Antiderivatives of a function

$\emptyset \neq D \subseteq \mathbb{R}$
$f: D \to \mathbb{R}$
$J \subseteq D$ is an interval

f has antiderivatives on J if $\exists\, F: J \to \mathbb{R}$ s.t. $\to$ F is differential on J
$\to F'(x) = f(x)\ ,\ \forall x \in J$

F is called an antiderivative of f on J

The set of all antiderivatives of f on J:
$\int f(x)\,dx = F(x) + \mathcal{C}$ $\longrightarrow$ the set of all constants
$\hookrightarrow$ an antiderivative

Theorem:
Each two different antiderivatives of f are different through a constant.

$J \subseteq \mathbb{R}$ an interval
$f: J \to \mathbb{R}$ a function
$F_1, F_2 : J \to \mathbb{R}$ two antiderivatives of f

$\Longrightarrow$ $\exists\, c \in \mathbb{R}$ s.t. $\forall x \in J$ $\quad F_2(x) = F_1(x) + c$

Proof:
F_1, F_2 differentiable $\Rightarrow$ $F_2 - F_1$ differentiable $\Rightarrow$ $(F_2 - F_1)'(x) = 0$ $\quad f(x) - f(x) = 0$
$\Downarrow$ $\quad \forall x \in [a, b]$
$F_2 - F_1 = 0$

Theorem:
(concerning the antiderivative when the function is continuous)

$J \subseteq \mathbb{R}$ an interval
$x_0 \in J$
$f: J \to \mathbb{R}$ LRI
f continuous at x_0

$\Longrightarrow$ $\forall a \in J$, the function $F: J \to \mathbb{R}$, $F(x) = \int_a^x f(t)\,dt$, $\forall x \in J$
is $\to$ differential at x_0
$\to F'(x_0) = f(x_0)$

Theorem:
$(\exists)$

$J \subseteq \mathbb{R}$ an interval
$f: J \to \mathbb{R}$ continuous
$a \in J$

$\Longrightarrow$ The function $F: J \to \mathbb{R}$, $F(x) = \int_a^x f(t)\,dt$, $\forall x \in J$
is an antiderivative of f on J and $F(a) = 0$.

Theorem:

$J \subseteq \mathbb{R}$ an interval
$f: J \to \mathbb{R}$ continuous
$a \in J$

$\Longrightarrow$ The function $F: J \to \mathbb{R}$, is an antiderivative of f on J s.t $F(a) = 0$
then $F(x) = \int_a^x f(t)\,dt$, $\forall x \in J$.

Leibniz-Newton theorem:

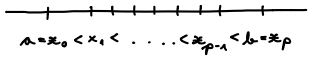
$f: [a, b] \to \mathbb{R}$ a function
f RI on $[a, b]$
f has antiderivatives on $[a, b]$

$\Longrightarrow$ $\forall F: [a, b] \to \mathbb{R}$ an antiderivative of f

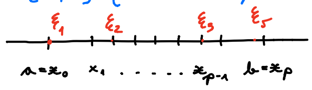
$\int_a^b f(x)\,dx = F(b) - F(a)$.

Partitions

$a < b \in \mathbb{R}$ | The ordered system $a = x_0 < x_1 < \ldots < x_p = b$ is called a partition of the segment $[a, b]$.



$\mathcal{P}art\,[a, b] = \{ \Delta : \Delta \to \text{is a partition of } [a, b] \}$



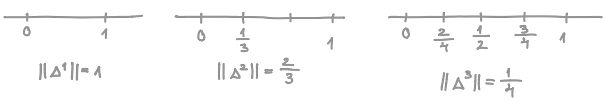
Each ordered system $\xi = (\xi_1, \ldots, \xi_p)$ s.t. $\forall i = \overline{1,p}$ $x_{i-1} \leq \xi_i \leq x_i$ is a system of intermediate points associated to the partition Δ.

$IP(\Delta)$ is the set of all systems of intermediate points associated to Δ.

Norm of a partition

$\Delta = (x_0, \ldots, x_p) \in \mathcal{P}art\,[a, b]$ | $\|\Delta\| = \max\{ x_i - x_{i-1} : i = \overline{1,p} \}$ is called the NORM of the partition.

Examples:



Remark:

- → the more points a partition has, the less its norm is
- → if we consider a sequence of partitions $(\Delta^n) \subseteq \mathcal{P}art\,[a, b]$, it makes sense that $\lim\limits_{n \to \infty} \|\Delta^n\| = 0$
- → the points of a partition should NOT always be evenly spread, however we can see the following property

P: $\forall \varepsilon > 0$, $\exists \Delta \in \mathcal{P}art\,[a, b]$ s.t. $\|\Delta\| < \varepsilon$

Proof:

$$\left.\begin{matrix} \varepsilon > 0 \\ b - a > 0 \end{matrix}\right\} \iff 0 < \frac{b-a}{\varepsilon} \xrightarrow[\text{Th.}]{\text{Archimedes}} \exists\, n_\varepsilon \in \mathbb{N} \text{ s.t. } \frac{b-a}{\varepsilon} < n_\varepsilon \iff \frac{b-a}{n_\varepsilon} < \varepsilon$$

$\frac{b-a}{n_\varepsilon} \to h$

$x_0 = a$
$<$
$x_1 = a + h$
$<$
$x_2 = a + 2 \cdot h$
$<$
$\vdots$
$<$
$x_{n_\varepsilon - 1} = a + (n_\varepsilon - 1) h$
$<$
$x_{n_\varepsilon} = a + n_\varepsilon h = a + \cancel{n_\varepsilon} \cdot \frac{b-a}{\cancel{n_\varepsilon}} = a + b - a = b$

$\implies \Delta = (a, a+h, a+2h, \ldots, a + n_\varepsilon \cdot h) \in \mathcal{P}art\,[a, b]$

$\|\Delta\| = h < \varepsilon$

The Riemann Sum

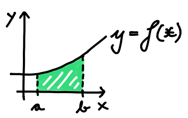
$a < b \in \mathbb{R}$
$\Delta = (x_0, \dots, x_p) \in \text{Part } [a,b]$
$\xi = (\xi_1, \dots, \xi_p) \in \mathcal{P}(\Delta)$
$f : [a,b] \to \mathbb{R}$ a function

The Riemann Sum associated to → the function f
→ the partition Δ
→ the system of p ξ

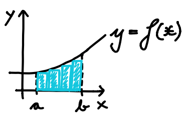
is $\sigma(f, \Delta, \xi) = \sum_{i=1}^{p} f(\xi_i)(x_i - x_{i-1})$

The Riemann Integral

Let $\int_a^b f(x)\,dx$ a definite integral.

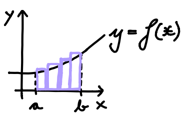


In the Riemann Integral we aproximate that area in two ways:



↑

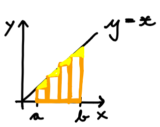
break it up into
the sums of the areas
of little rectangles
(underestimate for
the correct answer)



↑

break it up into
the sums of the areas
of big rectangles
(overestimate for
the correct answer)

e.g. $\int_a^b x\,dx$



$$\int_a^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \delta(a+(i-1)\delta), \quad \delta = \frac{b-a}{n}$$

$$\begin{aligned}
S_n = \sum_{i=1}^{n} \delta(a+(i-1)\delta) &= \sum_{i=1}^{n} \delta(a-\delta) + \delta^2 i = \\
&= \sum_{i=1}^{n} \delta(a-\delta) + \delta^2 \sum_{i=1}^{n} i = \\
&= n\delta(a-\delta) + \delta^2\left(\frac{n(n+1)}{2}\right) = \\
&= (b-a)\left(a - \frac{b-a}{n}\right) + \frac{(b-a)^2}{n^2} \cdot \frac{n^2+n}{2}
\end{aligned}$$

$$\lim_{n\to\infty} S_n = (b-a)a - 0 + \frac{(b-a)^2}{2} \lim_{n\to\infty}\Big(\underset{\downarrow\atop 1}{\frac{n^2}{n^2}} + \underset{\downarrow\atop 0}{\frac{n}{n^2}}\Big)$$

$$\int_a^b x\,dx = (b-a)a + \frac{(b-a)^2}{2} = \cancel{ba} - a^2 + \frac{1}{2}b^2 - \cancel{ab} + \frac{a^2}{2}$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$$

Riemann Integrable functions

$a < b \in \mathbb{R}$
$f : [a,b] \to \mathbb{R}$ a function

f is said to be Riemann Integrable on $[a,b]$ if:

→ $\forall (\Delta^n) \subseteq Part[a,b]$ with $\lim_{n\to\infty} \|\Delta^n\| = 0$
(sequence of partitions)

→ $\forall (\xi^n)$ s.t. $\xi^n \in IP(\Delta^n)$
(sequence of intermediate points)

$\underline{\exists \lim_{n\to\infty} \sigma(f, \Delta^n, \xi^n)} = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i^n)(x_i^n - x_{i-1}^n) \underline{\in \mathbb{R}}$

in this case, it is called the Riemann Integral of f on $[a,b]$ and it is denoted by $\int_a^b f(x)\,dx$.

Remark:

→ if a function is Riemann Integrable on $[a,b]$, the limit is unique.

P_1 | $f : [a,b] \to \mathbb{R}$, R.i. ; $f(x) \geq 0$, $\forall x \in [a,b]$ $\Rightarrow \int_a^b f(x)\,dx \geq 0$

Proof:

$$\lim_{\|\Delta^n\|\to 0} \sigma(f, \Delta^n, \xi^n) = \lim_{\|\Delta^n\|\to 0} \overbrace{f(\xi_i^n)}^{\geq 0}\overbrace{(x_i^n - x_{i-1}^n)}^{\geq 0} \geq 0$$

P_2 | $f : [a,b] \to \mathbb{R}$, R.i. ; $f(x) \leq g(x)$, $\forall x \in [a,b]$ $\Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Proof:

$$m \leq f(x) \leq M \ , \quad m, n \in \mathbb{R} \quad \Rightarrow \quad m\cdot(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

The Riemann-Stieltjes Integral

$\int_a^b f\,dg \quad : f, g : [a,b] \to \mathbb{R}$

f continuous
g differentiable
g' is R.i.

$\Rightarrow \int_a^b f(x)\,\underline{dg(x)} = \int_a^b f(x)\cdot g'(x)\,dx$

$$\lim_{m\to\infty} \sigma(f, g, \Delta^n, \xi^n) = \underbrace{\sum_{n=1}^{\infty} f(\xi_i)\cdot(g(x_i) - g(x_{i-1}))}_{\text{the Riemann Stieltjes Sum}}$$

Properties:

→ liniarity with respect too both functions

$$\int_a^b (f_1 + f_2)\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg$$

$$\int_a^b f\,d(g_1 + g_2) = \int_a^b f\,dg_1 + \int_a^b f\,dg_2$$

→ additivity

$$\int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg \quad , \; c \in (a,b)$$

→ *reversability*

$$\int_a^b g\,df = f(b)g(b) - f(a)g(a) - \int_a^b f\,dg$$

Exercises :

1) $\int_0^1 x\,dx^2$ $\qquad$ $f, g : [0,1] \to \mathbb{R}$, $f(x) = x \to$ cont.

$g(x) = x^2 \to g'(x) = 2x \to$ diff.

$\to$ cont. $\to$ R.I.

$\|$

$$\int_0^1 x \cdot (2x)\,dx = 2 \cdot \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}(1-0) = \frac{2}{3}$$

2) $\int_0^{\pi} x\,d\cos x = \int_0^{\pi} x(\cos x)'\,dx = x\cos x\Big|_0^{\pi} - \int_0^{\pi} x'\cos x\,dx = -\pi - \int_0^{\pi} x' \cdot (-\cos x)\,dx =$

$\int fg' = fg - \int f'g$

$= -\pi + \int_0^{\pi} \cos x\,dx =$

$= -\pi + \sin x\Big|_0^{\pi} =$

$= -\pi + 0 =$

$= -\pi$

3) $\int_0^{\pi} (x+1)\,d(\cos x + \sin x) = \int_0^{\pi} (x+1)(\cos x + \sin x)'\,dx = \int_0^{\pi} (x+1)(\cos x)'\,dx + \int_0^{\pi} (x+1)(\sin x)'\,dx =$

$= (x+1)(\cos x)\Big|_0^{\pi} - \int_0^{\pi} \cos x\,dx + (x+1)(\sin x)\Big|_0^{\pi} - \int_0^{\pi} \sin x\,dx =$

$= -(\pi+1) - 1 - \int_0^{\pi} \cos x\,dx - \int_0^{\pi} \sin x\,dx =$

$= -(\pi+1) - 1 - \sin x\Big|_0^{\pi} + \cos x\Big|_0^{\pi} =$

$= -(\pi+1) - 1 - 1 - 1 = -\pi - 1 - 1 - 1 - 1 = -\pi - 4$

4) $J = \int \sqrt{x^2-4}\,dx$, $\quad x \in (-2,2)$

$J = \int (x)'\sqrt{x^2-4}\,dx = x\sqrt{x^2-4} - \int x(\sqrt{x^2-4})'\,dx =$

$= x\sqrt{x^2-4} - \int x((x^2-4)^{\frac{1}{2}})'\,dx =$

$= x\sqrt{x^2-4} - \int x \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{\sqrt{x^2-4}}\,dx =$

$= x\sqrt{x^2-4} - \int \frac{x^2}{\sqrt{x^2-4}}\,dx =$

$= x\sqrt{x^2-4} - \int \frac{x^2-4+4}{\sqrt{x^2-4}}\,dx =$

$= x\sqrt{x^2-4} - \int \frac{x^2-4}{\sqrt{x^2-4}}\,dx - \int \frac{4}{\sqrt{x^2-4}}\,dx =$

$= x\sqrt{x^2-4} - J - 4 \cdot \int \frac{1}{\sqrt{x^2-4}}\,dx =$

$= x\sqrt{x^2-4} - J - 4\ln(x + \sqrt{x^2-2^2} + \mathcal{C}$

$$2J = x\sqrt{x^2-4} - 4\ln\left(x + \sqrt{x^2-2^2}\right) + \mathcal{C}$$

$$J = \frac{x\sqrt{x^2-4} - 4\ln\left(x + \sqrt{x^2-2^2}\right)}{2} + \mathcal{C}$$

1. $n \in \mathbb{N}$

$$\Delta_n = \left(0, \frac{1+n}{n^2+n}, \frac{2+n}{n^2+n}, \dots, \frac{n^2+n}{n^2+n}\right)$$

$\Delta_n \in \mathrm{Part}[0,1]$

$\Delta_0 = 0$

$\Delta_1 = \frac{2}{2} = 1$

$\Delta_2 = \frac{4}{6} = \frac{2}{3}$

$\Delta_3 = \frac{6}{12} = 2$

$\|\Delta_n\|$

$\lim\limits_{n \to \infty} \|\Delta_n\|$