ANALYTIC GEOMETRY, PROBLEM SET 13

1. Find the intersection points between the line $d_2 : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

2. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $d : 9x + 2y - 24 = 0$.

3. Find the equation of the parabola having the focus $F(-7, 0)$ and the director line $x-7 = 0$.

4. Find the equation of the tangent line(s) to:

(1) the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $d_2 : 4x + 3y - 7 = 0$;
(2) the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $d_3 : 2x + 2y - 3 = 0$.

5. Find the equations of the tangent line(s) to:

(1) the hyperbola $\mathcal{H} : \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$ passing through $P_2(1, -5)$;
(2) the parabola $\mathcal{P} : y^2 - 36x = 0$, passing through $P_3(2, 9)$.

6. Consider the hiperbola $x^2 - \frac{y^2}{4} = 1$ and denote by F_1, F_2 its foci. Find the locus of all points M, situated on the hyperbola such that

(a) The angle $\angle F_1MF_2$ is right;
(b) The angle $\angle F_1MF_2$ is equal to $60°$.

7. From the point $P(-3, 12)$ we draw tangents to the parabola $y^2 = 10x$. Compute the distance from the point P to the chord of the parabola which is formed by the two contact points.

8. Find a relation between the coordinates of the point $P_0(x_0, y_0)$ such that there is no tangent from this point to the hiperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

9. Write down the formula for the isometry $\mathrm{Rot}_{90} : \mathcal{E}_2 \to \mathcal{E}_2$ which represents the rotation of center O (origin) and angle $90°$ in the trigonometric sense. Find the equation of the image under Rot_{90} of:

(a) The hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$;
(b) The parabola $y^2 - 8x = 0$.

Do the same for $t_{\overline{v}} \circ \mathrm{Rot}_{90}$, where $t_{\overline{v}} : \mathcal{E}_2 \to \mathcal{E}_2$ is the translation by $\overline{v}(1, 0)$.

9. In the LORAN (Long Range Navigation) radio navigation system, two radio stations located at A and B transmit simultaneous signals to a ship or an aircraft located at P. The onboard computer converts the time difference in receiving these signals into a distance difference $|PA| - |PB|$, and this, according to the definition of a hyperbola, locates the

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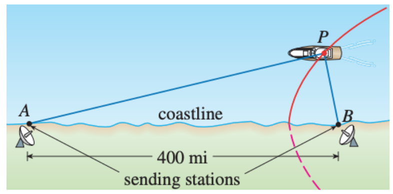
ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station B is located 400 mi due east of station A on a coastline. A ship received the signal from B 1200 micro-seconds (μs) before it received the signal from A.

(a) Assuming the radio signals travel at a speed of 0.2 miles per μs, find an equation of the hyperbola on which the ship lies.

(b) If the ship is due north of B, how far off the coastline is the ship?



1. Find the intersection points between the line $d_2 : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0.$

$$(*) \quad d_2 \cap \mathcal{H} : \left\{ \begin{array}{l} \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0 \\ 2x - y - 10 = 0 \quad \Rightarrow y = 2x - 10 \end{array} \right\} \Rightarrow \frac{x^2}{20} - \frac{(2x-10)^2}{5} - 1 = 0$$

$$\Rightarrow \quad x^2 - 4(2x-10)^2 - 20 = 0$$

$$\Rightarrow \quad x^2 - 4(4x^2 - 40x + 100) - 20 = 0$$

$$\Rightarrow \quad x^2 - 16x^2 + 160x - 420 = 0$$

$$\Rightarrow \quad 15x^2 - 160x - 420 = 0$$

$$\Rightarrow \quad 3x^2 - 32x + 84 = 0$$

$$\Delta = 32^2 - 4 \cdot 3 \cdot 84 = \overset{64}{2^{10}} - \overset{63}{2^4 \cdot 3^2 \cdot 7} = 2^4(2^6 - 3^2 \cdot 7) = 2^4$$

$$\Rightarrow \quad x_{1,2} = \frac{32 \pm 4}{6} = \frac{16 \pm 2}{3} = \begin{cases} 6 \\ \frac{14}{3} \end{cases} \Rightarrow y_{1,2} = \begin{cases} 2 \\ -\frac{2}{3} \end{cases}$$

$\Rightarrow$ $(2, 6)$ and $\left(\frac{14}{3}, -\frac{2}{3}\right)$ are solutions to the system $(*)$

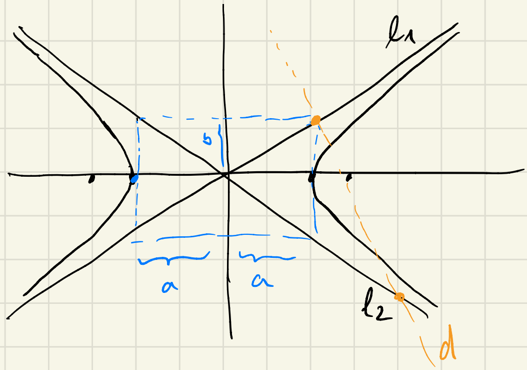
$\Rightarrow$ they are the intersection points.

2. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $d : 9x + 2y - 24 = 0$.

$$\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$$

$$\Downarrow \qquad \Downarrow$$

$$a = 2 \qquad b = 3$$



$\Rightarrow$ the asymptotes have equations

$$y = \pm \frac{b}{a} x$$

in our case

$$l_1 : y = \frac{3}{2} x \quad \text{and} \quad l_2 : y = -\frac{3}{2} x$$

$$l_1 \cap d : \begin{cases} 9x + 2y - 24 = 0 \\ y = \frac{3}{2} x \end{cases} \Rightarrow 9x + 3x - 24 = 0$$

$\Rightarrow 12x = 24$

$\Rightarrow x = 2$

$\Rightarrow y = 3$ $\Big\} \Rightarrow l_1 \cap d = \{(2,3)\}$

$$l_2 \cap d : \begin{cases} 9x + 2y - 24 = 0 \\ y = -\frac{3}{2} x \end{cases} \Rightarrow 9x - 3x - 24 = 0$$

$\Rightarrow 6x = 24$

$\Rightarrow x = 4$

$\Rightarrow y = -6$ $\Big\} \Rightarrow l_2 \cap d = \{(4,-6)\}$

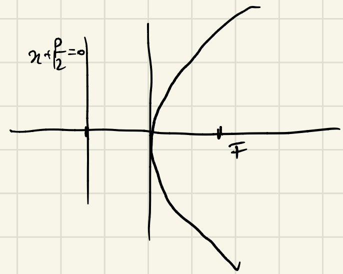
The triangle determined by l_1, l_2, d has vertices and $(0,0)$

$$\Rightarrow \text{area is} \quad \frac{1}{2} \left| \begin{vmatrix} 2 & 3 & 1 \\ 4 & -6 & 1 \\ 0 & 0 & 1 \end{vmatrix} \right|$$

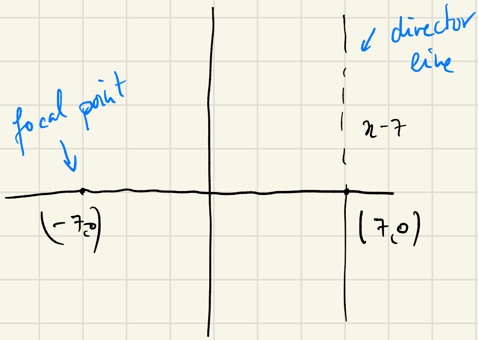
3. Find the equation of the parabola having the focus $F(-7,0)$ and the director line $x-7=0$.

a parabola with director line $x+\frac{p}{2}=0$ and focus $F(\frac{p}{2},0)$

has equation $y^2=2px$ and looks like this (approximately)



in our case we have



the picture is symetric relative to the y-axis

in order to flip points relative to the y-axis

we replace x by $-x$

so the parabola with director line $x-\frac{p}{2}=0$ and focus $F(-\frac{p}{2},0)$

has equation $y^2=-2px$

for us $p=14$ so the parabola is $y^2=-28p$

4. Find the equation of the tangent line(s) to:

(1) the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $d_2 : 4x + 3y - 7 = 0$;
(2) the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $d_3 : 2x + 2y - 3 = 0$.

(1) From the lecture, the tangent lines to $\mathcal{H}$ for a given angular coefficient (a.k.a slope) m is

$$y = mx \pm \sqrt{a^2m^2 - b^2}.$$

for $m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right)$.

- $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$ so, there is a tangent if $m \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

  $a = \sqrt{20}$ $\quad b = \sqrt{5}$

- $d_2 : 4x + 3y - 7 = 0 \iff y = -\frac{4}{3}x + \frac{7}{3} \Rightarrow$ slope is $\tilde{m} = -\frac{4}{3}$
- a line perpendicular to d_2 has slope $m = \frac{3}{4}$
- $m = \frac{3}{4} \in \left[\frac{1}{2}, \infty\right)$ So there is a tangent and we can apply the formula (two in fact)

or calculate this

$l_1 : y = \frac{3}{4}x + \sqrt{20 \cdot \left(\frac{3}{4}\right)^2 - 5}$ and $l_2 : y = \frac{3}{4}x - \sqrt{20 \cdot \left(\frac{3}{4}\right)^2 - 5}$

(2.) From the lecture the tangent line to $\mathcal{P}$ with slope m is

$$y = mx + \frac{p}{2m}.$$

for $m \neq 0$

- $\mathcal{P} : y^2 = 8x \Rightarrow p = 4$
- $d_3 : 2x + 2y - 3 = 0 \iff y = -x + \frac{3}{2} \Rightarrow$ slope $m = -1$

so, the tangent is $l : y = -x - 2$

5. Find the equations of the tangent line(s) to:

(1) the hyperbola $\mathcal{H} : \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$ passing through $P_2(1, -5)$;
(2) the parabola $\mathcal{P} : y^2 - 36x = 0$, passing through $P_3(2, 9)$.

(1) · the tangent line to $\mathcal{H}$ in the point $(x_0, y_0) \in \mathcal{H}$

is $\ell : \frac{x_0 x}{3} - \frac{y_0 y}{5} - 1 = 0$

· we need to find (x_0, y_0)

· since $(x_0, y_0) \in d$: $\frac{x_0}{3} - \frac{y_0(-5)}{5} - 1 = 0$

$$\Leftrightarrow \frac{x_0}{3} + y_0 - 1 = 0 \Rightarrow y_0 = 1 - \frac{x_0}{3}$$

· since $(x_0, y_0) \in \mathcal{H}$: $\frac{x_0^2}{3} - \frac{\left(1 - \frac{x_0}{3}\right)^2}{5} - 1 = 0$

$$\Leftrightarrow \frac{x_0^2}{3} - \frac{1}{5}\left(\frac{3 - x_0}{3}\right)^2 - 1 = 0$$

$$\Leftrightarrow \frac{x_0^2}{3} - \frac{1}{5 \cdot 9}(3 - x_0)^2 - 1 = 0 \quad \Big| \cdot 5 \cdot 9$$

$$\Leftrightarrow 15 x_0^2 - 9 + 6x_0 - x_0^2 - 45 = 0$$

$$\Leftrightarrow 14 x_0^2 + 6 x_0 - 54 = 0$$

$$\Leftrightarrow 7 x_0^2 + 3 x_0 - 27 = 0$$

$85 \,|\, 5$
$17 \,|$

84

$$\Delta = 9 + 4 \cdot 7 \cdot 27 = 9(1 + 4 \cdot 7 \cdot 3) = 9 \cdot 85$$

$$x_{1,2} = \frac{-3 \pm 3\sqrt{85}}{14}$$

$$\Rightarrow (x_0, y_0) \in \left\{ \left(\frac{-3 - 3\sqrt{85}}{14}, 1 + \frac{3 + \sqrt{85}}{14} \right), \left(\frac{-3 + 3\sqrt{85}}{14}, \frac{17 - \sqrt{85}}{14} \right) \right.$$

⇒ the two tangents are

$$l_1: \left(-\frac{1-\sqrt{85}}{14}\right)x - \frac{1}{5}\left(\frac{17-\sqrt{85}}{14}\right)y - 1 = 0$$

$$l_2: \ \ldots \frac{+\sqrt{85}}{\ldots} \ \ldots \ \frac{+\sqrt{85}}{\ldots} \ \ldots$$

Method 2:

For $\mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ the tangent lines with slope m have equations

$$y = mx \pm \sqrt{a^2m^2 - b^2} \qquad \text{if } m \in \left(-\infty, -\frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$$

in our case $a^2 = 3$ $b^2 = 5$ so

$$l_{1,2}: y = mx \pm \sqrt{3m^2 - b^2}$$

since $P_2(1,-5) \in l_{1,2}$ we have $-5 = m \pm \sqrt{3m^2 - 5}$

$$\Rightarrow -5 - m = \pm\sqrt{3m^2 - 5} \quad \Big| ()^2$$

$$\Rightarrow 25 + 10m + m^2 = 3m^2 - 5$$

$$\Rightarrow 2m^2 - 10m - 30 = 0 \quad \Rightarrow m^2 - 5m - 15 = 0$$

$$\Delta = 25 + 60 = 85$$

$$\Rightarrow m_{1,2} = \frac{5 \pm \sqrt{85}}{2}$$

$85 \mid 5$

17

⇒ we find two slopes

⇒ we have four tangent lines

$$y = \frac{5 \pm \sqrt{85}}{2} \pm \sqrt{3\left(\frac{5 \pm \sqrt{85}}{2}\right) + 5}$$

to finish the exercise one can check which of these lines contains $P(1,-5)$

(2) the parabola $\mathcal{P} : y^2 - 36x = 0$, passing through $P_3(2,9)$.

$$y^2 - 2px = 0 \Rightarrow p = 18$$

From the lecture: a tangent to $\mathcal{P}$ with contact point (x_0, y_0)

has equation $\quad y y_0 = p(x + x_0)$

so, in our case $\quad y y_0 = 18(x + x_0)$

- $P_3 \in \mathcal{P} \Rightarrow \quad 9 y_0 = 18(2 + x_0) \iff y_0 = 2x_0 + 4$
- $(x_0, y_0) \in \mathcal{P} \Rightarrow \quad (2x_0 + 4)^2 = 36 x_0$

$$4(x_0 + 2)^2 = 36 x_0 \quad \Big| :4$$

$$x_0^2 + 4x_0 + 4 = 9x_0$$

$$x_0^2 - 5x_0 + 4 = 0$$

$$\Delta = 25 - 16 = 9 \Rightarrow x_{0,1,2} = \frac{5 \pm 3}{2} \begin{matrix} \nearrow 4 \\ \searrow 1 \end{matrix}$$

$\Rightarrow (x_0, y_0) \in \{(1, 6), (4, 12)\}$

$\Rightarrow$ we found two tangents

$\ell_1 : \; 6y = 18(x + 1)$

$\ell_2 : \; 12y = 18(x + 4)$

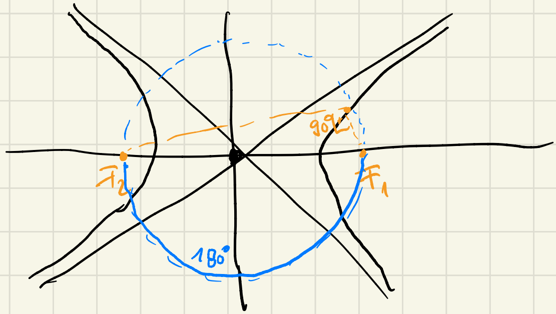
Obs: you can check, by plugging in the coordinates of P_2 in the eq of $\mathcal{H}$ (P_3 in the eq of $\mathcal{P}$) if the point is "inside or outside" the conic

if it is inside, then there are no tangents and you don't have to calculate anything.

6. Consider the hiperbola $x^2 - \frac{y^2}{4} = 1$ and denote by F_1, F_2 its foci. Find the locus of all points M, situated on the hyperbola such that

(a) The angle $\angle F_1MF_2$ is right;
(b) The angle $\angle F_1MF_2$ is equal to $60°$.

(a)



- the points M in the plane for which $\angle F_1MF_2$ is right lie on the circle with diameter F_1F_2
- so, the points that we are looking for can be obtained by intersecting $x^2 - \frac{y^2}{4} = 1$ with the circle centered in $(0,0)$ and radius c

Method II: as for the ellipse last week:

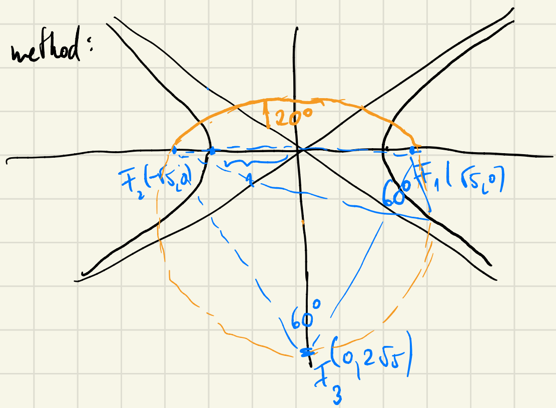
- express $\overrightarrow{MF_1}$ and $\overrightarrow{MF_2}$ with $M(x,y) \in \mathcal{H} : x^2 - \frac{y^2}{4} = 1$ for $y > 0$
- calculate $\frac{\overrightarrow{MF_1} \cdot \overrightarrow{MF_2}}{|MF_1| \cdot |MF_2|}$ (it will depend on x OR y)
- impose the condition $\cos \overrightarrow{MF_1} \cdot \overrightarrow{MF_2} = \cos 90° = 0$
- you will get two solutions in x (OR y)
  [these are one of the coordinates of the points that we are looking for]

6. Consider the hiperbola $x^2 - \frac{y^2}{4} = 1$ and denote by F_1, F_2 its foci. Find the locus of all points M, situated on the hyperbola such that

(a) The angle $\angle F_1MF_2$ is right;

(b) The angle $\angle F_1MF_2$ is equal to $60°$.

b.) Elementary method:



$x^2 - \frac{y^2}{4} = 1$

$\Downarrow \quad \Downarrow$

$a=1 \quad b=2$

$\Rightarrow c = \sqrt{5}$

$\Rightarrow$ focal distance $|F_1F_2| = 2\sqrt{5}$

The $\Delta F_1F_2F_3$ is equilateral

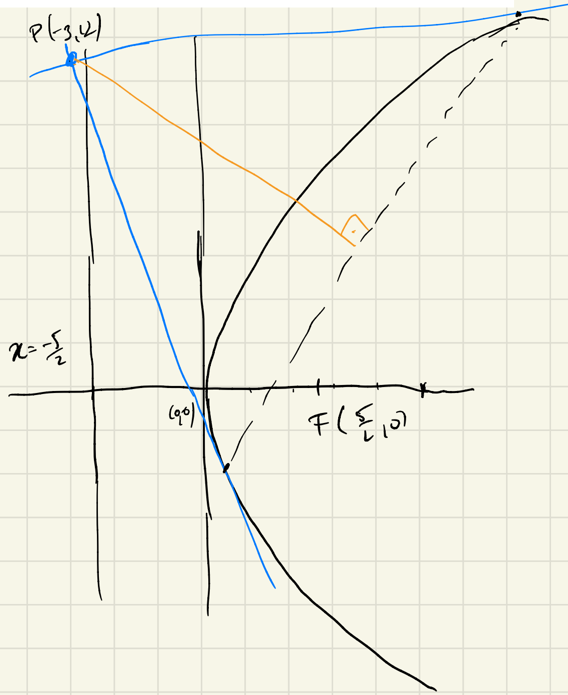
$\Rightarrow \forall$ points M on the circumcircle $\mathcal{C}$ of $\Delta F_1F_2F_3$

$m \measuredangle F_1MF_2 = 60°$

$\Rightarrow$ the points that we are looking for are the intersection of $\mathcal{C}$ with the hyperbola

Analytic method: as in method $\underline{\text{II}}$ of a.)

7. From the point $P(-3, 12)$ we draw tangents to the parabola $y^2 = 10x$. Compute the distance from the point P to the chord of the parabola which is formed by the two contact points.



$\mathcal{P}: y^2 = 10x \quad \Rightarrow p = 5$

a tangent to $\mathcal{P}$ in the point (x_0, y_0) has eq.

$$y y_0 = \overset{5}{p}(x + x_0)$$

- since $P(-3, 12) \in \mathcal{P}$

$$\Rightarrow \quad 12 y_0 = 5(-3 + x_0)$$

- since $(x_0, y_0) \in \mathcal{P}$

$$\Rightarrow \quad \left[\frac{5(-3 + x_0)}{12}\right]^2 = 10 x_0$$

$\rightsquigarrow$ solutions $x_{1,2} = \dots$

$\rightsquigarrow$ we obtain the two contact points for the parabola

$$Q_1(x_1, y_1) \text{ and } Q_2(x_2, y_2)$$

the distance from P to the chord $[Q_1 Q_2]$ is

$$d(P, [Q_1, Q_2])$$

9. Write down the formula for the isometry $\mathrm{Rot}_{90} : \mathcal{E}_2 \to \mathcal{E}_2$ which represents the rotation of center O (origin) and angle $90°$ in the trigonometric sense. Find the equation of the image under Rot_{90} of:

(a) The hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$;

(b) The parabola $y^2 - 8x = 0$.

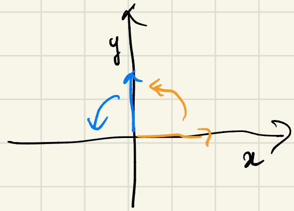
Do the same for $t_{\overline{v}} \circ \mathrm{Rot}_{90}$, where $t_{\overline{v}} : \mathcal{E}_2 \to \mathcal{E}_2$ is the translation by $\overline{v}(1,0)$.

the matrix of such a rotation is

$$\mathrm{Rot}_{90°} = \begin{pmatrix} \cos 90° & -\sin 90 \\ \sin 90° & \cos 90° \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

it changes the variables like this:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

notice that $\quad \frac{x^2}{4} - \frac{y^2}{9} = [x \ y] \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{new coords}}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ -x' \end{bmatrix}$$

so the eq. of the rotated hyperbola is obtained by making the variable change

$x \longleftrightarrow y'$

$y \longleftrightarrow -x'$

we obtain $$\frac{(y')^2}{4} - \frac{(x')^2}{9} = 1$$