ANALYTIC GEOMETRY, PROBLEM SET 6

Representations of the line in plane

1. Find the equation of the line passing through the intersection point of the lines $d_1 : 2x - 5y - 1 = 0$ and $d_2 : x + 4y - 7 = 0$ and through a point M which divides the segment $[AB]$, given by $A(4, -3)$ and $B(-1, 2)$, into the ratio $k = 2/3$.

2. Find the equation of the line passing through the intersection point of $d_1 : 3x - 2y + 5 = 0$, $d_2 : 4x + 3y - 1 = 0$ and intersecting the Oy axis at the point A with $OA = 3$.

3. Find the parametric equations of the line through P_1 and P_2, when

(1) $P_1(3, -2)$, $P_2(5, 1)$;
(2) $P_1(4, 1)$, $P_2(4, 3)$.

In each case, find the vector equation of the line passing through these points.

$$\begin{cases} x = -5 + 2t \\ y = 2 + 3t \end{cases}$$

4. Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.

5. Show that the equations $x = 3 - t$, $y = 1 + 2t$ and $x = -1 + 3t$, $y = 9 - 6t$. represent the same line. Write down a director vector for this line.

6. The points $M_1(1, 2)$, $M_2(3, 4)$ and $M_3(5, -1)$ are the midpoints of the sides of a triangle. Write down the equations of the lines determined by the sides of the triangle.

7. Given the line $d : 2x + 3y + 4 = 0$, find the equation of a line d_1 passing through the point $M_0(2, 1)$, in the following situations: a) d_1 is parallel with d; b) d_1 is orthogonal on d; c) the angle determined by d and d_1 is $\pi/4$.

8. The vertices of the triangle $\triangle ABC$ are the intersection points of the lines $d_1 : 4x + 3y - 5 = 0$, $d_2 : x - 3y + 10 = 0$, $d_3 : x - 2 = 0$. a) Find the coordinates of A, B and C. b) Find the equations of the median lines of the triangle. c) Find the equations of the heights of the triangle.

9. Find the coordinates of the symmetrical of the point $P(-5, 13)$ with respect to the line $d : 2x - 3y - 3 = 0$.

10. Find the coordinates of the point P on the line $d : 2x - y - 5 = 0$, for which the sum $AP + PB$ attains its minimum, when $A(-7, 1)$ and $B(-5, 5)$.

11. Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $4x - y + 2 = 0$, $x - 4y - 8 = 0$ and $x + 4y - 8 = 0$.

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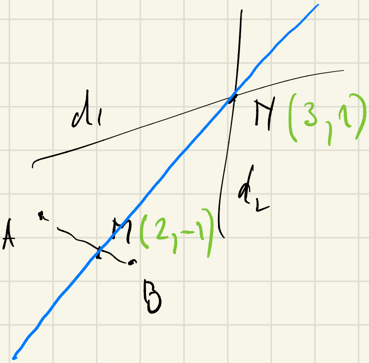
1. Find the equation of the line passing through the intersection point of the lines $d_1 : 2x - 5y - 1 = 0$ and $d_2 : x + 4y - 7 = 0$ and through a point M which divides the segment $[AB]$, given by $A(4, -3)$ and $B(-1, 2)$, into the ratio $k = 2/3$.

$M \in d_1 \cap d_2 \qquad M(x_M, y_M)$



- $M \in d_1 \Rightarrow \begin{cases} 2x_M - 5y_M - 1 = 0 \\ x_M + 4y_M - 7 = 0 \end{cases}$ $\rightsquigarrow$ solve to obtain x_M, y_M
  $M \in d_2 \Rightarrow$

- $M = ?$

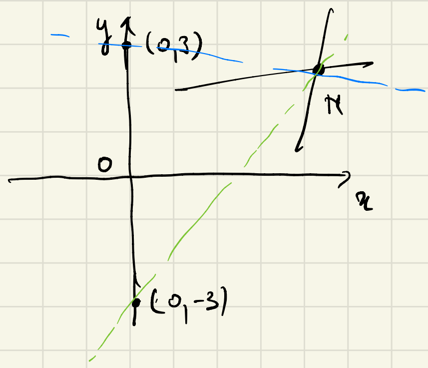
$$M\left(\frac{\overset{4}{x_1} + \overset{2/3}{k}\, x_2}{1+k}, \frac{\overset{-3}{y_1} + k y_2}{1+k}\right) = \ldots$$

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0 \iff \frac{x-3}{2-3} = \frac{y-1}{(-1)-1} \iff \ldots$$

$$2x - y - 5 = 0$$

2. Find the equation of the line passing through the intersection point of $d_1 : 3x - 2y + 5 = 0$, $d_2 : 4x + 3y - 1 = 0$ and intersecting the Oy axis at the point A with $OA = 3$.

$d_1 \cap d_2 \begin{cases} 3x - 2y + 5 = 0 \\ 4x + 3y - 1 = 0 \end{cases} \rightsquigarrow$



A is either $(0,3)$ or $(0,-3)$

two cases

3. Find the parametric equations of the line through P_1 and P_2, when

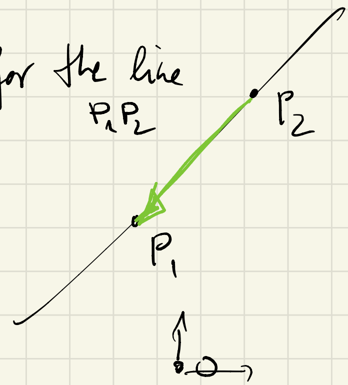
(1) $P_1(3,-2)$, $P_2(5,1)$;

(2) $P_1(4,1)$, $P_2(4,3)$.

In each case, find the vector equation of the line passing through these points.

(1) $\overline{P_2P_1}$ is a director vect. for the line P_1P_2

$$\overline{P_2P_1} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$



$$\begin{cases} x = 3 + (-2)t \\ y = -2 + (-3)t \end{cases} \quad t \in \mathbb{R}$$

$\overline{OP} = \overline{OP_1} + t\,\overline{P_2P_1}$

where $\overline{OP_1}\,(3,-2)$

and $\overline{P_2P_1}\,(-2,-3)$

and $t \in \mathbb{R}$

5. Show that the equations $x = 3 - t$, $y = 1 + 2t$ and $x = -1 + 3t$, $y = 9 - 6t$. represent the same line. Write down a director vector for this line.

Method I

$d_1 : \begin{cases} x = 3 - t_1 \\ y = 1 + 2t_1 \end{cases}$ $\qquad$ $d_2 : \begin{cases} x = -1 + 3t_2 \\ y = 9 - 6t_2 \end{cases}$ $\qquad$ $d_1 \overset{?}{=} d_2$

$\downarrow$

$(-1,9)$ is a point on d_2

$(-1,9) \in d_1 \Leftrightarrow \begin{cases} -1 = 3 - t_1 \\ 9 = 1 + 2t_1 \end{cases}$ has a solution. $\longrightarrow t_1 = 4$ $\longrightarrow t_1 = \frac{8}{2}$ ✓

$\Rightarrow$ yes $(-1,9) \in d_1$

do the same with $(3,1)$ to see that $(3,1) \in d_2$

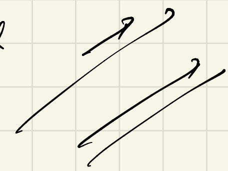
Method II

d_1 has director $v_1(-1,2)$

d_2 — " — $v_2(3,-6)$

are proportional $\Leftrightarrow$ are parallel $\Leftrightarrow$ $d_1 \parallel d_2$

if yes the d_1 and d_2 have two distinct pts in common $\Rightarrow d_1 = d_2$



notice that they have a point in common (eg. $(-1,9)$) $\Rightarrow d_1 = d_2$

Method III

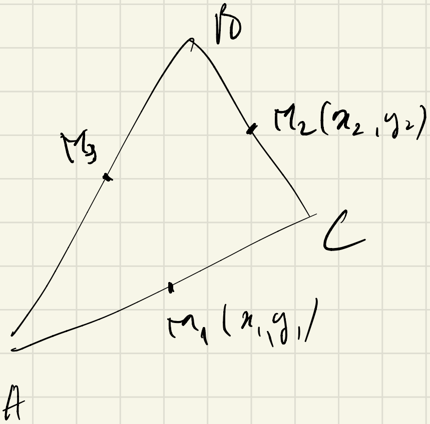
- find the general eq for d_1
- — " — d_2
- notice that they are equal up to a scalar

via the symm. eq.

$$d_1 : \frac{x-3}{-1} = \frac{y-1}{2} \iff \dots$$

$$d_2 : \frac{x+1}{3} = \frac{y-9}{-6} \iff \dots$$

6. The points $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ are the midpoints of the sides of a triangle. Write down the equations of the lines determined by the sides of the triangle.



Method I AB: $\begin{cases} x = \boxed{5} + t \cdot \boxed{2} \\ y = \boxed{-1} + t \cdot \boxed{2} \end{cases}$

$$P = M_3 + t\,\overline{M_1M_2}$$

AC, BC similar

Method II AB: $y = mx + k$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2}{2} = 1$$

AB: $y = x + k \ni M_3(5,-1)$ $\quad -1 = 5 + k \Rightarrow k = -6$

$\Rightarrow$ AB: $y = x - 6$

Method III

$$\left.\begin{array}{l} (1,2) = M_1 = \left(\frac{x_A + x_C}{2}, \frac{y_A + y_C}{2}\right) \\ (3,4) = M_2 = (\dots \\ (5,-1) = M_3 = (\dots \end{array}\right\} \Rightarrow \text{coords of } A, B, C$$

7. Given the line $d: 2x+3y+4=0$, find the equation of a line d_1 passing through the point $M_0(2,1)$, in the following situations: a) d_1 is parallel with d; b) d_1 is orthogonal on d; c) the angle determined by d and d_1 is $\pi/4$.

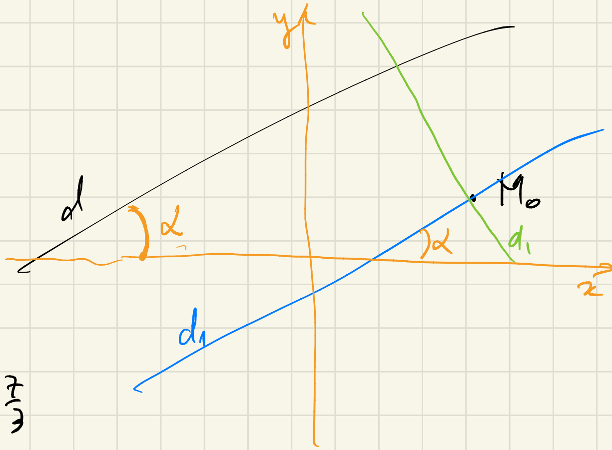
a.) $d: y = -\frac{2}{3}x - \frac{4}{3}$

$d_1 \parallel d \Rightarrow d_1: y = -\frac{2}{3}x + k$

$M_0(2,1) \in d_1 \Rightarrow 1 = -\frac{2}{3}\cdot 2 + k$

$\Rightarrow k = \frac{7}{3}$

$\Rightarrow d_1: y = -\frac{2}{3}x + \frac{7}{3}$



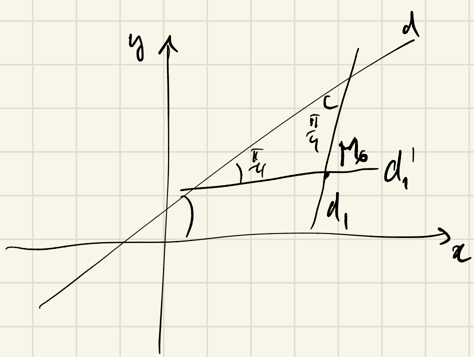
b.) $d_1: y = mx + k$

$d_1 \perp d \Rightarrow m = -(\text{slope of } d)^{-1} = -\left(\frac{-2}{3}\right)^{-1} = \frac{3}{2} = \tan\alpha$

$\Rightarrow d_1: y = \frac{3}{2}x + k$

$d_1 \ni M_0 \Rightarrow 1 = \frac{3}{2}\cdot 2 + k \to k \ldots$

c.)



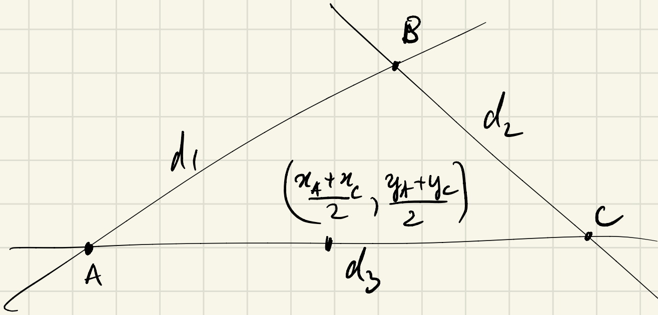
the slope of d_1 is $\tan\left(\alpha + \frac{\pi}{4}\right)$

— " — d_1' is $\tan\left(\alpha - \frac{\pi}{4}\right)$

$$\tan\left(\alpha \pm \frac{\pi}{4}\right) = \frac{\tan\alpha \pm \tan\frac{\pi}{4}}{1 \mp \underbrace{\tan\alpha}_{= -\frac{2}{3}} \underbrace{\tan\frac{\pi}{4}}_{=1}}$$

$\tan\frac{\pi}{4} = 1$

8. The vertices of the triangle $\triangle ABC$ are the intersection points of the lines $d_1 : 4x+3y-5 = 0$, $d_2 : x-3y+10 = 0$, $d_3 : x-2 = 0$. a) Find the coordinates of A, B and C. b) Find the equations of the median lines of the triangle. c) Find the equations of the heights of the triangle.



a.) coords of $A(x_A, y_A)$

$\{A\} = d_1 \cap d_3$

$A \in d_1 \Rightarrow (x_A, y_A)$ satisfy the eq $4x+3y-5=0$

$A \in d_3 \Rightarrow (x_A, y_A)$ is a sol to $x-2=0$ $\Big\} \Rightarrow$ A is the sol to the sys

$$\begin{cases} 4x+3y-5=0 \\ x-2=0 \end{cases}$$

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