Calculus on $\mathbb{R}$ ANCA GRAD, Ph.D., Lect. Prof.
Homework 3 **2021**

Sequences of Real Numbers- part 1

Exercise 1: Study the monotonicity, boundedness and convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, having the general term:

$$a)\quad x_n = \frac{3^n+5^n}{7^n},\quad b)\quad x_n = \frac{(-1)^n}{n},\quad c)\quad x_n = \frac{4^n}{n!},\quad d)\quad x_n = \frac{n}{n^2+1}.$$

Exercise2: Using the characterising theorem with ε prove that

$$a)\ \lim_{n\to\infty}\frac{2n}{n^2+1} = 0 \quad ?\ b)\ \lim_{n\to\infty}\frac{2n^2}{-2n+4} = -\infty.$$

Exercise 3: Compute the limit of the sequences of real numbers having the following general terms:

$$a)\ \frac{3^n+1}{5^n+1},\qquad b)\ \frac{9^n+(-3)^n}{9^{n-1}+3},\qquad c)\ \left(\sin\frac{\pi}{10}\right)^n,\qquad d)\sqrt{4n^2+2n+1}-2n,$$

$$e)\ \left(7+\frac{1-2n^3}{3n^4+2}\right)^2,\qquad f)\ \sqrt[3]{n^3+n+3}-\sqrt[3]{n^3+1},\qquad g)\ \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}},$$

$$h)\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n}\right).$$

? **Exercise 4:** Let $t\in\mathbb{R}$.

a) Prove that there exists an decreasing sequence of rational numbers converging to t.

b) Prove that there exists a increasing sequence of irrational numbers converging to t.

Exercise 5: Let $a > 0$ and let $x_0 \in \mathbb{R}$ be such that $0 < x_0 < \frac{1}{a}$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, defined recursively by:

$$x_{n+1} = 2x_n - ax_n^2, \forall n \in \mathbb{N}.$$

Study the convergence of the sequence by following the next steps:

a) Prove by induction that $x_n < \frac{1}{a}, \forall n \in \mathbb{N}$.

b) Prove by induction that $0 < x_n, \forall n \in \mathbb{N}$.

c) By using a) and b) prove that $(x_n)_{n\in\mathbb{N}}$ is increasing.

d) Compute the limit of the sequence.

Exercise 1: Study the monotonicity, boundedness and convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, having the general term:

$$a)\quad x_n = \frac{3^n+5^n}{7^n}, \qquad b)\quad x_n = \frac{(-1)^n}{n}, \qquad c)\quad x_n = \frac{4^n}{n!}, \qquad d)\quad x_n = \frac{n}{n^2+1}.$$

a) $x_n = \dfrac{3^n+5^n}{7^n}$

MONOTONICITY :

$$x_{n+1} - x_n = \frac{3^{n+1}+5^{n+1}}{7^{n+1}} - \frac{{}^{7)}\,3^n+5^n}{7^n} = \frac{3^{n+1}+5^{n+1}-7\cdot(3^n-5^n)}{7^{n+1}} = \frac{3^n(3-7)+5^n(5-7)}{7^{n+1}} =$$

$$= \frac{-4\cdot 3^n - 2\cdot 5^n}{7^{n+1}} = \frac{-(4\cdot 3^n + 2\cdot 5^n)}{7^{n+1}} < 0,\ \forall n\in\mathbb{N}$$

$\Rightarrow x_{n+1} - x_n < 0,\ \forall n\in\mathbb{N} \Rightarrow (x_n)_n$ DECREASING (1)

BOUNDEDNESS :

$$\left.\begin{array}{l} 3^n > 0,\ \forall n\in\mathbb{N} \\ 5^n > 0,\ \forall n\in\mathbb{N}\end{array}\right\} \Rightarrow \left.\begin{array}{l} 3^n+5^n > 0,\ \forall n\in\mathbb{N} \\ 7^n > 0,\ \forall n\in\mathbb{N}\end{array}\right\} \Rightarrow \frac{3^n+5^n}{7^n} > 0,\ \forall n\in\mathbb{N}$$

$\Rightarrow x_n > 0,\ \forall n\in\mathbb{N}$ (2)

$$x_n = \frac{3^n+5^n}{7^n} = \left(\frac{3}{7}\right)^n + \left(\frac{5}{7}\right)^n$$

$$\left.\begin{array}{l} \frac{3}{7} < 1 \Rightarrow \left(\frac{3}{7}\right)^n < 1,\ \forall n\in\mathbb{N} \\ \frac{5}{7} < 1 \Rightarrow \left(\frac{5}{7}\right)^n < 1,\ \forall n\in\mathbb{N}\end{array}\right\} \Rightarrow \left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n < 2,\ \forall n\in\mathbb{N}$$

$\Rightarrow x_n < 2,\ \forall n\in\mathbb{N}$ (3)

(2) & (3) $\Rightarrow 0 < x_n < 2\ \forall n\in\mathbb{N} \Rightarrow (x_n)_n$ *bounded* (4)

CONVERGENCE :

(1) & (4) $\Rightarrow (x_n)_n$ *convergent*, $\lim\limits_{n\to\infty} x_n = 0$

b) $x_n = \dfrac{(-1)^n}{n}$

MONOTONICITY:

If $n = 2k,\ k \in \mathbb{N}:\ x_{2k} = \dfrac{(-1)^{2k}}{2k} = \dfrac{1}{2k} > 0,\ \forall k \in \mathbb{N}$ (1)

If $n = 2k+1,\ k \in \mathbb{N}:\ x_{2k+1} = \dfrac{(-1)^{2k+1}}{2k+1} = \dfrac{-1}{2k+1} < 0,\ \forall k \in \mathbb{N}$ (2)

(1) & (2) $\Rightarrow (x_n)_n$ is not monotone

BOUNDEDNESS:

$$\left.\begin{array}{l} \text{If } n = 2k,\ k \in \mathbb{N}:\ x_{2k} = \dfrac{1}{2k} > 0,\ \forall k \in \mathbb{N} \\ \text{If } n = 2k+1,\ k \in \mathbb{N}:\ x_{2k+1} = \dfrac{-1}{2k+1} < 0,\ \forall k \in \mathbb{N} \end{array}\right\} \Rightarrow -1 < x_n < 1,\ \forall n \in \mathbb{N} \Rightarrow (x_n)_n \text{ is bounded}$$

CONVERGENCE:

$$\left.\begin{array}{l} \lim\limits_{k\to\infty} x_{2k} = 0 \\ \lim\limits_{k\to\infty} x_{2k+1} = 0 \end{array}\right\} \Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow (x_n)_n \text{ convergent}$$

c) $x_n = \dfrac{4^n}{n!}$

MONOTONICITY:

$$x_{n+1} - x_n = \frac{4^{n+1}}{(n+1)!} - \frac{\overset{n+1}{\diagup} 4^n}{n!} = \frac{4^{n+1} - 4^n(n+1)}{(n+1)!} = \frac{4^n(4-n-1)}{(n+1)!} = \frac{4^n(3-n)}{(n+1)!}$$

$$\left.\begin{array}{l} \Rightarrow x_{n+1} - x_n > 0,\ \forall n \in \mathbb{N} \text{ when } n < 3 \Rightarrow (x_n)_n \text{ INCREASING } \forall n \in \mathbb{N},\ n<3 \\ x_{n+1} - x_n \leq 0,\ \forall n \in \mathbb{N} \text{ when } n \geq 3 \Rightarrow (x_n)_n \text{ NON-INCREASING } \forall n \in \mathbb{N},\ n \geq 3 \end{array}\right\} \Rightarrow \text{NON-MONOTON (1)}$$

BOUNDEDNESS:

$x_1 \nearrow x_3 \searrow 0$

$$\left.\begin{array}{l} x_n > 0 \text{ for } n \in \mathbb{N} \Rightarrow \text{LOWER BOUNDED} \\ x_n \leq x_3 \text{ for } n \in \mathbb{N} \Rightarrow \text{UPPER BOUNDED} \end{array}\right\} \Rightarrow \text{BOUNDED (2)}$$

$\left(x_3 = \dfrac{64}{6} = \dfrac{32}{3}\right)$

CONVERGENCE:

$(x_n)_n$ is convergent, $\lim\limits_{n\to\infty} x_n = 0$

d) $x_n = \dfrac{n}{n^2+1}$

MONOTONICITY:

$$\frac{x_{n+1}}{x_n} = \frac{\dfrac{n+1}{(n+1)^2+1}}{\dfrac{n}{n^2+1}} = \frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} = \frac{(n+1)(n^2+1)}{n((n+1)^2+1)} = \frac{n^3+n^2+n+1}{n(n^2+2n+2)} =$$

$$= \frac{n^3+n^2+n+1}{n^3+2n^2+2n} < 1$$

$\Rightarrow x_{n+1} < x_n, \ \forall n \in \mathbb{N} \Rightarrow (x_n)_n$ DECREASING (1)

BOUNDEDNESS:

$$\left.\begin{array}{l} n > 0, \ \forall n \in \mathbb{N} \\ n^2+1 > 0, \ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow \frac{n}{n^2+1} > 0, \ \forall n \in \mathbb{N} \Rightarrow x_n > 0, \ \forall n \in \mathbb{N}$$

$$n \geq 1 \Rightarrow n^2 \geq n \Rightarrow n^2+1 \geq n+1 > n \Rightarrow \frac{1}{n^2+1} < \frac{1}{n} \ \Big|\cdot n \Rightarrow \frac{n}{n^2+1} < 1$$

$\forall n \in \mathbb{N}$

$\Rightarrow 0 < x_n < 1, \ \forall n \in \mathbb{N} \Rightarrow (x_n)_n$ is bounded (2)

CONVERGENCE:

(1) & (2) $\Rightarrow (x_n)_n$ is convergent, $\lim\limits_{n \to \infty} x_n = 0$

Exercise2: Using the characterising theorem with ε prove that

$$a) \lim_{n\to\infty} \frac{2n}{n^2+1} = 0 \qquad b) \lim_{n\to\infty} \frac{2n^2}{-2n+4} = -\infty.$$

a) $\lim_{n\to\infty} \frac{2n}{n^2+1} = 0$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $x_n = \frac{2n}{n^2+1}$

$\lim_{n\to\infty} x_n = 0 \iff \forall\, \varepsilon > 0,\ \exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \geq n_\varepsilon$, $|x_n - 0| < \varepsilon$

$\iff |x_n| < \varepsilon \iff \left|\frac{n}{n^2+1}\right| < \varepsilon$

We choose $\varepsilon > 0$ randomly.

$\frac{n}{n^2+1} < \varepsilon \iff n < n^2\varepsilon + \varepsilon \iff n^2\varepsilon - n + \varepsilon > 0$

$\Delta = 1 - 4\varepsilon^2$

$n_{1,2} = \frac{1 \pm \sqrt{1-4\varepsilon^2}}{2\varepsilon}$

$\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $n_\varepsilon > \frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon} \longrightarrow \frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon} > 0$

$\xRightarrow{\text{Archimedes}}$ $\exists\, n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon > \frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon}$

$\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $n_\varepsilon > \frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon} \iff \frac{n_\varepsilon}{n_\varepsilon^2+1} < \varepsilon$

$\forall\, n \geq n_\varepsilon \Rightarrow \frac{n}{n^2+1} \leq \frac{n_\varepsilon}{n_\varepsilon^2+1}$

$\Rightarrow \frac{n}{n^2+1} < \varepsilon \iff \left|\frac{n}{n^2+1}\right| < \varepsilon$

$\iff \left|\frac{n}{n^2+1} - 0\right| < \varepsilon$

For $\varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \geq n_\varepsilon$, $\left|\frac{n}{n^2+1} - 0\right| < \varepsilon$

$\Rightarrow$ the first relation holds for ε fixed randomly $\Rightarrow \lim_{n\to\infty} x_n = 0$

b) $\lim_{n\to\infty} \frac{2n^2}{-2n+4} = -\infty$

Exercise 3: Compute the limit of the sequences of real numbers having the following general terms:

$a)\ \frac{3^n+1}{5^n+1},\quad b)\ \frac{9^n+(-3)^n}{9^{n-1}+3},\quad c)\ \left(\sin\frac{\pi}{10}\right)^n,\quad d)\sqrt{4n^2+2n+1}-2n,$

$e)\ \left(7+\frac{1-2n^3}{3n^4+2}\right)^2,\quad f)\ \sqrt[3]{n^3+n+3}-\sqrt[3]{n^3+1},\quad g)\ \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}},$

$h)\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n}\right).$

a) $x_n = \dfrac{3^n+1}{5^n+1}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{3^n+1}{5^n+1} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{3^n(1+\overbrace{\tfrac{1}{3^n}}^{\to 0})}{5^n(1+\underbrace{\tfrac{1}{5^n}}_{\to 0})} = \lim_{n\to\infty}\underbrace{\left(\frac{3}{5}\right)^n}_{\to 0}\cdot 1 = 0$$

b) $x_n = \dfrac{9^n+(-3)^n}{9^{n-1}+3}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{9^n+(-3)^n}{9^{n-1}+3} = \lim_{n\to\infty}\frac{\cancel{9^n}\left[1+\overbrace{\left(\tfrac{-3}{9}\right)^n}^{\to 0}\right]}{\cancel{9^{n-1}}\left(1+\underbrace{\tfrac{3}{9^n}}_{\to 0}\right)} = \lim_{n\to\infty}\frac{9\cdot 1}{1} = 9$$

c) $x_n = \left(\sin\frac{\pi}{10}\right)^n$

$0<\frac{\pi}{10}<\frac{\pi}{2} \Rightarrow \sin\frac{\pi}{10}>0$

$0<\sin\frac{\pi}{10}<1$

$\lim_{n\to\infty}\left(\sin\frac{\pi}{10}\right)^n = 0$ $\quad\left(\text{rule used: } \lim_{n\to\infty} a^n = \begin{cases} 0, & a\in(-1,1)\\ 1, & a=1\\ \infty, & a>1\\ \nexists, & a<-1\end{cases}\right)$

d) $\sqrt{4n^2+2n+1}-2n$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sqrt{4n^2+2n+1}-2n\right) \overset{(\infty-\infty)}{=\!=} \lim_{n\to\infty}\frac{\cancel{4n^2}+2n+1-\cancel{4n^2}}{\sqrt{4n^2+2n+1}+2n} = \lim_{n\to\infty}\frac{2n+1}{\sqrt{4n^2+2n+1}+2n} =$$

$$= \lim_{n\to\infty}\frac{\cancel{n}(2+\overbrace{\tfrac{1}{n}}^{\to 0})}{\cancel{n}\left(\sqrt{4+\underbrace{\tfrac{2}{n}}_{\to 0}+\underbrace{\tfrac{1}{n^2}}_{\to 0}}+2\right)} = \lim_{n\to\infty}\frac{2}{\sqrt{4}+2} = \frac{2}{2+2} = \frac{1}{2}$$

e) $x_n = \left(7+\dfrac{1-2n^3}{3n^4+2}\right)^2$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(7+\frac{1-2n^{\cancel{3}}}{3n^{\cancel{4}}+2}\right)^2 = 7^2 = 49$$

f) $x_n = \sqrt[3]{n^3+n+3} - \sqrt[3]{n^3+1}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt[3]{n^3+n+3} - \sqrt[3]{n^3+1} = \lim_{n\to\infty} \frac{\cancel{n^3}+n+3-\cancel{n^3}-1}{\sqrt[3]{(n^3+n+3)^2}+\sqrt[3]{(n^3+n+3)(n^3+1)}+\sqrt[3]{(n^3+1)^2}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\underset{\to 0}{\frac{2}{n}}\right)}{\cancel{n^2}\left(\sqrt[3]{\left(1+\underset{\downarrow 0}{\frac{1}{n^2}}+\underset{\downarrow 0}{\frac{3}{n^3}}\right)^2}+\sqrt[3]{\left(1+\underset{\downarrow 0}{\frac{1}{n^2}}+\underset{\downarrow 0}{\frac{3}{n^3}}\right)\left(1+\underset{\downarrow 0}{\frac{1}{n^3}}\right)}+\sqrt[3]{\left(1+\underset{\downarrow 0}{\frac{1}{n^3}}\right)^2}\right)} = 0$$

g) $x_n = \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}} = \lim_{n\to\infty}\left(\frac{n^3\left(1+\underset{\to 0}{\frac{5}{n^3}}+\underset{\to 0}{\frac{1}{n}}\right)}{n^2\left(1-\underset{\downarrow 0}{\frac{1}{n^2}}\right)}\right)^{\frac{n^4\left(\underset{\to 0}{\frac{1}{n^4}}-5\right)}{n^4\left(6+\frac{1}{n^4}\right)}} \to 0 = (\infty)^{\frac{-5}{6}} = \left(\frac{1}{\infty}\right)^{\frac{5}{6}} = 0$$

h) $x_n = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdot\ldots\cdot\left(1-\frac{1}{n}\right)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\ldots\cdot\frac{\cancel{n-1}}{n} = \lim_{n\to\infty}\frac{1}{n} = 0$$

Exercise 4: Let $t \in \mathbb{R}$.

a) Prove that there exists an decreasing sequence of rational numbers converging to t.

b) Prove that there exists a increasing sequence of irrational numbers converging to t.

a) $x_n = \frac{t-1}{n}$, $n \in \mathbb{N}$

b) $x_n = \frac{t-1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$, $n \in \mathbb{N}$

$x_n = \frac{t-1}{\sqrt{n}}$, $n \in \mathbb{N} / \{ u \mid u = x^2, x \in \mathbb{N} \}$

??

Exercise 5: Let $a > 0$ and let $x_0 \in \mathbb{R}$ be such that $0 < x_0 < \frac{1}{a}$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, defined recursively by:

$$x_{n+1} = 2x_n - ax_n^2, \forall n \in \mathbb{N}.$$

Study the convergence of the sequence by following the next steps:

a) Prove by induction that $x_n < \frac{1}{a}, \forall n \in \mathbb{N}$.

b) Prove by induction that $0 < x_n, \forall n \in \mathbb{N}$.

c) By using a) and b) prove that $(x_n)_{n\in\mathbb{N}}$ is increasing.

d) Compute the limit of the sequence.

$a > 0$, $x_0 \in \mathbb{R}$, s.t. $0 < x_0 < \frac{1}{a}$

$(x_n)_{n\in\mathbb{N}}$ $\quad x_{n+1} = 2x_n - ax_n^2 \quad \forall n \in \mathbb{N}$

a) $x_n < \frac{1}{a}, \forall n \in \mathbb{N}$

$x_{n+1} = -a\left(x_n^2 - 2\cdot\frac{1}{a}x_n\right) = -a\left(x_n^2 - 2\cdot\frac{1}{a}x_n + \frac{1}{a^2} - \frac{1}{a^2}\right) = -a\left(\left(x_n - \frac{1}{a}\right)^2 - \frac{1}{a^2}\right)$

$\Rightarrow x_{n+1} = -a\left(x_n - \frac{1}{a}\right)^2 + \frac{1}{a} \Rightarrow x_{n+1} - \frac{1}{a} = -a\left(x_n - \frac{1}{a}\right)^2$

Let $(y_n)_{n\in\mathbb{N}}$ s.t. $y_n = x_n - \frac{1}{a}$ $\Bigg\} \Rightarrow y_{n+1} = -a\cdot y_n^2$, $a > 0$, $y_n^2 > 0, \forall n \in \mathbb{N}$ $\Bigg\} \Rightarrow$

$\Rightarrow y_{n+1} < 0, \forall n \in \mathbb{N}$

For $n = 0$: $y_1 = -ay_0^2 < 0$

For $n = 1$: $y_2 = -ay_1^2 < 0$

Suppose that $y_n < 0$. We shall prove that $y_{n+1} < 0$

$y_{n+1} = -a\cdot y_n^2 < 0, \forall n \in \mathbb{N}, a > 0$

$y_n < 0 \Rightarrow x_n - \frac{1}{a} < 0 \Rightarrow x_n < \frac{1}{a}$

b) $0 < x_n, \forall n \in \mathbb{N}$

$x_{n+1} = 2x_n - ax_n^2 = x_n(2 - ax_n)$

For $n = 0$: $x_1 = x_0(2 - ax_0)$

$x_0 < \frac{1}{a} \Rightarrow ax_0 < 1 \Rightarrow -1 < -ax_0 \,\big|+2 \Rightarrow 1 < 2 - ax_0 \Rightarrow 2 - ax_0 > 1 > 0$

$\Rightarrow \left.\begin{array}{l} 2 - ax_0 > 0 \\ x_0 > 0 \end{array}\right\} \Rightarrow x_0(2 - ax_0) > 0 \Rightarrow x_1 > 0$

For $n = 1$: $x_2 = x_1(2 - ax_1)$

its proved at a), $x_n < \frac{1}{a}, \forall n \in \mathbb{N} \Rightarrow x_1 < \frac{1}{a} \,\big|\cdot a > 0 \Rightarrow ax_1 < 1 \,\big|\cdot(-1) \Rightarrow -ax_1 > -1 \,\big|+2 \Rightarrow$

$\Rightarrow 2 - ax_1 > 1 > 0 \Rightarrow \left.\begin{array}{l} 2 - ax_1 > 0 \\ x_1 > 0 \end{array}\right\} \Rightarrow x_1(2 - ax_1) > 0 \Rightarrow x_2 > 0$

Suppose that $x_n > 0$. We shall prove that $x_{n+1} > 0$.

$x_{n+1} = x_n(2 - a x_n)$

$x_n < \frac{1}{a} \mid \cdot a > 0 \Rightarrow a x_n < 1 \mid \cdot(-1) \Rightarrow -a x_n > -1 \mid +2 \Rightarrow \left.\begin{array}{l} 2 - a x_n > 0 \\ x_n > 0 \end{array}\right\} \Rightarrow x_n(2 - a x_n) > 0 \Rightarrow x_{n+1} > 0 \Rightarrow x_n > 0, \forall n \in \mathbb{N}$

c) $(x_n)_{n\in\mathbb{N}}$ is increasing

$x_{n+1} - x_n = 2x_n - a x_n^2 - x_n = x_n - a x_n^2 = x_n(1 - a x_n)$

from a):

$x_n < \frac{1}{a} \mid \cdot a \Rightarrow a x_n < 1 \mid \cdot(-1) \Rightarrow -a x_n > -1 \mid +1 \Rightarrow \left.\begin{array}{l} 1 - a x_n > 0 \\ \text{from b)} \\ x_n > 0 \end{array}\right\} \Rightarrow x_n(1 - a x_n) > 0$

$\Rightarrow x_{n+1} - x_n > 0 \Rightarrow (x_n)_{n\in\mathbb{N}}$ is INCREASING

d) $\ell = ?$

$\left.\begin{array}{l} \text{From a) and b)}, \ 0 < x_n < \frac{1}{a}, \forall n \in \mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ is bounded} \\ \qquad a > 0 \\ \text{from c)} \\ (x_n)_{n\in\mathbb{N}} \text{ is increasing} \end{array}\right\} \Rightarrow (x_n)_{n\in\mathbb{N}}$ is $\Rightarrow$ CONVERGENT

$\Rightarrow \exists \lim_{n\to\infty} x_n = \ell \in \left(0, \frac{1}{a}\right)$

We replace x_n by ℓ in the recursive relation:

$\ell = 2\ell - a\ell^2 \Rightarrow \ell - a\ell^2 = 0 \Rightarrow \left.\begin{array}{l} \ell(1 - a\ell) = 0 \\ (x_n)_n \text{ INCREASING} \\ x_n > 0 \end{array}\right\} \Rightarrow \ell \neq 0 \ \ \& \ \ 1 - a\ell = 0 \Rightarrow \boxed{\ell = \frac{1}{a}}$

Excercise 3 (Do not turn this in, we will solve it together at the seminar)Fill in the following table:

Nr.	A	int A	bd A	cl A	ext A	Izo A	A’
1	$(-\infty, -1] \cup (2, +\infty)$	$(-\infty,1) \cup (2,+\infty)$	$\{-1,2\}$	$(-\infty,-1] \cup [2,+\infty)$	$(-1,2)$	$\emptyset$	$(-\infty,-1] \cup [2,+\infty)$
2	$(-1, 9] \cup [10, 20)$	$(-1,9) \cup (10,20)$	$\{-1,9,10,20\}$	$[-1,9] \cup [10,20]$	$(-\infty,-1) \cup (9,10) \cup (20,+\infty)$	$\emptyset$	$[-1,9] \cup [10,20]$
3	$\Big((-1, 9] \cup [10, 20)\Big) \cap \mathbb{N}$ $\{1,2,3,\ldots,19\}$	$\emptyset$	$\{1,2,3,\ldots,19\}$	$\{1,2,3,\ldots,19\}$	$(-\infty,1) \cup (1,2) \cup (2,3) \cup \ldots$ $\ldots \cup (19,+\infty)$	$\{1,2,3,\ldots,19\}$	$\emptyset$
4	$\{1, 2, 3\}$	$\emptyset$	$\{1,2,3\}$	$\{1,2,3\}$	$(-\infty,1) \cup (1,2) \cup (2,3) \cup (3,+\infty)$	$\{1,2,3\}$	$\emptyset$
5	$\mathbb{N}$	$\emptyset$	$\mathbb{N}$	$\mathbb{N}$	$(-\infty,1) \cup (1,2) \cup (2,3) \cup \ldots$ $\ldots \cup (m,m+1) \cup \ldots$	$\mathbb{N}$	$\emptyset$
6	$\mathbb{R} \setminus \{1, 2, 3\}$ $(-\infty,1) \cup (1,2) \cup (2,3) \cup (3,+\infty)$	$(-\infty,1) \cup (1,2) \cup (2,3) \cup$ $\cup (3,+\infty)$	$\{1,2,3\}$	$\mathbb{R}$	$\emptyset$	$\emptyset$	$\mathbb{R}$
7	$\mathbb{R} \setminus \mathbb{N}$ $(-\infty,1) \cup (1,2) \cup \ldots \cup (m,m+1) \cup \ldots$	$(-\infty,1) \cup (1,2) \cup (2,3) \cup \ldots$ $\ldots \cup (m,m+1) \cup \ldots$	$\{1,2,3,4,\ldots\}$	$\mathbb{R}$	$\emptyset$	$\emptyset$	$\mathbb{R}$
8	$\mathbb{Z}$ $\ldots,-m,\ldots,-1,0,1,2,3,\ldots,m,\ldots$	$\emptyset$	$\{\ldots,-m,\ldots,-1,0,1,2,\ldots\}$	$\mathbb{Z}$	$\ldots \cup (-m,-m+1) \cup \ldots \cup (-1,0)$ $\cup (0,1) \cup (1,2) \cup \ldots \cup (m,m+1) \cup \ldots$	$\mathbb{Z}$	$\emptyset$
9	$\mathbb{R} \setminus \mathbb{Z}$ $\ldots(-m,-m+1) \cup \ldots \cup (-1,0) \cup (0,1) \cup \ldots$	$\ldots(-m,-m+1) \cup \ldots \cup (-1,0) \cup$ $\cup (0,1) \cup (1,2) \cup \ldots \cup (m,m+1) \cup \ldots$	$\mathbb{Z}$	$\ldots[-m,-m+1] \cup \ldots \cup [-1,0] \cup$ $\cup [1,2] \cup \ldots \cup [m,m+1] \cup \ldots$	$\emptyset$	$\emptyset$	$\mathbb{R}$
10	$\mathbb{Q}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$	$\emptyset$	$\emptyset$	$\mathbb{R}$
11	$\mathbb{R} \setminus \mathbb{Q}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$	$\emptyset$	$\emptyset$	$\mathbb{R}$
12	$\mathbb{R}$	$(-\infty,+\infty)$	$\emptyset$	$\mathbb{R}$	$\emptyset$	$\emptyset$	$\mathbb{R}$

HW from LECTURE 3

$L2 : \ell = \infty \quad \infty = \lim_{n\to\infty} a_n \iff \forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon, \ x_n > \varepsilon$

PROOF:

$L2: \quad B(\infty, \varepsilon) = (\varepsilon, \infty]$

$\Rightarrow$ THE NECESSITY

We know: $\infty = \lim_{n\to\infty} a_n \overset{\text{def.}}{\iff} \forall \varepsilon > 0, \ \exists n_v \in \mathbb{N} \text{ s.t. } \forall n \geq n_v, \ a_n \in V_n$

We want: $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon, \ a_n < \varepsilon$ (1)

Choose an $\boxed{\varepsilon > 0}$ randomly

$a_n < \varepsilon \iff a_n \in B(\infty, \varepsilon) \in v(\infty)$

$\varepsilon > 0 \Rightarrow B(\infty, \varepsilon) \in v(\infty) \xRightarrow{(*)} \boxed{\exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon}, \ a_n \in B(\infty, \varepsilon)$

$\Updownarrow$

$\boxed{a_n < \varepsilon}$

$\Rightarrow$ 1 holds for ε, ε - random $\Big\} \Rightarrow$ (1) ✓

$\Leftarrow$ THE SUFFICIENCY

We know: (1)

We want: (*)

We start with: Choose $V \in v(\infty)$ randomly

$a_n \in V$

$\Big\downarrow$ def. of a neigh.

$\exists \underbrace{r > 0}$ s.t. $B(\infty, r) \subseteq V$

$\xRightarrow[\varepsilon := r]{(1)} \exists n_r \in \mathbb{N} \text{ s.t. } \forall n \geq n_r, \quad a_n < r$

$\Updownarrow$

$a_n \in B(\infty, r) \subseteq V$

$\Rightarrow a_n \in V$

$\Rightarrow$ (*) is true for V, V - random $\Big\} \Rightarrow$ (*) ✓

L3: $\ell = -\infty$ $\quad -\infty = \lim_{n\to\infty} a_n \Leftrightarrow \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon,\ x_n < -\varepsilon$

PROOF:

L3: $B(-\infty, \varepsilon) = [-\infty, -\varepsilon)$

$\Rightarrow$ THE NECESSITY

We know: $-\infty = \lim_{n\to\infty} a_n \overset{def.}{\Longleftrightarrow} \forall \varepsilon > 0,\ \exists n_V \in \mathbb{N}$ s.t. $\forall n \geq n_V,\ a_n \in V_n$

We want: $\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon,\ a_n < -\varepsilon$ (1)

Choose an $\boxed{\varepsilon > 0}$ randomly

$a_n < \varepsilon \Leftrightarrow a_n \in B(-\infty, \varepsilon) \in \mathcal{V}(-\infty)$

$\varepsilon > 0 \Rightarrow B(-\infty, \varepsilon) \in \mathcal{V}(-\infty) \overset{(*)}{\Longrightarrow} \boxed{\exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon},\ a_n \in B(-\infty, \varepsilon)$

$\Updownarrow$

$\boxed{a_n < \varepsilon}$

$\Rightarrow$ 1 holds for ε, ε-random $\Big\} \Rightarrow$ (1) ✓

$\Leftarrow$ THE SUFFICIENCY

We know: (1)

We want: (*)

We start with: Choose $V \in \mathcal{V}(-\infty)$ randomly

$a_n \in V$

$\big\downarrow$ def. of a neigh.

$\exists\ \underline{r > 0}$ s.t. $B(-\infty, r) \subseteq V$

$\xrightarrow[\varepsilon := r]{(1)} \exists n_r \in \mathbb{N}$ s.t. $\forall n \geq n_r,\ a_n < -r$

$\Updownarrow$

$a_n \in B(-\infty, r) \subseteq V$

$\Rightarrow a_n \in V$

$\Rightarrow$ (*) is true for V, V-random $\Big\} \Rightarrow$ (*) ✓