ANALYTIC GEOMETRY, PROBLEM SET 9

The line in 3D. Relative positions of lines and planes

1. Find the equation of the plane containing the points $P_1(2,-1,-3)$, $P_2(3,1,2)$ and parallel to the vector $\overline{a}(3,-1,-4)$.

2. Find the equation of the plane containing the perpendicular lines through $P(-2,3,5)$ on the planes $\pi_1 : 4x+y-3z+13=0$ and $\pi_2 : x-2y+z-11=0$.

3. Find the equation of the plane passing through the points A, B and C, where:
(a) $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$; (b) $A(3,2,1)$, $B(2,1,-1)$ and $C(-1,3,2)$.

4. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

5. Let d_1 and d_2 be two lines in $\mathcal{E}_3$, given by $d_1 : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6}$ and $d_2 : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-5}{-3}$.
(a) Find the parametric equations of d_1 and d_2;
(b) Prove that they intersect and find the coordinates of their intersection point;
(c) Find the equation of the plane determined by d_1 and d_2.

6. Given the lines $d_1 : x = 1+t, y = 1+2t, z = 3+t, t \in \mathbb{R}$ and $d_2 : x = 3+s, y = 2s, z = -2+s, s \in \mathbb{R}$, show that $d_1 \parallel d_2$ and find the equation of the plane determined by d_1 and d_2.

7. Find the parametric equations of the line $\begin{cases} -2x+3y+7z+2=0 \\ x+2y-3z+5=0 \end{cases}$.

8. Find the parametric equations of the line passing through $P_1(5,-2,1)$ and $P_2(2,4,2)$. Find the equations of the line passing through $P(6,4,-2)$ and parallel to the line $d : \frac{x}{2} = \frac{y-1}{-3} = \frac{z-5}{6}$.

9. Given the points $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,5)$, find the equation(s) of the internal bisector passing through the vertex A in the triangle ABC.

10. Find the equations of the line passing through the origin and parallel to the line given by the parametric equations: $x = t$, $y = -1+t$ and $z = 2$.

11. Given the lines $d_1 : x = 4-2t, y = 1+2t, z = 9+3t$ and $d_2 : \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-4}{2}$, find the intersection points between the two lines and the coordinate planes.

12. Let d_1 and d_2 be the lines given by $d_1 : x = 3+t, y = -2+t, z = 9+t, t \in \mathbb{R}$ and $d_2 : x = 1-2s, y = 5+s, z = -2-5s, s \in \mathbb{R}$.
a) Prove they are coplanar.

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b) Find the equation of the line passing through the point $P(4,1,6)$ and orthogonal on the plane determined by d_1 and d_2.

13. Prove that the intersection lines of the planes $\pi_1 : 2x-y+3z-5=0$, $\pi_2 : 3x+y+2z-1=0$ and $\pi_3 : 4x+3y+z+2=0$ are parallel.

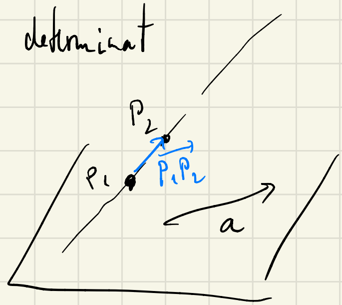
14. Verify that the lines $d_1 : \frac{x-3}{1} = \frac{y-8}{3} = \frac{z-3}{4}$ and $d_2 : \frac{x-4}{1} = \frac{y-9}{2} = \frac{z-9}{5}$ are coplanar and find the equation of the plane determined by the two lines.

15. Determine whether the line given by $x = 3+8t$, $y = 4+5t$, and $z = -3-t$, $t \in \mathbb{R}$ is parallel to the plane $x-3y+5z-12=0$.

1. Find the equation of the plane containing the points $P_1(2,-1,-3)$, $P_2(3,1,2)$ and parallel to the vector $\overline{a}(3,-1,-4)$.

Let π be the plane that we are looking for

- $\pi \ni P_1, P_2 \left(\Leftrightarrow \pi \supseteq P_1P_2\right) \Rightarrow \pi \ni P_1$ and $\pi \parallel \overrightarrow{P_1P_2}$
- π is also parallel to a
- so one can obtain the eq. of π with the determinant



coordinates of the point P_1

Components of $\overrightarrow{P_1P_2}$

Components of a

$$\begin{vmatrix} x-2 & y-(-1) & z-(-3) \\ 1 & 2 & 5 \\ 3 & -1 & -4 \end{vmatrix} = 0$$

$$\Leftrightarrow (x-2)\underbrace{\begin{vmatrix} 2 & 5 \\ -1 & -4 \end{vmatrix}}_{\substack{-8+5 \\ -3}} - (y+1)\underbrace{\begin{vmatrix} 1 & 5 \\ 3 & -4 \end{vmatrix}}_{\substack{-4-15 \\ -19}} + (z+3)\underbrace{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}}_{\substack{-1-6 \\ -7}} = 0$$

$$\Leftrightarrow -3x + 19y - 7z + 6 + 19 - 21 = 0$$

$$\Leftrightarrow \underbrace{-3x + 19y - 7z + 4 = 0}_{\text{an equation for } \pi}$$

2. Find the equation of the plane containing the perpendicular lines through $P(-2,3,5)$ on the planes $\pi_1 : 4x + y - 3z + 13 = 0$ and $\pi_2 : x - 2y + z - 11 = 0$.

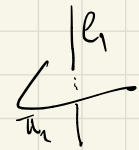
- Let π be the plane that we are looking for
- we know that π contains two lines:

$\nearrow$ l_1 which contains P and is perpendicular to π_1

$\searrow$ l_2 —— " —— P and —— " —— π_2

in math. terms $\nearrow$ $l_1 \ni P$ and $l_1 \perp \pi_1$

$\searrow$ $l_2 \ni P$ and $l_2 \perp \pi_1$



- $l_1 \perp \pi_1 \iff l_1$ parallel to vectors which are perpendicular to π_1

$\iff l_1$ parallel to $\underbrace{\text{the normal vectors of } \pi_1}$

all normal vectors to a plane in 3D are proportional

$\Updownarrow$

if you know one you know them all by rescaling

$\pi_1 : 4x + 1y - 3z + 13 = 0$

$(4, 1, -3) \longrightarrow$ components of 1 normal vector for π_1

so $l_1 \perp \pi_1 \iff n_1(4,1,-3)$ is a direction vector for l_1

similar $l_2 \perp \pi_2 \iff n_2(1,-2,1)$ is a direction vector for l_2

- since π contains $l_1 \Rightarrow \pi$ is parallel to $l_1 \Rightarrow \pi \parallel n_1$
- similar $\pi \parallel n_2$

so the eq. of π is

coordinates of P

$$\begin{array}{l} \\ \text{components of } n_1 \\ \text{components of } n_2 \end{array} \begin{vmatrix} x-(-2) & y-3 & z-5 \\ 4 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad (x+2)\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} - (y-3)\begin{vmatrix} 4 & -3 \\ 1 & 1 \end{vmatrix} + (z-5)\begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} = 0$$

$1-6$ $\qquad$ $4+3$ $\qquad$ $-8-1$

-5 $\qquad$ 7 $\qquad$ -9

$$\Leftrightarrow \quad -5x - 7y - 9z - 10 + 21 + 45 = 0$$

$$\Leftrightarrow \quad \underbrace{-5x - 7y - 9z + 56 = 0}_{\text{an equation for } \pi}$$

3. Find the equation of the plane passing through the points A, B and C, where:
(a) $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$; (b) $A(3,2,1)$, $B(2,1,-1)$ and $C(-1,3,2)$.

(a) from the course we know that the eq. is

$$\begin{array}{r}
\\
\text{coordinates of } A \rightarrow \\
\text{coordinates of } B \rightarrow \\
\text{coordinates of } C \rightarrow
\end{array}
\begin{vmatrix} x & y & z & 1 \\ -2 & 1 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 1 \end{vmatrix} = 0$$

$$\begin{array}{l} \Leftrightarrow \\ \ell_1 = \ell_1 - \ell_2 \\ \ell_3 = \ell_3 - \ell_2 \\ \ell_4 = \ell_4 - \ell_2 \end{array}
\underbrace{\begin{vmatrix} x-(-2) & y-1 & z-1 & 0 \\ -2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 3 & -1 & -2 & 0 \end{vmatrix}}_{\shortparallel} = 0$$

coordinates of A

$$\begin{array}{r}
\\
\text{components of } \vec{AB} \rightarrow \\
\text{components of } \vec{AC} \rightarrow
\end{array}
\begin{vmatrix} x+2 & y-1 & z-1 \\ 2 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix} = 0$$

$$\Leftrightarrow (x+2)\underbrace{\begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix}}_{\substack{-2+2 \\ 0}} - (y-1)\underbrace{\begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix}}_{\substack{-4-6 \\ -10}} + (z-1)\underbrace{\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}}_{\substack{-2-3 \\ -5}} = 0$$

$$\Leftrightarrow 10y - 5z - 10 + 5 = 0 \Leftrightarrow \underbrace{2y - z - 1 = 0}_{\text{an equation for } \pi}$$

b.) Let π be the plane that we are looking for

$\pi \ni A, B \Leftrightarrow \pi \supseteq AB \Rightarrow \pi \parallel \overrightarrow{AB}$

similar $\pi \parallel \overrightarrow{AC}$

so the eq. of π is

$$\begin{vmatrix} x-3 & y-2 & z-2 \\ -1 & -1 & -2 \\ -4 & 1 & 1 \end{vmatrix} = 0$$

$\Leftrightarrow \quad x + 7y - 3z - 14 = 0$

4. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

- A, B, C determine a plane π
- A, B, C, D are coplanar if $D \in \pi \iff$ the coordinates of D satisfy the equation of π

coordinates of A

$$\pi : \begin{vmatrix} x-1 & y-0 & z-(-1) \\ -1 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix} = 0$$

components of $\vec{AB}$

components of $\vec{AC}$

coordinates of D

$\iff \pi : \underbrace{2y - z - 1 = 0}_{\text{eq. of } \pi}$ $\quad \overset{?}{\ni} D \iff 0 \cdot 4 + 2 \cdot 2 - 1 \cdot 3 - 1 = 0$

true, so $D \in \pi$

Rem you don't need to write down the equation of π separately

A, B, C, D coplanar $\iff \vec{AB}, \vec{AC}, \vec{AD}$ are linearly dependent

$$\iff \begin{vmatrix} 3 & 2 & 4 \\ -1 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix} \overset{?}{=} 0$$

notice that the columns are proportional so the determinant is zero so the four points are collinear

5. Let d_1 and d_2 be two lines in $\mathcal{E}_3$, given by $d_1 : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6}$ and $d_2 : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-5}{-3}$.
(a) Find the parametric equations of d_1 and d_2;
(b) Prove that they intersect and find the coordinates of their intersection point;
(c) Find the equation of the plane determined by d_1 and d_2.

a) $P_1(1,-1,5)$ is a point on d_1

$$d_1 : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6} = \lambda \iff d_1 : \begin{cases} x = 1+2\lambda \\ y = -1-\lambda \\ z = 5+6\lambda \end{cases}$$

$\vec{v}_1(2,-1,6)$ is a direction vector for d_1

$$d_2 : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-5}{-3} = \mu \iff d_2 : \begin{cases} x = 1+\mu \\ y = -1+\mu \\ z = 5-3\mu \end{cases}$$

$\vec{v}_2(1,1,-3)$ is a direction vector for d_2

b) Method I d_1 and d_2 intersect if they have a point in common

but ... from the parametric equations we see that $P_1(1,-1,5)$ is a common point obtained for $\lambda=0$ and $\mu=0$

Method II if we don't recognize the common point, we can proceed as follows:

d_1 and d_2 intersect if they have a point in common

i.e. iff $\exists\ \lambda$ and μ such that

$$\begin{aligned} 1+2\lambda &= x = 1+\mu \\ -1-\lambda &= y = -1+\mu \\ 5+6\lambda &= z = 5-3\mu \end{aligned} \iff \begin{cases} 1+2\lambda = 1+\mu \\ -1-\lambda = -1+\mu \\ 5+6\lambda = 5-3\mu \end{cases} \iff \begin{cases} 2\lambda-\mu=0 \\ -\lambda-\mu=0 \\ 6\lambda+3\mu=0 \end{cases} \iff \lambda=\mu=0$$

Method III two lines in the 3-dimensional Euclidean space intersect iff they are coplanar

$\iff$ a plane built from the direction vectors of the two lines and a point on d_1 $\underbrace{\text{contains}}$ d_2

equivalently, contains one point on d_2 (why?)

c) the plane determined by the two lines is the plane containing the two lines
it is obtained with $P_1, \vec{v}_1, \vec{v}_2$ as in the previous exercises.

6. Given the lines $d_1 : x = 1 + t, y = 1 + 2t, z = 3 + t, t \in \mathbb{R}$ and $d_2 : x = 3 + s, y = 2s, z = -2 + s, s \in \mathbb{R}$, show that $d_1 \parallel d_2$ and find the equation of the plane determined by d_1 and d_2.

Method I.

the lines are $d_1 : \begin{cases} x = 1 + 1\cdot t \\ y = 1 + 2\cdot t \\ z = 3 + 1\cdot t \end{cases}$ and $d_2 : \begin{cases} x = 3 + 1\cdot s \\ y = 0 + 2\cdot s \\ z = -2 + 1\cdot s \end{cases}$

a direction vector for d_1 is $\rightarrow \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$

$\Downarrow$

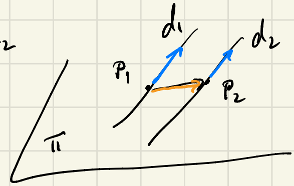
all non-zero scalar multiples of this vector are direction vectors for d_1

a direction vector for d_2 is $\rightarrow \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$

- since the direction vectors of d_1 and d_2 are proportional, the two lines are parallel

- plane determined by d_1 and d_2

Let π be this plane



$\left.\begin{array}{l} \text{the point } P_1(1,1,3) \text{ belongs to } d_1 \text{ (choose } t=0) \\ \\ \text{the point } P_2(3,0,-2) \text{ belongs to } d_2 \end{array}\right\} \Rightarrow P_1, P_2 \in \pi \Rightarrow \overrightarrow{P_1P_2} \parallel \pi$

since $d_1, d_2 \subseteq \pi$ the direction vectors of d_1 and d_2 are parallel to π

for example $\vec{v} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$ $\Rightarrow$

$\Rightarrow$ the eq. of π is

coordinates of P_1

$$\begin{array}{r} \\ \text{components of } \vec{v} \rightarrow \\ \text{components of } \overrightarrow{P_1P_2} \rightarrow \end{array} \begin{vmatrix} x-1 & y-1 & z-3 \\ 1 & 2 & 1 \\ 2 & -1 & -5 \end{vmatrix} = 0$$

$\Leftrightarrow \quad -9x + 7y - 5z + 17 = 0$

Method II (interpretation)

$\overrightarrow{P_1P_2} \times \vec{v}$ is a normal vector for π

$\hookrightarrow$ calculate the components (a, b, c) then $\pi : a(x-1) + b(y-1) + c(z-3) = 0$

7. Find ~~the~~ parametric equations of the line $\begin{cases} -2x+3y+7z+2=0 \\ x+2y-3z+5=0 \end{cases}$.

Rem there is not one unique or distinguished set of parametric equations for a line. Each line has many parametric equations.

- here we are interested in finding one set of parametric equations
- So we need to express x, y, z in terms of a parameter
  - in the given equations we can choose one of the variables to be the parameter, say y, and express the other two in terms of it

$$\begin{cases} -2x+3y+7z+2=0 \\ x+2y-3z+5=0 \end{cases} \longrightarrow x=-2y+3z-5$$

$$\xrightarrow{\ell_1+2\ell_2} 7y+z+12=0 \Rightarrow z=-7y-12 \quad (*)$$

$$\left.\begin{array}{l} x=-2y+3z-5 \\ z=-7y-12 \end{array}\right\} \Rightarrow x=-23y-41 \quad (**)$$

$$\begin{cases} x = -41-23y & (**) \\ y = 0+1\cdot y \\ z = -12-7y & (*) \end{cases}$$

this is one set of parametric equations for the given line part 1

8. Find ~~the~~ parametric equations of the line passing through $P_1(5,-2,1)$ and $P_2(2,4,2)$.
Find the equations of the line passing through $P(6,4,-2)$ and parallel to the line $d : \frac{x}{2} = \frac{y-1}{-3} = \frac{z-5}{6}$. (part 2)

For the first part:

- Let ℓ be the line that we are looking for

$\ell \ni P_1, P_2 \Rightarrow \ell = P_1P_2 \Rightarrow \overrightarrow{P_1P_2}$ is a direction vector for ℓ

so a set of parametric equations for ℓ are

$$\ell : \begin{cases} x = \boxed{5} + \boxed{(-3)} \cdot t \\ y = \boxed{-2} + \boxed{6} \cdot t \\ z = \boxed{1} + \boxed{1} \cdot t \end{cases}$$

coordinates of P_1 components of $\overrightarrow{P_1P_2}$

For part 2: $\quad d: \frac{x}{2} = \frac{y-1}{-3} = \frac{z-5}{6}$

$(2,-3,6)$ is a direction vector for d

Let ℓ be the line that we are looking for

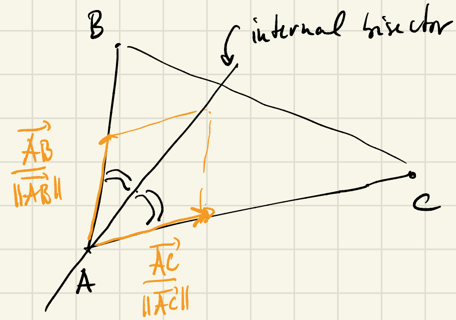
- $\ell \ni P(6,4,-2)$
- $\ell \parallel d \Rightarrow \ell$ and d have the same direction vectors

so a set of parametric equations for ℓ are

$$\ell : \begin{cases} x = \boxed{6} + \boxed{2} \cdot \lambda \\ y = \boxed{4} + \boxed{(-3)} \cdot \lambda \\ z = \boxed{-2} + \boxed{6} \cdot \lambda \end{cases}$$

$P \in \ell$ ↳ components of a direction vector

9. Given the points $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,5)$, find the equation(s) of the internal bisector passing through the vertex A in the triangle ABC.



$$\vec{AB} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 2 \\ 2 \\ 12 \end{bmatrix}$$

A direction vector for the angle bisector is

$$v = \frac{\vec{AB}}{\|\vec{AB}\|} + \frac{\vec{AC}}{\|\vec{AC}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\underbrace{\sqrt{4+4+144}}_{2\sqrt{38}}} \begin{bmatrix} 2 \\ 2 \\ 12 \end{bmatrix} = \frac{1}{2\sqrt{38}} \underbrace{\begin{bmatrix} 2\sqrt{38}+2 \\ 2 \\ 12 \end{bmatrix}}$$

this is also a direction vector (it is a scalar multiple of v)

so, a set of parametric equations for the interior bisector is

$$\begin{cases} x = 1 + \left(2\sqrt{38}+2\right) t \\ y = 2 + 2 \cdot t \\ z = -7 + 12 \cdot t \end{cases}$$

10. Find the equations of the line passing through the origin and parallel to the line given by the parametric equations: $x = t$, $y = -1 + t$ and $z = 2$.

$$d \begin{cases} x = t \\ y = -1 + t \\ z = 2 \end{cases} \quad \text{has } v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ as direction vector}$$

Let ℓ be the line that we are looking for.

$\ell \parallel d \Rightarrow v$ is a direction vector also for ℓ

So, a set of parametric equations for ℓ are

$$\begin{cases} x = 0 + t \\ y = 0 + t \\ z = 0 + 0 \end{cases}$$

passes through the origin

11. Given the lines $d_1 : x = 4 - 2t, y = 1 + 2t, z = 9 + 3t$ and $d_2 : \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-4}{2}$, find the intersection points between the two lines and the coordinate planes.

part 1. $d_1 : \begin{cases} x = 4-2t \\ y = 1+2t \\ z = 9+3t \end{cases}$

$d_1 \cap Oxy$: $Oxy : z=0$ $\quad 0 = 9+3t \Rightarrow t=-3$

$\Rightarrow x = 10$ and $y=5$

so $d_1 \cap Oxy = \{(10,5,0)\}$

$d_1 \cap Oyz$: $Oyz : x=0$ ... similar

$d_1 \cap Ozx$ similar

part 2 $d_2 : \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-4}{2}$

$d_2 \cap Oxy \ (z=0)$: $\frac{x-1}{2} = \frac{y+2}{3} = -2 \Rightarrow \begin{matrix} x=-5 \\ y=-8 \end{matrix}$

so $d_2 \cap Oxy = \{(-5,-8,0)\}$

$d_2 \cap Oyz$ and $d_2 \cap Ozx$ similar

12. Let d_1 and d_2 be the lines given by $d_1 : x = 3 + t, y = -2 + t, z = 9 + t, t \in \mathbb{R}$ and $d_2 : x = 1 - 2s, y = 5 + s, z = -2 - 5s, s \in \mathbb{R}$.

a) Prove they are coplanar.

$$d_1 : \begin{cases} x = 3 + t \\ y = -2 + t \\ z = 9 + t \end{cases} \qquad d_2 : \begin{cases} x = 1 - 2s \\ y = 5 + s \\ z = -2 - 5s \end{cases}$$

- let π be the plane containing d_1 and parallel to d_2

- d_1 and d_2 are coplanar if $d_2 \subseteq \pi$

$$\pi : \begin{vmatrix} x-3 & y-(-2) & z-9 \\ 1 & 1 & 1 \\ -2 & 1 & -5 \end{vmatrix} = 0 \iff \pi : -2x + y + z - 1 = 0$$

the point $P(1, 5, -2)$ from d_1 satisfies this equation $\Rightarrow P \in \pi \Rightarrow d_2 \subseteq \pi \Rightarrow d_1 \cup d_2 \subseteq \pi$ (they are coplanar)

b) Find ~~the equation~~ equations of the line passing through the point $P(4, 1, 6)$ and orthogonal on the plane determined by d_1 and d_2.

- from a) we know that the plane determined by d_1 and d_2 is $\pi : -2x + y + z - 1 = 0$

- a normal vector for π is $n(-2, 1, 1)$

- a line which is orthogonal to π has n as normal vector

- The line that we are looking for is orthogonal to π and passes through $P(4, 1, 6)$

So, a set of parametric equations for this line is

$$\begin{cases} x = 4 + (-2) \cdot t \\ y = 1 + 1 \cdot t \\ z = 6 + 1 \cdot t \end{cases}$$

13. Prove that the pairwise intersection lines of the planes $\pi_1 : 2x - y + 3z - 5 = 0$, $\pi_2 : 3x + y + 2z - 1 = 0$ and $\pi_3 : 4x + 3y + z + 2 = 0$ are parallel.

$\boxed{\text{Method I}}$ let $\ell_{12} = \pi_1 \cap \pi_2 \qquad \ell_{23} = \pi_2 \cap \pi_3 \qquad \ell_{13} = \pi_1 \cap \pi_2$

$\ell_{12} : \begin{cases} 2x - y + 3z - 5 = 0 \\ 3x + y + 2z - 1 = 0 \end{cases}$ this system has rank 2 $\Rightarrow$ it has infinitely many solutions

i.e. ℓ_{12} is a line

(*)

$\ell_{23} : \begin{cases} 3x + y + 2z - 1 = 0 \\ 4x + 3y + z + 2 = 0 \end{cases}$ similar this describes a line

$\ell_{13} : \begin{cases} 2x - y + 3z - 5 = 0 \\ 4x + 3y + z + 2 = 0 \end{cases}$ similar this describes a line

$\ell_{12}, \ell_{23}, \ell_{13}$ are parallel if ℓ_{12} is parallel to π_3

$\Leftrightarrow$ i.e. if $\ell_{12} \cap \pi_3 = \phi$ $\Leftrightarrow$ if the system has no solutions $\begin{cases} \ell_{12} : \begin{cases} 2x - y + 3z - 5 = 0 \\ 3x + y + 2z - 1 = 0 \end{cases} \\ \pi_3 \quad 4x + 3y + z + 2 = 0 \end{cases}$

$\Leftrightarrow$ rank of the extended matrix of the system is strictly bigger than the rank of the matrix of the system

this is true since $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 1 \end{vmatrix} = 0 \Downarrow$ rank $M < 3$ $\qquad \begin{vmatrix} 2 & -1 & -5 \\ 3 & 1 & -1 \\ 4 & 3 & 2 \end{vmatrix} = -5 \Downarrow$ rank $\overline{M} = 3$

$\boxed{\text{Method II}}$ Starting from the systems (*) find parametric equations for the three lines and check that the direction vectors are parallel

14. Verify that the lines $d_1 : \frac{x-3}{1} = \frac{y-8}{3} = \frac{z-3}{4}$ and $d_2 : \frac{x-4}{1} = \frac{y-9}{2} = \frac{z-9}{5}$ are coplanar and find the equation of the plane determined by the two lines.

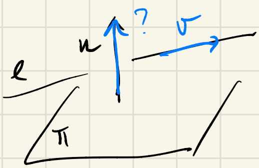
as in exercise 5

15. Determine whether the line given by $x = 3 + 8t$, $y = 4 + 5t$, and $z = -3 - t$, $t \in \mathbb{R}$ is parallel to the plane $x - 3y + 5z - 12 = 0$.

$$l : \begin{cases} x = 3 + 8t \\ y = 4 + 5t \\ z = -3 - t \end{cases} \Rightarrow v(8,5,-1) \text{ is a direction vector for } l$$

$$\pi : x - 3y + 5z - 12 = 0 \Rightarrow n(1,-3,5) \text{ is a normal vector for } \pi$$

$$l \parallel \pi \iff v \perp n \iff \underbrace{v \cdot n}_{=\, 8\cdot 1 + 5\cdot(-3) + (-1)\cdot 5 \,=\, -12 \,\neq\, 0} = 0$$



so $l \not\parallel \pi$