Calculus on $\mathbb{R}$ — ANCA GRAD PH.D., LECT. PROFF.
Homework 6 — **2021**

Series of real numbers - 2nd part

Exercise 1: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a)\sum_{n\geq 1}\frac{n+7}{\sqrt{n^2+7}},\quad b)\sum_{n\geq 1}\frac{1}{\sqrt[n]{n}},\quad c)\sum_{n\geq 1}\frac{1}{\sqrt[n]{n!}},\quad d)\sum_{n\geq 1}\left(1+\frac{1}{n}\right)^n.$$

Exercise 2: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a)\sum_{n\geq 1}\frac{2^n+3^n}{5^n},\quad b)\sum_{n\geq 1}\frac{2^n}{3^n+5^n}.$$

Exercise 3: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a)\sum_{n\geq 1}\frac{1}{2n-1},\quad b)\sum_{n\geq 1}\frac{1}{(2n-1)^2},\quad c)\sum_{n\geq 1}\frac{1}{\sqrt{4n^2-1}},\quad d)\sum_{n\geq 1}\frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}. \;?$$

Exercise 4: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a)\sum_{n\geq 1}\frac{100^n}{n!},\quad b)\sum_{n\geq 1}\frac{2^n n!}{n^n},\quad c)\sum_{n\geq 1}\frac{3^n n!}{n^n},\quad d)\sum_{n\geq 1}\frac{(n!)^2}{2^{n^2}},\quad e)\sum_{n\geq 1}\frac{n^2}{\left(2+\frac{1}{n}\right)^n}.$$

Exercise 5: Determine the nature (convergence or divergence), by discussing the value of the parameter $a>0$, of the following series of real numbers:

$$a)\sum_{n\geq 1}\frac{a^n}{n^n},\quad b)\sum_{n\geq 1}\left(\frac{n^2+n+1}{n^2}a\right)^n,\quad c)\sum_{n\geq 1}\frac{3^n}{2^n+a^n}.$$

Exercise 1: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a) \sum_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}}, \quad b) \sum_{n\geq 1} \frac{1}{\sqrt[n]{n}}, \quad c) \sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}}, \quad d) \sum_{n\geq 1} \left(1+\frac{1}{n}\right)^n.$$

a) $\sum_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}}$

$u_n = \frac{n+7}{\sqrt{n^2+7}}$ $\quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+7}{\sqrt{n^2+7}} = 1 \neq 0 \Rightarrow \sum_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}}$ is DIVERGENT

b) $\sum_{n\geq 1} \frac{1}{\sqrt[n]{n}}$ $\quad u_n = \frac{1}{\sqrt[n]{n}}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \overset{C-S}{=} \lim_{n\to\infty} \frac{1}{\frac{n+1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0 \Rightarrow \sum_{n\geq 1} \frac{1}{\sqrt[n]{n}}$ is DIVERGENT

c) $\sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}}$ $\quad u_n = \frac{1}{\sqrt[n]{n!}}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} \overset{C-S}{=} \lim_{n\to\infty} \frac{1}{\frac{(n+1)!}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow$ the series must be further investigated

We shall apply the comparison criterion.

Let $y_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{u_n}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[n]{n!}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$

$\sum u_n \sim \sum y_n \Rightarrow \sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}}$ is DIVERGENT

d) $\sum_{n\geq 1} \left(1+\frac{1}{n}\right)^n$ $\quad u_n = \left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0 \Rightarrow \sum_{n\geq 1} \left(1+\frac{1}{n}\right)^n$ is DIVERGENT

Exercise 2: Determine the nature (convergence or divergence) of the following series of real numbers:

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a) $\sum_{n\geq 1} \frac{2^n+3^n}{5^n}$ $\quad u_n = \frac{2^n+3^n}{5^n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n+3^n}{5^n} = \lim_{n\to\infty} \frac{3^n\left(\left(\frac{2}{3}\right)^n+1\right)}{5^n} = 0 \Rightarrow$ the series must be further investigated

$\sum_{n\geq 1} \frac{2^n+3^n}{5^n} = \sum_{n\geq 1} \left(\left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n\right)$

$$\left.\begin{array}{l} \left.\begin{array}{l} 2^n < 3^n \\ 3^n \leq 3^n \end{array}\right\} \Rightarrow 2^n + 3^n < 2\cdot 3^n \\ 5^n > 4^n \Rightarrow \frac{1}{5^n} < \frac{1}{4^n} \end{array}\right\} \Rightarrow \frac{2^n+3^n}{5^n} < 2\cdot \underbrace{\left(\frac{3^n}{4^n}\right)}_{\text{convergent}} \Rightarrow \sum_{n\geq 1} \frac{2^n+3^n}{5^n} \text{ is CONVERGENT}$$

b) $\sum_{n\geq 1} \frac{2^n}{3^n+5^n}$ $\quad u_n = \frac{2^n}{3^n+5^n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n}{5^n\left(\left(\frac{3}{5}\right)^n+1\right)} = 0 \Rightarrow$ the series must be further investigated

$\left.\begin{array}{l} 3^n \geq 3^n \\ 5^n > 3^n \end{array}\right\} \Rightarrow 3^n+5^n > 2\cdot 3^n \Rightarrow \frac{1}{3^n+5^n} < \frac{1}{2\cdot 3^n} \Rightarrow \frac{2^n}{3^n+5^n} < \frac{1}{2}\left(\frac{2}{3}\right)^n$

Choose $\alpha := \frac{1}{2}$ and $y_n = \left(\frac{2}{3}\right)^n$, y_n convergent $\Rightarrow \sum_{n\geq 1} \frac{2^n}{3^n+5^n}$ is CONVERGENT

Exercise 3: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a) \sum_{n\geq 1} \frac{1}{2n-1}, \quad b) \sum_{n\geq 1} \frac{1}{(2n-1)^2}, \quad c) \sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}, \quad d) \sum_{n\geq 1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}.$$

a) $\sum_{n\geq 1} \frac{1}{2n-1}$ $\quad u_n = \frac{1}{2n-1}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0 \Rightarrow$ the series must be further investigated

$\left.\begin{array}{l} 2n-1 \geq n \Rightarrow \frac{1}{2n-1} \leq \frac{1}{n} \\ \sum_{n\geq 1} \frac{1}{n} \text{ is divergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} \frac{1}{2n-1}$ is DIVERGENT

b) $\sum_{n\geq 1} \frac{1}{(2n-1)^2}$ $\quad u_n = \frac{1}{(2n-1)^2}$

$\left.\begin{array}{l} 2n-1 \geq n \Rightarrow (2n-1)^2 \geq n^2 \Rightarrow \frac{1}{(2n-1)^2} \leq \frac{1}{n^2} \\ \sum_{n\geq 1} \frac{1}{n^2} \text{ is convergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} \frac{1}{(2n-1)^2}$ is CONVERGENT

c) $\sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}$ $\quad u_n = \frac{1}{\sqrt{4n^2-1}}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{4n^2-1}} = 0$

$\left.\begin{array}{l} \sqrt{4n^2-1} < \sqrt{4n^2} \Rightarrow \frac{1}{\sqrt{4n^2-1}} > \frac{1}{\sqrt{4n^2}} = \frac{1}{2n} \\ \sum_{n\geq 1} \frac{1}{2n} = \frac{1}{2} \sum_{n\geq 1} \frac{1}{n} \text{ is divergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}$ is DIVERGENT

d) $\sum_{n\geq 1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}} = \sum_{n\geq 1} \frac{\sqrt[6]{(n^2+n)^3}}{\sqrt[6]{(n^5-n)^2}} = \sum_{n\geq 1} \sqrt[6]{\frac{(n^2+n)^3}{(n^5-n)^2}} = \sum_{n\geq 1} \sqrt[6]{\frac{n^{\cancel{3}}(n+1)^{\cancel{3}}}{\cancel{n^2}\cancel{(n+1)^2}(n-1)^2(n^2+1)^2}} = \sum_{n\geq 1} \sqrt[6]{\frac{n(n+1)}{(n+1)^2(n^2+1)^2}}$

$x_n = \sqrt[6]{\frac{n(n+1)}{(n-1)^2(n^2+1)^2}}$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} =$

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a) $\sum_{n\geq1}\frac{100^n}{n!}$, $x_n=\frac{100^n}{n!}$, $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{100^{n+1}}{(n+1)!}\cdot\frac{n!}{100^n}=\lim_{n\to\infty}\frac{100}{n+1}=0 \xRightarrow[\text{Consequence}]{\text{D'Alembert's}} \sum_{n\geq1}\frac{100^n}{n!}$ is CONVERGENT

b) $\sum_{n\geq1}\frac{2^n\cdot n!}{n^n}$, $x_n=\frac{2^n\cdot n!}{n^n}$, $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{2^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n\cdot n!}=\lim_{n\to\infty}\frac{2\cdot n^n}{(n+1)^n}=2\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=(1^\infty)=2\cdot\lim_{n\to\infty}\left(1+\frac{n}{n+1}-1\right)^n=$

$=2\lim_{n\to\infty}\left(1+\frac{n-n-1}{n+1}\right)^n=2\lim_{n\to\infty}\left(1+\frac{-1}{n+1}\right)^n=2\lim_{n\to\infty}\left(\left(1+\frac{-1}{n+1}\right)^{-(n+1)}\right)^{-\frac{1}{n+1}\cdot n}=2e^{\lim_{n\to\infty}\frac{-n}{n+1}}=$

$=2e^{-1}=\frac{2}{e}<1\Rightarrow\sum_{n\geq1}\frac{2^n\cdot n!}{n^n}$ is CONVERGENT

c) $\sum_{n\geq1}\frac{3^n\cdot n!}{n^n}$, $x_n=\frac{3^n\cdot n!}{n^n}$, $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{3^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{3^n\cdot n!}=\lim_{n\to\infty}3\cdot\left(\frac{n}{n+1}\right)^n=\frac{3}{e}>1\Rightarrow\sum_{n\geq1}\frac{3^n\cdot n!}{n^n}$ is DIVERGENT

d) $\sum_{n\geq1}\frac{(n!)^2}{2^{n^2}}$, $x_n=\frac{(n!)^2}{2^{n^2}}$, $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{((n+1)!)^2}{2^{(n+1)^2}}\cdot\frac{2^{n^2}}{(n!)^2}=\lim_{n\to\infty}\frac{(n+1)^2}{2^{(n+1)^2-n^2}}=\lim_{n\to\infty}\frac{(n+1)^2}{2^{(n+1-n)(n+1+n)}}=\lim_{n\to\infty}\frac{(n+1)^2}{2^{2n+1}}=\frac{1}{2}\lim_{n\to\infty}\left(\frac{n+1}{2^n}\right)^2=$

$=0<1\Rightarrow\sum_{n\geq1}x_n$ is CONVERGENT

e) $\sum_{n\geq1}\frac{n^2}{(2+\frac{1}{n})^n}=\sum_{n\geq1}\frac{n^2}{\frac{(2n+1)^n}{n^n}}=\sum_{n\geq1}\frac{n^{n+2}}{(2n+1)^n}$, $x_n=\frac{n^2}{(2+\frac{1}{n})^n}=\frac{n^{n+2}}{(2n+1)^n}$

$\lim_{n\to\infty}\frac{n^2}{(2+\frac{1}{n})^n}=0<1\Rightarrow\sum_{n\geq1}\frac{n^2}{(2+\frac{1}{n})^n}$ is CONVERGENT

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a) $\sum_{n\geq1}\frac{a^n}{n^n}=\sum_{n\geq1}\left(\frac{a}{n}\right)^n$, $a>0$

$x_n=\left(\frac{a}{n}\right)^n\Rightarrow\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{a^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a^n}=\lim_{n\to\infty}\frac{a}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{1}{e}\cdot\lim_{n\to\infty}\frac{a}{n+1}$

If $a\in(0,1)$: $x_n=\frac{1}{n^n}$, $a_n=a^n$

$(a_n)_n$ decreasing, $\lim_{n\to\infty}a_n=0$ $\Rightarrow\sum_{n\geq1}x_n\cdot a_n$ is CONVERGENT

If $a=1$: $\sum_{n\geq1}\frac{1}{n^n}=\sum_{n\geq1}\left(\frac{1}{n}\right)^n$

$\lim_{n\to\infty}\left(\frac{1}{n}\right)^n=\lim_{n\to\infty}e^{n\ln\left(\frac{1}{n}\right)}=\lim_{n\to\infty}e^{-n\ln n}=\lim_{n\to\infty}\left(\frac{1}{e}\right)^{n\ln n}=0$

$\left(\frac{1}{e}\right)^{n\ln n}<\left(\frac{1}{e}\right)^n<\underbrace{\left(\frac{1}{e}\right)^{n-1}}_{\text{convergent}}\Rightarrow\sum_{n\geq1}u_n$ is CONVERGENT

b) $\sum_{n\geq1}\left(\frac{n^2+n+1}{n^2}\cdot a\right)^n$, $u_n=\left(\frac{n^2+n+1}{n^2}\cdot a\right)^n$, $a>0$

If $a<0\Rightarrow\sum_{n\geq1}\left(\frac{n^2+n+1}{n^2}\cdot a\right)^n$ is DIVERGENT

If $a>0\Rightarrow$ If $a>1$: $\lim u_n=+\infty\Rightarrow\sum u_n$ is DIVERGENT

$n \to \infty$ $\qquad\qquad\qquad$ $n \geq 1$

If $a = 1$: $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \left(\frac{n^2+n+1}{n^2}\right)^n = e^{\lim\limits_{n\to\infty} \frac{n(n+1)}{n^2}} = e^1 > 0 \Rightarrow \sum\limits_{n\geq 1} u_n$ is DIVERGENT

If $a \in (0,1)$: $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \left(\frac{n^2+n+1}{n^2}\right)^n \cdot a^n = e \cdot 0 = 0$

We shall denote $a = \frac{1}{b}$, $b > 1$

$$u_n = \left(\frac{n^2+n+1}{n^2}\right)^n \cdot \left(\frac{1}{b}\right)^n < \underbrace{\frac{e}{b^{n-1}}}_{\text{convergent}}$$

convergent (geometric series with ratio $|q = \frac{1}{b}| < 1$)

$\Rightarrow \sum\limits_{n\geq 1} u_n$ is CONVERGENT