Example 1.4. Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbf{K}$. The map

$$\mathbf{V} \times \mathbf{V} \to \mathbf{V} \quad (\mathbf{a}, \mathbf{b}) = \mathbf{b} - \mathbf{a}$$

defines the structure of an affine space on $\mathbf{V}$ over itself. Thus, every vector space can be considered as an affine space over itself. If we want to view $\mathbf{V}$ as an affine space in this way, we denote it by $\mathbf{V}_a$.

- Axiom AS1 is satisfied:

  $\forall$ point $p \in V$ $\forall$ vector $v \in V$ the point $q = p + v$ is the unique point

  such that $\overrightarrow{pq} = p - q = v$

- Axiom AS2 is satisfied:

$$\overrightarrow{pq} + \overrightarrow{qr} \underset{\text{by. def}}{=} (q-p) + (r-q) \underset{\text{in } V}{=} r - p \underset{\text{by def.}}{=} \overrightarrow{pr}$$

Definition 1.10. Given $n + 1 \geq 2$ points $P_0, \ldots, P_n$ in an affine space $\mathbf{A}$, the affine subspace passing through P_0 and having associated vector subspace $\langle \overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}, \ldots, \overrightarrow{P_0P_n} \rangle$ is called the *subspace generated by* (or *span of*) $P_0, P_1, \ldots, P_n$. We denote it by $\langle P_0, P_1, \ldots, P_n \rangle$.

- In this definition P_0 appears to play a more important role than the other points $P_1, \ldots, P_n$
- The definition does not depend on the numbering of the points

  In other words, if we reshuffle the points we obtain the same affine subspace
- To see this, choose i with $0 \leq i \leq n$ and consider

  the affine space X passing through P_i and having associated vector subspace

$$W = \langle \overrightarrow{P_iP_0}, \overrightarrow{P_iP_2}, \dots, \overrightarrow{P_iP_{i-1}}, \overrightarrow{P_iP_{i+1}}, \dots, \overrightarrow{P_iP_n} \rangle$$

Notice that $\forall j \in \{0, \dots, n\}$

$$\overrightarrow{P_iP_j} = \overrightarrow{P_iP_0} + \overrightarrow{P_0P_j}$$

So $W = \langle \overrightarrow{P_iP_0}, \overrightarrow{P_iP_0} + \overrightarrow{P_0P_2}, \dots, \overrightarrow{P_iP_0} + \overrightarrow{P_0P_{i-1}}, \overrightarrow{P_iP_0} + \overrightarrow{P_0P_{i+1}}, \dots, \overrightarrow{P_iP_0} + \overrightarrow{P_0P_n} \rangle$

- Since $\overrightarrow{P_iP_0} \in W \Rightarrow \overrightarrow{P_0P_i} \in W$

  Since $\overrightarrow{P_iP_0}, \overrightarrow{P_iP_0} + \overrightarrow{P_0P_j} \in W \Rightarrow \overrightarrow{P_0P_j} \in W$

  Therefore $\langle \overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n} \rangle \subseteq W$ (1)

- Since the generators of W are linear combinations of the vectors $\overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n}$ it follows that

$$W \subseteq \langle \overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n} \rangle \quad (2)$$

- By (1) & (2) we have $W = \langle \overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n} \rangle$

- Returning to X, we showed that X and $\langle P_0, \dots, P_n \rangle$ have the same associated vector subspace

$$X = \{ P \in A : \overrightarrow{P_iP} \in W \}$$

$$\langle P_0, \dots, P_n \rangle = \{ P \in A : \overrightarrow{P_0P} \in W \}$$

but $\overrightarrow{P_iP} = \overrightarrow{P_0P} - \overrightarrow{P_iP_0}$ and $\overrightarrow{P_iP_0} \in W$ so $\overrightarrow{P_iP} \in W \Leftrightarrow \overrightarrow{P_0P} \in W$

Therefore $X = \langle P_0, \dots, P_n \rangle$

Proposition 1.11.

1. An affine subspace is determined by its associated vector subspace and any one of its points.
2. Let S be an affine subspace of $\mathbf{A}$ with associated vector subspace $\mathbf{W}$. Associating to any pair of points $P, Q \in S$ the vector $\overrightarrow{PQ}$ defines on S the structure of an affine space over $\mathbf{W}$.

1) • Let A be an affine space with associated vector space V

• Let S be the affine subspace of A passing through Q and having associated vector subspace $W \leq V$

• We have to show that S doesn't change if we change $Q \in S$

• Let $M \in S$ and let T be the aff. subsp. of A with ass. vector subspace W

• We need to show that $T = S$

(1) To see that $S \subseteq T$ notice that $\forall P \in S$

$$\overrightarrow{MP} = \overrightarrow{MQ} + \overrightarrow{QP} = -\underbrace{\overrightarrow{QM}}_{\in W} + \underbrace{\overrightarrow{QP}}_{\in W} \in W \Rightarrow P \in T$$

since $M, P \in S$

(2) To see that $T \subseteq S$ notice that $\forall P \in T$

$$\overrightarrow{QP} = \overrightarrow{QM} + \overrightarrow{MP} = -\underbrace{\overrightarrow{MQ}}_{\in W} + \underbrace{\overrightarrow{MP}}_{\in W} \in W \Rightarrow P \in S$$

since $M, P \in T$

2.) by 1.) $\forall P \in S$ $\quad S = \{ Q \in A : \vec{PQ} \in W \}$

$\Rightarrow \forall P, Q \in S \quad \vec{PQ} \in W$

Therefore, we have a map

$$S \times S \longrightarrow W$$

$$(P, Q) \longmapsto \vec{PQ}$$

which satisfies axioms AS1 and AS2, since they are satisfied in A.

Proposition 1.20. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$. A set $\mathbf{A}$ is an affine space if and only if there is a map $t : \mathbf{A} \times \mathbf{V} \to \mathbf{A}$ satisfying (AS1') and (AS2').

(AS1') For every $A, B \in \mathbf{A}$ there is a unique $\mathbf{a} \in \mathbf{V}$ such that

$$B = t(A, \mathbf{a}).$$

(AS2') For every $A \in \mathbf{A}$ and $\mathbf{a}, \mathbf{b} \in \mathbf{V}$ we have

$$t(t(A, \mathbf{a}), \mathbf{b}) = t(A, \mathbf{a} + \mathbf{b}).$$

1) If A is an affine space then there is a map with the desired properties:

- define $t(P, v) = Q$ where Q is the unique point (AS1) such that $\vec{PQ} = v$

- by (AS1) $\forall A \in A \ \forall a \in V \ \exists! B \in A$ s.t. $\vec{AB} = a$

this describes a bijection between V and A

$a \in V$ determines a unique $B \in A$

$\Leftrightarrow$ $B \in A$ determines a unique $a \in V$

$\Rightarrow \quad \forall A, B \in \mathbb{A} \; \exists!\, a \in \mathbb{V} : \overrightarrow{AB} = \vec{a}$ which is (AS1')

- fix $A \in \mathbb{A}$ and $a, b \in \mathbb{V}$

  $\exists!\, B \in \mathbb{A}$ s.t $t(A, a) = B \iff a = \overrightarrow{AB}$

  $\Rightarrow \exists!\, C \in \mathbb{A}$ st $t(t(A, a), b) = t(B, b) = C \iff b = \overrightarrow{BC}$

- by (AS2) $a + b = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \iff t(A, a+b) = C$

  so (AS2') follows.

2.) Conversely, given a vector space $\mathbb{V}$, a set $\mathbb{A}$ and a map $t : \mathbb{A} \times \mathbb{V} \to \mathbb{A}$ which satisfies (AS1') and (AS2')

there is an affine structure on $\mathbb{A}$:

- define $\mathbb{A} \times \mathbb{A} \to \mathbb{V}$ by $(A, B) \mapsto v \iff t(A, v) = B$
- reading the proof of 1) "backwards" one shows that t satisfies (AS1) and (AS2)