ANALYTIC GEOMETRY, PROBLEM SET 2

More polar coordinates

1. Let $ABCDEF$ be a regular hexagon with size length l. Find the polar coordinates of its vertices in each of the following cases:

a) The center of the hexagon O is chosen as *the pole* and the half-line $[OA$ is set as *the polar axis*.

b) The vertex A is chosen as *the pole* and the half-line $[AB$ is set as *the polar axis*.

2. Find the polar equation corresponding to the given Cartesian equation: a) $y = 5$; b) $x + 1 = 0$; c) $y = 7x$; d) $3x + 8y + 6 = 0$; e) $y^2 = -4x + 4$; f) $x^2 - 12y - 36 = 0$; g) $x^2 + y^2 = 36$; h) $x^2 - y^2 = 25$. Briefly give a geometric interpretation for the solutions to these equations.

3. Find the polar coordinates of the point $P \in \mathcal{E}_2$, whose rectangular (Cartesian) coordinates are $(1 + \cos\alpha, \sin\alpha)$, where $\alpha \in (0, 2\pi)$ is fixed.

Cylindrical and spherical (everything is in 3D here)

Warmp-up 1. In the cylindrical coordinate system, what do the following equations represent in $\mathcal{E}_3$?

a) $r = r_0$, where $r_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed;

c) $z = z_0$, where $z_0 \in \mathbb{R}$ is fixed.

Warm-up 2. In the spherical coordinate system, what do the following equations represent in $\mathcal{E}_3$?

a) $\rho = \rho_0$, where $\rho_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed;

c) $\varphi = \varphi_0$, where $\varphi_0 \in [0, \pi]$ is fixed.

4. Let $P_1(r_1, \theta_1, z_1)$ and $P_2(r_2, \theta_2, z_2)$ be points in $\mathcal{E}_3$ expressed using their cylindrical coordinates. Find the distance P_1P_2, as an expression of r_i, θ_i, z_i, where $i \in \{1, 2\}$.

5. Let $P_1(r_1, \theta_1, \varphi_1)$ and $P_2(r_2, \theta_2, \varphi_2)$ be points in $\mathcal{E}_3$, expressed using their spherical coordinates. Find the distance P_1P_2, as an expression of r_i, θ_i, φ_i, where $i \in \{1, 2\}$.

6. Determine, in cylindrical coordinates, the equation of the surface whose equation in rectangular coordinates is $z = x^2 + y^2 - 2x + y$.

7. Find the equation, in rectangular coordinates, of the surface whose equation in cylindrical coordinates is $r = 4\cos(\theta)$. Explain what the equation describes geometrically.

8. (Non-examinable) Three spheres are pairwise exterior tangent; a plane is tangent to these spheres at points A, B and C. Find the radii of the spheres in terms of a, b, c, representing the lengths of the sides of triangle ABC.

Date: October 2, 2021.

Time permitting...

9. Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that $\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}=\overline{0}$.

10. In the plane determined by the triangle ABC, let us consider the points M, N, P, Q such that $\overrightarrow{AM}=\frac{2}{3}\overrightarrow{AB}$, $2\overrightarrow{NA}+\overrightarrow{NC}=\overrightarrow{0}$, $\overrightarrow{AP}=\frac{2}{5}\overrightarrow{AB}$ and $3\overrightarrow{QA}+2\overrightarrow{QB}+\overrightarrow{QC}=\overrightarrow{0}$.

(1) Find $\alpha\in\mathbb{R}$ such that $\overrightarrow{QN}=\alpha\cdot\overrightarrow{QM}$.
(2) Find $\beta\in\mathbb{R}$ such that $\overrightarrow{CQ}=\beta\cdot\overrightarrow{QP}$.
(3) Find the value of the ratio $\frac{QA}{QR}$, where $AQ\cap BC=\{R\}$.

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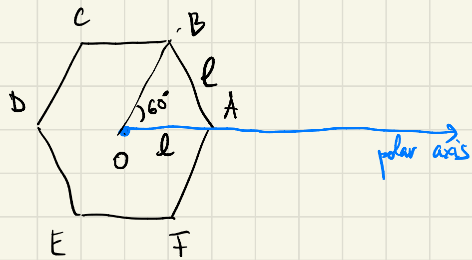
a)



(r, θ) PS

$A(l, 0)$

$B(l, \frac{\pi}{3})$

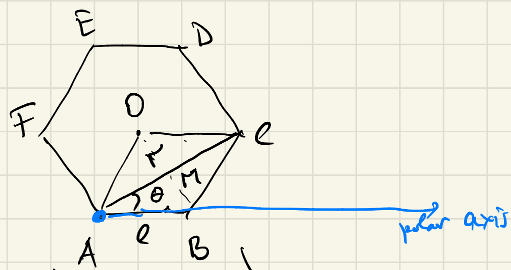
$C(l, \frac{2\pi}{3})$

$D(l, \pi)$

$E(l, \frac{4\pi}{3})$

$F(l, \frac{5\pi}{3})$

b)



(r, θ)

$A(0, 0)$

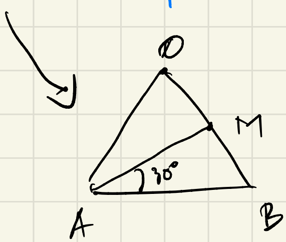
$B(l, 0)$

$C(l\sqrt{3}, \frac{\pi}{6})$

$D(2l, \frac{\pi}{3})$

$E(l\sqrt{3}, \frac{\pi}{2})$

$F(l, \frac{2\pi}{3})$



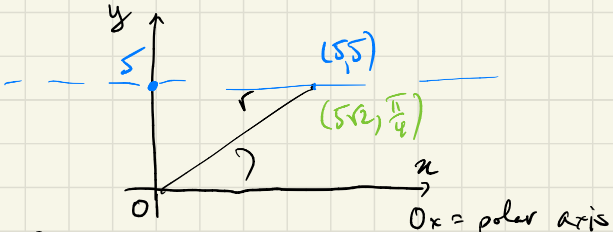
$(AM)^2 = AB^2 - MB^2$
$= l^2 - \frac{l^2}{4}$

$\frac{AC^2}{\cancel{4}} = \frac{3l^2}{\cancel{4}}$ $\quad AC^2 = 3l^2$

$AC = \sqrt{3} \cdot l$

2. Find the polar equation corresponding to the given Cartesian equation: a) $y = 5$; b) $x + 1 = 0$; c) $y = 7x$; d) $3x + 8y + 6 = 0$; e) $y^2 = -4x + 4$; f) $x^2 - 12y - 36 = 0$; g) $x^2 + y^2 = 36$; h) $x^2 - y^2 = 25$. Briefly give a geometric interpretation for the solutions to these equations.

a.) $y = 5$ (RS) $\rightsquigarrow$ ______ (PS)



RS (x, y) $\longleftarrow$ pts $\longrightarrow$ PS (r, θ)

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$r = \sqrt{x^2 + y^2}$

$\theta = \arctan\frac{y}{x} + k\pi$ where $k = \dots$ (see lect.)

corresponds to $r\sin\theta = 5$ by

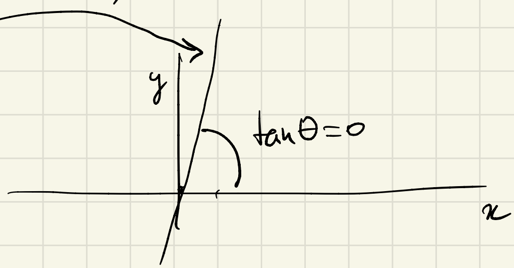
$5\sqrt{2} \cdot \underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} = 5$ ✓

b.) $x + 1 = 0 \rightsquigarrow r\cos\theta = -1$

$\Updownarrow$

$x = -1$

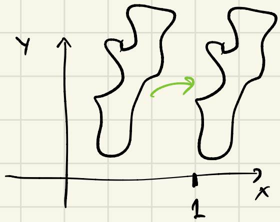
c.) $y = 7x \rightsquigarrow r\sin\theta = 7r\cos\theta \iff \tan\theta = 7$

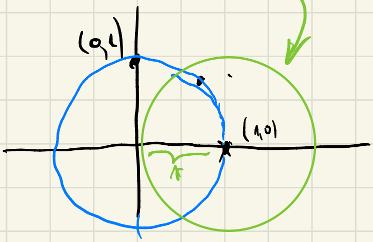


3. Find the polar coordinates of the point $P \in \mathcal{E}_2$, whose rectangular (Cartesian) coordinates are $(1+\cos\alpha, \sin\alpha)$, where $\alpha \in (0, 2\pi)$ is fixed.

$\{P(1+\cos\alpha, \sin\alpha) \mid \alpha \in (0, 2\pi)\}$ = "points of the form $(\cos\alpha, \sin\alpha)$ which we move by 1 along the x-axis"

"$\subseteq$" $\{Q(\cos\alpha, \sin\alpha) \mid \alpha \in (0, 2\pi)\}$ shifted by 1 to the right.



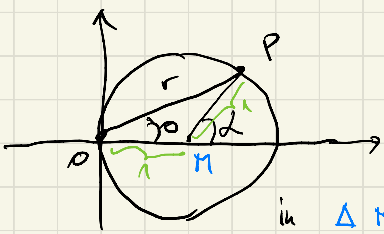


$P(r_p, \theta_p)$

$r_p = \sqrt{(1+\cos\alpha)^2 + \sin\alpha^2}$ $= 1$

$1 + 2\cos\alpha + \cos\alpha^2$

$r_p = \sqrt{2}\sqrt{1+\cos\alpha}$



$\theta_p = \arctan\frac{y}{x}$ if $x \neq 0$ else.... $\theta_p = \frac{\pi}{2}$ if $y>0$ and $\theta_p = \frac{3\pi}{2}$ if $y<0$

$= \arctan\left(\frac{\sin\alpha}{1+\cos\alpha}\right)$

in Δ MOP $\quad 180^\circ - 2\theta_p = 180^\circ - \alpha$

$\Rightarrow \alpha = 2\theta_p \quad \Rightarrow \theta_p = \frac{\alpha}{2}$

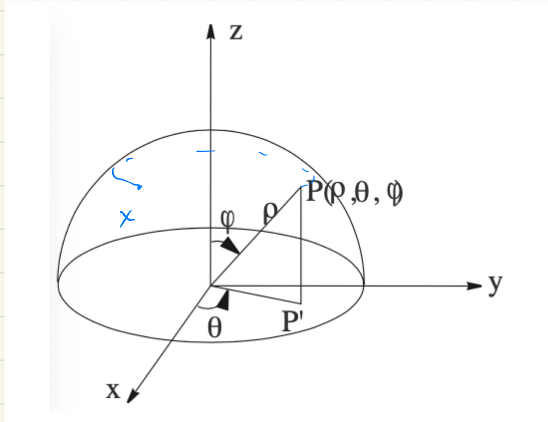
$\Rightarrow$ in PS : $P\left(\sqrt{2}\sqrt{1+\cos\alpha}, \frac{\alpha}{2}\right)$

Warm-up 2. In the spherical coordinate system, what do the following equations represent in $\mathcal{E}_3$?
a) $\rho = \rho_0$, where $\rho_0 \in \mathbb{R}_{\geq 0}$ is fixed; b) $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi)$ is fixed;
c) $\varphi = \varphi_0$, where $\varphi_0 \in [0, \pi]$ is fixed.

a.) $\rho = \rho_0$ for some $\rho_0 \in \mathbb{R}_{\geq 0}$

"ρ is constant (equal to ρ_0)"

↳ sphere of radius ρ_0



b.) "$\theta = \theta_0$" half-plane attached to the z-axis and having angle θ with Ox

5. Let $P_1(r_1, \theta_1, \varphi_1)$ and $P_2(r_2, \theta_2, \varphi_2)$ be points in $\mathcal{E}_3$, expressed using their spherical coordinates. Find the distance P_1P_2, as an expression of r_i, θ_i, φ_i, where $i \in \{1, 2\}$.

$$P_1P_2 = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2} \quad \text{if } P_i = (x_i, y_i, z_i) \text{ in RS}$$

$x_i = \rho_i \cos\theta_i \sin\varphi_i$, $y_i = \rho_i \sin\theta_i \sin\varphi_i$, $z_i = \rho_i \cos\varphi_i$ $\quad i = 1,2$

$$P_1P_2^{\,2} = (\overbrace{\rho_1 \cos\theta_1 \sin\varphi_1}^{x_1} - \overbrace{\rho_2 \cos\theta_2 \sin\varphi_2}^{x_2})^2 + (\rho_1 \sin\theta_1 \sin\varphi_1 - \rho_2 \sin\theta_2 \sin\varphi_2)^2 +$$
$$+ (\rho_1 \cos\varphi_1 - \rho_2 \cos\varphi_2)^2$$
$$= \rho_1^2 \cos\theta_1^2 \sin\varphi_1^2 - 2\rho_1\rho_2 \cos\theta_1 \sin\varphi_1 \cos\theta_2 \sin\varphi_2 + \rho_2^2 \cos\theta_2^2 \sin\varphi_2^2 +$$
$$\rho_1^2 \sin\theta_1^2 \sin\varphi_1^2 - 2\rho_1\rho_2 \sin\theta_1 \sin\varphi_1 \sin\theta_2 \sin\varphi_2 + \rho_2^2 \sin\theta_2^2 (\sin\varphi_2)^2$$
$$+ (\rho_1 \cos\varphi_1 - \rho_2 \cos\varphi_2)^2$$

$$= \rho_1^2 (\sin\varphi_1)^2 + \rho_2^2 (\sin\varphi_2)^2 - 2\rho_1\rho_2 \sin\varphi_1 \sin\varphi_2 (\underbrace{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}_{\cos(\theta_1-\theta_2)}) +$$

$$+ \rho_1^2 \cos\varphi_1^2 - 2\rho_1\rho_2\cos\varphi_1\cos\varphi_2 + \rho_2^2\cos\varphi_2^2$$

$$P_1P_2 = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \left(\sin\varphi_1 \sin\varphi_2 \cos(\theta_1 - \theta_2) + \cos\varphi_1\cos\varphi_2 \right)}$$

$$\parallel$$

$$P_2P_1$$