Calculus on $\mathbb{R}$ Lect. dr. Anca Grad
Homework 9 **2021**

Differentiable Functions

Exercise 1: Determine the n-th derivative of the following functions:

a) $f : (-1, \infty) \to \mathbb{R}$ defined by $f(x) = (1 + x)^r$, where $r \in \mathbb{R}$;

b) $f : (-1, \infty) \to \mathbb{R}$ defined by $f(x) = x \cdot \ln(1 + x)$;

c) $f : (-\infty, -1) \to \mathbb{R}$ defined by $f(x) = x \cdot \ln(1 - x)$;

d) $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \sqrt{3x + 4}$;

e) $f : (-\frac{1}{2}, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{\sqrt{2x+1}}$.

Exercise 2: Determine the n-th derivative of the following functions:

a) $f : \mathbb{R} \setminus \{-\frac{b}{a}\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{ax+b}$;

b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(ax + b)$;

c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(ax + b)$;

d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{ax+b}$.

Exercise 3: Compute the derivatives of the following functions

a) $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^x$;

b) $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^{\frac{1}{x}}$;

c) $f : (0, \pi) \to \mathbb{R}$ defined by $f(x) = \sin x^x$;

d) $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^{\sin x}$;

Exercise 4: Prove that $\frac{1}{x+1} < \ln(x + 1) - \ln x < \frac{1}{x}$ for all $x > 0$.

Exercise 5:

a) Prove that for all $n \in \mathbb{N}$ if holds

$$na^{n-1} < \frac{b^n - a^n}{b - a} < nb^{n-1}$$

for all $a, b \in (0, +\infty)$ with $a < b$.

Excercise 6:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x + |x - 1|$$

for all $x \in \mathbb{R}$.

a) Prove that f has side derivatives at $x_0 = 1$;

b) Compute the side derivates of f at $x_0 = 1$;

c) Is f differentiable on the left at $x_0 = 1$? What about on the right?

d) Does f have a derivative at $x_0 = 1$?

e) Is f differentiable at $x_0 = 1$?

Exercise 1: Determine the n-th derivative of the following functions:

a) $f:(-1,\infty)\to\mathbb{R}$ defined by $f(x)=(1+x)^r$, where $r\in\mathbb{R}$;

b) $f:(-1,\infty)\to\mathbb{R}$ defined by $f(x)=x\cdot\ln(1+x)$;

c) $f:(-\infty,-1)\to\mathbb{R}$ defined by $f(x)=x\cdot\ln(1-x)$;

d) $f:(-1,1)\to\mathbb{R}$ defined by $f(x)=\sqrt{3x+4}$;

e) $f:(-\frac{1}{2},\infty)\to\mathbb{R}$ defined by $f(x)=\frac{1}{\sqrt{2x+1}}$.

a) $f:(-1,\infty)\to\mathbb{R}$ $\quad f(x)=(1+x)^r,\ r\in\mathbb{R}$

→ if $r<0$: Denote $r=-t \Rightarrow f(x)=(1+x)^{-t}=\frac{1}{(1+x)^t}$

$$f'(x)=\left(\frac{1}{(1+x)^t}\right)'=\frac{-t}{(1+x)^{t+1}}$$

$$f''(x)=\frac{t(t+1)}{(1+x)^{t+2}}$$

Suppose $f^{(n)}(x)=\frac{(-1)^n\cdot t(t+1)\cdot\ldots\cdot(t+n-1)}{(1+x)^{t+n}}$

We shall prove that $f^{(n+1)}(x)=\frac{(-1)^{n+1}\cdot t(t+1)\cdot\ldots\cdot(t+n-1)(t+n)}{(1+x)^{t+n+1}}$

$$\left(f^{(n+1)}(x)\right)'=\left(\frac{(-1)^{n+1}\cdot t(t+1)\cdot\ldots\cdot(t+n-1)\cdot(t+n)}{(1+x)^{t+n+1}}\right)'=(-1)^n\cdot t(t+1)\cdot\ldots\cdot(t+n-1)\cdot\frac{-(t+n)}{(1+x)^{t+n+1}}=$$

$$=\frac{(-1)^{n+1}\cdot t(t+1)\cdot\ldots\cdot(t+n)}{(1+x)^{t+n+1}}$$

• if $r=0$: $f(x)=(1+x)^0=1\Rightarrow f^{(n)}(x)=0$

• if $r>0$: $f(x)=(1+x)^r$

$$f'(x)=r(1+x)^{r-1}$$

$$f''(x)=r(r-1)(1+x)^{r-2}$$

• if $r>m$: $f^{(m)}(x)=r(r-1)\cdot\ldots\cdot(r-(n-1))\cdot(1+x)^{r-m}$

• if $r<m$: $f^{(n)}(x)=r(r-1)\cdot\ldots\cdot(r-(n-2))(1+x)\Rightarrow f^{(n-1)}(x)=r(r-1)\cdot\ldots\cdot(r-(n-1))(1+x)$

$f^{(n)}(x)=0$

b) $f:(-1,+\infty)\to\mathbb{R}$ $\quad f(x)=x\ \ln(1+x)$

$$f^{(n)}(x)=\sum_{k=0}^{n}C_n^k\cdot(\ln(1+x))^{(n-k)}\cdot x^{(k)}=C_n^0\cdot(\ln(1+x))^{(n)}\cdot x+C_n^1(\ln(1+x))^{(n-1)}\cdot x'+C_n^2(\ln(1+x))^{(n-2)}\cdot x''+\ldots=$$

$$=(\ln(1+x))^{(n)}\cdot x+n\cdot(\ln(1+x))^{(n-1)}\cdot 1+0+\ldots$$

Let $g(x)=\ln(1+x)$, $g:(-1,+\infty)\to\mathbb{R}$

$g'(x)=\frac{1}{1+x}$ $\quad g''(x)=\frac{-1}{(1+x)^2}$ $\quad g^{(n)}(x)=\frac{(-1)^{n+1}(n+1)!}{(1+x)^n}$ (according to a))

$$f^{(n)}(x) = \frac{(-1)^{n+1}\cdot(n-1)!}{(1+x)^n}\cdot x + n\cdot\frac{(-1)^n\cdot(n-2)!}{(1+x)^{n-1}} = \cancel{\frac{(-1)^n\cdot(n-2)!}{(1+x)^{n-1}}\left(1+\frac{nx}{1+x}\right) = (-1)^n(n-2)!\,(x+}$$

c) $f:(-\infty,-1)\to\mathbb{R}\qquad f(x) = x\cdot\ln(1-x)$

$$f^{(n)}(x) = \sum_{k=0}^{n} C_n^k\cdot(\ln(1-x))^{(n-k)}\cdot x^{(k)} = (\ln(1-x))^{(n)}\cdot x + n\cdot(\ln(1-x))^{(n-1)}\cdot x + 0 + \dots$$

Let $g(x) = \ln(1-x)$, $\quad g:(-\infty,-1)\to\mathbb{R}$

$$g'(x) = \frac{-1}{1-x} = \frac{1}{x-1}\qquad g''(x) = \frac{-1}{(x-1)^2}\qquad g^{(n)}(x) = \frac{(n-1)!}{(x-1)^n}\cdot(-1)^{n+1}$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(x-1)^n}\cdot x + \frac{(-1)^n\cdot(n-2)!\cdot n}{(x-1)^{n-1}} = \frac{(-1)^n\cdot(n-2)!}{(x-1)^{n-1}}\cdot\left(\frac{-(n+1)}{x-1}\cdot x + n\right) =$$

$$= \frac{(-1)^n\cdot(n-2)!}{(x-1)^{n-1}}\cdot\frac{\cancel{-nx}+x+\cancel{nx}-n}{x-1} = \frac{(-1)^n\cdot(n-2)!\,(x-n)}{(x-1)^n}$$

d) $f:(-1,1)\to\mathbb{R}\qquad f(x) = \sqrt{3x+4} = \sqrt{3}\cdot\sqrt{x+\frac{4}{3}} = \sqrt{3}\left(x+\frac{4}{3}\right)^{\frac{1}{2}}$

$$f'(x) = \sqrt{3}\cdot\frac{1}{2}\cdot\left(x+\frac{4}{3}\right)^{-\frac{1}{2}}\qquad f''(x) = \frac{\sqrt{3}}{2}\cdot\frac{-1}{2}\cdot\left(x+\frac{4}{3}\right)^{-\frac{3}{2}}$$

Suppose $f^{(n)}(x) = \frac{\sqrt{3}}{2^n}\cdot(-1)^{n+1}\cdot 1\cdot 3\cdot\ldots\cdot(2(n-2)+1)\cdot\left(x+\frac{4}{3}\right)^{\frac{-(2n+1)}{2}}$

We shall prove that $f^{(n+1)}(x) = \frac{\sqrt{3}}{2^{n+1}}\cdot(-1)^{n+2}\cdot 1\cdot 3\cdot\ldots\cdot(2(n-1)+1)\cdot\left(x+\frac{4}{3}\right)^{\frac{-2n+1}{2}}$

$$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \frac{\sqrt{3}\cdot(-1)^{n+1}}{2^n}\cdot 1\cdot 3\cdot\ldots\cdot(2n-3)\cdot\frac{-(2n+1)}{2}\cdot\left(x+\frac{4}{3}\right)^{-\frac{2n+1}{2}}$$

e) $f:\left(-\frac{1}{2},+\infty\right)\to\mathbb{R}\qquad f(x) = \frac{1}{\sqrt{2x+1}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{x+\frac{1}{2}}} = \frac{1}{\sqrt{2}}\cdot\left(x+\frac{1}{2}\right)^{-\frac{1}{2}}$

$$f'(x) = \frac{1}{\sqrt{2}}\cdot\frac{-1}{2}\left(x+\frac{1}{2}\right)^{-\frac{3}{2}}\qquad f''(x) = \frac{1}{\sqrt{2}}\cdot\frac{1}{2^2}\cdot 3\left(x+\frac{1}{2}\right)^{-\frac{5}{2}}$$

Suppose that $f^{(n)}(x) = \frac{(-1)^{n+1}}{\sqrt{2}}\cdot\frac{1}{2^n}\cdot 1\cdot 3\cdot\ldots\cdot(2n-1)\cdot\left(x+\frac{1}{2}\right)^{\frac{-2n+1}{2}}$.

We shall prove that $f^{(n+1)}(x) = \frac{(-1)^{n+2}}{\sqrt{2}}\cdot\frac{1}{2^{n+1}}\cdot 1\cdot 3\cdot\ldots\cdot(2n-1)\cdot(2n+1)\cdot\left(x+\frac{1}{2}\right)^{\frac{-2n+3}{2}}$

$$f^{(n+1)}(x) = \frac{(-1)^{n+1}}{\sqrt{2}}\cdot\frac{1}{2^n}\cdot 1\cdot 3\cdot\ldots\cdot(2n-1)\cdot\frac{-(2n+1)}{2}\cdot\left(x+\frac{1}{2}\right)^{\frac{-2n+3}{2}}$$

Exercise 2: Determine the n-th derivative of the following functions:

a) $f:\mathbb{R}\setminus\{-\frac{b}{a}\}\to\mathbb{R}$ defined by $f(x)=\frac{1}{ax+b}$;

b) $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\sin(ax+b)$;

c) $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\cos(ax+b)$;

d) $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=e^{ax+b}$.

a) $f:\mathbb{R}\setminus\{-\frac{b}{a}\}\to\mathbb{R}\qquad f(x) = \frac{1}{ax+b} = \frac{1}{a}\cdot\frac{1}{x+\frac{b}{a}}$

$$f'(x) = \frac{1}{a}\cdot\frac{-1}{\left(x+\frac{b}{a}\right)^2}\qquad f''(x) = \frac{1}{a}\cdot\frac{-1}{\left(x+\frac{b}{a}\right)^3}$$

Suppose that $f^{(n)}(x) = \frac{1}{a}\cdot\frac{(-1)^n\cdot n!}{\left(x+\frac{b}{a}\right)^{n+1}}$.

We shall prove that $f^{(n+1)}x = \frac{1}{a} \cdot \frac{(-1)^{n+1}(n+1)!}{(x+\frac{b}{a})^{n+2}}$

$$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \frac{1}{a} \cdot (-1)^n \cdot n! \cdot \frac{-(n+1)}{(x+\frac{b}{a})^{n+2}} = \frac{(-1)^{n+1} \cdot (n+1)!}{a\left(x+\frac{b}{a}\right)^{n+2}}$$

b) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) = \sin(ax+b)$

$f'(x) = a\cos(ax+b)$ $\quad f''(x) = a^2 \cdot (-\sin(ax+b))$ $\quad f'''(x) = a^3 \cdot \cos(ax+b)$ $\quad f^{IV}(x) = a^4 \cdot \sin(ax+b)$

$$f^{(n)}(x) = \begin{cases} a^{4k} \cdot \sin(ax+b) &, n = 4k \\ a^{4k+1} \cdot \cos(ax+b) &, n = 4k+1 \\ -a^{4k+2} \cdot \sin(ax+b) &, n = 4k+2 \\ -a^{4k+3} \cdot \cos(ax+b) &, n = 4k+3 \end{cases} \quad , k \in \mathbb{R}$$

c) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) = \cos(ax+b)$

$f'(x) = -a\sin(ax+b)$ $\quad f''(x) = -a^2\cos(ax+b)$ $\quad f'''(x) = a^3\sin(ax+b)$ $\quad f^{IV}(x) = a^4\cos(ax+b)$

$$f^{n}(x) = \begin{cases} a^{4k}\cos(ax+b) &, n = 4k \\ -a^{4k+1}\sin(ax+b) &, n = 4k+1 \\ -a^{4k+2}\cos(ax+b) &, n = 4k+2 \\ a^{4k+3}\sin(ax+b) &, n = 4k+3 \end{cases} \quad , k \in \mathbb{R}$$

d) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) = e^{ax+b}$

$$f'(x) = a \cdot e^{ax+b} \Rightarrow f^{(n)}(x) = a^n \cdot e^{ax+b}$$

Exercise 3: Compute the derivatives of the following functions

a) $f:(0,\infty)\to\mathbb{R}$ defined by $f(x) = x^x$;

b) $f:(0,\infty)\to\mathbb{R}$ defined by $f(x) = x^{\frac{1}{x}}$;

c) $f:(0,\pi)\to\mathbb{R}$ defined by $f(x) = \sin x^x$;

d) $f:(0,\infty)\to\mathbb{R}$ defined by $f(x) = x^{\sin x}$;

a) $f:(0,\infty)\to\mathbb{R}$ $\quad f(x) = x^x = e^{x\ln x}$

$$f'(x) = e^{x\ln x}(\ln x + 1) = x^x(\ln x + 1)$$

b) $f:(0,\infty)\to\mathbb{R}$ $\quad f(x) = x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}$

$$f'(x) = e^{\frac{\ln x}{x}}\left(\frac{\frac{1}{x}\cdot x - \ln x}{x^2}\right) = x^{\frac{1}{x}} \cdot \frac{1-\ln x}{x^2} = x^{\frac{1-2x}{x}}(1-\ln x)$$

c) $f:(0,\pi)\to\mathbb{R}$ $\quad f(x) = \sin x^x$

$$f'(x) = \cos x^x \cdot (x^x)' = \cos(x^x) \cdot x^x(\ln x + 1)$$

d) $f:(0,+\infty)\to\mathbb{R}$ $\quad f(x) = x^{\sin x} = e^{\sin x \cdot \ln x}$

$$f'(x) = e^{\sin x \cdot \ln x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right)$$

Exercise 4: Prove that $\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$ for all $x > 0$.

$$\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x} \qquad \forall x \in \mathbb{R},\ x > 0$$

Let $f : (0, +\infty) \to \mathbb{R}$ $\quad f(x) = \ln x$

f continuous on $(0,+\infty)$ and differentiable on $(0,+\infty) \Rightarrow$

$\Rightarrow$ f continuous and differentiable on $[x, x+1]$, $\forall x > 0$ $\xrightarrow[\text{theorem}]{\text{Lagrange's}}$

$\Rightarrow$ $f(x+1) - f(x) = (\cancel{x}+1-\cancel{x}) \cdot f'(c)$, $c \in (x, x+1) \Rightarrow$

$\Rightarrow$ $\ln(x+1) - \ln x = f'(c)$, $c \in (x, x+1)$

$f'(c) = \frac{1}{c}$ $\qquad x < c < x+1$

$$\frac{1}{x+1} < \frac{1}{c} < \frac{1}{x}$$

$$\frac{1}{x+1} < f'(c) < \frac{1}{x}$$

$$\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$$

Exercise 5:

a) Prove that for all $n \in \mathbb{N}$ if holds

$$na^{n-1} < \frac{b^n - a^n}{b - a} < nb^{n-1}$$

for all $a, b \in (0, +\infty)$ with $a < b$.

a) $n \in \mathbb{N}$

$$n \cdot a^{n-1} < \frac{b^n - a^n}{b-a} < a \cdot b^{n-1}, \qquad a, b \in (0,+\infty),\ a < b$$

Let $f(x) = x^n$, $f: (0,+\infty) \to \mathbb{R}$

f continuous and derivable on $(0, +\infty) \Rightarrow$

$\Rightarrow$ f continuous and derivable on $[a, b]$, $a, b \in (0,+\infty)$, $a < b$

$\xrightarrow[\text{theorem}]{\text{Lagrange's}}$ $f(b) - f(a) = (b-a) f'(c)$, $c \in (a, b)$

$f'(c) = n \cdot c^{n-1}$

$a < c < b \Rightarrow a^{n-1} < c^{n-1} < b^{n-1} \Rightarrow n \cdot a^{n-1} < n \cdot c^{n-1} < n \cdot b^{n-1} \Rightarrow$

$$\Rightarrow n \cdot a^{n-1} < \frac{b^n - a^n}{b-a} < n \cdot b^{n-1}$$

Excercise 6:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x + |x - 1|$$

for all $x \in \mathbb{R}$.

a) Prove that f has side derivatives at $x_0 = 1$;

b) Compute the side derivates of f at $x_0 = 1$;

c) Is f differentiable on the left at $x_0 = 1$? What about on the right?

d) Does f have a derivative at $x_0 = 1$?

e) Is f differentiable at $x_0 = 1$?

$f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) = x+|x-1| = \begin{cases} x-(x-1), & x<1 \\ x+(x-1), & x\geq 1\end{cases} = \begin{cases} 1, & x<1 \\ 2x-1, & x\geq 1\end{cases}$

x	$-\infty$		1		$+\infty$
$x-1$		$- - - - - - -$	0	$+ + + + + +$	

a+b) $f_l'(1) = \lim\limits_{\substack{x\to 1\\ x<1}} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{\substack{x\to 1\\ x<1}} \dfrac{1-1}{x-1} = 0 \in \overline{\mathbb{R}} \Rightarrow f$ has a leftside derivative at $x_0=1$

$f_r'(1) = \lim\limits_{\substack{x\to 1\\ x>1}} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{\substack{x\to 1\\ x>1}} \dfrac{2x-1-1}{x-1} = \lim\limits_{\substack{x\to 1\\ x>1}} \dfrac{2x-2}{x-1} = \lim\limits_{\substack{x\to 1\\ x>1}} \dfrac{2\cancel{(x-1)}}{\cancel{(x-1)}} = 2 \in \overline{\mathbb{R}} \Rightarrow$

$\Rightarrow f$ has a rightside derivative at $x_0=1$

c) For $x\in(-\infty,1)$, $f(x)=1=$ constant $\to f$ is differentiable at any point $x_0\in(-\infty,1]$

$\Rightarrow f$ differentiable on the left at $x_0=1$

For $x\in[1,+\infty)$, $\left.\begin{array}{l} f(x)=2x-1 \\ f_r'(1)=2\in\mathbb{R}\end{array}\right\} \Rightarrow f$ differentiable on the right at $x_0=1$

d) $f_l'(1)=0\neq 2=f_r'(1) \Rightarrow f_l'(1)\neq f_r'(1) \Rightarrow f$ does not have a derivative at $x_0=1$

e) $f_l'(1) = 0 \neq 2 = f_r'(1) \Rightarrow f_l'(1)\neq f_r'(1) \Rightarrow f$ is not differentiable at $x_0=1$