Course 1

February 23, 2021

Evaluation : Final written exam – 5 problems in 2h / 2,5h

? Midterm test – 3 problems in 1h / 1,5h (20% weight in the final mark)

Chapter 1. Topology in $\mathbb{R}^n$

1. The Euclidean space $\mathbb{R}^n$

1.1. Definition (the vector space $\mathbb{R}^n$). Let $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, and let

$$\mathbb{R}^n := \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}.$$

In the set $\mathbb{R}^n$ we consider two operations defined as follows:

$$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \qquad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \longmapsto x + y \in \mathbb{R}^n$$

$$\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \qquad \forall (\alpha, x) \in \mathbb{R} \times \mathbb{R}^n \longmapsto \alpha \cdot x \in \mathbb{R}^n$$

Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ we set

$$x + y := (x_1 + y_1, \dots, x_n + y_n)$$

$$\alpha \cdot x := (\alpha x_1, \dots, \alpha x_n)$$

It is well-known that $(\mathbb{R}^n, +, \cdot)$ is a vector space $/ \mathbb{R}$.

The origin $0_n := (0, \dots, 0)$

The symmetric of $x = (x_1, \dots, x_n)$ is $-x = (-x_1, \dots, -x_n)$.

The elements of $\mathbb{R}$ are called <u>scalars</u>

The elements of $\mathbb{R}^n$ are considered to be $\longrightarrow$ n-tuples $(x_1, \dots, x_n)$

$\updownarrow$

points $P(x_1, \dots, x_n)$

$\updownarrow$

vectors $\overrightarrow{OP}$

We write αx for $\alpha \cdot x$

<u>1.2. Definition</u> (the standard basis in $\mathbb{R}^n$). Let

$$e_1 := (1, 0, 0, \dots, 0), \quad e_2 := (0, 1, 0, \dots, 0), \quad \dots, \quad e_n := (0, 0, \dots, 0, 1)$$

$\Rightarrow \{e_1, \dots, e_n\}$ is an algebraic basis in the vector space $\mathbb{R}^n$, called the standard / canonical basis in $\mathbb{R}^n$

Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n \implies x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$.

$n=2 \quad \{e_1, e_2\} = \{\vec{i}, \vec{j}\}$ $\qquad n=3 \quad \{e_1, e_2, e_3\} = \{\vec{i}, \vec{j}, \vec{k}\}$.

1.3. Definition (the inner/dot product in $\mathbb{R}^n$). Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ We define $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(1) $\langle x, y \rangle := x_1 y_1 + \dots + x_n y_n$

↳ the inner product between x and y

$$\boxed{\begin{aligned} \vec{v} &= v_1 \vec{i} + v_2 \vec{j} \\ \vec{w} &= w_1 \vec{i} + w_2 \vec{j} \\ \vec{v} \cdot \vec{w} &= v_1 w_1 + v_2 w_2 \end{aligned}}$$

Properties

1° $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad \forall x, y, z \in \mathbb{R}^n$

2° $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}, \ \forall x, y \in \mathbb{R}^n$

3° $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in \mathbb{R}^n$

4° $\langle x, x \rangle > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0_n\}$

From 1°, 2°, 3° $\Longrightarrow$ $\langle x, 0_n \rangle = \langle 0_n, x \rangle = 0 \quad \forall x \in \mathbb{R}^n$

$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad \forall x, y, z \in \mathbb{R}^n$

The notion of an inner product can be defined axiomatically as follows: let X be a vector space$/\mathbb{R}$. A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called an inner product on X if it satisfies:

(iP_1) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle \quad \forall x, y, z \in X;$

(iP_2) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle \quad \forall \alpha \in \mathbb{R}, \ \forall x, y \in X;$

(iP_3) $\langle x, y\rangle = \langle y, x\rangle \quad \forall x, y \in X$

(iP_4) $\langle x, x\rangle > 0 \quad \forall x \in X \setminus \{0_X\}$

The ordered pair $(X, \langle\cdot,\cdot\rangle)$ is called a prehilbertian space (David Hilbert)

From $1^\circ \div 4^\circ \Rightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ is a prehilbertian space, called the Euclidean space $\mathbb{R}^n$.

$\uparrow$ defined by (1)

Other examples of inner products:

a) In $\mathcal{M}_n(\mathbb{R})$ $\quad \langle A, B\rangle := tr(A \cdot B^t) \quad \forall A, B \in \mathcal{M}_n(\mathbb{R})$

$\langle A, B\rangle = tr(AB^t)$

$\langle B, A\rangle = tr(BA^t) = tr((AB^t)^t) = tr(AB^t) = \langle A, B\rangle$

$\langle A, A\rangle = tr(A \cdot A^t) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 > 0 \quad$ if $A \neq O_n$

$A = (a_{ij})$

b) In $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b] \}$

$$\langle f, g \rangle := \int_a^b f(x) g(x)\, dx \qquad \forall\ f, g \in C[a,b]$$

↳ is an inner product in $C[a,b]$

c) In $R[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is Riemann integrable over } [a,b] \}$

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx \qquad \forall\ f, g \in R[a,b]$$

↳ is <u>not</u> an inner product in $R[a,b]$

$f = ?$

$f \neq \theta$

$$\langle f, f \rangle = \int_a^b f^2(x)\, dx = 0$$

$$f(x) = \begin{cases} 0 & x \in (a,b] \\ 1 & x = a \end{cases}$$

<u>1.4. Theorem</u> (the Cauchy-Schwarz inequality). For all $x, y \in \mathbb{R}^n$ it holds

(2) $\quad |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$

<u>Proof</u> → seminar

$x = (x_1, \dots, x_n) \qquad y = (y_1, \dots, y_n)$

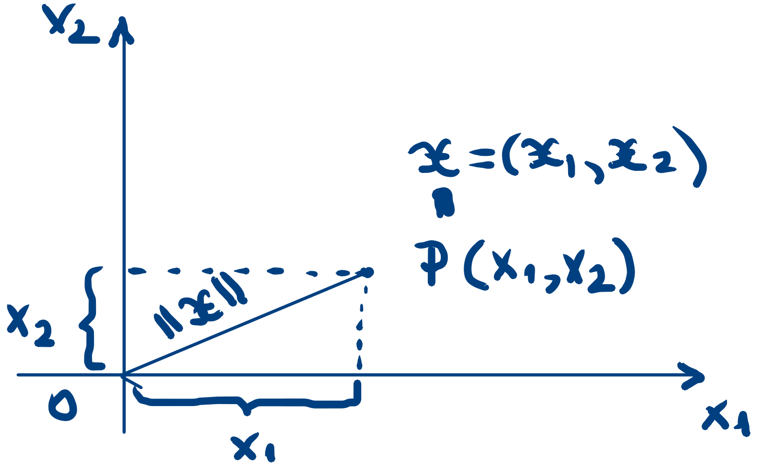
$$(2) \iff |x_1 y_1 + \dots + x_n y_n| \le \sqrt{x_1^2 + \dots + x_n^2} \cdot \sqrt{y_1^2 + \dots + y_n^2} \iff$$

$$\iff \boxed{(x_1 y_1 + \dots + x_n y_n) \le (x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2)}$$

<u>1.5. Definition</u> (the Euclidean norm in $\mathbb{R}^n$). Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we define

$$(3) \quad \|x\| := \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + \dots + x_n^2}$$

↳ the Euclidean norm of x



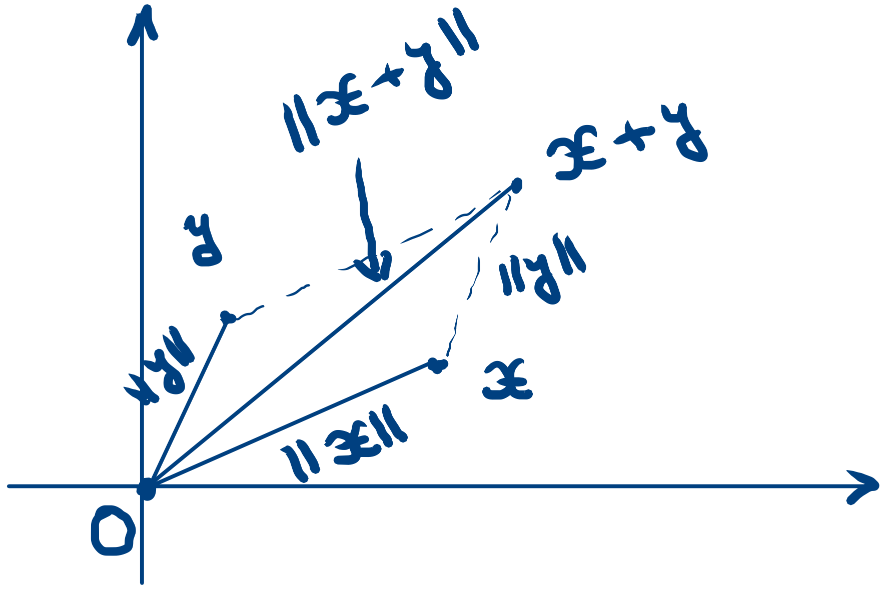
<u>Properties</u>

1° $\|x\| = 0 \iff x = 0_n$

2° $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall \alpha \in \mathbb{R}, \ \forall x \in \mathbb{R}^n$

3° $\|x + y\| \le \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$ $\Leftarrow$ Cauchy-Schwarz inequality

↳ triangle inequality



$\|x+y\| \leq \|x\| + \|y\|$

The notion of a norm can be defined axiomatically: let X be a vector space $/\mathbb{R}$

A function $\|\cdot\| : X \to [0,\infty)$ is called a norm on X if it satisfies

(N_1) $\quad \|x\| = 0 \iff x = 0_X$

(N_2) $\quad \|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall \alpha \in \mathbb{R}, \ \forall x \in X$

(N_3) $\quad \|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X.$

The ordered pair $(X, \|\cdot\|)$ is called a normed space.

By $1^\circ \div 3^\circ \implies (\mathbb{R}^n, \|\cdot\|)$ is a normed space

defined by (3)

Other examples of norms.

a) If $(X, \langle \cdot , \cdot \rangle)$ is a prehilbertian space $\Rightarrow$

$$\|x\| = \sqrt{\langle x, x\rangle} \quad \forall x \in X$$

↳ is a norm on X

$$\|A\| = \sqrt{tr(A \cdot A^t)} \quad \forall A \in \mathcal{M}_n(\mathbb{R}) \longleftarrow \text{a norm on } \mathcal{M}_n(\mathbb{R})$$

$$\|f\| = \sqrt{\int_a^b f^2(x)\,dx} \quad \forall f \in C[a,b] \longleftarrow \text{a norm on } C[a,b]$$

b) On $\mathbb{R}^n$

$\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ $\quad \|x\|_1 := |x_1| + \dots + |x_n| \quad \forall x = (x_1, \dots, x_n)$

↳ is a norm on $\mathbb{R}^n$, called the Minkowski norm on $\mathbb{R}^n$

$\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ $\quad \|x\|_\infty := \max\{|x_1|, \dots, |x_n|\} \quad \forall x = (x_1, \dots, x_n)$

↳ is a norm on $\mathbb{R}^n$, called the Tchebyshev norm on $\mathbb{R}^n$

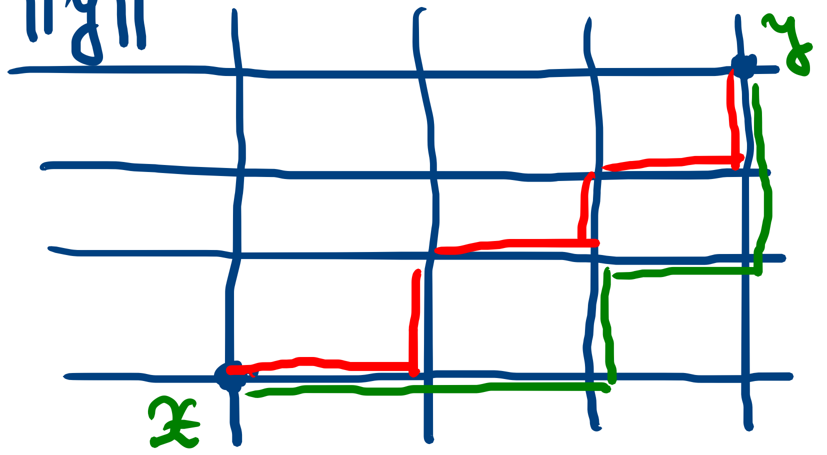
$\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not generated by an inner product

Given $x, y \in \mathbb{R}^n$, let $\alpha := m(\widehat{x, y})$. Assume that $x \neq 0_n$ and $y \neq 0_n$

By the Cauchy-Schwarz ineq $\Rightarrow$ $\dfrac{|\langle x, y\rangle|}{\|x\| \cdot \|y\|} \leq 1$ $\Rightarrow$

$$\Rightarrow -1 \leq \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|} \leq 1$$

$$\cos\alpha := \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|}$$



1.6. Definition (the Euclidean distance) Given $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define

$$d(x, y) := \|x - y\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

↳ the Euclidean distance between x and y

Other distances

$$d_1(x, y) := \|x - y\|_1 = |x_1 - y_1| + \dots + |x_n - y_n|$$ ← the taxicab / the Minkowski distance in $\mathbb{R}^n$

$$d_\infty(x, y) := \|x - y\|_\infty = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$ ← the Tchebyshev distance in $\mathbb{R}^n$

2. The topological structure of $\mathbb{R}^n$

2.1. Definition (balls). Let $a \in \mathbb{R}^n$, and let $r > 0$. We define

$$B(a,r) := \{x \in \mathbb{R}^n \mid d(x,a) < r\} = \{x \in \mathbb{R}^n \mid \|x-a\| < r\}$$

↳ the open ball about a, with radius r

$$\overline{B}(a,r) := \{x \in \mathbb{R}^n \mid d(x,a) \leq r\} = \{x \in \mathbb{R}^n \mid \|x-a\| \leq r\}$$

↳ the closed ball about a, with radius r

Let $\overline{B}_1(0_n, 1)$ = the closed Minkowski ball about 0_n, with radius 1

$\overline{B}_2(0_n, 1)$ = — " — Euclidean ball — " —

$\overline{B}_\infty(0_n, 1)$ = — " — Tchebyshev ball — " —

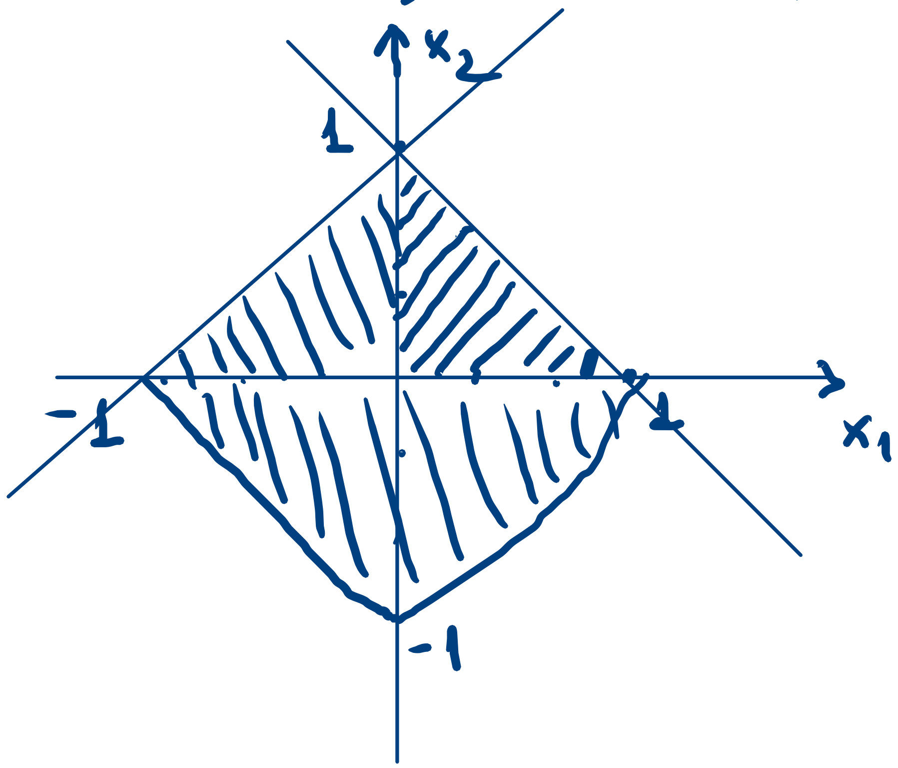
$n=1$ $\Rightarrow$ $\|x\| = \|x\|_1 = \|x\|_\infty = |x| \qquad \forall x \in \mathbb{R}$

$$\overline{B}_1(0,1) = \overline{B}_2(0,1) = \overline{B}_\infty(0,1) = \{x \in \mathbb{R} \mid |x-0| \leq 1\} = [-1,1]$$

$n=2$

$\overline{B}_1(0_2,1)=\{x\in\mathbb{R}^2 \mid \|x-0_2\|_1\le 1\}$

$=\{(x_1,x_2)\mid |x_1|+|x_2|\le 1\}$



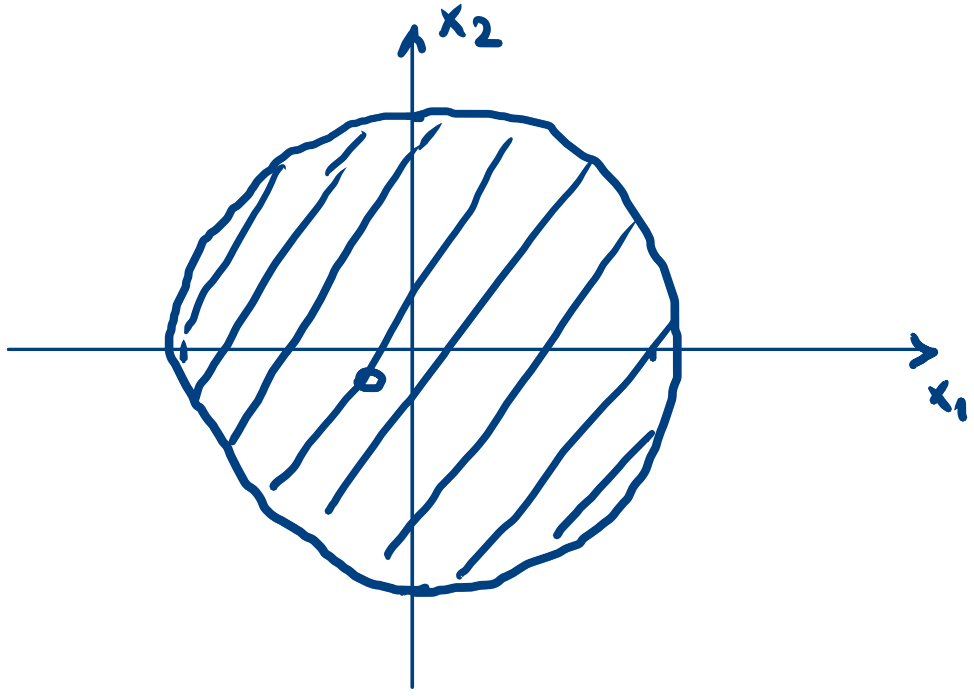
I $\quad x_1\ge 0,\ x_2\ge 0:\quad x_1+x_2\le 1$

$\boxed{x_1+x_2=1}$

II $\quad x_1\le 0,\ x_2\ge 0$

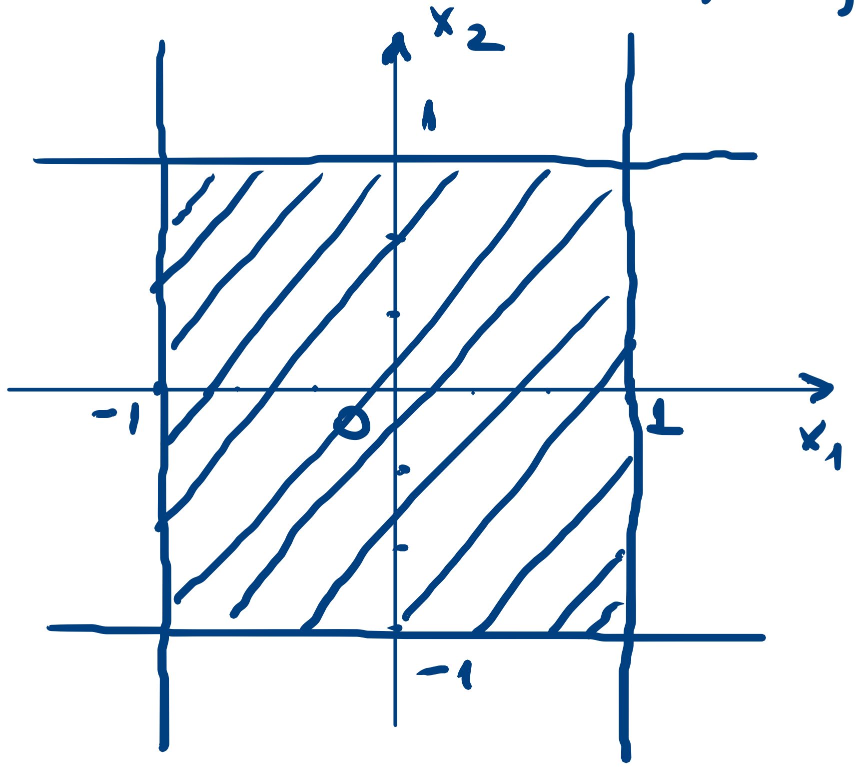
$\overline{B}_2(0_2,1)=\{x\in\mathbb{R}^2\mid \|x-0_2\|\le 1\}$

$=\{(x_1,x_2)\mid x_1^2+x_2^2\le 1\}$



$\overline{B}_\infty(0_2,1)=\{x\in\mathbb{R}^2\mid \|x-0_2\|_\infty\le 1\}$

$=\{(x_1,x_2)\mid \max\{|x_1|,|x_2|\}\le 1\}$



$\max\{|x_1|,|x_2|\}\le 1 \iff$

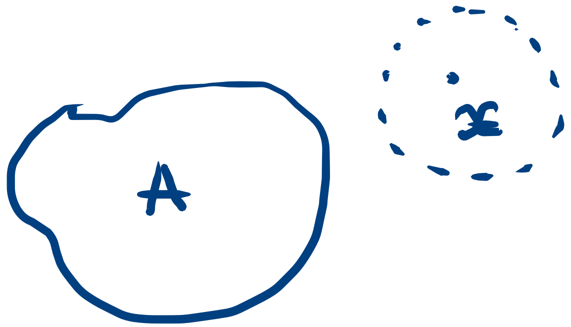
$\iff \underline{|x_1|\le 1}$ $\underline{\text{and}}$ $|x_2|\le 1$

2.2. Definition (neighbourhoods). Let $x \in \mathbb{R}^n$. A set $V \subseteq \mathbb{R}^n$ is said to be a neighbourhood of x if $\exists r > 0$ s.t. $B(x,r) \subseteq V$. We denote by $\mathcal{V}(x)$ = the family of all neighbourhoods of x.

2.3. Definition. Let $A \subseteq \mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Then x is said to be:

- an interior point of A if $A \in \mathcal{V}(x)$, i.e., $\exists r > 0$ s.t. $B(x,r) \subseteq A$
- an exterior point for A if $\mathbb{R}^n \setminus A \in \mathcal{V}(x)$, i.e., $\exists r > 0$ s.t. $B(x,r) \subseteq \mathbb{R}^n \setminus A$
- an adherent point for A if $\forall V \in \mathcal{V}(x)$: $V \cap A \neq \emptyset$
- a limit point for A if $\forall V \in \mathcal{V}(x)$: $V \cap A \setminus \{x\} \neq \emptyset$
- a boundary point for A if $\forall V \in \mathcal{V}(x)$: $V \cap A \neq 0$ and $V \cap (\mathbb{R}^n \setminus A) \neq \emptyset$
- an isolated point of A if $\exists V \in \mathcal{V}(x)$ s.t. $V \cap A = \{x\}$

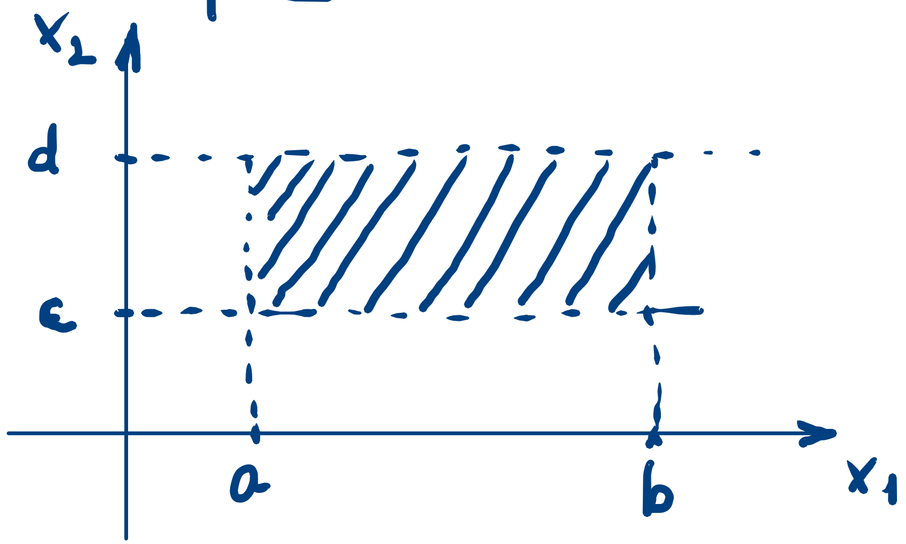
We denote by



$\underbrace{V \cap A}_{\emptyset} \setminus \{x\} = \emptyset$

int A = the set of all interior points of A
ext A = —— " —— exterior — " — for A
cl A = —— " —— adherent — " — for A
A' = —— " —— limit — " — for A
bd$A$ = —— " —— boundary — " — for A

Examples a) Let $A = (a,b) \times (c,d)$



$\operatorname{int} A = A$

$\operatorname{cl} A = [a,b] \times [c,d]$

$A' = [a,b] \times [c,d]$

$\operatorname{bd} A = ([a,b] \times \{c\}) \cup (\{b\} \times [c,d]) \cup ([a,b] \times \{d\}) \cup$
$\cup (\{a\} \times [c,d])$

b) Let $A = \{(x,y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q}\} \subseteq \mathbb{R}$

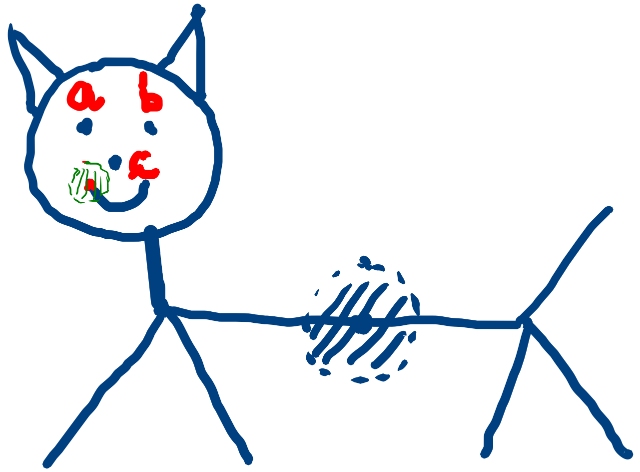
 $\notin A$

$\operatorname{int} A = \emptyset$

$\operatorname{cl} A = \mathbb{R}^2$

$A' = \mathbb{R}^2$

$\operatorname{bd} A = \mathbb{R}^2$

c) Let $A =$ 

$\subseteq \mathbb{R}^2$

$\operatorname{int} A = \emptyset$

$\operatorname{cl} A = A$

$A' = A \setminus \{a, b, c\}$

$\operatorname{bd} A = A$

2.4. Theorem. Given any subset A of $\mathbb{R}^n$, one has:

1° $\operatorname{int} A \subseteq A \subseteq \operatorname{cl} A$;

2° $\operatorname{cl} A = \mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A)$;

3° $\operatorname{bd} A = (\operatorname{cl} A) \cap \operatorname{cl}(\mathbb{R}^n \setminus A)$;

4° $\operatorname{cl} A = (\operatorname{int} A) \cup (\operatorname{bd} A)$;

5° $\operatorname{cl} A = A \cup A'$

Proof. At the seminar

2.5. Definition (open sets and closed sets). A set $A \subseteq \mathbb{R}^n$ is said to be:

- open if $\forall x \in A : A \in \mathcal{V}(x)$, i.e., $\forall x \in A\ \exists r > 0$ s.t. $B(x, r) \subseteq A$
- closed if $\mathbb{R}^n \setminus A$ is open.