ANALYTIC GEOMETRY, PROBLEM SET 5

Various problems with vectors

1. Given the vectors $\overline{a}(3, -1, -2)$ and $\overline{b}(1, 2, -1)$. Compute

$$\overline{a} \times \overline{b}, (2\overline{a} + \overline{b}) \times \overline{b} \text{ and } (2\overline{a} + \overline{b}) \times (2\overline{a} - \overline{b}).$$

2. Find the distances between the opposite sides of the parallelogram constructed on $\overrightarrow{AB}(6, 0, 2)$ şi $\overrightarrow{AC}(1.5, 2, 1)$.

3. Find the vector $\overline{p}$, knowing that $\overline{p}$ is perpendicular on $\overline{a}(2, 3, -1)$ and $\overline{b}(1, -1, 3)$ and its dot product with $\overline{c}(2, -3, 4)$ is equal to 51.

4. Given the points $A(1, -1, 2)$, $B(5, -6, 2)$ and $C(1, 3, -1)$, find the length of the altitude from the vertex B in the triangle $\triangle ABC$.

5. Given the vectors $\overline{a}(2, -3, 1)$, $\overline{b}(-3, 1, 2)$ and $\overline{c}(1, 2, 3)$, compute $(\overline{a} \times \overline{b}) \times \overline{c}$ and $\overline{a} \times (\overline{b} \times \overline{c})$.

6. Let $ABCD$ be a convex quadrilateral. Show that if the diagonal AC passes through the midpoint of the diagonal BD, then the triangles ACB and ACD have equal areas.

7. Prove that the points $A(1, 2, -1)$, $B(0, 1, 5)$, $C(-1, 2, 1)$ and $D(2, 1, 3)$ are situated in the same plane.

8. Find the volume of the tetrahedron which has $A(2, -1, 1)$, $B(5, 5, 4)$, $C(3, 2, 1)$ and $D(4, 1, 3)$ as vertices.

$\hookrightarrow Vol_{ABCD} = \frac{1}{6} \left|(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})\right|$

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be coplanar representatives of vectors with modulus 1 and such that A, B, C are on the same side of a line that passes through O. Show that $||\overline{OA}+\overline{OB}+\overline{OC}|| \geq 1$.

10. Let $ABCD$ be a quadrilateral and E, F the midpoints of $[AB]$ and $[CD]$. Denote by K, L, M and N the midpoints of the segments $[AF], [CE], [BF]$ and $[DE]$, respectively. Prove that $KLMN$ is a parallelogram.

Date: October 23, 2021.

I expect you are able to prove equalities as the ones below. Have a go at them!

12. Let $\overline{a}, \overline{b}, \overline{c}$ be vectors in $\mathcal{V}_3$. Prove the following formulae:

1. $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{a} \cdot \overline{b}) \cdot \overline{c} = \begin{vmatrix} \overline{b} & \overline{c} \\ \overline{a} \cdot \overline{b} & \overline{a} \cdot \overline{c} \end{vmatrix}$;
2. $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{b} \cdot \overline{c}) \cdot \overline{a} = \begin{vmatrix} \overline{b} & \overline{a} \\ \overline{b} \cdot \overline{c} & \overline{a} \cdot \overline{c} \end{vmatrix}$.
3. $(\overline{a} \times \overline{b}) \cdot (\overline{c} \times \overline{d}) = \begin{vmatrix} \overline{a} \cdot \overline{c} & \overline{a} \cdot \overline{d} \\ \overline{b} \cdot \overline{c} & \overline{b} \cdot \overline{d} \end{vmatrix}$; $\leftarrow$ $a(a_1, a_2, a_3)$, $b(b_1, b_2, b_3)$, $c(c_1, c_2, c_3)$, $d(d_1, d_2, d_3)$
4. $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = (\overline{a}, \overline{c}, \overline{d}) \cdot \overline{b} - (\overline{b}, \overline{c}, \overline{d}) \cdot \overline{a} = (\overline{a}, \overline{b}, \overline{d}) \cdot \overline{c} - (\overline{a}, \overline{b}, \overline{c}) \cdot \overline{d}$;
5. $(\overline{a} \times \overline{b}, \overline{b} \times \overline{c}, \overline{c} \times \overline{a}) = (\overline{a}, \overline{b}, \overline{c})^2$

$\overline{c}$ in 2.

1. Given the vectors $\bar{a}(3,-1,-2)$ and $\bar{b}(1,2,-1)$. Compute
$$\bar{a}\times\bar{b},\ (2\bar{a}+\bar{b})\times\bar{b} \text{ and } (2\bar{a}+\bar{b})\times(2\bar{a}-\bar{b}).$$

$$\bar{a}\times\bar{b} = (3i - j - 2k)\times(i + 2j - k)$$
$$= 3\underbrace{\cancel{i\times i}}_{0} + 6\underbrace{i\times j}_{k} + \dots$$

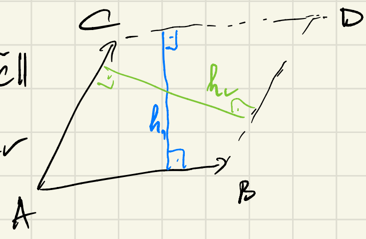
$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 5i + j + 7k$$

$$(2\bar{a}+\bar{b})\times\bar{b} = 2\bar{a}\times\bar{b} + \underbrace{\cancel{\bar{b}\times\bar{b}}}_{0} = 2\,\bar{a}\times\bar{b} = (10, 2, 14)$$

$$(2\bar{a}+\bar{b})\times(2\bar{a}-\bar{b}) = 4\underbrace{\cancel{\bar{a}\times\bar{a}}}_{0} - 2\bar{a}\times\bar{b} + 2\underbrace{\bar{b}\times\bar{a}}_{=\,-\bar{a}\times\bar{b}} + \underbrace{\cancel{\bar{b}\times\bar{b}}}_{0}$$
$$= -4\bar{a}\times\bar{b} = \dots$$

2. Find the distances between the opposite sides of the parallelogram constructed on $\overrightarrow{AB}(6,0,2)$ şi $\overrightarrow{AC}(1.5,2,1)$.

$$\text{Area}_{ABCD} = \|\overline{AB}\times\overline{AC}\| = h_1\cdot\|\overline{AB}\| = h_2\|\overline{AC}\|$$
$$\Rightarrow h_1 = \frac{\|\overline{AB}\times\overline{AC}\|}{\|\overline{AB}\|} \qquad h_2 = \text{similar}$$



3. Find the vector $\overline{p}$, knowing that $\overline{p}$ is perpendicular on $\overline{a}(2,3,-1)$ and $\overline{b}(1,-1,3)$ and its dot product with $\overline{c}(2,-3,4)$ is equal to 51.

1) $\left.\begin{array}{l}\bar p \perp \bar a \\ \bar p \perp \bar b\end{array}\right\}$ $\bar p \parallel \bar a \times \bar b \Rightarrow \bar a \times \bar b = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = i8 - j7 - 5k$

$\bar p \cdot \bar c = 51$

$\Rightarrow \bar p(\lambda 8, -\lambda 7, -5\lambda)$

$\Rightarrow 16\lambda + 21\lambda - 20\lambda = 51 \rightsquigarrow \lambda = \dots \Rightarrow \bar p$

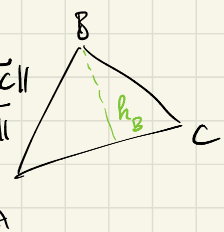
2.) $\bar p(x,y,z)$

$$\begin{array}{l} \bar p \cdot \bar a = 0 \iff \\ \bar p \cdot \bar b = 0 \iff \\ \bar p \cdot \bar c = 51 \iff \end{array} \left| \begin{array}{l} 2x + 3y - z = 0 \\ x - y + 3z = 0 \\ 2x - 3y + 4z = 0 \end{array} \right.$$

← solve this ....

4. Given the points $A(1,-1,2)$, $B(5,-6,2)$ and $C(1,3,-1)$, find the length of the altitude from the vertex B in the triangle $\triangle ABC$.

$$\text{area}_{\Delta ABC} = \frac{\|\overline{AB} \times \overline{BC}\|}{2} = \frac{h_B \cdot \|\overline{AC}\|}{2} \Rightarrow h_B = \frac{\|\overline{AB} \times \overline{BC}\|}{\|\overline{AC}\|}$$



5. Given the vectors $\overline{a}(2,-3,1)$, $\overline{b}(-3,1,2)$ and $\overline{c}(1,2,3)$, compute $(\overline{a}\times\overline{b})\times\overline{c}$ and $\overline{a}\times(\overline{b}\times\overline{c})$.

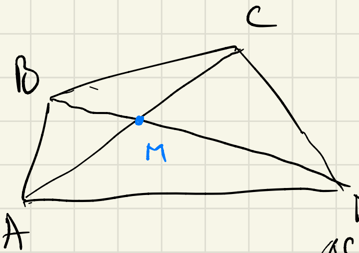
$$(\bar a \times \bar b) \times \bar c = \underbrace{(\bar a \cdot \bar c)}_{1}\bar b - \underbrace{(\bar b \cdot \bar c)}_{5} \cdot \bar a$$

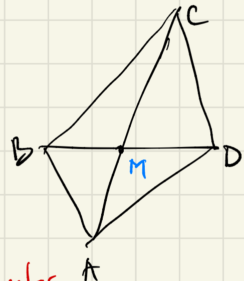
$\boxed{(\overline{u}\times\overline{v})\times\overline{w} = (\overline{u}\cdot\overline{w})\overline{v} - (\overline{v}\cdot\overline{w})\overline{u}}$.

6. Let $ABCD$ be a convex quadrilateral. Show that if the diagonal AC passes through the midpoint of the diagonal BD, then the triangles ACB and ACD have equal areas.

if M is the midpoint of BD

then $\text{area}_{\Delta ACB} \overset{(*)}{=} \text{area}_{\Delta ACD}$





$$(*) \iff \tfrac{1}{2}\|\overline{AC}\times\overline{AB}\| \overset{?}{=} \tfrac{1}{2}\|\overline{AC}\times\overline{AD}\|$$

$$(*) \Leftarrow \quad \overline{AC}\times\overline{AB} \overset{?}{=} -\overline{AC}\times\overline{AD}$$

$$\Updownarrow$$

$$\overline{AC}\times(\underbrace{\overline{AB}+\overline{AD}}_{2\overline{AM}}) = 0$$

$\Updownarrow$ because $\mathbb{V}$ is a real vector space

set of all vectors

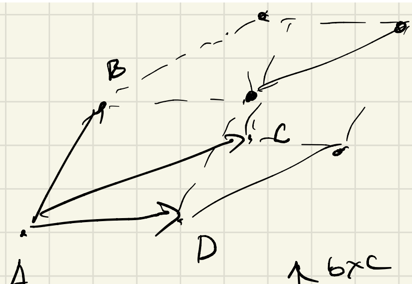
$$\overline{AC}\times\overline{AM} = 0$$ this is true since $\overline{AC} \parallel \overline{AM}$ since A, M, C colinear

rem $\quad 2\bar{v}=0 \iff \bar{v}=0$

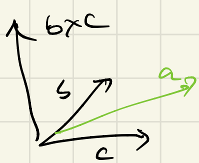
7. Prove that the points $A(1, 2, -1)$, $B(0, 1, 5)$, $C(-1, 2, 1)$ and $D(2, 1, 3)$ are situated in the same plane.

$\Updownarrow$ (no volume)

$$(\overline{AB}, \overline{AC}, \overline{AD}) = 0$$



$$(\bar{a}, \bar{b}, \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$



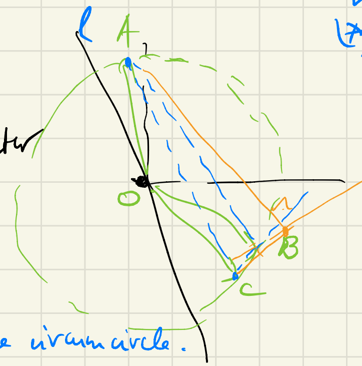
$\bar{a}\cdot(\bar{b}\times\bar{c}) = 0 \iff \bar{a} \perp \bar{b}\times\bar{c}$

$\bar{b}\times\bar{c} \perp \bar{b}, \bar{c}$

9. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be coplanar representatives of vectors with modulus 1 and such that A, B, C are on the same side of a line that passes through O. Show that $\underbrace{\|\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\| \geq 1}_{(*)}$.

Method 1. $\overline{OA}+\overline{OB}+\overline{OC}=\overline{OH}$

$\uparrow$ orthocenter



A, B, C on the same side of the line l

$\hookrightarrow$ $\sphericalangle B$ obtuse

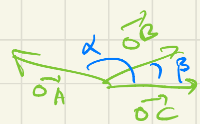
$\Rightarrow$ H lies outside of the circumcircle.

Method 2. $(*) \iff \|\overline{OA}+\overline{OB}+\overline{OC}\|^2 \geq 1$

A, B, C on the same side of l $\Rightarrow$ one of the angles $\overline{OA},\overline{OB}$ or $\overline{OA},\overline{OC}$ or $\overline{OB},\overline{OC}$ is acute

we may assume this is $\overline{OB},\overline{OC}$

$\downarrow$



choose the Ox-axis along on $\overline{OC}$

$$\overline{OA} = (\cos\alpha, \sin\alpha)$$
$$\overline{OB} = (\cos\beta, \sin\beta) \quad \text{for some } \alpha, \beta$$
$$\overline{OC} = (1, 0)$$

$+$

...

$$\|\overrightarrow{OA}+\overline{OB}+\overline{OC}\|^2 = (\cos\alpha+\cos\beta+1)^2 + (\sin\alpha+\sin\beta)^2$$

$$= 3 + \overbrace{2\cos\alpha\cos\beta} + 2\cos\alpha + \overbrace{2\cos\beta}$$
$$\underbrace{+2\sin\alpha\cdot\sin\beta}_{\geq 0}$$

$$= 3 + 2\underbrace{(\cos\alpha+1)}_{\geq 0}\underbrace{\cos\beta}_{\geq 0} + \underbrace{2\cos\alpha}_{\geq -2} + \underbrace{2\sin\alpha\cdot\sin\beta}_{\geq 0}$$

$\geqslant -1$