Seminar 4

March 16, 2021

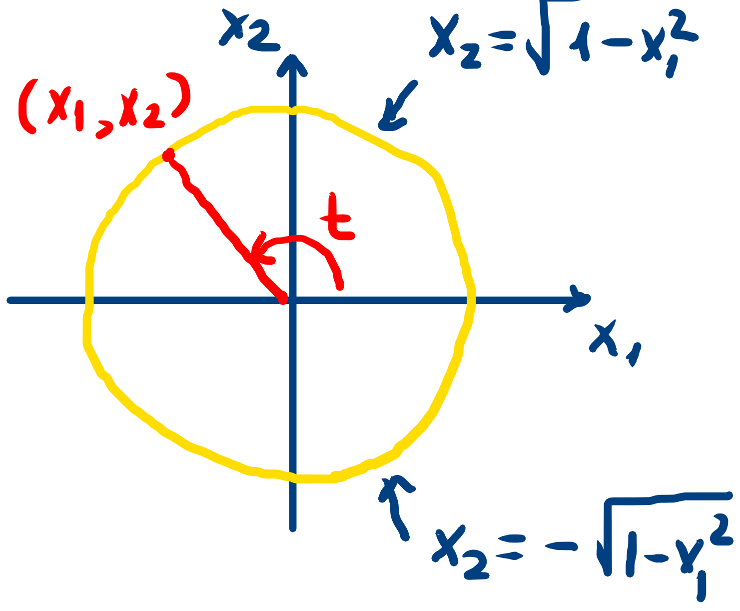
[1] Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping whose matrix is

$$[\varphi] = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}. \qquad \text{Determine } \|\varphi\| = ?$$

Solution. $\displaystyle \|\varphi\| = \max_{x_1^2+x_2^2=1} \|\varphi(x_1,x_2)\|$

$$\varphi(x_1,x_2) = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ -2x_1+3x_2 \end{pmatrix}$$

$$\varphi(x_1,x_2) = (x_1 - x_2, -2x_1+3x_2)$$



$$\|\varphi(x_1,x_2)\| = \sqrt{(x_1-x_2)^2+(-2x_1+3x_2)^2} = \sqrt{5x_1^2+10x_2^2-14x_1x_2}$$

$$\|\varphi\| = \max_{x_1^2+x_2^2=1} \sqrt{5x_1^2+10x_2^2-14x_1x_2}$$

$$\|\varphi\|^2 = \max_{x_1^2+x_2^2=1} \left(5x_1^2+10x_2^2-14x_1x_2\right)$$

$$x_1^2+x_2^2=1 \iff \exists\, t\in[0,2\pi) \text{ s.t. } x_1=\cos t,\ x_2=\sin t$$

$$\|\varphi\|^2 = \max_{t\in[0,2\pi)} \Big(\underbrace{5\cos^2 t + 10\sin^2 t - 14\sin t\cos t}_{f(t)}\Big)$$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$f(t) = 5\cos^2 t + 10\sin^2 t - 14\sin t\cos t$$
$$= 5 + 5\sin^2 t - 7\sin 2t$$
$$= 5 + 5\,\frac{1-\cos 2t}{2} - 7\sin 2t$$
$$= \frac{15}{2} - \frac{5}{2}\cos 2t - 7\sin 2t$$

$$a = -\frac{5}{2} \qquad b = -7$$

$$= \frac{15}{2} + a\cos 2t + b\sin 2t$$

$$a\cos x + b\sin x = c \quad \Big|\, : \sqrt{a^2+b^2}$$

$$= \frac{15}{2} + \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos 2t + \frac{b}{\sqrt{a^2+b^2}}\sin 2t\right)$$

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1 \Rightarrow \exists\,\alpha \in [0, 2\pi) \text{ s.t. } \frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$$
$$\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$$

$$f(t) = \frac{15}{2} + \sqrt{a^2+b^2}\left(\cos\alpha\cos 2t + \sin\alpha\sin 2t\right)$$

$$49 \times 4 = 196$$
$$196 + 25 = 221$$

$$= \frac{15}{2} + \sqrt{a^2+b^2}\cos(\alpha - 2t)$$

$$\Rightarrow \max_{t\in[0,2\pi)} f(t) = \frac{15}{2} + \sqrt{a^2+b^2} = \frac{15}{2} + \sqrt{\frac{25}{4} + 49} = \frac{15+\sqrt{221}}{2} = \|\varphi\|^2$$

$$\Rightarrow \|\varphi\| = \sqrt{\frac{15+\sqrt{221}}{2}}$$

$\boxed{2}$ Let $a_1, \dots, a_n \in \mathbb{R}$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be the linear mapping defined by

$$\varphi(x_1, \dots, x_n) := a_1 x_1 + \dots + a_n x_n .$$

Determine $\|\varphi\| = ?$

<u>Solution</u>. $\|\varphi\| = \max\limits_{x_1^2 + \dots + x_n^2 = 1} |\varphi(x_1, \dots, x_n)| = \max\limits_{x_1^2 + \dots + x_n^2 = 1} |a_1 x_1 + \dots + a_n x_n|$

Cauchy-Schwarz ineq. $\Rightarrow (a_1 x_1 + \dots + a_n x_n)^2 \leq (a_1^2 + \dots + a_n^2)\underbrace{(x_1^2 + \dots + x_n^2)}_{=1}$

$\Rightarrow |a_1 x_1 + \dots + a_n x_n| \leq \sqrt{a_1^2 + \dots + a_n^2} \quad \forall (x_1, \dots, x_n) \in S^{n-1}$

$\Rightarrow \|\varphi\| \leq \sqrt{a_1^2 + \dots + a_n^2} \qquad (1)$

In the Cauchy-Schwarz ineq. we have equality when $x_1 = \lambda a_1, \dots, x_n = \lambda a_n$

$x_1^2 + \dots + x_n^2 = 1 \iff \lambda^2 (a_1^2 + \dots + a_n^2) = 1 \iff \lambda = \pm \dfrac{1}{\sqrt{a_1^2 + \dots + a_n^2}}$

Let $x_1^* = \dfrac{a_1}{\sqrt{a_1^2 + \dots + a_n^2}}, \dots, x_n^* = \dfrac{a_n}{\sqrt{a_1^2 + \dots + a_n^2}} \Rightarrow (x_1^*)^2 + \dots + (x_n^*)^2 = 1$

$\Rightarrow \|\varphi\| \geq |a_1 x_1^* + \dots + a_n x_n^*| = \dfrac{a_1^2 + \dots + a_n^2}{\sqrt{a_1^2 + \dots + a_n^2}} = \sqrt{a_1^2 + \dots + a_n^2} \quad (2)$

By (1), (2) $\Rightarrow \|\varphi\| = \sqrt{a_1^2 + \dots + a_n^2}$.

$\boxed{3}$ Let $f:(0,\infty)\times\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x,y)=\arctan\frac{y}{x}$.
Prove that $\forall (x,y)\in(0,\infty)\times\mathbb{R}$: $\quad x\frac{\partial f}{\partial y}(x,y)-y\frac{\partial f}{\partial x}(x,y)=1$

Solution $\qquad \partial$ = curly d

$$\frac{\partial f}{\partial x}(x,y)=\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=-\frac{y}{\cancel{x^2}}\cdot\frac{\cancel{x^2}}{x^2+y^2}=-\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}=\frac{1}{x}\cdot\frac{x^2}{x^2+y^2}=\frac{x}{x^2+y^2}$$

$$\Rightarrow\ x\cdot\frac{\partial f}{\partial y}(x,y)-y\cdot\frac{\partial f}{\partial x}(x,y)=x\cdot\frac{x}{x^2+y^2}-y\left(-\frac{y}{x^2+y^2}\right)=\frac{x^2+y^2}{x^2+y^2}=1\ \checkmark$$

$\boxed{4}$ Let $f:(0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y,z)=(xy+z^2)\cos\frac{yz}{x^2}$.

Prove that $\forall\,(x,y,z)\in(0,\infty)\times\mathbb{R}^2$:

$$x\frac{\partial f}{\partial x}(x,y,z)+y\frac{\partial f}{\partial y}(x,y,z)+z\frac{\partial f}{\partial z}(x,y,z)=2f(x,y,z).$$

<u>Solution</u>.

$$x\;\Big|\;\frac{\partial f}{\partial x}(x,y,z)=y\cos\frac{yz}{x^2}+(xy+z^2)\left(-\sin\frac{yz}{x^2}\right)\cdot yz\cdot\left(-\frac{2}{x^3}\right)=y\cos\frac{yz}{x^2}+\frac{2yz(xy+z^2)}{x^3}\sin\frac{yz}{x^2}$$

$$y\;\Big|\;\frac{\partial f}{\partial y}(x,y,z)=x\cos\frac{yz}{x^2}+(xy+z^2)\left(-\sin\frac{yz}{x^2}\right)\cdot\frac{z}{x^2}=x\cos\frac{yz}{x^2}-\frac{z(xy+z^2)}{x^2}\sin\frac{yz}{x^2}$$

$$z\;\Big|\;\frac{\partial f}{\partial z}(x,y,z)=2z\cos\frac{yz}{x^2}+(xy+z^2)\left(-\sin\frac{yz}{x^2}\right)\cdot\frac{y}{x^2}=2z\cos\frac{yz}{x^2}-\frac{y(xy+z^2)}{x^2}\sin\frac{yz}{x^2}$$

$$\Rightarrow x\frac{\partial f}{\partial x}(x,y,z)+y\frac{\partial f}{\partial y}(x,y,z)+z\frac{\partial f}{\partial z}(x,y,z)=$$

$$=\cos\frac{yz}{x^2}\,(2xy+2z^2)+\sin\frac{yz}{x^2}\Bigg(\underbrace{\frac{2yz(xy+z^2)}{x^2}-\frac{yz(xy+z^2)}{x^2}-\frac{yz(xy+z^2)}{x^2}}_{=0}\Bigg)$$

$$=2(xy+z^2)\cos\frac{yz}{x^2}$$

$$=2f(x,y,z)$$

$\boxed{5}$ Let $r>0$, $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<r^2\}$, $f: A\to\mathbb{R}$

$$f(x,y)=2\ln\frac{r\sqrt{8}}{r^2-x^2-y^2}=2\ln(r\sqrt{8})-2\ln(r^2-x^2-y^2)$$

Prove that $\forall (x,y)\in A$: $\dfrac{\partial^2 f}{\partial x^2}(x,y)+\dfrac{\partial^2 f}{\partial y^2}(x,y)=e^{f(x,y)}$.

<u>Solution</u>

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x}(x,y)=-2\cdot\frac{-2x}{r^2-x^2-y^2}=\frac{4x}{r^2-x^2-y^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{4y}{r^2-x^2-y^2}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\left(\frac{4x}{r^2-x^2-y^2}\right)'_x=\frac{4(r^2-x^2-y^2)-4x(-2x)}{(r^2-x^2-y^2)^2}=\frac{4(r^2+x^2-y^2)}{(r^2-x^2-y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{4(r^2-x^2+y^2)}{(r^2-x^2-y^2)^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{8r^2}{(r^2-x^2-y^2)^2}=e^{f(x,y)} \quad \checkmark$$

$\boxed{6}$ Determine $\alpha \in \mathbb{R}$ s.t. $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}$, $f(x,y) = y^{\alpha} e^{-\frac{x^2}{4y}}$ satisfies

$$x^2 \frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial x}\left(x^2 \frac{\partial f}{\partial x}(x,y)\right) \qquad \forall\, (x,y) \in \mathbb{R} \times (0,\infty)$$

<u>Solution</u>. $\frac{\partial f}{\partial x}(x,y) = y^{\alpha} \cdot e^{-\frac{x^2}{4y}} \cdot \frac{-\cancel{2}x}{\cancel{4}_{2}y} = -\frac{1}{2} x y^{\alpha-1} e^{-\frac{x^2}{4y}} \quad \Big|\, x^2$

$$\Rightarrow x^2 \frac{\partial f}{\partial x}(x,y) = -\frac{1}{2} x^3 y^{\alpha-1} e^{-\frac{x^2}{4y}}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(x^2 \frac{\partial f}{\partial x}(x,y)\right) = \left(-\frac{1}{2} x^3 y^{\alpha-1} e^{-\frac{x^2}{4y}}\right)'_x = -\frac{1}{2} 3x^2 y^{\alpha-1} e^{-\frac{x^2}{4y}} - \frac{1}{2} x^3 y^{\alpha-1} \cdot e^{-\frac{x^2}{4y}} \cdot \frac{-\cancel{2}x}{\cancel{4}_{2}y}$$

$$= -\frac{3}{2} x^2 y^{\alpha-1} e^{-\frac{x^2}{4y}} + \frac{1}{4} x^4 y^{\alpha-2} e^{-\frac{x^2}{4y}} \qquad (1)$$

$$x^2 \Big|\; \frac{\partial f}{\partial y}(x,y) = \alpha y^{\alpha-1} e^{-\frac{x^2}{4y}} + y^{\alpha} e^{-\frac{x^2}{4y}} \cdot \left(-\frac{x^2}{4}\right) \cdot \left(-\frac{1}{y^2}\right) = \alpha y^{\alpha-1} e^{-\frac{x^2}{4y}} + \frac{1}{4} x^2 y^{\alpha-2} e^{-\frac{x^2}{4y}}$$

$$\Rightarrow x^2 \frac{\partial f}{\partial y}(x,y) = \alpha x^2 y^{\alpha-1} e^{-\frac{x^2}{4y}} + \frac{1}{4} x^4 y^{\alpha-2} e^{-\frac{x^2}{4y}} \qquad (2)$$

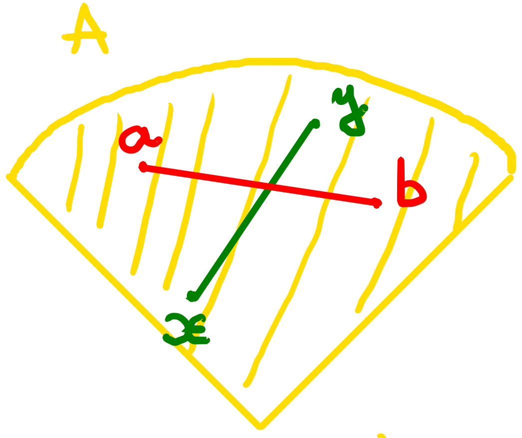
By (1) and (2) $\Rightarrow \alpha = -\frac{3}{2}$.

$\boxed{7}$ Let A be a compact convex subset of $\mathbb{R}^n$, and let $f: A \to \mathbb{R}$ be a continuous function. Prove that $f(A)$ is a compact interval.

<u>Solution</u>. We know that $f(A)$ is a compact set in $\mathbb{R}$

From Weierstrass Thm $\Rightarrow$ $\exists\, a \in A$ s.t. $f(a) = \min\limits_{x\in A} f(x)$

$$\exists\, b \in A \quad \text{s.t.} \quad f(b) = \max_{x\in A} f(x)$$



convex set

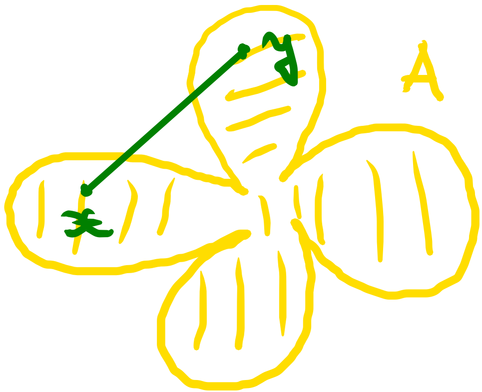
$$\Rightarrow \quad f(A) \subseteq [f(a), f(b)] \qquad (1)$$

<u>Definition</u>. A set $A \subseteq \mathbb{R}^n$ is called convex if

$$\forall\, x, y \in A \quad \forall\, t \in [0,1] \; : \; (1-t)x + ty \in A$$

Consider $g: [0,1] \to \mathbb{R}$, $\quad g(t) = f((1-t)a + tb)$

$$\left.\begin{array}{l} \Rightarrow g \text{ is continuous} \Rightarrow \operatorname{Im} g \text{ is an interval} \\ \operatorname{Im} g \subseteq f(A) \\ g(0) = f(a) \\ g(1) = f(b) \end{array}\right\} \Rightarrow [f(a), f(b)] \subseteq \operatorname{Im} g \subseteq f(A) \quad (2)$$



non-convex set

By (1), (2) $\Rightarrow$ $f(A) = [f(a), f(b)]$