M4GB: An Efficient Gröbner Basis Algorithm

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ALGANT-DOC Meeting, 15th May 2017

1 Introduction

2 Gröbner Basis

3 M4GB Algorithm

4 Performance Comparison

5 Solving MQ Challenges

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Problem (Multivariate Quadratic(MQ) problem)

- *Given* : $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, $\deg(f_i) = 2$
- *Problem* : *Find a* $(v_1, \ldots, v_n) \in \mathbb{F}^n$ *such that*

$$\begin{aligned} f_1(v_1, \ldots, v_n) &= 0 \\ &\vdots \\ f_m(v_1, \ldots, v_n) &= 0 \end{aligned}$$

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Open Public Challenge - MQChallenge

- Initiated at 2015
- Random and dense system
- Various parameters

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Parameter Choice

Require at least one month for Magma 2.19-9 to solve using Four 6-cores Intel(R) Xeon(R) CPU E5-4617 @ 2.9GHz and 1TB of RAM.

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`https://www.mqchallenge.org`

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This talk

Gröbner basis

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3. $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$

Monomial Ordering : Examples

Lexicographic

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- $\sum_i \alpha_i > \sum_i \beta_i$ OR
- $\sum_i \alpha_i = \sum_i \beta_i$ and the rightmost nonzero entry of $\alpha - \beta$ is negative

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$$F \subseteq \mathbb{F}[x_1, \ldots, x_n] \begin{cases} \mathrm{Tail}(F) = \cup_{f \in F} \mathrm{Tail}(f) \\ \mathrm{Term}(F) = \cup_{f \in F} \mathrm{Term}(f) \\ \mathrm{Mono}(F) = \cup_{f \in F} \mathrm{Mono}(f) \end{cases}$$

Polynomial Reduction

TODO

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$G \subseteq I$, $|G| < \infty$ that generates I is a Gröbner basis of I if,

$$\text{for any } f \in I, \exists g \in G \text{ s.t. } \mathrm{LT}(g) \mid \mathrm{LT}(f)$$

Gröbner basis and Solving System of Equations

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Lexicographic Ordering

$$
\begin{gathered}
g_1(x_1), \ldots, \\
g_2(x_1, x_2), \ldots, g_{k_1}(x_1, x_2) \\
g_{k_1+1}(x_1, x_2, x_3), \ldots, \\
g_{k_n}(x_1, \ldots, x_n)
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\end{aligned}
$$

Unique Solution in the Base Field

$$
\begin{aligned}
g_1 &= x_1 + c_1, \\
&\vdots \\
g_n &= x_n + c_n
\end{aligned}
$$

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Definition

$$\mathrm{Spoly}(f, g) = \frac{x^\gamma}{\mathrm{LT}(f)} \cdot f - \frac{x^\gamma}{\mathrm{LT}(g)} \cdot g.$$

Buchberger's Algorithm

Input: A finite ordered subset $F \subseteq \mathbb{F}[x_1, \ldots, x_n]$
Result: A Gröbner basis G such that $\langle G \rangle = \langle F \rangle$

$P \leftarrow \{\{p, q\} : \forall p, q \in F$ **and** $p \neq q\}$
$G \leftarrow F$
while $P \neq \{\}$ **do**
  $\{p, q\} \leftarrow \textsc{Select}(P)$
  $P \leftarrow P \setminus \{\{p, q\}\}$
  $r \leftarrow \textsc{FullReduce}(\mathsf{Spoly}(p, q), G)$
  if $r \neq 0$ **then**
    $P \leftarrow P \cup \{\{r, g\} : \forall g \in G\}$
    $G \leftarrow G \cup \{r\}$
return G

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$$r = x_1^3x_4 + x_2x_3^2 + 1$$

1. Maintain tail-reduced polynomials (during reduction and when a new element for the basis is found)
2. Identify polynomial with their leading monomial (i.e. no two polynomials in G that have equal leading monomial)

M4GB Reduction

$\textsc{MulFullReduce}(G, u, f)$

```
r ← 0
forall t ∈ Term(f) do
    t′ ← u · t
    if ∃g ∈ G : LT(g) | t′ then
        (G, g) ← GetReductor(G, t′)
        r ← r − (t′/LT(g)) · Tail(g)
    else
        r ← r + t′
return (G, r)
```

M4GB Reduction

$\textsc{MulFullReduce}(G, u, f)$

1. $r \leftarrow 0$
2. **forall** $t \in \mathrm{Term}(f)$ **do**
    3. $t' \leftarrow u \cdot t$
    4. **if** $\exists g \in G : \mathrm{LT}(g) \mid t'$ **then**
        5. $(G, g) \leftarrow \textsc{GetReductor}(G, t')$
        6. $r \leftarrow r - (t'/\mathrm{LT}(g)) \cdot \mathrm{Tail}(g)$
    7. **else**
        8. $r \leftarrow r + t'$
9. **return** (G, r)

$\textsc{GetReductor}(G, t)$

1. **if** $\exists g \in G : \mathrm{LM}(g) = \mathrm{LM}(t)$ **then**
    2. **return** (G, g)
3. $h \leftarrow \textsc{SelectReductor}(G, t)$
4. $(G, h) \leftarrow \textsc{MulFullReduce}(G, t/\mathrm{LT}(h), \mathrm{Tail}(h))$
5. $g \leftarrow t + h$
6. **return** $(G \cup \{g\}, g)$

$\textsc{UpdateReduce}(G, f)$

$H \leftarrow \{\mathrm{LC}(f)^{-1} \cdot f\}$
$Q \leftarrow \mathrm{Mono}(\mathrm{Tail}(G \cup H)) \setminus \mathrm{LM}(H)$

while $\exists u \in Q : \mathrm{LM}(f) \mid u$ **do**
  $u \leftarrow \max\{\mathfrak{m} \in Q : \mathrm{LM}(f) \mid \mathfrak{m}\}$
  $(G, h) \leftarrow \textsc{MulFullReduce}(G, u/\mathrm{LT}(f), \mathrm{Tail}(f))$
  $H \leftarrow H \cup \{u + h\}$
  $Q \leftarrow \mathrm{Mono}(\mathrm{Tail}(G \cup H)) \setminus \mathrm{LM}(H)$

while $H \neq \{\}$ **do**
  Select $h \in H$ such that $\mathrm{LM}(h) = \min \mathrm{LM}(H)$
  $H \leftarrow H \setminus \{h\}$
  $H \leftarrow \{g - ch : g \in H, c$ is a coefficient of $\mathrm{LM}(h)$ in $\mathrm{Tail}(g)\}$
  $G \leftarrow \{g - ch : g \in G, c$ is a coefficient of $\mathrm{LM}(h)$ in $\mathrm{Tail}(g)\}$
  $G \leftarrow G \cup \{h\}$

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- Comparison with existing implementations
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- Test cases
  1. Dense polynomials with coefficients in $\mathbb{F}_{31}$
  2. $m = 2n$ and $m = n + 1$.

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[2] Available at `https://github.com/nauotit/openf4`

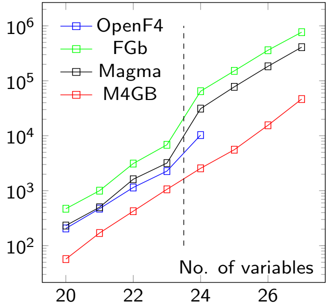
Benchmark for $m = 2n$

		Total CPU time (sec)			
n	m	M4GB	OpenF4	Magma	FGb
20	40	57	206	232	470
21	42	170	472	500	1002
22	44	424	1145	1617	3118
23	46	1060	2274	3185	6849
24	48	2556	10293	31168	64700
25	50	5575	-	77679	151653
26	52	15517	-	183629	360055
27	54	46548	-	409452	767543

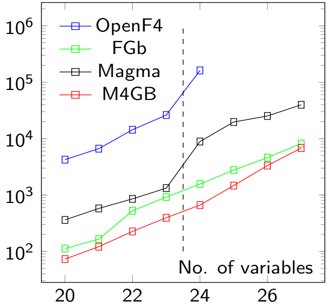
		Memory (MB)			
n	m	M4GB	FGb	Magma	OpenF4
20	40	73	112	362	4240
21	42	121	165	577	6640
22	44	226	525	859	14368
23	46	395	918	1324	26135
24	48	663	1561	8873	161945
25	50	1471	2765	19719	-
26	52	3328	4607	25197	-
27	54	6799	8180	39845	-

Graph for $m = 2n$

Total CPU time (sec)



Memory usage (MB)

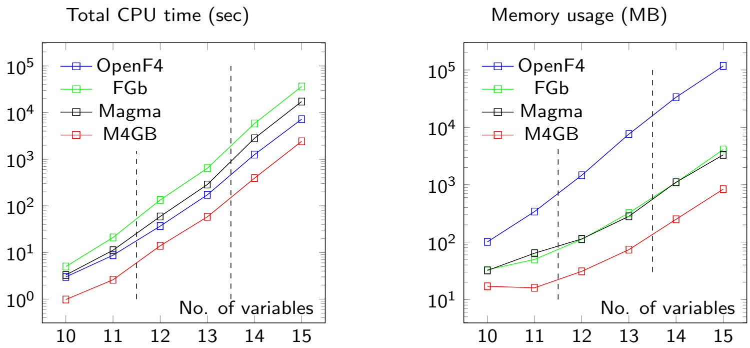


Benchmark for $m = n + 1$

		Total CPU time (sec)			
n	m	M4GB	OpenF4	Magma	FGb
10	11	0.98	2.99	3.29	5
11	12	2.6	8.73	11.172	21
12	13	13.92	36.76	59.08	134
13	14	58.18	172.49	286.4	642
14	15	393.19	1258	2810.75	5850
15	16	2424	7225	17265.5	36361

		Memory (MB)			
n	m	M4GB	FGb	Magma	OpenF4
10	11	17	33	32	101
11	12	16	50	64	341
12	13	31	112	114	1463
13	14	74	323	281	7622
14	15	250	1098	1104	33460
15	16	837	4118	3320	117396

Graph for $m = n + 1$



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Solving Type V and VI of MQ Challenge

- Hybrid approach : trade-off between exhaustive search and computing Gröbner bases
- Idea :
  1. Select a random vector $(a_1, \dots, a_{n-m}) \in \mathbb{F}_q^{n-m}$
  2. Construct a new system with $n = m$

$$\tilde{F} = \{f(x_1, \dots, x_m, a_1, \dots, a_{n-m}) : \forall f \in F\}$$

  3. Select $k \in \{1, \dots, m\}$ and construct q^k subsystems from $\tilde{F}$ by substituting k variables with all elements of $\mathbb{F}_q^k$.
  4. Each subsystem generated can be solved in parallel.

Computational Resources

A) Desktop machine with Intel(R) Core(TM) i7-2600K CPU @ 3.40GHz and 16GB RAM

Computational Resources

A) Desktop machine with Intel(R) Core(TM) i7-2600K CPU @ 3.40GHz and 16GB RAM

B) NUMA machine with two nodes of Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz and 128GB RAM each.

Solved Challenges

Type	n/m	Machine Used	# Node	Duration

Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16			
V	25/17			
V	27/18			

Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17			
V	27/18			

Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17	B	1	$\approx$ 46.33 hours
V	27/18	B	2	$\approx$ 10.9 days

Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17	B	1	$\approx$ 46.33 hours
V	27/18	B	2	$\approx$ 10.9 days
VI	24/16			
VI	25/17			
VI	27/18			
VI	28/19			

Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17	B	1	$\approx$ 46.33 hours
V	27/18	B	2	$\approx$ 10.9 days
VI	24/16	A	1	$\approx$ 1.2 hours
VI	25/17			
VI	27/18			
VI	28/19			

Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17	B	1	$\approx$ 46.33 hours
V	27/18	B	2	$\approx$ 10.9 days
VI	24/16	A	1	$\approx$ 1.2 hours
VI	25/17	B	1	$\approx$ 9.87 hours
VI	27/18	B	1	$\approx$ 31.48 hours
VI	28/19	B	2	$\approx$ 7.61 days

https://github.com/cr-marcstevens/m4gb

Question ?