M4GB: An Efficient Gröbner Basis Algorithm

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Problem

- $\mathbb{F}$ - A Field
- n - number of variables
- m - number of equations

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Problem (Multivariate Quadratic(MQ) problem)

- *Given :* $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, $\deg(f_i) = 2$
- *Problem : Find a* $(v_1, \ldots, v_n) \in \mathbb{F}^n$ *such that*

$$\begin{aligned} f_1(v_1, \ldots, v_n) &= 0 \\ &\vdots \\ f_m(v_1, \ldots, v_n) &= 0 \end{aligned}$$

Why MQ Problem ?

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Open Public Challenge - MQChallenge

- Initiated at 2015
- Random and dense system
- Various parameters

MQ Challenge Types

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$			
$m \approx 2/3n$			

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	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
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Parameter Choice

Require at least one month for Magma 2.19-9 to solve using Four 6-cores Intel(R) Xeon(R) CPU E5-4617 @ 2.9GHz and 1TB of RAM.

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	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
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`https://www.mqchallenge.org`

Solving MQ-problem

- Linearization

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This talk

Gröbner basis

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Definition (Monomial Ordering)

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2. Let $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$ and $x = (x_1, \ldots, x_n)$

$$x^\alpha > x^\beta \Rightarrow x^\gamma x^\alpha > x^\gamma x^\beta$$

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$$x^\alpha > x^\beta \Rightarrow x^\gamma x^\alpha > x^\gamma x^\beta$$

3. $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$

Monomial Ordering : Examples

Lexicographic

$$x^{\alpha} >_{\text{lex}} x^{\beta} \Leftrightarrow \text{the leftmost nonzero entry of } \alpha - \beta \text{ is positive}$$

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- $\sum_i \alpha_i > \sum_i \beta_i$ OR
- $\sum_i \alpha_i = \sum_i \beta_i$ and the rightmost nonzero entry of $\alpha - \beta$ is negative

Notations

$\mathbb{F}[x_1, \ldots, x_n]$ together with $>$

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$$F \subseteq \mathbb{F}[x_1, \ldots, x_n] \begin{cases} \mathrm{Tail}(F) = \cup_{f \in F}\{\mathrm{Tail}(f)\} \\ \mathrm{Term}(F) = \cup_{f \in F}\mathrm{Term}(f) \\ \mathrm{Mono}(F) = \cup_{f \in F}\mathrm{Mono}(f) \end{cases}$$

Gröbner Basis : Definition

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$G \subseteq I$, $|G| < \infty$ that generates I is a Gröbner basis of I if,

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Definition

$I \neq \{0\}$ be an ideal of $\mathbb{F}[x_1, \ldots, x_n]$
$G \subseteq I$, $|G| < \infty$ that generates I is a Gröbner basis of I if,

$$\text{for any } f \in I, \exists g \in G \text{ s.t. } \mathrm{LT}(g) \mid \mathrm{LT}(f)$$

Gröbner Basis and Solving System of Equations

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Lexicographic Ordering

$$\begin{aligned}
&g_1(x_1), \ldots, \\
&g_2(x_1, x_2), \ldots, g_{k_1}(x_1, x_2) \\
&g_{k_1+1}(x_1, x_2, x_3), \ldots, \\
&g_{k_n}(x_1, \ldots, x_n)
\end{aligned}$$

Gröbner Basis and Solving System of Equations

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&g_1(x_1), \ldots, \\
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&g_{k_1+1}(x_1, x_2, x_3), \ldots, \\
&g_{k_n}(x_1, \ldots, x_n)
\end{aligned}
$$

Unique Solution in the Base Field

$$
\begin{aligned}
g_1 &= x_1 + c_1, \\
&\vdots \\
g_n &= x_n + c_n
\end{aligned}
$$

Question

Given $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, how to compute the Gröbner basis G of $\langle f_1, \ldots, f_m \rangle$?

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Given $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$*, how to compute the Gröbner basis G of* $\langle f_1, \ldots, f_m \rangle$ *?*

How to obtain new element in the ideal ?

Question

Given $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, how to compute the Gröbner basis G of $\langle f_1, \ldots, f_m \rangle$?

How to obtain new element in the ideal ?

1. Generate polynomials of higher degree

Question

Given $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, *how to compute the Gröbner basis* G *of* $\langle f_1, \ldots, f_m \rangle$ *?*

How to obtain new element in the ideal ?

1. Generate polynomials of higher degree
2. Division algorithm in $\mathbb{F}[x_1, \ldots, x_n]$ (computing remainder)

Generate higher degree polynomial : S-Polynomial

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- $x^\gamma = \mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))$

Generate higher degree polynomial : S-Polynomial

- $f, g \in \mathbb{F}[x_1, \ldots, x_n]$ with $f \neq 0, g \neq 0$
- $x^\gamma = \mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))$

Definition

$$\mathrm{Spoly}(f,g) = \frac{x^\gamma}{\mathrm{LT}(f)} \cdot f - \frac{x^\gamma}{\mathrm{LT}(g)} \cdot g.$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$xy + z \,\big)\, \overline{x^2y^2 + z^4 + xy}$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy \\
\hline
xy + z & x^2y^2 + z^4 + xy
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy \\
\hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\hline
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & \quad xy \\
\cline{2-2}
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\cline{2-2}
 & z^4 - xyz + xy
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy \qquad\qquad\qquad\qquad r \\
\hline
xy + z & x^2y^2 + z^4 + xy \\
 & \underline{x^2y^2 + xyz} \\
 & z^4 - xyz + xy
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy \\
\cline{2-2}
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\cline{2-2}
 & z^4 - xyz + xy
\end{array}
\qquad\qquad
\begin{array}{c}
r \\
 \\
 \\
z^4
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy \\
\cline{2-2}
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\cline{2-2}
 & z^4 - xyz + xy \\
 & z^4 \\
\cline{2-2}
\end{array}
\qquad
\begin{array}{c}
r \\
\\
\\
\\
z^4
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy \\
\hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\hline
 & z^4 - xyz + xy \\
 & z^4 \\
\hline
 & -xyz + xy \\
\hline
\\
\hline
\\
\hline
\\
\hline
\end{array}
\qquad
\begin{array}{c}
r \\
\\
\\
z^4
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy - z \\
\hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\hline
 & z^4 - xyz + xy \\
 & z^4 \\
\hline
 & -xyz + xy \\
 & \\
\hline
 & \\
\hline
 & \\
\hline
 & \\
\hline
\end{array}
\qquad
\begin{array}{c}
r \\
\\
\\
z^4
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Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy - z \\ \hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\ \hline
 & z^4 - xyz + xy \\
 & z^4 \\ \hline
 & -xyz + xy \\
 & -xyz - z^2 \\ \hline
 & \\ \hline
 & \\ \hline
 & \\ \hline
\end{array}
\qquad
\begin{array}{c}
r \\
\\
\\
z^4
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l l}
 & xy - z & \quad r \\
\hline
xy + z & x^2y^2 + z^4 + xy & \\
 & x^2y^2 + xyz & \\
\hline
 & z^4 - xyz + xy & \quad z^4 \\
 & z^4 & \\
\hline
 & -xyz + xy & \\
 & -xyz - z^2 & \\
\hline
 & xy + z^2 &
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy - z + 1 \\
\hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\hline
 & z^4 - xyz + xy \\
 & z^4 \\
\hline
 & -xyz + xy \\
 & -xyz - z^2 \\
\hline
 & xy + z^2 \\
 & \\
\hline
 & \\
 & \\
\hline
 & \\
 & \\
\hline
\end{array}
\qquad
\begin{array}{c}
r \\
\\
\\
z^4
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l l}
 & xy - z + 1 & \qquad r \\
\hline
xy + z & x^2y^2 + z^4 + xy & \\
 & x^2y^2 + xyz & \\
\hline
 & z^4 - xyz + xy & \qquad z^4 \\
 & z^4 & \\
\hline
 & -xyz + xy & \\
 & -xyz - z^2 & \\
\hline
 & xy + z^2 & \\
 & xy + z & \\
\hline
 & & \\
\hline
 & & \\
\hline
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy - z + 1 \qquad\qquad r \\
\hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\
\hline
 & z^4 - xyz + xy \qquad\qquad z^4 \\
 & z^4 \\
\hline
 & -xyz + xy \\
 & -xyz - z^2 \\
\hline
 & xy + z^2 \\
 & xy + z \\
\hline
 & z^2 - z \\
 & \\
\hline
 & \\
\hline
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy - z + 1 \\ \hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\ \hline
 & z^4 - xyz + xy \\
 & z^4 \\ \hline
 & -xyz + xy \\
 & -xyz - z^2 \\ \hline
 & xy + z^2 \\
 & xy + z \\ \hline
 & z^2 - z \\
 & \\ \hline
 & \\ \hline
\end{array}
\qquad
\begin{array}{l}
r \\
\\
\\
z^4 \\
\\
\\
\\
\\
\\
z^4 + z^2
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l}
 & xy - z + 1 \\ \hline
xy + z & x^2y^2 + z^4 + xy \\
 & x^2y^2 + xyz \\ \hline
 & z^4 - xyz + xy \\
 & z^4 \\ \hline
 & -xyz + xy \\
 & -xyz - z^2 \\ \hline
 & xy + z^2 \\
 & xy + z \\ \hline
 & z^2 - z \\
 & z^2 \\ \hline
 & \\ \hline
\end{array}
\qquad
\begin{array}{c}
r \\
\\
\\
z^4 \\
\\
\\
\\
\\
\\
z^4 + z^2 \\
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l l}
 & xy - z + 1 & r \\
\hline
xy + z & x^2y^2 + z^4 + xy & \\
 & x^2y^2 + xyz & \\
\hline
 & z^4 - xyz + xy & z^4 \\
 & z^4 & \\
\hline
 & -xyz + xy & \\
 & -xyz - z^2 & \\
\hline
 & xy + z^2 & \\
 & xy + z & \\
\hline
 & z^2 - z & z^4 + z^2 \\
 & z^2 & \\
\hline
 & -z & \\
\hline
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l l}
 & \quad xy - z + 1 & \qquad r \\
\hline
xy + z & x^2y^2 + z^4 + xy & \\
 & x^2y^2 + xyz & \\
\hline
 & z^4 - xyz + xy & \qquad z^4 \\
 & z^4 & \\
\hline
 & -xyz + xy & \\
 & -xyz - z^2 & \\
\hline
 & xy + z^2 & \\
 & xy + z & \\
\hline
 & z^2 - z & \qquad z^4 + z^2 \\
 & z^2 & \\
\hline
 & -z & \qquad z^4 + z^2 - z \\
 & & \\
\hline
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{r|l l}
\multicolumn{1}{r}{} & xy - z + 1 & r \\
\cline{2-2}
xy + z & x^2y^2 + z^4 + xy & \\
\multicolumn{1}{r}{} & x^2y^2 + xyz & \\
\cline{2-2}
\multicolumn{1}{r}{} & z^4 - xyz + xy & z^4 \\
\multicolumn{1}{r}{} & z^4 & \\
\cline{2-2}
\multicolumn{1}{r}{} & -xyz + xy & \\
\multicolumn{1}{r}{} & -xyz - z^2 & \\
\cline{2-2}
\multicolumn{1}{r}{} & xy + z^2 & \\
\multicolumn{1}{r}{} & xy + z & \\
\cline{2-2}
\multicolumn{1}{r}{} & z^2 - z & z^4 + z^2 \\
\multicolumn{1}{r}{} & z^2 & \\
\cline{2-2}
\multicolumn{1}{r}{} & -z & z^4 + z^2 - z \\
\multicolumn{1}{r}{} & -z & \\
\cline{2-2}
\end{array}
$$

Division Algorithm in $\mathbb{F}[x_1, \ldots, x_n]$: Example

$$
\begin{array}{rl@{\qquad\qquad}l}
 & xy - z + 1 & r \\ \cline{2-2}
xy + z \;\Big| & x^2y^2 + z^4 + xy & \\
 & x^2y^2 + xyz & \\ \cline{2-2}
 & z^4 - xyz + xy & z^4 \\
 & z^4 & \\ \cline{2-2}
 & -xyz + xy & \\
 & -xyz - z^2 & \\ \cline{2-2}
 & xy + z^2 & \\
 & xy + z & \\ \cline{2-2}
 & z^2 - z & z^4 + z^2 \\
 & z^2 & \\ \cline{2-2}
 & -z & z^4 + z^2 - z \\
 & -z & \\ \cline{2-2}
 & 0 &
\end{array}
$$

Division Algorithm / Polynomial Reduction

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Theorem (Division Algorithm)

Every $f \in \mathbb{F}[x_1, \ldots, x_n]$ *can be written as*

$$f = q_1 g_1 + \ldots + q_t g_t + r$$

and

$$r = \begin{cases} 0 \\ \textit{none of terms of } r \textit{ is divisible by any of } \mathrm{LT}(g_i) \end{cases}$$

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$$r \leftarrow \textsc{FullReduce}(f, G)$$

Buchberger's Algorithm

Input: A finite ordered subset $F \subseteq \mathbb{F}[x_1, \ldots, x_n]$
Result: A Gröbner basis G such that $\langle G \rangle = \langle F \rangle$

$G \leftarrow F$
repeat
    $G' \leftarrow G$
    forall $(p, q) \in G' \times G'$, $p \neq q$ **do**
        $r \leftarrow \textsc{FullReduce}(\mathrm{Spoly}(p, q), G')$
        if $r \neq 0$ **then**
            $G \leftarrow G \cup \{r\}$
until $G = G'$;
return G

Table of Contents

- Gröbner bases algorithms follows the same principle

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- Differs in reduction techniques

Example : Standard Reduction vs M4GB Reduction

$\mathbb{F}_2[x_1, x_2, x_3, x_4]$ with *degrevlex* ordering

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M4GB Main Strategy

Maintain tail-reduced polynomials

M4GB Reduction

$\textsc{MulFullReduce}(G, u, f)$

$r \leftarrow 0$
forall $t \in \mathrm{Term}(f)$ **do**
  $t' \leftarrow u \cdot t$
  if $\exists g \in G : \mathrm{LT}(g) \mid t'$ **then**
    $(G, g) \leftarrow \textsc{GetReductor}(G, t')$
    $r \leftarrow r - (t'/\mathrm{LT}(g)) \cdot \mathrm{Tail}(g)$
  else
    $r \leftarrow r + t'$
return (G, r)

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    $r \leftarrow r + t'$
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$\textsc{GetReductor}(G, t)$

if $\exists g \in G : \mathrm{LM}(g) = \mathrm{LM}(t)$ **then**
  return (G, g)
$h \leftarrow \textsc{SelectReductor}(G, t)$
$(G, h) \leftarrow$
  $\textsc{MulFullReduce}(G, t/\mathrm{LT}(h), \mathrm{Tail}(h))$
$g \leftarrow t + h$
return $(G \cup \{g\}, g)$

$$\textsc{UpdateReduce}(G, f)$$

$H \leftarrow \{\mathsf{LC}(f)^{-1} \cdot f\}$
$Q \leftarrow \mathsf{Mono}(\mathsf{Tail}(G \cup H)) \setminus \mathsf{LM}(H)$

while $\exists u \in Q : \mathsf{LM}(f) \mid u$ **do**
  $u \leftarrow \max\{\mathfrak{m} \in Q : \mathsf{LM}(f) \mid \mathfrak{m}\}$
  $(G, h) \leftarrow \textsc{MulFullReduce}(G, u/\mathsf{LT}(f), \mathsf{Tail}(f))$
  $H \leftarrow H \cup \{u + h\}$
  $Q \leftarrow \mathsf{Mono}(\mathsf{Tail}(G \cup H)) \setminus \mathsf{LM}(H)$

while $H \neq \{\}$ **do**
  Select $h \in H$ such that $\mathsf{LM}(h) = \min \mathsf{LM}(H)$
  $H \leftarrow H \setminus \{h\}$
  $H \leftarrow \{g - ch : g \in H, c$ is a coefficient of $\mathsf{LM}(h)$ in $\mathsf{Tail}(g)\}$
  $G \leftarrow \{g - ch : g \in G, c$ is a coefficient of $\mathsf{LM}(h)$ in $\mathsf{Tail}(g)\}$
  $G \leftarrow G \cup \{h\}$

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- Implemented using `C++11`

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- Test cases
  1. Dense polynomials with coefficients in $\mathbb{F}_{31}$
  2. $m = 2n$ and $m = n + 1$.

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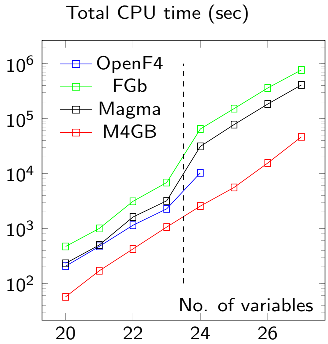
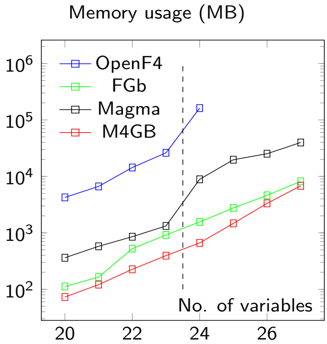
Benchmark for $m = 2n$

		Total CPU time (sec)			
n	m	M4GB	OpenF4	Magma	FGb
20	40	57	206	232	470
21	42	170	472	500	1002
22	44	424	1145	1617	3118
23	46	1060	2274	3185	6849
24	48	2556	10293	31168	64700
25	50	5575	-	77679	151653
26	52	15517	-	183629	360055
27	54	46548	-	409452	767543

		Memory (MB)			
n	m	M4GB	FGb	Magma	OpenF4
20	40	73	112	362	4240
21	42	121	165	577	6640
22	44	226	525	859	14368
23	46	395	918	1324	26135
24	48	663	1561	8873	161945
25	50	1471	2765	19719	-
26	52	3328	4607	25197	-
27	54	6799	8180	39845	-

Graph for $m = 2n$



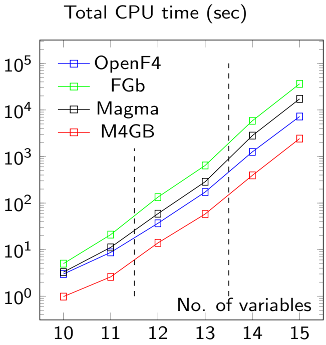


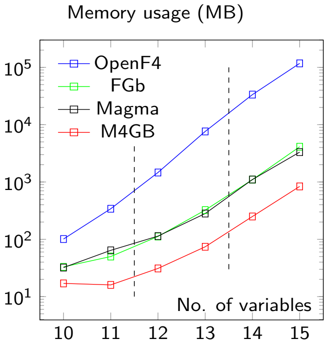
Benchmark for $m = n + 1$

		Total CPU time (sec)			
n	m	M4GB	OpenF4	Magma	FGb
10	11	0.98	2.99	3.29	5
11	12	2.6	8.73	11.172	21
12	13	13.92	36.76	59.08	134
13	14	58.18	172.49	286.4	642
14	15	393.19	1258	2810.75	5850
15	16	2424	7225	17265.5	36361

		Memory (MB)			
n	m	M4GB	FGb	Magma	OpenF4
10	11	17	33	32	101
11	12	16	50	64	341
12	13	31	112	114	1463
13	14	74	323	281	7622
14	15	250	1098	1104	33460
15	16	837	4118	3320	117396

Graph for $m = n + 1$





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Solved MQ Challenges

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
	$n \geq 55$	$n \geq 35$	$n \geq 34$
$m \approx 2/3n$	IV	V	VI
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    4. Generate q^k subsystems

$$\{\tilde{F}(x_1, \ldots, x_{n-k}, v_1, \ldots, v_k) : (v_1, \ldots, v_k) \in \mathbb{F}_q^k\}$$

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  5. Compute Gröbner basis of each subsystem

Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

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VI	27/18	B	1	$\approx$ 31.48 hours
VI	28/19	B	2	$\approx$ 7.61 days

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https://github.com/cr-marcstevens/m4gb

Question ?