1. Let $\mathbf{A}$ be an affine space and S and T two affine subspaces of $\mathbf{A}$. Show that if $S \subseteq T$ then $S \parallel T$.

2. In $\mathbf{A}^4(\mathbb{R})$ consider the affine subspaces

$$\alpha = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} \quad \beta = \langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \rangle + \begin{bmatrix}1\\3\\0\\0\end{bmatrix} \quad \gamma = \langle \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix} \rangle + \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \delta = \langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \rangle.$$

Which of the following is true?

1.	$\alpha \in \beta$	5.	$\beta \parallel \delta$
2.	$\alpha \in \gamma$	6.	$\gamma \parallel \delta$
3.	$\alpha \in \delta$	7.	$\beta \subseteq \gamma$
4.	$\beta \parallel \gamma$	8.	$\gamma \subseteq \delta$

3. Consider the following affine subspaces of $\mathbf{A}^4(\mathbb{R})$

$$Y : \begin{cases} x_1 + x_3 - 2 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 &= 0 \end{cases}$$

$$Z : \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 + x_3 - 3x_4 &= -1 \\ x_1 - x_2 + 3x_4 &= 3 \end{cases}$$

1. Determine the dimensions of Y and Z.
2. What are the parametric equations of the two affine subspaces?
3. Is $Y \parallel Z$?

4. In $\mathbf{A}^n(\mathbb{R})$ $(n \geq 2)$ consider the line

$$L = P + \langle \begin{bmatrix}v_1\\ \vdots \\ v_n\end{bmatrix} \rangle \quad \text{and the hyperplane} \quad H : \alpha_1 x_1 + \cdots + \alpha_n x_n + \beta = 0.$$

Show that $L \parallel H$ if and only if

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

5. Let $\mathbf{K}$ be a finite field. Determine

1. the number of points in an affine subspace of $\mathbf{A}^n(\mathbf{K})$,
2. the number of lines passing through a given point of $\mathbf{A}^n(\mathbf{K})$ and
3. the number of hyperplanes passing through a given point of $\mathbf{A}^n(\mathbf{K})$.

6. Which of the following are affine subspaces?

1. $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 - 2 = 0\} \subseteq \mathbf{A}^3(\mathbb{R})$.
2. $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 0\} \subseteq \mathbf{A}^3(\mathbb{R})$.
3. $C = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^4 + x_2 - 2x_3 + x_4 = 0\} \subseteq \mathbf{A}^4(\mathbb{R})$.

7. For $n, m \in \mathbb{N}$ let $a_1, \dots, a_n$, $b_1, \dots, b_m \in \mathbb{R}$. Fix a function $g : \mathbb{R} \to \mathbb{R} \in \mathcal{C}^\infty(\mathbb{R})$. Which of the following are affine subspaces?

1. $A = \{f \in \mathcal{C}^\infty(\mathbb{R}) : a_n f^{(n)} + a_{n-1} f^{(n-1)} + \dots + a_1 f' + g = 0\} \subseteq \mathcal{C}^\infty(\mathbb{R})_a$.
2. $B = \{f \in \mathcal{C}^\infty(\mathbb{R}) : f^3 - 5f^2 + 6f = 0\} \subseteq \mathcal{C}^\infty(\mathbb{R})_a$.

8. Let H be the affine plane in $\mathbf{A}^3(\mathbb{C})$ with equation $2x + y - 1 = 0$. In each of the following cases calculate the coordinates of $\mathrm{Pr}_{H,\mathbf{u}}(x, y, z)$ where $(x, y, z) \in \mathbf{A}^3$ and $\mathrm{Pr}_{H,\mathbf{u}} : \mathbf{A}^3 \to H$ is the projection in the direction of $\mathbf{u} \in \mathbb{C}^3$, where

1. $\mathbf{u} = (1, 0, 0)$
2. $\mathbf{u} = (i, 0, 0)$
3. $\mathbf{u} = (2i, i, 1)$

9. Show that the hyperplane $H = \mathbf{a} + \langle \mathbf{v}_1, \dots, \mathbf{v}_{n-1} \rangle$ of $\mathbf{A}^n(\mathbb{R})$ is described by the equation

$$\begin{vmatrix} x_1 - a_1 & \dots & x_n - a_n \\ v_{1,1} & \dots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n-1,1} & \dots & v_{n-1,n} \end{vmatrix} = 0$$

where $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,n})$ and $\mathbf{a} = (a_1, \dots, a_n)$.

10. In an affine n-dimensional space X let H be a hyperplane and Y a d-dimensional affine subspace. Show that for $d \in \{1, \dots, n-1\}$ exactly one of the following holds

1. $\dim(H \cap Y) = d - 1$,
2. $H \parallel Y$.

11. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space and let $\mathbf{W}$ be a vector subspace of $\mathbf{V}$. Show that $\mathbf{V}/\mathbf{W}$ has a structure of a vector space. (*Hint.* this is a quotient of vector spaces, in particular it is a quotient of groups.)

1. Let **A** be an affine space and S and T two affine subspaces of **A**. Show that if $S \subseteq T$ then $S \parallel T$.

Let V be the vector space associated to A

Let W be the vector subspace of V associated to S

Let U be the vector subspace of V associated to T

Recall that $W = \{ \overrightarrow{PQ} : P, Q \in S \}$

$$U = \{ \overrightarrow{PQ} : P, Q \in T \}$$

$$\Rightarrow \left. \begin{array}{l} U \subseteq W \\ W \text{ is a vector subspace of } V \\ U \text{ is a vector subspace of } V \end{array} \right\} \Rightarrow U \text{ is a vector subspace of } W$$

$$\Rightarrow S \parallel T$$

2. In $\mathbf{A}^4(\mathbb{R})$ consider the affine subspaces

$$\alpha = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} \quad \beta = \langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \rangle + \begin{bmatrix}1\\3\\0\\0\end{bmatrix} \quad \gamma = \langle \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix} \rangle + \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \delta = \langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \rangle.$$

Which of the following is true?

1. $\alpha \in \beta$
2. $\alpha \in \gamma$
3. $\alpha \in \delta$
4. $\beta \parallel \gamma$
5. $\beta \parallel \delta$
6. $\gamma \parallel \delta$
7. $\beta \subseteq \gamma$
8. $\gamma \subseteq \delta$

1.) $\alpha \in \beta$ if there exists $t \in \mathbb{R}$ such that $\alpha = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\3\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

this is equivant to the existance of $t \in \mathbb{R}$ such that $\begin{cases} 2 = 1+t \\ 1 = 3+t \\ 2 = t \\ 1 = t \end{cases}$

from the last two equations we see that there is no such $t \Rightarrow \alpha \notin \beta$

Rem a different, but equivalent way of thinking about this is:

$\alpha \in \beta \iff \forall$ point $P \in \beta$ $\quad \overrightarrow{\alpha P} \in \langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \rangle$

2.) $\alpha \in \gamma$? Let $P = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \in \gamma$. Then $\alpha \in \gamma \iff \overrightarrow{P\alpha} \in \langle \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix} \rangle$

i.e. $\alpha \in \gamma \iff \overrightarrow{P\alpha} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix}$ are linearly dependent

$$\iff \operatorname{rank}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & -1 \\ 1 & 0 & 2 & -2 \end{pmatrix} \leq 2$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & -1 \\ 1 & 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -3 \\ 0 & -1 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rank} A = 2 \quad \Rightarrow \alpha \in \gamma$$

Rem a different but equivalent way of thinking about this is the following

$\alpha \in \gamma \iff \exists\, t,s \in \mathbb{R}$ such that $\begin{bmatrix}2\\1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\1\\3\\-1\end{bmatrix} + s\begin{bmatrix}1\\0\\2\\-2\end{bmatrix} \iff \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = t\begin{bmatrix}2\\1\\3\\-1\end{bmatrix} + s\begin{bmatrix}1\\0\\2\\-2\end{bmatrix}$ $(*)$

$(*)$ is a system of 4 equations in the unknowns s,t

the augmented matrix of this system is

$$\tilde{C}\begin{pmatrix}2&1&1\\1&0&1\\3&2&1\\-1&-2&1\end{pmatrix} \sim \begin{pmatrix}1&0&1\\2&1&1\\3&2&1\\-1&-2&1\end{pmatrix} \sim \begin{pmatrix}1&0&1\\0&1&-1\\0&2&-2\\0&-2&2\end{pmatrix} \sim \begin{pmatrix}1&0&1\\0&1&-1\\0&0&0\\0&0&0\end{pmatrix}$$

so the system $(*)$ is compatible with a unique solution $(t,s) = (1,-1)$

3.) $\alpha \in \delta$? Since δ is a hyperplane passing through the origin

$$\alpha \in \delta \iff \overrightarrow{O\alpha} \in \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\rangle$$

$\hookrightarrow$ not possible since the last coordinate has to be 0

4.) $\beta \parallel \gamma$? $W_\beta = \left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\rangle$ $W_\gamma = \left\langle \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix} \right\rangle$ so $\beta \parallel \gamma \iff W_\beta \subseteq W_\gamma$ or $W_\gamma \subseteq W_\beta$

since $\dim \gamma = \dim W_\gamma = 2 > \dim W_\beta = \dim \beta$

$\beta \parallel \gamma \iff W_\beta \subseteq W_\gamma$

$\iff \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix}$ are linearly dependent

we checked in 2.) that this is the case

so $\beta \parallel \gamma$

5.) $\beta \parallel \delta \iff W_\beta \subseteq \overset{\gamma}{\overset{\shortparallel}{W_\gamma}} = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\rangle$ which is not possible because of the last coordinate

so $\beta \nparallel \delta$

6.) $\gamma \parallel \delta \iff W_\gamma \subseteq \overset{\gamma}{\overset{\shortparallel}{W_\gamma}}$ which is not possible because of the last coordinate

so $\gamma \nparallel \gamma$

7.) $\beta \subseteq \gamma$? By 4) $\beta \parallel \gamma$ so $\beta \subseteq \gamma \iff$ a point in β lies in γ

$\iff \begin{bmatrix}1\\3\\0\\0\end{bmatrix} \in \gamma$ which is true $\Rightarrow \beta \subseteq \gamma$

So $\beta \subseteq \gamma \iff \begin{bmatrix} 0 \\ 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in W_\gamma = \left\langle \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \right\rangle$

$\iff \operatorname{rank} \begin{pmatrix} 0 & 3 & -1 & 0 \\ 2 & 1 & 3 & -1 \\ 1 & 0 & 2 & -2 \end{pmatrix} \leq 2$

$$\begin{pmatrix} 0 & 3 & -1 & 0 \\ 2 & 1 & 3 & -1 \\ 1 & 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 2 & 1 & 3 & -1 \\ 0 & 3 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 3 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & -9 \end{pmatrix}$$ has rank 3 $\Rightarrow \beta \not\subseteq \gamma$

8) $\beta \subseteq \delta$? If $\beta \subseteq \delta$ then $\beta \parallel \delta$ (by exercise 1) but by 5) $\beta \nparallel \delta$ so $\beta \not\subseteq \delta$

3. Consider the following affine subspaces of $\mathbf{A}^4(\mathbb{R})$

$$Y : \begin{cases} x_1 + x_3 - 2 = 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 = 0 \end{cases} \quad (*)$$

$$Z : \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 = 1 \\ x_2 + x_3 - 3x_4 = -1 \\ x_1 - x_2 + 3x_4 = 3 \end{cases} \quad (**)$$

1. Determine the dimensions of Y and Z.
2. What are the parametric equations of the two affine subspaces?
3. Is $Y \parallel Z$?

1) • Let A be the matrix of the system $(*)$ and $\tilde{A}$ the augmented matrix

$$\tilde{A} = \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 2 & -1 & 1 & 3 & 1 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & -1 & -1 & 3 & -3 \end{array} \right) \Rightarrow \operatorname{rank}(\tilde{A}) = 2$$

$\operatorname{rank}(A) = 2$

$\Rightarrow$ the system is compatible $\Rightarrow Y \neq \emptyset$

$\dim(Y) = \dim \mathbb{A}^4(\mathbb{R}) - \operatorname{rank}(A) = 4 - 2 = 2$

• Let B be the matrix of the system $(**)$ and $\tilde{B}$ the augmented matrix

$$\tilde{B} = \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & -1 & 0 & +3 & 3 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & -2 & -2 & 6 & 2 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

So $\operatorname{rank} B = \operatorname{rank} \tilde{B} = 2$

$\dim(Z) = \dim \mathbb{A}^4(\mathbb{R}) - \operatorname{rank}(A) = 4 - 2 = 2$

2.) We obtain parametric equations by finding all solutions to (*) and (**)

$$\tilde{A} \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & -1 & -1 & 3 & -3 \end{array}\right) \sim \left(\begin{array}{cccc c} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 1 \\ 1 \end{matrix} & \begin{matrix} 0 \\ -3 \end{matrix} & \begin{matrix} 2 \\ 3 \end{matrix} \end{array}\right) \iff \begin{cases} x_1 + x_3 = 2 \\ x_2 + x_3 - 3x_4 = 3 \end{cases}$$

$$\iff \begin{cases} x_1 = 2 - x_3 \\ x_2 = 3 - x_3 + 3x_4 \end{cases} \iff \begin{cases} x_1 = 2 - x_3 \\ x_2 = 3 - x_3 + 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

we choose x_3, x_4 as free variables

a parametrization for Y

$$\tilde{B} \sim \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \iff \begin{cases} x_1 + x_3 = 2 \\ x_2 + x_3 - x_4 = -1 \end{cases}$$

$$\iff \begin{cases} x_1 = 2 - x_3 \\ x_2 = -1 - x_3 + x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Is $Y \parallel Z$? $\quad W_Y = \left\langle \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\rangle$ (this follows from the parametrization)

$W_Z = W_Y$ (from the parametrization of Z)

$\Rightarrow Y \parallel Z$, yes.

4. In $\mathbf{A}^n(\mathbb{R})$ $(n \geq 2)$ consider the line

$$L = P + \langle \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \rangle \quad \text{and the hyperplane} \quad H : \alpha_1 x_1 + \cdots + \alpha_n x_n + \beta = 0. \quad (*)$$

Show that $L \parallel H$ if and only if

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

- Let W_L be the direction of L and let $W_H \leq \mathbb{R}^n$ be the vector subspace associated to H
- We have $W_L = \langle \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \rangle$ and $W_H : \alpha_1 x_1 + \dots + \alpha_n x_n = 0$ the homogeneous equation corresponding to $(*)$
- $L \parallel H \iff W_L \subseteq W_H$ or $W_H \subseteq W_L$
- Since $\dim L = 1 \leq n-1 = \dim H$, we have $L \parallel H \iff W_L \subseteq W_H$
- Since W_L is generated by $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, we have $L \parallel H \iff \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in W_H$
- Clearly, $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in W \iff \alpha_1 v_1 + \dots + \alpha_n v_n = 0$

<u>Rem</u> forget about normal vectors until further notice

5. Let **K** be a finite field. Determine

1. the number of points in an affine subspace of $\mathbf{A}^n(\mathbf{K})$,
2. the number of lines passing through a given point of $\mathbf{A}^n(\mathbf{K})$ and
3. the number of hyperplanes passing through a given point of $\mathbf{A}^n(\mathbf{K})$.

Let $q = |\mathbb{K}|$, i.e $\mathbb{K}$ has q elements

1. Let X be an affine subspace of $\mathbb{A}^n(\mathbb{K})$

Recall that as a set $\mathbb{A}^n(\mathbb{K}) = \mathbb{K}^n$ and

that the affine subspaces of $\mathbb{K}^n$ are

$q + W$ for some $q \in \mathbb{K}^n$ and some vector subspace W

moreover W is the associated vector subspace of $q+W$

so $\dim(q+W) = \dim W$ and $|q+W| = |W|$

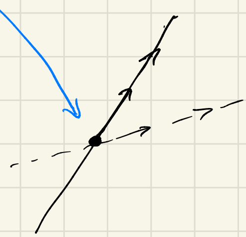
since W is itself a vector space over $\mathbb{K}$, it has $q^{\dim W}$ elements

$$\Rightarrow |q+W| = q^{\dim W}$$

So, $|X| = q^{\dim X}$

2. # lines passing through a given point =

$$= \frac{\#\text{ non-zero vectors}}{\#\text{ vectors proportional to a given vector}} =$$



$$= \frac{\#\{(v_1,\dots,v_n)\in\mathbb{K}^n : \text{not all } v_i \text{ are zero}\}}{\#\{t(v_1,\dots,v_n) : t\in\mathbb{K},\ t\neq 0\}} =$$

$$= \frac{q^n-1}{q-1} \qquad \text{since} \quad n=\dim\mathbb{K}^n$$

3. # hyperplanes passing through a given point =

if we choose the given point to be the origin of a coordinate system in $\mathbb{A}^n(\mathbb{K})$

$= \#\ (n-1)$-dimensional vector subspaces of $\mathbb{K}^n$ =

an $(n-1)$-dim. vector subspace of $\mathbb{K}^n$ is described up to a non-zero constant by an equation of the form $a_1x_1+\dots+a_nx_n=0$ where not all a_n are zero

$$= \frac{\#\{(a_1,\dots,a_n)\ a_i\in\mathbb{K} \text{ not all zero}\}}{\#\text{ non-zero constants}}$$

$$= \frac{q^n-1}{q-1} \qquad \text{where} \quad n=\dim\mathbb{K}^n$$

6. Which of the following are affine subspaces?

1. $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 - 2 = 0\} \subseteq \mathbf{A}^3(\mathbb{R})$.
2. $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 0\} \subseteq \mathbf{A}^3(\mathbb{R})$.
3. $C = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^4 + x_2 - 2x_3 + x_4 = 0\} \subseteq \mathbf{A}^4(\mathbb{R})$.

Recall that $A^n(\mathbb{R})$ is $\mathbb{R}^n$ viewed as an affine space

1) A is an affine subspace because it is described by one linear eq.
it is in fact a plane in $A^3(\mathbb{R})$

2.) Notice that

$$x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3 = (x_1 - x_2 - x_3)^2$$

which is zero if and only if $x_1 - x_2 - x_3 = 0$

so $B = \{(x_1, x_2, x_3) \in A^3(\mathbb{R}) : x_1 - x_2 - x_3 = 0\}$

so B is a plane in $A^3(\mathbb{R})$

3.) $C : x_1^4 + x_2 - 2x_3 + x_4 = 0$

looks suspicious

notice that $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in C$ so, if C is an affine space it is in fact a vector subspace of $\mathbb{R}^4$

notice that $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \in C$ but $2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \notin C$

$\Rightarrow$ C is not a vector subspace of $\mathbb{R}^4$

$\Rightarrow$ C is not an affine subspace of $A^4(\mathbb{R})$ by

7. For $n, m \in \mathbb{N}$ let $a_1, \dots, a_n, b_1, \dots, b_m \in \mathbb{R}$. Fix a function $g : \mathbb{R} \to \mathbb{R} \in C^\infty(\mathbb{R})$. Which of the following are affine subspaces?

1. $A = \{f \in C^\infty(\mathbb{R}) : a_n f^{(n)} + a_{n-1} f^{(n-1)} + \dots + a_1 f' + g = 0\} \subseteq C^\infty(\mathbb{R})_a$.
2. $B = \{f \in C^\infty(\mathbb{R}) : f^3 - 5f^2 + 6f = 0\} \subseteq C^\infty(\mathbb{R})_a$.

Recall: • $C^\infty(\mathbb{R})$ is the set of functions f for which all derivatives $f^{(n)}$ exist $(\forall n)$

• $C^\infty(\mathbb{R})$ is a real vector space with

$$(af + bg)(x) = a \cdot f(x) + b \cdot g(x) \quad \forall x$$
$$\forall f, g \in C^\infty(\mathbb{R}), \ \forall a, b \in \mathbb{R}$$

• We denoted by $C^\infty(\mathbb{R})_a$ the vector space $C^\infty(\mathbb{R})$ viewed as an affine space

• $(af + bg)^{(n)} = a f^{(n)} + b g^{(n)}$ n-th derivative of a linear combination of functions

1.) $a_n f^{(n)} + a_{n-1} f^{(n-1)} + \dots + a_1 f'$ is linear in f since

$\forall \alpha, \beta \in \mathbb{R}, \ \forall f_1, f_2 \in C^\infty(\mathbb{R})$

$$a_n(\alpha f_1 + \beta f_2)^{(n)} + a_{n-1}(\alpha f_1 + \beta f_2)^{(n-1)} + \dots + a_1(\alpha f_1 + \beta f_2)' =$$
$$= \alpha\left(a_n f_1^{(n)} + a_{n-1} f_1^{(n-1)} + \dots + a_1 f_1'\right) + \beta\left(a_n f_2^{(n)} + a_{n-1} f_2^{(n-1)} + \dots + a_1 f_2'\right)$$

$\Rightarrow a_n f^{(n)} + a_{n-1} f^{(n-1)} + \dots + a_1 f' = g$ is a linear equation in f

$\Rightarrow A$ is a hyperplane in $C^\infty(\mathbb{R})$

2.) $B: f^3 - 5f^2 + 6f = 0$

$\Leftrightarrow f(f-2)(f-3) = 0$

$\Leftrightarrow f(x)(f(x)-2)(f(x)-3) = 0 \quad \forall \ x \in \mathbb{R}$

So, all functions in B take values in $\{0,2,3\}$

The constant function $f \equiv 0 \in B$ (this is the zero element in $C^\infty(\mathbb{R})$)

So if B is an affine subspace it is a vector subspace of $C^\infty(\mathbb{R})$

Notice that the constant function $g \equiv 2 \in B$

but $2 \cdot g \equiv 4 \notin B$

$\Rightarrow$ B is not a vector subspace of $C^\infty(\mathbb{R})$

$\Rightarrow$ B is not an affine subspace of $C^\infty(\mathbb{R})_a$ by

8. Let H be the affine plane in $\mathbf{A}^3(\mathbb{C})$ with equation $2x + y - 1 = 0$. In each of the following cases calculate the coordinates of $\mathrm{Pr}_{H,\mathbf{u}}(x,y,z)$ where $(x,y,z) \in \mathbf{A}^3$ and $\mathrm{Pr}_{H,\mathbf{u}} : \mathbf{A}^3 \to H$ is the projection in the direction of $\mathbf{u} \in \mathbb{C}^3$, where

1. $\mathbf{u} = (1,0,0)$
2. $\mathbf{u} = (i,0,0)$
3. $\mathbf{u} = (2i,i,1)$

1. • A line passing through (x_0, y_0, z_0) in the direction of $u = (1,0,0)$ has parametric equations $\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot t \quad t \in \mathbb{R}$

• let $H : 2x + y - 1 = 0$ be the given plane

• $\ell \cap H$ is obtained for those values $t \in \mathbb{R}$ which satisfy

$$2(x_0 + t) + y_0 - 1 = 0 \iff 2t + 2x_0 + y_0 - 1 = 0$$

$$\iff t = \frac{1 - 2x_0 - y_0}{2}$$

$\Rightarrow \ell \cap H$ is the point $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \frac{1 - 2x_0 - y_0}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - x_0 - y_0 \\ y_0 \\ z_0 \end{bmatrix}$

$$\Rightarrow \mathrm{Pr}_{H,u}(x,y,z) = \begin{bmatrix} 1 - x - y \\ y \\ z \end{bmatrix}$$

2. and 3. are similar.