**Calculus on $\mathbb{R}$** Lect. dr. Anca Grad
**Homework 4** **2021**

# Sequences of real numbers - 2nd part

**Exercise 1:** Study the nature (covergence or divergence) of the following sequence of real numbers.

$$x_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\dots\left(1 - \frac{1}{n^2}\right),\ n \geq 2.$$

In case it is convergent, compute its limit.

**Exercise 2:** Determine the limits of the following sequences of real numbers, having as general term:

$$a)\quad \frac{3^n}{4^n}, \qquad b)\quad \frac{2^n + (-2)^n}{3^n}, \qquad c)\quad \frac{5-n^3}{n^2+1}, \qquad d)\quad \left(2 + \frac{4^n + (-5)^n}{7^n+1}\right)^{2n^3-n^2},$$

$$e)\quad \frac{1+2+\ldots+n}{n^2}, \qquad f)\quad \left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}}, \qquad g)\quad (\cos(-2013))^n,$$

$$h)\quad \left(\frac{n^5+3n+1}{2n^5-n^4+3}\right)^{\frac{3n-n^4}{n^3+1}}.$$

**Exercise 3:** Determine the limits of the following sequences of real numbers, having as general term:

$$a)\quad \left(1 + \frac{1}{-n^3+3n}\right)^{n^2-n^3}, \quad b)\quad (3n^2+5)ln\left(1 + \frac{1}{n^2}\right),$$

$$c)\frac{n^n}{1^1 + 2^2 + ... + n^n}$$

$$d)\frac{x_1 + 2x_2 + ... + nx_n}{n^2},$$

when $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, with the limit $x \in \mathbb{R}$.

**Exercise 4:** Determine the limits of the following sequences of real numbers, having as general term:

$$a)x_n = \frac{a^n - a^{-n}}{a^n + a^{-n}}, \quad a \neq 0$$

$$b) y_n = \frac{a^n + b^n}{a^{n+1} + b^{n+1}}, \quad a \neq -b$$

$$c) z_n = \frac{1 + a + ... + a^n}{1 + b + ... + b^n}, \quad a, b > 0.$$

**Exercise 1:** Study the nature (covergence or divergence) of the following sequence of real numbers.

$$x_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\dots\left(1 - \frac{1}{n^2}\right),\ n \geq 2.$$

In case it is convergent, compute its limit.

$$x_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\dots\left(1 - \frac{1}{n^2}\right),\ n \geqslant 2$$

$$x_n = \left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\dots\left(\frac{n^2-1}{n^2}\right) = \frac{(2-1)(2+1)}{2^2} \cdot \frac{(3-1)(3+1)}{3^2} \cdot \dots \cdot \frac{(n-1)(n+1)}{n^2} =$$

$$= \prod_{k=2}^{n} \frac{k^2-1}{k^2} = \prod_{k=2}^{n} \frac{(k-1)(k+1)}{k^2} = \prod_{k=2}^{n} \frac{k-1}{k} \cdot \prod_{k=2}^{n} \frac{k+1}{k} =$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \dots \cdot \frac{\cancel{n-1}}{n} \cdot \frac{\cancel{3}}{2} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \dots \cdot \frac{n+1}{\cancel{n}} =$$

$$= \frac{1}{n} \cdot \frac{n+1}{2} =$$

$$= \frac{n+1}{2n}$$

MONOTONICITY:

$$\frac{x_{n+1}}{x_n} = \frac{\frac{n+2}{2(n+1)}}{\frac{n+1}{2n}} = \frac{n+2}{\cancel{2}(n+1)} \cdot \frac{\cancel{2}n}{n+1} = \frac{n(n+2)}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} < 1 \Rightarrow x_{n+1} < x_n,\ \forall n \in \mathbb{N}, n \geqslant 2$$

$$x_{n+1} - x_n = \frac{n+2}{2(n+1)} - \frac{n+1}{2n} = \frac{\cancel{n^2} + \cancel{2n} - \cancel{n^2} - \cancel{2n} - 1}{2n(n+1)} = \frac{-1}{2n(n+1)} < 0 \Rightarrow x_{n+1} < x_n \Rightarrow (x_n)_n \text{ DECREASING } (1)$$

BOUNDEDNESS:

$$x_n = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} > \frac{1}{2},\ \forall n \in \mathbb{N}, n \geqslant 2$$

$$n \geqslant 2 \Rightarrow \frac{1}{2n} \leqslant \frac{1}{4} \ \Big|+\frac{1}{2}$$

$$\frac{1}{2n} + \frac{1}{2} \leqslant \frac{3}{4} \Rightarrow x_n \leqslant 1,\ \forall n \in \mathbb{N}, n \geqslant 2$$

$$\Rightarrow \frac{1}{2} < x_n < \frac{3}{4} \Rightarrow (x_n)_n \text{ is BOUNDED } (2)$$

$$(1) \,\&\, (2) \Rightarrow (x_n)_n \text{ is CONVERGENT} \Rightarrow \exists \lim_{n\to\infty} x_n = \frac{1}{2}$$

**Exercise 2:** Determine the limits of the following sequences of real numbers, having as general term:

a) $\frac{3^n}{4^n}$, b) $\frac{2^n+(-2)^n}{3^n}$, c) $\frac{5-n^3}{n^2+1}$, d) $\left(2+\frac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2}$,

e) $\frac{1+2+\ldots+n}{n^2}$, f) $\left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}}$, g) $(\cos(-2013))^n$,

h) $\left(\frac{n^5+3n+1}{2n^5-n^4+3}\right)^{\frac{3n-n^4}{n^3+1}}$.

a) $\lim\limits_{n\to\infty} \frac{3^n}{4^n} = \lim\limits_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$

b) $\lim\limits_{n\to\infty} \frac{2^n+(-2)^n}{3^n} = \lim\limits_{n\to\infty} \left(\left(\frac{2}{3}\right)^n + \left(-\frac{2}{3}\right)^n\right) = 0+0 = 0$

c) $\lim\limits_{n\to\infty} \frac{5-n^3}{n^2+1} = -\infty$

d) $\lim\limits_{n\to\infty} \left(2+\frac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2} = \lim\limits_{n\to\infty} \Bigg(2+\underbrace{\frac{4^n+(-5)^n}{7^n+1}}_{0}\Bigg)^{n^3(2-\overbrace{\frac{1}{n}}^{0})} = +\infty$

e) $\lim\limits_{n\to\infty} \frac{1+2+\ldots+n}{n^2} = \lim\limits_{n\to\infty} \frac{(1+n)\cdot n}{2}\cdot\frac{1}{n^2} = \lim\limits_{n\to\infty} \frac{n+n^2}{2n^2} = \frac{1}{2}$

f) $\lim\limits_{n\to\infty} \left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}} = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$

g) $\lim\limits_{n\to\infty} (\cos(-2013))^n = \lim\limits_{n\to\infty} (\cos 2013)^n = 0$

$[0 < \cos 2013 < 1]$

h) $\lim\limits_{n\to\infty} \left(\frac{n^5+3n+1}{2n^5-n^4+3}\right)^{\frac{3n-n^4}{n^3+1}} = \left(\frac{1}{2}\right)^0 = 1$

**Exercise 3:** Determine the limits of the following sequences of real numbers, having as general term:

$$a)\quad \left(1+\frac{1}{-n^3+3n}\right)^{n^2-n^3},\quad b)\quad (3n^2+5)ln\left(1+\frac{1}{n^2}\right),$$

$$c)\frac{n^n}{1^1+2^2+...+n^n}$$

$$d)\frac{x_1+2x_2+...+nx_n}{n^2},$$

when $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, with the limit $x\in\mathbb{R}$.

a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{-n^3+3n}\right)^{n^2-n^3} = (1^\infty) = \lim\limits_{n\to\infty}\left(\left(1+\frac{1}{-n^3+3n^2}\right)^{-n^3+3n^2}\right)^{\frac{1}{-n^3+3n^2}\cdot(n^2-n^3)} = e^{\lim\limits_{n\to\infty}\frac{n^2-n^3}{-n^3+3n^2}} = e^{\frac{-1}{-1}} = e^1 = e$

b) $\lim\limits_{n\to\infty}(3n^2+5)\ln\left(1+\frac{1}{n^2}\right) = \lim\limits_{n\to\infty}\ln\left(1+\frac{1}{n^2}\right)^{(3n^2+5)} = \ln\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{3n^2+5} =$

$= \ln\lim\limits_{n\to\infty}\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n^2}\cdot(3n^2+5)} = \ln e^{\lim\limits_{n\to\infty}\frac{3n^2+5}{n^2}} = \ln e^3 = 3$

c) $\lim\limits_{n\to\infty}\frac{n^n}{1^1+2^2+...+n^n} \underset{=}{\overset{C-S}{=}} \lim\limits_{n\to\infty}\frac{(n+1)^{n+1}-n^n}{\cancel{1+2^2}+...+(n+1)^2-\cancel{1-2^2}-...-\cancel{n^n}} = \lim\limits_{n\to\infty}\frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}} =$

$= 1-\lim\limits_{n\to\infty}\frac{n^n}{n^{n+1}} = 1-0 = 1$

d) $\lim\limits_{n\to\infty}\frac{x_1+2x_2+...+nx_n}{(n+1-n)(n+1+n)} \underset{=}{\overset{C-S}{=}} \lim\limits_{n\to\infty}\frac{x_1+2x_2+...+nx_n+(n+1)x_{n+1}-x_1-x_2-...-nx_n}{(n+1)^2-n^2} =$

$= \lim\limits_{n\to\infty}\frac{(n+1)x_{n+1}}{(n+1-n)(n+1+n)} = \lim\limits_{n\to\infty}\frac{n+1}{2n+1}\cdot x_{n+1} = \frac{x}{2}$

$\left[\lim\limits_{n\to\infty}x_n = x\in\mathbb{R}\right]$

**Exercise 4:** Determine the limits of the following sequences of real numbers, having as general term:

$$a) x_n = \frac{a^n - a^{-n}}{a^n + a^{-n}}, \quad a \neq 0$$

$$b) y_n = \frac{a^n + b^n}{a^{n+1} + b^{n+1}}, \quad a \neq -b$$

$$c) z_n = \frac{1 + a + \dots + a^n}{1 + b + \dots + b^n}, \quad a, b > 0.$$

a) $x_n = \dfrac{a^n - a^{-n}}{a^n + a^{-n}}, \quad a \neq 0$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a^n - a^{-n}}{a^n + a^{-n}} = \lim_{n\to\infty} \frac{a^n - \frac{1}{a^n}}{a^n + \frac{1}{a^n}} = \lim_{n\to\infty} \frac{a^{2n} - 1}{a^{2n} + 1}$$

CASE $\underline{I}$ : $a > 1$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a^{2n} - 1}{a^{2n} + 1} = \lim_{n\to\infty} \frac{a^{2n}\left(1 - \overbrace{\frac{1}{a^{2n}}}^{\to 0}\right)}{a^{2n}\left(1 + \underbrace{\frac{1}{a^{2n}}}_{\to 0}\right)} = 1$$

CASE $\underline{II}$ : $a = 1$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1^n - 1^{-n}}{1^n + 1^{-n}} = 0$$

CASE $\underline{III}$ : $|a| < 1 \Rightarrow -1 < a < 1, \ a \neq 0$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a^n - a^{-n}}{a^n + a^{-n}} = \lim_{n\to\infty} \frac{\frac{1}{t^n} - t^n}{\frac{1}{t^n} + t^n} = \lim_{n\to\infty} \frac{1 - t^{2n}}{1 + t^{2n}} = -1$$

Let $t = a^{-1} > 1$

CASE $\underline{IV}$ : $a = -1$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{(-1)^{2n} - 1}{(-1)^{2n} + 1} = \lim_{n\to\infty} \frac{1 - 1}{1 + 1} = 0$$

CASE $\underline{V}$ : $a < -1$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a^n - a^{-n}}{a^n + a^{-n}} = \lim_{n\to\infty} \frac{a^{2n} - 1}{a^{2n} + 1} = \lim_{n\to\infty} \frac{(a^2)^n - 1}{(a^2)^n - 1} = 1$$

$a < -1 \Rightarrow a^2 > 1$

The solution is : $\lim_{n\to\infty} x_n = \begin{cases} 1, & a \in (-\infty; -1) \\ 0, & a = -1 \\ -1, & a \in (-1; 1) \\ 0, & a = 1 \\ 1, & a \in (1; +\infty) \end{cases} = \begin{cases} 1, & a \in (-\infty; -1) \cup (1; +\infty) \\ 0, & a \in \{-1; 1\} \\ -1, & a \in (-1; 1) \end{cases}$

b) $y_n = \dfrac{a^n + b^n}{a^{n+1} + b^{n+1}}$, $a \neq -b$

CASE $\underline{I}$ : $a > b$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}} = \lim_{n\to\infty} \frac{a^n\left(1+\left(\frac{b}{a}\right)^n\right)}{a^{n+1}\left(1+\left(\frac{1}{b}\right)^{n+1}\right)} = \frac{1}{a}, \quad a \neq 0$$

CASE $\underline{II}$ : $a < b$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}} = \lim_{n\to\infty} \frac{b^n\left(\left(\frac{a}{b}\right)^n+1\right)}{b^{n+1}\left(\left(\frac{a}{b}\right)^n+1\right)} = \frac{1}{b}, \quad b \neq 0$$

CASE $\underline{III}$ : $a = b \neq 0$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{\cancel{2a^n}}{\cancel{2}a^{\cancel{n+1}}} = \frac{1}{a}, \quad a \neq 0$$

CASE $\underline{IV}$ : $a = 0$, $b \neq 0$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{b^n}{b^{n+1}} = \frac{1}{b}, \quad b \neq 0$$

CASE $\underline{V}$ : $a \neq 0$, $b = 0$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{a^n}{a^{n+1}} = \frac{1}{a}, \quad a \neq 0$$

The solution is : $\lim\limits_{n\to\infty} y_n = \begin{cases} \frac{1}{a}, & a>b \text{ and } a=b\neq 0 \text{ and } a\neq 0, b=0 \\ \frac{1}{b}, & a<b \text{ and } a=0, b\neq 0 \end{cases}$

c) $z_n = \dfrac{1+a+\ldots+a^n}{1+b+\ldots+b^n}$, $a, b > 0$

CASE $\underline{I}$ : $a = b = 1$

$$\lim_{n\to\infty} z_n = \frac{n+1}{n+1} = 1$$

CASE $\underline{II}$ : $a = 1$, $b \neq 1$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{n+1}{1+b+\ldots+b^n} = \lim_{n\to\infty} \frac{n+1}{\frac{b^{n+1}-1}{b-1}} = \lim_{n\to\infty} (b-1)\cdot\frac{n+1}{b^{n+1}-1}$$

If $b > 1 \Rightarrow \lim\limits_{n\to\infty} b^{n+1} = +\infty \Rightarrow (b-1)\lim\limits_{n\to\infty}\frac{n+1}{b^{n+1}-1} = 0 \Rightarrow \lim\limits_{n\to\infty} z_n = 0$

If $b \in (0,1) \Rightarrow \lim\limits_{n\to\infty} b^{n+1} = 0 \Rightarrow \lim\limits_{n\to\infty} \underbrace{(b-1)}\cdot\frac{n+1}{b^{n+1}-1} = (b-1)\cdot(-\infty) = +\infty \Rightarrow \lim\limits_{n\to\infty} z_n = +\infty$

$$\begin{bmatrix} b<1 \mid -1 \\ b-1<0 \end{bmatrix} \quad 0$$

CASE $\overline{\underline{III}}$ : $a \neq 1,\ b = 1$

$$\lim_{n\to\infty} z_n = \lim_{n\to 0} \frac{1+a+a^2+\ldots+a^n}{n+1} \overset{C\text{-}S}{=\!=} \lim_{n\to\infty} \frac{\cancel{1+a+\ldots+a^n}+a^{n+1}\cancel{-1-a-\ldots-a^n}}{\cancel{n}+2-\cancel{n}-1} = \lim_{n\to\infty} a^{n+1}$$

If $a \in (0,1) \Rightarrow \lim\limits_{n\to\infty} a^{n+1} = 0 \longrightarrow \lim\limits_{n\to\infty} z_n = 0$

If $a \in (1,+\infty) \Longrightarrow \lim\limits_{n\to\infty} a^{n+1} = +\infty \Longrightarrow \lim\limits_{n\to\infty} z_n = -\infty$

CASE $\overline{\underline{IV}}$ : $a \neq 1,\ b \neq 1$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{1+a+\ldots+a^n}{1+b+\ldots+b^n} = \lim_{n\to\infty} \frac{\frac{a^{n+1}-1}{a-1}}{\frac{b^{n+1}-1}{b-1}} = \lim_{n\to\infty} \frac{b-1}{a-1}\cdot\frac{a^{n+1}-1}{b^{n+1}-1} = \frac{b-1}{a-1}\lim_{n\to\infty}\frac{a^{n+1}-1}{b^{n+1}-1}$$

If $a\in(0,1)$ & $b\in(0,1) \Rightarrow \lim\limits_{n\to\infty} z_n = \dfrac{b-1}{a-1}\lim\limits_{n\to\infty}\dfrac{\overset{\nearrow 0}{a^{n+1}}-1}{\underset{\searrow 0}{b^{n+1}}-1} = \dfrac{b-1}{a-1}$

If $a\in(0,1)$ & $b\in(1,+\infty) \Rightarrow \lim\limits_{n\to\infty} z_n = \dfrac{b-1}{a-1}\lim\limits_{n\to\infty}\dfrac{\overset{\to 0}{a^{n+1}}-1}{b^{n+1}-1} = 0$

If $a \in (1,+\infty)$ & $b\in(0,1) \Rightarrow \lim\limits_{n\to\infty} z_n = \dfrac{b-1}{a-1}\lim\limits_{n\to\infty}\dfrac{a^{n+1}-1}{\underset{\downarrow\, 0}{b^{n+1}}-1} = \dfrac{\overbrace{b-1}^{<0}}{\underbrace{a-1}_{>0}}\cdot(-\infty) = +\infty$

$\Downarrow$ $a-1>0$ $\qquad$ $\Downarrow$ $b-1<0$

If $a\in(1,+\infty)$ & $b\in(1,+\infty) \Rightarrow \lim\limits_{n\to\infty} z_n = \dfrac{b-1}{a-1}\lim\limits_{n\to\infty}\dfrac{a^{n+1}-1}{b^{n+1}-1}$

If $a > b \Rightarrow \lim\limits_{n\to\infty} z_n = +\infty$

If $a < b \Rightarrow \lim\limits_{n\to\infty} z_n = 0$

The solution is :
$$\lim_{n\to\infty} z_n = \begin{cases} 0, & a=1,\ b>1 \\ +\infty, & a=1,\ b\in(0,1) \\ 0, & a\in(0,1),\ b=1 \\ -\infty, & a\in(1,+\infty),\ b=1 \\ \frac{b-1}{a-1}, & a,b\in(0,1) \\ 0, & a\in(0,1),\ b\in(1,+\infty) \\ +\infty, & a\in(1,+\infty),\ b\in(0,1) \\ +\infty, & a,b\in(1,+\infty),\ a>b \\ 0, & a,b\in(1,+\infty),\ a<b \end{cases} = \begin{cases} -\infty, & a\in(1,+\infty),\ b=1 \\ 0, & a=1,\ b>1 \text{ or } a\in(0,1),\ b=1 \text{ or } a\in(0,1),\ b\in(1,+\infty) \text{ or } a,b\in(1,+\infty),\ a<b \\ \frac{b-1}{a-1}, & a,b\in(0,1) \\ +\infty, & a=1,\ b\in(0,1) \text{ or } a\in(1,+\infty),\ b\in(0,1) \text{ or } a,b\in(1,+\infty),\ a>b \end{cases}$$