ANALYTIC GEOMETRY, PROBLEM SET 3

Warm-up 1. What are the conditions that vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ should satisfy to ensure the existence of a triangle ABC such that $\overrightarrow{AB} \in a$, $\overrightarrow{BC} \in \overline{b}$ and $\overrightarrow{CA} \in \overline{c}$?

1. On the sides of a triangle ABC, one constructs the parallelograms $ABB'A''$, $BCC'B''$, $CAA'C''$. Show that one can construct a triangle MNP such that $\overrightarrow{MN} \in \overline{A'A''}$, $\overrightarrow{NP} \in \overline{B'B''}$ and $\overrightarrow{PM} \in \overline{C'C''}$.

2. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA} + \overline{PB} + \overline{PC} + \overline{PD} = \overline{0}$.

3. In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 2\overline{OM}$.

4. Consider, in the 3-dimensional space, the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Prove that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a new parallelogram.

5. Let ABC be a triangle and a, b, c the lengths of its sides, respectively. If A_1 is the intersection point of the internal bisector of the angle $\angle A$ and BC and M is an arbitrary point, show that

$$\overline{MA_1} = \frac{b}{b+c}\overline{MB} + \frac{c}{b+c}\overline{MC}$$

6. If G is the centroid (center of mass) of a triangle ABC in the plane and O is a given point, then

$$\overline{OG} = \frac{\overline{OA} + \overline{OB} + \overline{OC}}{3}$$

7. Let ABC be a triangle, H its orthocenter, O the circumcenter (center of the circumcircle), G the centroid of the triangle and A' the point on the circumcircle diametrically opposed to A. Then:

(1) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$; (**Sylvester's formula**)
(2) $\overline{HB} + \overline{HC} = \overline{HA'}$;
(3) $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{HO}$;
(4) $\overline{HA} + \overline{HB} + \overline{HC} = 3\overline{HG}$;
(5) the points H, G, O are collinear and $2GO = HG$. (the **Euler line**)

Date: October 11, 2021.

(W1.) Necessary conditions.

$\exists$ a triangle ABC s.t. $\overrightarrow{AB} \in \bar{a}$, $\overrightarrow{BC} \in \bar{b}$ and $\overrightarrow{CA} \in \bar{c}$.

- $\bar{a}, \bar{b}, \bar{c}$ are non-zero
- No 2 of them are collinear.
- $\bar{a} + \bar{b} + \bar{c} = 0$.

- Claim: These are also sufficient.

Suppose
$$\begin{cases} \bullet\ \bar{a}, \bar{b}, \bar{c} \text{ are non-zero} \\ \bullet\ \text{No 2 of them are collinear.} \\ \bullet\ \bar{a} + \bar{b} + \bar{c} = 0. \end{cases}$$

Let $\overrightarrow{AB} \in \bar{a}$. $(A \neq B)$

$\exists C \in \mathcal{E}_2$ s.t. $\overrightarrow{BC} \in \bar{b}$. $(B \neq C)$

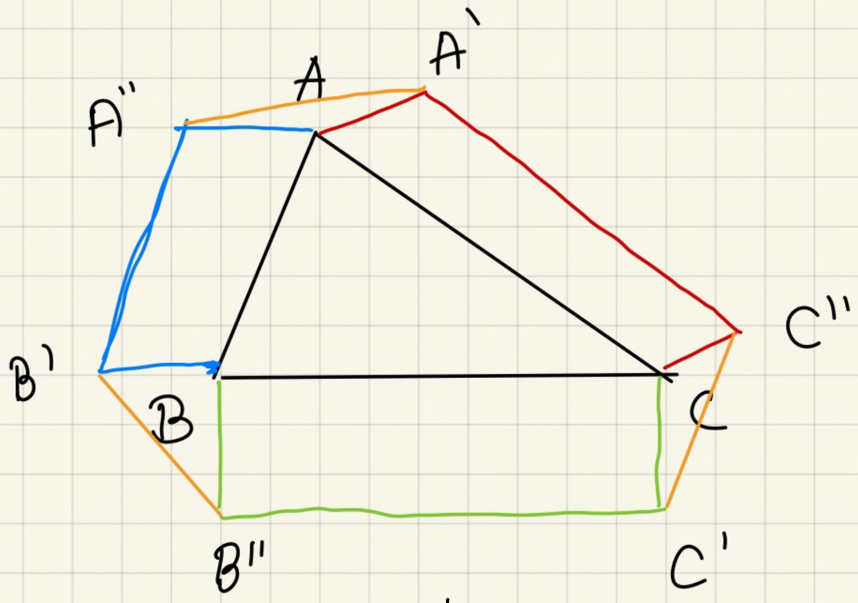
The points A, B, C form a non-degenerate $\triangle$.

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \bar{0}$$

Since $\bar{a} + \bar{b} + \bar{c} = \bar{0} \Rightarrow \overrightarrow{CA} \in \bar{c}$.

$\boxtimes$

1. On the sides of a triangle ABC, one constructs the parallelograms $ABB'A''$, $BCC'B''$, $CAA'C''$. Show that one can construct a triangle MNP such that $\overrightarrow{MN} \in \overline{A'A''}$, $\overrightarrow{NP} \in \overline{B'B''}$ and $\overrightarrow{PM} \in \overline{C'C''}$.



- We want to show that

$$\overline{A'A''} + \overline{B'B''} + \overline{C'C''} = \bar{0}.$$

Observe that:

$$\overline{A'A''} = \overline{AA''} - \overline{AA'}$$

$$\overline{B'B''} = \overline{BB''} - \overline{BB'}$$

$$\overline{C'C''} = \overline{CC''} - \overline{CC'}.$$

Summing up,

$$\overline{A'A''} + \overline{B'B''} + \overline{C'C''} = \cancel{\overline{AA''}} + \cancel{\overline{BB''}} + \cancel{\overline{CC''}} - \cancel{\overline{BB'}} - \cancel{\overline{CC'}} - \cancel{\overline{AA'}}$$

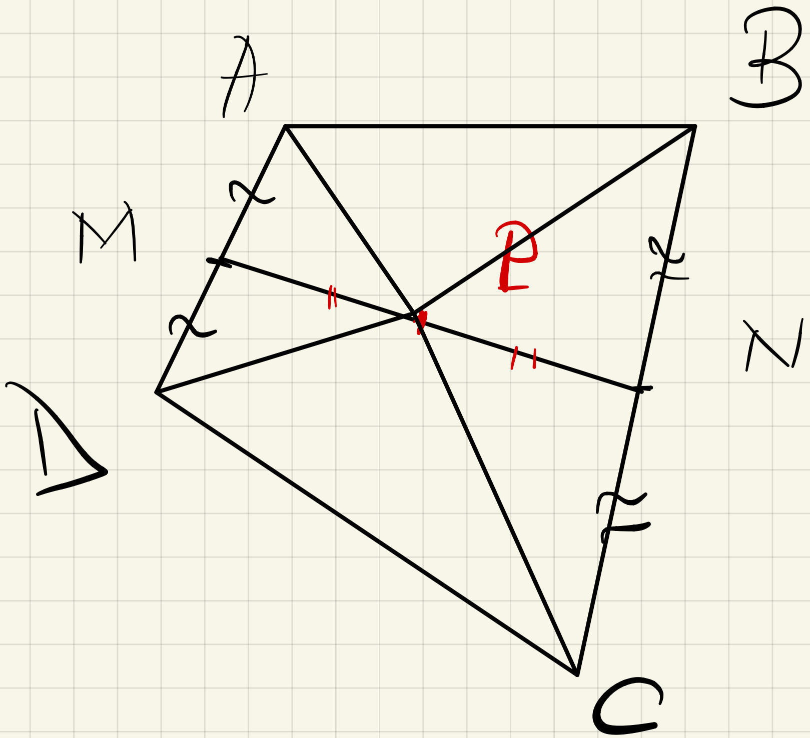
$$= \overline{0}$$

↑ use the parallelograms.

<u>Exercise</u>: Check that:

$\overline{A'A''}$, $\overline{B'B''}$, $\overline{C'C''}$ are non-zero and no 2 of them are collinear.

2. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}=\overline{0}$.

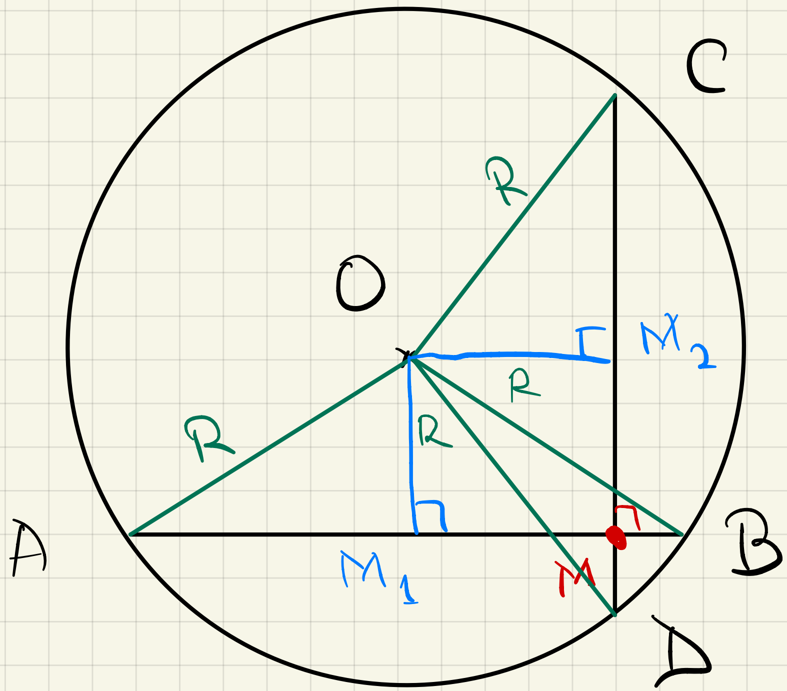


Notice that:

$$\left.\begin{aligned}\overline{PA}+\overline{PD} &= 2\cdot\overline{PM}\\ \overline{PB}+\overline{PC} &= 2\cdot\overline{PN}\end{aligned}\right\}$$

$$\therefore\ \overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}=2\left(\overline{PM}+\overline{PN}\right)=\overline{0}.$$

3. In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that $\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD} = 2\overline{OM}$.



Let M_1, M_2 be the midpoints of $[AB]$ and $[CD]$.

$$\overline{OA} + \overline{OB} = 2\cdot\overline{OM_1}$$

$$\overline{OD} + \overline{OC} = 2\cdot\overline{OM_2}$$

$$\therefore\ \overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 2\cdot\left(\overline{OM_1} + \overline{OM_2}\right).$$

- OM_1MM_2 is a rectangle.

$\Rightarrow \overline{OM_1} + \overline{OM_2} = \overline{OM}$.

The conclusion follows. $\boxed{\Xi}$

4. Consider, in the 3-dimensional space, the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Prove that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a new parallelogram.

Let C_1, C_2, C_3, C_4 be the midpoints of $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$.

- It's enough to show that $\overline{C_1C_2} = \overline{C_4C_3}$.

Let $O \in \mathcal{E}_3$ be the origin.

$$\overline{OC_i} = \frac{1}{2} \cdot \left(\overline{OA_i} + \overline{OB_i}\right), \quad \forall i = \overline{1,4}.$$

$$\overline{C_1C_2} = \overline{OC_2} - \overline{OC_1} = \frac{1}{2}\left(\overline{OA_2} + \overline{OB_2} - \overline{OA_1} - \overline{OB_1}\right)$$

$$= \frac{1}{2}\left(\underline{\overline{A_1A_2}} + \underline{\underline{\overline{B_1B_2}}}\right)$$

$$\overline{C_4C_3} = \overline{OC_3} - \overline{OC_4} = \frac{1}{2}\left(\overline{OA_3} + \overline{OB_3} - \overline{OA_4} - \overline{OB_4}\right)$$

$$= \frac{1}{2}\left(\underline{\overline{A_4A_3}} + \underline{\underline{\overline{B_4B_3}}}\right)$$

$$\therefore \quad \overline{C_1C_2} = \overline{C_4C_3} \Rightarrow \begin{cases} C_1C_2 = C_4C_3 \\ \text{and} \\ C_1C_2 \parallel C_4C_3 . \end{cases}$$

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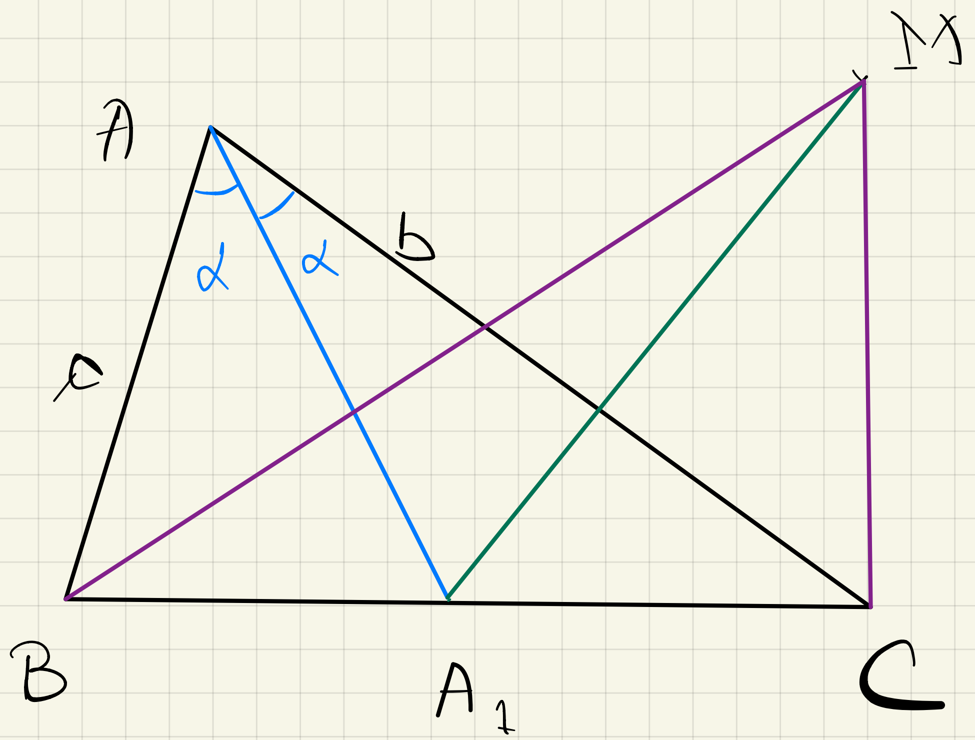
5. Let ABC be a triangle and a, b, c the lengths of its sides, respectively. If A_1 is the intersection point of the internal bisector of the angle $\angle A$ and BC and M is an arbitrary point, show that

$$\overline{MA_1} = \frac{b}{b+c}\overline{MB} + \frac{c}{b+c}\overline{MC}$$



The T. of the bisector gives that

$$\frac{K}{1} = \frac{A_1B}{A_1C} \overset{\text{Proof}}{=} \frac{[ABA_1]}{[ACA_1]} = \frac{\cancel{\frac{1}{2}} \cdot c \cdot \cancel{AA_1} \cdot \cancel{\sin(\alpha)}}{\cancel{\frac{1}{2}} \cdot b \cdot \cancel{AA_1} \cdot \cancel{\sin(\alpha)}}$$

$$\overline{MA_1} = \frac{1}{1+k}\overline{MB} + \frac{k}{k+1}\cdot\overline{MC}.$$

$$= \frac{b}{b+c}\cdot\overline{MB} + \frac{c}{b+c}\cdot\overline{MC}.$$

$\boxtimes$

7. Let ABC be a triangle, H its orthocenter, O the circumcenter (center of the circumcircle), G the centroid of the triangle and A' the point on the circumcircle diametrically opposed to A. Then:

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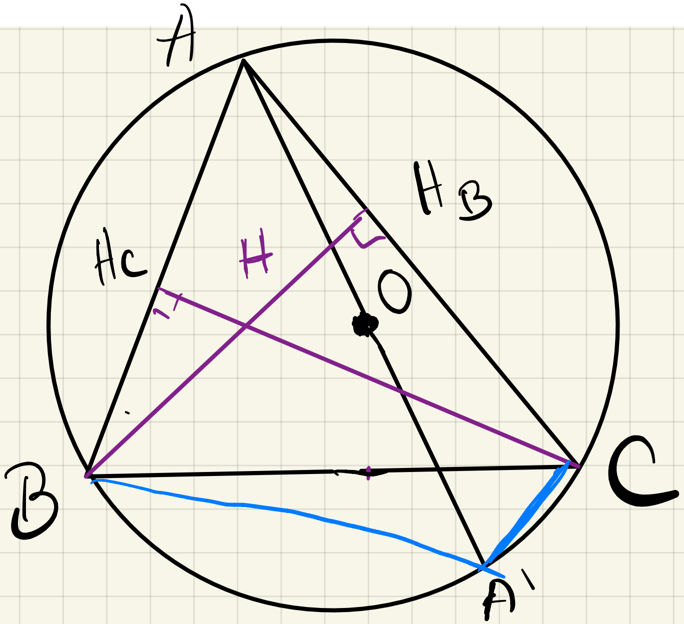
(1)



O - circumcenter
H - orthocenter.

$m(\measuredangle A'CA) = \frac{\pi}{2}$ (since $[AA']$ is a diameter)

$\therefore$ $BH \parallel A'C$ (both are $\perp$ on AC).

$m(\measuredangle ABA') = \frac{\pi}{2}$ (since $[AA']$ is a diameter)

$\therefore$ $CH \parallel A'B$ (both are $\perp$ on AB).

$\therefore$ $BHCA'$ is a parallelogram.

$$\therefore \overline{BH} = \overline{A'C} = \overline{A'O} + \overline{OC}$$

$$= \overline{OA} + \overline{OC}$$

$$\overline{BH} = \overline{OA} + \overline{OC} \quad \Big| + \overline{OB}$$

$$\therefore \underbrace{\overline{OB} + \overline{BH}}_{\overline{OH}} = \overline{OA} + \overline{OB} + \overline{OC}.$$

$\blacksquare$