Course 9 April 20, 2021

1. Integration over generalized rectangles (closed cells)

1.1. Definition. A set $T \subseteq \mathbb{R}^n$ is called a generalized rectangle (closed cell) if there exist real numbers $a_1 < b_1$, $a_2 < b_2$, ..., $a_n < b_n$ s.t.

$$T = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

The positive real number, defined by

$$m(T) := (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

is called the measure / content / volume of T

$T = [a_1, b_1]$ $\qquad m(T) = b_1 - a_1$ $\leftarrow$ the length of T

$T = [a_1, b_1] \times [a_2, b_2]$ $\qquad m(T) = (b_1 - a_1)(b_2 - a_2)$ $\leftarrow$ the area of T

$T = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ $\Rightarrow$ $m(T) = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$ $\leftarrow$ the volume of T

1.2. Definition Let $[a, b]$ be an interval, and let $P = (a = x_0 < x_1 < \cdots < x_k = b)$ be a partition of $[a, b]$. We identify P with the set

$$P = \{ [x_0, x_1], \ldots, \underbrace{[x_{j-1}, x_j]}_{:= X_j}, \ldots, [x_{k-1}, x_k] \}$$

$P = \{ x_1, x_2, \dots, x_k \}$

$\|P\| = \max\limits_{1 \le j \le k} (x_j - x_{j-1})$ ← the mesh (norm) of the partition P

Let $T = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be a generalized rectangle in $\mathbb{R}^n$

$P_1 = \{ x_{1,1}, x_{2,1}, \dots, x_{k_1,1} \}$ be a partition of $[a_1, b_1]$

$P_2 = \{ x_{1,2}, x_{2,2}, \dots, x_{k_2,2} \}$ be a partition of $[a_2, b_2]$

$\vdots$

$P_n = \{ x_{1,n}, x_{2,n}, \dots, x_{k_n,n} \}$ —— " —— $[a_n, b_n]$

We set

$$P_1 \times P_2 \times \cdots \times P_n = \{ x_{j_1,1} \times x_{j_2,2} \times \cdots \times x_{j_n,n} \mid j_1 = \overline{1, k_1},\ j_2 = \overline{1, k_2}, \dots, j_n = \overline{1, k_n} \}$$

A set π is said to be a partition of the generalized rectangle T if $\exists$ partitions P_1 of $[a_1, b_1]$, ..., P_n of $[a_n, b_n]$ s.t. $\pi = P_1 \times P_2 \times \cdots \times P_n$.

$$\|\pi\| := \max \{ \|P_1\|, \|P_2\|, \dots, \|P_n\| \}$$

↳ the norm / mesh of π

We number the generalized rectangles $x_{j_1,1} \times x_{j_2,2} \times \cdots \times x_{j_n,n}$ and denote them by $T_1, T_2, \dots, T_p$ ($p = k_1 k_2 \cdots k_n$). We write

$$\pi = \{ T_1, T_2, \dots, T_p \}$$

Example $n=2$ $T=[a_1,b_1]\times[a_2,b_2]$

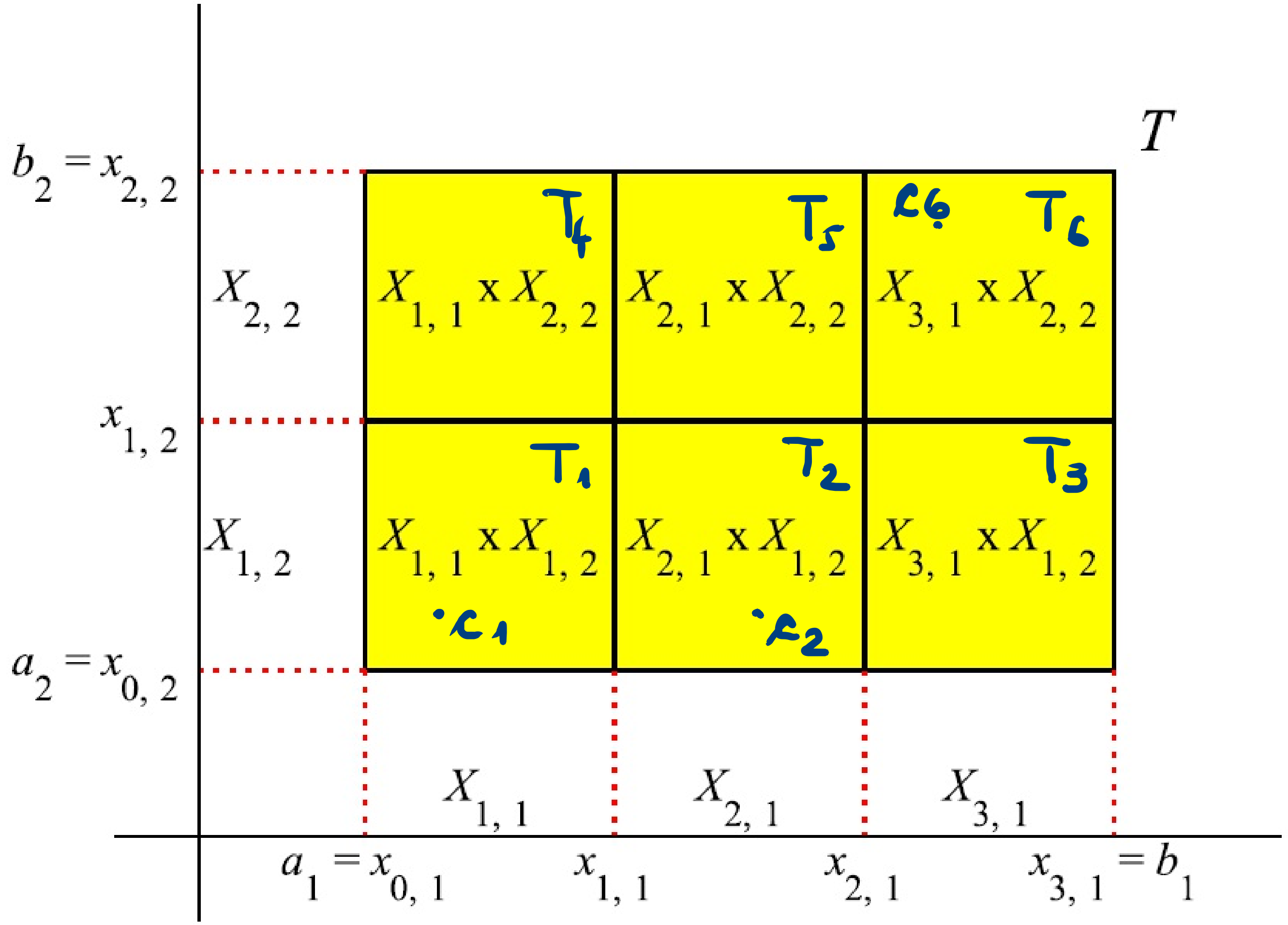
$P_1 = (a_1 = x_{0,1} < x_{1,1} < x_{2,1} < x_{3,1} = b_1)$ partition of $[a_1,b_1]$

$= \{X_{1,1}, X_{2,1}, X_{3,1}\}$

$P_2 = (a_2 = x_{0,2} < x_{1,2} < x_{2,2} = b_2)$ partition of $[a_2,b_2]$

$= \{X_{1,2}, X_{2,2}\}$

$\Rightarrow \pi = P_1 \times P_2 = \{X_{1,1}\times X_{1,2}, X_{2,1}\times X_{1,2}, X_{3,1}\times X_{1,2}, X_{1,1}\times X_{2,2}, X_{2,1}\times X_{2,2}, X_{3,1}\times X_{2,2}\}$ $\rightarrow$ partition of the rectangle T



$\pi = \{T_1, T_2, \ldots, T_6\}$

1.3. Definition Let $T = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a generalized rectangle in $\mathbb{R}^n$.

$\pi = \{T_1, \ldots, T_p\}$ be a partition of T

$\xi = \{c_1, \ldots, c_p\}$ be a set of points in $\mathbb{R}^n$ s.t.

$c_1 \in T_1, \ldots, c_p \in T_p$

The ordered pair $(\pi, \xi) = \dot{\pi}$ is said to be a tagged partition of T.

1.4. Definition Let T be a generalized rectangle in $\mathbb{R}^n$, let (π, ξ) be a tagged partition of T, $\pi = \{T_1, \ldots, T_p\}$, $\xi = \{c_1, \ldots, c_p\}$, and let $f : T \to \mathbb{R}$

We define

$$\sigma(f, \pi, \xi) := \sum_{j=1}^{p} f(c_j) \cdot m(T_j)$$

$\hookrightarrow$ the Riemann sum associated with f and (π, ξ)

1.5. Definition Let $T \subseteq \mathbb{R}^n$ be a generalized rectangle, $f : T \to \mathbb{R}$.

The function f is said to be Riemann integrable over T if $\exists\, I \in \mathbb{R}$ s.t.

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. for every tagged partition (π, ξ) of T, with $\|\pi\| < \delta$ we have $\quad |\sigma(f, \pi, \xi) - I| < \varepsilon$.

If f is Riemann integrable over T $\Rightarrow$ the number I with the above property is unique. It is called the Riemann integral of f over T and is denoted by $\int_T f(x)\,dx$, $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \cdots dx_n$

$n=2$ $\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dx\,dy$ $n=3$ $\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)\,dx\,dy\,dz$

1.6. Theorem. If T is a generalized rectangle in $\mathbb{R}^n$ and $f: T \to \mathbb{R}$ is Riemann integrable over T, then f is bounded.

1.7. Definition. A set $A \subseteq \mathbb{R}$ is said to have Lebesgue measure zero if $\forall \varepsilon > 0$ there exists a sequence (T_k) of generalized rectangles in $\mathbb{R}^n$ s.t.

$$A \subseteq \bigcup_{k=1}^{\infty} T_k \quad \text{and} \quad \sum_{k=1}^{\infty} m(T_k) < \varepsilon.$$

1.8. Theorem (The Lebesgue criterion of Riemann integrability). Let T be a generalized rectangle in $\mathbb{R}^n$ and let $f: T \to \mathbb{R}$. Then

f is Riemann integrable over T $\iff$ $\begin{cases} \cdot\ f \text{ is bounded} \\ \cdot\ \mathrm{disc}(f) \text{ has Lebesgue measure zero} \end{cases}$

$\hookrightarrow$ the set of all discontinuity points of f

2. Computation of multiple integrals by means of iterated integrals

Let $\left.\begin{array}{l} S \subseteq \mathbb{R}^m \\ T \subseteq \mathbb{R}^n \end{array}\right|$ be generalized rectangles $\Rightarrow$ $S \times T$ is a generalized rectangle in $\mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$

$f : S \times T \to \mathbb{R}$

$\forall x \in S$ we consider $f_x : T \to \mathbb{R}$ $\quad f_x(y) := f(x,y)$ $\quad f_x = f(x, \cdot)$

$\forall y \in T$ we consider $f_y : S \to \mathbb{R}$ $\quad f_y(x) := f(x,y)$ $\quad f_y = f(\cdot, y)$

The function f is called partially integrable over $S \times T$ if

$\forall x \in S$ $\quad f_x$ is Riemann integrable over T

$\forall y \in T$ $\quad f_y$ is ———— " ———— S

In this case we can define

$F_1 : S \to \mathbb{R}$ $\qquad F_1(x) := \int_T f_x(y)\, dy = \int_T f(x,y)\, dy$

$F_2 : T \to \mathbb{R}$ $\qquad F_2(y) := \int_S f_y(x)\, dx = \int_S f(x,y)\, dx$

If F_1 is Riemann integrable over S, then we can consider its integral

$$\int_S F_1(x)\,dx = \int_S \left(\int_T f(x,y)\,dy \right) dx$$

If F_2 is Riemann integrable over T, the we can consider its integral

$$\int_T F_2(y)\,dy = \int_T \left(\int_S f(x,y)\,dx \right) dy$$

iterated integrals of f

<u>2.1. Theorem</u> (G. Fubini) Let $S \subseteq \mathbb{R}^m$, $T \subseteq \mathbb{R}^n$ be generalized rectangles $f : S \times T \to \mathbb{R}$ s.t. f is Riemann integrable and partially integrable over $S \times T$.

Then F_1 is Riemann integrable over S

F_2 ——— " ——— T

and

$$\int_{S\times T} f(x,y)\,dx\,dy = \int_S F_1(x)\,dx = \int_T F_2(y)\,dy.$$

<u>Proof</u>. Let $I := \int_{S\times T} f(x,y)\,dx\,dy$.

? F_1 is Riemann integrable over S and $\int_S F_1(x)\,dx = I$

(F_2 is Riemann integrable over T and $\int_T F_2(y)\,dy = I$ $\to$ analogously)

$\Longleftrightarrow$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall (\pi_1, \xi)$ tagged partition of S with $\|\pi_1\| < \delta$ we have

$$|\sigma(F_1, \pi_1, \xi) - I| < \varepsilon$$

Let $\varepsilon > 0$

Since that f is R-integrable over $S \times T$ and $\int_{S \times T} f(x,y)\,dx\,dy = I \Rightarrow$

$\Rightarrow \exists \delta > 0$ s.t. $\forall (\pi, \xi)$ tagged partition of $S \times T$ with $\|\pi\| < \delta$:

$$|\sigma(f, \pi, \xi) - I| < \frac{\varepsilon}{2}$$

Let (π_1, ξ) be an arbitrary tagged partition of S with $\|\pi_1\| < \delta$

$\pi_1 = \{S_1, \dots, S_p\}$ $\quad \xi = \{a_1, \dots, a_p\}$ $\quad a_j \in S_j$, $j = \overline{1,p}$

Since f is partially integrable over $S \times T \Rightarrow$

$\Rightarrow \forall i \in \{1, \dots, p\}$ $\quad f_{a_i}$ is R-integrable over T and $\int_T f_{a_i}(y)\,dy = F_1(a_i)$

$\Rightarrow \forall i \in \{1, \dots, p\}$ $\exists \delta_i > 0$ s.t. $\forall (\pi_2, \eta)$ tagged partition of T with $\|\pi_2\| < \delta_i$ we have

$$|\sigma(f_{a_i}, \pi_2, \eta) - F_1(a_i)| < \varepsilon' = \frac{\varepsilon}{2m(S)}$$

Choose a tagged partition (π_2, η) of T s.t. $\|\pi_2\| < \min\{\delta, \delta_1, \dots, \delta_p\}$

$\pi_2 = \{T_1, \dots, T_q\}$ $\quad \eta = \{b_1, \dots, b_q\}$ $\quad b_j \in T_j$ $\quad j = \overline{1,q}$

Let $\quad \pi = \{ S_i \times T_j \mid i=\overline{1,p},\ j=\overline{1,q} \}$

$\quad\quad \xi = \{ (a_i, b_j) \mid i=\overline{1,p},\ j=\overline{1,q} \}$

$\Rightarrow (\pi, \xi)$ is a tagged partition of $S \times T$ and $\|\pi\| < \underbrace{\delta}_{< \frac{\varepsilon}{2}}$

We have

$$|\sigma(F_1, \pi_1, \xi) - I| \le |\sigma(F_1, \pi_1, \xi) - \sigma(f, \pi, \xi)| + \overbrace{|\sigma(f, \pi, \xi) - I|}^{< \frac{\varepsilon}{2}}$$

$$< \frac{\varepsilon}{2} + \Big| \sum_{i=1}^{p} F_1(a_i)\, m(S_i) - \sum_{i=1}^{p} \sum_{j=1}^{q} \underbrace{f(a_i, b_j)}_{= f_{a_i}(b_j)}\, \underbrace{m(S_i \times T_j)}_{= m(S_i)\, m(T_j)} \Big|$$

$$\Rightarrow |\sigma(F_1, \pi_1, \xi) - I| < \frac{\varepsilon}{2} + \Big| \sum_{i=1}^{p} \Big(F_1(a_i) - \underbrace{\sum_{j=1}^{q} f_{a_i}(b_j)\, m(T_j)}_{} \Big) m(S_i) \Big|$$

$$\le \frac{\varepsilon}{2} + \sum_{i=1}^{p} \underbrace{\Big| F_1(a_i) - \overset{\shortparallel}{\sigma}(f_{a_i}, \pi_2, \eta) \Big|}_{< \varepsilon'} m(S_i)$$

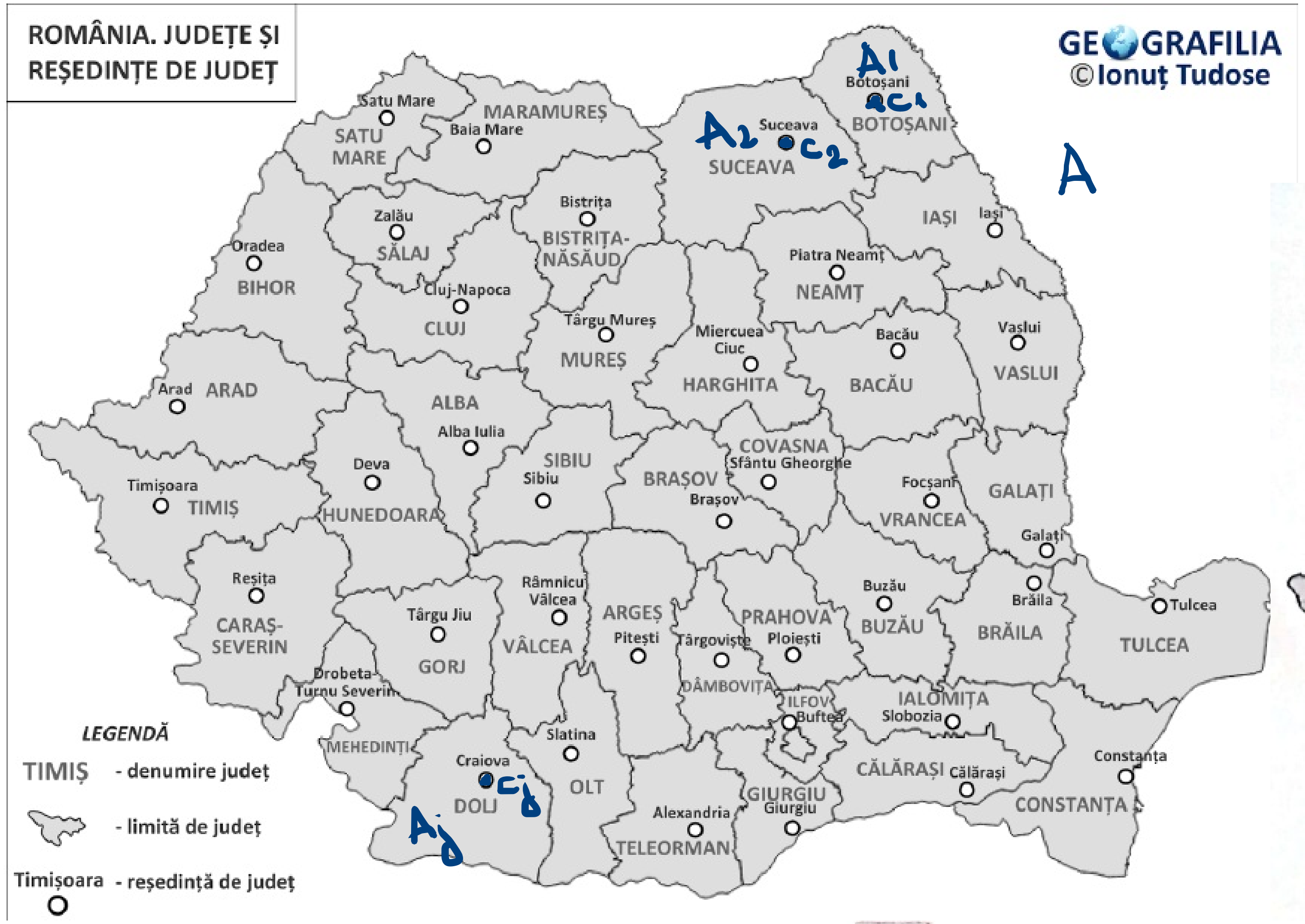
$$< \frac{\varepsilon}{2} + \varepsilon' \underbrace{\sum_{i=1}^{p} m(S_i)}_{= m(S)} = \frac{\varepsilon}{2} + \varepsilon'\, m(S) = \varepsilon$$

__Corollary__ If $S \subseteq \mathbb{R}^m$, $T \subseteq \mathbb{R}^n$ are generalized rectangles, $f: S\times T \to \mathbb{R}$ is a continuous function, then

$$\int_{S\times T} f(x,y)\,dx\,dy = \int_S F_1(x)\,dx = \int_T F_2(y)\,dy$$

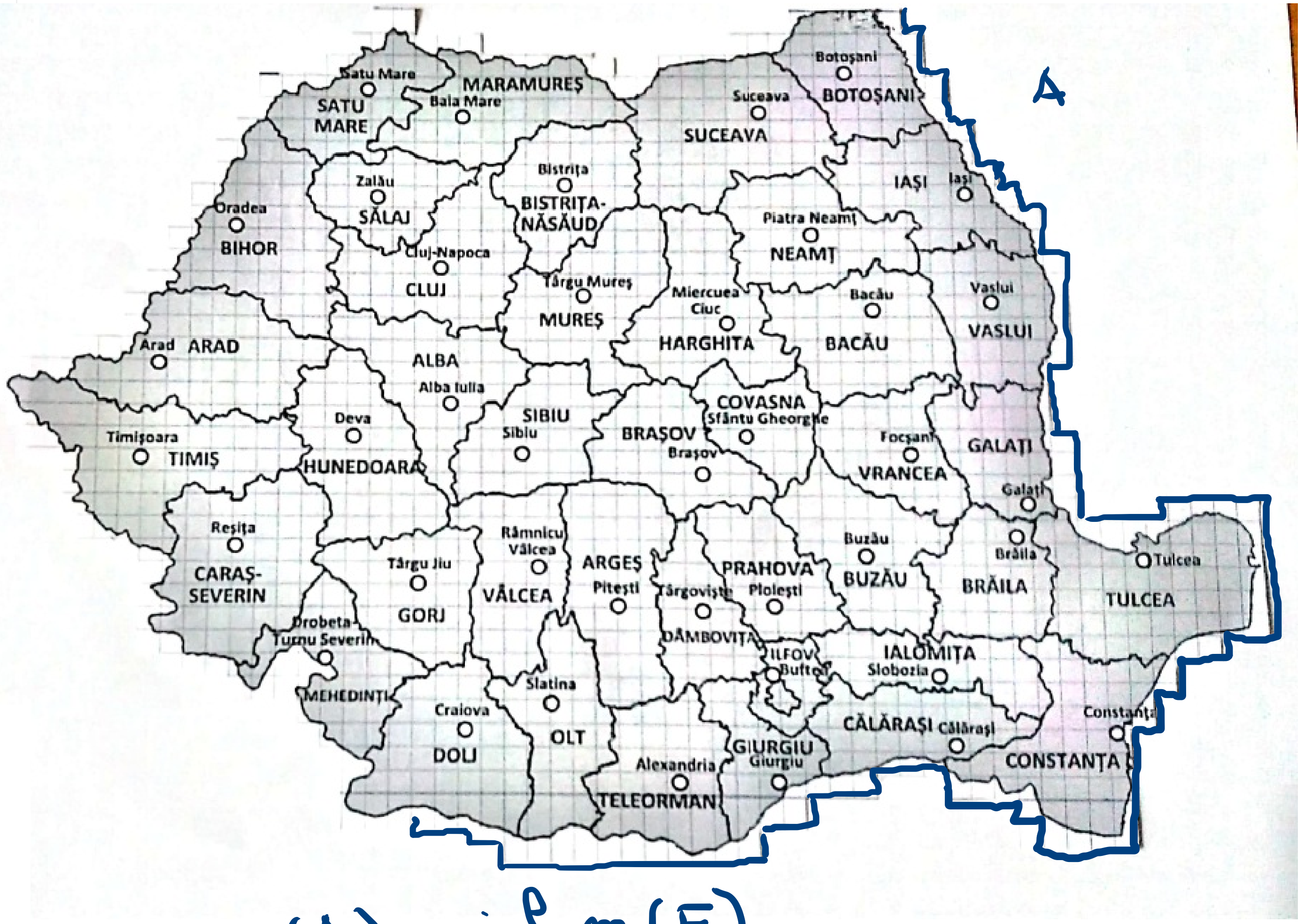
$$\Updownarrow$$

$$\underbrace{\int_{S\times T} f(x,y)\,dx\,dy}_{m+n \text{ variables}} = \underbrace{\int_S \Big(\underbrace{\int_T f(x,y)\,dy}_{n \text{ variables}}\Big)dx}_{m \text{ variables}} = \underbrace{\int_T \Big(\underbrace{\int_S f(x,y)\,dx}_{m \text{ variables}}\Big)dy}_{n \text{ variables}}$$

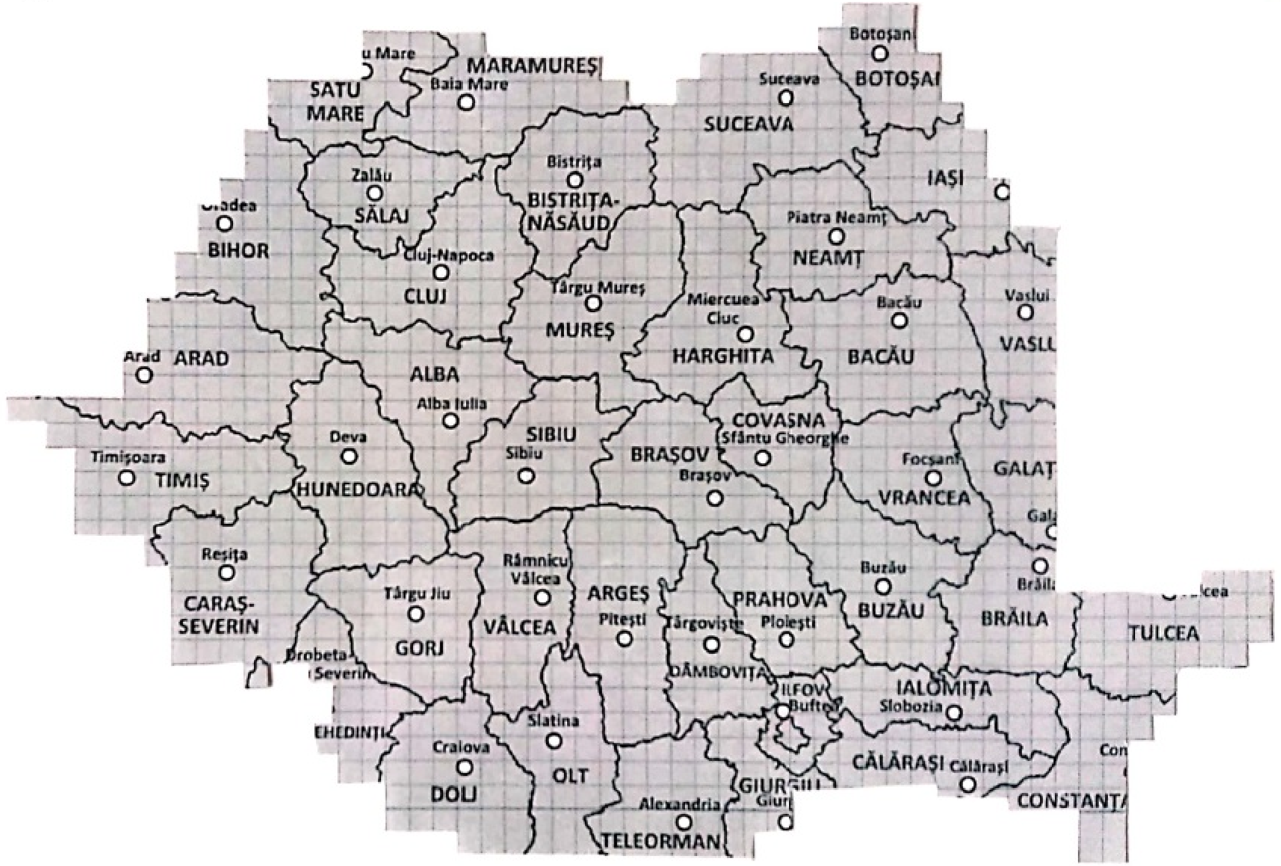


$$\sigma(f,\mathcal{I},\xi) = \sum_{j=1}^{p} f(c_j)\cdot m(A_j)$$

?



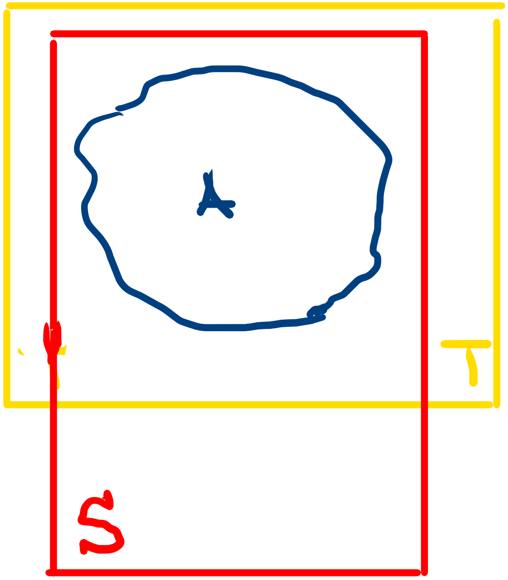
$$m_e(A) = \inf_{A\subseteq E} m(E)$$



$$m_i(A) = \sup_{E\subseteq A} m(E)$$

A measurable $\iff m_e(A) = m_i(A)$

3. Integration over bounded sets in $\mathbb{R}^n$



Let $A \subseteq \mathbb{R}^n$ be a bounded set and let $f : A \to \mathbb{R}$. We define

$\bar{f} : \mathbb{R}^n \to \mathbb{R}$

$$\bar{f}(x) := \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R}^n \setminus A \end{cases}$$

↳ the null extension of f

It can be proved that if $\exists$ a generalized rectangle T in $\mathbb{R}^n$ s.t. $A \subseteq T$ and $\bar{f}|_T$ is Riemann integrable over T, then for every other generalized rectangle S in $\mathbb{R}^n$ s.t. $A \subseteq S$, $\bar{f}|_S$ is R-integrable over S, too and one has

$$\int_T \bar{f}(x)dx = \int_S \bar{f}(x)dx$$

Due to the above remark we can define R-integrability over A as follows: a function $f : A \to \mathbb{R}$ is called R-integrable over A if $\exists T \subseteq \mathbb{R}^n$ generalized rectangle s.t. $A \subseteq T$ and $\bar{f}|_T$ is R-integrable over T. In this case the real number $\int_T \bar{f}(x)dx$ is called the Riemann integral of f over A and it is denoted by $\int_A f(x)dx$, $\underbrace{\int \cdots \int_A}_{n} f(x_1, \dots, x_n)dx_1 \dots dx_n$

$n=2$ $\qquad \iint_A f(x,y)\,dx\,dy$ $\qquad\qquad n=3$ $\qquad \iiint_A f(x,y,z)\,dx\,dy\,dz$

<u>Remark</u>

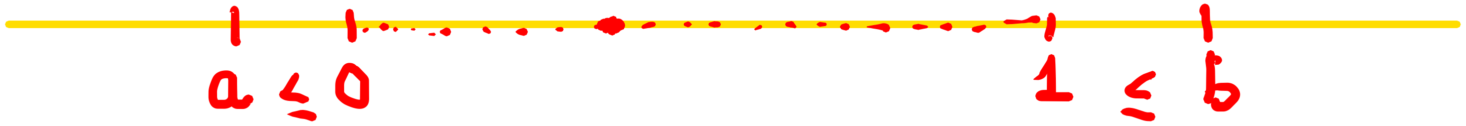
- $\left.\begin{array}{l}\text{If } T \subseteq \mathbb{R}^n \text{ is a generalized rectangle} \\ f: T \to \mathbb{R} \text{ is a continuous function}\end{array}\right\} \Rightarrow f$ is R-integrable over T
- $\left.\begin{array}{l}\text{If } A \subseteq \mathbb{R}^n \text{ is a bounded set} \\ f: A \to \mathbb{R} \text{ is a continuous bounded function}\end{array}\right\} \not\Rightarrow f$ is R-integrable over A

<u>Counterexample</u>. $\quad A = [0,1] \cap \mathbb{Q}$

$f: A \to \mathbb{R}, \quad f(x) = 1 \quad \forall x \in A$

Obviously f is continuous and bounded. However f is not R-integrable over A.
Suppose that f is R-integrable over $A \Rightarrow \exists$ a compact interval $[a,b]$ s.t. $A \subseteq [a,b]$ and $\bar{f}\big|_{[a,b]}$ is R-integrable over $[a,b]$

But $\operatorname{disc}(\bar{f}) = [0,1]$ which does not have Lebesgue measure zero $\Leftarrow$ Lebesgue's criterion



A bounded set $A \subseteq \mathbb{R}^n$ is said to be Jordan-measurable if the constant function 1 is R-integrable over A

$\downarrow$

the null extension $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R}^n \setminus A \end{cases}$ $\qquad \chi = \text{chi}$

$\hookrightarrow$ the characteristic function of A

A is Jordan measurable $\iff$ $\exists\, T \subseteq \mathbb{R}^n$ generalized rectangle s.t. $A \subseteq T$ and χ_A is R-integrable over T

We define

$$m(A) := \int_A dx = \int \dots \int_A dx_1 \dots dx_n$$

$\hookrightarrow$ the Jordan measure of A

$n=2$ $\qquad m(A) = \iint_A dx\,dy$ $\quad \leftarrow$ area of A

$n=3$ $\qquad m(A) = \iiint_A dx\,dy\,dz$ $\quad \leftarrow$ volume of A.