M4GB: An Efficient Gröbner Bases Algorithm

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ISSAC 2017 – Kaiserslautern, 27th July 2017

Motivation

Multivariate Quadratic (MQ) Problem

Given quadratic polynomials $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, find a $v = (v_1, \ldots, v_n) \in \mathbb{F}^n$ such that $f_i(v_1, \ldots, v_n) = 0$ for all $i \in \{1, \ldots, m\}$

1. Gröbner Bases Algorithms
2. M4GB Algorithm
3. Fukuoka MQ Challenges
4. Implementation Results
5. Solving MQ Challenges

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Buchberger's Algorithm (Buchberger, 1965)

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Output : A Gröbner basis G of $\langle f_1, \ldots, f_m \rangle$

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    $G' \leftarrow G$
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        $r \leftarrow \textsc{FullReduce}(\mathrm{Spoly}(h_1, h_2), G')$
        **if** $r \neq 0$
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Improvement

Detect zero reduction in advance – Buchberger's 1st and 2nd criterion (e.g. using Gebauer-Möller installation)

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Improvement of F_4 : SIMPLIFY function

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- Requires all previously constructed intermediate matrices and their corresponding (reduced) row echelon form
- Rewriting reductors

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Advantages

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1. Parallel reduction of S-polynomials with efficient linear algebra

Disadvantages

1. Normal selection strategy $\Rightarrow$ Many critical pairs processed $\Rightarrow$ Large intermediate matrices
2. The cost of having SIMPLIFY function : high memory consumption

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- The term of f corresponding to LM(g) will be eliminated
- Less monomials in Tail(g) $\Rightarrow$ less operations

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M4GB Main Strategy

The tail of every polynomial must be fully reduced

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- Two main differences :
  1. M4GB Reduction : prioritizes reduction on tail of reductors (recursive)
     - MULFULLREDUCE(G, t, f)
  2. Reduction on tail of all polynomials using new element found in the ideal
     - UPDATEREDUCE(G, P, f)

M4GB Reduction

$\textsc{MulFullReduce}(G, t, f)$

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$\textsc{MulFullReduce}(G, t, f)$
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2. for all $s \in \mathrm{Term}(f)$
```

M4GB Reduction

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MulFullReduce($G, t, f$)
$h \leftarrow 0$
for all $s \in \mathrm{Term}(f)$
    $r \leftarrow t \cdot s$
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M4GB Reduction

$\textsc{MulFullReduce}(G, t, f)$

```
$h \leftarrow 0$
for all $s \in \text{Term}(f)$
    $r \leftarrow t \cdot s$
    if $\exists g \in G : \text{LT}(g) \mid r$ then
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for all $s \in \mathrm{Term}(f)$
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    if $\exists g \in G : \mathrm{LT}(g) \mid r$ then
        $(G, g) \leftarrow \textsc{GetReductor}(G, r)$
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for all $s \in \mathrm{Term}(f)$
    $r \leftarrow t \cdot s$
    if $\exists g \in G : \mathrm{LT}(g) \mid r$ then
        $(G, g) \leftarrow \textsc{GetReductor}(G, r)$
        $h \leftarrow h - (r/\mathrm{LT}(g)) \cdot \mathrm{Tail}(g)$
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    else
        $h \leftarrow h + r$
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    else
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return (G, h)
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$\textsc{GetReductor}(G, r)$

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M4GB Reduction

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MulFullReduce($G, t, f$)

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for all $s \in \mathrm{Term}(f)$
    $r \leftarrow t \cdot s$
    if $\exists g \in G : \mathrm{LT}(g) \mid r$ then
        $(G, g) \leftarrow$ GetReductor($G, r$)
        $h \leftarrow h - (r/\mathrm{LT}(g)) \cdot \mathrm{Tail}(g)$
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GetReductor($G, r$)

if $\exists g \in G : \mathrm{LM}(g) = \mathrm{LM}(r)$ then
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$\textsc{GetReductor}(G, r)$

if $\exists g \in G : \mathrm{LM}(g) = \mathrm{LM}(r)$ then
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$h \leftarrow 0$
for all $s \in \text{Term}(f)$
    $r \leftarrow t \cdot s$
    if $\exists g \in G : \text{LT}(g) \mid r$ then
        $(G, g) \leftarrow \textsc{GetReductor}(G, r)$
        $h \leftarrow h - (r/\text{LT}(g)) \cdot \text{Tail}(g)$
    else
        $h \leftarrow h + r$
return (G, h)
```

```
$\textsc{GetReductor}(G, r)$

if $\exists g \in G : \text{LM}(g) = \text{LM}(r)$ then
    return $(G, g)$
$f \leftarrow \textsc{ReduceSel}(G, r)$
```

M4GB Reduction

```
MulFullReduce$(G, t, f)$

$h \leftarrow 0$
for all $s \in \text{Term}(f)$
    $r \leftarrow t \cdot s$
    if $\exists g \in G : \text{LT}(g) \mid r$ then
        $(G, g) \leftarrow$ GetReductor$(G, r)$
        $h \leftarrow h - (r/\text{LT}(g)) \cdot \text{Tail}(g)$
    else
        $h \leftarrow h + r$
return (G, h)
```

```
GetReductor$(G, r)$

if $\exists g \in G : \text{LM}(g) = \text{LM}(r)$ then
    return $(G, g)$
$f \leftarrow$ ReduceSel$(G, r)$
$(G, h) \leftarrow$ MulFullReduce$(G, r/\text{LT}(f), \text{Tail}(f))$
```

M4GB Reduction

```
$\textsc{MulFullReduce}(G, t, f)$

$h \leftarrow 0$
for all $s \in \mathrm{Term}(f)$
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```
$\textsc{GetReductor}(G, r)$

if $\exists g \in G : \mathrm{LM}(g) = \mathrm{LM}(r)$ then
    return $(G, g)$
$f \leftarrow \textsc{ReduceSel}(G, r)$
$(G, h) \leftarrow \textsc{MulFullReduce}(G, r/\mathrm{LT}(f), \mathrm{Tail}(f))$
return $(G \cup \{r + h\}, r + h)$
```

UPDATEREDUCE(G, P, f)

$\textsc{UpdateReduce}(G, P, f)$

1. $H \leftarrow \{\mathrm{LC}(f)^{-1} \cdot f\}$
2. $S \leftarrow \mathrm{Mono}(\mathrm{Tail}(G \cup H)) \setminus \mathrm{LM}(H)$

UpdateReduce(G, P, f)

```
1  $H \leftarrow \{\mathrm{LC}(f)^{-1} \cdot f\}$
2  $S \leftarrow \mathrm{Mono}(\mathrm{Tail}(G \cup H)) \setminus \mathrm{LM}(H)$
3  while $\exists u \in S : \mathrm{LM}(f) \mid u$ do
```

$\textsc{UpdateReduce}(G, P, f)$

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1 $H \leftarrow \{\mathrm{LC}(f)^{-1} \cdot f\}$
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3 while $\exists u \in S : \mathrm{LM}(f) \mid u$ do
4     Find the largest monomial $u \in S$ s.t. $\mathrm{LM}(f) \mid u$
```

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    $(G, h) \leftarrow \textsc{MulFullReduce}(G, u/\mathrm{LT}(f), \mathrm{Tail}(f))$
```

$\textsc{UpdateReduce}(G, P, f)$

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6     $H \leftarrow H \cup \{u + h\}$
```

$\textsc{UpdateReduce}(G, P, f)$

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    $H \leftarrow H \cup \{u + h\}$
    $S \leftarrow \mathrm{Mono}(\mathrm{Tail}(G \cup H)) \setminus \mathrm{LM}(H)$
while $H \neq \{\}$ do
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while $H \neq \{\}$ do
    Select $h \in H$ such that $\mathrm{LM}(h) = \min(\mathrm{LM}(H))$
    $H \leftarrow H \setminus \{h\}$
```

UpdateReduce(G, P, f)

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    $H \leftarrow H \setminus \{h\}$
    $H \leftarrow \{g - ch : g \in H, c$ is the coefficient of $\mathrm{LM}(h)$ in $\mathrm{Tail}(g)\}$
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UPDATEREDUCE(G, P, f)

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    $G \leftarrow \{g - ch : g \in G, c$ is the coefficient of $\mathrm{LM}(h)$ in $\mathrm{Tail}(g)\}$
    $G \leftarrow G \cup \{h\}$
```

Implementation Specific Notes

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M4GB is efficient when polynomials in G are maintained based on the uniqueness of their leading monomial i.e., if $f, g \in G$ s.t. $\mathrm{LM}(f) = \mathrm{LM}(g)$ then $f = g$

Implementation Specific Notes

M4GB is efficient when polynomials in G are maintained based on the uniqueness of their leading monomial i.e., if $f, g \in G$ s.t. $\mathrm{LM}(f) = \mathrm{LM}(g)$ then $f = g$

- M : associative array that maintain all polynomials

Implementation Specific Notes

M4GB is efficient when polynomials in G are maintained based on the uniqueness of their leading monomial i.e., if $f, g \in G$ s.t. $\mathrm{LM}(f) = \mathrm{LM}(g)$ then $f = g$

- M : associative array that maintain all polynomials
- L : a set of monomials that mark which polynomials in M that constitute a minimal basis

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2 M4GB Algorithm

3 Fukuoka MQ Challenges

4 Implementation Results

5 Solving MQ Challenges

MQ Challenges

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- A series of open public challenge to solve MQ problems over finite field.

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`https://www.mqchallenge.org`

MQ Challenge Types

	I	II	III
	IV	V	VI

MQ Challenge Types

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
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	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
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MQ Challenge Types

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
$n \approx 1.5m$	IV	V	VI

MQ Challenge Types

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
	$n \geq 55$	$n \geq 35$	$n \geq 34$
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MQ Challenge Types

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
	$n \geq 55$	$n \geq 35$	$n \geq 34$
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	$m \geq 55$	$m \geq 16$	$m \geq 16$

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	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
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Parameter Choice

MQ Challenge Types

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$m = 2n$	I	II	III
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Parameter Choice

- Time to solve : at least one month

MQ Challenge Types

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
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Parameter Choice

- Time to solve : at least one month
- Using Magma 2.19-9

MQ Challenge Types

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
	$n \geq 55$	$n \geq 35$	$n \geq 34$
$n \approx 1.5m$	IV	V	VI
	$m \geq 55$	$m \geq 16$	$m \geq 16$

Parameter Choice

- Time to solve : at least one month
- Using Magma 2.19-9
- CPU Used : Four 6-cores Intel(R) Xeon(R) CPU E5-4617 @ 2.9GHz and 1TB of RAM

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- Implemented using `C++11`

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  1. FGb C Interface - Implementation by Jean-Charles Faugère[1]
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---

[1] Available at `http://www-polsys.lip6.fr/~jcf/FGb/C/index.html`

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- Test cases
    1. Dense quadratic polynomials with coefficients in $\mathbb{F}_{31}$
    2. $m = 2n$ and $m = n + 1$.

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[1] Available at `http://www-polsys.lip6.fr/~jcf/FGb/C/index.html`

[2] Available at `https://github.com/nauotit/openf4`

Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40				
21	42				
22	44				
23	46				
24	48				
25	50				
26	52				
27	54				

		Memory Usage Ratio			
n	m				
20	40				
21	42				
22	44				
23	46				
24	48				
25	50				
26	52				
27	54				

Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40	1			
21	42	1			
22	44	1			
23	46	1			
24	48	1			
25	50	1			
26	52	1			
27	54	1			

		Memory Usage Ratio			
n	m				
20	40				
21	42				
22	44				
23	46				
24	48				
25	50				
26	52				
27	54				

Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40	1	3.61		
21	42	1	2.78		
22	44	1	2.70		
23	46	1	2.15		
24	48	1	4.03		
25	50	1	-		
26	52	1	-		
27	54	1	-		

		Memory Usage Ratio			
n	m				
20	40				
21	42				
22	44				
23	46				
24	48				
25	50				
26	52				
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Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40	1	3.61	4.07	
21	42	1	2.78	2.94	
22	44	1	2.70	3.81	
23	46	1	2.15	3.00	
24	48	1	4.03	12.19	
25	50	1	-	13.93	
26	52	1	-	11.83	
27	54	1	-	8.8	

		Memory Usage Ratio			
n	m				
20	40				
21	42				
22	44				
23	46				
24	48				
25	50				
26	52				
27	54				

Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40	1	3.61	4.07	8.25
21	42	1	2.78	2.94	5.89
22	44	1	2.70	3.81	7.35
23	46	1	2.15	3.00	6.46
24	48	1	4.03	12.19	25.31
25	50	1	-	13.93	27.2
26	52	1	-	11.83	23.20
27	54	1	-	8.8	16.49

		Memory Usage Ratio			
n	m				
20	40				
21	42				
22	44				
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24	48				
25	50				
26	52				
27	54				

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		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
20	40				
21	42				
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27	54				

Benchmark for $m = 2n$: Ratio

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n	m	M4GB	OpenF4	Magma	FGb
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25	50	1	-	13.93	27.2
26	52	1	-	11.83	23.20
27	54	1	-	8.8	16.49

		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
20	40	1			
21	42	1			
22	44	1			
23	46	1			
24	48	1			
25	50	1			
26	52	1			
27	54	1			

Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40	1	3.61	4.07	8.25
21	42	1	2.78	2.94	5.89
22	44	1	2.70	3.81	7.35
23	46	1	2.15	3.00	6.46
24	48	1	4.03	12.19	25.31
25	50	1	-	13.93	27.2
26	52	1	-	11.83	23.20
27	54	1	-	8.8	16.49

		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
20	40	1	1.53		
21	42	1	1.36		
22	44	1	2.32		
23	46	1	2.32		
24	48	1	2.35		
25	50	1	1.88		
26	52	1	1.38		
27	54	1	1.2		

Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40	1	3.61	4.07	8.25
21	42	1	2.78	2.94	5.89
22	44	1	2.70	3.81	7.35
23	46	1	2.15	3.00	6.46
24	48	1	4.03	12.19	25.31
25	50	1	-	13.93	27.2
26	52	1	-	11.83	23.20
27	54	1	-	8.8	16.49

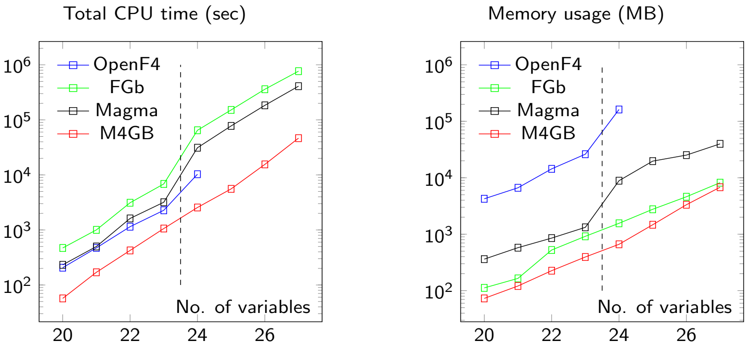
		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
20	40	1	1.53	4.96	
21	42	1	1.36	4.77	
22	44	1	2.32	3.8	
23	46	1	2.32	3.35	
24	48	1	2.35	13.38	
25	50	1	1.88	13.4	
26	52	1	1.38	7.57	
27	54	1	1.2	5.86	

Benchmark for $m = 2n$: Ratio

		CPU time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
20	40	1	3.61	4.07	8.25
21	42	1	2.78	2.94	5.89
22	44	1	2.70	3.81	7.35
23	46	1	2.15	3.00	6.46
24	48	1	4.03	12.19	25.31
25	50	1	-	13.93	27.2
26	52	1	-	11.83	23.20
27	54	1	-	8.8	16.49

		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
20	40	1	1.53	4.96	58.08
21	42	1	1.36	4.77	54.88
22	44	1	2.32	3.8	63.57
23	46	1	2.32	3.35	66.16
24	48	1	2.35	13.38	244.26
25	50	1	1.88	13.4	-
26	52	1	1.38	7.57	-
27	54	1	1.2	5.86	-

Graph for $m = 2n$



Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11				
11	12				
12	13				
13	14				
14	15				
15	16				

		Memory Usage Ratio			
n	m				
10	11				
11	12				
12	13				
13	14				
14	15				
15	16				

Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11	1			
11	12	1			
12	13	1			
13	14	1			
14	15	1			
15	16	1			

		Memory Usage Ratio			
n	m				
10	11				
11	12				
12	13				
13	14				
14	15				
15	16				

Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11	1	3.05		
11	12	1	3.36		
12	13	1	2.64		
13	14	1	2.96		
14	15	1	3.2		
15	16	1	2.98		

		Memory Usage Ratio			
n	m				
10	11				
11	12				
12	13				
13	14				
14	15				
15	16				

Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11	1	3.05	3.36	
11	12	1	3.36	4.3	
12	13	1	2.64	4.24	
13	14	1	2.96	4.92	
14	15	1	3.2	7.15	
15	16	1	2.98	7.12	

		Memory Usage Ratio			
n	m				
10	11				
11	12				
12	13				
13	14				
14	15				
15	16				

Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11	1	3.05	3.36	5.1
11	12	1	3.36	4.3	8.08
12	13	1	2.64	4.24	9.63
13	14	1	2.96	4.92	11.03
14	15	1	3.2	7.15	14.88
15	16	1	2.98	7.12	15

		Memory Usage Ratio			
n	m				
10	11				
11	12				
12	13				
13	14				
14	15				
15	16				

Benchmark for $m = n + 1$: Ratio

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		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
10	11				
11	12				
12	13				
13	14				
14	15				
15	16				

Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
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13	14	1	2.96	4.92	11.03
14	15	1	3.2	7.15	14.88
15	16	1	2.98	7.12	15

		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
10	11	1			
11	12	1			
12	13	1			
13	14	1			
14	15	1			
15	16	1			

Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11	1	3.05	3.36	5.1
11	12	1	3.36	4.3	8.08
12	13	1	2.64	4.24	9.63
13	14	1	2.96	4.92	11.03
14	15	1	3.2	7.15	14.88
15	16	1	2.98	7.12	15

		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
10	11	1	1.94		
11	12	1	3.12		
12	13	1	3.61		
13	14	1	4.36		
14	15	1	4.39		
15	16	1	4.92		

Benchmark for $m = n + 1$: Ratio

		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11	1	3.05	3.36	5.1
11	12	1	3.36	4.3	8.08
12	13	1	2.64	4.24	9.63
13	14	1	2.96	4.92	11.03
14	15	1	3.2	7.15	14.88
15	16	1	2.98	7.12	15

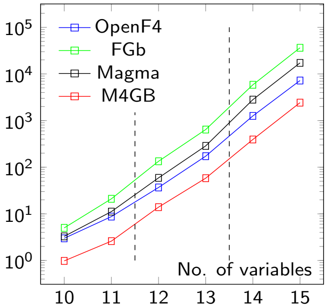
		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
10	11	1	1.94	1.88	
11	12	1	3.12	4	
12	13	1	3.61	3.68	
13	14	1	4.36	3.8	
14	15	1	4.39	4.42	
15	16	1	4.92	3.97	

Benchmark for $m = n + 1$: Ratio

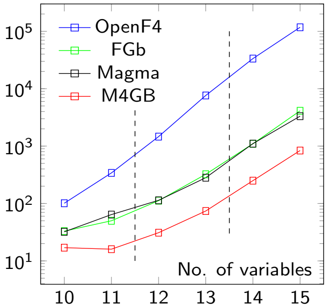
		CPU Time Ratio			
n	m	M4GB	OpenF4	Magma	FGb
10	11	1	3.05	3.36	5.1
11	12	1	3.36	4.3	8.08
12	13	1	2.64	4.24	9.63
13	14	1	2.96	4.92	11.03
14	15	1	3.2	7.15	14.88
15	16	1	2.98	7.12	15

		Memory Usage Ratio			
n	m	M4GB	FGb	Magma	OpenF4
10	11	1	1.94	1.88	5.94
11	12	1	3.12	4	21.31
12	13	1	3.61	3.68	47.19
13	14	1	4.36	3.8	103
14	15	1	4.39	4.42	133.84
15	16	1	4.92	3.97	140.26

Graph for $m = n + 1$



Total CPU time (sec)



Memory usage (MB)

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3. Fukuoka MQ Challenges
4. Implementation Results
5. Solving MQ Challenges

Solved MQ Challenges

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
	$n \geq 55$	$n \geq 35$	$n \geq 34$
$n \approx 1.5m$	IV	V	VI
	$m \geq 55$	$m \geq 16$	$m \geq 16$

Solved MQ Challenges

	$\mathbb{F}_2$	$\mathbb{F}_{2^8}$	$\mathbb{F}_{31}$
$m = 2n$	I	II	III
	$n \geq 55$	$n \geq 35$	$n \geq 34$
$n \approx 1.5m$	IV	V	VI
	$m \geq 55$	$m \geq 16$	$m \geq 16$

Solving Strategy (Bettale, Faugère and Perret, 2009)

F

$(n > m)$

Solving Strategy (Bettale, Faugère and Perret, 2009)

$$\bar{F} = F(x_1, \ldots, x_m, a_1, \ldots, a_{n-m})$$

$$\begin{tikzcd} F \arrow[r] & \bar{F} \end{tikzcd}$$

$(n > m)$ $(\#\mathrm{var} = m)$

Solving Strategy (Bettale, Faugère and Perret, 2009)

$$\bar{F} = F(x_1, \ldots, x_m, a_1, \ldots, a_{n-m})$$

$$\begin{tikzcd}
 & & \bar{F}_1 \\
 & & \bar{F}_2 \\
F \arrow[r] & \bar{F} \arrow[uur] \arrow[ur] \arrow[ddr] & \\
(n > m) & (\#\mathrm{var} = m) & \vdots \\
 & & \bar{F}_{q^k}
\end{tikzcd}$$

Solving Strategy (Bettale, Faugère and Perret, 2009)

$\bar{F} = F(x_1, \ldots, x_m, a_1, \ldots, a_{n-m})$

$$\begin{tikzcd}
 & & \bar{F}_1 \arrow[r] & \bar{G}_1 \\
 & & \bar{F}_2 \arrow[r] & \bar{G}_2 \\
F \arrow[r] & \bar{F} \arrow[uur] \arrow[ur] \arrow[ddr] & & \\
(n > m) & (\#\mathrm{var} = m) & \vdots & \vdots \\
 & & \bar{F}_{q^k} \arrow[r] & \bar{G}_{q^k}
\end{tikzcd}$$

Solving Strategy (Bettale, Faugère and Perret, 2009)

$\bar{F} = F(x_1, \ldots, x_m, a_1, \ldots, a_{n-m})$

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 & & \bar{F}_1 \arrow[r] & \bar{G}_1 \\
F \arrow[r] & \bar{F} \arrow[ur] \arrow[r] \arrow[ddr] & \bar{F}_2 \arrow[r] & \bar{G}_2 \\
(n > m) & (\#\text{var} = m) & \vdots & \vdots \\
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\end{tikzcd}$$

$\bar{G}_i$ is the reduced Gröbner basis of $\bar{F}_i$

Solving Strategy (Bettale, Faugère and Perret, 2009)

$$\bar{F} = F(x_1, \ldots, x_m, a_1, \ldots, a_{n-m})$$

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F \arrow[r] & \bar{F} \arrow[ur] \arrow[r] \arrow[ddr] & \bar{F}_2 \arrow[r] & \bar{G}_2 \\
(n > m) & (\#\mathrm{var} = m) & \vdots & \vdots \\
 & & \bar{F}_{q^k} \arrow[r] & \bar{G}_{q^k}
\end{tikzcd}$$

$\bar{G}_i$ is the reduced Gröbner basis of $\bar{F}_i$

Assume $\bar{F}$ has a unique solution in $\mathbb{F}_q^m \Rightarrow \exists i \in \{1, \ldots, q^k\}$ such that

$$\bar{G}_i = \{x_1 + c_1, \ldots, x_m + c_m : c_1, \ldots, c_m \in \mathbb{F}_q\}$$

Solving Strategy (Bettale, Faugère and Perret, 2009)

$\bar{F} = F(x_1, \ldots, x_m, a_1, \ldots, a_{n-m})$

$$\begin{tikzcd}
& & \bar{F}_1 \arrow[r] & \bar{G}_1 \\
& & \bar{F}_2 \arrow[r] & \bar{G}_2 \\
F \arrow[r] & \bar{F} \arrow[uur] \arrow[ur] \arrow[ddr] & & \\
(n > m) & (\#\mathrm{var} = m) & \vdots & \vdots \\
& & \bar{F}_{q^k} \arrow[r] & \bar{G}_{q^k}
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$\bar{G}_i$ is the reduced Gröbner basis of $\bar{F}_i$

Assume $\bar{F}$ has a unique solution in $\mathbb{F}_q^m \Rightarrow \exists i \in \{1, \ldots, q^k\}$ such that

$$\bar{G}_i = \{x_1 + c_1, \ldots, x_m + c_m : c_1, \ldots, c_m \in \mathbb{F}_q\}$$

Solution

$$(-c_1, \ldots, -c_m, a_1, \ldots, a_{n-m}) \in \mathbb{F}_q^n$$

Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

B) Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz and 128GB RAM (NUMA)

Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16			
V	25/17			
V	27/18			

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

B) Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz and 128GB RAM (NUMA)

Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17			
V	27/18			

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

B) Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz and 128GB RAM (NUMA)

Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17	B	1	$\approx$ 46.33 hours
V	27/18	B	2	$\approx$ 10.9 days

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

B) Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz and 128GB RAM (NUMA)

Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17	B	1	$\approx$ 46.33 hours
V	27/18	B	2	$\approx$ 10.9 days
VI	24/16			
VI	25/17			
VI	27/18			
VI	28/19			

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

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Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	$\approx$ 9.3 hours
V	25/17	B	1	$\approx$ 46.33 hours
V	27/18	B	2	$\approx$ 10.9 days
VI	24/16	A	1	$\approx$ 1.2 hours
VI	25/17			
VI	27/18			
VI	28/19			

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

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Summary of Solved Challenges

Type	n/m	Machine Used	# Node	Duration
V	24/16	A	1	≈ 9.3 hours
V	25/17	B	1	≈ 46.33 hours
V	27/18	B	2	≈ 10.9 days
VI	24/16	A	1	≈ 1.2 hours
VI	25/17	B	1	≈ 9.87 hours
VI	27/18	B	1	≈ 31.48 hours
VI	28/19	B	2	≈ 7.61 days

A) Intel(R) Core(TM) i7-2600K CPU @3.40GHz and 16GB RAM (Desktop)

B) Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz and 128GB RAM (NUMA)

New Record

Type	n/m	Machine Used	# Node	Duration
VI	30/20	B	2	$\approx$ 11.32 days

Future Work

- Implementation for sparse system of equations
- Vectorization / Parallelization using GPU
- Larger finite field
- Adapting signature in M4GB

`https://github.com/cr-marcstevens/m4gb`

Question ?