Attainable forms of Assouad spectra

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ABSTRACT. Let $d \in \mathbb{N}$ and let $\varphi\colon (0,1) \to [0,d]$. We prove that there exists a set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^\theta F = \varphi(\theta)$ for all $\theta \in (0,1)$ if and only if for every $0 < \lambda < \theta < 1$,

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta - \lambda)\varphi\Big(\frac{\lambda}{\theta}\Big).$$

In particular, the following behaviours which have not previously been witnessed in any examples are possible: the Assouad spectrum can be non-monotonic on every open set, and can fail to be Hölder in a neighbourhood of 1.

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1. Introduction

The Assouad dimension is a particular notion of dimension which captures the scaling properties of the "thickest" part of a set. This in contrast to the more usual notions of box (or Hausdorff) dimension, which are in some sense a global measurement of scaling. The Assouad dimension was originally introduced in [1] to study the embedding theory of metric spaces. More recently, the Assouad dimension has received a significant amount of attention in the literature: see, for example, the books by Mackay and Tyson on conformal geometry [15], Robinson on embedding theory [16], and Fraser on Assouad dimension in fractal geometry [8]

If the box dimension and the Assouad dimension of a set agree, this implies that the set has a large amount of spatial regularity. For instance, this is the case for any Ahlfors-regular subset of $\mathbb{R}^d$. However, the box dimension and Assouad dimension can be distinct for many naturally-occurring sets, such as self-conformal sets with overlaps or self-affine sets. In order to obtain a more fine-grained understanding of the Assouad dimension in this situation, the *Assouad spectrum* was introduced by Fraser and Yu in [13]. This is a notion of dimension parametrized by a variable $\theta \in (0,1)$, which approaches the box dimension as θ approaches 0 and the (quasi-)Assouad dimension as θ approaches 1. We refer the reader to [8] for a general introduction to Assouad-type dimensions.

Besides being a useful bi-Lipschitz invariant and an important notion of fractal dimension in its own right, the Assouad spectrum provides more refined information about the Assouad dimension itself. As a result, the Assouad spectrum has been explicitly studied for a wide range of examples (see, for example, [2, 5, 10, 11, 12, 13]). This relationship has also been useful in applications outside of fractal geometry. For instance, the Assouad spectrum plays an important role in the work by Roos and Seeger [17] on L^p bounds for spherical maximal operators. The Assouad spectrum has also been used to obtain bounds for quasiconformal distortion in geometric mapping theory [6].

In this paper, rather than consider explicit examples and applications of the Assouad spectrum, we focus on the general question of classification: what constraints on a function $\varphi\colon (0,1) \to [0,d]$ guarantee that there is a set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^\theta F = \varphi(\theta)$ for all $\theta \in (0,1)$?

1.1. Classifying Assouad spectra.

Classifying Assouad spectra. We fix $d \in \mathbb{N}$ and work in $\mathbb{R}^d$ with the Euclidean norm. We write $B(x,\delta)$ to denote the open ball centred at x with radius δ. If F is a bounded subset of $\mathbb{R}^d$, for $\delta > 0$, we let $N_\delta(F)$ denote the least number of balls of radius δ required to cover F. Then, for $\theta \in (0,1)$, the *Assouad spectrum* of $F \subset \mathbb{R}^d$ is given by

$$\dim_{\mathrm{A}}^\theta F = \inf\Big\{\alpha : (\exists C > 0)(\forall 0 < \delta \leq 1)(\forall x \in F)\ N_{\delta^{1/\theta}}(F \cap B(x,\delta)) \leq C\Big(\frac{\delta}{\delta^{1/\theta}}\Big)^\alpha\Big\}.$$

In general, $\lim_{\theta\to 0}\dim_{\mathrm{A}}^{\theta} F = \overline{\dim}_{\mathrm{B}} F$, and $\lim_{\theta\to 1}\dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{qA}} F$ [9], where $\dim_{\mathrm{qA}} F$ denotes the quasi-Assouad dimension of F as introduced by Lü and Xi [14]. Like the Assouad dimension, the Assouad spectrum measures the worst-case local scaling of the set, but the Assouad spectrum specifies the relationship between the small and large scales.

The main result of this paper is to give a complete classification of possible forms of Assouad spectra.

Theorem A. *Let $d\in\mathbb{N}$ and let $\varphi\colon(0,1)\to[0,d]$ be a function. Then there exists $F\subset\mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F=\varphi(\theta)$ for all $\theta\in(0,1)$ if and only if for every $0<\lambda<\theta<1$,*

$$0\le(1-\lambda)\varphi(\lambda)-(1-\theta)\varphi(\theta)\le(\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big). \tag{1.1}$$

The proof of this result is given in Section 3. For a geometric interpretation of the bound (b), we refer the reader to Section 2.1. That (a) implies (b) is well-known (see, for example, [8, Theorem 3.3.1]); the reverse implication is proven in Theorem 3.5.

We can interpret the first inequality in (1.1) as a growth rate constraint, and the second inequality as an oscillation constraint. In fact, the second inequality is always satisfied when φ is increasing (the short argument is given in Lemma 2.6), which yields the following corollary.

Corollary B. *Let $d\in\mathbb{N}$ and let $\varphi\colon(0,1)\to[0,d]$ be an increasing function. Then there exists a set $F\subset\mathbb{R}^d$ with $\dim_{\mathrm{A}}^{\theta} F=\varphi(\theta)$ if and only if $\theta\mapsto(1-\theta)\varphi(\theta)$ is decreasing.*

We can also obtain results for the *upper Assouad spectrum*, which is defined by bounding the lower scale from above, rather than specifying the relationship precisely:

$$\overline{\dim}_{\mathrm{A}}^{\theta} F=\inf\Big\{\alpha:(\exists C>0)(\forall 0<\delta\le 1)(\forall 0<\delta'\le\delta^{1/\theta})(\forall x\in F)$$
$$N_{\delta'}(B(x,\delta))\le C\Big(\frac{\delta}{\delta'}\Big)^{\alpha}\Big\}.$$

The upper Assouad spectrum is closely related to the Assouad spectrum: in fact $\overline{\dim}_{\mathrm{A}}^{\theta} F=\sup_{0<\theta'<\theta}\dim_{\mathrm{A}}^{\theta'} F$ by [9, Theorem 2.1]. Combining this with Corollary 2.7 gives a full characterization of the upper Assouad spectrum (the details are given in Section 3.5).

Corollary C. *Let $d\in\mathbb{N}$ and let $\varphi\colon(0,1)\to[0,d]$ be an arbitrary function. Then the following are equivalent:*

(a) There exists a set $F\subset\mathbb{R}^d$ such that $\overline{\dim}_{\mathrm{A}}^{\theta} F=\varphi(\theta)$ for all $\theta\in(0,1)$.
(b) $\varphi(\theta)$ is increasing and $\theta\mapsto(1-\theta)\varphi(\theta)$ is decreasing.
(c) φ is the supremum of functions of the form $\theta\mapsto f(\theta)/(1-\theta)$ where

$$f(\theta)=\begin{cases}\kappa(1-c) & :0<\theta\le c\\ \kappa(1-\theta) & :c<\theta<1\end{cases}$$

for $c\in(0,1)$ and $\kappa\in[0,d]$.

Beyond giving a full classification, Theorem A also clarifies many of the properties of the Assouad spectrum: certain observations which might *a priori* depend on explicit properties of the Assouad spectrum in fact only require the bound (1.1). For instance, the observation that if $\overline{\dim}_{\mathrm{B}} F = 0$ then $\dim_{\mathrm{A}}^{\theta} F = 0$ only requires the fact that $\lim_{\theta\to 0} \dim_{\mathrm{A}}^{\theta} F = \overline{\dim}_{\mathrm{B}} F$ along with the general bound (see Proposition 2.3).

We note that the 2-parameter family of functions in Corollary C consists corresponds to the Assouad spectra of sets with upper box dimension $\kappa(1-c)$, quasi-Assouad dimension κ, and Assouad spectrum as large as possible. In [17], such sets are called *quasi-Assouad regular*.

Having completed the classification, in Section 4 we construct some exceptional sets. Our first result concerns Hölder regularity.

Theorem D. *There is a compact set $F \subset \mathbb{R}$ such that $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ is not Hölder in any neighbourhood of 1.*

In fact, there is no lower control on the rate at which $\dim_{\mathrm{A}}^{\theta} F$ approaches $\dim_{\mathrm{qA}} F$. See Section 4.1 for the details. This result is sharp: in Proposition 2.4, we prove that $\dim_{\mathrm{A}}^{\theta} F$ is (uniformly) Lipschitz on $(0, 1-\delta)$ for all $\delta > 0$, with constants depending only on δ and the ambient dimension d. This observation, along with Theorem D, provides a complete answer to [13, Question 9.2].

Finally, we address the question of monotonicity. In [8, Question 17.7.1], Fraser conjectures that the Assouad spectrum must be monotonic in some neighbourhood of 1. This was originally conjectured in [9, Conjecture 2.4]. We provide a strong negative answer to this question: we show that Assouad spectra that are non-monotonic on any open set are dense in the set of all possible upper Assouad spectra.

Theorem E. *For any $\epsilon > 0$ and function φ satisfying one of the equivalent constraints in Corollary C, there is a compact set $F \subset \mathbb{R}$ such that $\phi(\theta) = \dim_{\mathrm{A}}^{\theta} F$ is non-monotonic on any open subset of $(0, 1)$ and $\|\phi - \varphi\|_{\infty} < \epsilon$.*

Since $\dim_{\mathrm{A}}^{\theta} F$ is Lipschitz on $(0, 1-\delta)$ for every $\delta > 0$, if φ is non-constant then by Rademacher's theorem φ must have strictly positive derivative on a set with positive Lebesgue measure. This is sharp: using similar techniques as used in the proof of Theorem E, one can construct examples of sets with quasi-Assouad dimension d, box dimension arbitrarily close to 0, and Assouad spectrum that is strictly decreasing on a dense open subset of $(0, 1)$ with Lebesgue measure arbitrarily close to 1. We leave the details of such a construction to the interested reader.

1.2. **Rate constraints and the relationship with intermediate dimensions.** The intermediate dimensions are a different notion of dimension spectrum introduced in [7] that interpolate between the Hausdorff and box dimensions. In [3], the author and Banaji fully classify the possible forms of the intermediate dimensions. For simplicity, in the discussion that follows we assume that the upper and lower intermediate dimensions coincide and denote the common value by $\dim_{\theta} F$. We refer the reader to [3] for precise statements of the results in full generality.

Recall that the (upper right) Dini derivative of a function $f\colon \mathbb{R}\to\mathbb{R}$ at x is given by

$$D^+f(x) = \limsup_{\epsilon\searrow 0} \frac{f(x+\epsilon)-f(x)}{\epsilon}. \tag{1.2}$$

We then recall the following result:

Theorem 1.1 ([3]). *Let $g\colon (0,1)\to[0,d]$. Then there exists a non-empty bounded set $F\subset\mathbb{R}^d$ with $\dim_\theta F = g(\theta)$ if and only if*

$$0 \le D^+g(\theta) \le \frac{g(\theta)(d-g(\theta))}{d\cdot\theta} \tag{1.3}$$

for all $\theta\in(0,1)$.

On the other hand, if $\varphi\colon(0,1)\to[0,d]$, Corollary C gives that there exists $F\subset\mathbb{R}^d$ such that $\overline{\dim}_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ if and only if

$$0 \le D^+\varphi(\theta) \le \frac{\varphi(\theta)}{1-\theta} \tag{1.4}$$

for all $\theta\in(0,1)$. In particular, when $\dim_{\mathrm{B}} F < d$, $D^+g(\theta) \le \frac{g(\theta)}{\theta}\cdot(d-\dim_{\mathrm{B}} F)/d$, so an arbitrary function which is the intermediate dimension of some set can be transformed to be the upper Assouad spectrum of a set through multiplication by a constant, reflection, and translation—and vice versa.

1.3. **Structure and outline of the paper.** In Section 2, we study the family of functions $\mathcal{A}_d$ (see Definition 2.1) which satisfy the bound (1.1) for some fixed $d\in\mathbb{N}$. This is the family which we will prove is the set of possible forms of Assouad spectra for subsets of $\mathbb{R}^d$. First, in Proposition 2.3, we establish a number of basic properties of such functions. The Assouad spectrum has been known to satisfy these properties, but here we only require the bound (1.1) and do not require any geometric properties of the Assouad spectrum itself. Then in Section 2.3, we establish the growth rate bounds and the corresponding Lipschitz constraints.

Now, in Section 3, we prove Theorem A. The general bound (a) implies (b) is standard, and follows by a straightforward covering argument: for completeness, we give the details in Proposition 3.1. To see that (b) implies (a), we will construct a homogeneous Moran set with prescribed Assouad spectrum using the techniques from [3]. This result is encapsulated in Proposition 3.3, where for a function satisfying certain derivative constraints, there exists a homogeneous Moran set such that the Assouad spectrum is given by a convenient formula. It then remains to choose such a function carefully, which is done in Theorem 3.5. In the remainder of the section, we construct families of monotonic and non-monotonic Assouad spectra in Section 3.3, prove closure under suprema in Section 3.4, and complete the proof of Corollary C.

To conclude, we use the classification result to construct examples of sets with exceptional Assouad spectra. The result proving Hölder failure at 1 is described in Section 4.1. Then in Section 4.2 we use the general family of non-monotonic spectra

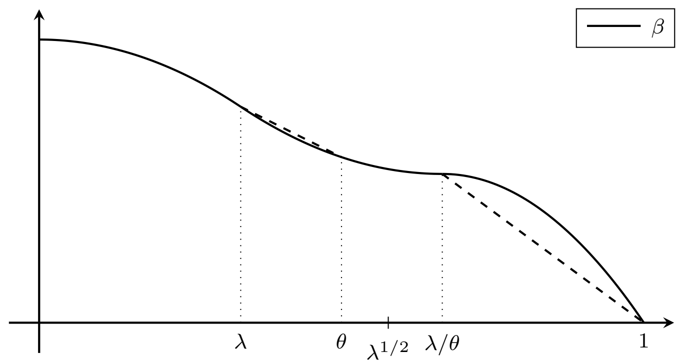

FIGURE 1. A plot of $\beta(\theta) = (1-\theta)\varphi(\theta)$ where $\varphi \in \mathcal{A}_d$, and the lines with slopes corresponding to (2.2).

from Section 3.3 to construct a set with Assouad spectra which is not monotonic on any open subset of $(0,1)$.

2. FORMS OF THE FAMILY OF FUNCTIONS $\mathcal{A}_d$

We first define the family of functions $\mathcal{A}_d$, which we will prove in Section 3 are the possible forms of the maps $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ for sets $F \subset \mathbb{R}^d$.

Definition 2.1. Let $\mathcal{A}_d$ denote the set of functions $\varphi\colon (0,1) \to [0,d]$ where for any $0 < \lambda < \theta < 1$,

$$0 \le (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \le (\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big). \tag{2.1}$$

In this section, we study properties of the family $\mathcal{A}_d$ directly: we emphasize that we do not require any geometric facts about the Assouad spectrum itself.

In Proposition 2.3, we will prove that functions in $\mathcal{A}_d$ are uniformly continuous. Thus, we will embed $\mathcal{A}_d$ in $C([0,1])$ by defining $\varphi(0) = \lim_{\theta\to 0}\varphi(\theta)$ and $\varphi(1) = \lim_{\theta\to 1}\varphi(\theta)$. We will use this notation once we prove uniform continuity.

2.1. **Rescaling and a geometric interpretation of the bound.**

Given $\varphi \in \mathcal{A}_d$, define $\beta(\theta) = (1-\theta)\varphi(\theta)$. In (2.1), the first inequality implies that $\beta(\theta)$ is decreasing, and the second states that for all $0 < \lambda < \theta < 1$,

$$\frac{\beta(\lambda) - \beta(\theta)}{\theta - \lambda} \le \frac{\beta\big(\frac{\lambda}{\theta}\big)}{1 - \frac{\lambda}{\theta}}. \tag{2.2}$$

The left hand side is the negative of the slope of the line passing through $(\lambda, \beta(\lambda))$ and $(\theta, \beta(\theta))$, and the right hand side is the negative of the slope of the line passing through $(\lambda/\theta, \beta(\lambda/\theta)$ and $(1,0)$. The secants in this constraint for a function β are depicted in Figure 1.

2.2. **Basic properties.** In this section, we collect various properties of the family $\mathcal{A}_d$. First, we observe the following useful lemma which was essentially proven in [13, Remark 3.8]. Here, we obtain it as a direct consequence of (2.1). Heuristically, this lemma states that the function $\varphi(\theta)$ is "almost increasing", up to some possible local oscillations.

Lemma 2.2. *Let $\varphi \in \mathcal{A}_d$. Given $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$,*

$$\varphi(\theta_1) \leq \max\left\{\varphi\Big(\frac{\theta_1}{\theta_2}\Big), \varphi\Big(\frac{\theta_2}{\theta_3}\Big), \ldots, \varphi\Big(\frac{\theta_{n-1}}{\theta_n}\Big), \varphi(\theta_n)\right\}.$$

In particular, for any $n \in \mathbb{N}$ and $\theta \in (0,1)$, $\varphi(\theta) \leq \varphi(\theta^{1/n})$.

Proof. Let $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$. Applying (2.1) to each pair θ_i, θ_{i+1},

$$(1-\theta_1)\varphi(\theta_1) \leq (1-\theta_n)\varphi(\theta_n) + \sum_{k=2}^{n}(\theta_k - \theta_{k-1})\varphi\Big(\frac{\theta_{k-1}}{\theta_k}\Big)$$

from which the result follows. Taking $\theta_i = \theta^{\frac{n-i+1}{n}}$ for each $i = 1, \ldots, n$, observe that $\theta_{k-1}/\theta_k = \theta^{1/n}$ and $\theta_n = \theta^{1/n}$ so that $\varphi(\theta) \leq \varphi(\theta^{1/n})$. □

We now have the following essential properties of $\mathcal{A}_d$. All of these properties have been previously observed for the Assouad spectrum, but the main point here is that these properties only depend on the family $\mathcal{A}_d$ and not on other properties of the Assouad spectrum. Some of these properties will be used in the proof of Theorem 3.5, so we cannot formally depend on the corresponding results for the Assouad spectrum. We draw on ideas from [9, 13].

Proposition 2.3. *Let $\varphi \in \mathcal{A}_d$ be arbitrary. Then the following properties hold:*

(i) The limits $\varphi(0) := \lim_{\theta\to 0}\varphi(\theta)$ and $\varphi(1) := \lim_{\theta\to 1}\varphi(\theta)$ exist.
(ii) Each $\varphi \in \mathcal{A}_d$ is uniformly continuous.
(iii) $\varphi(0) = \inf_{\theta\in(0,1)}\varphi(\theta)$ and $\varphi(1) = \sup_{\theta\in(0,1)}\varphi(\theta)$.
(iv) For any $\theta_0 \in (0,1)$, if $\varphi(\theta_0) = \varphi(1)$, then $\varphi(\theta_0) = \varphi(\theta)$ for all $\theta_0 < \theta < 1$.
(v) If $\varphi(0) = 0$, then $\varphi(\theta) = 0$ for all θ.

Proof. First, we show that $\varphi(\theta)$ is continuous on $(0,1)$. For $0 < \theta_1 < \theta_2 < 1$ we have $\theta_1 < \theta_1/\theta_2 < 1$, so applying (2.1) we obtain

$$(1-\theta_2)\varphi(\theta_2) \leq (1-\theta_1)\varphi(\theta_1) \leq \frac{\theta_1}{\theta_2}(1-\theta_2)\varphi(\theta_2) + \Big(1 - \frac{\theta_1}{\theta_2}\Big)\varphi\Big(\frac{\theta_1}{\theta_2}\Big). \tag{2.3}$$

This implies that

$$|\varphi(\theta_1) - \varphi(\theta_2)| \leq \frac{\varphi(\theta_1/\theta_2)}{\theta_2(1-\theta_1)}|\theta_2 - \theta_1|.$$

Since $\varphi(\theta_1/\theta_2) \leq d$, it follows that $\varphi(\theta)$ is Lipschitz on any closed subinterval of $(0,1)$, and therefore continuous on $(0,1)$.

Now consider (i). Observe that $(1-\theta)\varphi(\theta)$ is a bounded decreasing function of θ, so $\lim_{\theta\to 0}(1-\theta)\varphi(\theta)$ exists so $\lim_{\theta\to 0}\varphi(\theta)$ exists as well. To see that $\lim_{\theta\to 1}\varphi(\theta)$ exists, we use the proof from [9, Section 3.2]. Set $L = \limsup_{\theta\to 1}\varphi(\theta)$ and let $\epsilon > 0$. Since $\varphi(\theta)$ is continuous, we can find $0 < u < v < 1$ such that $\varphi(\theta) > L - \epsilon$ for all $\theta \in [u, v]$. Thus by Lemma 2.2, with

$$X \coloneqq \bigcup_{n=1}^{\infty}[u^{1/n}, v^{1/n}]$$

we have $\varphi(\theta) > L - \epsilon$ for all $\theta \in X$. But $v^{1/n} \geq u^{1/(n+1)}$ for all $n \geq n_0$ with $\frac{n_0}{n_0+1} \geq \frac{\log v}{\log u}$, so in fact $(u^{1/n_0}, 1) \subset X$. Thus $\lim_{\theta\to 1}\varphi(\theta)$ exists as well. In particular, combining the existence of endpoint limits with continuity of φ on $(0, 1)$, (ii) also follows immediately.

To see (iii), if $\theta_1 \in (0, 1)$, then $\theta_n = \theta_1^{1/n}$ is a sequence converging monotonically to 1 with $\varphi(\theta_n) \geq \varphi(\theta_1)$ by Lemma 2.2. Thus $\varphi(1) \geq \varphi(\theta_1)$. Similarly $\varphi(\theta_1^n) \leq \varphi(\theta_1)$ for any $n \in \mathbb{N}$, and $\lim_{n\to\infty}\theta_1^n = 0$. But θ_1 was arbitrary, giving (iii).

Now we see (iv). Suppose $\varphi(1) = \varphi(\theta_1)$ for some $0 < \theta_1 < 1$. By (2.3),

$$(1-\theta_1)\varphi(1) - (1-\theta_2)\varphi(\theta_2) \leq (\theta_2-\theta_1)\varphi(\theta_1/\theta_2) \leq (\theta_2-\theta_1)\varphi(1)$$

since $\varphi(\theta_1/\theta_2) \leq \varphi(1)$ by (iii). This implies that $\varphi(1) \leq \varphi(\theta_2)$, so (iv) follows.

To see (v), if $\varphi(0) = 0$, then $\lim_{\theta\to 0}(1-\theta)\varphi(\theta) = 0$. But $(1-\theta)\varphi(\theta)$ is a decreasing function of θ, so $(1-\theta)\varphi(\theta) = 0$ for all $\theta \in (0, 1)$, i.e. $\varphi(\theta) = 0$ for all $\theta \in (0, 1)$. $\square$

2.3. **Rate constraints and increasing functions.** Now, we obtain bounds on growth rates of functions in $\mathcal{A}_d$. We recall that the Dini derivative is defined in (1.2). We obtain the following regularity property for functions $\varphi \in \mathcal{A}_d$.

Proposition 2.4. *Let $\varphi \in \mathcal{A}_d$ be arbitrary and $\theta \in (0, 1)$. Then*

$$-\frac{\varphi(1)-\varphi(\theta)}{1-\theta} \leq D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}$$

In particular, φ is Lipschitz on $[0, 1-\delta]$ for any $\delta > 0$.

Proof. The first inequality in (2.1) is equivalent to saying that $\beta(\theta) = (1-\theta)\varphi(\theta)$ is decreasing. Since φ is continuous by Proposition 2.3, by [4, Corollary 11.4.2] β is decreasing if and only if $D^+\beta(\theta) = -\varphi(\theta) + (1-\theta)D^+\varphi(\theta) \leq 0$, or equivalently

$$D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}.$$

This gives the upper bound.

To obtain the lower bound, let $0 < \lambda < \theta < \lambda^{1/2} < 1$ be arbitrary. By (2.1),

$$-\varphi(\lambda/\theta) \leq \frac{\beta(\lambda)-\varphi(\theta)}{\lambda-\theta}$$

and passing to the limit

$$-\varphi(1) \leq D^+\beta(\lambda) = -\varphi(\theta) + (1-\theta)D^+\varphi(\theta).$$

$\square$

Remark 2.5. In Section 4.1, we will see that, in general, elements of $\mathcal{A}_d$ need not be Lipschitz (in fact, not even Hölder) on the entire interval $[0,1]$.

Now, we obtain the following result concerning increasing functions.

Lemma 2.6. *If $\varphi\colon (0,1)\to[0,d]$ is increasing, then $\varphi\in\mathcal{A}_d$ if and only if*

$$D^+\varphi(\theta)\leq\frac{\varphi(\theta)}{1-\theta}. \tag{2.4}$$

Proof. The forward direction is Proposition 2.4. To obtain the reverse implication, let $0<\lambda<\theta<1$. Since φ is increasing, if $\theta\leq\lambda/\theta$, then $\varphi(\lambda)\leq\varphi(\theta)\leq\varphi(\lambda/\theta)$ and

$$0\leq(1-\lambda)\varphi(\lambda)-(1-\theta)\varphi(\theta)\leq(\theta-\lambda)\varphi(\theta)\leq(\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big)$$

and if $\lambda/\theta\leq\theta$,

$$0\leq(1-\lambda)\varphi(\lambda)-(1-\theta)\varphi(\theta)\leq(1-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big)-(1-\theta)\varphi\Big(\frac{\lambda}{\theta}\Big)=(\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big)$$

which is (2.1). $\square$

We obtain the following convenient application, which we use to characterize the upper Assouad spectra.

Corollary 2.7. *Let $\varphi\in\mathcal{A}_d$. Then $\overline{\varphi}\in\mathcal{A}_d$ where*

$$\overline{\varphi}(\theta)=\sup_{0<\theta'\leq\theta}\varphi(\theta').$$

Proof. As proven in Lemma 2.6, since $\overline{\varphi}(\theta)$ is increasing, we only need to verify that $D^+\overline{\varphi}(\theta)\leq\overline{\varphi}(\theta)/(1-\theta)$. Since $\varphi(\theta)\leq\overline{\varphi}(\theta)$, it suffices to show $D^+\overline{\varphi}\leq\max\{D^+\varphi,0\}$.

Fix θ_0 and let $(\theta_n)_{n=1}^\infty\to\theta_0$ be strictly decreasing. Passing to a subsequence if necessary, we may assume $\overline{\varphi}(\theta_n)>\overline{\varphi}(\theta_0)$ for all n; otherwise $D^+\overline{\varphi}(\theta_0)\leq 0$. Thus for each n there is $\theta_0<\theta_n'\leq\theta_n$ be such that $\varphi(\theta_n')=\overline{\varphi}(\theta_n)$. Thus

$$\frac{\overline{\varphi}(\theta_n)-\overline{\varphi}(\theta_0)}{\theta_n-\theta_0}\leq\frac{\varphi(\theta_n')-\varphi(\theta_0)}{\theta_n'-\theta_0}\leq D^+\varphi(\theta_0).$$

But $(\theta_n)_{n=1}^\infty$ was an arbitrary sequence, so the result follows. $\square$

3. Classifying the forms of Assouad spectra

In this section, we prove the main classification result, Theorem A.

3.1. Bounding the Assouad spectrum.

We recall the following general bounds, which are given in [13, Proposition 3.4] and [8, Theorem 3.3.1]. We include the details here for completeness.

Proposition 3.1. *For any set $F\subset\mathbb{R}^d$, the function $\varphi(\theta)=\dim_{\mathrm{A}}^\theta F$ is in $\mathcal{A}_d$.*

Proof. Let $0 < \theta_1 < \theta_2 < 1$ and let $\epsilon > 0$ be arbitrary. For $\delta > 0$ sufficiently small, since $B(x, \delta^{\theta_2}) \subset B(x, \delta^{\theta_1})$ for all $x \in F$,

$$\begin{aligned}\sup_{x\in F} N_\delta(F \cap B(x, \delta^{\theta_1})) &\geq \sup_{x\in F} N_\delta(F \cap B(x, \delta^{\theta_2})) \\ &\geq \left(\frac{\delta^{\theta_2}}{\delta}\right)^{(\varphi(\theta_2)-\epsilon)} \\ &= \left(\delta^{\theta_1-1}\right)^{(\varphi(\theta_2)-\epsilon)\left(\frac{1-\theta_2}{1-\theta_1}\right)}\end{aligned}$$

which proves that $(1-\theta_1)\varphi(\theta_1) \geq (1-\theta_2)(\varphi(\theta_2)-\epsilon)$. This gives the lower inequality in (2.1).

To obtain the upper inequality, by covering $B(x, \delta^{\theta_1})$ by balls with radius δ^{θ_2},

$$\sup_{x\in F} N_\delta(F \cap B(x, \delta^{\theta_1})) \leq \sup_{x\in F} N_{\delta^{\theta_2}}(F \cap B(x, \delta^{\theta_1})) \sup_{x\in F} N_\delta(F \cap B(x, \delta^{\theta_2})).$$

This implies for all $\delta > 0$ sufficiently small

$$\begin{aligned}\sup_{x\in F} N_\delta(F \cap B(x, \delta^{\theta_1})) &\leq \left(\frac{\delta^{\theta_1}}{\delta^{\theta_2}}\right)^{\varphi(\theta_1/\theta_2)+\epsilon} \left(\frac{\delta^{\theta_2}}{\delta}\right)^{\varphi(\theta_2)+\epsilon} \\ &= \left(\delta^{\theta_1-1}\right)^{(\varphi(\theta_1/\theta_2)+\epsilon)\left(\frac{\theta_2-\theta_1}{1-\theta_1}\right)+(\varphi(\theta_2)+\epsilon)\left(\frac{1-\theta_2}{1-\theta_1}\right)}\end{aligned}$$

which implies that

$$(1-\theta_1)\varphi(\theta_1) \leq (\theta_2-\theta_1)(\varphi(\theta_1/\theta_2)+\epsilon) + (1-\theta_2)(\varphi(\theta_2)+\epsilon)$$

as required. □

3.2. **Constructing sets with prescribed spectra.** Now for any $\varphi \in \mathcal{A}_d$ we construct a homogeneous Moran set C such that $\dim_{\mathrm{A}}^{\theta} C = \varphi(\theta)$ for all $\theta \in (0,1)$. The techniques here are based on ideas first introduced by the author and Banaji in [3]: we refer the reader to that paper for more details on this general technique.

We first recall the notion of homogeneous Moran sets from [3]. The construction is analogous to the usual 2^d-corner Cantor set, except that the subdivision ratios need not be the same at each level.

Let $\mathcal{I} = \{0,1\}^d$, set $\mathcal{I}^* = \bigcup_{n=0}^{\infty} \mathcal{I}^n$, and denote the word of length 0 by $\varnothing$. Let $\boldsymbol{r} = (r_n)_{n=1}^{\infty} \subset (0, 1/2]$ and for each n and $\boldsymbol{i} \in \mathcal{I}$, define $S_{\boldsymbol{i}}^n \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$S_{\boldsymbol{i}}^n(x) \coloneqq r_n x + b_{\boldsymbol{i}}^n$$

where $b_{\boldsymbol{i}}^n \in \mathbb{R}^d$ has

$$(b_{\boldsymbol{i}}^n)^{(j)} = \begin{cases} 0 & : \boldsymbol{i}^{(j)} = 0 \\ 1 - r_n & : \boldsymbol{i}^{(j)} = 1 \end{cases}.$$

Given $\sigma = (\boldsymbol{i}_1, \ldots, \boldsymbol{i}_n) \in \mathcal{I}^n$, we write $S_\sigma = S^1_{\boldsymbol{i}_1} \circ \cdots \circ S^n_{\boldsymbol{i}_n}$. Then set

$$C = C(\boldsymbol{r}) \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{I}^n} S_\sigma([0,1]^d).$$

We refer to the set C as a *homogeneous Moran set*.

Given $\delta > 0$, let $k = k(\delta)$ be such that $r_1 \cdots r_k \leq \delta < r_1 \cdots r_{k-1}$. We then define

$$s(\delta) = s_{\boldsymbol{r}}(\delta) \coloneqq \frac{k(\delta) \cdot d \log 2}{-\log \delta}.$$

Heuristically, $s(\delta)$ is the best candidate for the box dimension of C at scale δ.

Definition 3.2. Let $0 \leq \lambda < \alpha \leq d$ and let $\mathcal{G}(\lambda, \alpha)$ denote the set of functions $g\colon (0,\infty) \to (\lambda, \alpha)$ satisfying

$$\lambda - (\lambda - g(y)) \exp(-t) \leq g(y+t) \leq \alpha - (\alpha - g(y)) \exp(-t)$$

for any $y > 0$ and $t > 0$.

This family is the same as the family defined in [3, Definition 3.1] (the argument is given in [3, Lemma 3.2]).

We first establish a general technique to prescribe Assouad spectra for homogeneous Moran sets.

Proposition 3.3. *Let $d \in \mathbb{N}$ and $g \in \mathcal{G}(0, d)$. Then there exists a homogeneous Moran set C such that*

$$\dim_{\mathrm{A}}^{\theta} C = \limsup_{x \to \infty} \frac{g\left(x + \log \frac{1}{\theta}\right) - \theta g(x)}{1 - \theta}. \tag{3.1}$$

Proof. If

$$\limsup_{x \to \infty} (D^+ g(x) + g(x)) = 0,$$

then

$$\limsup_{x \to \infty} \frac{g\left(x + \log \frac{1}{\theta}\right) - \theta g(x)}{1 - \theta} = 0$$

and we can define the Moran set $C(\boldsymbol{r})$ where $\boldsymbol{r}$ is a sequence converging monotonically to 0. Otherwise, performing an appropriate translation of g which does not change (3.1), [3, Lemma 3.4] provides a sequence $\boldsymbol{r} \subset (0, 1/2]$ so that

$$|s_{\boldsymbol{r}}(\exp(-\exp(x))) - g(x)| \leq d \log(2) \cdot \exp(-x). \tag{3.2}$$

To obtain (3.1), we first note that it follows directly from the definition that

$$\dim_{\mathrm{A}}^{\theta} C = \limsup_{\delta \to 0} \sup_{x \in C} \frac{\log N_{\delta^{1/\theta}}(C \cap B(x, \delta))}{(1 - 1/\theta) \log \delta}.$$

Observe that there is some constant $M > 0$ such that $B(x, \delta)$ intersects at most M cylinders in level $k(\delta)$. In particular, $C \cap B(x, \delta)$ can be covered by $M \cdot 2^{d(k(\delta^{1/\theta}) - k(\delta))}$

balls of radius $\delta^{1/\theta}$. On the other hand, $C \cap B(x,\delta)$ contains an interval in level $k(\delta)$, and therefore contains a δ-separated subset of size $2^{d(k(\delta^{1/\theta})-1-k(\delta))}$. Thus there is a constant $M' > 0$ so that

$$M' \cdot 2^{k(\delta^{1/\theta})-k(\delta)} \leq \sup_{x\in C} N_\delta^{1/\theta}(C \cap B(x,\delta)) \leq M \cdot 2^{k(\delta^{1/\theta})-k(\delta)}$$

and therefore

$$\begin{aligned}\limsup_{\delta\to 0}\sup_{x\in C}\frac{\log N_{\delta^{1/\theta}}(C\cap B(x,\delta))}{(1-1/\theta)\log\delta} &= \limsup_{\delta\to 0}\frac{\theta(k(\delta^{1/\theta})-k(\delta))\cdot d\log 2}{(1-\theta)\cdot(-\log\delta)}\\ &= \limsup_{\delta\to 0}\frac{s(\delta^{1/\theta})-\theta\cdot s(\delta)}{1-\theta}.\end{aligned}$$

Applying (3.2) then yields the desired formula. □

Definition 3.4. Given a sequence of continuous functions $(f_k)_{k=1}^\infty$ each defined on some interval $[0,a_k]$, the *concatenation* of $(f_k)_{k=1}^\infty$ is the function $f\colon (0,\sum_{k=1}^\infty a_k)\to\mathbb{R}$ given as follows: for each $x\in\mathbb{R}^+$ with $\sum_{j=0}^{k-1} a_j < x \leq \sum_{j=0}^{k} a_j$ where $a_0 = 0$ we define

$$f(x) = f_k\left(x - \sum_{j=0}^{k-1} a_j\right).$$

The concatenation of a finite tuple of functions is defined similarly.

Now, we can prove that (b) implies (a) in Theorem A. For the convenience of the reader, we also give an explicit description of the construction technique in $\mathbb{R}$. Note that, in the proof of Theorem 3.5, the precise choice of the contractions $(r_i)_{i=1}^\infty \subset (0,1/2]$ is concealed in the application of [3, Lemma 3.4] in Proposition 3.3.

Let $\varphi \in \mathcal{A}_1$ be some fixed function. Fix some small constant δ_1. Then we will inductively choose constants $r_1^{(n)},\ldots,r_{m_n}^{(n)}$ in $(0,1/2]$ for each $n\in\mathbb{N}$ so that for each $1\leq i\leq m_n$,

$$2^j \approx \left(\frac{1}{r_1^{(n)}\cdots r_j^{(n)}}\right)^{\varphi(\theta)} \tag{3.3}$$

where θ is such that $\delta_n^{1/\theta}\approx \delta_n\cdot r_1^{(n)}\cdots r_j^{(n)}$, and m_n satisfies $\delta_n\cdot r_1^{(n)}\cdots r_{m_n}^{(n)}\approx\delta_n^n$. Then take R_n very small, and set $\delta_{n+1} = \delta_n\cdot r_1^{(n)}\cdots r_{m_n}^{(n)}\cdot R_n$. Now let C denote the Moran set corresponding to the sequence

$$(\delta_1, r_1^{(1)},\ldots,r_{m_1}^{(1)},R_1,r_1^{(2)},\ldots,r_{m_2}^{(2)},R_2,\ldots).$$

Now for $n\in\mathbb{N}$ and $x\in C$, $N_{\delta_n^{1/\theta}}(C\cap B(x,\delta_n))\approx 2^j$ where $\delta_n^{1/\theta}\approx\delta_n\cdot r_1^{(n)}\cdots r_j^{(n)}$. In particular, (3.3) guarantees that the Assouad spectrum of C with respect to θ at scale δ_n is precisely $\varphi(\theta)$, for all n sufficiently large so that $1/n\leq\theta$.

The main details of the proof are to show (1) that such a choice of the constants r_i is possible, and (2) that for fixed θ and sufficiently small scales δ not of the form δ_n, the Assouad spectrum at θ of C at scale δ is at most $\varphi(\theta)$.

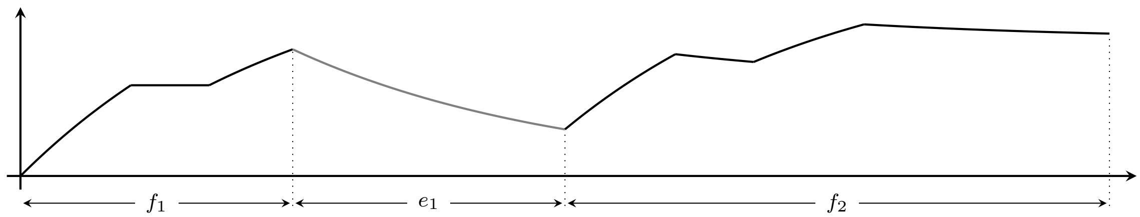

FIGURE 2. The concatenation of (f_1, e_1, f_2) corresponding to a function $\phi \in \mathcal{C}_d$ defined in Section 3.3.

Theorem 3.5. *Let $\varphi \in \mathcal{A}_d$ be arbitrary. Let α be such that $\varphi(1) \leq \alpha \leq d$. Then there exists a homogeneous Moran set $C \subseteq \mathbb{R}^d$ such that $\dim_{\mathrm{A}} C = \alpha$ and, for all $\theta \in (0,1)$,*

$$\dim_{\mathrm{A}}^{\theta} C = \varphi(\theta).$$

Proof. We may assume $\alpha > 0$, or the result is immediate. We will prove the result for the Assouad spectrum, and then explain how to modify the proof to accommodate the Assouad dimension as well.

First, we apply some convenient rescaling to $\varphi(\theta)$. Given $y \in (0, \infty)$, $\exp(-y) \in (0,1)$ so we may define

$$\xi(y) = (1 - \exp(-y))\varphi(\exp(-y)).$$

In particular, given $0 < y_1 < y_2 < \infty$, it follows that $0 < \exp(-y_2) < \exp(-y_1) < 1$ so

$$\begin{aligned} 0 &\leq (1 - \exp(-y_2))\varphi(\exp(-y_2)) - (1 - \exp(-y_1))\varphi(\exp(-y_1)) \\ &\leq \exp(-y_1)\left(1 - \exp(-(y_2 - y_1))\right)\varphi(\exp(-(y_2 + y_1))) \end{aligned}$$

or equivalently

$$0 \leq \xi(y_2) - \xi(y_1) \leq \exp(-y_1)\xi(y_2 - y_1). \tag{3.4}$$

Moreover, observe that $\varphi(1) = \lim_{y \to 0} \varphi(\exp(-y))$ so $\lim_{y \to 0} \xi(y) = 0$, and similarly $\lim_{y \to \infty} \xi(y) = \varphi(0)$. In particular ξ is continuous, increasing, and bounded.

Now for $z \in (0, \alpha)$, let ξ_z denote the function

$$\xi_z(y) = \xi(y) + \exp(-y)z$$

and similarly $\Psi_z(y) = \exp(-y)z$. We note that $\xi_z(0) = \Psi_z(0) = z$.

Now, choose constants w_n, z_n such that the functions $f_n := \xi_{z_n}|_{[0,n]}$ and $e_n := \Psi_{w_n}|_{[0,n]}$ satisfy $f_n(n) = e_n(0)$ and $e_n(n) = f_{n+1}(0)$ for all $n \in \mathbb{N}$. Then, let g be the infinite concatenation of the sequence

$$\left(f_1, e_1, f_2, e_2, \ldots\right).$$

This construction is illustrated in Figure 2.

First, let us verify that $g \in \mathcal{G}(0,\varphi(1)) \subset \mathcal{G}(0,\alpha)$. Let $n \in \mathbb{N}$. Note that $e_n \in \mathcal{G}(\varphi(0),\varphi(1))$ since the e_n are differentiable with $e_n'(x) = \varphi(0) - e_n(x)$. Next let $0 < y < y+t < \infty$. First observe that

$$\xi(y+t) \leq \xi(t) + \xi(y)\exp(-t) \leq (1-\exp(-t))\varphi(1) + \xi(y)\exp(-t)$$

by (3.4) and (iii) in Proposition 2.3. Thus

$$\begin{aligned} f_n(y+t) &= \xi(y+t) + \exp(-(y+t))z_n \\ &\leq (1-\exp(-t))\varphi(1) + \xi(y)\exp(-t) + \exp(-(y+t))z_n \\ &= (1-\exp(-t))\varphi(1) + f_n(y)\exp(-t) \end{aligned}$$

as required. To obtain the other bound, since ξ is increasing,

$$\begin{aligned} f_n(y+t) &= \xi(y+t) + \exp(-(y+t))z_n \\ &\geq \xi(y)\exp(-t) + \exp(-y)\exp(-t)z_n \\ &= f_n(y)\exp(-t). \end{aligned}$$

Now, let C denote the Moran set corresponding to the function g. Let $\theta \in (0,1)$: we must show that $\dim_{\mathrm{A}}^\theta C = \varphi(\theta)$. Let $\tau = \log(1/\theta)$. By Proposition 3.3, it suffices to show

$$\varphi(\theta) = \limsup_{x\to\infty} \frac{g(x+\tau) - \theta g(x)}{1-\theta}. \tag{3.5}$$

For $n \in \mathbb{N}$ set $x_n = 2\sum_{i=1}^{n-1} i$ and let $N \in \mathbb{N}$ be sufficiently large so that $N \geq \tau + 1$.

Now if $n \geq N$, $g(x_n + \tau) = f_n(\tau)$ and $g(x_n) = z_n$ so that

$$\frac{g(x_n+\tau) - \theta g(x_n)}{1-\theta} = \frac{(1-\theta)\varphi(\theta) + \theta z_n - \theta z_n}{1-\theta} = \varphi(\theta).$$

This gives the lower bound in (3.5).

It remains to see the upper bound. We first observe for all $y > 0$ and $z \in \mathbb{R}$ that

$$\frac{\xi_z(y+\tau) - \theta\xi_z(y)}{1-\theta} \leq \varphi(\theta).$$

Indeed, expanding the definition of ξ_z and applying (3.4),

$$\begin{aligned} \xi_z(y+\tau) - \theta\xi_z(y) &= \xi(y+\tau) + \exp(-(y+\tau))z - \exp(-\tau)(\xi(y) + \exp(-y)z) \\ &= \xi(y+\tau) - \exp(-\tau)\xi(y) \\ &\leq \xi(\tau) = (1-\theta)\varphi(\theta). \end{aligned}$$

Now let $x \geq x_N$ be arbitrary and let n be such that $x \in [x_n - (n-1), x_n + n]$. First note that for $y \in [x_n - (n-1), x_n + 2n]$, $g(y) = \exp(-(y - x_n))z_n + \phi(y)$ where

$$\phi(y) = \begin{cases} 0 & : x_n - (n-1) \leq y \leq x_n \\ \xi(y - x_n) & : x_n \leq y \leq x_n + n \\ \xi(n)\exp(-(y - x_n + n)) & : x_n + n \leq y \leq x_n + 2n \end{cases}$$

by choice of the constants w_n and z_n. If $x \in [x_n, x_n + n]$, since $x + \log(1/\theta) \leq x_n + 2n$ and $g(y) \leq \xi_{z_n}(y - x_n)$ for all $y \in [x_n, x_n + 2n]$, the prior computation shows that $g(x+\tau) - \theta g(x) \leq (1-\theta)\varphi(\theta)$. Otherwise, $x \in [x_n - (n-1), x_n]$. If $x + \tau \leq x_n$, then $g(x+\tau) - \theta g(x) = 0 \leq (1-\theta)\varphi(\theta)$, and if $x_n < x + \tau \leq x_n + n$, then

$$g(x+\tau) - \theta g(x) = \xi(x + \tau - x_n) \leq \xi(\tau)$$

since ξ is increasing. Thus (3.5) holds, finishing the proof.

In order to obtain the result for the Assouad dimension as well, we modify the construction as follows. Define functions $u_n\colon [0, 1/n] \to (0, \alpha)$ by the rule $u_n(x) = \alpha - (\alpha - q_n)\exp(-x)$. Choosing the constants q_n appropriately and modifying the constants w_n and z_n, the concatenation $\tilde{g}$ of the sequence

$$(f_1, e_1, u_1, f_2, e_2, u_2, \ldots)$$

is continuous and $\tilde{g} \in G(0, \alpha)$ since $\alpha \geq \varphi(1)$. Since the u_n are supported on intervals with lengths converging to 0, the same arguments as before yield the correct bounds for $\dim_{\mathrm{A}}^{\theta} C$ up to an error decaying to 0 as n goes to infinity. On the other hand, the same arguments as given in [3, Lemma 3.7 and Theorem 3.9] give that $\dim_{\mathrm{A}} C = \alpha$. We leave the precise details to the reader. □

3.3. **Families of monotonic and non-monotonic spectra.** In this section, we define two general parametrized families of functions in $\mathcal{A}_d$. The first is a 2-parameter family composed of increasing functions, and the second is a 3-parameter family composed of functions which (outside of degenerate cases) are non-monotonic.

3.3.1. *Monotonic spectra.* Let

$$M_d = \big\{(\kappa, c) : 0 \leq \kappa \leq d, 0 < c < 1\big\}$$

and for $\boldsymbol{i} = (\kappa, c) \in M_d$, we may define

$$f_{\boldsymbol{i}}(\theta) = \begin{cases} \kappa(1-c) & : \theta \in [0, c] \\ \kappa(1-\theta) & : \theta \in [c, 1] \end{cases}.$$

Then, let

$$\mathcal{M}_d \coloneqq \Big\{\theta \mapsto \frac{f_{\boldsymbol{i}}(\theta)}{1-\theta} : \boldsymbol{i} \in M_d\Big\}. \tag{3.6}$$

A direct argument shows that each $\varphi \in \mathcal{M}_d$ is an increasing element of $\mathcal{A}_d$.

3.3.2. *Non-monotonic spectra.* This family generalizes the example considered in [8, Theorem 3.4.16]. Let

$$C_d = \big\{(\kappa, c_1, c_2) : 0 \leq \kappa \leq d, 0 < c_1 \leq c_2 \leq c_1^{1/2} < 1\big\}.$$

Suppose $\boldsymbol{c} = (\kappa, c_1, c_2) \in C_d$. If $c_1 = 0$, let $h_{\boldsymbol{c}}(\theta) = \kappa(1-\theta)$ for $\theta \in [0, 1]$. Otherwise, $c_2 \leq c_1/c_2$. Thus we may define $h = h_{\boldsymbol{c}}\colon [0, 1] \to [0, d]$ to be the unique continuous

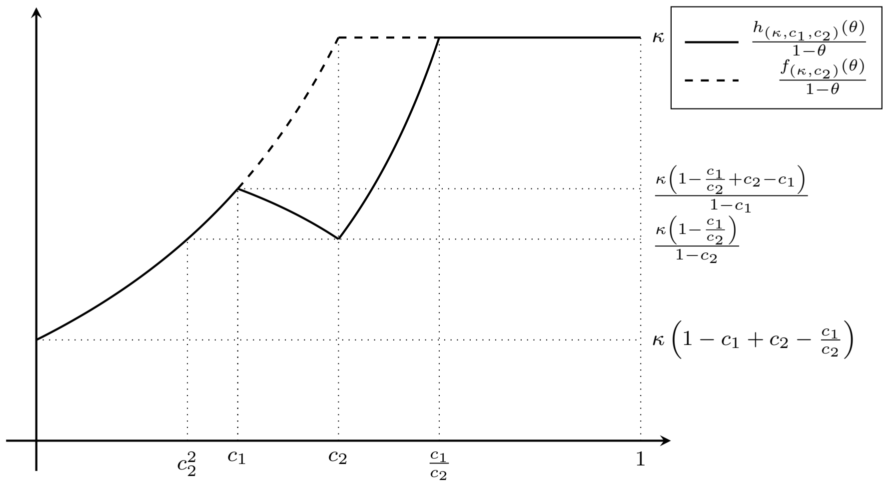

FIGURE 3. A plot of $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ and $f_{\boldsymbol{i}}(\theta)/(1-\theta)$ where $\boldsymbol{c} = (\kappa, c_1, c_2)$ and $\boldsymbol{i} = (\kappa, c_2)$.

function which has slope 0 on $[0, c_1] \cup [c_2, c_1/c_2]$, has slope $-\kappa$ on $[c_1, c_2] \cup [c_1/c_2, 1]$, and satisfies $h(1) = 0$. Now, let

$$\mathcal{C}_d = \left\{\theta \mapsto \frac{h_{\boldsymbol{c}}(\theta)}{1-\theta} : \boldsymbol{c} \in C_d\right\} \tag{3.7}$$

We note that $h_{\boldsymbol{c}}$ satisfies a certain rescaling invariance: for $c_2^2 \leq \theta \leq c_2$,

$$h_{\boldsymbol{c}}(\theta) - h_{\boldsymbol{c}}\left(\frac{\theta}{c_2}\right) = \kappa(c_2 - c_1).$$

In particular, $h_{\boldsymbol{c}}(c_2^2)/(1-c_2^2) = h_{\boldsymbol{c}}(c_2)/(1-c_2)$.

There are degenerate cases: if $c_2 = c_1^{1/2}$, then $h_{(\kappa,c_1,c_2)} = f_{(\kappa,c_1)}$ and if $c_1 = c_2$ or $\kappa = 0$, then $h_{\kappa,c_1,c_2} = 0$. Otherwise, $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ is strictly increasing on $[0, c_1]$ and $[c_2, c_1/c_2]$, constant on $[c_1/c_2, 1]$, and strictly decreasing on $[c_1, c_2]$. A plot of the function $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ for non-degenerate parameters is given in Figure 3.

Proposition 3.6. *For any $d \in \mathbb{N}$, $\mathcal{C}_d \subset \mathcal{A}_d$.*

Proof. Fix $0 < \theta_1, \theta_2 < 1$ and $\boldsymbol{c} = (\kappa, c_1, c_2) \in C_d$. We may assume $0 < c_1 < 1$. Since $h_{\boldsymbol{c}}$ is decreasing, it suffices to show that

$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} \leq h_{\boldsymbol{c}}\left(\frac{\lambda}{\theta}\right) \tag{3.8}$$

for all $0 < \lambda < \theta < \lambda^{1/2} < 1$. Since (3.8) is invariant under scaling by a positive factor, we may assume $\kappa = 1$. We prove this result in cases depending on the positions of λ and θ.

If $\lambda \in [0, c_2^2] \cup [c_2, 1]$, then $h_{\boldsymbol{c}}(\lambda)/(1-\lambda) \leq h_{\boldsymbol{c}}(\theta)/(1-\theta)$ for all $\lambda < \theta$. In particular, as argued in Lemma 2.6, (3.8) holds for all such λ. Moreover, suppose we know that the bound holds at c_1. Since $h_{\boldsymbol{c}}$ is the constant function on $[c_2^2, c_1]$, this implies the bound on $[c_2^2, c_1]$. Moreover, for $\lambda \in [c_1, c_2]$, since $h_{\boldsymbol{c}}(c_1/\theta) - h_{\boldsymbol{c}}(\lambda/\theta) \leq (c_1 - \lambda)/\theta$,

$$h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta) \leq \theta h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big) - (c_1 - \lambda) \leq \theta h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big).$$

Thus it suffices to establish the bounds for $\lambda = c_1$. Write $g(\theta) = (h_{\boldsymbol{c}}(c_2) - h_{\boldsymbol{c}}(\theta))/\theta$: we must show that $g(\theta) \leq h_{\boldsymbol{c}}(c_1/\theta)$.

(1) If $\theta \in [c_1, c_2]$, then $c_1/\theta \geq c_1/c_2$ and

$$g(\theta) = 1 - \frac{c_1}{\theta} = h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big).$$

(2) If $\theta \in [c_2, c_1/c_2]$ then $c_1/\theta \in [c_2, c_1/c_2]$ and

$$g(\theta) = \frac{c_2 - c_1}{\theta} \leq 1 - \frac{c_1}{c_2} = h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big).$$

(3) If $\theta \in [c_1/c_2, 1]$, then $c_1/\theta \in [c_1, c_2]$ and

$$g(\theta) = (\theta - c_1) + \Big(c_2 - \frac{c_1}{c_2}\Big) \leq \Big(1 - \frac{c_1}{\theta}\Big) + \Big(c_2 - \frac{c_1}{c_2}\Big) = h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big).$$

This treats all the cases $0 < \lambda < \theta < 1$, as required. □

3.4. **Closure under suprema.** In this section, we prove that $\mathcal{A}_d$ is closed under taking suprema. This essentially follows since $\mathcal{A}_d$ is uniformly Lipschitz on $[0, 1-\delta]$ for any $\delta > 0$.

Proposition 3.7. *Let $(\varphi_j)_{j\in\mathcal{J}}$ be some family of elements in $\mathcal{A}_d$. Then $\sup_{j\in\mathcal{J}} \varphi_j \in \mathcal{A}_d$.*

Proof. Let $f = \sup_{j\in\mathcal{J}} \varphi_j$. Get a sequence $J_1 \subset J_2 \subset \cdots \subset \mathcal{J}$ such that each J_n is finite and with

$$f_n \coloneqq \max\{\varphi_i : i \in J_n\}$$

that $f = \lim_{n\to\infty} f_n$ pointwise. An easy computation shows that if $\varphi_1, \varphi_2 \in \mathcal{A}_d$, then $\max\{\varphi_1, \varphi_2\} \in \mathcal{A}_d$; in particular, each $f_n \in \mathcal{A}_d$.

We first show that $f \in C([0,1])$. Since $(f_n)_{n=1}^\infty$ is monotonically increasing, by the Arzelà-Ascoli Theorem, it suffices to show that $(f_n)_{n=1}^\infty$ is uniformly bounded and uniformly equicontinuous. Uniform boundedness is immediate, so we must verify uniform equicontinuity.

Set $b = \lim_{n\to\infty} f_n(1)$ and let N be sufficiently large so that $f_N(1) > b - \epsilon/2$. Since f_N is continuous, get $\delta > 0$ so that $f_N(y) > f_N(1) - \epsilon/2$ for all $y \in [1-\delta, 1]$. Then $|f_n(x) - f_n(y)| \leq \epsilon$ whenever $x, y \in [1-\delta, 1]$. Finally, since each $f_n \in \mathcal{A}_d$, f_n is uniformly Lipschitz on $[0, 1-\delta]$ as proven in Proposition 2.4. It follows that $(f_n)_{n=1}^\infty$ is uniformly equicontinuous on $[0,1]$.

Thus $f \in C([0,1])$. To verify that $f \in \mathcal{A}_d$, let $0 < \lambda < \theta < 1$ be arbitrary. Then for any $\epsilon > 0$, get n such that $\|f_n - f\|_\infty \leq \epsilon$ so that

$$\begin{aligned}(1-\lambda)f(\lambda) - (1-\theta)f(\theta) &\leq (1-\lambda)(f_n(\lambda)+\epsilon) - (1-\theta)(f_n(\theta)-\epsilon)\\ &\leq (\theta-\lambda)f_n(\lambda/\theta) + 2\epsilon\\ &\leq (\theta-\lambda)f(\lambda/\theta) + 3\epsilon\end{aligned}$$

for any $\epsilon > 0$, so the inequality holds. The lower inequality follows identically. □

Remark 3.8. Note that $\mathcal{A}_d$ is not compact: for example, consider the functions $\varphi_n(\theta) = \min\{c_n/(1-\theta), 1\}$ with constants $c_n > 0$. If $\lim_{n\to\infty} c_n = 0$, then φ_n converges pointwise to the function which is 0 on $[0,1)$ and 1 at 1, and hence has no uniformly convergent subsequence. However, a simple modification of the above proof gives that for every $\delta > 0$, the restriction of $\mathcal{A}_d$ to $C([0,1-\delta])$ is compact.

3.5. **Characterization of upper Assouad spectra.** We conclude this section with the proof of Corollary C. We recall that the family $\mathcal{M}_d$ is defined in Section 3.3.

Proof of Corollary C. To see that (a) implies (b), if $F \subset \mathbb{R}^d$ has $\dim_{\mathrm{A}}^\theta F = \varphi(\theta)$ and $\overline{\dim}_{\mathrm{A}}^\theta F = \overline{\varphi}(\theta)$, by [9, Theorem 2.1],

$$\overline{\varphi}(\theta) = \sup_{0<\theta'\leq\theta} \varphi(\theta')$$

so by Corollary 2.7, $\overline{\varphi} \in \mathcal{A}_d$ so $\theta \mapsto (1-\theta)\overline{\varphi}(\theta)$ is decreasing. Of course, $\overline{\varphi}$ is increasing as well.

Next, (b) is equivalent to saying that for each $\lambda \in (0,1)$,

$$f_{(\overline{\varphi}(\lambda),\lambda)}(\theta) \leq \overline{\varphi}(\theta)$$

for all $\theta \in (0,1)$, with equality at $\theta = \lambda$. Since $\overline{\varphi} \in \mathcal{A}_d$ by Lemma 2.6, $\overline{\varphi}$ is uniformly continuous on $(0,1)$ and therefore $\overline{\varphi} = \sup_{\lambda\in\mathcal{Q}} f_{(\overline{\varphi}(\lambda),\lambda)}$ for any countable dense subset $\mathcal{Q} \subset (0,1)$. This implies (c).

Finally, to see (c) implies (a), since $\mathcal{A}_d$ is closed under suprema by Proposition 3.7, if $\overline{\varphi}(\theta) = \sup_{f\in\mathcal{F}} f(\theta)$ for some $\mathcal{F} \subset \mathcal{M}_d$, then $\overline{\varphi} \in \mathcal{A}_d$. Thus the result follows by Theorem 3.5. □

4. Examples with exceptional Assouad spectra

4.1. **Hölder failure at 1.** Here, we prove the following result which states that there is no control of the rate at which Assouad spectrum approaches the quasi-Assouad dimension.

Proposition 4.1. *Let $f : [0,1] \to [0,d]$ be an increasing function with $f(0) > 0$. Then there exists a compact set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^\theta F \leq f(\theta)$ for all $\theta \in (0,1)$ and $\lim_{\theta\to 1} \dim_{\mathrm{A}}^\theta F = f(1)$.*

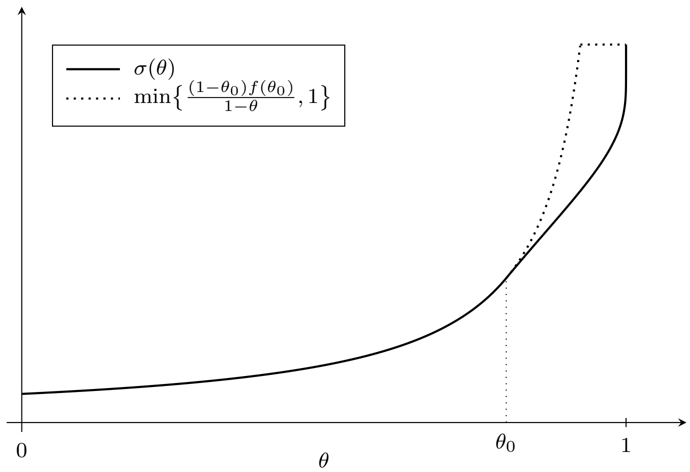

FIGURE 4. Plot of a spectrum which is not Hölder at 1, along with the general upper bound.

Proof. Let $h(\theta) = \inf_{0<\theta'\leq\theta}(1-\theta')f(\theta')$. By definition, $h(\theta)$ is decreasing, $h(\theta) \leq (1-\theta)f(\theta)$, and since $f(\theta) \leq d$, $\lim_{\theta\to 1} h(\theta) = 0$. Observe also that $\varphi(\theta) = h(\theta)/(1-\theta)$ is increasing, so $\varphi \in \mathcal{A}_d$ and $\varphi(\theta) \leq f(\theta)$.

To see that $\lim_{\theta\to 1}\varphi(\theta) = f(1)$, note that $(1-\theta)f(\theta) \geq (1-\theta)(f(1)-\epsilon)$ for any $\epsilon > 0$ and θ sufficiently close to 1. Since $\lim_{\theta\to 1}(1-\theta)f(\theta) = 0$ and $(1-\theta)f(\theta) > 0$ for all $\theta \in [0,1)$ since $f(0) > 0$, it follows that $h(\theta)/(1-\theta) \geq f(1) - \epsilon$ for some θ.

Finally, Theorem 3.5 gives a compact set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F = h(\theta)/(1-\theta)$. Since $\varphi(\theta) \geq h(\theta)/(1-\theta)$, so the claim follows. $\square$

We also consider an explicit example. Consider the function $f(\theta) = 1 + \frac{1}{\log(1-\theta)}$; note that f is not Hölder at 1. A direct computation shows that there is some minimal $\theta_0 \in (0,1)$ so that $(1-\theta)f(\theta)$ is decreasing on $[\theta_0, 1]$. Thus if we define

$$\sigma(\theta) = \begin{cases} \frac{(1-\theta_0)f(\theta_0)}{1-\theta} & : 0 < \theta \leq \theta_0 \\ f(\theta) & : \theta_0 < \theta < 1 \end{cases}$$

then σ is a continuous increasing function of θ with $(1-\theta)\sigma(\theta)$ decreasing. Thus by Lemma 2.6, $\sigma \in \mathcal{A}_1$. A plot of $\sigma(\theta)$ and the upper bound $\min\{(1-\theta_0)f(\theta_0)/(1-\theta), 1\}$ is given in Figure 4.

4.2. **Non-monotonicity on any open set.** In this section, we prove that Assouad spectra which are non-monotonic on every open subset of $(0,1)$ are dense in the set of upper Assouad spectra. Throughout this section, we fix an increasing $\varphi \in \mathcal{A}_d$, and write $\beta(\theta) = (1-\theta)\varphi(\theta)$.

We recall that the functions $f_{(\kappa,c)}$ for $(\kappa, c) \in M_d$ and $h_{(\kappa,c_1,c_2)}$ for $(\kappa, c_1, c_2) \in C_d$ are defined in Section 3.3, and moreover $h_{(\kappa,c_1,c_1^{1/2})} = f_{(\kappa,c_1)}$. We also recall from the proof of Corollary C in Section 3.5 that for each $\lambda \in (0,1)$,

$$f_{(\varphi(\lambda),\lambda)}(\theta) \leq \varphi(\theta)$$

for all $\theta \in (0,1)$, with equality at $\lambda = \theta$.

Fix $0 < \lambda < 1$ and for $0 \leq y \leq \beta(\lambda)$, define

$$c(\lambda, y) = \frac{\lambda + y/\varphi(\lambda) - 1 + \sqrt{(\lambda + y/\varphi(\lambda) - 1)^2 + 4\lambda}}{2}.$$

The constraint on y ensures that $\boldsymbol{c}(\lambda, y) \coloneqq (\lambda, c(\kappa, \lambda, y)) \in \mathcal{C}_d$. Note that $c(\lambda, y)$ is chosen precisely so that

$$h_{\boldsymbol{c}(\lambda,y)}(\lambda) = y.$$

A direct argument shows that for each $\lambda \in (0,1)$, $\{h_{\boldsymbol{c}(\lambda,y)} : y \in [0, \beta(\lambda)]\}$ is a continuously parametrized and monotonically increasing family of functions in y with

$$\lim_{y \to \beta(\lambda)} h_{\boldsymbol{c}(\lambda,y)} = f_{(\varphi(\lambda),\lambda)}.$$

In the following proof, we will choose a family of values y_λ for $\lambda \in \mathcal{D}$ where $\mathcal{D}$ is the set of dyadic rationals in $(0,1)$ so that with $f = \sup_{\lambda \in \mathcal{D}} h_{\boldsymbol{c}(\lambda, y_\lambda)}$, for each $\lambda \in \mathcal{D}$,

(1) $f(\lambda) = h_{\boldsymbol{c}(\lambda,y_\lambda)}(\lambda)$,
(2) $D^- f(\lambda) = D^- h_{\boldsymbol{c}(\lambda,y_\lambda)}$, and
(3) $D^+ f(\lambda) = D^+ h_{\boldsymbol{c}(\lambda,y_\lambda)}$.

From this, the result follows since $y_\lambda < \beta(\lambda)$ gives that

$$D^-\Big(\frac{h_{\boldsymbol{c}(\lambda,y_\lambda)}(\lambda)}{1-\lambda}\Big) > 0 \qquad \text{and} \qquad D^+\Big(\frac{h_{\boldsymbol{c}(\lambda,y_\lambda)}(\lambda)}{1-\lambda}\Big) < 0$$

so $\theta \mapsto f(\theta)/(1-\theta)$ is non-monotonic at each $\lambda \in \mathcal{D}$.

Theorem 4.2. *Let $\varphi \in \mathcal{A}_d$ be increasing. Then for any $\epsilon > 0$, there exists $F = F_\epsilon \subset \mathbb{R}^d$ such that $\phi(\theta) = \dim_{\mathrm{A}}^\theta F$ is non-monotonic on every open subset of $(0,1)$ and $\|\phi - \varphi\|_\infty \leq \epsilon$.*

Proof. Since every increasing function of $\mathcal{A}_d$ can be uniformly approximated by a strictly increasing function in $\mathcal{A}_d$, we may assume that φ is strictly increasing. As usual, define $\beta(\theta) = (1-\theta)\varphi(\theta)$. By Theorem 3.5, it suffices to construct a function $f \in \mathcal{A}_d$ satisfying the desired properties.

First define $\omega : (0,1) \to [0,d]$ by the rule

$$\omega(\theta) = f_{(\varphi(\theta^2),\theta^2)}(\theta).$$

Since φ is strictly increasing, $\omega(\theta) < \beta(\theta)$ for all $\theta \in (0,1)$. Since β is decreasing, there is a continuous decreasing function $\psi \colon [0,1] \to [0,d]$ so that for all $\theta \in (0,1)$

$$\max\{\omega(\theta), \beta(\theta) - \epsilon(1-\theta)\} \leq \psi(\theta) < \beta(\theta). \tag{4.1}$$

Now for $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{D}_n = \{j \cdot 2^{-n} : j \in \mathbb{Z}\} \cap (0,1)$ so that $\mathcal{D} = \bigcup_{n=1}^\infty \mathcal{D}_n$ is a dense subset of $(0,1)$. We will inductively choose constants $\psi(\lambda) < y_\lambda < \beta(\lambda)$ for $\lambda \in \mathcal{D}_n$ and $n \in \mathbb{N}$. When $n = 1$, $\mathcal{D}_1 = \{1/2\}$ and choose $\psi(\lambda_{1/2}) < y_{1/2} < \beta(\lambda_{1/2})$.

Now assume we have chosen constants y_λ for all $\lambda \in \mathcal{D}_n$. Set

$$\beta_n = \max\{h_{\boldsymbol{c}(\lambda,y_\lambda)} : \lambda \in \mathcal{D}_n\}.$$

First suppose $\ell_1 < \lambda < \ell_2$ where $\ell_1, \ell_2 \in \mathcal{D}_n$. We will successively constrain y_λ as follows. First, let y_λ be so that

$$\max\{\beta_n(\lambda), \psi(\lambda), y_{\ell_2}\} < y_\lambda < \min\Big\{y_{\ell_1}, \beta(\lambda), \frac{y_{\ell_1} + y_{\ell_2}}{2}\Big\}. \tag{4.2}$$

Now, since $y_{\ell_1} > \psi(\ell_1)$, there is some maximal $\ell_1 < \ell_0 < \ell_2$ such that for all $\theta \in [\ell_1, \ell_0]$,

$$\max\{\beta_n(\theta), \psi(\theta)\} = h_{(\varphi(\ell_1), \ell_1, \ell_1^{1/2})}(\theta).$$

Let m be minimal so that $\ell_1 + 2^{-n-m} \leq \ell_0$ and let y_λ be so that

$$\frac{y_{\ell_1} - y_\lambda}{2^{-n-m}} \leq \varphi(\ell_1) + \frac{1}{n+m}. \tag{4.3}$$

Similarly, since $y_{\ell_2} > \psi(\ell_2)$, there is some minimal $\ell_0 \leq \ell_0' < \ell_2$ such that for all $\theta \in [\ell_0', \ell_2]$,

$$\max\{\beta_n(\theta), \psi(\theta)\} = y_{\ell_2}.$$

Let k be minimal so that $\ell_2 - 2^{-n-m} \geq \ell_0'$. If $k = 1$, let y_λ be so that

$$\frac{y_\lambda - y_{\ell_2}}{2^{-n-k}} \leq \frac{1}{n+k}$$

and if $k > 1$, let y_λ be so that

$$\frac{h_{\boldsymbol{c}(\lambda,y_\lambda)}(\ell_0') - y_{\ell_2}}{2^{-n-k}} \leq \frac{1}{n+k}.$$

Such a choice is possible since φ is increasing.

If instead $\lambda = 1 - 2^{-n-1}$, simply choose y_λ so that

$$\max\{\beta_n(\lambda), \psi(\lambda), y_{\ell_2}\} < y_\lambda < \beta(\lambda). \tag{4.4}$$

and if $\lambda = 2^{-n-1}$, choose y_λ so that

$$\max\{\beta_n(\lambda), \psi(\lambda)\} < y_\lambda < \min\big\{y_{\ell_1}, \beta(\lambda)\big\}. \tag{4.5}$$

Finally, set $\nu = \lim_{n\to\infty} \beta_n$. Then

$$\psi(\theta) < \nu(\theta) < \beta(\theta)$$

and with $\phi(\theta) = \nu(\theta)/(1-\theta)$, $\phi \in \mathcal{A}_d$ by Proposition 3.7.

Fix $\lambda \in \mathcal{D}$. We first note that (4.2), (4.4), and (4.5) guarantee that

$$(1-\lambda)\phi(\lambda) = h_{\boldsymbol{c}(\lambda,y_\lambda)}(\lambda).$$

In particular, $0 \leq \varphi(\lambda) - (1-\lambda)\phi(\lambda) \leq \epsilon$, so $\|\phi - \varphi\|_\infty \leq \epsilon$.

It remains to show that ϕ is non-monotonic on every open set: since $y_\lambda < \beta(\lambda)$,

$$\theta \mapsto h_{\boldsymbol{c}(\lambda,y_\lambda)}(\theta)/(1-\theta) \in \mathcal{C}_d \setminus \mathcal{M}_d$$

so it suffices to show that $D^{-}\nu(\lambda) = 0$ and $D^{+}\nu(\lambda) = -\varphi(\lambda)$, where D^- denotes the upper left Dini derivative.

We prove that $D^+\nu(\lambda) = \varphi(\lambda)$: the argument that $D^-\nu(\lambda) = 0$ follows similarly. We first note that $D^+\nu(\lambda) \geq -\varphi(\lambda)$ since $D^+\beta_n(\lambda) = -\varphi(\lambda)$ for all n sufficiently large. To obtain the converse bound, since $y_\lambda < (y_{\ell_1} + y_{\ell_2})/2$ from (4.2), it suffices to prove that there is a monotonic sequence $(\lambda_k)_{k=1}^{\infty}$ converging to λ from the right such that $\lim_{k\to\infty}(y_\lambda - y_{\lambda_k})/(\lambda_k - \lambda) = \varphi(\lambda)$. But such a sequence is guaranteed by (4.3) (take a sequence of λ_k for which $m = 1$). □

The construction in the above proof is quite flexible: the constants y_λ for $\lambda \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ can be chosen from an open set of parameters. This motivates the following question.

Question 4.3. *Are "typical" elements of $\mathcal{A}_d$ non-monotonic? Does the set of functions $\varphi \in \mathcal{A}_d$ where φ is non-monotonic on every open subset of $(0,1)$ form a residual subset of $\mathcal{A}_d$?*

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