ANALYTIC GEOMETRY, PROBLEM SET 9

The line in 3D. Relative positions of lines and planes

1. Find the equation of the plane containing the points $P_1(2,-1,-3)$, $P_2(3,1,2)$ and parallel to the vector $\overline{a}(3,-1,-4)$.

2. Find the equation of the plane containing the perpendicular lines through $P(-2,3,5)$ on the planes $\pi_1 : 4x+y-3z+13=0$ and $\pi_2 : x-2y+z-11=0$.

3. Find the equation of the plane passing through the points A, B and C, where:
(a) $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$; (b) $A(3,2,1)$, $B(2,1,-1)$ and $C(-1,3,2)$.

4. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

5. Let d_1 and d_2 be two lines in $\mathcal{E}_3$, given by $d_1 : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6}$ and $d_2 : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-5}{-3}$.
(a) Find the parametric equations of d_1 and d_2;
(b) Prove that they intersect and find the coordinates of their intersection point;
(c) Find the equation of the plane determined by d_1 and d_2.

6. Given the lines $d_1 : x = 1+t, y = 1+2t, z = 3+t, t \in \mathbb{R}$ and
$d_2 : x = 3+s, y = 2s, z = -2+s, s \in \mathbb{R}$, show that $d_1 \parallel d_2$ and find the equation of the plane determined by d_1 and d_2.

7. Find the parametric equations of the line $\begin{cases} -2x+3y+7z+2=0 \\ x+2y-3z+5=0 \end{cases}$.

8. Find the parametric equations of the line passing through $P_1(5,-2,1)$ and $P_2(2,4,2)$. Find the equations of the line passing through $P(6,4,-2)$ and parallel to the line $d : \frac{x}{2} = \frac{y-1}{-3} = \frac{z-5}{6}$.

9. Given the points $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,5)$, find the equation(s) of the internal bisector passing through the vertex A in the triangle ABC.

10. Find the equations of the line passing through the origin and parallel to the line given by the parametric equations: $x = t$, $y = -1+t$ and $z = 2$.

11. Given the lines $d_1 : x = 4-2t, y = 1+2t, z = 9+3t$ and $d_2 : \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-4}{2}$, find the intersection points between the two lines and the coordinate planes.

12. Let d_1 and d_2 be the lines given by $d_1 : x = 3+t, y = -2+t, z = 9+t, t \in \mathbb{R}$ and $d_2 : x = 1-2s, y = 5+s, z = -2-5s, s \in \mathbb{R}$.

a) Prove they are coplanar.

Date: November 21, 2021.

b) Find the equation of the line passing through the point $P(4, 1, 6)$ and orthogonal on the plane determined by d_1 and d_2.

13. Prove that the intersection lines of the planes $\pi_1 : 2x-y+3z-5 = 0$, $\pi_2 : 3x+y+2z-1 = 0$ and $\pi_3 : 4x + 3y + z + 2 = 0$ are parallel.

14. Verify that the lines $d_1 : \frac{x-3}{1} = \frac{y-8}{3} = \frac{z-3}{4}$ and $d_2 : \frac{x-4}{1} = \frac{y-9}{2} = \frac{z-9}{5}$ are coplanar and find the equation of the plane determined by the two lines.

15. Determine whether the line given by $x = 3 + 8t$, $y = 4 + 5t$, and $z = -3 - t$, $t \in \mathbb{R}$ is parallel to the plane $x - 3y + 5z - 12 = 0$.

1. Find the equation of the plane containing the points $P_1(2,-1,-3)$, $P_2(3,1,2)$ and parallel to the vector $\overline{a}(3,-1,-4)$.

$P_1(2,-1,3)$
$\overline{P_1P_2}(1,2,5)$
$\overline{a}(3,-1,-4)$.

Let π be the plane passing through P_1 and $\parallel$ to $\overline{P_1P_2}$, $\overline{a}$.

$$\pi : \begin{vmatrix} x-2 & y+1 & z-3 \\ 1 & 2 & 5 \\ 3 & -1 & -4 \end{vmatrix} = 0$$

2. Find the equation of the plane containing the perpendicular lines through $P(-2,3,5)$ on the planes $\pi_1 : 4x+y-3z+13=0$ and $\pi_2 : x-2y+z-11=0$.

$\overline{n}_1(4,1,-3)$, $\overline{n}_2(1,-2,1)$ - normal vectors on π_1 and π_2.

$P \in \pi$.
$\overline{n}_1, \overline{n}_2 \parallel \pi$

$$\pi : \begin{vmatrix} x+2 & y-3 & z-5 \\ 4 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

3. Find the equation of the plane passing through the points A, B and C, where:
(a) $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$; (b) $A(3,2,1)$, $B(2,1,-1)$ and $C(-1,3,2)$.

MI

$\overrightarrow{AB}(2,1,2)$

$\overrightarrow{AC}(3,-1,-2)$

$$\pi : \begin{vmatrix} x+2 & y-1 & z-1 \\ 2 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix} = 0.$$

MII

$$\pi : \begin{vmatrix} x & y & z & 1 \\ -2 & 1 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 1 \end{vmatrix} = 0.$$

4. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

- Write the equation of π passing through A, B and C.
- Check if the coordinates of D satisfy this equation.

5. Let d_1 and d_2 be two lines in $\mathcal{E}_3$, given by $d_1 : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6}$ and $d_2 : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-5}{-3}$.
(a) Find the parametric equations of d_1 and d_2;
(b) Prove that they intersect and find the coordinates of their intersection point;
(c) Find the equation of the plane determined by d_1 and d_2.

(a) For d_1 : $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6} = t \in \mathbb{R}$.

$$d_1 : \begin{cases} x = 1 + 2t \\ y = -1 - t \\ z = 5 + 6 \cdot t \end{cases}, \quad t \in \mathbb{R}.$$

For d_2 : $\frac{x-1}{1} = \frac{y+1}{1} = \frac{z-5}{3} = s \in \mathbb{R}$.

$$d_2 : \begin{cases} x = 1 + s \\ y = -1 + s \\ z = 5 + 3 \cdot s \end{cases}, \quad s \in \mathbb{R}.$$

(b) For $t = 0$, we get $P(1,-1,5) \in d_1$ and for $s = 0$, $P(1,-1,5) \in d_2$.

(c) Let π be the sought-after plane.

$P(1,-1,5) \in \pi$.

$\overline{v_1}(2,-1,6), \overline{v_2}(1,1,3) \parallel \tilde{\pi}$.

$$\pi : \begin{vmatrix} x-1 & y+1 & z-5 \\ 2 & -1 & 6 \\ 1 & 1 & 3 \end{vmatrix} = 0.$$

6. Given the lines $d_1 : x = 1 + t, y = 1 + 2t, z = 3 + t, t \in \mathbb{R}$ and $d_2 : x = 3 + s, y = 2s, z = -2 + s, s \in \mathbb{R}$, show that $d_1 \parallel d_2$ and find the equation of the plane determined by d_1 and d_2.

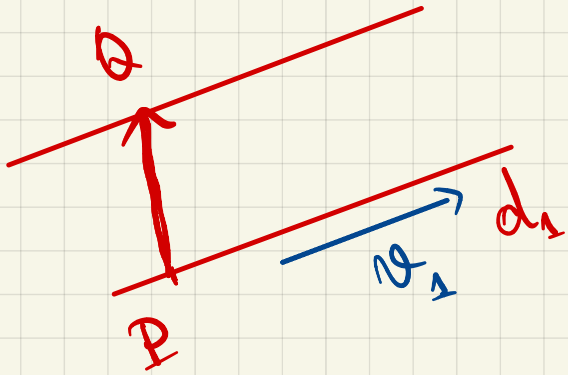
$\overline{v}_1(1, 2, 1)$ – d. v. of d_1

$\overline{v}_2(1, 2, 1)$ – d. v. of d_2.

$\overline{v}_1 \parallel \overline{v}_2$, so $d_1 \parallel d_2$.

$P(1, 1, 3) \in d_1$

$Q(3, 0, -2) \in d_2$



$\overline{PQ}(2, -1, -5) \parallel \pi$ and $\overline{PQ} \nparallel \overline{v}_1$

$\overline{v}_1(1, 2, 1) \parallel \pi$

$$\pi : \begin{vmatrix} x-1 & y-1 & z-3 \\ 2 & -1 & -5 \\ 1 & 2 & 1 \end{vmatrix} = 0.$$

7. Find the parametric equations of the line $\begin{cases} -2x + 3y + 7z + 2 = 0 \\ x + 2y - 3z + 5 = 0 \end{cases}$.

Let $\boxed{z = t} \in \mathbb{R}$.

$$\begin{cases} -2x + 3y = -2 - 7 \cdot t \\ x + 2y = 3 \cdot t - 5 \quad | \cdot 2 \end{cases}$$

$+$

$$7y = -12 - t$$

$$\therefore \quad y = -\frac{12}{7} - \frac{t}{7}.$$

$$x = 3t - 5 - 2 \cdot \left(-\frac{12}{7} - \frac{t}{7}\right)$$

$$x = -5 + \frac{24}{7} + \left(3 + \frac{2}{7}\right) \cdot t$$

$$\begin{cases} x = -\frac{11}{7} + \frac{23}{7} \cdot t \\ y = -\frac{12}{7} - \frac{t}{7} \\ z = t \end{cases}$$

A d. v. is $\overline{v}\left(\frac{23}{7}, -\frac{1}{7}, 1\right)$.

Alternatively we can use

$$\overline{v'} = 7 \cdot \overline{v}\,(23, -1, 7).$$

8(a) Find the parametric equations of the line passing through $P_1(5,-2,1)$ and $P_2(2,4,2)$.
(b) Find the equations of the line passing through $P(6,4,-2)$ and parallel to the line $d: \frac{x}{2} = \frac{y-1}{-3} = \frac{z-5}{6}$.

(a) Let $\ell_1 = P_1P_2$.

$\overline{P_1P_2}(-3, 6, 1)$

$$\ell_1 : \begin{cases} x = 5 - 3\cdot t \\ y = -2 + 6\cdot t \\ z = 1 + t \end{cases}, \quad t \in \mathbb{R}.$$

Symmetric equations

$$t = \frac{x-5}{-3} \overset{(1)}{=} \frac{y+2}{6} \overset{(2)}{=} \frac{z-1}{1}$$

General equations

$$\begin{cases} (1) \quad 6x - 30 = -3y - 6 \\ (2) \quad y + 2 = 6z - 6 \end{cases}$$

(b) $\overline{v}(2,-3,6)$ is d.v. of d.

Symmetric equation is

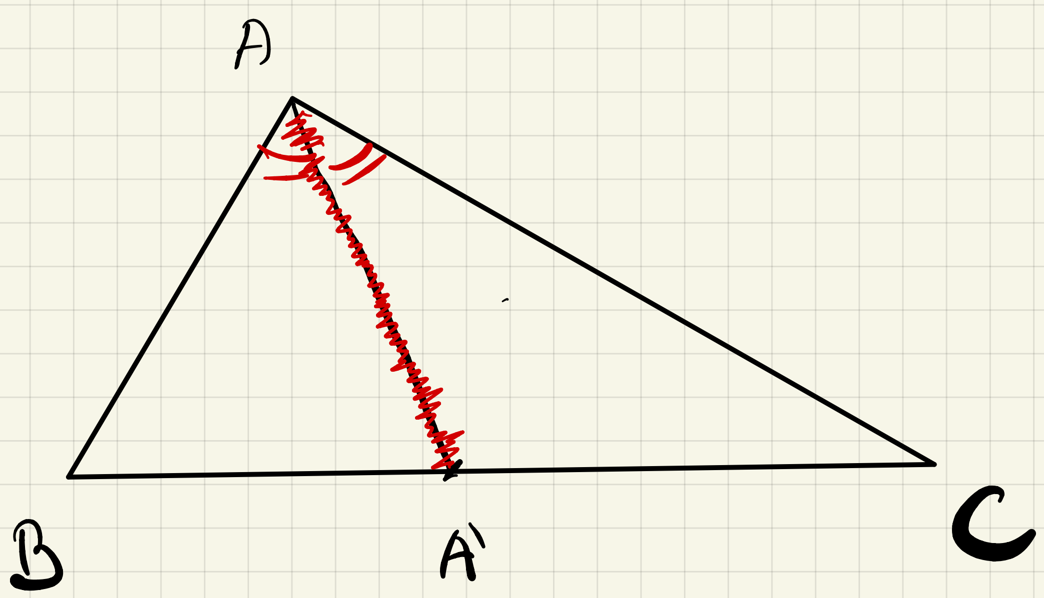
$$\frac{x-6}{2} = \frac{y-4}{-3} = \frac{z+2}{6}$$

Parametric equations:

$$\begin{cases} x = 6 + 2t \\ y = 4 + (-3)\cdot t \\ z = -2 + 6\cdot t \end{cases}, \quad t \in \mathbb{R}.$$

$-3 \qquad 6$

9. Given the points $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,5)$, find the equation(s) of the internal bisector passing through the vertex A in the triangle ABC.



Want to find a director vector for AA'

$\overline{AB}(1,0,0)$, so $\|\overline{AB}\| = 1$.

$\overline{AC}(2,2,12)$, so $\|\overline{AC}\| = \sqrt{4+4+144}$

$= \sqrt{152} = 2\sqrt{38}$.

$\overline{v} := 2\sqrt{38} \cdot \overline{AB} + \overline{AC}$

$\overline{v}(2+2\sqrt{38},\ 2,\ 12)$ - is a d.v. for AA'.

$$AA' : \begin{cases} x = 1 + (2+2\sqrt{38}) \cdot t \\ y = 2 + 2 \cdot t \\ z = -7 + 12 \cdot t \end{cases}, \quad t \in \mathbb{R}.$$