Proposition 4.1. Let π and π' be two planes in $\mathbf{A}$ with Cartesian equations

$$\begin{aligned} ax+by+cz+d &= 0 \\ a'x+b'y+c'z+d' &= 0 \end{aligned} \tag{4.6}$$

respectively. Then

1. π and π' are parallel if and only if the matrix

$$\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} \tag{4.7}$$

has rank 1.

- The proposition discusses the intersection of π and π' which in coordinates is described by $\underbrace{\text{the system (4.6)}}$

$\hookrightarrow$ has matrix of coeff. $M = \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix}$ and $\tilde{M} = \begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \end{bmatrix}$ is the extended matrix

- The associated vector subspaces are given by the homogeneous equations

$W: ax+by+cz=0$

$W': a'x+b'y+c'z=0$

$\pi \parallel \pi' \iff W = W' \iff$ the homogeneous equations are proportional $\iff \operatorname{rank} M = 1$

2. If the matrix (4.7) has rank 1 then π and π' are disjoint if the augmented matrix

$$\begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \end{bmatrix} \tag{4.8}$$

has rank 2; if its rank is 1 then they coincide.

If $\operatorname{rank} M = 1$ $(\iff \pi \parallel \pi')$ then

either

- $\operatorname{rank} \tilde{M} = 2 \iff$ system (4.6) is incompatible $\iff \pi \cap \pi' = \phi$
- $\operatorname{rank} \tilde{M} = 1 \Rightarrow$ system (4.6) has a sol $\iff \pi = \pi'$ since $W = W'$

3. If π and π' are not parallel then they intersect, and $\pi \cap \pi'$ is a line; this occurs if and only if the matrix (4.7) has rank 2.

$\iff \operatorname{rank} M = 2 \Rightarrow \operatorname{rank} \tilde{M} = 2 \Rightarrow$ system (4.6) compatible and has a 1-parameter family of solutions

Proposition 4.2. Let ℓ be a line with parametric equations (4.3) and Cartesian equations (4.4). Let π'' be a plane with equation

$$a''x + b''y + c''z + d'' = 0. \tag{4.9}$$

Then

1. ℓ and π'' are parallel if and only if

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0 \tag{4.10}$$

   or equivalently, if and only if

$$a''v_x + b''v_y + c''v_z = 0. \tag{4.11}$$

$$\ell : \begin{cases} x = x_Q + tv_x \\ y = y_Q + tv_y \\ z = z_Q + tv_z \end{cases} \quad (4.3)$$

$$\ell : \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases} \quad (4.4)$$

$$v_x = \begin{vmatrix} b & c \\ b' & c' \end{vmatrix}, \quad v_y = -\begin{vmatrix} a & c \\ a' & c' \end{vmatrix} \quad \text{and} \quad v_z = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}.$$

- If $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0$

then the system $\begin{cases} ax + by + cz = 0 \\ a'x + b'y + c'z = 0 \\ a''x + b''y + c''z = 0 \end{cases}$ (*)

is equivalent to the system $\begin{cases} a'x + b'y + c'z = 0 \\ a''x + b''y + c''z = 0 \end{cases}$

which describes the direction of ℓ $\Rightarrow$ every direction vector of ℓ is a solution to (*)

In particular $\forall$ dir. vector of ℓ satisfies $ax + by + cz = 0$ which is the eq. of the vector subspace W'' associated to π'' $\Rightarrow \ell \parallel \pi''$

- $(4.10) \Leftrightarrow (4.11)$ since $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = a'' \underbrace{\begin{vmatrix} b & c \\ b' & c' \end{vmatrix}}_{v_x} - b'' \underbrace{\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}}_{v_y} + c'' \underbrace{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}_{v_z}$

2. If (4.10) is satisfied, then $\ell \subseteq \pi''$ if and only if the matrix

$$\begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{bmatrix} \tag{4.12}$$

   has rank 2, otherwise they are disjoint (and the matrix has rank 3).

$\Leftrightarrow \ell \parallel \pi$ so, either

- they have a point in common $\Leftrightarrow \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ a''x + b''y + c''z + d'' = 0 \end{cases}$ has a sol. $\Leftrightarrow$ rank of (4.12) $= 2$
- they don't have a point in common $\Leftrightarrow$ rank of (4.12) is 3

3. If ℓ and π'' are not parallel, then they are incident, and $\ell \cap \pi''$ consists of only one point; this occurs if and only if

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \neq 0 \tag{4.13}$$

or equivalently, if and only if,

$$a'' v_x + b'' v_y + c'' v_z \neq 0. \tag{4.14}$$

- as noted for 1. (4.13) and (4.14) are equivalent

- If $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \neq 0 \Rightarrow$ rank of matrix of coefficients of $\left\{ \begin{array}{l} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \\ a''x + b''y + c''z + d'' = 0 \end{array} \right.$ is 3 $\left.\begin{array}{l} \\ \end{array}\right\} \ell$, π''

  $\Rightarrow$ ℓ and π have a point in common, the coordinates of which are the solutions to the sys

Proposition 4.4. Two lines ℓ_1 and ℓ_2 in **A** are coplanar if and only if one of the following conditions is satisfied:

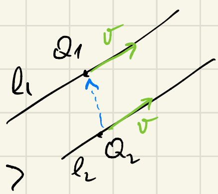
1. ℓ_1 and ℓ_2 are parallel.
2. ℓ_1 and ℓ_2 are incident.

In particular, two lines ℓ_1 and ℓ_2 in **A** are coplanar if and only if they are not skew.

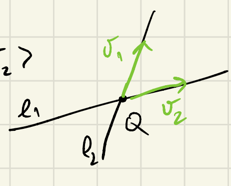
- Two lines in an affine plane are either incident (they meet in a point) or parallel

  $\Rightarrow$ two coplanar lines are either incident or parallel

- Conversely



  - if $\ell_2 = \ell_1$ clearly they are coplanar
  - if $\ell_2 \neq \ell_1$ and $\ell_1 \parallel \ell_2$ then $\ell_1, \ell_2 \in Q_1 + \langle v, \overrightarrow{Q_1Q_2} \rangle$
  - if $\ell_2 \neq \ell_1$ and $\ell_1 \cap \ell_2 \neq 0$ then $\ell_1, \ell_2 \in Q + \langle v_1, v_2 \rangle$



Proposition 4.5. Let ℓ and ℓ_1 be two lines in **A** and suppose that ℓ has Cartesian equations (4.4) and ℓ_1 has equations

$$\ell_1 : \begin{cases} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_1'x + b_1'y + c_1'z + d_1' &= 0 \end{cases}. \tag{4.15}$$

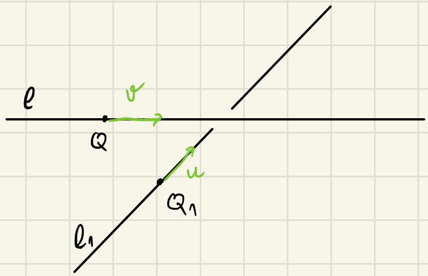
Let $Q(x_0, y_0, z_0) \in \ell$ and $Q_1(x_1, y_1, z_1) \in \ell_1$, let $\mathbf{v}(v_x, v_y, v_z)$ and $\mathbf{u}(u_x, u_y, u_z)$ be direction vectors for ℓ and ℓ_1 respectively. The following conditions are equivalent

1. ℓ and ℓ' are coplanar;
2. $\begin{vmatrix} x_0 - x_1 & y_0 - y_1 & z_0 - z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0;$ $(*)$
3. $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a_1 & b_1 & c_1 & d_1 \\ a_1' & b_1' & c_1' & d_1' \end{vmatrix} = 0.$ $(**)$

$$\ell : \begin{cases} ax + by + cz + d &= 0 \\ a'x + b'y + c'z + d' &= 0 \end{cases} \quad (4.4)$$

1. ⇒ 2. If $\ell \parallel \ell'$ ⇒ v and u are proportional
⇒ $(*)$

If $\ell \cap \ell' \neq \phi$ ⇒ $\vec{QQ_1} \in \langle v, u \rangle$
⇒ $\vec{QQ_1}, v, u$ are linearly dependant
⇒ $(*)$



2. ⇒ 1. If $(*)$ is true then either
- the last two rows in $(*)$ are proportional
⇒ $\langle v \rangle = \langle u \rangle$
⇒ $\ell \parallel \ell_1$
- the first row is a linear combination of the last two
⇒ $\vec{QQ_1} \in \langle v, u \rangle$
⇒ $Q_1 \in Q + \langle v, u \rangle = \pi$
⇒ $\ell, \ell_1 \in \pi$

3 ⇒ 1. • If the $(*)$ system
$$\begin{cases} \ell : \begin{cases} ax + by + cz + d &= 0 \\ a'x + b'y + c'z + d' &= 0 \end{cases} \\ \ell_1 : \begin{cases} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_1'x + b_1'y + c_1'z + d_1' &= 0 \end{cases} \end{cases}$$
is compatible then $\ell \cap \ell_1 \neq \phi$ ⇒ ℓ, ℓ_1 coplanar

• Let $R = \operatorname{rank}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a_1 & b_1 & c_1 & d_1 \\ a_1' & b_1' & c_1' & d_1' \end{pmatrix}$ and $r = \operatorname{rank}\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a_1 & b_1 & c_1 \\ a_1' & b_1' & c_1' \end{pmatrix}$

• notice that $R \geq 2$ and $r \geq 2$ ⇒ if $R = 2$ ⇒ $r = 2$ ⇒ System $(*)$ is compatible

• Suppose (*) is incompatible $\Leftrightarrow$ $r \underset{\neq}{<} R$ $\Leftrightarrow$

$\Downarrow$

$R \geq 3$ so $R = 3$ or 4 but $R \neq 4$ by $(**)$

$\Downarrow$

$R = 3$ and $r = 2$

• since $r = 2$ $\Rightarrow$ the last two rows in $\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a_1 & b_1 & c_1 \\ a_1' & b_1' & c_1' \end{pmatrix}$ are linear combinations of the first two rows

$\Rightarrow$ the homogeneous systems describing directions of ℓ and ℓ_1 are equivalent

$\Rightarrow \langle v \rangle = \langle u \rangle$

$\Rightarrow \ell_1 \parallel \ell_2$

$\Rightarrow \ell_1, \ell_2$ coplanar

(1) $\Rightarrow$ (3) If $\ell \cap \ell_1 \neq \emptyset$ then the system (*) is compatible $\Rightarrow R \leq 3 \Rightarrow (**)$

If $\ell \parallel \ell_1$ $\Rightarrow$ rank $\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a_1 & b_1 & c_1 \\ a_1' & b_1' & c_1' \end{pmatrix} = 2$ $\Rightarrow R \leq 3 \Rightarrow (**)$

Proposition 4.7. If $\pi_1 : ax + by + cz + d = 0$ and $\pi_2 : a'x + b'y + c'z + d' = 0$ are two distinct planes [redacted] in the pencil Π_ℓ, then Π_ℓ consists of planes having equations of the form

$$\pi_{\lambda,\mu} : \lambda(ax + by + cz + d) + \mu(a'x + b'y + c'z + d') = 0.$$

where $\lambda, \mu \in \mathbf{K}$ not both zero.

proof similar to the analogue statement for pencils of lines