Course 7 April 6, 2021

8.1. Definition Let $A, B \subseteq \mathbb{R}^n$ be open sets. A function $f : A \to B$ is called:

- a diffeomorphism if f is bijective, diff on A, with f^{-1} diff. on B
- a C^1-diffeomorphism if f is bijective, of class C^1 on A, with f^{-1} of class C^1 on B

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + \sin x$ bijective $x + \sin x = y$ $x = f^{-1}(y)$

8.2. Theorem (The inverse function theorem) Let A be an open subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}^n$ be a C^1 function on A, and let $a \in A$ s.t. $df(a)$ is bijective. Then $\exists U \in \mathcal{V}(a)$, U open s.t. $U \subseteq A$ and

- $f(U)$ is an open set
- the co-restriction $\tilde{f} : U \to f(U)$, $\tilde{f}(x) := f(x)$ $\forall x \in U$ is a C^1-diffeom.

8.3. Corollary Let $A, B \subseteq \mathbb{R}^n$ be open sets, and let $f : A \to B$ be a C^1 bijective function s.t. $df(x)$ is bijective $\forall x \in A$. Then f is a C^1-diffeomorphism.

Example. Prove that $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) := (x^3+x,\ y-x^2)$ $f(x,y) = (x^3+x,\ y-x^2)$, is a C^1-diffeom.

Solution. ~~X~~ a) f is bijective

b) f is of class C^1 on $\mathbb{R}^2$

c) $df(x,y)$ is bijective $\forall\ (x,y) \in \mathbb{R}^2$

a) ~~X~~ f is bijective $\iff \forall (u,v) \in \mathbb{R}\ \exists!\ (x,y) \in \mathbb{R}^2$ s.t. $f(x,y) = (u,v)$

Let $(u,v) \in \mathbb{R}^2$

$$f(x,y) = (u,v) \iff (*) \begin{cases} x^3 + x = u \\ y - x^2 = v \end{cases}$$

$x^3+x=u$ has a unique solution $x_0 \in \mathbb{R}$

$\Downarrow$

the system $(*)$ has the unique solution $(x_0,\ x_0^2+v)$

b) f is a C^1-function because

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= (3x^2+1,\ -2x) \\ \frac{\partial f}{\partial y}(x,y) &= (0,\ 1) \end{aligned}\right| \text{ continuous on } \mathbb{R}^2$$

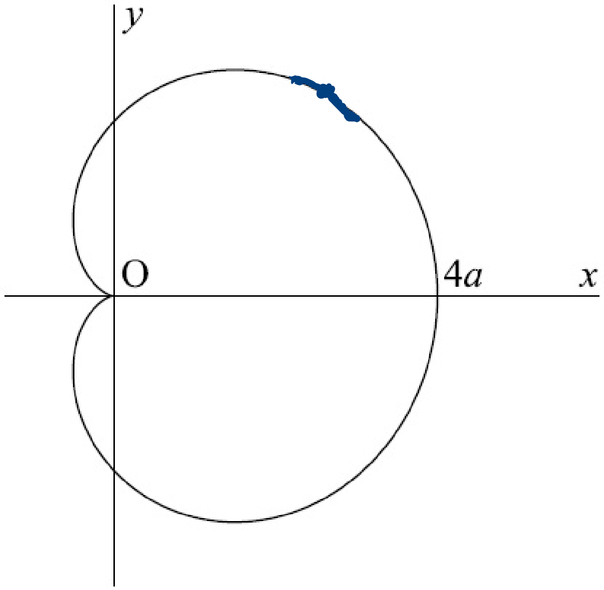
c) $\forall (x,y) \in \mathbb{R}^2$ $df(x,y)$ is bijective $\iff [df(x,y)] = J(f)(x,y)$ is invertible

$$\det J(f)(x,y) = \begin{vmatrix} 3x^2+1 & 0 \\ -2x & 1 \end{vmatrix} = 3x^2+1 > 0$$

9. The implicit function theorem

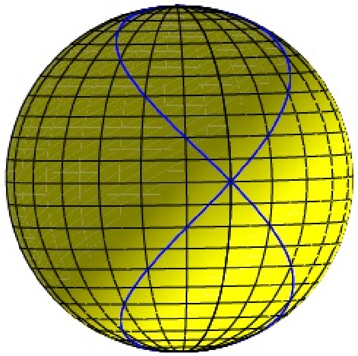
The circle : $x^2 + y^2 = 1$

The cardioid : $x^2+y^2 = 2a\left(x + \sqrt{x^2+y^2}\right)$



The Viviani window :

$$\begin{cases} x^2+y^2+z^2 = a^2 \\ x^2+y^2 = ax \end{cases}$$



The sphere : $x^2+y^2+z^2 = a^2$ $z = \pm\sqrt{a^2-y^2-z^2}$

Consider a function $F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $F = (F_1, \dots, F_m)$ and let

$$S = \{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid F(x,y) = 0_m \}$$

$$= \left\{ (x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^n \times \mathbb{R}^m \;\middle|\; (*) \begin{cases} F_1(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ \vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \end{cases} \right\}.$$

$$S_1 = \{ x \in \mathbb{R}^n \mid \exists\, y \in \mathbb{R}^m \text{ s.t. } F(x,y) = 0_m \}$$

Let $(a,b) \in S$, i.e. $F(a,b) = 0_m$

One says that the system $F(x,y) = 0_m \iff (*)$ defines the variables $y_1, \dots, y_m$ as functions of $x_1, \dots, x_n$ around (a,b) if

$\exists\, U \in \mathcal{V}(a)$ $\exists\, V \in \mathcal{V}(b)$ s.t. $\forall\, x \in U \cap S_1$ the set

$$\{ y \in V \mid F(x,y) = 0_m \} \qquad (**)$$

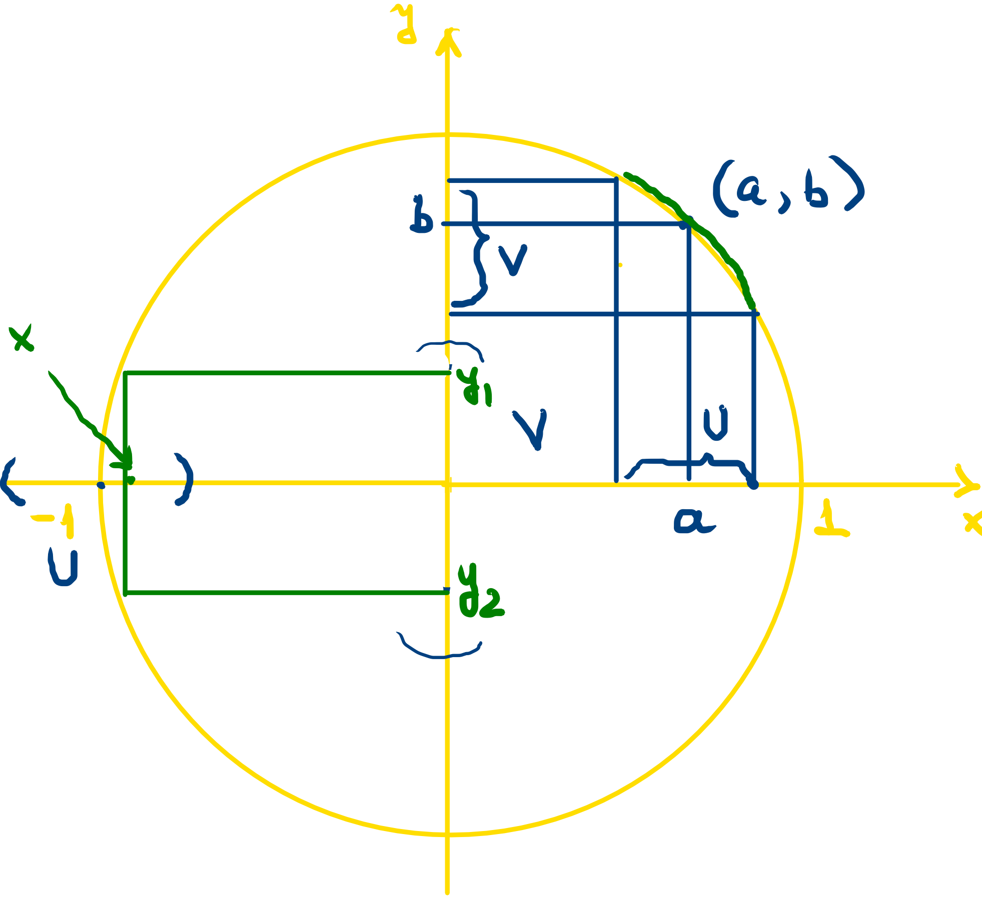
contains exactly one element

We can define $f : U \to V$, $f(x) :=$ the unique element of $(**)$

$$\forall\, x \in U \cap S_1 : \qquad F(x, f(x)) = 0_m$$

Example $\quad F: \mathbb{R}\times\mathbb{R} \to \mathbb{R} \qquad F(x,y) = x^2+y^2-1$

$$S = \{(x,y) \mid x^2+y^2-1=0\} \qquad S_1 = [-1,1] \qquad (a,b)\in S$$



- If $a \neq \pm 1 \Rightarrow$ the equation $F(x,y)=0$ defines y as a function of x around (a,b)

  $y = \sqrt{1-x^2}$ if $b>0$

  $y = -\sqrt{1-x^2}$ if $b<0$

- If $a = \pm 1 \Rightarrow$ the equation $F(x,y)=0$ does not define y as a function of x around $(\pm 1, 0)$

9.1. Theorem (The implicit function theorem) Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a C^1-function on $\mathbb{R}^n \times \mathbb{R}^m$ and let $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ be a point s.t $F(a,b) = 0_m$ and

$$\frac{\partial(F_1, \dots, F_m)}{\partial(y_1, \dots, y_m)}(a,b) = \begin{vmatrix} \frac{\partial F_1}{\partial y_1}(a,b) & \dots & \frac{\partial F_1}{\partial y_m}(a,b) \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial y_1}(a,b) & \dots & \frac{\partial F_m}{\partial y_m}(a,b) \end{vmatrix} \neq 0$$

Then $\exists\, U \in \mathcal{V}(a)$, U - open, $\exists\, V \in \mathcal{V}(b)$, V - open

$\exists\, f : U \to V$ a function of class C^1 on U s.t.

$f(a) = b$

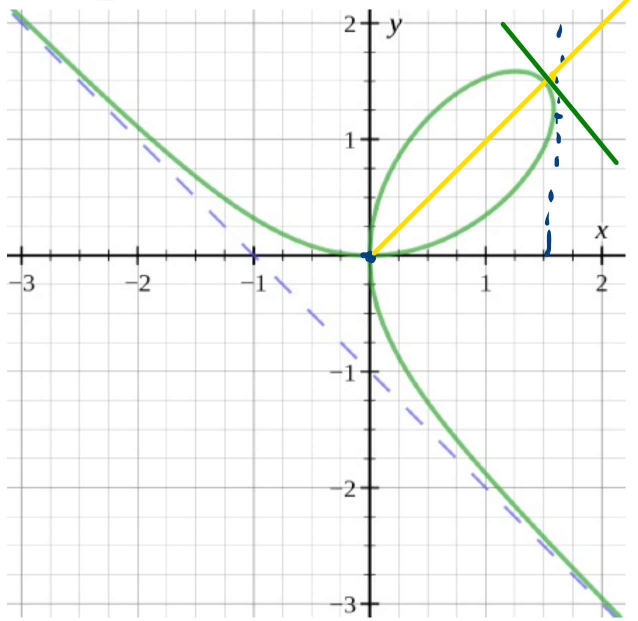
$F(x, f(x)) = 0_m \quad \forall\, x \in U. \qquad (*)$

Moreover, the neighbourhoods U and V can be chosen s.t. there exists only one function $f : U \to V$ satisfying $(*)$.

Example. Let $a>0$. The folium (leaf) of Descartes is the plane curve whose implicit equation is $x^3+y^3-3axy=0$.

Determine all points on the folium around which one can express y as a function of x. Write the eq. of the tangent line to the folium at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Solution.



$(x_0,y_0)=?$ $\qquad F(x,y)=x^3+y^3-3axy$

$$\begin{cases} F(x_0,y_0)=0 \\ \frac{\partial F}{\partial y}(x_0,y_0)=0 \end{cases} \qquad \begin{cases} x_0^3+y_0^3-3ax_0y_0=0 \\ 3y_0^2-3ax_0=0 \end{cases}$$

$$x_0=\frac{y_0^2}{a} \Rightarrow \frac{y_0^6}{a^3}+y_0^3-3\cancel{a}\cdot\frac{y_0^2}{\cancel{a}}\cdot y_0=0$$

$$y_0^6=2a^3y_0^3 \Rightarrow y_0=0 \Rightarrow x_0=0$$

or

$$y_0^3=2a^3 \Rightarrow \begin{aligned} y_0&=a\sqrt[3]{2} \\ x_0&=a\sqrt[3]{4} \end{aligned}$$

By IFT $\Rightarrow \exists\, U\in\mathcal{V}\left(\frac{3a}{2}\right)$ open $U\subseteq\mathbb{R}$

$\exists\, f:U\to\mathbb{R}$ of class C^1 on U s.t.

$f\left(\frac{3a}{2}\right)=\frac{3a}{2}$

$F(x,f(x))=x^3+f^3(x)-3axf(x)=0 \quad \forall x\in U \;\Big|\; \frac{d}{dx}$

$\Rightarrow \quad 3x^2 + 3f^2(x)f'(x) - 3af(x) - 3axf'(x) = 0 \qquad \forall\, x \in U$

Letting $x = \frac{3a}{2}$ and taking into account that $f\left(\frac{3a}{2}\right) = \frac{3a}{2} \Rightarrow$

$$\frac{27a^2}{4} + \frac{27a^2}{4} f'\left(\frac{3a}{2}\right) - \frac{9a^2}{2} - \frac{9a^2}{2} f'\left(\frac{3a}{2}\right) = 0 \Rightarrow f'\left(\frac{3a}{2}\right) = -1$$

$$y - \frac{3a}{2} = -\left(x - \frac{3a}{2}\right) \qquad \boxed{x + y - 3a = 0}$$

10. Constrained extrema

$A \subseteq \mathbb{R}^n \qquad f : A \to \mathbb{R} \qquad F = (F_1, \dots, F_m) : A \to \mathbb{R}^m$

$$C := \{ x \in A \mid F(x) = 0_m \}$$

$$= \left\{ (x_1, \dots, x_n) \in A \;\middle|\; \begin{cases} F_1(x_1, \dots, x_n) = 0 \\ \vdots \\ F_m(x_1, \dots, x_n) = 0 \end{cases} \right\}$$

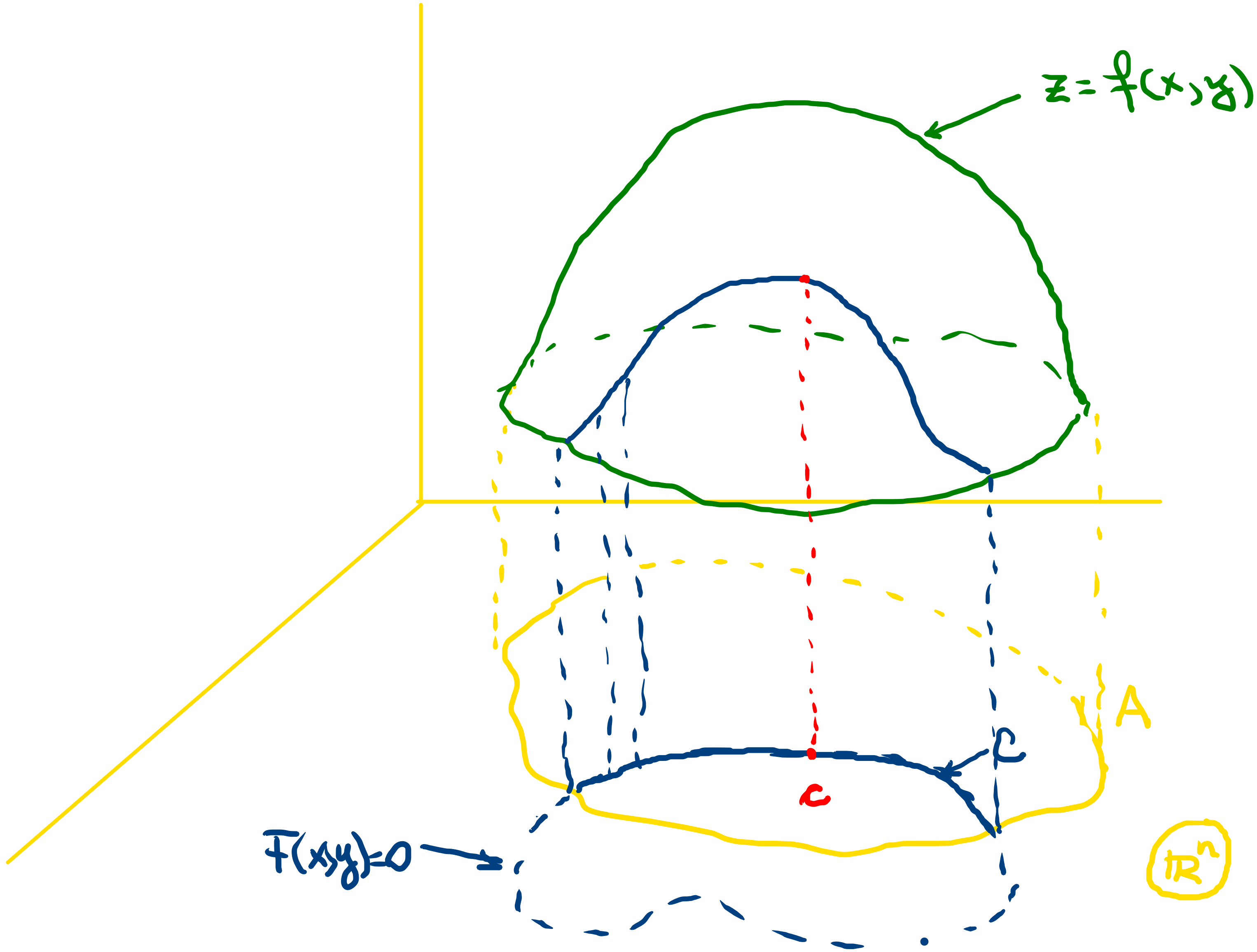
Assume that $m < n$

We are looking for local extrema of $f\big|_C$

$n=2$ $\quad m=1$

$C=\{(x,y) \mid F(x,y)=0\}$

the implicit equation of a curve



c is a local maximum for $f|_C$, but c is not a local extremum for f

Every local extremum of the restriction $f|_C$ is called a constrained extremum

<u>10.1. Theorem</u> (The method of Lagrange multipliers) Let $A \subseteq \mathbb{R}^n$ be an open set, $f : A \to \mathbb{R}$, $F = (F_1, \dots, F_m) : A \to \mathbb{R}^n$ be functions of class C^1 on A.
If $m < n$ and $c \in C$ is a constrained extremum for s.t.

$$\operatorname{rank} J(F)(c) = m,$$

then $\exists\, \lambda_0 \in \mathbb{R}^m$ with the property the (c, λ_0) is a critical point for the Lagrange function $L : A \times \mathbb{R}^m \to \mathbb{R}$,

$$L(x, \lambda) := f(x) + \langle \lambda, F(x) \rangle =$$

$$= f(x) + \lambda_1 F_1(x) + \dots + \lambda_m F_m(x) \qquad \forall\, x \in A, \ \forall\, \lambda \in \mathbb{R}^m$$

$\lambda_0 = (\lambda_{01}, \dots, \lambda_{0m})$ → Lagrange multipliers.

Goal : to provide sufficient conditions for a critical point to be a local extremum

11. Second order partial derivatives

11.1. Definition $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}$, $i, j \in \{1, \dots, n\}$

Suppose that $\exists V \in \mathcal{V}(a)$, V - open, $V \subseteq A$ s.t.

(i) f is partially diff wrto x_i at each point of V;

(ii) the function

$$\forall x \in V \mapsto \frac{\partial f}{\partial x_i}(x) \in \mathbb{R} \qquad (*)$$

is partially diff. wrto x_j at a

Then f is said to be twice partially diff wrto (x_i, x_j) at a.
The partial derivative wrto x_j of the function $(*)$ is called the second order partial derivative of f wrto (x_i, x_j) at a

Notation:

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(a)$$

$$f''_{x_i x_j}(a) = \left(f'_{x_i}\right)'_{x_j}(a)$$

$i = j$

$$\frac{\partial^2 f}{\partial x_i^2}(a) \qquad f''_{x_i x_i}(a)$$

f can have at most n^2 second order partial derivatives at a

$$H(f)(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(a) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(a) \\ \cdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(a) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{pmatrix}$$

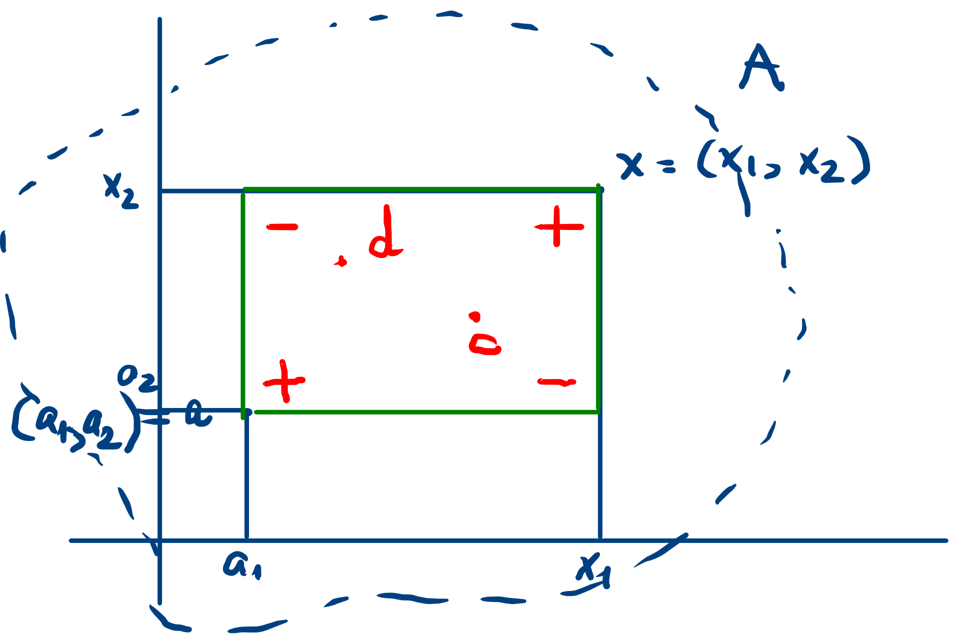
$\to$ the Hessian matrix of f at a

! $i \neq j$ $\qquad \frac{\partial^2 f}{\partial x_j \partial x_i}(a) \neq \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$

11.2. Theorem (A. Clairaut, H. Schwarz). Let $A \subseteq \mathbb{R}^n$ be an open set, $i, j \in \{1, \dots, n\}$, $i < j$, $f : A \to \mathbb{R}$ s.t. f is twice partially differentiable at each point in A wrt to both (x_i, x_j) and (x_j, x_i). If $\frac{\partial^2 f}{\partial x_j \partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} : A \to \mathbb{R}$ are continuous at some point $a \in A$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$$

Proof $n = 2$ $i = 1$ $j = 2$. Let $\varepsilon > 0$. Since $\frac{\partial^2 f}{\partial x_2 \partial x_1}$ and $\frac{\partial^2 f}{\partial x_1 \partial x_2}$



are continuous at $a = (a_1, a_2) \Rightarrow$

$\Rightarrow \exists \delta > 0$ s.t. $B(a, \delta) \subseteq A$ and $\forall (x_1, x_2) \in B(a, \delta)$:

$$\left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1, x_2) - \frac{\partial^2 f}{\partial x_2 \partial x_1}(a_1, a_2) \right| < \varepsilon$$

$$\left| \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a_1, a_2) \right| < \varepsilon$$

Choose $x=(x_1,x_2)\in B(a,\delta)$ s.t. $x_1>a_1$ and $x_2>a_2$ and define

$$\alpha = f(a_1,a_2)+f(x_1,x_2)-f(x_1,a_2)-f(a_1,x_2)$$

Then $\alpha = g(x_1)-g(a_1)$ where $g:[a_1,x_1]\to \mathbb{R}$
$g(t)=f(t,x_2)-f(t,a_2)$

Lagrange MVT $\Rightarrow \exists c_1\in(a_1,x_1)$ s.t. $g(x_1)-g(a_1)=(x_1-a_1)g'(c_1)$
But $g'(t)=\frac{\partial f}{\partial x_1}(t,x_2)-\frac{\partial f}{\partial x_1}(t,a_2)$ $\Bigg\} \Rightarrow$

$$\Rightarrow \alpha = g(x_1)-g(a_1) = (x_1-a_1)\left[\frac{\partial f}{\partial x_1}(c_1,x_2)-\frac{\partial f}{\partial x_1}(c_1,a_2)\right]$$

Let $g_1:[a_2,x_2]\to\mathbb{R}$, $g_1(t)=\frac{\partial f}{\partial x_1}(c_1,t)$

Lagrange MVT $\Rightarrow \exists c_2\in(a_2,x_2)$ s.t. $g_1(x_2)-g_1(a_2)=(x_2-a_2)g_1'(c_2)$

But $g_1'(t)=\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right)(c_1,t)=\frac{\partial^2 f}{\partial x_2\partial x_1}(c_1,t)$ $\Bigg\} \Rightarrow$

$$\Rightarrow \alpha = (x_1-a_1)(x_2-a_2)\frac{\partial^2 f}{\partial x_2 \partial x_1}(c_1,c_2) \quad (1)$$

$c=(c_1,c_2)\in(a_1,x_1)\times(a_2,x_2)$

Similarly $\alpha = (x_1 - a_1)(x_2 - a_2) \frac{\partial^2 f}{\partial x_1 \partial x_2}(d_1, d_2)$ $\quad$ (2) $\quad d = (d_1, d_2) \in (a_1, x_1) \times (a_2, x_2)$

By (1), (2) $\Rightarrow \frac{\partial^2 f}{\partial x_2 \partial x_1}(c) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(d)$

$$\Rightarrow \left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right| = \left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) - \frac{\partial^2 f}{\partial x_2 \partial x_1}(c) + \frac{\partial^2 f}{\partial x_1 \partial x_2}(d) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right|$$

$$\leq \underbrace{\left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) - \frac{\partial^2 f}{\partial x_2 \partial x_1}(c) \right|}_{< \varepsilon} + \underbrace{\left| \frac{\partial^2 f}{\partial x_1 \partial x_2}(d) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right|}_{< \varepsilon}$$

$\|c - a\| \leq \|x - a\| < \delta$

$< 2\varepsilon \qquad \forall \varepsilon > 0$

$\varepsilon \downarrow 0 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(a)$