ANALYTIC GEOMETRY, PROBLEM SET 5

Various problems with vectors

1. Given the vectors $\overline{a}(3, -1, -2)$ and $\overline{b}(1, 2, -1)$. Compute

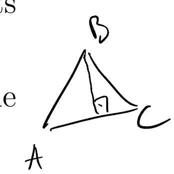
$$\overline{a} \times \overline{b}, (2\overline{a} + \overline{b}) \times \overline{b} \text{ and } (2\overline{a} + \overline{b}) \times (2\overline{a} - \overline{b}).$$

2. Find the distances between the opposite sides of the parallelogram constructed on $\overrightarrow{AB}(6, 0, 2)$ şi $\overrightarrow{AC}(1.5, 2, 1)$.

as in 2.

3. Find the vector $\overline{p}$, knowing that $\overline{p}$ is perpendicular on $\overline{a}(2, 3, -1)$ and $\overline{b}(1, -1, 3)$ and its dot product with $\overline{c}(2, -3, 4)$ is equal to 51.

4. Given the points $A(1, -1, 2)$, $B(5, -6, 2)$ and $C(1, 3, -1)$, find the length of the altitude from the vertex B in the triangle $\triangle ABC$.



5. Given the vectors $\overline{a}(2, -3, 1)$, $\overline{b}(-3, 1, 2)$ and $\overline{c}(1, 2, 3)$, compute $(\overline{a} \times \overline{b}) \times \overline{c}$ and $\overline{a} \times (\overline{b} \times \overline{c})$.

6. Let $ABCD$ be a convex quadrilateral. Show that if the diagonal AC passes through the midpoint of the diagonal BD, then the triangles ACB and ACD have equal areas.

7. Prove that the points $A(1, 2, -1)$, $B(0, 1, 5)$, $C(-1, 2, 1)$ and $D(2, 1, 3)$ are situated in the same plane.

8. Find the volume of the tetrahedron which has $A(2, -1, 1)$, $B(5, 5, 4)$, $C(3, 2, 1)$ and $D(4, 1, 3)$ as vertices. $\hookrightarrow Vol_{ABCD} = \frac{1}{6}(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}) = \dots$

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be coplanar representatives of vectors with modulus 1 and such that A, B, C are on the same side of a line that passes through O. Show that $||\overline{OA} + \overline{OB} + \overline{OC}|| \geq 1$.

10. Let $ABCD$ be a quadrilateral and E, F the midpoints of $[AB]$ and $[CD]$. Denote by K, L, M and N the midpoints of the segments $[AF], [CE], [BF]$ and $[DE]$, respectively. Prove that $KLMN$ is a parallelogram.

$\overline{KL} + \overline{KN} = \overline{KM}$

Date: October 23, 2021.

I expect you are able to prove equalities as the ones below. Have a go at them!

12. Let $\overline{a}, \overline{b}, \overline{c}$ be vectors in $\mathcal{V}_3$. Prove the following formulae:

1. $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{a} \cdot \overline{b}) \cdot \overline{c} = \begin{vmatrix} \overline{b} & \overline{c} \\ \overline{a} \cdot \overline{b} & \overline{a} \cdot \overline{c} \end{vmatrix}$;
2. $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{b} \cdot \overline{c}) \cdot \overline{a} = \begin{vmatrix} \overline{b} & \overline{a} \\ \overline{b} \cdot \overline{c} & \overline{a} \cdot \overline{c} \end{vmatrix}$.
3. $(\overline{a} \times \overline{b}) \cdot (\overline{c} \times \overline{d}) = \begin{vmatrix} \overline{a} \cdot \overline{c} & \overline{a} \cdot \overline{d} \\ \overline{b} \cdot \overline{c} & \overline{b} \cdot \overline{d} \end{vmatrix}$;
4. $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = (\overline{a}, \overline{c}, \overline{d}) \cdot \overline{b} - (\overline{b}, \overline{c}, \overline{d}) \cdot \overline{a} = (\overline{a}, \overline{b}, \overline{d}) \cdot \overline{c} - (\overline{a}, \overline{b}, \overline{c}) \cdot \overline{d}$;
5. $(\overline{a} \times \overline{b}, \overline{b} \times \overline{c}, \overline{c} \times \overline{a}) = (\overline{a}, \overline{b}, \overline{c})^2$

$a(a_1, a_2, a_3)$
$b(b_1, b_2, b_3)$
$c(\ldots)$
$d(\quad)$

use 2, $c \leftrightarrow c \times d$

use 4.

$\left[(\bar{a} \times \bar{b}) \times (\bar{b} \times \bar{c})\right] \cdot (\bar{c} \times \bar{a})$

1. Given the vectors $\overline{a}(3,-1,-2)$ and $\overline{b}(1,2,-1)$. Compute

$$\overline{a}\times\overline{b},\ (2\overline{a}+\overline{b})\times\overline{b} \text{ and } (2\overline{a}+\overline{b})\times(2\overline{a}-\overline{b}).$$

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = i(1+4) - j(-3+2) + k(6+1) = (5,1,7)$$

$$\bar{a}\times\bar{b} = (3i - j - 2k)\times(i + 2j - k) \qquad \_\times\_ : V^3\times V^3 \to V^3$$

$$= \underbrace{3\cancel{i\times i}}_{0} + 6\underbrace{i\times j}_{k} + \dots$$

$$(2\bar{a}+\bar{b})\times\bar{b} = (2\bar{a})\times\bar{b} + \underbrace{\cancel{\bar{b}\times\bar{b}}}_{0} = 2(\bar{a}\times\bar{b}) = (10,2,14)$$

$$(2\bar{a}+\bar{b})\times(2\bar{a}-\bar{b}) = 2\bar{a}\times(2\bar{a}-\bar{b}) + \bar{b}\times(2\bar{a}-\bar{b})$$

$$= \underbrace{\cancel{(2\bar{a})\times(2\bar{a})}}_{0} - (2\bar{a})\times\bar{b} + \bar{b}\times(2\bar{a}) - \underbrace{\cancel{\bar{b}\times\bar{b}}}_{0}$$

$$= 2\underbrace{\bar{b}\times\bar{a}}_{-\bar{a}\times\bar{b}} - 2\bar{a}\times\bar{b}$$

$$= -4\,\bar{a}\times\bar{b} = (-20,-4,-28)$$

3. Find the vector $\overline{p}$, knowing that $\overline{p}$ is perpendicular on $\overline{a}(2,3,-1)$ and $\overline{b}(1,-1,3)$ and its dot product with $\overline{c}(2,-3,4)$ is equal to 51.

Method 1

$$\bar{p}\perp\bar{a} \text{ and } \bar{b} \Rightarrow \bar{p}\parallel\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} =$$

$$= i(9-1) - j(6+1) + k(-2-3)$$

$$= (8,-7,-5)$$

$\Leftrightarrow \bar{p} = \lambda\,\bar{a}\times\bar{b}$ for some $\lambda\in\mathbb{R}$

$\Leftrightarrow \bar{p} = (8\lambda, -7\lambda, -5\lambda)$

$$\bar{p}\cdot\bar{c} = 51 \Leftrightarrow 16\lambda + 21\lambda - 20\lambda = 51 \Rightarrow \lambda = \frac{51}{17} = 3$$

$\Rightarrow \quad \overline{p} = (24, -21, -15)$

Method 2

$\overline{p}(x,y,z)$

$\overline{p} \perp \overline{a} \Leftrightarrow \overline{p} \cdot \overline{a} = 0$

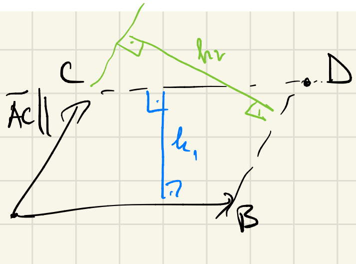
$\overline{p} \perp \overline{b} \Leftrightarrow \overline{p} \cdot \overline{b} = 0$

$\overline{p} \cdot \overline{c} = 51$

$$\begin{cases} 2x + 3y - z = 0 \\ x - y + 3z = 0 \\ 2x - 3y - 4z = 0 \end{cases} \rightarrow \dots x, y, z.$$

2. Find the distances between the opposite sides of the parallelogram constructed on $\overrightarrow{AB}(6,0,2)$ şi $\overrightarrow{AC}(1.5,2,1)$.

$\text{area}_{ABDC} = \| \overline{AB} \times \overline{AC} \| = h_1 \cdot \|\overline{AB}\| = h_2 \cdot \|\overline{AC}\|$



$\Rightarrow h_1 = \dfrac{\| \overline{AB} \times \overline{AC} \|}{\| \overline{AB} \|} = \dots$

$h_2 =$ similar

5. Given the vectors $\overline{a}(2,-3,1)$, $\overline{b}(-3,1,2)$ and $\overline{c}(1,2,3)$, compute $(\overline{a} \times \overline{b}) \times \overline{c}$ and $\overline{a} \times (\overline{b} \times \overline{c})$.

$\boxed{(\overline{u} \times \overline{v}) \times \overline{w} = (\overline{u} \cdot \overline{w})\overline{v} - (\overline{v} \cdot \overline{w})\overline{u}}$.

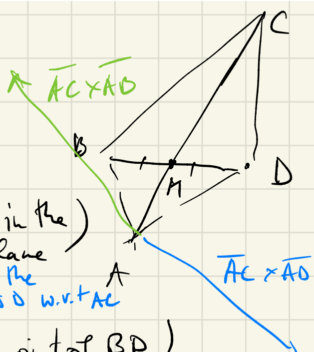
$(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{b} \cdot \overline{c}) \cdot \overline{a}$

6. Let $ABCD$ be a convex quadrilateral. Show that if the diagonal AC passes through the midpoint of the diagonal BD, then the triangles ACB and ACD have equal areas.

$\text{area}_{\Delta ACB} = \text{area}_{\Delta ACD}$

$\Leftrightarrow \dfrac{\| \overline{AC} \times \overline{AB} \|}{\cancel{2}} = \dfrac{\| \overline{AC} \times \overline{AD} \|}{\cancel{2}}$

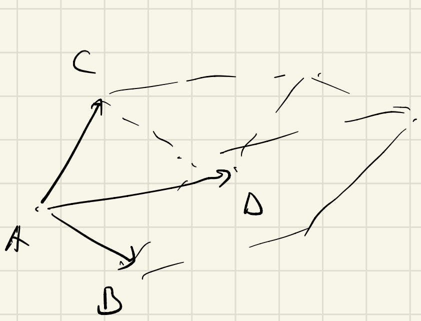
$\Leftrightarrow \overline{AC} \times \overline{AB} = - \overline{AC} \times \overline{AD}$ (since A, B, C, D in the same plane and B is on the other side as D w.r.t AC)



$\Leftrightarrow \overline{AC} \times (\underbrace{\overline{AB} + \overline{AD}}_{2\overline{AM}}) = 0$ (since M is the midpoint of BD)

$\Leftrightarrow \quad \overline{AC} \times \cancel{2}\,\overline{AM} = 0$ which is true since $\vec{AM} \parallel \overline{AC}$ since A, M, C colinear.

7. Prove that the points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are situated in the same plane.



$\Updownarrow$

$$\mathrm{Vol}_{ABCD} = 0$$

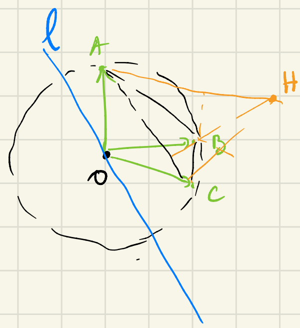
$\Updownarrow$

$$(\overline{AB}, \overline{AC}, \overline{AD}) = 0$$

$\parallel$

$$\begin{array}{c} \overline{AB} \\ \overline{AC} \\ \overline{AD} \end{array} \begin{vmatrix} -1 & -1 & 6 \\ \to & & \\ \to & & \end{vmatrix} \overset{?}{=} 0$$

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be coplanar representatives of vectors with modulus 1 and such that A, B, C are on the same side of a line that passes through O. Show that $\|\overline{OA}+\overline{OB}+\overline{OC}\| \geq 1$. (*)



I. for the ΔABC O is circumcenter

$\Rightarrow \quad \overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$

where H = ortocenter

A, B, C are on the same side of l

$\Rightarrow \hat{ABC}$ obtuse

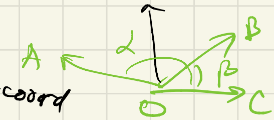
$\Rightarrow$ the orthocenter lies outside the circumcircle

II. $\overset{?}{\Rightarrow}$ one of the angles $A\hat{O}B$, $A\hat{O}C$, $B\hat{O}C$ is acute

we may assume WLOG that it is $B\hat{O}C$

• choose $\overline{OC}$ to be i

• we may assume A, B have positive y-coord



then $\overline{OA} = (\cos\alpha, \sin\alpha)$

$\overline{OB} = (\cos\beta, \sin\beta)$

$\overline{OC} = (1, 0)$

$$(*) \Leftrightarrow \|\underbrace{\overline{OA} + \overline{OB} + \overline{OC}}\|^2 \geq 1$$

$$\Leftrightarrow (\cos\alpha + \cos\beta + 1)^2 + (\sin\alpha + \sin\beta)^2 \geq 1$$

$$\Leftrightarrow 3 + 2\cos\alpha\cos\beta + \overbrace{2\cos\alpha}^{\geq -2} + 2\cos\beta + \overbrace{2\sin\alpha\sin\beta}^{\geq 0} \geq 1$$

$$2\underbrace{(\cos\alpha + 1)}_{\geq 0}\underbrace{\cos\beta}_{\geq 0}$$