ANALYTIC GEOMETRY, PROBLEM SET 10

Mostly angles in 3D.

1. Show that the line $d = \begin{cases} x = 0 \\ y = t \\ z = t \end{cases}$ is contained inside the plane $6x + 4y - 4z = 0$.

2. Determine whether the line given by $x = 3 + 8t$, $y = 4 + 5t$, and $z = -3 - t$, $t \in \mathbb{R}$ is parallel to the plane $x - 3y + 5z - 12 = 0$.

3. Prove that the lines $d_1 : \begin{cases} x = 1 + 4t \\ y = 5 - 4t \\ z = -1 + 5t \end{cases}, t \in \mathbb{R}$ and $d_2 : \begin{cases} x = 2 + 8t \\ y = 4 - 3t \\ z = 5 + t \end{cases}, t \in \mathbb{R}$ are skew.

4. Find the parametric equations of the line passing through $(5, 0, -2)$ and parallel to the planes $x - 4y + 2z = 0$ and $2x + 3y - z + 1 = 0$.

5. Find the equation of the plane containing the point $P(2, 0, 3)$ and the line $d : \begin{cases} x = -1 + t \\ y = t \\ z = -4 + 2t \end{cases}$.

6. Let $M_1(2, 1, -1)$ and $M_2(-3, 0, 2)$ be two points. Find:

a) the equation of the bundle of planes passing through M_1 and M_2;
b) the plane π from the bundle, which is orthogonal on xOy;
c) the plane ρ from the bundle, which is orthogonal on π.

7. Find the angle determined by d_1 and d_2, when: a) $d_1 : x = 4-t, y = 3+2t, z = -2t, t \in \mathbb{R}$ and $d_2 : x = 5 + 2s, y = 1 + 3s, z = 5 - 6s, s \in \mathbb{R}$.
b) $d_1 : \frac{x-1}{2} = \frac{y+5}{7} = \frac{z-1}{-1}$ and $d_2 : \frac{x+3}{-2} = \frac{y-9}{1} = \frac{z}{4}$.

8. Find the angle determined by the planes $\pi_1 : x-\sqrt{2}y+z-1 = 0$ and $\pi_2 : x+\sqrt{2}y-z+3 = 0$.

9. Find the coordinates of the orthogonal projection of the point P(2, 1, 1) on the plane $\pi : x + y + 3z + 5 = 0$.

10. Determine the orthogonal projection of the point $A(1, 3, 5)$ on the line which is given as the intersection of the planes $2x + y + z - 1 = 0$ and $3x + y + 2z - 3 = 0$.

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1. Show that the line $d : \begin{cases} x = 0 \\ y = t \\ z = t \end{cases}$ is contained inside the plane $6x + 4y - 4z = 0$.

$$6 \cdot 0 + 4 \cdot t - 4 \cdot t = 0, \quad \forall t \in \mathbb{R}.$$

Since all the points on d have the form $P_t(0, t, t)$ for $t \in \mathbb{R}$, we showed that all these points are contained in the given plane.

$\blacksquare$

2. Determine whether the line given by $x = 3 + 8t$, $y = 4 + 5t$, and $z = -3 - t$, $t \in \mathbb{R}$ is parallel to the plane $x - 3y + 5z - 12 = 0$.

- The director vector of the line is $\bar{v}(8, 5, -1)$.
- The normal vector to the plane is $\bar{n}(1, -3, 5)$

$$\bar{v} \cdot \bar{n} = 8 \cdot 1 - 15 - 5 = -12.$$

$$\left[\ \bar{a} \cdot \bar{b} = 0 \iff \bar{a} \perp \bar{b}\ \right]$$

$\therefore$ $\bar{v}$ and $\bar{n}$ are not perpendicular.

$\therefore$ $\bar{v}$ is not parallel to the plane.

3. Prove that the lines $d_1 : \begin{cases} x = 1 + 4t \\ y = 5 - 4t \\ z = -1 + 5t \end{cases}, t \in \mathbb{R}$ and $d_2 : \begin{cases} x = 2 + 8t \\ y = 4 - 3t \\ z = 5 + t \end{cases}, t \in \mathbb{R}$ are skew.

Step 1 $\quad \bar{v}_1 (4, -4, 5)$

$\bar{v}_2 (8, -3, 1)$

$\bar{v}_1 \nparallel \bar{v}_2$, so $d_1 \nparallel d_2$.

Step 2. If $d_1 \cap d_2 \neq \emptyset$, then $\exists\, t_1, t_2 \in \mathbb{R}$ such that

$$\begin{cases} 1 + 4t_1 = 2 + 8 \cdot t_2 & (1) \\ 5 - 4t_1 = 4 - 3t_2 & (2) \\ -1 + 5t_1 = 5 + t_2 & (3) \end{cases}$$

From (3), $t_2 = -6 + 5t_1$.

Replace in (2).

$$5 - 4 \cdot t_1 = 4 - 3(-6 + 5t_1)$$

$\therefore\ 19 = -11 t_1 \Rightarrow t_1 = -\frac{19}{11}$.

$$t_2 = -6 - \frac{95}{11}$$

See if these values satisfy (1)

$$1+4\cdot t_1 = 1-\frac{4\cdot 19}{11} = \frac{11-76}{11}$$

$$= \frac{-65}{11}.$$

$$2+8\cdot t_2 = 2-48-8\cdot\frac{95}{11}$$

$$= -46-\frac{8\cdot 95}{11}.$$

Clearly $1+4\cdot t_1 \neq 2+8\cdot t_2$.

We showed that $d_1 \nparallel d_2$ and $d_1 \cap d_2 = \emptyset$, so d_1 and d_2 are skew.

4. Find the parametric equations of the line passing through $(5, 0, -2)$ and parallel to the planes $x - 4y + 2z = 0$ and $2x + 3y - z + 1 = 0$.

We want to find $\overline{v}(a, b, c)$, the director vector of the line.

$$\begin{cases} \overline{n_1}(1, -4, 2) \\ \overline{n_2}(2, 3, -1) \end{cases}$$ are the normal vectors of the gireen planes.

We can choose $\overline{v} := \overline{n_1} \times \overline{n_2}$.

$$\overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & -4 & 2 \\ 2 & 3 & -1 \end{vmatrix} = 4\overline{i} + 3\overline{k} + 4\overline{j} + 8\overline{k} - 6i + \overline{j}$$

$$= -2\overline{i} + 5\overline{j} + 11\overline{k}$$

$$\overline{v}(-2, 5, 11)$$

The parametric equations of the line are

$$\begin{cases} x = 5 + (-2) \cdot t \\ y = 0 + 5 \cdot t \\ z = -2 + 11 \cdot t \end{cases}, \quad t \in \mathbb{R}.$$

$\blacksquare$

5. Find the equation of the plane containing the point $P(2,0,3)$ and the line $d: \begin{cases} x=-1+t \\ y=t \\ z=-4+2t \end{cases}$.

$\bar{v}(1,1,2)$ - director vector of d.

$Q(-1,0,-4) \in d$ (obtained at $t=0$).

$\overline{PQ}(-3,0,-7)$.

$$\pi: \begin{vmatrix} x-2 & y-0 & z-3 \\ 1 & 1 & 2 \\ -3 & 0 & -7 \end{vmatrix} = 0.$$

Alternatively: $\bar{n} := \bar{v} \times \overline{PQ} = (-7,1,3)$.

Then,

$$\pi: -7(x-2)+1\cdot(y-0)+3\cdot(z-3)=0$$

8. Find the angle determined by the planes $\pi_1: x-\sqrt{2}y+z-1=0$ and $\pi_2: x+\sqrt{2}y-z+3=0$.

$\bar{n}_1(1,-\sqrt{2},1)$ and $\bar{n}_2(1,\sqrt{2},-1)$

$\overline{n_1}\cdot\overline{n_2} = 1-2-1=-2$.

$\quad = (|\bar{n}_1|\cdot|\bar{n}_2|\cdot\cos(\measuredangle\,\bar{n}_1,\bar{n}_2))$

$\measuredangle(\pi_1,\pi_2) = \pi - \measuredangle(\bar{n}_1,\bar{n}_2)$

$$\measuredangle(\overline{n_1}, \overline{n_2}) = \arccos\left(\frac{\overline{n_1} \cdot \overline{n_2}}{|\overline{n_1}| \cdot |\overline{n_2}|}\right)$$

$$= \arccos\left(\frac{-2}{2 \cdot 2}\right)$$

$$= \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

$$\therefore \measuredangle(\pi_1, \pi_2) = \pi - \frac{2\pi}{3} = \frac{\pi}{3}.$$

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11. Determine the equations of the planes which pass through the points $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the Oz axis.

12. Find the equations of the projection of the line $d: \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$ on the plane $\pi: x + 2y - z = 0$.

13. Find the angle determined by the lines $d_1: \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases}$ and $d_2: \begin{cases} x - y - z - 1 = 0 \\ x - y + 2x + 1 = 0 \end{cases}$.

14. Find the angle determined by the planes $\pi_1: x+3y+2z+1 = 0$ and $\pi_2: 3x+2y-z-6 = 0$.

15. Find the angle determined by the plane xOy and the line M_1M_2, where $M_1(1,2,3)$ and $M_2(-2,1,4)$.

$\pi_{xOy}: \quad z = 0.$

$\bar{n}_{xOy}(0, 0, 1)$

The director vector of the line is:

$\bar{v}_{M_1M_2} = \overline{M_1M_2}(-3, -1, 1)$

$\sphericalangle(\bar{n}_{xOy}, \bar{v}_{M_1M_2}) < 90^\circ.$

$\bar{n}_{xOy} \cdot \bar{v}_{M_1M_2} = 1 > 0.$

$\Rightarrow \sphericalangle(M_1M_2, \pi_{xOy}) = \frac{\pi}{2} - \sphericalangle(\bar{n}_{xOy}, \bar{v}_{M_1M_2})$

$$\sphericalangle\left(\bar{n}_{xOy},\ \bar{r}_{M_1M_2}\right) = \arccos\left(\frac{1}{\sqrt{11}}\right)$$

$$\sphericalangle\left(M_1M_2,\ \Pi_{xOy}\right) = \frac{\pi}{2} - \arccos\left(\frac{\sqrt{11}}{11}\right).$$

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