Proposition 8.5. Let $\mathbf{V}$ be a finite dimensional vector space with basis $\boldsymbol{e} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Associating to each bilinear form its matrix with respect to $\boldsymbol{e}$ gives rise to a bijection between the set $\mathrm{Bil}(\mathbf{V})$ of bilinear forms on $\mathbf{V}$ and the set $\mathrm{Mat}_{n\times n}(\mathbf{K})$. This bijection induces a bijection of the set of symmetric bilinear forms with the set of symmetric matrices.

- To each bilinear form b we have associated a matrix A (w.r.t. a basis)
- To see that b determines A take two vectors

$$v = x_1 e_1 + \dots + x_n e_n \qquad w = y_1 e_1 + \dots + y_n e_n$$

then
$$b(v,w) = b(x_1 e_1 + \dots + x_n e_n, y_1 e_1 + \dots + y_n e_n)$$
$$= b(x_1 e_1, y_1 e_1 + \dots + y_n e_n) + \dots + b(x_n e_n, y_1 e_1 + \dots + y_n e_n)$$
$$= x_1 b(e_1, y_1 e_1 + \dots + y_n e_n) + \dots + x_n b(e_n, y_1 e_1 + \dots + y_n e_n)$$
$$= x_1 [b(e_1, y_1 e_1) + \dots + b(e_1, y_n e_n)] + \dots + x_n [b(e_n, y_1 e_1) + \dots + b(e_n, y_n e_n)]$$
$$= x_1 [y_1 b(e_1, e_1) + \dots + y_n b(e_1, e_n)] + \dots + x_n [y_1 b(e_n, e_1) + \dots + y_n b(e_n, e_n)]$$
$$= \sum_{i,j=1}^{n} x_i y_j b(e_i, e_j) = x^t A y$$

where $x = (x_1, \dots, x_n)$ $y = (y_1, \dots, y_n)$ are the coordinates of v and w respectively.

- We show next that a matrix A determines a bilinear form w.r.t the basis e

Define b by
$$b(v,w) = \sum_{i,j=1}^{n} a_{ij} x_i y_j = x^t A y \qquad \forall v(x_1, \dots, x_n) \quad \forall w(y_1, \dots, y_n)$$

then
$$b(v+v', w) = (x+x')^t A y = x^t A y + x'^t A y = b(v,w) + b(v',w)$$
$$b(v, w+w') = x^t A (y+y') = x^t A y + x^t A y' = b(v,w) + b(v,w')$$
$$b(cv, w) = (c \cdot x)^t A y = c x^t A y = c b(v,w)$$
$$b(v, cw) = x^t A (cy) = c x^t A y = c b(v,w)$$

It is also clear that the matrix associated to this bilinear form is A

- Moreover $b(w,v) = y^t A x = x^t A y = b(v,w) \quad \forall v, w \iff A^t = A$

Proposition 8.7. Let **V** be a **K**-vector space of dimension n. Two matrices represent the same bilinear form b on **V** with respect to two bases if and only if they are congruent.

- The bijection in Proposition 8.6 depends on the choice of the basis

  $\rightarrow$ a bilinear form is represented in general by two distinct matrices with respect to two distinct bases

- Let $b: V \times V \to \mathbb{K}$ be a bilinear form and $e = \{e_1, \dots, e_n\}$ $f = \{f_1, \dots, f_n\}$ two bases

- Let $A = (a_{ij})$ with $a_{ij} = b(e_i, e_j)$ and $B = (b_{ij})$ with $b_{ij} = b(f_i, f_j)$

- Let
  $$v = x_1 e_1 + \dots + x_n e_n = x_1' f_1 + \dots + x_n' f_n$$
  $$w = y_1 e_1 + \dots + y_n e_n = y_1' f_1 + \dots + y_n' f_n$$
  be two arbitrary vectors

- In each of the basis we have $b(v,w) = x^t A y = x'^t B y'$

- Let $M = M_{e,f}$ be the base change matrix from f to e then $x = Mx'$ and $y = My'$

  $$\Rightarrow x'^t B y' = (Mx')^t A (My') = (x')^t M^t A M y' \quad \forall x', y' \in \mathbb{K}^n$$

  $$\Rightarrow B = M^t A M \quad (A \text{ and } B \text{ are congruent})$$

- Conversely, if A is the matrix of b w.r.t e and $M \in GL_n(\mathbb{K})$

  $\Rightarrow \exists$ basis f s.t. $M = M_{e,f}$

  So, if $B = M^t A M$ then B is the matrix of b w.r.t f (by the $\overbrace{}^{\text{above calculation}}$)

Proposition 8.9. Let $\mathbf{V}$ be a finite dimensional $\mathbf{K}$-vector space and let $b : \mathbf{V}\times\mathbf{V} \to \mathbf{K}$ be a bilinear form. The following properties are equivalent:

1. b is non-degenerate.
2. For every $\mathbf{v} \neq 0$ in $\mathbf{V}$ there is a $\mathbf{w} \in \mathbf{V}$ for which $b(\mathbf{v},\mathbf{w}) \neq 0$.
3. For every $\mathbf{w} \neq 0$ in $\mathbf{V}$ there is a $\mathbf{v} \in \mathbf{V}$ for which $b(\mathbf{v},\mathbf{w}) \neq 0$.

- Fix a basis $e = \{e_1, \dots, e_n\}$ of V
- Let A be the matrix of b w.r.t e

$\boxed{1. \Rightarrow 2.}$ If $\operatorname{rank} A = n$ and $x \neq (0,\dots,0) \Rightarrow x^t A \neq (0,\dots,0)$

$$\Rightarrow \exists y \in \mathbb{K}^n : x^t A y \neq 0$$

$\boxed{2. \Rightarrow 1.}$ By hypothesis $\forall x \neq (0,\dots,0)\ \exists y \in \mathbb{K}^n$ s.t $x^t A y \neq 0$

$$\Rightarrow x^t A \neq (0,\dots,0) \quad \forall x \neq (0,\dots,0)$$

$$\Rightarrow \operatorname{rank} A = n$$

$\boxed{1. \Leftrightarrow 3.}$ is proved similarly.

Proposition 8.13. Let $\mathbf{V}$ be a vector space over $\mathbf{K}$ with symmetric bilinear form $b : \mathbf{V}\times\mathbf{V} \to \mathbf{K}$. The quadratc form q associated to b satisfies the following two condition:

$$\begin{aligned} q(c\mathbf{v}) &= c^2 q(\mathbf{v}) \\ 2b(\mathbf{v},\mathbf{w}) &= q(\mathbf{v}+\mathbf{w}) - q(\mathbf{v}) - q(\mathbf{w}) \end{aligned}$$

for every $c \in \mathbf{K}$ and every $\mathbf{v},\mathbf{w} \in \mathbf{V}$.

- First property follows from BF3
- For the second property

$$\begin{aligned} q(v+w) - q(v) - q(w) &= b(v+w, v+w) - b(v,v) - b(w,w) \\ &= b(v,w) + b(w,v) \\ &= 2b(v,w) \end{aligned}$$

Theorem 8.16. Let $\mathbf{V}$ be a $\mathbf{K}$-vector space and let b be a symmetric bilinear form on $\mathbf{V}$. Then there exists a diagonalizing basis for b. Equivalently, any symmetric matrix is congruent to a diagonal matrix.

<u>Proof 1</u>

- Induction on $n = \dim V$
- If $n=1$ there is nothing to prove
- Suppose $n \geq 2$ and that any symmetric bilinear form on a vector space of dimension $< n$ is diagonalizable
  - if b is the zero form, there is nothing to prove
  - Suppose b is not the zero form

$$\Rightarrow \exists\, e_1 \in V \text{ s.t. } b(e_1, e_1) \neq 0$$

$$\Rightarrow V = \langle e_1 \rangle \oplus e_1^{\perp}$$

$$\Rightarrow \dim(e_1^{\perp}) = n-1$$

- Consider b' the restriction of b to $e_1^{\perp}$

by the induction hypothesis b' is diagonalizable

i.e $\exists \{e_1, \dots, e_n\}$ a basis of $e_1^{\perp}$ such that the matrix of b' is diagonal w.r.t this basis

then $e = \{e_1, e_2, \dots e_n\}$ is a basis of V

since $\langle e_2, \dots, e_n \rangle = e_1^{\perp}$ $\quad$ $e_1, e_2 \dots e_n$ are linearly indep.

Moreover $b(e_1, e_j) = 0 \;\forall\, j = 2, \dots, n$

and $b(e_i, e_j) = b'(e_i, e_j) = 0 \;\forall\, i \neq j$ in $\{2, \dots, n\}$

$\Rightarrow e = \{e_1, e_2, \dots, e_n\}$ is a diagonalizing basis for b.

Proof 2 · Induction on $n = \dim V$

- If $n=1$ there is nothing to prove

- Suppose $n \geq 2$ and that any symmetric bilinear form on a vector space of dimension $< n$ is diagonalizable

- if b is the zero form, there is nothing to prove

- Suppose b is not the zero form

- Choose a basis $e = \{e_1, \dots, e_n\}$

- If e is a diagonalizing basis then we are done

- Else we may obtain another basis $f = \{f_1, \dots, f_n\}$ such that $b(f_1, f_1) \neq 0$

Indeed: if $\exists\, i = 1,n$ such that $b(e_i, e_i) \neq 0$ it suffices to exchange b_1 and b_i

Else, if $b(e_i, e_i) = 0 \;\forall i = 1,n$

there must exist $i \neq j$ such that $b(e_i, e_j) \neq 0$ (else b is the zero form)

so we may exchange them with e_1 and e_2 such that $b(e_1, e_2) = 0$

then we may choose $f = \{b_1 + b_2, b_2, \dots, b_n\}$

- Now, with respect to f the quadratic form q associated to b has the form

$$q(v(y_1, \dots, y_n)) = h_{11} y_1^2 + 2 \sum_{i=2}^{n} h_{1i} y_1 y_i + \sum_{i,j=2}^{n} h_{ij} y_i y_j \qquad (*)$$

where $h_{ij} = b(f_i, f_j)$.

- Since $h_{11} = b(f_1, f_1) \neq 0$ we can rewrite $(*)$ as

$$q(v(y_1, \dots, y_n)) = h_{11}\left(y_1 + \sum_{i=2}^{n} h_{11}^{-1} h_{1i} y_i\right)^2 + (\text{terms not involving } y_1)$$

- Now we can change coordinates as follows

$$z_1 = y_1 + \sum_{i=2}^{n} h_{11}^{-1} h_{1i} y_i \quad z_2 = y_2, \dots, z_n = y_n$$

- This corresponds to a change of basis from f to $g = \{g_1, \dots, g_n\}$ given by $g_1 = f_1$ and for $i > 1$ $g_i = f_i - h_{11}^{-1} h_{1i} f_1$

- With respect to g the quadratic form q is

$$q(v(z_1, \dots, z_n)) = h_{11} z_1^2 + q'(z_2, \dots, z_n)$$

where q' is a homogeneous polynomial of degree 2 in $z_2 \dots z_n$

$\Rightarrow$ it defines a quadratic form on $\langle g_2, \dots, g_n \rangle$

By the inductive hypothesis $\langle g_2, \dots, g_n \rangle$ has a basis $e_2', \dots, e_n'$ which diagonalizes q'

$\Rightarrow$ $g_1, e_2', \dots, e_n'$ is a diagonalizing basis for q.

a) **Theorem 8.17** (Sylvester). Let **V** be a real vector space of dimension n, and let $b : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ be a symmetric bilinear form on **V** of rank $r \le n$. Then there is an integer p with $0 \le p \le r$ depending only on b, and a basis $\boldsymbol{e} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of **V** with respect to which the matrix associated to b has the form

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_{r-p} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.1}$$

where 0 denotes zero matrices of appropriate sizes.

b.) Equivalently, every symmetric matrix $A \in \mathrm{Mat}_{n\times n}(R)$ is congruent to a diagonal matrix of the form (8.1) in which $r = \mathrm{rank}(A)$ and p depends only on A.

- Here $\mathbb{K} = \mathbb{R}$
- It is clear that the two statements (a.) and b.)) are equivalent
- We prove the first one:
- By theorem 8.16 $\exists$ a basis $e = \{e_1, \ldots, e_n\}$ for which

$$q(v(y_1, \ldots, y_n)) = a_{11} y_1^2 + \ldots + a_{nn} y_n^2 \quad \forall v \in V$$

- The number of non-zero a_{ii} is equal to the rank of q
- After possibly reordering the basis, we can suppose that the first p coefficients are positive, the next $r-p$ are negative, and the remaining $n-r$ are zero.

$\Rightarrow$ $a_{11} = \alpha_1^2, \ldots, a_{pp} = \alpha_p^2$

$a_{p+1,p+1} = -\alpha_{p+1}^2, \ldots, a_{rr} = -\alpha_r^2$ for appropriate $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ which we can choose positive

$a_{r+1,r+1} = \ldots = a_{nn} = 0$

- consider the basis $f_1 = e_1/\alpha_1, \ldots, f_r = e_r/\alpha_r, f_{r+1} = e_{r+1}, \ldots, f_n = e_n$

then $b(f_i, f_i) = b\left(\frac{e_i}{\alpha_i}, \frac{e_i}{\alpha_i}\right) = \frac{1}{\alpha_i^2} b(e_i, e_i) = \frac{\alpha_i^2}{\alpha_i^2} = 1$ for $i \le p$

Similarly one checks that $b(f_i, f_i) = -1$ if $i = p+1 \ldots r$ and
$b(f_i, f_i) = 0$ if $i = r \ldots n$

So, the matrix of b w.r.t f is as in (8.1)

and the associated quadratic form is

(**) $$q(v(x_1, \ldots, x_n)) = x_1^2 + \ldots + x_p^2 - x_{p-1}^2 - \ldots - x_r^2 \quad \forall v \in V$$

- It remains to show that p depends only on b and not on the basis f.
- Suppose that with respect to another basis $g = \{g_1, \ldots, g_n\}$, q is

(***) $$q(v(z_1, \ldots, z_n)) = z_1^2 + \ldots + z_t^2 - z_{t+1}^2 - \ldots - z_r^2$$

- If $p \neq t$ we may suppose that $t < q$
- Consider the vector subspaces $S = \langle e_1, \ldots, e_p \rangle$

$T = \langle b_{t+1}, \ldots, b_n \rangle$

- $\dim(S) + \dim(T) > n \Rightarrow S \cap T \neq \langle 0 \rangle$

$\Rightarrow \exists\, v \in S \cap T$ s.t $v \neq 0$. Then

$$v = x_1 f_1 + \ldots + x_p f_p = z_{t+1} b_{t+1} + \ldots + z_n b_n$$

Since $v \neq 0 \overset{(**)}{\Longrightarrow} q(v) = x_1^2 + \ldots + x_p^2 > 0$

$\overset{(***)}{\Longrightarrow} q(v) = -z_{t+1}^2 - \ldots - z_r^2 < 0$

$\Bigg\}$ this is a contradiction $\Downarrow$ so $t = p$