x + C$$

Let $F:(0,1]\to\mathbb{R}$, $F(x)=x\ln x - x$ be an antiderivative of f

$$\lim_{\substack{a\to 0\\ a>0}}\int_a^1 f(x)\,dx = \lim_{\substack{a\to 0\\ a>0}}\left(F(1)-F(a)\right) = \lim_{\substack{a\to 0\\ a>0}}\left(-1-(a\ln a - a)\right) = -1-\lim_{\substack{a\to 0\\ a>0}}(a\ln a - a) = -1-\lim_{\substack{a\to 0\\ a>0}} a\ln a = -1-\lim_{\substack{a\to 0\\ a>0}}\frac{\ln a}{\frac{1}{a}} =$$

$$= -1-\left(\frac{\infty}{\infty}\right) l'H,\ -1-\lim_{\substack{a\to 0\\ a>0}}\frac{\frac{1}{a}}{-\frac{1}{a}} = -1+\lim_{\substack{a\to 0\\ a>0}} a = -1\in\mathbb{R}$$

d)

$$f:[0,1)\to\mathbb{R}\quad f(x)=\frac{\arcsin x}{\sqrt{1-x^2}}.$$

$$\int\frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{\arcsin^2 x}{2} + C$$

Let $F:[0,1)\to\mathbb{R}$, $F(x)=\frac{\arcsin^2 x}{2}$ be an antiderivative of f

$$\lim_{\substack{a\to 1\\ a<1}}\int_0^a f(x)\,dx = \lim_{\substack{a\to 1\\ a<1}}\left(F(a)-F(0)\right) = \lim_{\substack{a\to 1\\ a<1}}\frac{\arcsin^2 a}{2} = \frac{\arcsin^2 1}{2} = \frac{\pi^2}{8}\in\mathbb{R} \Rightarrow \text{convergence case}$$

e)

$$f:(0,1]\to\mathbb{R} \quad f(x)=\frac{\ln x}{\sqrt{x}}.$$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 2\int \frac{\sqrt{x}}{x}\,dx = 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

Let $F:(0,1)\to\mathbb{R}$ be an antiderivative of f, $F(x)=2\sqrt{x}(\ln x-2)$

$$\lim_{\substack{a\to 0\\ a>0}}\int_a^1 f(x)\,dx = \lim_{\substack{a\to 0\\ a>0}}(F(1)-F(a)) = \lim_{\substack{a\to 0\\ a>0}}\left(-4-2\sqrt{a}(\ln a-2)\right) = -4-2\lim_{\substack{a\to 0\\ a>0}}(\sqrt{a}\cdot\ln a-2) =$$

$$= -4-2\lim_{\substack{a\to 0\\ a>0}}\sqrt{a}\cdot\ln a = -4-2\lim_{\substack{a\to 0\\ a>0}}\frac{\ln a}{\frac{1}{\sqrt{a}}} \overset{\text{l'H}}{=} -4-2\lim_{\substack{a\to 0\\ a>0}}\frac{\cancel{\frac{1}{a}}}{-\frac{1}{\cancel{a}}\cdot\frac{1}{2\sqrt{a}}} = -4+2\lim_{\substack{a\to 0\\ a>0}}2\sqrt{a} = -4+0\in\mathbb{R}$$

$\Rightarrow$ convergence case

f)

$$f:[e,\infty)\to\mathbb{R} \quad f(x)=\frac{1}{x\cdot(\ln x)^3}.$$

$$\int u'(x)\cdot u(x)^n\,dx = \frac{(u(x))^{n+1}}{n+1}+C \quad \forall\, n\neq -1$$

$$\int u'(x)\cdot u(x)^{-1} = \ln|u(x)| + C$$

$$\int \frac{1}{(\ln x)^3}\cdot\frac{1}{x} = \frac{(\ln x)^{-2}}{-2}+c = \frac{-1}{2\ln^2 x}+c$$

Let $F:[e,+\infty)\to\mathbb{R}$, $F(x)=\frac{1}{2\ln^2 x}$ be an antiderivative of f

$$\lim_{a\to+\infty}\int_e^a f(x)\,dx = \lim_{a\to+\infty}(F(a)-F(e)) = \lim_{a\to+\infty}\left(-\frac{1}{2\ln^2 a}+\frac{1}{2}\right) = \frac{1}{2}-\lim_{a\to+\infty}\frac{1}{2\ln^2 a} = \frac{1}{2}-0 = \frac{1}{2}\in\mathbb{R} \Rightarrow$$ convergent case

g)

$$f:\left(\frac{1+\sqrt{3}}{2},2\right]\to\mathbb{R} \quad f(x)=\frac{1}{x\sqrt{2x^2-2x-1}}.$$

$$\mathcal{I} = \int f(x)\,dx = \int\frac{1}{x\sqrt{2x^2-2x-1}}\,dx = \int\frac{1}{x^2\sqrt{2-\frac{2}{x}-\frac{1}{x^2}}}\,dx$$

$$\left.\begin{aligned}\frac{1}{x}&=t\\ -\frac{1}{x^2}\,dx&=dt\end{aligned}\right\} \Rightarrow \mathcal{I}' = -\int\frac{dt}{\sqrt{2-2t-t^2}} = -\int\frac{dt}{\sqrt{2-(t^2+2t+1-1)}} = -\int\frac{dt}{\sqrt{3-(t+1)^2}} =$$

$$= \arcsin\frac{t+1}{\sqrt{3}} \Rightarrow \mathcal{I} = -\arcsin\frac{\frac{1}{x}+1}{\sqrt{3}}$$

$$\mathcal{I} = -\arcsin\frac{1+x}{x\sqrt{3}}+C$$

Let $F:\left(\frac{1+\sqrt{3}}{2},2\right]\to\mathbb{R}$, $F(x)=-\arcsin\frac{1+x}{x\sqrt{3}}$ be an antiderivative of f

$$\lim_{\substack{a\to\frac{1+\sqrt{3}}{2}\\ a>\frac{1+\sqrt{3}}{2}}}\int_a^2 f(x)\,dx = \lim_{\substack{a\to\frac{1+\sqrt{3}}{2}\\ a>\frac{1+\sqrt{3}}{2}}}(F(2)-F(a)) = \lim_{\substack{a\to\frac{1+\sqrt{3}}{2}\\ a>\frac{1+\sqrt{3}}{2}}}\left(-\arcsin\frac{\sqrt{3}}{2}+\arcsin\frac{1+a}{a\sqrt{3}}\right) = -\frac{\pi}{2}+\arcsin\frac{1+\frac{1+\sqrt{3}}{2}}{\sqrt{3}+3} =$$

$$= -\frac{\pi}{2}+\arcsin\frac{\cancel{3+\sqrt{3}}}{\cancel{\sqrt{3}+3}} = -\frac{\pi}{2}+\frac{\pi}{2} = 0$$

h)

$$f:[0,\infty)\to\mathbb{R}, f(x)=\frac{\pi}{2}-\operatorname{arctg}x.$$

$$\int\left(\frac{\pi}{2}-\operatorname{arctg}x\right)dx=\frac{\pi}{2}\cdot x-\int \operatorname{arctg}x\,dx=\frac{\pi}{2}\cdot x-x\operatorname{arctg}x+\int\frac{x}{1+x^2}dx=\frac{\pi}{2}\cdot x-x\operatorname{arctg}x+\frac{1}{2}\ln(x^2+1)+C$$

Let $F:[0,+\infty)\to\mathbb{R}$, $F(x)=\frac{\pi x}{2}-x\operatorname{arctg}x+\frac{1}{2}\ln(x^2+1)$ be an antiderivative of f

$$\lim_{a\to+\infty}\int_0^a f(x)=\lim_{a\to\infty}(F(a)-F(0))=\lim_{a\to\infty}\left(\frac{\pi a}{2}-\operatorname{arctg}a+\frac{1}{2}\ln(a^2+1)\right)=$$

$$=\lim_{a\to\infty}\left(a\cdot\frac{\frac{\pi}{2}-\operatorname{arctg}a}{tg\left(\frac{\pi}{2}-\operatorname{arctg}a\right)}\cdot tg\left(\frac{\pi}{2}-\operatorname{arctg}a\right)+\frac{1}{2}\ln(a^2+1)\right)=$$

$$=\lim_{a\to+\infty}\left(\frac{\frac{\pi}{2}-\operatorname{arctg}a}{tg\left(\frac{\pi}{2}-\operatorname{arctg}a\right)}\cdot\frac{\cancel{a}}{\cancel{a}}+\frac{1}{2}\ln(a^2+1)\right)=1+\infty=+\infty\Rightarrow \text{divergence case}$$

i)

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=\frac{1}{1+x^2},$$

$$\int\frac{1}{1+x^2}dx=\operatorname{arctg}x+C$$

Let $F:\mathbb{R}\to\mathbb{R}$, $F(x)=\operatorname{arctg}x$ be an antiderivative of f

$$\lim_{x\to-\infty}F(x)=\lim_{x\to-\infty}\operatorname{arctg}x=-\frac{\pi}{2}$$

$$\lim_{x\to+\infty}F(x)=\lim_{x\to+\infty}\operatorname{arctg}x=\frac{\pi}{2}$$

j)

$$f:\left(\frac{1}{3},3\right]\to\mathbb{R}\quad f(x)=\frac{1}{\sqrt[3]{3x-1}}$$

$$J=\int\frac{1}{\sqrt[3]{3x-1}}dx$$

$$\left.\begin{array}{l}3x-1=t^3\\ 3dx=3t^2dt\end{array}\right]\Rightarrow J'=\int\frac{t^2dt}{t}=\frac{t^2}{2}\Rightarrow J=\frac{\sqrt[3]{(3x-1)^2}}{2}+C$$

Let $F:\left(\frac{1}{3},3\right]\to\mathbb{R}$, $F(x)=\frac{\sqrt[3]{(3x-1)^2}}{2}$ be an antiderivative of f

$$\lim_{\substack{a\to\frac{1}{3}\\a>\frac{1}{3}}}\int_a^3 f(x)=\lim_{\substack{a\to\frac{1}{3}\\a>\frac{1}{3}}}(F(3)-F(a))=\lim_{\substack{a\to\frac{1}{3}\\a>\frac{1}{3}}}\left(2-\frac{\sqrt[3]{(3a-1)^2}}{2}\right)=2-\frac{\sqrt[3]{(3\cdot\frac{1}{3}-1)^2}}{2}=2-0=2\in\mathbb{R}\Rightarrow \text{convergence case}$$

k)

$$f:[1,\infty)\to\mathbb{R}\quad f(x)=\frac{x}{(1+x^2)^2}.$$

$$\int\frac{x}{(1+x^2)^2}dx=\frac{1}{2}\int\frac{2x}{(1+x^2)^2}dx=\frac{1}{2}\cdot\frac{-1}{1+x^2}+C$$

Let $F:[1,+\infty)\to\mathbb{R}$, $F(x)=\frac{-1}{2(1+x^2)}$ be an antiderivative of f

$$\lim_{a\to+\infty}\int_1^a f(x)dx=\lim_{a\to+\infty}(F(a)-F(1))=\lim_{a\to+\infty}\left(\frac{-1}{2(1+a^2)}+\frac{1}{4}\right)=\frac{1}{4}-0=\frac{1}{4}\in\mathbb{R}\Rightarrow \text{convergence case}$$

a)

$$f : [1, \infty) \to \mathbb{R} \quad f(x) = \frac{1}{\cancel{x}\sqrt{1+x^2}}$$

$$\lim_{x \to +\infty} x^p \cdot \frac{1}{x\sqrt{1+x^2}} = \lim_{x \to +\infty} \frac{x^p}{x^2\sqrt{1+\frac{1}{x^2}}} \overset{\text{for } p=2}{=\!=} 1 \in (0,+\infty) \Rightarrow$$ convergent improper integrability

b)

$$f : [0, \frac{\pi}{2}) \to \mathbb{R} \quad f(x) = \frac{1}{\cos x}$$

Problematic boundary points: $\frac{\pi}{2}$

$$\left.\begin{array}{l}\lim\limits_{\substack{x \to \frac{\pi}{2} \\ x < \frac{\pi}{2}}} \left(\frac{\pi}{2} - x\right)^p \cdot \frac{1}{\cos x} = \lim\limits_{\substack{x \to \frac{\pi}{2} \\ x < \frac{\pi}{2}}} \frac{\frac{\pi}{2} - x}{\sin(\frac{\pi}{2} - x)} = 1 \\ (\text{for } p = 1)\end{array}\right\} \Rightarrow$$ divergent improper integrability

c)

$$f : (0, \infty) \to \mathbb{R} \quad f(x) = \left(\frac{arctgx}{x}\right)^2$$

$$\lim_{\substack{x \to 0 \\ x > 0}} (x-0)^p \cdot \frac{arctg^2 x}{x^2} = \lim_{\substack{x \to 0 \\ x > 0}} x^p \cdot \frac{arctg^2 x}{x^2} \overset{\text{for } p=0}{=\!=} 1 \in (0,+\infty) \Rightarrow$$ convergent improper integrability

$$\lim_{x \to +\infty} x^p \cdot \frac{arctg^2 x}{x^2} \overset{\text{for } p=2}{=\!=} \frac{\pi}{4}^2 \in (0,+\infty) \to$$ convergent improper integrability

(not sure...)

d)

$$f : (1, \infty) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$$

$$x^p \cdot \frac{\ln^2 x}{x\sqrt{x^2-1}}$$

$$\lim_{\substack{x \to 1 \\ x > 1}} (x-1)^p \cdot \frac{\ln^2 x}{x^2(x^2-1)} = \lim_{\substack{x \to 1 \\ x > 1}} (x-1)^p \cdot \left[\frac{\ln(1+x-1)}{x-1}\right]^2 \cdot (x-1)^{\cancel{2}} \cdot \frac{1}{x^2\cancel{(x-1)}(x+1)} =$$

$$= \lim_{\substack{x \to 1 \\ x > 1}} (x-1)^{p-1} \cdot \left(\frac{\ln^2(1+x-1)}{\cancel{(x-1)^2}}\right)^{\cancel{2}} \cdot \frac{1}{x^2(x+1)} =$$

$$= \lim_{\substack{x \to 1 \\ x > 1}} (x-1)^{p-1} \cdot \left(\frac{\ln^2(1+x-1)}{\cancel{x-1}}\right)^{\cancel{2}} \cdot \frac{1}{x^2(x+1)} \overset{\text{for } p=1}{=\!=} \frac{1}{2} \in (0,+\infty) \Rightarrow$$ divergent improper integrability

$$\lim_{x \to +\infty} x^p \cdot \frac{\ln^2 x}{x^2(x^2-1)} = \lim_{x \to +\infty} x^{p-2} \cdot \frac{\ln^2 x}{x^2-1} \overset{\text{for } p=4}{=\!=} \frac{1}{2} \in (0,+\infty) \Rightarrow$$ convergent improper integrality

e)

$$f : (0, 1) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2$$

$$\lim_{x \downarrow a} (x-a)^p f(x)$$

$a = 0$

$$\lim_{\substack{x \to 1 \\ x < 1}} (1-x)^p \cdot \frac{\ln^2 x}{x^2(x-1)(x+1)} = \frac{1}{2} \lim_{\substack{x \to 1 \\ x < 1}} (1-x)^p \cdot \frac{\ln^2(1+x-1)}{(x-1)^2} \cdot \frac{(x-1)^{\cancel{2}}}{x^2\cancel{(x-1)}} = \frac{1}{2} \lim_{\substack{x \to 1 \\ x < 1}} (1-x)^p \cdot (x-1) = \frac{1}{2} \lim_{\substack{x \to 1 \\ x < 1}} (-1)^p \cdot (x-1)^{p+1} \to$$

Exercise 2: Study the improper integrability (with the help of the comparison criteria) for the following functions. (Notice that in case of convergence, this criteria does not provide us the directly with the value of the improper integral, since the criteria mainly assures the nature).

For the examples within this exercise, the following steps should be followed:

Step 1: Determine the problematic boundary points of the domain,

Step 2: Compute

$$\lim_{x\uparrow b}(b-x)^p f(x)$$

and set p such that the value of the limit to belong to $\in(0,\infty)$, for $f:[a,b)\to[0,\infty)$.

!! $f:(a,b]\to[0,\infty)$

If the domain is open at în a, then we compute

$$L=\lim_{x\downarrow a}(x-a)^p f(x)$$

and if the domain is upper unbounde and we compute (deci $[a,\infty)$)

$$L=\lim_{x\to\infty}x^p f(x).$$

Step 3: For the first two cases, if $p<1$ we have convergent improper integrability, while in the third one, improper integrability is convergent if $p>1$.

$f:(1,\infty)\to\mathbb{R}$ $\quad f(x)=\left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$

b)

$$f:[0,\frac{\pi}{2})\to\mathbb{R}\quad f(x)=\frac{1}{\cos x}$$

→ $f(x)$ is positive >0 → $f(x)>0$

$f(x)$ is positive >0 because $\frac{1}{\cos 0}=\frac{1}{1}=1$ and it decreases until $\frac{\pi}{2}$, but it never reaches 0, because it's open in $\frac{\pi}{2}$

Problematic boundary points: $\frac{\pi}{2}$

$$\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}\left(\frac{\pi}{2}-x\right)^p\cdot\frac{1}{\cos x}=\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}\frac{\frac{\pi}{2}-x}{\sin(\frac{\pi}{2}-x)}=1 \quad (\text{for } p=1)$$

>0

$\Rightarrow$ divergent improper integrability

For $p=1 \Rightarrow \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}\frac{\frac{\pi}{2}-x}{\cos x}=\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}\frac{-1}{-\sin x}=1\in(0,+\infty)$ We may apply the comparison

L'Hôpital

→ the improper integral is divergent

d)

$$f:(1,\infty)\to\mathbb{R}\quad f(x)=\left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$$

$f:(1,\infty)\to\mathbb{R}$ $\quad f(x)=\left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$

→ (a,b)

→ $(a,+\infty)$

$\to (a, \underline{b})$

$\to (a, +\infty) = (1, +\infty)$

$$L = \lim_{x\to\infty} x^p \cdot \frac{\ln^2 x}{x^2(x^2-1)} = \lim_{x\to\infty} x^p \cdot \frac{\ln^2 x}{x^2(x-1)(x+1)}$$

$p = \alpha$

$$L = \lim_{x\to\infty} \cancel{x^2} \cdot \frac{\ln^2 x}{\cancel{x^2}(x-1)(x+1)} = \lim_{x\to\infty} \frac{\ln^2 x}{(x-1)(x+1)} = \lim_{x\to\infty} \frac{\ln^2 x}{x^2-1} = \underbrace{0 < +\infty}_{p=2>1}$$

$$\left.\begin{array}{l} \Rightarrow C.\ (2, +\infty) \\ C\ (1, 2) \end{array}\right\} \Rightarrow C.\ (1, +\infty)$$

e)

$$f : (0,1) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2$$

$f : (0, 1) \to \mathbb{R} \quad f(x) = \left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2$

(a, b)

$\int_{0+}^{\frac{1}{2}}$ and $\int_{\frac{1}{2}}^{1-}$

$(0, \frac{1}{2}]$ $\quad$ $[\frac{1}{2}, 1)$

$\downarrow$ $\quad$ $\downarrow$

similarly to ex. d) and b)

$$L = \lim_{\substack{x\to 0\\ x>0}} x^p \cdot f(x) = \lim_{\substack{x\to 0\\ x>0}} x^p \cdot \left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2 = \lim_{\substack{x\to 0\\ x>0}} x^p \cdot \frac{\ln x}{x^2(1-x^2)}$$

$\downarrow$

$$\left.\begin{array}{l} C \\ D \end{array}\right\} \Rightarrow D \text{ (overall)}$$

for $p = 0$

$$L = \lim_{\substack{x\to 0\\ x>0}} x^0 \cdot \frac{\ln x}{x^2(1-x^2)} = \lim_{\substack{x\to 0\\ x>0}} \frac{\ln x}{x^2(1-x^2)} = \infty$$

for $p = 1$

$$\left.L = \lim_{\substack{x\to 0\\ x>0}} x \cdot \frac{\ln x}{x^2(1-x^2)} = \lim_{\substack{x\to 0\\ x>0}} \frac{\ln x}{x-x^3} = \infty \quad p = 1 \geq 1\right\} \Rightarrow D$$

IMPROPER INTEGRALS

EXERCISE 1.

STEP 1 → do the nondeterminate integral of f

STEP 2 → choose an antiderivative of f, ($C=0$)

STEP 3 → do the limit of the antiderivative

a) $f:(-1,1)\to\mathbb{R}$ $f(x)=\frac{1}{\sqrt{1-x^2}}$

ST 1: $\int f(x)dx=\int\frac{1}{\sqrt{1-x^2}}dx=\arcsin x+C$

ST 2: $\arcsin x+0$

ST 3: $\lim\limits_{x\to 1}\arcsin x=\frac{\pi}{2}$

$l=\lim\limits_{x\to -1}\arcsin -1=-\frac{\pi}{2}\Rightarrow$ The limit exists, and it is finite $\Rightarrow$

$\Rightarrow$ we have a case of convergence, we can calculate the improper integral

$$\int_{-1^+}^{1^-}\frac{1}{\sqrt{1-x^2}}dx=\int_{-1^+}^{0}\frac{1}{\sqrt{1-x^2}}dx+\int_{0}^{1^-}\frac{1}{\sqrt{1-x^2}}dx$$

$$\int_{-1^+}^{0}\frac{1}{\sqrt{1-x^2}}dx=\lim_{\substack{t\to-1\\t>-1}}\int_t^0\frac{1}{\sqrt{1-x^2}}dx=$$

$$=\lim_{\substack{t\to-1\\t>-1}}\arcsin x\Big|_t^0=\lim_{\substack{t\to-1\\t>-1}}\arcsin 0-\arcsin t=$$

$$=\lim_{\substack{t\to-1\\t>-1}}0-\arcsin t=-\arcsin(-1)=-\left(-\frac{\pi}{2}\right)=\frac{\pi}{2}$$

$$\int_0^{1^-}\frac{1}{\sqrt{1-x^2}}dx=\lim_{\substack{t\to1\\t<1}}\int_0^t\frac{1}{\sqrt{1-x^2}}dx=\lim_{\substack{t\to1\\t<1}}\arcsin t-\arcsin 0=$$

$= \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$$\int_{-1^+}^{1^-} \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

b) $f : [1, +\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x(x+1)}$

$$\int \frac{1}{x(x+1)} dx = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx =$$

$= \ln|x| - \ln|x+1| + c$

$F(x) = \ln|x| - \ln|x+1| + 0 = \ln|x| - \ln|x+1|$

$$\lim_{x \to \infty} \ln|x| - \ln|x+1| = \lim_{x \to \infty} \ln\left|\frac{x}{x+1}\right| = \lim_{x \to \infty} \ln\left(\frac{x}{x+1}\right) =$$

$= \ln \lim_{x \to \infty} \frac{x}{x+1} = \ln 1 = 0 \Rightarrow$ the limit exists and it is finite

$$\int_1^{\infty} \frac{1}{x(x+1)} dx = \lim_{t \to \infty} \int_1^t \frac{1}{x(x+1)} dx = \lim_{t \to \infty} \ln|x| - \ln|x+1| \Big|_1^t =$$

$$= \lim_{t \to \infty} \ln\left|\frac{x}{x+1}\right| \Big|_1^t = \lim_{t \to \infty} \ln\left(\frac{x}{x+1}\right)\Big|_1^t =$$

$$= \lim_{t \to \infty} \left(\ln\left(\frac{t}{t+1}\right) - \ln\left(\frac{1}{1+1}\right) \right) =$$

$$= \lim_{t \to \infty} \ln\left(\frac{t}{t+1}\right) - \ln\left(\frac{1}{2}\right) =$$

$$= \ln \lim_{t \to \infty} \left(\frac{t}{t+1}\right) - \ln\left(\frac{1}{2}\right) = \ln 1 - \ln\left(\frac{1}{2}\right) =$$

$= \ln 1 - \ln 1 + \ln 2 = \ln 2.$

c) $f : (0, 1] \to \mathbb{R}$, $f(x) = \ln x$

$$\int \ln x \, dx = \int x' \ln x \, dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x$$

$F(x) = x \ln x - x$

$$\int_{0^+}^{1} \ln x \, dx = \lim_{\substack{t \to 0 \\ t > 0}} \int_t^1 \ln x \, dx = \lim_{\substack{t \to 0 \\ t > 0}} (x \ln x - x) \Big|_t^1 =$$

$$= \lim_{\substack{t \to 0 \\ t > 0}} (1\ln 1 - 1) - (t\ln t - t) =$$

$$= \lim_{\substack{t \to 0 \\ t > 0}} (0 - 1) - (t\ln t - t) = -1 - \lim_{\substack{t \to 0 \\ t > 0}} t\ln t + \lim_{\substack{t \to 0 \\ t > 0}} t =$$

$$= -1 - \lim_{\substack{t \to 0 \\ t > 0}} t\ln t + 0 =$$

$$= -1 - \lim_{\substack{t \to 0 \\ t > 0}} \frac{\ln t}{\frac{1}{t}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} -1 - \lim_{\substack{t \to 0 \\ t > 0}} \frac{\frac{1}{\cancel{t}}}{-\frac{1}{t^{\cancel{2}}}} =$$

$$= -1 + \lim_{\substack{t \to 0 \\ t > 0}} t = -1 + 0 = -1$$

d) $f : [0,1) \to \mathbb{R}$ $\quad f(x) = \dfrac{\arcsin x}{\sqrt{1-x^2}}$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \int \arcsin x \cdot \frac{1}{\sqrt{1-x^2}} dx =$$

$$= \int \arcsin x (\arcsin x)' dx =$$

$$= \frac{(\arcsin x)^2}{2} + C.$$

$$\lim_{\substack{t \to 1 \\ t < 1}} \frac{(\arcsin t)^2}{2} = \frac{(\arcsin 1)^2}{2} = \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{\pi^2}{8} \Rightarrow$$

$\Rightarrow$ it exists and it is finite

$$\int_0^{1^-} \frac{\arcsin x \, dx}{\sqrt{1-x^2}} = \lim_{\substack{t \to 1 \\ t < 1}} \int_0^t \frac{\arcsin x}{\sqrt{1-x^2}} dx =$$

$$= \lim_{\substack{t \to 1 \\ t < 1}} \frac{(\arcsin x)^2}{2} \Big|_0^t = \lim_{\substack{t \to 1 \\ t < 1}} \frac{(\arcsin t)^2}{2} - \frac{(\arcsin 0)^2}{2} =$$

$$= \frac{(\arcsin 1)^2}{2} - 0 = \frac{\pi^2}{8}$$

e) $f:(0,1]\to\mathbb{R}$ $f(x)=\frac{\ln x}{\sqrt{x}}$

$$\int\frac{\ln x}{\sqrt{x}}dx=\int x^{-\frac{1}{2}}\cdot\ln x\,dx=\int(2\sqrt{x})'\cdot\ln x\,dx=$$

$$=2\sqrt{x}\ln x-\int 2\sqrt{x}\cdot\frac{1}{x}dx=2\sqrt{x}\ln x-2\int\frac{1}{\sqrt{x}}dx=$$

$$=2\sqrt{x}\ln x-2\int x^{-\frac{1}{2}}dx=2\sqrt{x}\ln x-2\cdot 2\sqrt{x}=$$

$$=2\sqrt{x}(\ln x-2)+C$$

$$\lim_{\substack{x\to0\\x>0}}2\sqrt{x}(\ln x-2)=2\lim_{\substack{x\to0\\x>0}}\frac{\ln x-2}{\frac{1}{\sqrt{x}}}\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}}$$

$$=2\lim_{\substack{x\to0\\x>0}}\frac{\frac{1}{x}-0}{-\frac{1}{2}x^{-\frac{3}{2}}}=2\cdot-2\lim_{\substack{x\to0\\x>0}}x^{-1}\cdot x^{\frac{3}{2}}=$$

$$=-4\lim_{\substack{x\to0\\x>0}}x^{\frac{3}{2}-1}=-4\lim_{\substack{x\to0\\x>0}}x^{\frac{1}{2}}=$$

$$=-4\lim_{\substack{x\to0\\x>0}}\sqrt{x}=-4\cdot\sqrt{0^+}=0$$ - it exists and it is finite

$$\int_{0^+}^{1}\frac{\ln x\,dx}{\sqrt{x}}=\lim_{\substack{t\to0\\t>0}}\int_t^1\frac{\ln x}{\sqrt{x}}dx=\lim_{\substack{t\to0\\t>0}}2\sqrt{x}(\ln x-2)\Big|_t^1=$$

$$=\lim_{\substack{t\to0\\t>0}}2\sqrt{1}(\ln 1-2)-2\sqrt{t}(\ln t-2)=$$

$$=2\cdot(-2)-2\lim_{\substack{t\to0\\t>0}}\sqrt{t}(\ln t-2)=-4-2\lim_{\substack{t\to0\\t>0}}\frac{\ln t-2}{\frac{1}{\sqrt{t}}}\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}}$$

$$=-4-2\lim_{\substack{t\to0\\t>0}}\frac{\frac{1}{t}-0}{-\frac{1}{2}\cdot\frac{1}{t\sqrt{t}}}=-4+4\lim_{\substack{t\to0\\t>0}}\sqrt{t}=-4+4\cdot0=-4$$

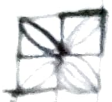
f) $f:[e,\infty)\to\mathbb{R}$ $f(x)=\frac{1}{x(\ln x)^3}$

$$\int\frac{1}{x(\ln x)^3}dx=\int\frac{(\ln x)'}{(\ln x)^3}dx=\quad\frac{\ln x)^4}{4}+C$$

$$\lim_{x\to\infty} \frac{(\ln x)^4}{4} = \frac{1}{4}\lim_{x\to\infty}(\ln x)^4 =$$

$$= \frac{1}{4}\cdot\infty^4 = \infty$$, so the limit exists but it is not finite

g) $f:\left[\frac{1+\sqrt{3}}{2},2\right]\to\mathbb{R},\ f(x)=\frac{1}{x\sqrt{2x^2-2x-1}}$



$$\int \frac{1}{x\sqrt{2x^2-2x-1}}\,dx = \int \frac{1}{x\sqrt{\left(\sqrt{2}x-\frac{1}{\sqrt{2}}\right)^2-\frac{3}{2}}}\,dx =$$

$$= \int \frac{1}{x\frac{1}{\sqrt{2}}\sqrt{(2x-1)^2-3}}\,dx = \sqrt{2}\int \frac{1}{x\sqrt{(2x-1)^2-3}}\,dx$$

$u=2x-1$

$x=\frac{u+1}{2} \Rightarrow dx=\frac{1}{2}\,du$

$$I=\int \frac{1}{\frac{u+1}{2}\sqrt{u^2-3}}\cdot\frac{1}{2}\,du = \int \frac{du}{(u+1)\sqrt{u^2-3}} =$$

h) $f:[0,\infty)\to\mathbb{R},\ f(x)=\frac{\pi}{2}-\operatorname{arctg}x$

$$\int\frac{\pi}{2}-\operatorname{arctg}x\,dx=\int\frac{\pi}{2}dx-\int\operatorname{arctg}x\,dx$$

$$=\frac{\pi}{2}\cdot x-\int x'\operatorname{arctg}x\,dx=\frac{\pi}{2}x-x\operatorname{arctg}x+\int x\cdot\frac{1}{x^2+1}dx$$

$$=\frac{\pi}{2}x-x\operatorname{arctg}x+\frac{1}{2}\int\frac{2x\,dx}{x^2+1}=\frac{\pi}{2}x-x\operatorname{arctg}x+\frac{1}{2}\int\frac{1}{u}du=$$

$u=x^2+1\Rightarrow du=2x\,dx$

$$=\frac{\pi}{2}x-x\operatorname{arctg}x+\frac{1}{2}\cdot\ln(x^2+1)+C$$

$$\lim_{x\to\infty}\frac{\pi}{2}x-x\operatorname{arctg}x+\frac{1}{2}\ln(x^2+1)=\frac{\pi}{2}\cdot\infty-\frac{\pi}{2}+\frac{1}{2}\cdot\infty=\infty$$

it exists but it isn't finite

i) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\frac{1}{1+x^2}$

$$\int\frac{1}{1+x^2}dx=\operatorname{arctg}x+C$$

$\lim_{x\to\infty}\operatorname{arctg}x=\frac{\pi}{2}$

$\lim_{x\to-\infty}\operatorname{arctg}x=-\frac{\pi}{2}$

$\Rightarrow$ both limits exists and they are finite.

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\int_{-\infty}^{0}\frac{1}{1+x^2}dx+\int_{0}^{\infty}\frac{1}{1+x^2}dx=$$

$$=\lim_{t\to\infty}\int_t^0\frac{1}{1+x^2}dx+\lim_{z\to\infty}\int_0^z\frac{1}{1+x^2}dx=$$

$$=\lim_{t\to-\infty}\operatorname{arctg}x\Big|_t^0+\lim_{z\to\infty}\operatorname{arctg}x\Big|_0^z=$$

$$=\lim_{t\to\infty}(\operatorname{arctg}0-\operatorname{arctg}t)+\lim_{z\to\infty}(\operatorname{arctg}z-\operatorname{arctg}0)=$$

$$=-\lim_{t\to-\infty}\operatorname{arctg}t+\lim_{z\to\infty}\operatorname{arctg}z=-\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

j) $f:\left(\frac{1}{3},3\right] \to \mathbb{R} \quad f(x)=\frac{1}{\sqrt[3]{3x-1}}$

$$\int \frac{1}{\sqrt[3]{3x-1}}\,dx = \frac{1}{3}\int \frac{(3x-1)'}{(3x-1)^{\frac{1}{3}}}\,dx =$$

$$= \frac{1}{3}\cdot\frac{(3x-1)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{1}{\cancel{3}}\,\frac{(3x-1)^{\frac{2}{3}}}{\frac{2}{\cancel{3}}} + C = \frac{(3x-1)^{\frac{2}{3}}}{2} + C$$

$$\lim_{\substack{x\to\frac{1}{3}\\ x>\frac{1}{3}}} \frac{(3x-1)^{\frac{2}{3}}}{2} = \frac{\left(\cancel{3}\cdot\frac{1^+}{\cancel{3}}-1\right)^{\frac{2}{3}}}{2} = \frac{\left(0^+\right)^{\frac{2}{3}}}{2} = 0 \Rightarrow$$

$\Rightarrow$ it exists and it is finite

$$\int_{\frac{1}{3}^+}^{3} \frac{1}{\sqrt[3]{3x-1}}\,dx = \lim_{\substack{t\to\frac{1}{3}\\ t>\frac{1}{3}}} \int_{t}^{3} \frac{1}{\sqrt[3]{3x-1}}\,dx =$$

$$= \lim_{\substack{t\to\frac{1}{3}\\ t>\frac{1}{3}}} \left.\frac{(3x-1)^{\frac{2}{3}}}{2}\right|_{t}^{3} = \lim_{\substack{t\to\frac{1}{3}\\ t>\frac{1}{3}}} \frac{(3\cdot 3-1)^{\frac{2}{3}}}{2} - \frac{(3t-1)^{\frac{2}{3}}}{2} =$$

$$= \frac{(9-1)^{\frac{2}{3}}}{2} - \lim_{\substack{t\to\frac{1}{3}\\ t>\frac{1}{3}}} \frac{(3t-1)^{\frac{2}{3}}}{2} = \frac{8^{\frac{2}{3}}}{2} - 0 =$$

$$= \frac{2^{\cancel{3}\cdot\frac{2}{\cancel{3}}}}{2} - 0 = \frac{2\cdot\cancel{2}}{\cancel{2}} - 0 = 2$$

k) $f:[1,\infty)\to\mathbb{R} \quad f(x)=\frac{x}{(1+x^2)^2}$

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}\int \frac{2x}{(1+x^2)^2}\,dx = \frac{1}{2}\int \frac{(1+x^2)'}{(1+x^2)^2}\,dx =$$

$$= \frac{1}{2}\cdot\frac{(1+x^2)^{-2+1}}{-2+1} = \frac{1}{2}\cdot\frac{(1+x^2)^{-1}}{-1} = -\frac{1}{2(1+x^2)} + C$$

$$\lim_{x\to\infty} -\frac{1}{2}\cdot\frac{1}{(1+x^2)} = -\frac{1}{2}\cdot\frac{1}{\infty} = 0$$

$$\int_1^\infty \frac{x}{(1+x^2)^2}dx = \lim_{t\to\infty} \int_1^t \frac{x}{(1+x^2)^2}dx = \lim_{t\to\infty} -\frac{1}{2(1+x^2)}\Big|_1^t =$$

$$= \lim_{t\to\infty} -\frac{1}{2(1+1^2)} + \frac{1}{2(1+t^2)} = -\frac{1}{2\cdot 2} + \lim_{t\to\infty} \frac{1}{2(1+t^2)} =$$

$$= -\frac{1}{4} + 0 = -\frac{1}{4}$$

EXERCISE 2

a) $f: [1,\infty) \to \mathbb{R}$ $f(x) = \frac{1}{5\sqrt{1+x^2}}$

$$L = \lim_{x\to\infty} x^P \cdot \frac{1}{5\sqrt{1+x^2}} = \frac{1}{5} \lim_{x\to\infty} \frac{x^P}{\sqrt{1+x^2}} =$$

$$= \frac{1}{5} \cdot \lim_{x\to\infty} \frac{x^P}{x\sqrt{\frac{1}{x^2}+1}} = \frac{1}{5} \cdot 1 \Rightarrow L \in (0,\infty) \Leftrightarrow$$

$$\Leftrightarrow p = 1$$

$\Rightarrow L = \frac{1}{5} \in (0,\infty) \Rightarrow$ apply the table therefore $\int_1^\infty f(x)\,dx$ is D.

b) $f: [0, \frac{\pi}{2}) \to \mathbb{R}$

$f(x) = \frac{1}{\cos x}$ $\int \frac{1}{\cos x} dx = \int \frac{1}{\sqrt{}}$

$$\lim_{\substack{x\to\frac{\pi}{2} \\ x<\frac{\pi}{2}}} (1-x)^P \cdot \frac{1}{\cos x}$$

c) $f:(0,\infty)\to\mathbb{R}\quad f(x)=\left(\frac{\operatorname{arctg}x}{x}\right)^2$

$$\int\left(\frac{\operatorname{arctg}x}{x}\right)^2dx=\int\frac{1}{x^2}\cdot(\operatorname{arctg}x)^2dx=$$

$$=-\int-\frac{1}{x^2}(\operatorname{arctg}x)^2dx=$$

$$=-\int\left(\frac{1}{x}\right)'(\operatorname{arctg}x)^2dx=-\left[\frac{1}{x}(\operatorname{arctg}x)^2-\int\frac{1}{x}\cdot 2\operatorname{arctg}x\cdot\frac{1}{x^2+1}\right.$$

$$=-\frac{1}{x}(\operatorname{arctg}x)^2+\int\frac{2}{x(x^2+1)}\operatorname{arctg}x\,dx\ ??$$

Going to ask

ex 1: g)

ex 2: c) d) e)

$$f(x) = \frac{1}{\sin x + \cos x - 1}$$

$$\int f(x) = \int \frac{1}{\sin x + \cos x - 1} = \int \frac{1}{\frac{2\tan\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)+1} + \frac{1-\tan^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)+1} - 1}\, dx = \int \frac{1}{\frac{2t}{t^2+1} - \frac{1-t^2}{t^2+1} - 1} \cdot \frac{2}{1+t^2}\, dt =$$

Take $t = \frac{x}{2} \rightarrow dx = \frac{2}{1+t^2}\, dt$

$$= \int \frac{\frac{2}{1+t^2}}{\frac{2t-1-t^2-t^2-1}{t^2+1}}\, dt =$$

$$= \int \frac{\cancel{t^2+1}}{\underbrace{2t-1-t^2-t^2-1}_{2(\dots)}} \cdot \frac{\cancel{2}}{\cancel{1+t^2}}\, dt$$

$$= \int \frac{1}{t-1-t^2}\, dt =$$

$$= \int \frac{1}{-t^2+t-1}\, dt =$$

$$= (-1)\int \frac{1}{t^2-t+1}\, dt =$$

$$= (-1)\int \frac{1}{(t^2-1)-t}\, dt =$$

$$= (-1)\left(\int \frac{1}{t^2-1}\, dt - \int \frac{1}{t}\, dt\right) =$$

$$= (-1)\cdot\left[\left(\frac{1}{2\cdot 1}\cdot \ln\left|\frac{t-1}{t+1}\right|\right) - \left(\ln|t|\right)\right] =$$

$$= (-1)\cdot \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + \ln|t| =$$

$$= \ln|t| - \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| =$$

$$= \ln\left|tg\left(\frac{x}{2}\right)\right| - \frac{1}{2}\ln\left|\frac{tg\left(\frac{x}{2}\right)-1}{tg\left(\frac{x}{2}\right)+1}\right|$$

ii on $(0, \frac{\pi}{2}]$

$[a, b]$

and set p such that the value of the limit to belong to $\in (0, \infty)$, for $f : [a, b) \rightarrow [0, \infty)$.

If the domain is open at in a, then we compute

$$L = \lim_{x \downarrow a} (x-a)^p f(x)$$

$$L = \lim_{\substack{x \to 0 \\ x > 0}} (x)^p \frac{1}{\sin x + \cos x - 1} =$$

$$= \lim_{\substack{x \to 0 \\ x > 0}} \frac{x^p}{\sin x + \cos x - 1}$$

Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval.

Then $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x) + c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x) = \int f(x)\,dx$) then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\,f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)\,f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)\,f[u(x)] - v'(x)\,f[v(x)]$$

Properties

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^a f(x)\,dx = 0$

$\int_a^b c\,dx = c(b-a)$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any value of c.

If $f(x) \ge g(x)$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $f(x) \ge 0$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m \le f(x) \le M$ on $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = k\,x + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \cos u\,du = \sin u + c$

$\int \sin u\,du = -\cos u + c$

$\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$

$\int \csc u \cot u\,du = -\csc u + c$

$\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$

Standard Integration Techniques

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

***u* Substitution :** The substitution $u = g(x)$ will convert $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using $du = g'(x)\,dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2\cos(x^3)\,dx$

$u = x^3 \;\Rightarrow\; du = 3x^2dx \;\Rightarrow\; x^2dx = \frac{1}{3}du$

$x = 1 \;\Rightarrow\; u = 1^3 = 1 \;::\; x = 2 \;\Rightarrow\; u = 2^3 = 8$

$$\int_1^2 5x^2\cos(x^3)\,dx = \int_1^8 \tfrac{5}{3}\cos(u)\,du$$
$$= \tfrac{5}{3}\sin(u)\Big|_1^8 = \tfrac{5}{3}\left(\sin(8) - \sin(1)\right)$$

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.

Ex. $\int x\mathbf{e}^{-x}\,dx$

$u = x \quad dv = \mathbf{e}^{-x} \quad\Rightarrow\quad du = dx \quad v = -\mathbf{e}^{-x}$

$$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} + c$$

Ex. $\int_3^5 \ln x\,dx$

$u = \ln x \quad dv = dx \quad\Rightarrow\quad du = \frac{1}{x}dx \quad v = x$

$$\int_3^5 \ln x\,dx = x\ln x\Big|_3^5 - \int_3^5 dx = \left(x\ln(x) - x\right)\Big|_3^5$$
$$= 5\ln(5) - 3\ln(3) - 2$$

Products and (some) Quotients of Trig Functions

For $\int \sin^n x\cos^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2 x = 1 - \cos^2 x$, then use the substitution $u = \cos x$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2 x = 1 - \sin^2 x$, then use the substitution $u = \sin x$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n x\sec^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \sec x$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2 x = 1 + \tan^2 x$, then use the substitution $u = \tan x$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd.** Each integral will be dealt with differently.

Trig Formulas **:** $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

Ex. $\int \tan^3 x\sec^5 x\,dx$

$$\int \tan^3 x\sec^5 x\,dx = \int \tan^2 x\sec^4 x\tan x\sec x\,dx$$
$$= \int \left(\sec^2 x - 1\right)\sec^4 x\tan x\sec x\,dx$$
$$= \int \left(u^2 - 1\right)u^4\,du \qquad (u = \sec x)$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + c$$

Ex. $\int \frac{\sin^5 x}{\cos^3 x}\,dx$

$$\int \frac{\sin^5 x}{\cos^3 x}\,dx = \int \frac{\sin^4 x\sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}\,dx$$
$$= \int \frac{(1-\cos^2 x)^2\sin x}{\cos^3 x}\,dx \qquad (u = \cos x)$$
$$= -\int \frac{(1-u^2)^2}{u^3}\,du = -\int \frac{1-2u^2+u^4}{u^3}\,du$$
$$= \tfrac{1}{2}\sec^2 x + 2\ln\left|\cos x\right| - \tfrac{1}{2}\cos^2 x + c$$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b}\sin\theta$	$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b}\sec\theta$	$\sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b}\tan\theta$
$\cos^2\theta = 1 - \sin^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\sec^2\theta = 1 + \tan^2\theta$

Ex. $\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$

$x = \frac{2}{3}\sin\theta \Rightarrow dx = \frac{2}{3}\cos\theta\,d\theta$

$\sqrt{4-9x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\left|\cos\theta\right|$

Recall $\sqrt{x^2} = |x|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

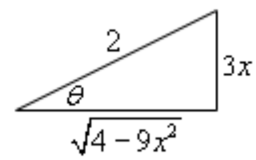
$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

In this case we have $\sqrt{4-9x^2} = 2\cos\theta$.

$$\int \frac{16}{\frac{4}{9}\sin^2\theta\,(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta = \int \frac{12}{\sin^2\theta}\,d\theta$$

$$= \int 12\csc^2 d\theta = -12\cot\theta + c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin\theta = \frac{3x}{2}$ so,



From this we see that $\cot\theta = \frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx = -\frac{4\sqrt{4-9x^2}}{x} + c$$

Partial Fractions : If integrating $\int \frac{P(x)}{Q(x)}\,dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Ex. $\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx = \int \frac{4}{x-1} + \frac{3x+16}{x^2+4}\,dx$$

$$= \int \frac{4}{x-1} + \frac{3x}{x^2+4} + \frac{16}{x^2+4}\,dx$$

$$= 4\ln|x-1| + \tfrac{3}{2}\ln\left(x^2+4\right) + 8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2 + 13x = (A+B)x^2 + (C-B)x + 4A - C$$

Set coefficients equal to get a system and solve to get constants.

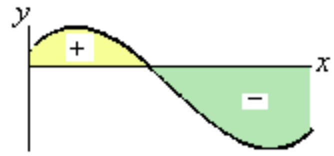
$$A+B=7 \qquad C-B=13 \qquad 4A-C=0$$

$$A=4 \qquad B=3 \qquad C=16$$

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example : $7x^2+13x = A\left(x^2+4\right) + (Bx+C)(x-1)$. Chose *nice* values of x and plug in. For example if $x=1$ we get $20 = 5A$ which gives $A=4$. This won't always work easily.

Applications of Integrals

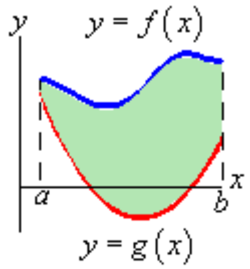
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



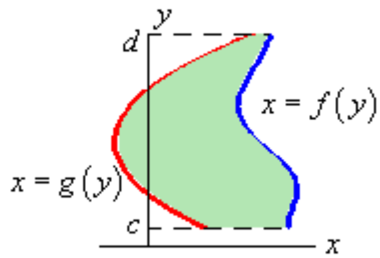
Area Between Curves : The general formulas for the two main cases for each are,

$$y=f(x) \Rightarrow A=\int_a^b [\text{upper function}]-[\text{lower function}]\,dx \quad \& \quad x=f(y) \Rightarrow A=\int_c^d [\text{right function}]-[\text{left function}]\,dy$$

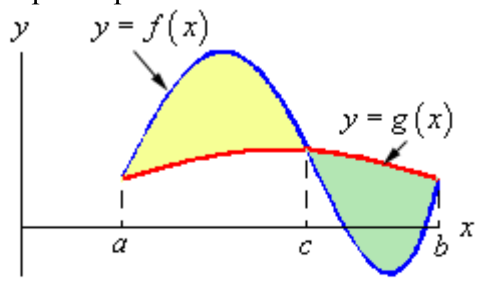
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$$A=\int_a^b f(x)-g(x)\,dx$$



$$A=\int_c^d f(y)-g(y)\,dy$$



$$A=\int_a^c f(x)-g(x)\,dx+\int_c^b g(x)-f(x)\,dx$$

Volumes of Revolution : The two main formulas are $V=\int A(x)\,dx$ and $V=\int A(y)\,dy$. Here is some general information about each method of computing and some examples.

Rings

$$A=\pi\left((\text{outer radius})^2-(\text{inner radius})^2\right)$$

Limits: x/y of right/bot ring to x/y of left/top ring

Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Cylinders

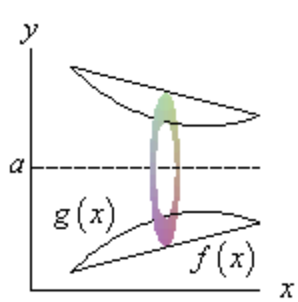
$$A=2\pi\,(\text{radius})\,(\text{width / height})$$

Limits : x/y of inner cyl. to x/y of outer cyl.

Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

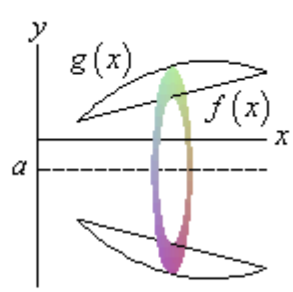
Ex. Axis : $y=a>0$



outer radius : $a-f(x)$

inner radius : $a-g(x)$

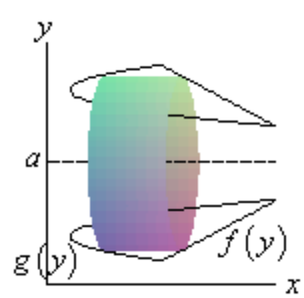
Ex. Axis : $y=a\le 0$



outer radius: $|a|+g(x)$

inner radius: $|a|+f(x)$

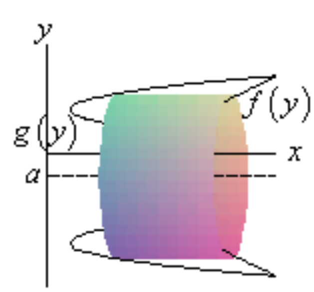
Ex. Axis : $y=a>0$



radius : $a-y$

width : $f(y)-g(y)$

Ex. Axis : $y=a\le 0$



radius : $|a|+y$

width : $f(y)-g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y=a\le 0$ case with $a=0$. For vertical axis of rotation ($x=a>0$ and $x=a\le 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \le x \le b$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,

$L = \int_a^b ds$ $\quad$ $SA = \int_a^b 2\pi y\,ds$ (rotate about *x*-axis) $\quad$ $SA = \int_a^b 2\pi x\,ds$ (rotate about *y*-axis)

where *ds* is dependent upon the form of the function being worked with as follows.

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if $y = f(x),\ a \le x \le b$ $\qquad$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ if $x = f(t), y = g(t),\ a \le t \le b$

$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if $x = f(y),\ a \le y \le b$ $\qquad$ $ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ if $r = f(\theta),\ a \le \theta \le b$

With surface area you *may* have to substitute in for the *x* or *y* depending on your choice of *ds* to match the differential in the *ds*. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn't exist or has infinite value. This is typically a Calc II topic.

Infinite Limit

1. $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
2. $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
3. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ provided BOTH integrals are convergent.

Discontinuous Integrand

1. Discont. at *a*: $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
2. Discont. at b : $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
3. Discontinuity at $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided both are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a,\infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ conv. then $\int_a^\infty g(x)\,dx$ conv.
2. If $\int_a^\infty g(x)\,dx$ divg. then $\int_a^\infty f(x)\,dx$ divg.

Useful fact : If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x)\,dx$ and a *n* (must be even for Simpson's Rule) define $\Delta x = \frac{b-a}{n}$ and divide $[a,b]$ into *n* subintervals $[x_0,x_1]$, $[x_1,x_2]$, … , $[x_{n-1},x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$, x_i^* is midpoint $[x_{i-1},x_i]$

Trapezoid Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)++2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Simpson's Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$