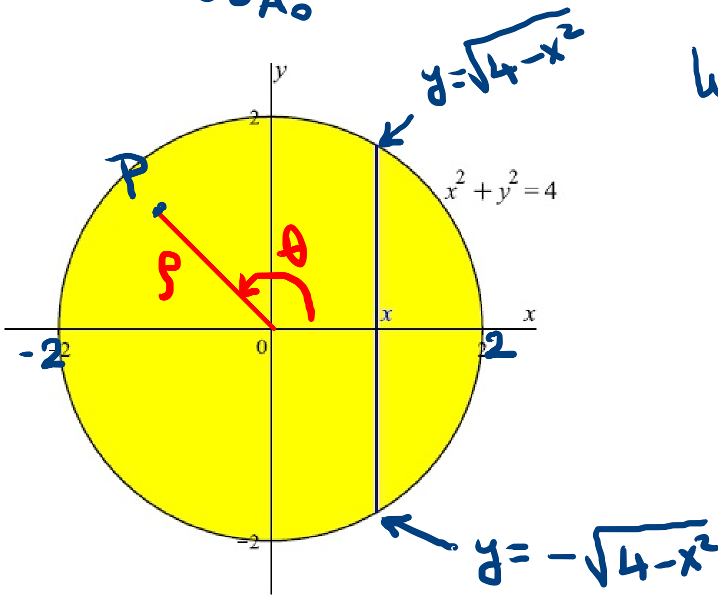
Seminar 10

April 27, 2021

$$I = \iint_{A_0} (x^2+y^2)(4-x^2-y^2)\,dx\,dy$$



With Fubini's theorem

$$I = \int_{x=-2}^{x=2} \left(\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (x^2+y^2)(4-x^2-y^2)\,dy \right) dx$$

HORRIBLE !

In order to evaluate the double integral, we pass to polar coordinates:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \rho\in[0,2] \quad \theta\in[0,2\pi] \qquad \frac{\partial(x,y)}{\partial(\rho,\theta)} = \rho$$

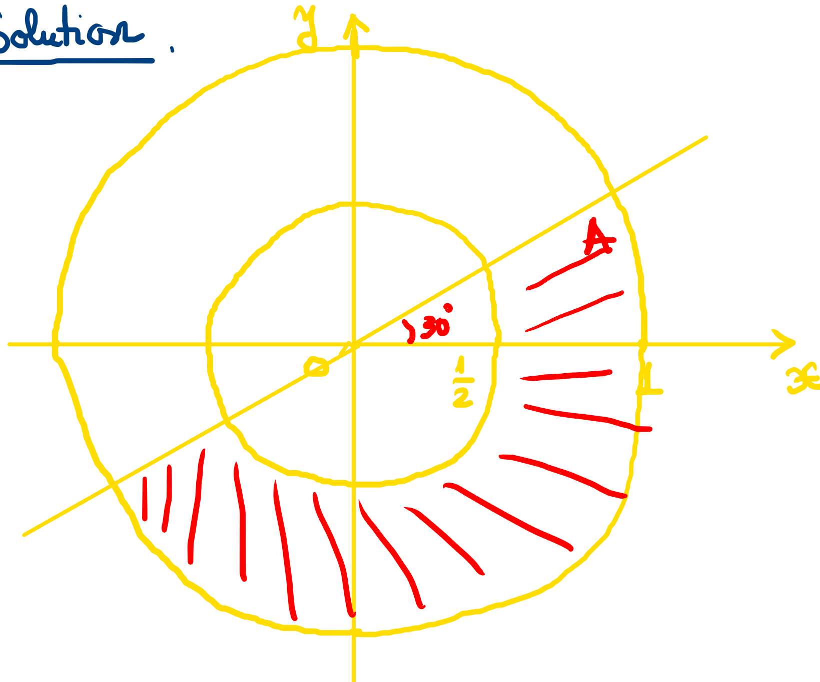
$$I = \int_{\rho=0}^{\rho=2}\int_{\theta=0}^{\theta=2\pi} \underbrace{(\rho^2\cos^2\theta+\rho^2\sin^2\theta)}_{\rho^2}\underbrace{(4-\rho^2\cos^2\theta-\rho^2\sin^2\theta)}_{4-\rho^2}\cdot\rho\,d\rho\,d\theta$$

$$I = \left(\int_0^2 \rho^3(4-\rho^2)\,d\rho\right)\left(\int_0^{2\pi} d\theta\right) \Rightarrow \boxed{I = \frac{32\pi}{3}}$$

$\boxed{1}$ Evaluate $I = \iint_A x\sqrt{1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$, if

$$A := \{\,(x,y)\in\mathbb{R}^2 \mid \sqrt{3}x-3y\ge 0,\ 1\le 4(x^2+y^2)\le 4\,\}. \quad \Big|\,\frac{1}{4}$$

Solution.



$\sqrt{3}x - 3y \ge 0$ $\qquad$ $\sqrt{3}x - 3y = 0$

$\sqrt{3}x = 3y \qquad \boxed{y = \frac{\sqrt{3}}{3}x}$

$$\frac{1}{4} \le x^2+y^2 \le 1$$

We pass to polar coordinates

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \rho\in\left[\frac12, 1\right] \qquad \frac{\partial(x,y)}{\partial(\rho,\theta)} = \rho$$

$$\theta \in \left[0, \frac{\pi}{6}\right] \cup \left[\frac{7\pi}{6}, 2\pi\right]$$

or

$$\theta \in \left[-\frac{5\pi}{6}, \frac{\pi}{6}\right]$$

$$I = \int_{\rho=\frac12}^{\rho=1}\int_{\theta=-\frac{5\pi}{6}}^{\frac{\pi}{6}} \rho\cos\theta\sqrt{1-\rho^2}\cdot\rho\, d\rho\, d\theta = \left(\int_{\frac12}^{1}\rho^2\sqrt{1-\rho^2}\,d\rho\right)\underbrace{\left(\int_{-\frac{5\pi}{6}}^{\pi/6}\cos\theta\, d\theta\right)}_{\sin\theta\Big|_{-\frac{5\pi}{6}}^{\pi/6} = 1}$$

$$\rho = \sin t$$

$$I = \frac14\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2 t\cdot\cos^2 t\, dt = \frac14\int_{\pi/6}^{\pi/2}\sin^2 2t\, dt = \frac14\int_{\pi/6}^{\pi/2}\frac{1-\cos 4t}{2}\, dt$$

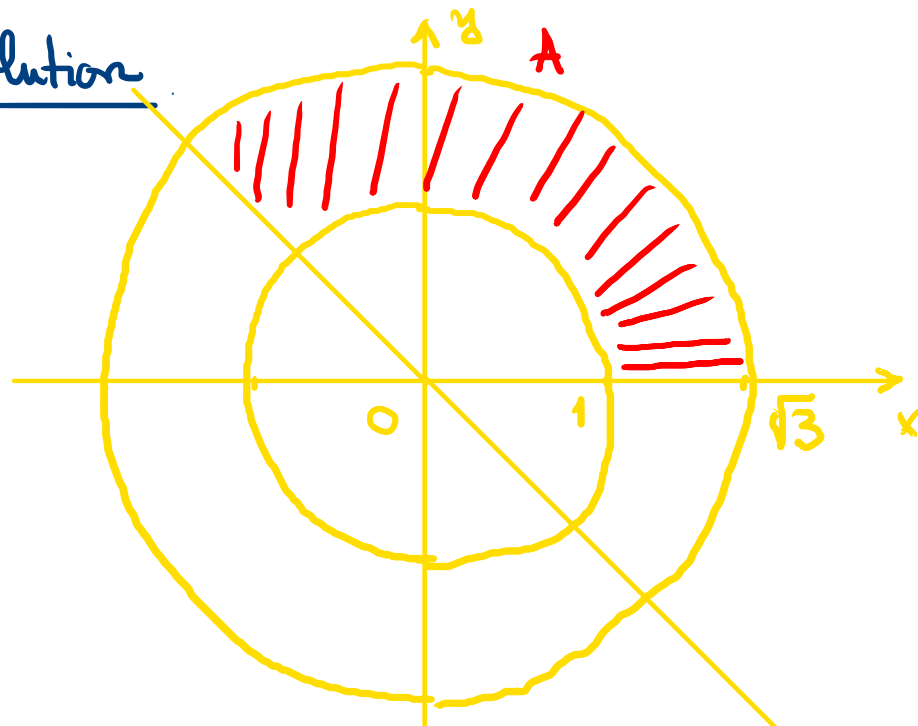
$$I = \frac{1}{8}\left(t - \frac{\sin 4t}{4}\right)\Bigg|_{\pi/6}^{\pi/2} = \frac{\sqrt{3}}{64} + \frac{\pi}{24}.$$

$$(x-a)^2 + (y-b)^2 = r^2$$

$\boxed{2}$ Evaluate $I = \displaystyle\iint_A \frac{x^2}{x^2+3y^2}\,dx\,dy$, if

$$A := \{(x,y) \in \mathbb{R}^2 \mid 1 \le x^2+y^2 \le 3,\ x+y \ge 0,\ y \ge 0\}.$$

Solution



$x+y=0 \Rightarrow y=-x$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \rho \in [1, \sqrt{3}] \qquad \theta \in \left[0, \frac{3\pi}{4}\right] \qquad \frac{\partial(x,y)}{\partial(\rho,\theta)} = \rho$$

$$I = \int_{\rho=1}^{\rho=\sqrt{3}} \int_{\theta=0}^{\theta=\frac{3\pi}{4}} \frac{\cancel{\rho^2}\cos^2\theta}{\cancel{\rho^2}\cos^2\theta + 3\cancel{\rho^2}\sin^2\theta} \cdot \rho\, d\rho\, d\theta =$$

$$= \underbrace{\left(\int_1^{\sqrt{3}} \rho\, d\rho\right)}_{\frac{1}{2}\rho^2\Big|_{\rho=1}^{\rho=\sqrt{3}} = 1} \left(\int_0^{\frac{3\pi}{4}} \frac{\cos^2\theta}{\cos^2 + 3\sin^2\theta}\, d\theta\right)$$

$$I = \int_{0+0}^{\frac{3\pi}{4}} \frac{\cos^2\theta}{\cos^2\theta + 3\sin^2\theta}\, d\theta = -\int_{\infty}^{-1} \frac{\frac{t^2}{\cancel{t^2+1}}}{\frac{t^2}{\cancel{t^2+1}} + 3\cdot\frac{1}{\cancel{t^2+1}}} \cdot \frac{1}{t^2+1}\, dt$$

$$\operatorname{ctg}\theta = t \qquad \frac{\cos^2\theta}{\sin^2\theta} = t^2 \qquad \frac{\cos^2\theta}{\cos^2\theta + \sin^2\theta} = \frac{t^2}{t^2+1} \qquad \cos^2\theta = \frac{t^2}{t^2+1}$$

$$\theta = \operatorname{arccotan} t \qquad \sin^2\theta = \frac{1}{t^2+1}$$

$$d\theta = -\frac{1}{1+t^2}\, dt$$

$$I = \int_{-1}^{\infty} \frac{t^2}{(t^2+3)(t^2+1)}\, dt$$

$$\cancel{\frac{t^2}{(t^2+3)(t^2+1)} = \frac{At+B}{t^2+3} + \frac{Ct+D}{t^2+1}} \qquad \frac{y}{(y+3)(y+1)} = \frac{A}{y+3} + \frac{B}{y+1}$$

$$y = t^2$$

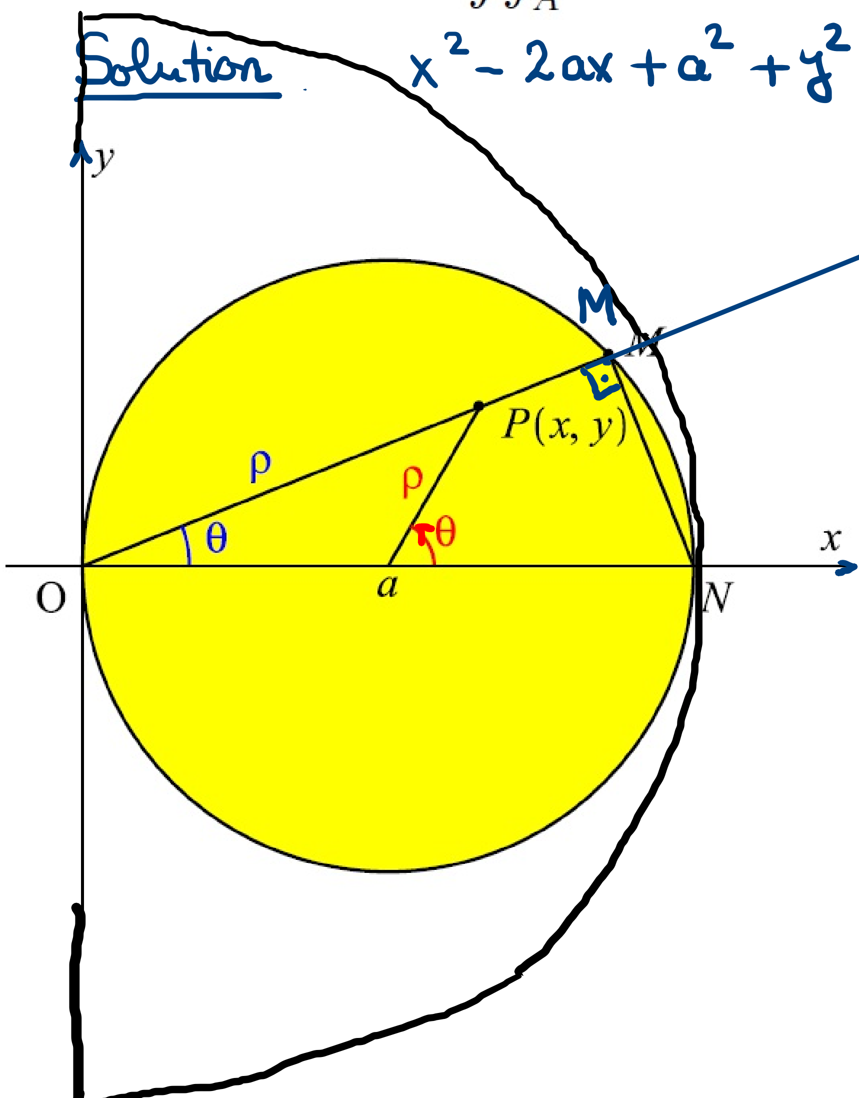
$$I = \frac{\pi}{24}(8\sqrt{3} - 9).$$

(Homework) Evaluate $\displaystyle\iint_A \frac{y}{x+y+\sqrt{x^2+y^2}}\,\mathrm{d}x\mathrm{d}y,$ if

$$A := \{\,(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 4,\ x+y\ge 0,\ y\ge 0\,\}.$$

$\boxed{3}$ Evaluate $I = \displaystyle\iint_A \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$, if $A := \{\,(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2ax\,\}$, $a>0$.

Solution $x^2-2ax+a^2+y^2-a^2\le 0 \iff (x-a)^2+y^2\le a^2$



First method: $\begin{cases} x-a=\rho\cos\theta \\ y=\rho\sin\theta\end{cases}$ $\quad \rho\in[0,a]$, $\theta\in[0,2\pi]$

$$\frac{\partial(x,y)}{\partial(\rho,\theta)}=\rho$$

$$I=\int_{\rho=0}^{\rho=a}\int_{\theta=0}^{\theta=2\pi}\sqrt{(a+\rho\cos\theta)^2+\rho^2\sin^2\theta}\cdot\rho\,d\rho\,d\theta$$

$$=\int_{\rho=0}^{\rho=a}\int_{\theta=0}^{\theta=2\pi}\rho\sqrt{a^2+2a\rho\cos\theta+\rho^2}\,d\rho\,d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\left(\int_{\rho=0}^{\rho=a}\rho\sqrt{a^2+2a\rho\cos\theta+\rho^2}\,d\rho\right)d\theta$$

ABANDON DU TRAVAIL

$$\cos\theta=\frac{OM}{ON}=\frac{OM}{2a}$$

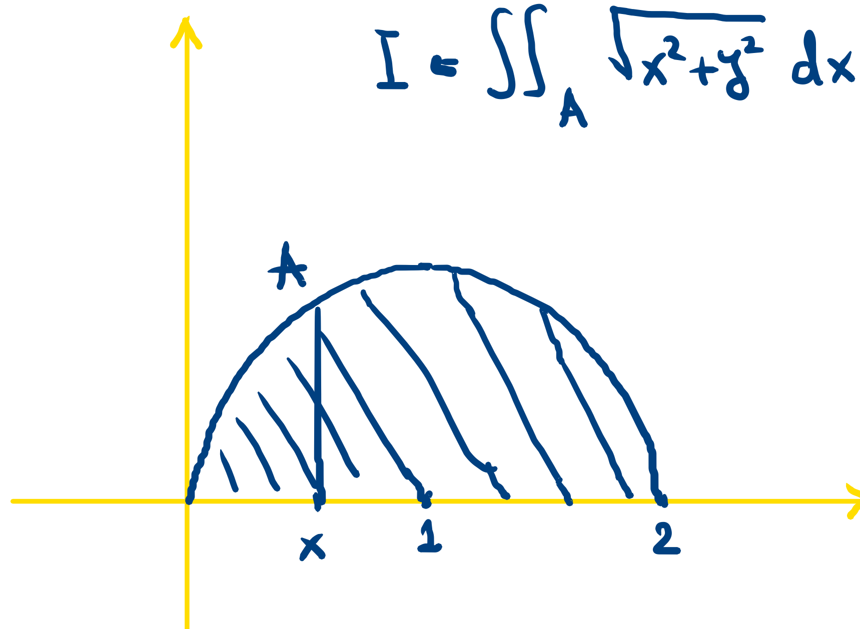
<u>Second method</u>: $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\quad \cancel{\rho \in [0, 2a]} \quad \cancel{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$ $\quad \rho \in [0, 2a\cos\theta] \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$I = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left(\int_{\rho=0}^{\rho=2a\cos\theta} \rho \cdot \rho \, d\rho \right) d\theta = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{\rho^3}{3} \Bigg|_{\rho=0}^{\rho=2a\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8a^3}{3} \cos^3\theta \, d\theta = \frac{32a^3}{9}.$$

$\boxed{4}$ Evaluate $I = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \right) dx.$

$$I = \iint_A \sqrt{x^2+y^2} \, dx\,dy$$



<u>Solution</u>.

$y = \sqrt{2x - x^2} \qquad y^2 = 2x - x^2$

$(x-1)^2 + y^2 = 1$

$$\int_{-1}^{1} f(x)\,dx \overset{?}{=} 2\int_0^1 f(x)\,dx$$

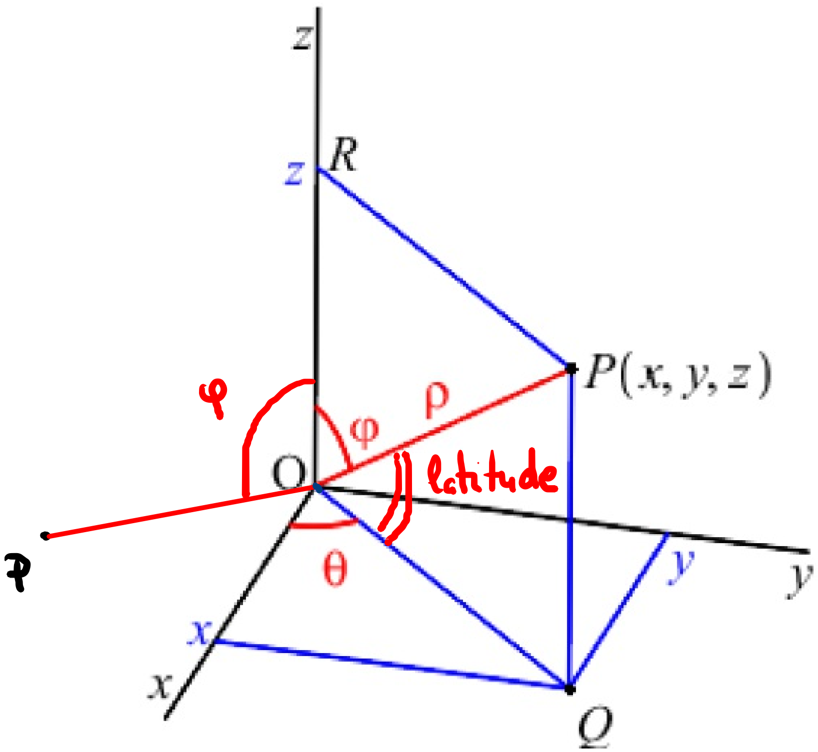
(Homework) Evaluate $\displaystyle\int_0^{1/2}\left(\int_{\sqrt{3}y}^{\sqrt{1-y^2}}\sqrt{x^2+y^2}\,\mathrm{d}x\right)\mathrm{d}y.$

$\boxed{5}$ Evaluate $\displaystyle I = \iiint_A \frac{1}{\sqrt{x^2+y^2+(z-2)^2}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if

$$A := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\}.$$

<u>Solution</u>. We pass to spherical coordinates :

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$



$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2\sin\varphi$$

$\rho\in[0,1]$

$\varphi\in[0,\pi]$

$\theta\in[0,2\pi]$

$$I = \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi} \int_{\theta=0}^{\theta=2\pi} \frac{1}{\sqrt{\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta + (\rho\cos\varphi - 2)^2}} \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi} \int_{\theta=0}^{\theta=2\pi} \frac{\rho^2 \sin\varphi}{\sqrt{\rho^2 - 4\rho\cos\varphi + 4}} \, d\rho \, d\varphi \, d\theta$$

$$= \left(\int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi} \frac{\rho^2 \sin\varphi}{\sqrt{\rho^2 - 4\rho\cos\varphi + 4}} \, d\rho \, d\varphi \right) \underbrace{\left(\int_0^{2\pi} d\theta \right)}_{2\pi}$$

$$= 2\pi \int_{\rho=0}^{\rho=1} \left(\int_{\varphi=0}^{\varphi=\pi} \frac{\rho^2 \sin\varphi}{\sqrt{\rho^2 - 4\rho\cos\varphi + 4}} \, d\varphi \right) d\rho = \cancel{2}\pi \int_{\rho=0}^{\rho=1} \left(\rho \int_{t=|\rho-2|}^{t=\rho+2} \frac{\frac{1}{\cancel{2}} \cancel{t} \, dt}{\cancel{t}} \right) d\rho$$

$$\sqrt{\rho^2 - 4\rho\cos\varphi + 4} = t$$

$$\rho^2 - 4\rho\cos\varphi + 4 = t^2$$

$$4\rho \sin\varphi \, d\varphi = 2t\,dt \quad \Big| \cdot \frac{1}{4} \qquad \rho \sin\varphi \, d\varphi = \frac{1}{2} t \, dt$$

$$I = \pi \int_{\rho=0}^{\rho=1} \rho \cdot t \Big|_{t=2-\rho}^{t=2+\rho} d\rho = \pi \int_0^1 \rho \cdot 2\rho \, d\rho = 2\pi \cdot \frac{\rho^3}{3} \Big|_0^1 = \boxed{\frac{2\pi}{3}}$$

$\boxed{6}$ Evaluate $I = \iiint_A \frac{1}{(x^2+y^2+z^2+1)^2}\,dx\,dy\,dz$, if $A : \begin{cases} x^2+y^2+z^2 \le 1 \\ z \geq 0 \end{cases}$

Solution We pass to spherical coordinates

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho\cos\varphi \end{cases} \qquad \begin{array}{l} \rho \in [0,1] \\ \varphi \in \left[0, \frac{\pi}{2}\right] \\ \theta \in [0, 2\pi] \end{array} \qquad \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2 \sin\varphi$$

$$I = \int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi} \frac{1}{(\rho^2+1)^2}\cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \left(\int_0^1 \frac{\rho^2}{(\rho^2+1)^2}\,d\rho\right) \underbrace{\left(\int_0^{\pi/2} \sin\varphi\, d\varphi\right)}_{= -\cos\varphi\,\big|_0^{\pi/2} = 1} \underbrace{\left(\int_0^{2\pi} d\theta\right)}_{2\pi} =$$

$$\frac{\rho}{(\rho^2+1)^2} = \left(-\frac{1}{2}\cdot\frac{1}{\rho^2+1}\right)'$$

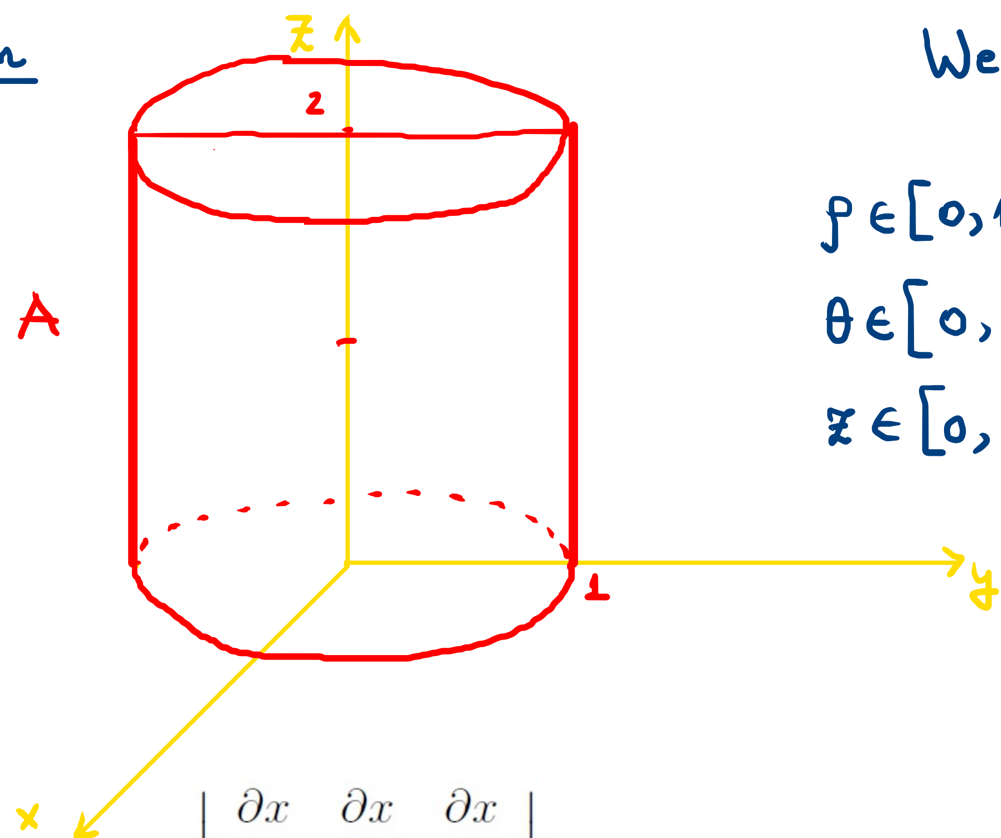
$$= 2\pi \cdot \int_0^1 \rho \left(-\frac{1}{2(\rho^2+1)}\right)' d\rho = \cancel{2}\pi \cdot \rho\left(-\frac{1}{\cancel{2}(\rho^2+1)}\right)\Bigg|_0^1 + \cancel{2}\pi \int_0^1 \frac{1}{\cancel{2}(\rho^2+1)}\,d\rho$$

$$= -\frac{\pi}{2} + \pi \arctan\rho\,\Big|_0^1 = \frac{\pi^2}{4} - \frac{\pi}{2} = \boxed{\frac{\pi^2-2\pi}{4}}$$

$\boxed{7}$ Evaluate $I = \displaystyle\iiint_A \frac{1}{\sqrt{x^2+y^2+(3-z)^2}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if

$$A := \{\,(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le 1,\ 0 \le z \le 2\,\}.$$

Solution

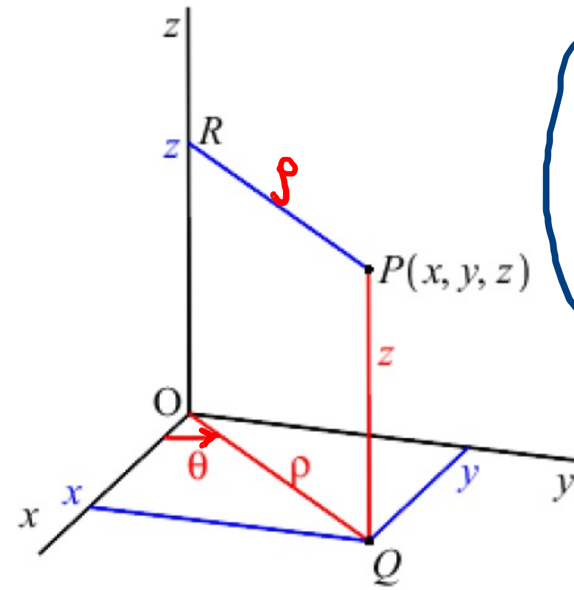


We pass to cylindrical coordinates

$\rho \in [0,1]$

$\theta \in [0,2\pi]$

$z \in [0,2]$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

x, y, z = Cartesian coordinates of P

ρ, θ, z = cylindrical coordinates of P

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{\rho}$$