# Limits

## Definitions

**Precise Definition :** We say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

**"Working" Definition :** We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close to $L$ as we want by taking $x$ sufficiently close to $a$ (on either side of $a$) without letting $x = a$.

**Right hand limit :** $\lim_{x\to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

**Left hand limit :** $\lim_{x\to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

**Limit at Infinity :** We say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to $L$ as we want by taking $x$ large enough and positive.

There is a similar definition for $\lim_{x\to -\infty} f(x) = L$ except we require $x$ large and negative.

**Infinite Limit :** We say $\lim_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking $x$ sufficiently close to $a$ (on either side of $a$) without letting $x = a$.

There is a similar definition for $\lim_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

## Relationship between the limit and one-sided limits

$\lim_{x\to a} f(x) = L \;\Rightarrow\; \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ $\qquad$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \;\Rightarrow\; \lim_{x\to a} f(x) = L$

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \;\Rightarrow\; \lim_{x\to a} f(x)$ Does Not Exist

## Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and $c$ is any number then,

1. $\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

## Basic Limit Evaluations at $\pm\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and $x^r$ is real for negative $x$ then $\lim_{x\to-\infty} \frac{b}{x^r} = 0$
5. $n$ even : $\lim_{x\to\pm\infty} x^n = \infty$
6. $n$ odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. $n$ even : $\lim_{x\to\pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. $n$ odd : $\lim_{x\to\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. $n$ odd : $\lim_{x\to-\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

# Evaluation Techniques

## Continuous Functions

If $f(x)$ is continuous at $a$ then $\lim_{x\to a} f(x) = f(a)$

## Continuous Functions and Composition

$f(x)$ is continuous at $b$ and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$

## Factor and Cancel

$$\lim_{x\to 2} \frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2} \frac{(x-2)(x+6)}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{x+6}{x} = \frac{8}{2} = 4$$

## Rationalize Numerator/Denominator

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{x^2-81} = \lim_{x\to 9} \frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}}$$

$$= \lim_{x\to 9} \frac{9-x}{(x^2-81)(3+\sqrt{x})} = \lim_{x\to 9} \frac{-1}{(x+9)(3+\sqrt{x})}$$

$$= \frac{-1}{(18)(6)} = -\frac{1}{108}$$

## Combine Rational Expressions

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

## L'Hospital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ $a$ is a number, $\infty$ or $-\infty$

## Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)}$ factor largest power of $x$ in $q(x)$ out of both $p(x)$ and $q(x)$ then compute limit.

$$\lim_{x\to-\infty} \frac{3x^2-4}{5x-2x^2} = \lim_{x\to-\infty} \frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)} = \lim_{x\to-\infty} \frac{3-\frac{4}{x^2}}{\frac{5}{x}-2} = -\frac{3}{2}$$

## Piecewise Function

$\lim_{x\to-2} g(x)$ where $g(x) = \begin{cases} x^2+5 & \text{if } x < -2 \\ 1-3x & \text{if } x \ge -2 \end{cases}$

Compute two one sided limits,

$$\lim_{x\to-2^-} g(x) = \lim_{x\to-2^-} x^2+5 = 9$$

$$\lim_{x\to-2^+} g(x) = \lim_{x\to-2^+} 1-3x = 7$$

One sided limits are different so $\lim_{x\to-2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x\to-2} g(x)$ would have existed and had the same value.

# Some Continuous Functions

Partial list of continuous functions and the values of $x$ for which they are continuous.

1. Polynomials for all $x$.
2. Rational function, except for $x$'s that give division by zero.
3. $\sqrt[n]{x}$ ($n$ odd) for all $x$.
4. $\sqrt[n]{x}$ ($n$ even) for all $x \ge 0$.
5. $\mathbf{e}^x$ for all $x$.
6. $\ln x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all $x$.
8. $\tan(x)$ and $\sec(x)$ provided $x \ne \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \ne \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

# Intermediate Value Theorem

Suppose that $f(x)$ is continuous on [$a$, $b$] and let $M$ be any number between $f(a)$ and $f(b)$. Then there exists a number $c$ such that $a < c < b$ and $f(c) = M$.

# Limits of functions

$\emptyset \neq A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ | $\lim\limits_{x\to a} f(x)$, $a \in A' \to$ the set of the accumulation (points in $\overline{\mathbb{R}}$) of $A$

The set of the accumulation points of $A$ is the collection of all the limits of the sequences in $A\setminus\{a\}$, who have a limit.

## Characterisation theorem for the accumulation points

$A \neq \emptyset \subseteq \mathbb{R}$ | Then $a \in A' \Longleftrightarrow \exists (a_n) \subseteq A\setminus\{a\}$ with $\lim\limits_{n\to\infty} a_n = a$ (*)

Proof:

"$\Rightarrow$"

We know that $a \in A'$

$\overset{def}{\Longleftrightarrow} \forall V \in \mathcal{V}(a)$, $V \cap A\setminus\{a\} \neq \emptyset$

CASE 1: $a \in \mathbb{R}$

$\left.\begin{array}{l} \overset{def}{\Longleftrightarrow} \forall V \in \mathcal{V}(a),\ V\cap A\setminus\{a\} \neq \emptyset \\ \forall n \in \mathbb{N}\ \ B(a, \frac{1}{n}) \in \mathcal{V}(a) \end{array}\right\} \Rightarrow \forall n \in \mathbb{N}, B(a, \frac{1}{n}) \cap A\setminus\{a\} \neq \emptyset \Leftrightarrow \forall n \in \mathbb{N}, \exists t_n \in B(a, \frac{1}{n}) \cap A\setminus\{a\}$

$\Leftrightarrow \exists (t_n) \subseteq A\setminus\{a\}$ a sequence, $\forall n \in \mathbb{N}$ $t_n \in B(a, \frac{1}{n})$

$\Rightarrow \forall n \in \mathbb{N}$, $0 \leq |t_n - a| \leq \frac{1}{n}$

$\lim\limits_{n\to\infty}(t_n - a) = 0 \Leftrightarrow \lim\limits_{n\to\infty} t_n = a$

CASE 2: $a = \infty \in A'$

$\left.\begin{array}{l} \overset{def}{\Longleftrightarrow} \forall V \in \mathcal{V}(\infty),\ V\cap A\setminus\{\infty\} \neq \emptyset \\ \forall n \in \mathbb{N}\ \ B(\infty, n) \in \mathcal{V}(\infty) \end{array}\right\} \Rightarrow \forall n \in \mathbb{N}, B(\infty, n) \cap A\setminus\{\infty\} \neq \emptyset$

$\Longleftrightarrow \exists (s_n) \subseteq A\setminus\{\infty\}$ a sequence s.t. $\forall n \in \mathbb{N}$, $s_n \in B(\infty, n)$

$\Longleftrightarrow \forall n \in \mathbb{N}$, $s_n > n$

$\Longleftrightarrow \lim\limits_{n\to\infty} s_n = \infty \Rightarrow$ (*) true

CASE 3: $a = -\infty \in A'$

$\left.\begin{array}{l} \overset{def}{\Longleftrightarrow} \forall V \in \mathcal{V}(-\infty),\ V\cap A\setminus\{-\infty\} \neq \emptyset \\ \forall n \in \mathbb{N}\ \ B(-\infty, n) \in \mathcal{V}(-\infty) \end{array}\right\} \Rightarrow \forall n \in \mathbb{N}, B(-\infty, n) \cap A\setminus\{-\infty\} \neq 0$

$\Longleftrightarrow \exists (s_n) \subseteq A\setminus\{-\infty\}$ a sequence s.t. $\forall n \in \mathbb{N}$, $s_n \in B(-\infty, n)$

$\Longleftrightarrow \forall n \in \mathbb{N}$, $s_n < -n$

$\Longleftrightarrow \lim\limits_{n\to\infty} s_n = -\infty \Rightarrow$ (*) true

"$\Leftarrow$"

We know (*)

We want: $a \in A' \iff \forall V \in \mathcal{V}(a),\ V \cap A \setminus \{a\} \neq \emptyset$

Choose $V \in \mathcal{V}(a)$ randomly

(*) we know $\underline{\exists (a_n) \subseteq A \setminus \{a\}}$ s.t. $\lim_{n\to\infty} a_n = a$

$V \in \mathcal{V}(a)$

$\overset{def}{\Longrightarrow} \exists n_V \in \mathbb{N}$ s.t. $\forall n \geq n_V$ it holds $\underline{a_n \in V}$

$$\left.\begin{array}{l} a_{n_V} \in V \\ a_{n_V} \in A \setminus \{a\} \end{array}\right\} \Rightarrow a_{n_V} \in V \cap A \setminus \{a\} \Rightarrow$$

$\Rightarrow V \cap A \setminus \{a\} \neq \emptyset$

$V$ chosen randomly $\Rightarrow a \in A'$ true

# Limit at a point

$\emptyset \neq A \subseteq \mathbb{R}$
$f: A \to \mathbb{R}$
$a \in A'$ (in $\mathbb{R}$)

$f$ is said to have a limit at $a$ if

$\forall (a_n) \subseteq A \setminus \{a\}$ with $\lim_{n\to\infty} a_n = a$, it $\exists \lim_{n\to\infty} f(a_n) \in \overline{\mathbb{R}}$

## Theorem of the limits uniques

If a function has a limit at a point, it is **unique**.

$\exists \lim_{x\to a} f(x) \in \overline{\mathbb{R}} \iff \exists\, l \in \overline{\mathbb{R}}$ s.t. $\forall (a_n) \subseteq A \setminus \{a\}$ with $\lim_{n\to\infty} a_n = a$ it holds that $\lim_{n\to\infty} f(a_n) = l$

Each function has <u>at most</u> a unique limit at a point $a \in A'$, denoted by $\lim_{x\to a} f(x) \in \overline{\mathbb{R}}$.

Proof:

$a \in A' \overset{\text{CHAR.}}{\underset{\text{T. OF AC.P.}}{\iff}} \exists (t_n) \subseteq A \setminus \{a\}$ with $\lim_{n\to\infty} t_n = a$

$\overset{\text{def}}{\Downarrow}$ $f$ has a limit at $a$ if

$\exists \underbrace{\lim_{n\to\infty} f(t_n)}_{\text{not } l} \in \overline{\mathbb{R}}$

Consider a diffe. sequence $(u_n) \subseteq A \setminus \{a\}$ with $\lim_{n\to\infty} u_n = a$

$\overset{\text{def}}{\Downarrow}$ $f$ has a limit at $a$ if

$\exists \underbrace{\lim_{n\to\infty} f(u_n)}_{\text{not } y} \in \overline{\mathbb{R}}$

Assuming that $l \neq y$ we construct the sequence $z_n := \begin{cases} t_n : n = 2k \\ u_n \cdot n = 2k-1 \end{cases}, k \in \mathbb{N}$

$\Rightarrow \exists \lim_{n\to\infty} z_n = a$. In conclusion $(z_n) \subseteq A \setminus \{a\}$ and $\lim_{n\to\infty} z_n = a$

$\overset{\text{def}}{\Downarrow}$

$\exists \underbrace{\lim_{n\to\infty} f(z_n)}_{\text{not } z} \in \overline{\mathbb{R}}$

$(f(t_n)) = (f(z_{2k}))$ and $(f(u_n)) = (f(z_{2k-1}))$ are subsequences of $(f(z_n))$

$\Rightarrow z = y = l \Rightarrow \lightning\ y = l \rightarrow$ the limit is <u>UNIQUE</u>

## Characterization theorem with neighbourhoods of the limit

$\emptyset \neq A \subseteq \mathbb{R}$
$f: A \to \mathbb{R}$
$a \in A'$

$\ell = \lim_{x \to a} f(x) \in \overline{\mathbb{R}} \iff \forall V \in v(\ell), \ \exists U \in v(a)$ s.t.

$\forall x \in U \cap A \setminus \{a\}$ it holds $f(x) \in V$

This theorem is split in 9 ≠ cases when $\begin{cases} V \text{ is replaced } B(\ell, \varepsilon) \ (\ell \in \mathbb{R}, +\infty, -\infty) \\ U \text{ is replaced } B(a, \delta) \ (a \in \mathbb{R}, -\infty, +\infty) \end{cases}$

## Characterization theorem with $\varepsilon$ and $\delta$ of the limit

$\emptyset \neq A \subseteq \mathbb{R}$
$f: A \to \mathbb{R}$
$a \in A'$

$\ell = \lim_{x \to a} f(x) \iff \forall \varepsilon > 0, \ \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with

a) $a, \ell \in \mathbb{R} \to |x-a| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$

b) $a \in \mathbb{R}, \ell = \infty \to |x-a| < \delta \Rightarrow f(x) > \varepsilon$

c) $a \in \mathbb{R}, \ell = -\infty \to |x-a| < \delta \Rightarrow f(x) < -\varepsilon$

d) $a = +\infty, \ell = -\infty \to x > \delta \Rightarrow f(x) < -\varepsilon$

e) $a = -\infty, \ell = \infty \to x < -\delta \Rightarrow |f(x) - \ell| < \varepsilon$

## Theorem with the side limits

$\exists \lim_{\substack{x \to a \\ x < a}} f(x)$ and $\lim_{\substack{x \to a \\ x > a}} f(x)$ and $= \ell = f(a) \Rightarrow \exists \lim_{x \to a} f(x) = \ell$

# Limits of Functions

Recall the following:

$$\lim_{x\to\infty} q^x = \begin{cases} +\infty & : q > 1 \\ 1 & : q = 1 \\ 0 : & |q| < 1 \\ \nexists & : q \leq 1 \end{cases}$$

$$\lim_{x\to\infty} q^x = q^{x_0}, \forall q \in (0,\infty) \quad and \quad x_0 \in \mathbb{R}$$

$$\lim_{x\to x_0} \log_a x = \log_a x_0, \forall a \in (0,\infty) \setminus \{1\}, x_0 > 0.$$

$$\lim_{x\to\infty} \log_a x = \begin{cases} +\infty & : a > 1 \\ -\infty & : 0 < a < 1 \end{cases}$$

$$\lim_{x\to 0} \frac{q^x - 1}{x} = \ln q, \forall q > 0 \quad and \quad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

**Exercise 1:** Compute the limits of the following functions at the specified points:

$$a)\lim_{x\to\infty} x\cos^2\frac{x+2}{x} \quad b)\lim_{x\to 1}\frac{x}{x^2+1} \quad c)\lim_{x\to-\infty}\frac{x^2+5}{x^3} \quad d)\lim_{x\to\infty}\frac{(x+2)(2x+1)}{x^2+3x+5}$$

$$e)\lim_{x\to 1}\frac{x^2-1}{x^3-1} \quad f)\lim_{x\to 2}\left(\frac{1}{2-x}-\frac{2x}{4-x^2}\right)$$

$$g)\lim_{x\to 1}\frac{1+x+x^2+...+x^n-(n+1)}{x-1}, n\in\mathbb{N} \quad h)\lim_{x\to 1}\frac{x+x^2+...+x^n-n}{x+x^2+...+x^m-m}, \forall m,n\in\mathbb{N}.$$

$$i)\lim_{x\to 27}\frac{x-27}{\sqrt[3]{x}-3} \quad j)\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

$$k)\lim_{x\to\infty}\left(\sqrt[3]{ax^3+x^2+bx+c}-(bx+c)\right)\forall a,b,c>0.$$

**Exercise 2:** Compute the limits of the following functions at the specified points:

$$a)\lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}} \quad b)\lim_{x\to 0}\left(\frac{3\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}}$$

$$c)\ \lim_{x\to 0}(1+\cos x)^{\frac{1}{x^2}} \quad d)\ \lim_{x\to 0}(e^x - x + 1)^{\frac{1}{1-\cos x}}$$

$$e)\ \lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} \quad f)\ \lim_{x\to\infty}\left(\frac{x+7}{x}\right)^x$$

## Exercise 3:

$$a)\ \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sin^2 x+\sin^2 2x+\ldots+\sin^2 nx\right)^{\frac{1}{n^3x^2}}\right]$$

$$a)\ \lim_{n\to\infty}\left[\lim_{x\to 0}(1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx))^{\frac{1}{n^2x}}\right]$$

**Exercise 4:** Compute the following limits:

$$a)\ \lim_{x\to 0}\frac{e^{2x}-1}{3x}; \quad b)\ \lim_{x\to 0}\frac{e^x-\cos x}{3x}.$$

**Exercise 1:** Compute the limits of the following functions at the specified points:

$a)\ \lim\limits_{x\to\infty} x\cos^2\dfrac{x+2}{x}$

$$\lim_{x\to\infty} x\cos^2\left(\frac{x+2}{x}\right) = \lim_{x\to\infty} \underset{\downarrow\ \infty}{x}\cdot\cos^2\overbrace{\left(1+\underset{\downarrow\ 0}{\frac{2}{x}}\right)}^{=1} = \infty\cdot\underbrace{\cos^2 1}_{>0} = \infty$$

$b)\ \lim\limits_{x\to 1}\dfrac{x}{x^2+1}$

$$\lim_{x\to 1}\frac{x}{x^2+1} = \frac{1}{1^2+1} = \frac{1}{2}$$

$c)\ \lim\limits_{x\to -\infty}\dfrac{x^2+5}{x^3}$

$$\lim_{x\to-\infty}\frac{x^2+5}{x^3} = \lim_{x\to-\infty}\frac{\cancel{x^2}}{x^{\cancel{3}}}+\frac{5}{x^3} = \lim_{x\to-\infty}\frac{1}{x}+\frac{5}{x^3} = \frac{1}{-\infty}+\frac{5}{-\infty} = 0+0 = 0$$

$d)\ \lim\limits_{x\to\infty}\dfrac{(x+2)(2x+1)}{x^2+3x+5}$

$$\lim_{x\to\infty}\frac{(x+2)(2x+1)}{x^2+3x+5} = \lim_{x\to\infty}\frac{2\underline{x^2}+5x+2}{\underline{x^2}+3x+5} = 2$$

$e)\ \lim\limits_{x\to 1}\dfrac{x^2-1}{x^3-1}$

$$\lim_{x\to 1}\frac{x^2-1}{x^3-1} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x^2+x+1)} = \lim_{x\to 1}\frac{(x+1)}{(x^2+x+1)} = \frac{1+1}{1^2+1+1} = \frac{2}{3}$$

$a^2-b^2 = (a-b)(a+b)$

$a^3\pm b^3 = (a\pm b)(a^2\mp ab+b^2)$

$(a\pm b)^3 = a^3\pm b^3\pm 3ab(a\pm b)$

$(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

$f)\ \lim\limits_{x\to 2}\left(\dfrac{1}{2-x}-\dfrac{2x}{4-x^2}\right)$

$$\lim_{x\to 2}\left(\frac{1}{2-x}-\frac{2x}{4-x^2}\right) = \lim_{x\to 2}\left(\frac{\overset{(2+x)}{1}}{2-x}-\frac{2x}{(2-x)(2+x)}\right) = \lim_{x\to 2}\frac{2+x-2x}{(2-x)(2+x)} = \lim_{x\to 2}\frac{\cancel{2-x}}{\cancel{(2-x)}(2+x)} = \lim_{x\to 2}\frac{1}{2+x} = \frac{1}{4}$$

$g)\ \lim\limits_{x\to 1}\dfrac{1+x+x^2+...+x^n-(n+1)}{x-1}, n\in\mathbb{N}$

$$\lim_{x\to 1}\frac{1+x+x^2+\ldots+x^n-(n+1)}{x-1} = \lim_{n\to\infty}\frac{(1-1)+(x-1)+(x^2-1)+\ldots+(x^n-1)}{x-1} =$$

$n\in\mathbb{N}$

$$= \cancel{x}+\cancel{x^2}+\cancel{x^3}+\ldots+\cancel{x^{n-1}}+x^n-1-\cancel{x}-\cancel{x^2}-\ldots-\cancel{x^{n-1}} = x^n-1$$

$$= \lim_{x\to 1} \frac{(x-1) + (x-1)(x+1) + (x-1)(x^2+x+1) + \ldots + (x-1)(1+x+x^2+\ldots+x^{n-1})}{x-1} =$$

$$= \lim_{x\to 1} \frac{(x-1)}{(x-1)}\left(x+1+x^2+x+1+\ldots+1+x+x^2+\ldots+x^{n-1}\right) =$$

$$= \lim_{x\to 1}\left(1+1+x+1+x+x^2+\ldots+1+x+x^2+\ldots+x^{n-1}\right) =$$

$$= \underbrace{1}_{1}+\underbrace{1+1}_{2}+\underbrace{1+1+1}_{3}+\ldots+\underbrace{1+1+1+\ldots+1}_{n} =$$

$$= 1+2+3+\ldots+n =$$

$$= \frac{(1+n)\cdot n}{2} =$$

$$= \frac{n^2+n}{2},\ \forall n\in\mathbb{N}$$

$h)\ \lim\limits_{x\to 1} \dfrac{x+x^2+...+x^n-n}{x+x^2+...+x^m-m}, \forall m,n\in\mathbb{N}.$

$$\lim_{x\to 1}\frac{x+x^2+\ldots+x^n-n}{x+x^2+\ldots+x^m-m} = \lim_{x\to 1}\frac{(x-1)+(x^2-1)+\ldots+(x^n-1)}{(x-1)+(x^2-1)+\ldots+(x^m-1)} = \lim_{x\to\infty}\frac{\cancel{(x-1)}(1+x+1+x^2+x+1+\ldots+1+x+x^2+\ldots+x^{n-1})}{\cancel{(x-1)}(1+x+1+x^2+x+1+\ldots+1+x+x^2+\ldots+x^{m-1})}$$

$\forall m,n\in\mathbb{N}$

$$= \lim_{x\to 1}\frac{(1+1+x+1+x+x^2+\ldots+1+x+x^2+\ldots+x^{n-1})}{(1+1+x+1+x+x^2+\ldots+1+x+x^2+\ldots+x^{m-1})}$$

$$= \frac{\overbrace{1}^{1}+\overbrace{1+1}^{2}+\overbrace{1+1+1}^{3}+\ldots+\overbrace{1+1+\ldots+1}^{n\ \text{ori}}}{\underbrace{1}_{1}+\underbrace{1+1}_{2}+\underbrace{1+1+1}_{3}+\ldots+\underbrace{1+1+\ldots+1}_{m\ \text{ori}}} =$$

$$= \frac{1+2+\ldots+n}{1+2+\ldots+m} = \frac{(1+n)\cdot n}{\cancel{2}}\cdot\frac{\cancel{2}}{(1+m)m} = \frac{(1+n)\cdot n}{(1+m)m}\quad \forall m,n\in\mathbb{N}$$

$i)\ \lim\limits_{x\to 27} \dfrac{x-27}{\sqrt[3]{x}-3}$

$$\lim_{x\to 27}\frac{x-27}{\sqrt[3]{x}-3} = \lim_{x\to 27}\frac{x-27}{\sqrt[3]{x}-\sqrt[3]{27}} = \lim_{x\to 27}\frac{((\sqrt[3]{x})^2+9-3\sqrt[3]{x})\cancel{(x-27)}}{\cancel{x-27}} =$$

$(\sqrt[3]{x}-\sqrt[3]{9})$

$(\sqrt[3]{x})^2+(\sqrt[3]{27})^2-\sqrt[3]{x}\sqrt[3]{27}$

$a^3-b^3=(a-b)(a^2-b^2+ab)$

$$= \lim_{x\to 27}(\sqrt[3]{x})^2+9-3\sqrt[3]{x} =$$

$$= 9+9-3\cdot 3 =$$

$$= 9$$

$j)\ \lim\limits_{x\to 1} \dfrac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

l.c.m.$(3,4)=12$

$t^{12}=x$

$t=\sqrt[12]{x}$

$$\Rightarrow \lim_{t\to 1}\frac{\sqrt[3]{t^{12}}-1}{\sqrt[4]{t^{12}}-1} = \lim_{t\to 1}\frac{t^4-1}{t^3-1} = \frac{\cancel{(t-1)}(t^3+t^2+t+1)}{\cancel{(t-1)}(t^2+t+1)} = \frac{4}{3}$$

$k)\ \lim_{x\to\infty}\left(\sqrt[3]{ax^3+x^2+bx+c}-(bx+c)\right)\ \forall a,b,c>0.$

$$\lim_{x\to\infty}\left(\sqrt[3]{ax^3+x^2+bx+c}-(bx+c)\right)=\lim_{x\to\infty} x\left[\sqrt[3]{a+\frac{1}{x}+\frac{b}{x^2}+\frac{c}{x^3}}-b-\frac{c}{x}\right]=$$

$\forall a,b,c>0$

$$=\infty\cdot\left(\sqrt[3]{a}-b\right)=\begin{cases}\infty: & \sqrt[3]{a}>b\\ \infty\cdot 0: & \sqrt[3]{a}=b\iff \underline{a=b^3}\\ \infty: & \sqrt[3]{a}<b\end{cases}$$

CASE: $a=b^3$

$$\lim_{x\to\infty}\left(\sqrt[3]{b^3x^3+x^2+bx+c}-(bx+c)\right]=\lim_{x\to\infty}\left(\underbrace{\sqrt[3]{b^3x^3+x^2+bx+c}}_{u}-\underbrace{\sqrt[3]{(bx+3)^3}}_{v}\right)=$$

$$=\lim_{x\to\infty}\frac{u^3-v^3}{u^2+uv+v^2}=\lim_{x\to\infty}\frac{b^3x^3+x^2+bx+c-(bx+c)^3}{\sqrt[3]{(b^3x^3+x^2+bx+c)^2}+\sqrt[3]{(b^3x+x^2+bx+c)(bx+c)^3}+\sqrt[3]{(bx+c)^{3\cdot 2}}}$$

$$=\lim_{x\to\infty}\frac{\cancel{b^3x^3}+x^2+bx+c-(\cancel{b^3x^3}+3b^2x^2c+3bxc^2+c^3)}{x^2\left(\underset{\substack{\downarrow\\0}}{\sqrt[3]{b^6+\dots}}+\underset{\substack{\downarrow\\0}}{\sqrt[3]{b^6+\dots}}+\underset{\substack{\downarrow\\0}}{\sqrt[3]{b^6+\dots}}\right)}=\begin{cases}\frac{1-3b^2c}{3b^2} & : 1-3b^2c=0\\ 0 & : c=\frac{1}{3b}\end{cases}$$

**Exercise 2:** Compute the limits of the following functions at the specified points:

$a)\ \lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}}$

$$\lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}}=0^{\frac{5}{2}}=0$$

$b)\ \lim_{x\to 0}\left(\frac{3\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}}$

$$\lim_{x\to 0}\left(\frac{3\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}}=\lim_{x\to 0}\left(3\underbrace{\frac{\sin x}{x}}_{=1}-\underbrace{\frac{\tan x}{x}}_{=1}\right)^{\underbrace{\frac{\sin x}{x}}_{=1}+2}=$$

$$=(3\cdot 1-1)^{1+2}=2^3=8$$

$c)\ \lim_{x\to 0}(1+\cos x)^{\frac{1}{x^2}}$

$$\lim_{x\to 0}(1+\cos x)^{\frac{1}{x^2}}=(1+1)^{\frac{1}{0+}}=2^{\infty}=\infty$$

$d)\ \lim_{x\to 0}(e^x-x+1)^{\frac{1}{1-\cos x}}$

$$\lim_{x\to 0}(e^x-x+1)^{\frac{1}{\underbrace{1-\cos x}_{=0}}}=\lim_{x\to 0}(e^x-x+1)^{\frac{1}{1-1+2\sin^2\frac{x}{2}}}=\lim_{x\to 0}(e^x-x+1)^{\frac{1}{2\sin^2\frac{x^2}{2}}}=$$

$$=\lim_{x\to 0}(e^x-x+1)^{\frac{1}{\cancel{2}\frac{\sin^2\frac{x^2}{2}}{\frac{x^2}{2}}\cdot\frac{x^2}{\cancel{2}}}}=\lim_{x\to 0}(e^x-x+1)^{\frac{1}{x^2}}=(1+0+1)^{\frac{1}{0+}}=2^{\infty}=\infty$$

$e)\ \lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$

$$\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} = \lim_{x\to\infty}\left((1+\sin x)^{\frac{1}{\sin x}}\right)^{\sin x\cdot\frac{1}{x}} = e^{\lim\limits_{x\to\infty}\frac{\sin x}{x}} = e^1 = e$$

$f)\ \lim\limits_{x\to\infty}\left(\frac{x+7}{x}\right)^x$

$$\lim_{x\to\infty}\left(\frac{x+7}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^{\frac{x}{7}\cdot\frac{7}{x}\cdot x} = e^{\lim\limits_{x\to\infty}\frac{7\cancel{x}}{\cancel{x}}} = e^7$$

## Exercise 3:

$$a)\ \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sin^2 x+\sin^2 2x+\ldots+\sin^2 nx\right)^{\frac{1}{n^3x^2}}\right]$$

$$a)\ \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx)\right)^{\frac{1}{n^2x}}\right]$$

## Exercise 4: Compute the following limits:

$$a)\ \lim_{x\to 0}\frac{e^{2x}-1}{3x};\quad b)\ \lim_{x\to 0}\frac{e^x-\cos x}{3x}.$$

1) $\lim\limits_{n\to\infty} \dfrac{1^5+2^5+\dots+n^5}{n^6}$

$$\lim_{n\to\infty} \frac{1^5+2^5+\dots+n^5}{n^6} = \lim_{n\to\infty} \frac{\cancel{n^2}(n+1)^2(2n^2+2n-1)}{12\, n^{\cancel{6}}} = \lim_{n\to\infty} \frac{(n+1)^2(2n^2+2n-1)}{12\, n^3} = \frac{\overbrace{(n^2+2n+1)(2n^2+2n-1)}^{\cancel{2n^4}}}{\underset{6}{\cancel{12}}\,\cancel{n^3}} = \lim_{x\to\infty} x = \infty$$

$$\sum_{k=0}^{n} k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

$$\lim_{n\to\infty} \frac{\cancel{n^5}\left(\overset{\to 0}{\frac{1^5}{n^5}} + \overset{\to 0}{\frac{2^5}{n^5}} + \dots + 1\right)}{\cancel{n^5}\cdot n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

2) *Define and give an example for the limit of a function at a point $a \in \overline{\mathbb{R}}$.*

Let $\emptyset \neq A \subseteq \mathbb{R}$
$f : A \to \mathbb{R}$
$a \in A'$ (in $\mathbb{R}$).

$f$ is said to have a limit at $a$ if $\forall (a_n) \subseteq A\setminus\{a\}$ with $\lim\limits_{n\to\infty} a_n = a$,
$\exists \lim\limits_{n\to\infty} f(a_n) \in \overline{\mathbb{R}}$.

Let $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \frac{1}{x}$ a function

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$$

# STUDY LIMITS

e.g. $f: A \to \mathbb{R}$

$f(x) = \frac{1}{x}$

Study $\lim_{x \to a} f(x)$ for each $a \in A' \setminus A$

$A = \mathbb{R} \setminus \{0\} = \mathbb{R}^* \Rightarrow A' = \overline{\mathbb{R}} \Rightarrow A' \setminus A = \{\infty, 0, -\infty\}$

$\underline{\mathrm{I}}$ $a = 0$ $\quad \lim_{x \to 0} \frac{1}{x}$

We will prove $\nexists \lim_{x \to 0} \frac{1}{x}$ with the help of the definition

Choose: $a_n := \frac{1}{n} \quad \forall n \in \mathbb{N} \Rightarrow \lim_{n \to \infty} a_n = 0$

and

$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \frac{1}{a_n} = \lim_{n \to \infty} \frac{1}{\frac{1}{n}} = \lim_{n \to \infty} n = \infty$

$b_n := \frac{-1}{n} \quad \forall n \in \mathbb{N} \Rightarrow \lim_{n \to \infty} b_n = 0$

and

$\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} \frac{1}{-\frac{1}{n}} = \lim_{n \to \infty} -n = -\infty$

$\neq$

$\Rightarrow \nexists \lim_{x \to 0} f(x)$

$\boxed{\mathrm{I}}$ $\boxed{a = -\infty}$

Consider a random sequence $(a_n) \subseteq \mathbb{R}^* \setminus \{-\infty\} = \mathbb{R}^*$ with $\lim_{n \to \infty} a_n = -\infty$.

$? \ \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \frac{1}{a_n} = \frac{1}{-\infty} = 0 \ (\exists)$

$(a_n)$ - random

$\xRightarrow{\text{def}} \exists \lim_{x \to -\infty} f(x) = 0.$

$\underline{\mathrm{II}}$ $\boxed{a = \infty}$ - similarly $\exists \lim_{x \to \infty} f(x) = 0.$

Study. $\lim_{x \to \infty} \sin x$ and $\lim_{x \to 0} \sin x$.

⊙ $\boxed{\lim_{x \to 0} \sin x = 0}$ . Choose $(a_n) \subseteq \mathbb{R}$ with $\lim_{n \to \infty} a_n = 0$

$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \sin a_n = \sin 0 = 0$

$(a_n)$ random

$\Rightarrow \exists \lim_{x \to 0} \sin x$

⊙ $\nexists \lim_{x \to \infty} \sin x$. Choose • $(a_n) \subseteq \mathbb{R}$ $\quad a_n = 2n\pi$ $\quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} 2n\pi = \infty$ ✓

$f(a_n) = \sin a_n = \sin(2n\pi) = \cancel{0} \xrightarrow[n \to \infty]{} 0$

• $(b_n) \subseteq \mathbb{R}$ $\quad b_n = 2n\pi + \frac{\pi}{2}$ $\quad \lim_{n \to \infty} b_n = \infty$

$f(b_n) = \sin(b_n) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$

$\Rightarrow \lim_{n \to \infty} f(b_n) = 1$

$\neq$

$\Rightarrow \nexists \lim_{x \to \infty} f(x)$.

1) Study the limit in $a$, for each $a \in A' \setminus A$, in the given functions:

a) $f : A \to \mathbb{R}$

$f(x) = \frac{1}{x}$

$A = \mathbb{R} \setminus \{0\} = \mathbb{R}^* \Rightarrow A' = \overline{\mathbb{R}} \Rightarrow A' \setminus A = \{-\infty, 0, \infty\}$

I $a = -\infty$

Consider a random sequence $(a_n) \subseteq \mathbb{R}^* \setminus \{-\infty\} = \mathbb{R}^*$ with $\lim\limits_{n \to \infty} a_n = -\infty$

$\left.\begin{array}{l} \lim\limits_{n \to -\infty} f(a_n) = \lim\limits_{n \to -\infty} \frac{1}{a_n} = \frac{1}{-\infty} = 0 \ (\exists) \\ (a_n) \text{ chosen randomly} \end{array}\right\} \overset{\text{def.}}{\Rightarrow} \exists \lim\limits_{n \to -\infty} f(x) = 0$

II $a = \infty$

Similarly, $\exists \lim\limits_{x \to \infty} f(x) = \frac{1}{\infty} = 0$

III $a = 0$

$\lim\limits_{x \to 0} \frac{1}{x} \to$ We will prove $\nexists \lim\limits_{x \to 0} \frac{1}{x}$ with the help of the definition.

Choose: $a_n := \frac{1}{n} \ \forall n \in \mathbb{N} \Rightarrow \lim\limits_{n \to \infty} a_n = 0$ and $\lim\limits_{n \to \infty} f(a_n) = \lim\limits_{n \to \infty} \frac{1}{a_n} =$

$= \lim\limits_{n \to \infty} \frac{1}{\frac{1}{n}} = \lim\limits_{n \to \infty} n = \infty$ (1)

$b_n := -\frac{1}{n} \ \forall n \in \mathbb{N} \Rightarrow \lim\limits_{n \to \infty} b_n = 0$ and $\lim\limits_{n \to \infty} f(b_n) = \lim\limits_{n \to \infty} -\frac{1}{\frac{1}{n}} =$

$= \lim\limits_{n \to \infty} -n = -\infty$ (2)

(1) and (2) are diff. therefore $\Rightarrow \nexists \lim\limits_{x \to 0} \frac{1}{x}$

**Example**: Study:

- $\boxed{\lim_{x\to 0} x \sin\frac{1}{x}} = 0$
- $\boxed{\lim_{x\to 0} x \sin x} = 0$ $\quad \boxed{(-\infty)\sin(-\infty)} \;\nexists\; \boxed{\infty \sin \infty}$
- $\nexists \lim_{x\to 0} \frac{1}{x} \sin\frac{1}{x}$
- $\boxed{\lim_{x\to 0} \frac{1}{x}\sin x} = \lim_{x\to 0}\frac{\sin x}{x} = 1$

This is a classical exercise in which we use the negation of the definition

$\exists\, (t_n) \subseteq \mathbb{R}$ s.t. $\lim_{n\to\infty} t_n = 0$
$\exists\, (s_n) \subseteq \mathbb{R}$ s.t. $\lim_{n\to\infty} s_n = 0$
but for which $\lim_{n\to\infty} f(t_n) \neq \lim_{n\to\infty} f(s_n)$.

$c_n := \frac{1}{n} \quad \sin\frac{1}{c_n} = \sin\frac{1}{\frac{1}{n}} = \boxed{\sin n} \rightsquigarrow ?$

$t_n = 2n\pi,\ \forall n\in\mathbb{N} \quad \lim_{n\to\infty} t_n = \lim_{n\to\infty} 2n\pi = \infty$, $\quad f(t_n) = (2n\pi)\sin(2n\pi) = 2n\pi\cdot 0 = 0$

$\Rightarrow (f(t_n))_{n\in\mathbb{N}}$ is the constant sequence 0

$s_n = 2n\pi + \frac{\pi}{2},\ \forall n\in\mathbb{N}$

$\Rightarrow \lim_{n\to\infty} f(t_n) = 0$

$\hookrightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2n\pi + \frac{\pi}{2}\right) = \infty$, $\quad f(s_n) = \left(2n\pi + \frac{\pi}{2}\right)\underbrace{\sin\left(2n\pi+\frac{\pi}{2}\right)}_{=1} = 2n\pi + \frac{\pi}{2}$

$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty}\left(2n\pi + \frac{\pi}{2}\right) = \infty$ $\quad \neq$

$\Rightarrow f$ does not have a limit at 0.

$(\nexists)$

**Example 4**: Study:

- $\lim_{x\to\infty} x\sin x = \lim_{t\to 0}\frac{1}{t}\sin\frac{1}{t}$
- $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0}\frac{\sin t}{t}$
- $\lim_{x\to\infty} \frac{1}{x}\sin x = 0$ $\quad \lim_{t\to 0} t\cdot\sin\frac{1}{t} = 0$
- $\lim_{x\to\infty}\frac{1}{x}\sin\frac{1}{x} = 0$

- $\lim_{x\to\infty}\frac{1}{x}\sin\frac{1}{x} = 0$

At limits of functions, pay attention to where $x$ $\boxed{\text{goes}}$