Proposition 3.1. Let ℓ and ℓ' be two lines in $\mathbf{A}$ with Cartesian equations

$$\ell : ax + by + c = 0 \quad \text{and} \quad \ell' : a'x + b'y + c = 0.$$

Then

1. ℓ and ℓ' are parallel if and only if the matrix

$$\begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \tag{3.3}$$

   has rank 1, that is, if and only if $ab' - a'b = 0$.

1) The directions of ℓ and ℓ' are determined by the homogeneous equations

$$W_\ell : ax + by = 0 \quad \text{and} \quad W_{\ell'} : a'x + b'y = 0$$

$$\ell \parallel \ell' \iff W_\ell = W_{\ell'} \iff \exists \lambda \in \mathbb{R} \setminus \{0\} : \ a' = \lambda a \ \text{ and } \ b' = \lambda b$$

$$\iff \operatorname{rank} \begin{bmatrix} a & b \\ a' & b \end{bmatrix} = 1$$

2. If ℓ and ℓ' are parallel, then they are disjoint or they coincide accordingly as the matrix

$$\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} \tag{3.4}$$

   has rank 2 or 1.

- The matrix $\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix}$ has rank 1 $\iff$ the two equations of ℓ and ℓ' are proportional

$$\iff \operatorname{rank} \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} = 1$$

- If $\operatorname{rank} \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} = 2$ and $\operatorname{rank} \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} = 1$

  then the system $\begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{cases}$ has no solution

$$\Rightarrow \ell \cap \ell' = \emptyset \Rightarrow \ell \parallel \ell'$$

3. ℓ and ℓ' have precisely one point in common if and only if the matrix (3.3) has rank 2. In this case the point $\ell \cap \ell'$ has coordinates

$$x_0 = \frac{cb' - c'b}{ab' - a'b}, \quad y_0 = \frac{ac' - a'c}{ab' - a'b}.$$

$\operatorname{rank}\begin{bmatrix} a & b \\ a' & b' \end{bmatrix} = 2 \iff$ the system $\begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{cases}$ has a unique solution

the unique solution (x_0, y_0) are the coordinates of the intersection point $\ell \cap \ell'$

By Cramer's rule x_0, y_0 have the indicated expressions

Proposition 3.3. If $\ell_1 : ax + by + c = 0$ and $\ell_2 : a'x + b'y + c' = 0$ are two distinct lines in the pencil $\mathcal{L}_Q$, then $\mathcal{L}_Q$ consists of lines having equations of the form

$$\ell_{\lambda,\mu} : \lambda(ax + by + c) + \mu(a'x + b'y + c') = 0. \qquad (*)$$

where $\lambda, \mu \in \mathbf{K}$ not both zero.

- Since $Q = Q(x_0, y_0) \in \ell_1, \ell_2$ the coordinats of Q satisfy the equations of ℓ_1, ℓ_2

  $\Rightarrow$ the coordinates of Q satisfy $(*)$ $\forall \lambda, y$

  $\Rightarrow Q \in \ell_{\lambda,\mu} \quad \forall \lambda, \mu$

- if $\ell_3 : a''x + b''y + c'' = 0$ is a line containing Q but different from ℓ_1, ℓ_2

then $\begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0 \\ a''x + b''y + c'' = 0 \end{cases}$ has a solution $\Rightarrow \operatorname{rank}\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \leq 2$

$\Rightarrow \det\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} a'' \\ b'' \\ c'' \end{bmatrix}$ is a linear combination of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$.

Theorem 3.6 (Thales)**.** Let H, H' and H'' be three distinct parallel lines in an affine plane $\mathbf{A}$, and let ℓ_1 and ℓ_2 be two lines not parallel to H, H', H''. For $i = 1, 2$ let

$$\begin{aligned} P_i &= \ell_i \cap H \\ P_i' &= \ell_i \cap H' \\ P_i'' &= \ell_i \cap H'' \end{aligned}$$

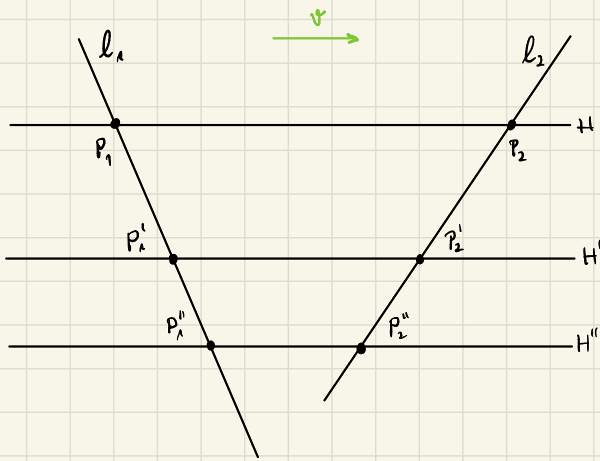
and let $k_1, k_2 \in \mathbf{K}$ be such that

$$\overrightarrow{P_iP_i''} = k_i\, \overrightarrow{P_iP_i'} \qquad (*)$$

Then $k_1 = k_2$.

If $\ell_1 = \ell_2$ the theorem is trivial.

So suppose $\ell_1 \neq \ell_2$ (see Fig.)



If $P_1 = P_2$ we may interchange H and H''

So we may assume $P_1 \neq P_2$

Let $v = \overrightarrow{P_1P_2}$.

$$\overrightarrow{P_1'P_2'} - \cancel{\overrightarrow{P_1'P_2}} - (\overrightarrow{P_1P_2} + \cancel{\overrightarrow{P_2P_1'}})$$

Then $$\overrightarrow{P_2P_2'} - \overrightarrow{P_1P_1'} = \overrightarrow{P_1'P_2'} - \overrightarrow{P_1P_2} \overset{(**)}{=} \alpha v$$ and

$$\overrightarrow{P_2P_2''} - \overrightarrow{P_1P_1''} \overset{(**)}{=} \overrightarrow{P_1''P_2''} - \overrightarrow{P_1P_2} = \beta v$$ for some $\alpha, \beta \in \mathbb{K}$

If $\alpha = 0$ then $\ell_1 \parallel \ell_2$

$\Rightarrow \beta = 0$ else $v \parallel \ell_1$ and $v \parallel \ell_2$ by which contradicts the hypotheses

Then we have $\overrightarrow{P_2P_2''} \overset{(*)}{=} k_2\overrightarrow{P_2P_2'} \overset{\substack{\alpha=0\\(**)}}{=} k_2\,\overrightarrow{P_1P_1'}$ and $\overrightarrow{P_1P_1''} \overset{(*)}{=} k_1\overrightarrow{P_1P_1'}$

$\beta = 0 \Rightarrow \overrightarrow{P_1P_1''} \overset{(**)}{=} \overrightarrow{P_2P_2''}$

$\Rightarrow \quad k_2\,\overrightarrow{P_1P_1'} = \overrightarrow{P_2P_2''} = \overrightarrow{P_1P_1''} = k_1\,\overrightarrow{P_1P_1'}$

$\Rightarrow \quad k_1 = k_2$

If $\alpha \neq 0$ then

$$\overrightarrow{P_2P_2''} - \overrightarrow{P_1P_1''} = \beta v = \frac{\beta}{\alpha}(\alpha v) = \frac{\beta}{\alpha}\overrightarrow{P_2P_2'} - \frac{\beta}{\alpha}\overrightarrow{P_1P_1'} \qquad (****)$$

but also $\overrightarrow{P_2P_2''} - \overrightarrow{P_1P_1''} \overset{(*)}{=} k_2\overrightarrow{P_2P_2'} - k_1\overrightarrow{P_1P_1} \qquad (*****)$

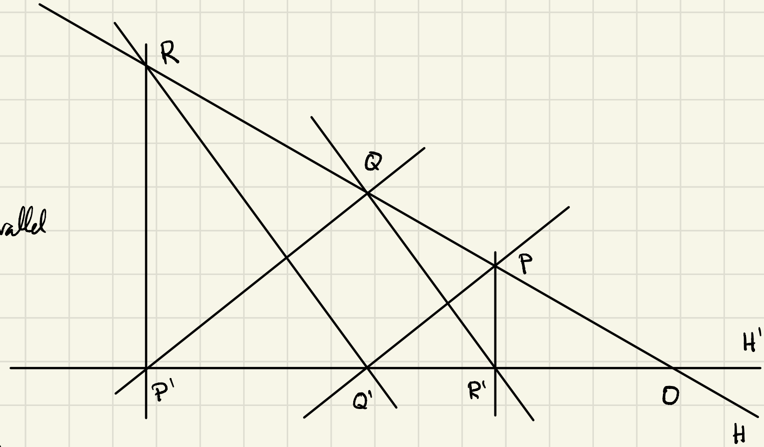
since $\alpha \neq 0$ $\overrightarrow{P_1P_1'}$ and $\overrightarrow{P_2P_2'}$ are not parallel

i.e. $\overrightarrow{P_1P_1'}$ and $\overrightarrow{P_2P_2'}$ are linearly independent

$\Rightarrow$ the coefficients of the vector $\overrightarrow{P_2P_2''} - \overrightarrow{P_1P_1''}$ in $(****)$ and $(*****)$ are unique

$$\Rightarrow \quad k_2 = \frac{\beta}{\alpha} = k_1$$

Theorem 3.7 (Pappus). Let H, H' be two distinct lines in an affine plane $\mathbf{A}$. Let $P, Q, R \in H$ and $P', Q', R' \in H$ be distinct points, none of which lies at the intersection $H \cap H'$. If $\langle P, Q' \rangle \parallel \langle P', Q \rangle$ and $\langle Q, R' \rangle \parallel \langle Q', R \rangle$ and then $\langle P, R' \rangle \parallel \langle P', R \rangle$.



- Suppose H and H' are not parallel

Let $H \cap H' = O$ (see Fig)

by Thales' theorem $\exists\, h, k \in \mathbb{R} \setminus \{0\}$ such that

$$\overrightarrow{OP'} = k\overrightarrow{OQ} \qquad \overrightarrow{OQ} = k\overrightarrow{OP} \qquad \text{since } \langle P, Q' \rangle \parallel \langle P', Q \rangle$$

$$\overrightarrow{OQ'} = h\overrightarrow{OR'} \qquad \overrightarrow{OR} = h\overrightarrow{OQ} \qquad \text{since } \langle R, Q' \rangle \parallel \langle R', Q \rangle$$

Therefore $\overrightarrow{PR'} = \overrightarrow{OR'} - \overrightarrow{OP} = h^{-1}\overrightarrow{OQ'} - k^{-1}\overrightarrow{OQ} = \frac{1}{hk}\left(k\overrightarrow{OQ'} - h\overrightarrow{OQ}\right)$

$$\overrightarrow{RP'} = \overrightarrow{OP'} - \overrightarrow{OR} = k\overrightarrow{OQ} - h\overrightarrow{OQ}$$

$$\Rightarrow \overrightarrow{RP'} = hk \cdot \overrightarrow{PR'} \Rightarrow \langle R, P' \rangle \parallel \langle P, R' \rangle$$

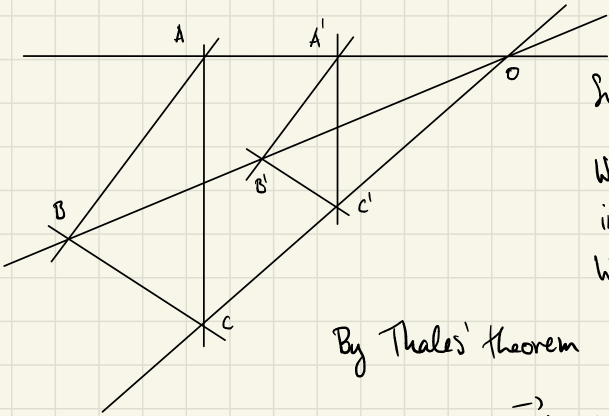
- If $H \parallel H'$ then $\overrightarrow{PQ} = \overrightarrow{Q'P'}$ since $\langle P, Q \rangle \parallel \langle Q', P' \rangle$

and $\overrightarrow{QR} = \overrightarrow{R'Q'}$ since $\langle Q, R \rangle \parallel \langle Q', R' \rangle$

$$\Rightarrow \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{Q'P'} + \overrightarrow{R'Q'} = \overrightarrow{R'P'}$$

$$\Rightarrow \langle R, P' \rangle \parallel \langle P, R' \rangle$$

Theorem 3.8 (Desargues)**.** Let $A, B, C, A', B', C' \in \mathbf{A}$ be points such that no three (***) are collinear, and such that $\langle A, B\rangle \parallel \langle A', B'\rangle$, $\langle B, C\rangle \parallel \langle B', C'\rangle$ and $\langle A, C\rangle \parallel \langle A', C'\rangle$. Then the three lines $\langle A, A'\rangle$, $\langle B, B'\rangle$ and $\langle C, C'\rangle$ are either parallel or have a point in common.



$XY = \langle X, Y \rangle$

Suppose AA', BB' and CC' are not parallel

We have to show that they intersect in a point.

We may assume that $AA' \cap BB' = \{O\}$

By Thales' theorem applied to AB and $A'B'$ $\exists\, k \in \mathbb{K}$ s.t.

$$\vec{OA'} = k\vec{OA} \quad \text{and} \quad \vec{OB'} = k\vec{OB}$$

Let $C'' = OC \cap A'C'$. By Thales' theorem applied to AC and $A'C'$

$$(*) \quad \vec{OC''} = k\vec{OC} \quad \text{since} \quad \vec{OA'} = k\vec{OA}$$

Let $C''' = OC \cap B'C'$ By Thales' theorem applied to BC and $B'C'$

$$(**) \quad \vec{OC'''} = k\vec{OC} \quad \text{since} \quad \vec{OB'} = k\vec{OB}$$

By (*) and (**) $\vec{OC''} = k\vec{OC} = \vec{OC'''}$

$$\left.\begin{array}{l} \Rightarrow \quad C'' = C''' \in OC \cap A'C' \cap B'C' \\ \text{but } A'C' \neq B'C' \text{ else } A', B', C' \text{ collinear \& contradicting } (***) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad C' = C'' = C'''$$

$$\Rightarrow \quad CC' = CC'' \ni O \in AA' \cap BB'$$