ANALYTIC GEOMETRY, PROBLEM SET 11

Mostly distances in 3D.

1. Find the distance from the point $P(1,2,-1)$ to the line $d: x = y = z$.

2. Find the distance from $P(3,1,-1)$ to the plane $\pi: 22x + 4y - 20z - 45 = 0$.

3. Find the distance between the planes
$\pi_1: 2x - 3y + 4z - 7 = 0$ and $\pi_2: 4x - 6y + 8z - 3 = 0$.

4. Find the distance between the lines $d_1: \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}$ and $d_2: \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$.

5. Find the distance between the lines $d_1: x = 1 - 2t$, $y = 3t$, $z = -2t + t$, where $t \in \mathbb{R}$ and $d_2: x = 7 + 4s$, $y = 5 - 6s$, $z = 4 - 2s$, where $s \in \mathbb{R}$.

6. Show that the line
$d: \frac{x+1}{1} = \frac{y-3}{2} = \frac{z}{-1}$ and the plane $\pi: 2x - 2y - 2z + 3 = 0$ are parallel and find the distance between them.

7. Given the point $P(6,-5,5)$ and the plane $\pi: 2x - 3y + z - 4 = 0$, find the coordinates of the symmetric P' of the point P with respect to the plane π.

8. Consider the point $P(4,3,10)$ the line $d: \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{5}$. Find the coordinates of the symmetric point P' of P with respect to the line d.

Use "bundle of planes".

11.[From the previous set.] Determine the equations of the planes which pass through the points $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the Oz axis.

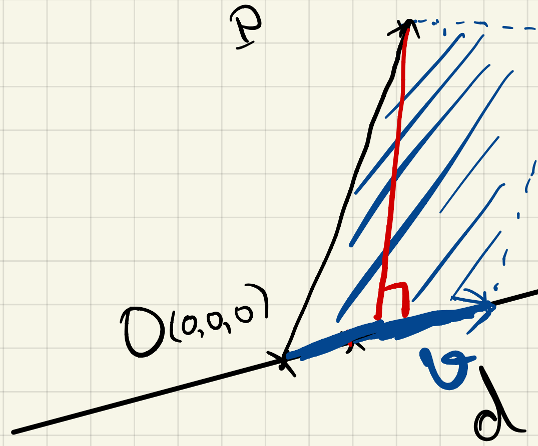
12. Find the geometric locus of the lines passing through a given point and having a constant distance to a given line.

The setup of the next problem is in the Euclidean plane $\mathcal{E}_2$.

13. In each of the following situations, find the equation of the circle:
a) of diameter [AB], where $A(1,2)$ and $B(-3,-1)$; b) of center $I(2,-3)$ and radius $R = 7$;
c) of center $I(-1,2)$ and which passes through $A(2,6)$;
d) centered at the origin and tangent to $d: 3x - 4y + 20 = 0$; e) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $d: 3x - y - 2 = 0$;
f) determined by $A(1,1)$, $B(1,-1)$ and $C(2,0)$; g) tangent to both $d_1: 2x + y - 5 = 0$ and $d_2: 2x + y + 15 = 0$, if the tangency point with d_1 is $M(3,1)$.

Date: December 12, 2021.

1. Find the distance from the point $P(1,2,-1)$ to the line $d: x=y=z$.



- $\overline{OP}\,(1,2,-1)$
- $\overline{v}\,(1,1,1)$ – director vector of d.

$$\therefore \quad d(P,d) = \frac{\text{Area (parallelogram)}}{\|\overline{v}\|}$$

$$= \frac{\|\overline{v} \times \overline{OP}\|}{\|\overline{v}\|}.$$

2. Find the distance from $P(3,1,-1)$ to the plane $\pi: 22x+4y-20z-45=0$.

$$d(P,\pi) = \frac{|22\cdot 3+4\cdot +20-45|}{\sqrt{22^2+16+400}} = \ldots\ldots$$

<u>Note</u>: Have a look at the proof (textbook, pag. 73).

3. Find the distance between the planes
$\pi_1 : 2x - 3y + 4z - 7 = 0$ and $\pi_2 : 4x - 6y + 8z - 3 = 0$.

$$\left.\begin{array}{l}\overline{n}_1(2,-3,4)\\ \overline{n}_2(4,-6,8)\end{array}\right\} \quad \overline{n}_2 = 2\cdot\overline{n}_1, \text{ so } \pi_1 \parallel \pi_2.$$

$d(\pi_1, \pi_2) = d(P, \pi_2)$, where

P is any point on π_1.

Choose $P(0,-1,1) \in \pi_1$.

$$d(P,\pi_2) = \frac{|0+6+8-3|}{\sqrt{16+36+64}} = \frac{11}{\sqrt{116}}.$$

4. Find the distance between the lines $d_1 : \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}$ and $d_2 : \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$.

$\overline{v}_1(2,3,1)$

$\overline{v}_2(3,4,3)$

- Clearly $d_1 \nparallel d_2$. Next step is to check that

$$\boxed{d_1 \cap d_2 = \emptyset}$$

- d_1 and d_2 are skew lines.
- We find 2 planes π_1, π_2 s.t.

$$\begin{cases} \pi_1 \parallel \pi_2 \text{ and} \\ d_1 \subset \pi_1 \text{ and } d_2 \subset \pi_2. \end{cases}$$

- We choose $\overline{n}_1 = \overline{n}_2 = \overline{v}_1 \times \overline{v}_2$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} = 5\overline{i} - 3\overline{j} - \overline{k}$$

$\overline{n}_1 = \overline{n}_2\,(5, -3, -1)$.

$P_1(1, -1, 0) \in \pi_1$ and $P_2(-1, 0, 1) \in \pi_2$.

$$\therefore \begin{cases} \pi_1 : \; 5(x-1) - 3(y+1) - 1 \cdot z = 0 \\ \pi_2 : \; 5(x+1) - 3y - (z-1) = 0. \end{cases}$$

$$d(d_1, d_2) = d(\pi_1, \pi_2) = d(P_1, \pi_2)$$

See Problem 2

<u>Note</u>: Please read the theory in [textbook, pag. 76].

5. Find the distance between the lines $d_1 : x = 1 - 2t,\ y = 3t,\ z = -2 + t$, where $t \in \mathbb{R}$ and $d_2 : x = 7 + 4s,\ y = 5 - 6s,\ z = 4 - 2s$, where $s \in \mathbb{R}$.

- $\overline{v}_1(-2, 3, 1)$
- $\overline{v}_2(4, -6, -2)$

$\overline{v}_2 = -2 \cdot \overline{v}_1$, so $d_1 \parallel d_2$.

Take $P_1(1, 0, -2)$.

Then $d(d_1, d_2) = d(P_1, d_2)$.

In order to compute this, apply the method given in Problem 1.

6. Show that the line
$d : \frac{x+1}{1} = \frac{y-3}{2} = \frac{z}{-1}$ and the plane $\pi : 2x - 2y - 2z + 3 = 0$ are parallel and find the distance between them.

$\overline{v}(1, 2, -1)$ - director vector of d.

$\overline{n}(2, -2, -2)$ - normal vector of π.

$\overline{v} \cdot \overline{n} = 2 - 4 + 2 = 0.$

$\overline{v} \perp \overline{n}$, therefore $d \parallel \pi$.

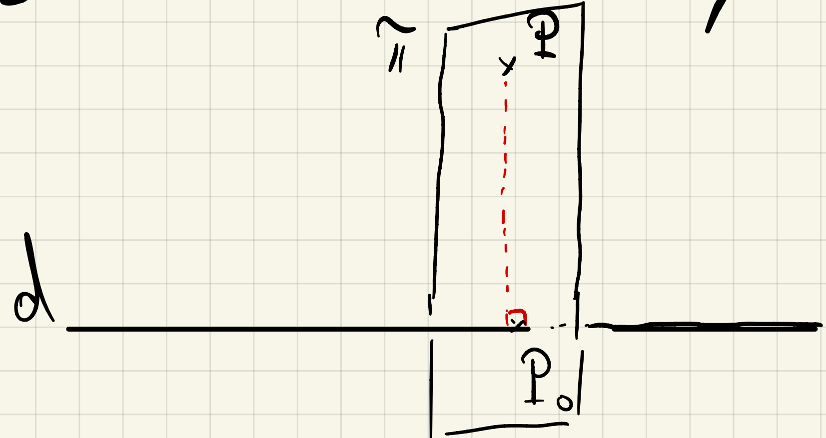
$d(d, \pi) = d(P, \pi)$, where

$P(-1, 3, 0)$.

Problem 7 (Exercise)

8. Consider the point $P(4, 3, 10)$ the line $d : \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{5}$. Find the coordinates of the symmetric point P' of P with respect to the line d.

- Find $P_0 = \mathrm{Proj}_d(P)$.
- After that, notice that P_0 is the midpoint of $[PP']$, so one can find P'!



$P(4,3,10) \in \tilde{\pi}$.

$\bar{v}(2,4,5)$ is the d.v. of d, so $\bar{v}$ is the normal vector on $\tilde{\pi}$.

$$\tilde{\pi}: \quad 2(x-4) + 4\cdot(y-3) + 5\cdot(z-10) = 0.$$

$d : \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{5}. = t$

$$d: \begin{cases} x = 1+2t \\ y = 2+4\cdot t \\ z = 3+5\cdot t \end{cases}, \quad t \in \mathbb{R}.$$

$$d \cap \pi: \quad 2(2t-3) + 4(4t-1) + 5\cdot(5t-7) = 0$$

$$\Leftrightarrow \quad 37t = 45$$

$$\Leftrightarrow \quad \boxed{t = \frac{45}{37}}$$

$$P_0\left(1 + 2\cdot\frac{45}{37},\ 2 + 4\cdot\frac{45}{37},\ 3 + 5\cdot\frac{45}{37}\right).$$

P_0 is the midpoint of $[PP']$.

13. In each of the following situations, find the equation of the circle:
a) of diameter $[AB]$, where $A(1,2)$ and $B(-3,-1)$; b) of center $I(2,-3)$ and radius $R=7$;
c) of center $I(-1,2)$ and which passes through $A(2,6)$;
d) centered at the origin and tangent to $d: 3x-4y+20=0$; e) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $d: 3x-y-2=0$;
f) determined by $A(1,1)$, $B(1,-1)$ and $C(2,0)$; g) tangent to both $d_1: 2x+y-5=0$ and $d_2: 2x+y+15=0$, if the tangency point with d_1 is $M(3,1)$.

a) The center of the circle is $I\left(-1, \frac{1}{2}\right)$.

The radius $R = \dfrac{d(A,B)}{2} = \dfrac{\sqrt{25}}{2} = \dfrac{5}{2}$.

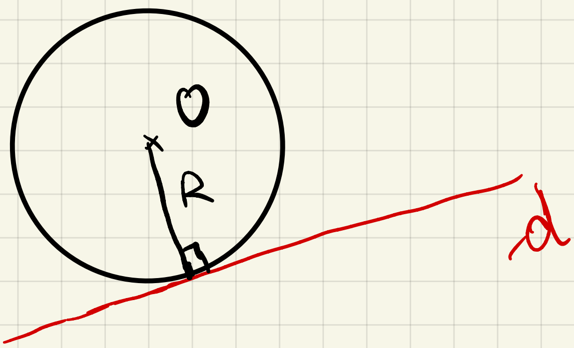
$\mathcal{C}\left(I, \frac{5}{2}\right): (x+1)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{25}{4}$.

b) $\mathcal{C}(I, 7): (x-2)^2 + (y+3)^2 = 49$.

c) $R = d(A, I) = \sqrt{3^2+4^2} = 5$

$\mathcal{C}(I, 5): (x+1)^2 + (y-2)^2 = 25$

d)

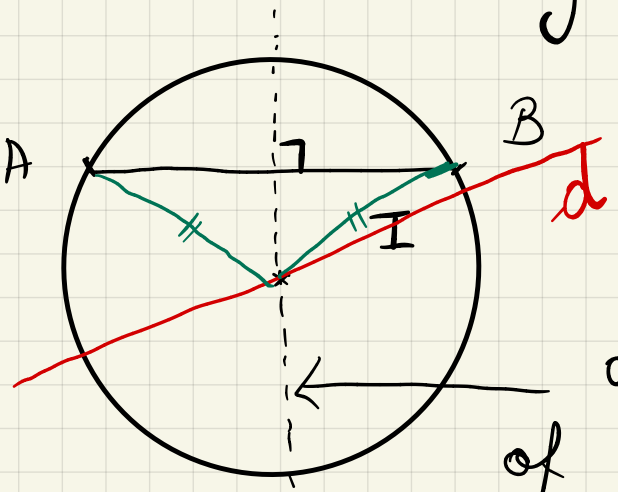


$R = d(O, d) = \dfrac{|20|}{5} = 4$.

$\mathcal{C}(O, 4): x^2+y^2 = 16$.

e) passing through $A(3,1)$, $B(-1,3)$

having the center on $d: 3x - y - 2 = 0$.



d' perp. bisect. of $[AB]$.

Find $\{I\} = d \cap d'$.