Seminar 3 March 9, 2021

$\boxed{1}$ Determine the following limits:

1) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\frac{1}{xy}$;

2) $\displaystyle\lim_{(x,y)\to(0,2)} \frac{\sin(xy)}{x}$;

3) $\displaystyle\lim_{(x,y)\to(a,a)} \frac{\frac{x+y}{2}-\frac{2xy}{x+y}}{\frac{x+y}{2}-\sqrt{xy}}$, $\quad a>0$;

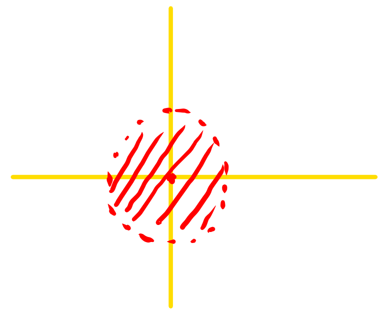
4) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$;

5) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$;

6) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy}$;

Solution 1) Let $f: A \to \mathbb{R}$, $f(x,y) = (x^2+y^2)\sin\frac{1}{xy}$

$A = \{(x,y) \mid x \neq 0 \text{ and } y \neq 0\}$



$$\lim_{\substack{x\to0\\y\to0}} f(x,y) = 0 \iff \lim_{\substack{x\to0\\y\to0}} |f(x,y)-0| = 0$$

$$0 \le |f(x,y)-0| = (x^2+y^2)\,\overbrace{\left|\sin\frac{1}{xy}\right|}^{\le 1} \le x^2+y^2 \xrightarrow[\substack{x\to0\\y\to0}]{} 0$$

2) $$\lim_{\substack{x\to0\\y\to2}} \frac{\sin(xy)}{x} = \lim_{\substack{x\to0\\y\to2}} \frac{\sin(xy)}{xy}\cdot y$$

$$= \lim_{\substack{x\to0\\y\to2}} \frac{\sin(xy)}{xy} \cdot \underbrace{\lim_{\substack{x\to0\\y\to2}} y}_{=2} = 2\lim_{t\to0}\frac{\sin t}{t} = 2$$

$t = xy$

3) $$\lim_{\substack{x\to a\\ y\to a}} \frac{\frac{x+y}{2}-\frac{2xy}{x+y}}{\frac{x+y}{2}-\sqrt{xy}} = \lim_{\substack{x\to a\\ y\to a}} \frac{(x+y)^2-4xy}{\cancel{2}(x+y)}\cdot\frac{\cancel{2}}{x+y-2\sqrt{xy}} = \lim_{\substack{x\to a\\ y\to a}} \frac{(x-y)^2}{x+y}\cdot\frac{1}{(\sqrt{x}-\sqrt{y})^2}$$

$$= \lim_{\substack{x\to a\\ y\to a}} \frac{\cancel{(\sqrt{x}-\sqrt{y})^2}(\sqrt{x}+\sqrt{y})^2}{(x+y)\cancel{(\sqrt{x}-\sqrt{y})^2}} = \frac{(2\sqrt{a})^2}{2a} = \boxed{2}$$

$$x-y=(\sqrt{x})^2-(\sqrt{y})^2$$

4) $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2} \qquad f(x,y)=\frac{x^2-y^2}{x^2+y^2}$$

$$\left.\begin{array}{ll} \left(\frac{1}{k},\frac{1}{k}\right)\xrightarrow{k\to\infty}(0,0) & f\left(\frac{1}{k},\frac{1}{k}\right)=0\xrightarrow{k\to\infty}0 \\ \left(\frac{2}{k},\frac{1}{k}\right)\xrightarrow{k\to\infty}(0,0) & f\left(\frac{2}{k},\frac{1}{k}\right)=\frac{\frac{3}{k^2}}{\frac{5}{k^2}}=\frac{3}{5}\longrightarrow\frac{3}{5} \end{array}\right\} \Rightarrow \nexists \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2-y^2}{x^2+y^2}$$

<u>**Met II**</u>

$y=mx$ $$\lim_{x\to 0} f(x,mx)=\lim_{x\to 0}\frac{x^2-m^2x^2}{x^2+m^2x^2}=\frac{1-m^2}{1+m^2}$$ depends on m

$$\Downarrow$$

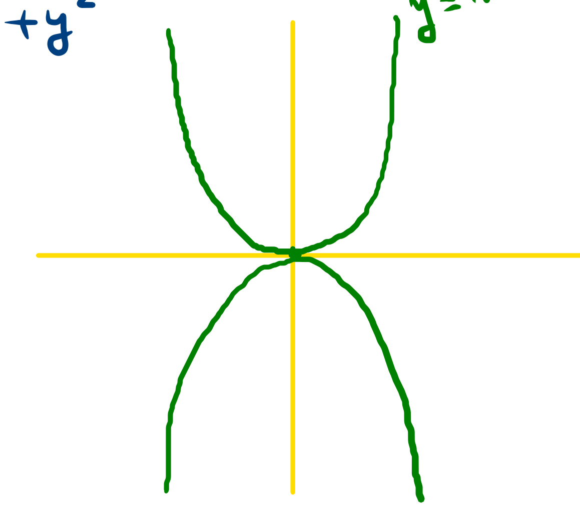
$$\nexists \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$$

5) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^3+y^3}{x^2+y^2}$ $\qquad f(x,y)=\dfrac{x^3+y^3}{x^2+y^2}$

$$0 \le |f(x,y)-0| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x|^3+|y|^3}{x^2+y^2} = \frac{|x|^3}{x^2+y^2}+\frac{|y|^3}{x^2+y^2}$$

$$= \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1}\cdot|x| + \underbrace{\frac{y^2}{x^2+y^2}}_{\le 1}\cdot|y|$$

$$\le |x|+|y| \xrightarrow[y\to 0]{x\to 0} 0$$



$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+y^3}{x^2+y^2} = 0$$

6) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^3+y^3}{xy}$ $\qquad f(x,y)=\dfrac{x^3+y^3}{xy}$

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x^3+m^3x^3}{mx^2} = 0 \text{ independent of } m \Longrightarrow ?$$

$$\left(\frac{1}{k},\frac{1}{k}\right) \longrightarrow (0,0) \qquad f\left(\frac{1}{k},\frac{1}{k}\right) = \frac{2/k^3}{1/k^2} = \frac{2}{k} \xrightarrow{k\to\infty} 0$$

$$\left(\frac{1}{k},\frac{1}{k^2}\right) \longrightarrow (0,0) \qquad f\left(\frac{1}{k},\frac{1}{k^2}\right) = \frac{\frac{1}{k^3}+\frac{1}{k^6}}{\frac{1}{k^3}} = 1+\frac{1}{k^3} \xrightarrow{k\to\infty} 1$$

$$\Rightarrow \nexists \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$$

$$\lim_{x\to 0} f(x, mx^2) = \lim_{x\to 0} \frac{x^3 + m^3 x^6}{m x^3} = \frac{1}{m} \text{ depends on } m \Rightarrow \nexists \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$$

! $\lim_{x\to 0} f(x, mx^2)$ does not depend on m $\Rightarrow$?

$\boxed{2}$ Find the following limits:

1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = 0.$

2) $\displaystyle\lim_{(x,y)\to(0,0)} (1+x^2y^2)^{-\frac{1}{x^2+y^2}};$

3) $\displaystyle\lim_{(x_1,\ldots,x_n)\to 0_n} \frac{x_1\cdots x_n}{x_1^2+\cdots+x_n^2},\qquad n\in\mathbb{N};$

$$\frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$

$$\frac{x^2+y^2}{2} \le \sqrt{\frac{x^4+y^4}{2}}$$

$$\frac{(x^2+y^2)^2}{2} \le x^4+y^4 \Rightarrow \frac{1}{x^4+y^4} \le \frac{2}{(x^2+y^2)^2}$$

Rezolvare 1) Let $f(x,y) = \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}$

$$0 \le |f(x,y) - 0| = \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} \le$$

$$\le \frac{2}{(x^2+y^2)^2}\cdot e^{-\frac{1}{x^2+y^2}} \xrightarrow[y\to 0]{x\to 0} 0$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{2}{(x^2+y^2)^2}\, e^{-\frac{1}{x^2+y^2}} = \lim_{t\to\infty} 2t^2 e^{-t}$$

$$t = \frac{1}{x^2+y^2}$$

$$= \lim_{t\to\infty} \frac{2t^2}{e^t} = 0$$

3) Let $$\ell_n = \lim_{(x_1,\dots,x_n)\to 0_n} \frac{x_1 x_2 \cdots x_n}{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$n=1$ $$\ell_1 = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \quad \nexists$$

$n=2$ $$\ell_2 = \lim_{\substack{x_1\to 0\\ x_2\to 0}} \frac{x_1x_2}{x_1^2+x_2^2} \quad \nexists \text{ (see the course)}$$

$n\geq 3$ $$f(x_1,x_2,\dots,x_n) = \frac{x_1x_2\cdots x_n}{x_1^2+x_2^2+\cdots+x_n^2}$$

$$0 \leq |f(x_1,x_2,\dots,x_n) - 0| = \frac{|x_1|\,|x_2|\cdots|x_n|}{x_1^2+x_2^2+\cdots+x_n^2} \leq \frac{(x_1^2+\cdots+x_n^2)^{\frac{n-2}{2}}}{n^{n/2}}$$

$$0 \quad (x_1,\dots,x_n)\to 0_n$$

$$\sqrt[n]{x_1^2\cdot x_2^2\cdots x_n^2} \leq \frac{x_1^2+x_2^2+\cdots+x_n^2}{n} \Big|^{n} \Rightarrow x_1^2x_2^2\cdots x_n^2 \leq \frac{(x_1^2+x_2^2+\cdots+x_n^2)^n}{n}\Big|^{1/2}$$

$$\Rightarrow |x_1|\cdot|x_2|\cdots|x_n| \leq \frac{(x_1^2+\cdots+x_n^2)^{n/2}}{n^{n/2}} \Big| \cdot \frac{1}{x_1^2+\cdots+x_n^2}$$

R: $$\ell_n \begin{cases} \nexists & \text{pt } n=1,2 \\ 0 & \text{pt. } n\geq 3 \end{cases}$$

$\boxed{3}$ Let B be a closed subset of $\mathbb{R}^n$, and let $f_1, \dots, f_p, g_1, \dots, g_q : B \to \mathbb{R}$ be continuous functions on B. Prove that the set

$$A := \{ x \in B \mid f_i(x) = 0 \quad \forall i = \overline{1,p}, \quad g_j(x) \leq 0 \quad \forall j = \overline{1,q} \}$$

is closed.

<u>Solution</u>. $\divideontimes$ A closed $\Leftrightarrow$ $\forall (x_k)$ convergent sequence of points in A we have $\lim\limits_{k\to\infty} x_k \in A$.

Let (x_k) be an arbitrary convergent seq. of points in A, and let $x := \lim\limits_{k\to\infty} x_k$

$$\divideontimes \quad x \in A \Leftrightarrow \begin{cases} \cdot\ x \in B \\ \cdot\ f_i(x) = 0 & \forall i = \overline{1,p} \\ \cdot\ g_j(x) \leq 0 & \forall j = \overline{1,q} \end{cases}$$

$\cdot$ $x_k \in A \Rightarrow x_k \in B \quad \forall k \geq 1$, B closed $\Big| \Rightarrow x \in B$

$\cdot$ Fix $i \in \{1, \dots, p\}$. Since $x_k \in A \Rightarrow f_i(x_k) = 0 \quad \forall k \geq 1$

f_i continuous at x $\Rightarrow \lim\limits_{k\to\infty} \overbrace{f_i(x_k)}^{=0} = f_i(x) \Rightarrow f_i(x) = 0$

$\cdot$ Fix $j \in \{1, \dots, q\}$

$x_k \in A \Rightarrow g_j(x_k) \leq 0 \quad \forall k \geq 1 \quad \Big| k \to \infty$

$\Rightarrow \underbrace{\lim\limits_{k\to\infty} g_j(x_k)}_{= g_j(x)} \leq 0 \quad \Rightarrow g_j(x) \leq 0.$

$A = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1 \} \to$ <u>closed</u>

$$\Updownarrow$$

$$f(x,y,z) = x^2+y^2+z^2-1=0$$

$A = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1,\ x+y+z=1 \} \to$ closed

$$\Updownarrow$$

$$g(x,y,z) = x+y+z-1=0.$$

$\boxed{4}$ Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function, and let G be an open subset of $\mathbb{R}^m$. Prove that $f^{-1}(G)$ is an open subset of $\mathbb{R}^n$.

<u>Solution</u>. $f^{-1}(G) = \{ x \in \mathbb{R}^n \mid f(x) \in G \}$

$f^{-1}(G)$ open $\iff$ $\forall a \in f^{-1}(G)\ \exists r>0$ s.t. $B(a,r) \subseteq f^{-1}(G)$

Let $a \in f^{-1}(G) \Rightarrow \left.\begin{array}{l} f(a) \in G \\ G \text{ open} \end{array}\right| \Rightarrow \exists \varepsilon > 0$ s.t. $B(f(a),\varepsilon) \subseteq G$

f continuous at $a \Rightarrow \exists r>0$ s.t. $\forall x \in \mathbb{R}^n$ with $\|x-a\|<r$: $\|f(x)-f(a)\|<\varepsilon$

$\Rightarrow \forall x \in B(a,r)$: $f(x) \in B(f(a),\varepsilon) \subseteq G \iff x \in f^{-1}(G)$ $\Rightarrow B(a,r) \subseteq f^{-1}(G)$

$\boxed{5}$ Let $f:[a,b]\longrightarrow\mathbb{R}$ be a function, and let $G_f := \{(x,f(x)) \mid x\in[a,b]\}$ denote its graph. Prove that f is continuous on $[a,b]$ $\Longleftrightarrow$ the set G_f is compact in $\mathbb{R}^2$

<u>Solution</u> $\boxed{\Rightarrow}$ Assume that f is continuous on $[a,b]$.

~~?~~ G_f compact $\Longleftrightarrow$ G_f sequentially compact

Let (z_k) be an arbitrary sequence in G_f

$\forall k\geq 1$ z_k has the form $z_k=(x_k, f(x_k))$ where $x_k\in[a,b]$.

$\Rightarrow$ (x_k) is a sequence in $[a,b]$

$\Rightarrow$ $\exists (x_{k_j})$ subseq. of (x_k) and $\exists\, x\in[a,b]$ s.t. $\lim\limits_{j\to\infty} x_{k_j}=x$

Since f is continuous at x $\Rightarrow$ $\lim\limits_{j\to\infty} f(x_{k_j})=f(x)$

$\Rightarrow \lim\limits_{j\to\infty}(x_{k_j}, f(x_{k_j}))=(x,f(x))$

$\lim\limits_{j\to\infty} z_{k_j}=(x,f(x))\in G_f$

$\boxed{\Leftarrow}$ Assume that G_f is compact in $\mathbb{R}^2$.

Suppose that $\exists x_0 \in [a,b]$ s.t. f is not continuous at x_0

$\Rightarrow \exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x \in [a,b]$ with $|x - x_0| < \delta$ s.t. $|f(x) - f(x_0)| \geq \varepsilon$

In particular, for $\delta = 1/k$ $\Rightarrow$

$\Rightarrow \forall k \geq 1$ $\exists x_k \in [a,b]$ with $|x_k - x_0| < 1/k$ s.t. $|f(x_k) - f(x_0)| \geq \varepsilon$

$\left.\begin{array}{l} \Rightarrow ((x_k, f(x_k))) \text{ is a seq. in } G_f \\ G_f \text{ is compact} \end{array}\right\} \Rightarrow \exists (k_j)$ strictly increasing sequence of natural numbers $\exists (x, f(x)) \in G_f$ s.t.

$\lim_{j\to\infty} (x_{k_j}, f(x_{k_j})) = (x, f(x)) \Rightarrow \lim_{j\to\infty} x_{k_j} = x$ and $\lim_{j\to\infty} f(x_{k_j}) = f(x)$

Since $|x_k - x_0| < \frac{1}{k}$ $\forall k \geq 1 \Rightarrow \lim_{k\to\infty} x_k = x_0$

$\Downarrow$

$x = x_0$

$\Downarrow$

$\lim_{j\to\infty} f(x_{k_j}) = f(x_0) \not\Leftarrow |f(x_{k_j}) - f(x_0)| \geq \varepsilon$ $\forall j \geq 1$