1. Establish which of the following triples of points in $\mathbf{A}^3(\mathbb{C})$ are collinear:

1. $\{(2,1,-3),(1,-1,2),(3/2,0,-1/2)\}$
2. $\{(\mathbf{i},0,0),(1+\mathbf{i},2\mathbf{i},1),(1,2,-\mathbf{i})\}$

2. In each of the following, find the value (if it exists) of the real parameter m for which the triple of points is collinear in $\mathbf{A}^3(\mathbb{R})$

1. $\{(2,-1,2),(1,1,1),(4,-m+1,4)\}$
2. $\{(3,0,0),(0,1,1),(m,m,m)\}$

3. After checking, for each of the following, that the points are not collinear, find parametric and Cartesian equations for the planes determined by the points

1. $\{(2,\sqrt{2},1),(1,1,\sqrt{2}),(0,0,1)\}$
2. $\{(5,-1,0),(1,1,\sqrt{5}),(-3,1,\pi/2)\}$

4. In each of the following, find a Cartesian equation of the plane in $\mathbf{A}^3(\mathbb{C})$ passing through Q and parallel to the plane π

1. $Q=(-1,2,2)$, $\pi : x+2y+3z+1=0$
2. $Q=(\mathbf{i},\mathbf{i},\mathbf{i})$, $\pi : 2x-y=0$

5. For each of the following, determine whether or not the three planes belong to the same pencil

1. $x-y+z=0$, $-x+3y-5z+2=0$, $y-2z+1=0$
2. $2x-3y+3=0$, $x-y+6=0$, $x-3z=-1$

6. For each of the following, find parametric and Cartesian equations for the line in $\mathbf{A}^3(\mathbb{R})$ passing through the point Q and parallel to the vector $\mathbf{v}$.

1. $Q=(1,1,0)$, $\mathbf{v}=(2,-1,\sqrt{2})$
2. $Q=(-2,2,-2)$, $\mathbf{v}=(1,1,0)$

7. Find parametric equations for each of the following lines in $\mathbf{A}^3(\mathbb{C})$

1. $x-\mathbf{i}y=0$, $2y+z+1=0$
2. $3x+z-1=0$, $y+z-5=0$

8. For each of the following, find parametric equations for the line in $\mathbf{A}^3(\mathbb{C})$ passing through the point Q and parallel to the line ℓ.

1. $Q(1,1,0)$, $\ell : x-\mathbf{i}y=0, z+1=0$
2. $Q(2,1,-5)$, $\ell : y=2, x=\mathbf{i}z+7$

9. In each of the following, find a Cartesian equation of the plane in $\mathbf{A}^3(\mathbb{R})$ passing through Q and parallel to the lines ℓ and ℓ':

1. $Q(1,-1,-2)$, $\ell : x-y=1, x+z=5$, $\ell' : x=1, z=2$
2. $Q(0,1,3)$, $\ell : x+y=-5, x-y+2z=0$, $\ell : 2x-2y=1, x-y+2z=1$

10. In each of the following, determine whether the lines ℓ and ℓ' are skew or coplanar. If they are coplanar, find whether they are incident or parallel, and then, after checking that they are distinct, find a Cartesian equation for the plane containing them,

1. $\ell : x=1+t, y=-t, z=2+2t$, $\ell' : x=2-t, y=-1+3t, z=t$
2. $\ell : 2x+y+1=0, y-z=2$, $\ell' : x=2-t, y=3+2t, z=1$

11. In each of the following, find the relative positions of the line ℓ and the plane π in $\mathbf{A}^3(\mathbb{R})$, and, if they are incident, determine the point of intersection.

1. $\ell : x=1+t, y=2-2t, z=1-4t$, $\pi : 2x-y+z-1=0$
2. $\ell : x=2-t, y=1+2t, z=-1+3t$, $\pi : 2x+2y-z+1=0$

12. In each of the following, find a Cartesian equation for the plane in $\mathbf{A}^3(\mathbb{R})$ containing the point Q and the line ℓ.

1. $Q=(3,3,1)$, $\ell : x=2+3t, y=5+t, z=1+7t$
2. $Q=(2,1,0)$, $\ell : x-y+1=0, 3x+5z-7=0$

13. In each of the following, find Cartesian equations for the line ℓ in $\mathbf{A}^3(\mathbb{R})$ passing through Q, contained in the plane π and intersecting the line ℓ'

1. $Q=(1,1,0)$, $\pi : 2x-y+z-1=0$, $\ell' : x=2-t, y=2+t, z=t$
2. $Q=(-1,-1,-1)$, $\pi : x+y+z+3=0$, $\ell' : x-2z+4=0, 2y-z=0$

14. In each of the following, find Cartesian equations for the line ℓ in $\mathbf{A}^3(\mathbb{R})$ passing through Q and coplanar to the lines ℓ' and ℓ''. Furthermore, establisch whether ℓ meets or is parallel to ℓ' and ℓ''

1. $Q=(1,1,2)$, $\ell' : 3x-5y+z=-1, 2x-3z=-9$, $\ell'' : x+5y=3, 2x+2y-7z=-7$
2. $Q=(2,0,-2)$, $\ell' : -x+3y=2, x+y+z=-1$, $\ell'' : x=2-t, y=3+5t, z=-t$

15. In each of the following, find the value of the real parameter k for which the lines ℓ and ℓ' are coplanar. Find a Cartesian equation for the plane that contains them, and find the point of intersection whenever they meet

1. $\ell : x=k+t, y=1+2t, z=-1+kt$, $\ell' : x=2-2t, y=3+3t, z=1-t$
2. $\ell : x=3-t, y=1+2t, z=k+t$, $\ell' : x=1+t, y=1+2t, z=1+3t$

16. Find a Cartesian equation for the plane π in $\mathbf{A}^3(\mathbb{R})$ which contains the line of intersection of the two planes

$$x+y=3 \quad \text{and} \quad 2y+3z=4$$

and is parallel to the vector $\mathbf{v}=(3,-1,2)$.

1. Establish which of the following triples of points in $\mathbf{A}^3(\mathbb{C})$ are collinear:

1. $\{\overset{A}{(2,1,-3)},\overset{B}{(1,-1,2)},\overset{C}{(3/2,0,-1/2)}\}$
2. $\{(\mathbf{i},0,0),(1+\mathbf{i},2\mathbf{i},1),(1,2,-\mathbf{i})\}$

1. A, B, C collinear $\iff$ $\vec{AB}$, $\vec{BC}$ linearly dependent

$$\iff \frac{1-2}{\frac{3}{2}-1} = \frac{-1-1}{1} = \frac{2+3}{-\frac{1}{2}-2}$$

$$\frac{-1}{\frac{1}{2}} = \frac{-2}{1} = \frac{5}{-\frac{5}{2}} \quad \checkmark \ \checkmark$$

2. $\dfrac{1}{-i} = \dfrac{2i}{2-2i} = \dfrac{1}{-i-1}$ ↯ not collinear

2. In each of the following, find the value (if it exists) of the real parameter m for which the triple of points is collinear in $\mathbf{A}^3(\mathbb{R})$

1. $\{\overset{A}{(2,-1,2)},\overset{B}{(1,1,1)},\overset{C}{(4,-m+1,4)}\}$
2. $\{(3,0,0),(0,1,1),(m,m,m)\}$

Are the component of $\vec{AB}$ and $\vec{AC}$ proportional?

1. $\dfrac{-1}{2} = \dfrac{2}{-m+2} = \dfrac{-1}{2} \iff 4 = m-2 \iff m=2$

2. $\dfrac{-3}{m-3} = \dfrac{1}{m} = \dfrac{1}{m} \iff -3m = m-3 \iff m = \dfrac{3}{4}$

3. After checking, for each of the following, that the points are not collinear, find parametric and Cartesian equations for the planes determined by the points

1. $\{\overset{A}{(2,\sqrt{2},1)},\overset{B}{(1,1,\sqrt{2})},\overset{C}{(0,0,1)}\}$

2. $\{(5,-1,0),(1,1,\sqrt{5}),(-3,1,\pi/2)\}$

1. $\vec{CA} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$ $\quad \vec{CB} = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2}-1 \end{bmatrix}$ $\longrightarrow$ not proportional

$$\langle A,B,C\rangle : \begin{vmatrix} x & y & z-1 \\ 2 & \sqrt{2} & 0 \\ 1 & 1 & \sqrt{2}-1 \end{vmatrix} = 0$$

$$\iff x(2-\sqrt{2}) - y(2\sqrt{2}-2) + (z-1)(2-\sqrt{2}) = 0 \iff \dots = 0$$

2. $\vec{AB} = \begin{bmatrix} -4 \\ 2 \\ \sqrt{5} \end{bmatrix}$ $\quad \vec{AC} = \begin{bmatrix} -8 \\ 2 \\ \pi/2 \end{bmatrix}$ $\quad$ not proportional

$$\langle A,B,C\rangle : \begin{vmatrix} x-5 & y+1 & z \\ -4 & 2 & \sqrt{5} \\ -8 & 2 & \pi/2 \end{vmatrix} = 0$$

$$\iff (x-5)(\pi-2\sqrt{5}) - (y+1)(-2\pi+8\sqrt{5}) + z(-8+16) = 0$$

$$\iff \dots = 0$$

4. In each of the following, find a Cartesian equation of the plane in $\mathbf{A}^3(\mathbb{C})$ passing through Q and parallel to the plane π

1. $Q = (-1, 2, 2)$, $\pi : x + 2y + 3z + 1 = 0$
2. $Q = (\mathbf{i}, \mathbf{i}, \mathbf{i})$, $\pi : 2x - y = 0$

1. The vector subspace associated to π is $W: x + 2y + 3z = 0$

Let π' be the plane containing Q and parallel to π

Then $\pi' : x + 2y + 3z + d = 0$ for some $d \in \mathbb{C}$

Since $Q \in \pi'$: $-1 + 4 + 6 + d = 0 \Rightarrow d = -9$

$\Rightarrow \pi' : x + 2y + 3z - 9 = 0$

2. The vector subspace associated to π is $W: 2x - y = 0$

Let π' be the plane containing Q and parallel to π

Then $\pi' : 2x - y + d = 0$ for some $d \in \mathbb{C}$

Since $Q \in \pi'$: $2i - i + d = 0 \Rightarrow d = -i$

$\Rightarrow \pi' : 2x - y - i = 0$

5. For each of the following, determine whether or not the three planes belong to the same pencil

1. $x - y + z = 0$, $-x + 3y - 5z + 2 = 0$, $y - 2z + 1 = 0$
2. $2x - 3y + 3 = 0$, $x - y + 6 = 0$, $x - 3z = -1$

The three planes belong to the same pencil if they intersect in one line

$\iff$ the system with the three equations has solutions depending on one parameter

$\iff$ the rank of the matrix of the system and the rank of the augmented matrix are equal to 2

1.
$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 3 & -5 & 2 \\ 0 & 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank of matrix and augmented matrix is 2

$\Rightarrow$ the three planes are in the same pencil

2.
$$\begin{pmatrix} 2 & -3 & 0 & 3 \\ 1 & -1 & 0 & 6 \\ 1 & 0 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 6 \\ 0 & -1 & 0 & -9 \\ 0 & 1 & -3 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 6 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -14 \end{pmatrix}$$

$\Rightarrow$ rank of matrix and augmented matrix is 3

$\Rightarrow$ the three planes intersect in one point

$\Rightarrow$ the three planes are not in the same pencil

6. For each of the following, find parametric and Cartesian equations for the line in $\mathbf{A}^3(\mathbb{R})$ passing through the point Q and parallel to the vector $\mathbf{v}$.

1. $Q = (1,1,0)$, $\mathbf{v} = (2,-1,\sqrt{2})$
2. $Q = (-2,2,-2)$, $\mathbf{v} = (1,1,0)$

1. $$\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ \sqrt{2} \end{bmatrix} \quad t \in \mathbb{R}$$

$$\Rightarrow \quad t = \frac{x-1}{2} = \frac{y-1}{-1} = \frac{z}{\sqrt{2}}$$

$$\ell : \begin{cases} \frac{x-1}{2} = \frac{y-1}{-1} \\ \frac{x-1}{2} = \frac{z}{\sqrt{2}} \end{cases}$$

2. $$\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad t \in \mathbb{R}$$

$$\operatorname{rank} \begin{pmatrix} x+2 & y-2 & z+2 \\ 1 & 1 & 0 \end{pmatrix} = 1$$

$$\Leftrightarrow \ell : \begin{cases} x+2 = y-2 \\ z+2 = 0 \end{cases} \quad \Leftrightarrow \ell : \begin{cases} x - y = -4 \\ z = -2 \end{cases}$$

7. Find parametric equations for each of the following lines in $\mathbf{A}^3(\mathbb{C})$

1. $x - \mathbf{i}y = 0,\ 2y + z + 1 = 0$
2. $3x + z - 1 = 0,\ y + z - 5 = 0$

1. $\ell : \begin{cases} x - iy = 0 \\ 2y + z + 1 = 0 \end{cases}$ $\quad \begin{pmatrix} 1 & -i & 0 \\ 0 & 2 & 1 \end{pmatrix}$

$\Rightarrow$ a direction vector for ℓ is

$$\left(\begin{vmatrix} -i & 0 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -i \\ 0 & 2 \end{vmatrix} \right) = (-i, -1, 2)$$

Notice that $P(0, 0, -1) \in \ell$

$$\Rightarrow \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -i \\ -1 \\ 2 \end{bmatrix}$$

2. $\ell : \begin{cases} 3x + z - 1 = 0 \\ y + z - 5 = 0 \end{cases}$ $\quad \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\Rightarrow$ a dir. vect. for ℓ is

$$\left(\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} \right) = (-1, -3, 3)$$

Notice that $P(1/3, 5, 0) \in \ell$

$$\Rightarrow \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \quad t \in \mathbb{R}$$

as discussed during the lecture

Rem: There is a second method of obtaining parametric eq.

8. For each of the following, find parametric equations for the line in $\mathbf{A}^3(\mathbb{C})$ passing through the point Q and parallel to the line ℓ.

1. $Q(1,1,0)$, $\ell : x - \mathbf{i}y = 0, z+1 = 0$
2. $Q(2,1,-5)$, $\ell : y = 2, x = \mathbf{i}z + 7$

1. $$\ell : \begin{cases} x - iy = 0 \\ z + 1 = 0 \end{cases} \qquad \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \left(\begin{vmatrix} -i & 0 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -i \\ 0 & 0 \end{vmatrix} \right)$$

$= (-i, -1, 0)$ is a dir vector for ℓ

$$\Rightarrow \tilde{\ell} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -i \\ -1 \\ 0 \end{bmatrix} \quad \forall t \in \mathbb{C}$$

2. $$\ell : \begin{cases} y = 2 \\ x = iz + 7 \end{cases} \iff \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$$

dir vector for ℓ

$$\Rightarrow \tilde{\ell} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + t \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \quad \forall t \in \mathbb{C}$$

9. In each of the following, find a Cartesian equation of the plane in $\mathbf{A}^3(\mathbb{R})$ passing through Q and parallel to the lines ℓ and ℓ':

1. $Q(1,-1,-2)$, $\ell : x-y=1, x+z=5$, $\ell' : x=1, z=2$
2. $Q(0,1,3)$, $\ell : x+y=-5, x-y+2z=0$, $\ell : 2x-2y=1, x-y+2z=1$

1. $\ell : \begin{cases} x-y=1 \\ x+z=5 \end{cases} \iff \ell : \begin{cases} x=x \\ y=-1+x \\ z=5-x \end{cases} \iff \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + x \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad x \in \mathbb{R}$

$\ell' : \begin{cases} x=1 \\ z=2 \end{cases} \iff \ell' : \begin{cases} x=1 \\ y=y \\ z=2 \end{cases} \iff \ell' : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad y \in \mathbb{R}$

$\Rightarrow$ the plane parallel to ℓ and ℓ' and passing through Q has eq.:

$$\begin{vmatrix} x-1 & y+1 & z+2 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$$\iff (x-1)\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} - (y+1)\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} + (z+2)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 0$$

$$\iff x-1+z+2=0 \iff x+z+1=0$$

2. $\ell : \begin{cases} x+y=-5 \\ x-y+2z=0 \end{cases} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} \qquad \ell' : \begin{cases} 2x-2y=1 \\ x-y+2z=1 \end{cases} \quad \begin{pmatrix} 2 & -2 & 0 \\ 1 & -1 & 2 \end{pmatrix}$

$\Rightarrow$ the plane parallel to ℓ and ℓ' and passing through Q has eq.:

$$\begin{vmatrix} x-0 & y-1 & z-3 \\ \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 & -\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2 & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \\ \begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix} = 4 & -\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = -4 & \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = 0 \end{vmatrix} = 0 \iff \begin{vmatrix} x & y-1 & z-3 \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

$$\iff x-(y-1)+2(z-3)=0$$

$$\iff x-y+2z-5=0$$

10. In each of the following, determine whether the lines ℓ and ℓ' are skew or coplanar. If they are coplanar, find whether they are incident or parallel, and then, after checking that they are distinct, find a Cartesian equation for the plane containing them,

1. $\ell : x = 1 + t, y = -t, z = 2 + 2t$, $\ell' : x = 2 - t, y = -1 + 3t, z = t$
2. $\ell : 2x + y + 1 = 0, y - z = 2$, $\ell' : x = 2 - t, y = 3 + 2t, z = 1$

1. $\ell : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $\qquad$ $\ell' : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

ℓ and ℓ' coplanar $\iff$ $\begin{vmatrix} 2-1 & -1-0 & 0-2 \\ 1 & -1 & 2 \\ -1 & 3 & 1 \end{vmatrix} = 0$

$-1 - 6 + 2 + 2 + 1 - 6 = -8 \neq 0 \Rightarrow$ ℓ and ℓ' are not coplanar

$\Rightarrow$ since the direction vectors are not proportional, the two lines are skew

2. $\ell : \begin{cases} 2x + y + 1 = 0 \\ y - z = 2 \end{cases}$ $\qquad$ $\ell' : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ $\Rightarrow t = 2 - x$, $\Rightarrow t = \frac{y-3}{2}$

Notice that $(1, -3, 5) \in \ell$ $\qquad$ $\ell' : \begin{cases} 4 - 2x = y - 3 \\ z = 1 \end{cases} \iff \begin{cases} 2x + y - 7 = 0 \\ z = 1 \end{cases}$

ℓ and ℓ' are coplanar $\iff$ $\begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 2 & 1 & 0 & -7 \\ 0 & 0 & 1 & -1 \end{vmatrix} = 0$

$$= \begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 1 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 & -2 \\ 0 & 0 & -8 \\ 0 & 1 & -1 \end{vmatrix} = 2 \begin{vmatrix} 0 & -8 \\ 1 & -1 \end{vmatrix} \neq 0 \Rightarrow$$ ℓ and ℓ' are not coplanar

$\ell : \begin{cases} x = -1/2 - 1/2\, y \\ y = y \\ z = -2 + y \end{cases} \iff \ell : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ -2 \end{pmatrix} + y \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$

The direction vectors of ℓ and ℓ' are not proportional

$\Rightarrow$ ℓ is not parallel to ℓ', the two lines are skew

11. In each of the following, find the relative positions of the line ℓ and the plane π in $\mathbf{A}^3(\mathbb{R})$, and, if they are incident, determine the point of intersection.

1. $\ell : x = 1+t, y = 2-2t, z = 1-4t,\ \pi : 2x - y + z - 1 = 0$
2. $\ell : x = 2-t, y = 1+2t, z = -1+3t,\ \pi : 2x + 2y - z + 1 = 0$

1. $\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{Q} + t \underbrace{\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}}_{v}$ the associated vector subspace to π is

$\xrightarrow{?}$ $W : 2x - y + z = 0$

$2 + 2 - 4 = 0 \Rightarrow v \in W$

$\Rightarrow \langle v \rangle \subseteq W$

$\Rightarrow \ell \parallel \pi$

moreover $Q \in \pi$ since $2 - 2 + 1 - 1 = 0$

$\Rightarrow \ell \subseteq \pi$

Rem you can also replace $x = 1+t$, $y = 2-2t$, $z = 1-4t$ in the eq. of π to find out that it is satisfied $\forall t$

2. $\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t \underbrace{\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}}_{v}$ the associated vector subspace to π is

$\xrightarrow{?}$ $W : 2x + 2y - z = 0$

$-2 + 4 - 3 \neq 0 \Rightarrow \ell \not\parallel \pi$

$\ell \cap \pi : \ 2(2-t) + 2(1+2t) - (-1+3t) + 1 = 0$

$4 - 2t + 2 + 4t + 1 - 3t + 1 = 0$

$8 - t = 0 \quad \Rightarrow t = 8 \quad \Rightarrow \ell \cap \pi = \left\{ \begin{bmatrix} 6 \\ 17 \\ 23 \end{bmatrix} \right\}$

12. In each of the following, find a Cartesian equation for the plane in $\mathbf{A}^3(\mathbb{R})$ containing the point Q and the line ℓ.

π

1. $Q = (3,3,1)$, $\ell : x = 2+3t, y = 5+t, z = 1+7t$
2. $Q = (2,1,0)$, $\ell : x - y + 1 = 0, 3x + 5z - 7 = 0$

1. $$\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{\overset{\|}{P}}{\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}} + t \underset{\overset{\|}{v}}{\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}} \qquad \overrightarrow{QP} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$Q \in \pi$, $\ell \subseteq \pi$
$\overrightarrow{QP}$ and v are linearly indep $\Big\} \Rightarrow \langle \overrightarrow{QP}, v \rangle$ is the vector subspace associated to the plane

$$\pi : \begin{vmatrix} x-3 & y-3 & z-1 \\ 1 & -2 & 0 \\ 3 & 1 & 7 \end{vmatrix} = 0 \iff -14(x-3) - 7(y-3) + 7(z-1) = 0$$

$$\iff 2x - 6 + y - 3 - z + 1 = 0$$

$$\iff 2x + y - z - 8 = 0$$

2. $$\ell : \begin{cases} x - y + 1 = 0 \\ 3x + 5z - 7 = 0 \end{cases} \qquad \ell \in \pi \Rightarrow \pi \in \mathcal{L}_\ell$$

We use a reduced pencil

$$\pi : x - y + 1 + t(3x + 5z - 7) = 0$$

$$Q \in \pi \Rightarrow 2 - 1 + 1 + t(6-7) = 0 \iff 2 = t$$

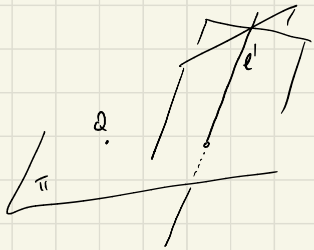
$$\Rightarrow \pi : x - y + 1 + 2(3x + 5z - 7) = 0$$

$$\Rightarrow \pi : 7x - y + 10z - 13 = 0$$

13. In each of the following, find Cartesian equations for the line ℓ in $\mathbf{A}^3(\mathbb{R})$ passing through Q, contained in the plane π and intersecting the line ℓ'

1. $Q = (1,1,0)$, $\pi : 2x - y + z - 1 = 0$, $\ell' : x = 2 - t, y = 2 + t, z = t$
2. $Q = (-1,-1,-1)$, $\pi : x + y + z + 3 = 0$, $\ell' : x - 2z + 4 = 0, 2y - z = 0$

1. $\ell' : \begin{cases} x = 2 - t \\ y = 2 + t \\ z = t \end{cases} \Rightarrow t = 2 - x = y - 2 = z$



$\Rightarrow \ell' : \begin{cases} x + y - 4 = 0 \\ y - z - 2 = 0 \end{cases}$

$\ell = \pi \cap \underbrace{\text{plane containing } \ell' \text{ and passing through } Q}_{\text{let } \pi' \text{ be this plane}}$

$\ell' \subseteq \pi'$, we use a reduced pencil $\pi' : x + y - 4 + t(y - z - 2) = 0$

$Q \in \pi'$ $\quad 2 - 4 + t(1 - 2) = 0 \iff t = -2$

$\Rightarrow \pi' : x - y + 2z = 0$

$\Rightarrow \ell = \pi \cap \pi' : \begin{cases} 2x - y + z - 1 = 0 \\ x - y + 2z = 0 \end{cases}$

2. $\ell' : \begin{cases} x - 2z + 4 = 0 \\ 2y - z = 0 \end{cases}$

$\ell' \subseteq \pi'$ and we use a reduced pencil

$\pi' : x - 2z + 4 + t(2y - z) = 0$

$Q \in \pi'$ $\quad 5 + t(-1) = 0 \iff t = 5$

$\Rightarrow \pi' : x - 2z + 4 + 5(2y - z) = 0$

$x + 10y - 7z + 4 = 0$

$\Rightarrow \ell = \pi \cap \pi' : \begin{cases} x + y + z + 3 = 0 \\ x + 10y - 7z + 4 = 0 \end{cases}$

14. In each of the following, find Cartesian equations for the line ℓ in $\mathbf{A}^3(\mathbb{R})$ passing through Q and coplanar to the lines ℓ' and ℓ''. Furthermore, establish whether ℓ meets or is parallel to ℓ' and ℓ''

1. $Q=(1,1,2)$, $\ell': 3x-5y+z=-1, 2x-3z=-9$, $\ell'': x+5y=3, 2x+2y-7z=-7$
2. $Q=(2,0,-2)$, $\ell': -x+3y=2, x+y+z=-1$, $\ell'': x=2-t, y=3+5t, z=-t$

1. For a line ℓ to be coplanar to ℓ' and ℓ'', ℓ' and ℓ'' have to be coplanar

$$\ell': \begin{cases} 3x-5y+z=-1 \\ 2x-3z=-9 \end{cases} \qquad \ell'': \begin{cases} x+5y=3 \\ 2x+2y-7z=-7 \end{cases}$$

$$\begin{pmatrix} 3 & -5 & 1 & 1 \\ 2 & 0 & -3 & 9 \\ 1 & 5 & 0 & -3 \\ 2 & 2 & -7 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & 0 & -3 \\ 0 & -20 & 1 & 10 \\ 0 & -10 & -3 & 15 \\ 0 & -8 & -7 & 13 \end{pmatrix} \leftarrow \text{has determinant } 4$$

$\Downarrow$

rank of extended matrix of system $=4$

$\Downarrow$

lines ℓ' and ℓ'' are not coplanar

they are skew

$\Rightarrow \nexists\, \ell$

2. $\ell': \begin{cases} -x+3y=2 \\ x+y+z=-1 \end{cases}$ $\begin{pmatrix} -1 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ $\ell' \ni (-2,0,1) \Rightarrow \ell': \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$

$\hookrightarrow$ has dir. vector $\left(\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \right) = (3,1,-4)$

$$\ell'': \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$$

ℓ' and ℓ'' coplanar if $\begin{vmatrix} -2-2 & 0-3 & 1-0 \\ 3 & 1 & -4 \\ -1 & 5 & -1 \end{vmatrix} = 0$ $\quad \begin{vmatrix} -4 & -3 & 1 \\ 3 & 1 & -4 \\ -1 & 5 & -1 \end{vmatrix} = 4+15-12+1-9-80 = -81 \neq 0$

$\Rightarrow \ell'$ and ℓ'' are not coplanar, they are skew

$\Rightarrow \nexists\, e$

15. In each of the following, find the value of the real parameter k for which the lines ℓ and ℓ' are coplanar. Find a Cartesian equation for the plane that contains them, and find the point of intersection whenever they meet

1. $\ell : x = k + t, y = 1 + 2t, z = -1 + kt,\ \ell' : x = 2 - 2t, y = 3 + 3t, z = 1 - t$
2. $\ell : x = 3 - t, y = 1 + 2t, z = k + t,\ \ell' : x = 1 + t, y = 1 + 2t, z = 1 + 3t$

$$\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix} \qquad \ell' : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

ℓ and ℓ' coplanar if $\begin{vmatrix} 2-k & 2 & 2 \\ 1 & 2 & k \\ -2 & 3 & -1 \end{vmatrix} = 0$

$$\Leftrightarrow (2-k)(-2-3k) - 2(-1+2k) + 2 \cdot 7 = 0$$

$$\Leftrightarrow -4 - 6k + 2k + 3k^2 + 2 - 4k + 14 = 0$$

$$\Leftrightarrow 3k^2 - 8k + 12 = 0$$

$\Delta = 64 - 12^2 < 0 \Rightarrow$ there are no values k for which ℓ and ℓ' are coplanar

2. $$\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \ell' : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

ℓ and ℓ' coplanar if $\begin{vmatrix} 2 & 0 & k-1 \\ -1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$

$$\Leftrightarrow 8 + (k-1)(-4) = 0$$

$$\Leftrightarrow k - 1 = 2 \Leftrightarrow k = 3$$

16. Find a Cartesian equation for the plane π in $\mathbf{A}^3(\mathbb{R})$ which contains the line of intersection of the two planes

$$x + y = 3 \quad \text{and} \quad 2y + 3z = 4$$

and is parallel to the vector $\mathbf{v} = (3, -1, 2)$.

We use a reduced pencil

$$\pi: \quad x + y - 3 + \lambda(2y + 3z - 4) = 0 \qquad \lambda \in \mathbb{R}$$

The vector subspace associated to π is

$$W: \quad x + y + \lambda(2y + 3z) = 0$$

$$v \in W \iff 2 + \lambda(-2 + 6) = 0 \iff \lambda = -\frac{1}{2}$$

$$\Rightarrow \pi: \ 2x + 2y - 6 - 2y - 3z + 4 = 0$$

$$\pi: \ 2x - 3z - 2 = 0$$