ANALYTIC GEOMETRY, PROBLEM SET 4

Projections. Dot product. Cross product.

1. Find the orthogonal projection $pr_{\overline{u}}(\overline{v})$, where $\overline{v} = 10\overline{a} + 2\overline{b}$, $\overline{u} = 5\overline{a} - 12\overline{b}$, if $\overline{a} \perp \overline{b}$ and $||\overline{a}|| = ||\overline{b}|| \neq 0$.

2. Using the dot product, prove the **Cauchy-Buniakowski-Schwarz** inequality, i.e. show that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

3. For a tetrahedron $ABCD$, show that $\cos(\widehat{\overline{AB}, \overline{CD}}) = \frac{AD^2+BC^2-AC^2-BD^2}{2AB \cdot CD}$. (the 3D version of the **cosine theorem**)

4. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds: $9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$.

* **5.** Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ and $A'B'$, respectively, are concurrent. Then the perpendicular lines through A', B', C' on BC, CA and AB, respectively are also concurrent. (Steiner's theorem on **orthologic triangles**)

6. Find the area of the plane triangle having the vertices $A(1, 0, 1)$, $B(0, 2, 3)$, $C(2, 1, 0)$.

7. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \overline{a}$, $\overline{CA} = \overline{b}$ and $\overline{AB} = \overline{c}$ if and only if $\overline{a} \times \overline{b} = \overline{b} \times \overline{c} = \overline{c} \times \overline{a}$.

8. Find a vector orthogonal on both $\overline{u}$ and $\overline{v}$, if:
a) $\overline{u} = -7\overline{i} + 3\overline{j} + \overline{k}$ and $\overline{v} = 2\overline{i} + 4\overline{k}$ b) $\overline{u} = (-1, -1, -1)$ and $\overline{v} = (2, 0, 2)$.

9. Let a, b, and c denote the lengths of the sides of $\triangle ABC$. We write O for its circumcenter, R for the length of its circumradius, H for its orthocenter and G for the centroid. Show that
a) $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$; b) $OG^2 = R^2 - 1/9(a^2 + b^2 + c^2)$.

Try it using ideas in 4.

Date: October 14, 2021.

* Not examinable.

1. Find the orthogonal projection $pr_{\overline{u}}(\overline{v})$, where $\overline{v} = 10\overline{a} + 2\overline{b}$, $\overline{u} = 5\overline{a} - 12\overline{b}$, if $\overline{a} \perp \overline{b}$ and $||\overline{a}|| = ||\overline{b}|| \neq 0$.

$$pr_{\overline{u}}(\overline{v}) = \frac{\overline{v} \cdot \overline{u}}{\|u\|^2} \cdot \overline{u}$$

$$= \frac{(10\overline{a} + 2\overline{b}) \cdot (5\overline{a} - 12\overline{b})}{\|5\overline{a} - 12\overline{b}\|^2} \cdot (5\overline{a} - 12\overline{b})$$

$$\overline{v} \cdot \overline{u} = (10\overline{a} + 2\overline{b}) \cdot (5\overline{a} - 12\overline{b})$$

$$= 50\|\overline{a}\|^2 - 24 \cdot \|\overline{b}\|^2$$

$$\|\overline{u}\|^2 = \overline{u} \cdot \overline{u} = (5\overline{a} - 12\overline{b}) \cdot (5\overline{a} - 12\overline{b})$$

$$= 25\|\overline{a}\|^2 + 144\|\overline{b}\|^2.$$

$$pr_{\bar{u}}(\bar{v}) = \frac{50\|\bar{a}\|^2 - 24\|\bar{b}\|^2}{25\|\bar{a}\|^2 + 144\|\bar{b}\|^2} \cdot (5\bar{a} - 12\bar{b})$$

$$= \frac{16\cancel{\|\bar{a}\|^2}}{169\cancel{\|\bar{a}\|^2}} (5\bar{a} - 12\bar{b})$$

$$= \left(\frac{4}{13}\right)^2 \cdot (5\bar{a} - 12\bar{b}).$$

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2. Using the dot product, prove the **Cauchy-Buniakowski-Schwarz** inequality, i.e. show that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

Let $\bar{v}_1(a_1, a_2, a_3)$, $\bar{v}_2(b_1, b_2, b_3) \in V_3$.

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 = (\bar{v}_1 \cdot \bar{v}_2)^2$$

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)=\|\overline{v}_1\|^2\cdot\|\overline{v}_2\|^2.$$

We know that

$$(\overline{v}_1\cdot\overline{v}_2)^2=(\|\overline{v}_1\|\cdot\|\overline{v}_2\|\cdot\cos\theta)^2,\quad\text{where}$$

$$\theta=\sphericalangle(\overline{v}_1,\overline{v}_2)\in[0,\pi].$$

Since $\cos\theta\in[-1,1]$, then $\cos^2\theta\in[0,1]$.

So $(\overline{v}_1\cdot\overline{v}_2)^2\leq\|\overline{v}_1\|^2\cdot\|\overline{v}_2\|^2$.

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<u>Alternative proof</u>:

Take $\overline{v}_1(a_1,a_2,a_3)$, $\overline{v}_2(b_1,b_2,b_3)$.

Let $x\in\mathbb{R}$.

Obviously, $\|\overline{v}_1-x\cdot\overline{v}_2\|^2\geq 0\ \forall x\in\mathbb{R}$.

$$(\overline{v}_1 - x\cdot\overline{v}_2)\cdot(\overline{v}_1 - x\cdot\overline{v}_2) \geq 0 \quad \forall x \in \mathbb{R}.$$

$$\underbrace{\|\overline{v}_1\|^2}_{\in \mathbb{R}} - \underbrace{2(\overline{v}_1 \cdot \overline{v}_2)}_{\in \mathbb{R}} \cdot x + \underbrace{\|\overline{v}_2\|^2}_{\in \mathbb{R}} \cdot x^2 \geq 0, \boxed{\forall x \in \mathbb{R}}.$$

$$\Rightarrow \quad \Delta \leq 0$$

$$4\cdot(\overline{v}_1\cdot\overline{v}_2)^2 - 4\cdot\|\overline{v}_1\|^2\cdot\|\overline{v}_2\|^2 \leq 0.$$

$$\Leftrightarrow (\overline{v}_1\cdot\overline{v}_2)^2 \leq \|\overline{v}_1\|^2\cdot\|\overline{v}_2\|^2$$

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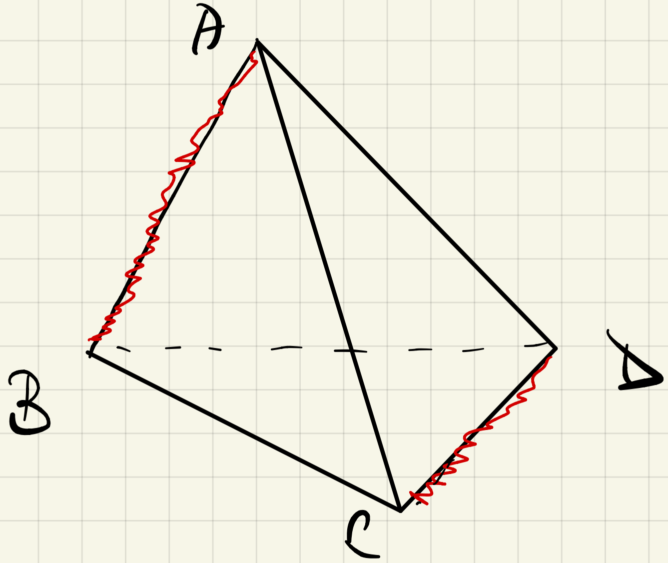
3. For a tetrahedron $ABCD$, show that $\cos(\widehat{\overline{AB},\overline{CD}}) = \frac{AD^2+BC^2-AC^2-BD^2}{2AB\cdot CD}$. (the 3D version of the **cosine theorem**)

<u>Remark</u>:
- $\overline{AB}\cdot\overline{CD} = \|\overline{AB}\|\cdot\|\overline{CD}\|\cdot\cos(\widehat{\overline{AB},\overline{CD}})$.
- It's enough to show $AD^2+BC^2-AC^2-BD^2 = 2\cdot\overline{AB}\bullet\overline{CD}$



$$AD^2 + BC^2 - AC^2 - BD^2 = \overline{AD}\cdot\overline{AD} + \overline{BC}\cdot\overline{BC} - \overline{AC}\cdot\overline{AC} - \overline{BD}\cdot\overline{BD}$$

$$= (\overline{AD} - \overline{AC})\cdot(\overline{AD} + \overline{AC}) + (\overline{BC} - \overline{BD})\cdot(\overline{BC} + \overline{BD})$$

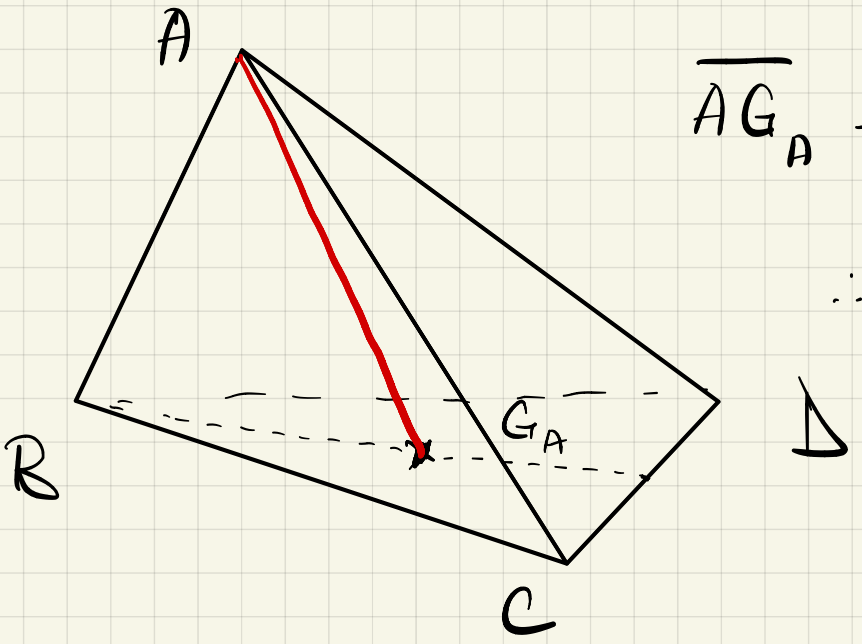
$$= \overline{CD}\cdot(\overline{AD} + \overline{AC}) + \overline{DC}\cdot(\overline{BC} + \overline{BD})$$

$$= \overline{CD}\cdot(\overline{AD} + \overline{AC} + \overline{CB} + \overline{DB})$$

$$= \overline{CD}\cdot(\overline{AB} + \overline{AB}) = 2\cdot\overline{AB}\cdot\overline{CD}.$$

centroid

4. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds: $9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$.



$$\overline{AG_A} = \frac{1}{3}\left(\overline{AB} + \overline{AC} + \overline{AD}\right)$$

$$\therefore\ 3 \cdot \overline{AG_A} = \overline{AB} + \overline{AC} + \overline{AD}$$

$$9\,AG_A^2 = \left(3\overline{AG_A}\right)\cdot\left(3\overline{AG_A}\right) = \left(\overline{AB} + \overline{AC} + \overline{AD}\right)\cdot\left(\overline{AB} + \overline{AC} + \overline{AD}\right)$$

$$= AB^2 + AC^2 + AD^2 + 2\cdot\overline{AB}\cdot\overline{AC} + 2\cdot\overrightarrow{AB}\cdot\overline{AD} + 2\cdot\overline{AC}\cdot\overline{AD}.$$

$$2 \cdot \overline{AB} \cdot \overline{AC} = AB \cdot AC \cdot \cos(\measuredangle BAC)$$

$$\overset{(*)}{=} AB^2 + AC^2 - BC^2$$

(*) Cosine theorem in ΔABC.

Similarly, $2 \cdot \overline{AB} \cdot \overline{AD} = AB^2 + AD^2 - BD^2$

$$2 \cdot \overline{AC} \cdot \overline{AD} = AC^2 + AD^2 - CD^2$$

Replacing, one gets

$$9 \cdot AG_A^2 = 3\left(AB^2 + AC^2 + AD^2\right) - BC^2 - BD^2 - CD^2.$$

$\blacksquare$

6. Find the area of the plane triangle having the vertices $A(1,0,1)$, $B(0,2,3)$, $C(2,1,0)$.

$\overline{AB}(-1, 2, 2)$

$\overline{AC}(1, 1, -1)$

We saw in the lecture that

$$\text{Area}_{\Delta ABC} = \frac{1}{2} \cdot \| \overline{AB} \times \overline{AC} \| .$$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 2 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -2 \cdot \bar{i} - \bar{k} + 2\bar{j} - 2\bar{k} - 2\bar{i} - \bar{j}$$

$$= -4 \cdot \bar{i} + \bar{j} - 3\bar{k} .$$

$\overline{AB} \times \overline{AC}(-4, 1, -3)$

$$\|\overline{AB}\times\overline{AC}\| = \sqrt{16+1+9} = \sqrt{26}.$$

$$\text{Area}_{\Delta ABC} = \frac{\sqrt{26}}{2}.$$

$\blacksquare$

7. Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \bar{a}$, $\overline{CA} = \bar{b}$ and $\overline{AB} = \bar{c}$ if and only if $\bar{a}\times\bar{b} = \bar{b}\times\bar{c} = \bar{c}\times\bar{a}$.

In particular, $\bar{a}, \bar{b}, \bar{c} \neq \bar{0}$.

$\bar{a}, \bar{b}, \bar{c}$ have representatives that form a triangle if and only if $\bar{a}+\bar{b}+\bar{c} = \bar{0}$.

(see previous seminar).

"$\Rightarrow$" Assume $\bar{a}+\bar{b}+\bar{c} = \bar{0}$. $\Big| \times \bar{b}$

By distributivity,

$$\bar{a} \times \bar{b} + \underbrace{\bar{b} \times \bar{b}}_{\bar{0}} + \bar{c} \times \bar{b} = \underbrace{\bar{0} \times \bar{b}}_{=\bar{0}}$$

$$\bar{a} \times \bar{b} + \bar{c} \times \bar{b} = \bar{0}$$

$$\Rightarrow \quad \bar{a} \times \bar{b} = \underbrace{-\bar{c} \times \bar{b} = \bar{b} \times \bar{c}}_{\times \text{ is anti-commutative.}} \qquad (1)$$

Similarly, starting with $\bar{a} + \bar{b} + \bar{c} = \bar{0}$ and taking $\times \bar{c}$, we get

$$\bar{a} \times \bar{c} = \bar{c} \times \bar{b} \Rightarrow$$

$$\boxed{\bar{c} \times \bar{a} = \bar{b} \times \bar{c}} \qquad (2).$$

"$\Leftarrow$" Assume $\bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$.

$$\bar{a} \times \bar{b} = \bar{b} \times \bar{c} \Rightarrow$$

$$\bar{a} \times \bar{b} + \bar{c} \times \bar{b} = \bar{0} \quad \Big| + \underbrace{\bar{b} \times \bar{b}}_{= \bar{0}}$$

$$\bar{a} \times \bar{b} + \bar{b} \times \bar{b} + \bar{c} \times \bar{b} = \bar{0}.$$

$\therefore$ By distributivity wrt to addition:

$$(\bar{a} + \bar{b} + \bar{c}) \times \bar{b} = \bar{0}.$$

$\therefore$ $\bar{a} + \bar{b} + \bar{c}$ is collinear to $\bar{b}$.

Similarly, one can get that

$\bar{a} + \bar{b} + \bar{c}$ is collinear to $\bar{c}$.

Since $\bar{b}, \bar{c}$ are not collinear,

we get $\bar{a} + \bar{b} + \bar{c} = \bar{0}$.

$\boxtimes$