ANALYTIC GEOMETRY, PROBLEM SET 4

Projections. Dot product. Cross product.

1. Find the orthogonal projection $pr_{\overline{u}}(\overline{v})$, where $\overline{v} = 10\overline{a} + 2\overline{b}$, $\overline{u} = 5\overline{a} - 12\overline{b}$, if $\overline{a} \perp \overline{b}$ and $||\overline{a}|| = ||\overline{b}|| \neq 0$.

2. Using the dot product, prove the **Cauchy-Buniakowski-Schwarz** inequality, i.e. show that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

3. For a tetrahedron $ABCD$, show that $\cos(\widehat{\overline{AB}, \overline{CD}}) = \frac{AD^2+BC^2-AC^2-BD^2}{2AB \cdot CD}$. (the 3D version of the **cosine theorem**)

4. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds: $9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$.

5. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ and $A'B'$, respectively, are concurrent. Then the perpendicular lines through A', B', C' on BC, CA and AB, respectively are also concurrent. (Steiner's theorem on **orthologic triangles**)

6. Find the area of the plane triangle having the vertices $A(1, 0, 1)$, $B(0, 2, 3)$, $C(2, 1, 0)$.

7. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \overline{a}$, $\overline{CA} = \overline{b}$ and $\overline{AB} = \overline{c}$ if and only if $\overline{a} \times \overline{b} = \overline{b} \times \overline{c} = \overline{c} \times \overline{a}$.

8. Find a vector orthogonal on both $\overline{u}$ and $\overline{v}$, if:
a) $\overline{u} = -7\overline{i} + 3\overline{j} + \overline{k}$ and $\overline{v} = 2\overline{i} + 4\overline{k}$ b) $\overline{u} = (-1, -1, -1)$ and $\overline{v} = (2, 0, 2)$.

9. Let a, b, and c denote the lengths of the sides of $\triangle ABC$. We write O for its circumcenter, R for the length of its circumradius, H for its orthocenter and G for the centroid. Show that
a) $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$; b) $OG^2 = R^2 - 1/9(a^2 + b^2 + c^2)$.

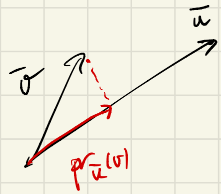
I $w(x,y,z)$ $\quad w \perp u \Leftrightarrow w \cdot u = 0 \Leftrightarrow \begin{cases} -7x + 3y + z = 0 \\ 2x + 4z = 0 \end{cases}$ .....
$\quad w \perp v \Leftrightarrow w \cdot v = 0 \Leftrightarrow$

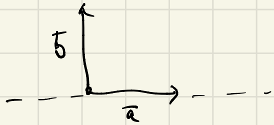
II $w(x,y,z)$ $\quad w \perp u$ and $w \perp v \Leftrightarrow w \parallel u \times w$
↳ calculate this.

Date: October 14, 2021.

1. Find the orthogonal projection $pr_{\overline{u}}(\overline{v})$, where $\overline{v} = 10\overline{a} + 2\overline{b}$, $\overline{u} = 5\overline{a} - 12\overline{b}$, if $\overline{a} \perp \overline{b}$ and $||\overline{a}|| = ||\overline{b}|| \neq 0$.

$$pr_{\bar u}(\bar v) = \frac{\bar v \cdot \bar u}{\|\bar u\|^2}\,\bar u$$





$$\bar v \cdot \bar u = \left(\|\bar v\| \cdot \|\bar u\| \cos(\bar v, \bar u)\right)$$

$$= (10\bar a + 2\bar b)\cdot(5\bar a - 12\bar b) \qquad \|\bar b\|^2 = \|\bar a\|^2$$

$$= 50\bar a^2 - 120\underset{0}{\cancel{\bar a\cdot\bar b}} + 10\underset{0}{\cancel{\bar b\cdot\bar a}} - 24\bar b^2 = 26\,\bar a^2$$

$$\Big(\bar a^2 = a\cdot a = \underset{x_a^2+y_a^2+z_a^2}{\|\bar a\|^2}\Big)$$

$$\Rightarrow pr_{\bar u}(v) = \frac{26\cancel{\bar a^2}}{\underset{?}{c}\cancel{\bar a^2}}(5\bar a - 12\bar b)$$

$$\|\bar u\|^2 = (5\bar a - 12\bar b)^2 = \dots = \underset{?}{c}\,\bar a^2$$

2. Using the dot product, prove the **Cauchy-Buniakowski-Schwarz** inequality, i.e. show that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

$v_a(a_1, a_2, a_3)$

$v_b(b_1, b_2, b_3)$

I $\quad (v_a \cdot v_b)^2 \le v_a^2 \cdot v_b^2$

$$\cancel{\|v_a\|^2}\cdot\cancel{\|v_b\|^2}\cdot\cos(v_a\, v_b) \le \cancel{\|v_a\|^2}\cdot\cancel{\|v_b\|^2}$$

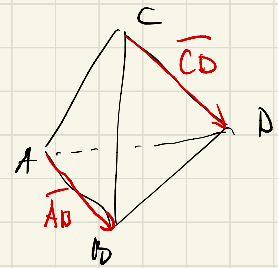
$$\cos(v_a v_b) \le 1$$

II $\quad$ Lagrange's identity

3. For a tetrahedron $ABCD$, show that $\cos(\widehat{\overline{AB},\overline{CD}}) \overset{(*)}{=} \frac{AD^2+BC^2-AC^2-BD^2}{2AB\cdot CD}$. (the 3D version of the **cosine theorem**)

$$\| $$

$$\frac{\overline{AB}\cdot\overline{CD}}{\|\overline{AB}\|\cdot\|\overline{CD}\|}$$



So $(*) \iff 2\,\overline{AB}\cdot\overline{CD} \overset{(**)}{=} \overline{AD}^2+\overline{BC}^2-\overline{AC}^2-\overline{BD}^2$

$$\overline{AB}\cdot\overline{CD} = (\overline{AD}+\overline{DB})\cdot(\overline{CA}+\overline{AD})$$

$$= \overline{AD}\cdot\overline{CA} + \overline{AD}^2 + \underbrace{\overline{DB}\cdot\overline{CA}}_{(\overline{DC}+\overline{CD})(\overline{CB}+\overline{BA})} + \overline{DB}\cdot\overline{AD}$$

$$\overline{DC}\cdot\overline{CB} + \overline{DC}\cdot\overline{BA} + \overline{CB}^2 + \overline{CB}\cdot\overline{BA}$$

$$= \overline{AD}^2 + \overline{BC}^2 + \overline{AD}\cdot\overline{CA} + \overline{DB}\cdot\overline{AD} + \overline{DC}\cdot\overline{CB} + \underbrace{\overline{DC}\cdot\overline{BA} + \overline{CB}\cdot\overline{BA}}_{\overline{DB}\cdot\overline{BA}}$$

$$\underbrace{\overline{DB}\cdot\overline{BD}}_{-\overline{BD}^2}$$

$$\overline{DB} = -\overline{BD}$$

$$= \overline{AD}^2 + \overline{BC}^2 - \overline{BD}^2 + \overline{AD}\cdot\overline{CA} + \overline{DC}\cdot\overline{CB}$$

$$\overline{AD}\cdot\overline{CA} = (\overline{AC}+\overline{CD})\overline{CA}$$

$$\underbrace{\overline{AC}\cdot\overline{CA}}_{-\overline{AC}^2} + \overline{CD}\cdot\overline{CA}$$

$$\overline{CA} = -\overline{AC}$$

$$= \overline{AD}^2 + \overline{DC}^2 - \overline{BD}^2 - \overline{AC}^2 + \overline{CD}\cdot(\underbrace{\overline{CA}+\overline{BC}}_{\overline{BA}})$$

$$\overline{AB}\cdot\overline{CD} = \overline{AD}^2 + \overline{DC}^2 - \overline{BD}^2 - \overline{AC}^2 - \overline{AB}\cdot\overline{CD}$$

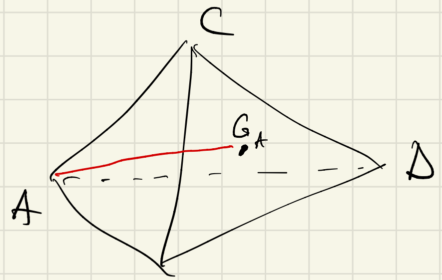
$\Rightarrow$ (**) is true.

4. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds: $9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$.

$\forall$ pt O

$$\overrightarrow{\overset{}{\underset{A}{\cancel{O}}}G} = \frac{\overline{\overset{A}{\cancel{O}}C} + \overline{\overset{A}{\cancel{O}}B} + \overline{\overset{A}{\cancel{O}}D}}{3}$$

$$\overrightarrow{AG}^2 = \left(\frac{\overline{AC} + \overline{AB} + \overline{AD}}{3}\right)^2$$



$$\Rightarrow 9AG^2 = \left(\overline{AC} + \overline{AB} + \overline{AD}\right)^2$$

$$= \overline{AC}^2 + \overline{AB}^2 + \overline{AD}^2 + \overbrace{2\overline{AC}\cdot\overline{AB}}^{\|\overline{AB}\|^2+\|\overline{AC}\|^2-\|\overline{CB}\|^2} + \underbrace{2\overline{AC}\cdot\overline{AD}}_{\cdots} + \underbrace{2\overline{AB}\cdot\overline{AD}}_{\cdots}$$

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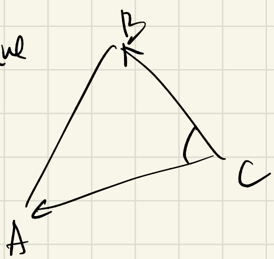
law of cosine



$$\|\overline{AB}\|^2 = \|\overline{CA}\|^2 + \|\overline{CB}\|^2 - 2\underbrace{\|\overline{CB}\|\cdot\|\overline{CA}\|}\cdot\cos\widehat{\overline{CA}\,\overline{CB}}$$

$$\cos\widehat{\overline{CA}\,\overline{CB}} = \frac{\overline{CA}\cdot\overline{CB}}{\|\overline{CA}\|\cdot\|\overline{CB}\|}$$

$$\overline{CA}\cdot\overline{CB} = \frac{1}{2}\left(\underset{\overline{CA}^2}{\|\overline{CA}\|^2} + \|\overline{CB}\|^2 - \|\overline{AB}\|^2\right)$$

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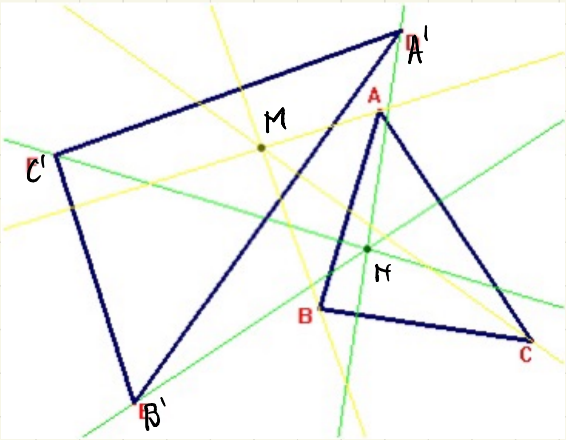
5. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ and $A'B'$, respectively, are concurrent. Then the perpendicular lines through A', B', C' on BC, CA and AB, respectively are also concurrent.
(Steiner's theorem on **orthologic triangles**)



We know

(1) $\overline{AM} \cdot \overline{B'C'} = 0$

(2) $\overline{BM} \cdot \overline{C'A'} = 0$

(3) $\overline{CM} \cdot \overline{A'B} = 0$

Let N be the intersection of the perp. line from A' on BC with the perp. line from B' on CA

we know that (4) $\overline{A'N} \cdot \overline{BC} = 0$

(5) $\overline{B'N} \cdot \overline{CA} = 0$

We need to show that N lies on the perp line from C' on AB

$\iff$ (6) $\overline{C'N} \cdot \overline{AB} = 0$

$$\iff \left(\overline{C'A'} + \overline{A'N}\right) \cdot \left(\overline{AC} + \overline{CB}\right) = 0$$

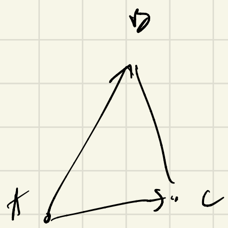
by (4)

$$\iff \underbrace{\overline{C'A'} \cdot \overline{AC} + \overline{C'A'} \cdot \overline{CB}}_{\overline{C'A'} \cdot \overline{AB}} + \underbrace{\overline{A'N} \cdot \overline{AC}}_{\underbrace{\left(\overline{A'B'} + \overline{B'N}\right) \cdot \overline{AC}}_{\overset{(5)}{=} \overline{A'B'} \cdot \overline{AC}}} + \cancel{\overline{A'N} \cdot \overline{CB}}^{\,0} = 0$$

$$(6) \iff \overline{C'A'} \cdot \overline{AB} + \overline{A'B'} \cdot \overline{AC} = 0$$

finish this by showing that (*) follows from (1) (2) (3)

6. Find the area of the plane triangle having the vertices $A(1,0,1)$, $B(0,2,3)$, $C(2,1,0)$.



$\frac{1}{2}\|\overline{AB}\times\overline{AC}\| = \text{Area } \Delta ABC$

$\overline{AB}(-1,2,2)$ $\overline{AC}(1,1,-1)$

$$\overline{AB}\times\overline{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & 2 \\ 1 & 1 & -1 \end{vmatrix} = \ldots$$

7. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \overline{a}$, $\overline{CA} = \overline{b}$ and $\overline{AB} = \overline{c}$ if and only if $\overline{a}\times\overline{b} = \overline{b}\times\overline{c} = \overline{c}\times\overline{a}$. (*)

(*) this triangle exists $\Leftrightarrow$ $\bar{a}+\bar{b}+\bar{c}=0$

so we need to show that $\bar{a}+\bar{b}+\bar{c}=0 \Leftrightarrow \bar{a}\times\bar{b}=\bar{b}\times\bar{c} = \bar{c}\times\bar{a}$

if $\bar{a}+\bar{b}+\bar{c}=0 \;\Big|\; \times\bar{a}$ then.

$\cancel{\bar{a}\times\bar{a}}^{\,0}+\bar{b}\times\bar{a}+\bar{c}\times\bar{a}=0$

$\bar{b}\times\bar{a}+\bar{c}\times\bar{a}=0$ $\qquad \bar{c}\times\bar{a}=-\bar{b}\times\bar{a}=\bar{a}\times\bar{b}$

....

if $\bar{a}\times\bar{b}=\bar{b}\times\bar{c}=\bar{c}\times\bar{a}$ then

$\bar{a}\times\bar{b}+\bar{c}\times\bar{b}=0 \Rightarrow (\bar{a}+\bar{c})\times\bar{b}=0 \Rightarrow \bar{a}+\bar{c}\parallel\bar{b}$

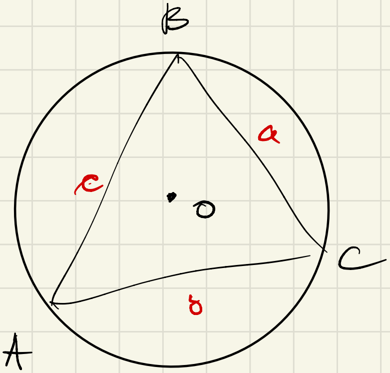
i.e $\bar{a}+\bar{c}=\alpha\bar{b}$

similar $\longrightarrow$ $\bar{a} + \bar{b} = \beta \cdot \bar{c}$

$\bar{b} + \bar{c} = \gamma \cdot \bar{a}$

. . . . you find that $\alpha = \beta = \gamma = -1$

9. Let a, b, and c denote the lengths of the sides of $\triangle ABC$. We write O for its circumcenter, R for the length of its circumradius, H for its orthocenter and G for the centroid. Show that
a) $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$; b) $OG^2 = R^2 - 1/9(a^2 + b^2 + c^2)$.



a.) $OH^2 = ?$

$$\overrightarrow{OH}^2 = (\overline{OA} + \overline{OB} + \overline{OC})^2$$

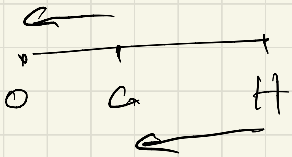
$$= \overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2 + \underbrace{2\,\overline{OA} \cdot \overline{OB}}_{\shortparallel} + \ldots$$

$$R^2 + R^2 + R^2 + \|\overline{OA}\|^2 + \overline{OB}^2 - \overline{AB}^2 + \ldots$$

$$R^2 \quad R^2 - c^2$$

b.) $\overline{OG}^2 = ?$

$$2\overrightarrow{GO} = \overrightarrow{HG}$$



$$\overline{OG}^2 = \left(\frac{\overline{OH}}{3}\right)^3 = \ldots$$