Fourier decay outside sparse frequencies

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ABSTRACT. Let μ be a finite Borel measure on $\mathbb{R}^d$ with Fourier transform $\widehat{\mu}$. We say that μ has *Fourier decay outside sparse frequencies* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every $R \geq 1$, the set

$$\left\{\xi \in \mathbb{R}^d : |\xi| \leq R \text{ and } |\widehat{\mu}(\xi)| \geq R^{-\delta}\right\}$$

can be covered by $O_\varepsilon(R^\varepsilon)$ balls of radius 1.

In this note, we explain how Fourier decay outside sparse frequencies follows from certain L^2-smoothing estimates on the measure μ. By Shmerkin's inverse theorem for L^2 norms in $\mathbb{R}^d$, L^2-smoothing is implied by quantitative non-concentration on proper affine subspaces. In particular, if μ is self-similar, then it has Fourier decay outside sparse frequencies if and only if $\operatorname{supp} \mu$ is not contained in a hyperplane.

1. INTRODUCTION

1.1. Polynomial Fourier decay outside sparse frequencies. Let $M(\mathbb{R}^d)$ denote the space of finite complex measures equipped with the norm

$$\|\mu\| = |\mu|(\mathbb{R}^d)$$

where $|\mu|$ is the total variation. The Fourier transform of μ is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i x \cdot \xi} \, \mathrm{d}\mu(x).$$

We are interested in the following control on the decay rate of the Fourier transform.

Definition 1.1. The measure μ has *Fourier decay outside sparse frequencies* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every $R \geq 1$, the set

$$\left\{\xi \in \mathbb{R}^d : \|\xi\| \leq R \text{ and } |\widehat{\mu}(\xi)| \geq R^{-\delta}\right\}$$

can be covered by $O_\varepsilon(R^\varepsilon)$ balls of radius 1.

Heuristically, μ has Fourier decay outside sparse frequencies if for every $\varepsilon > 0$ there is a $\delta > 0$ so that the Fourier transform $\widehat{\mu}$ has polynomial Fourier decay with exponent δ outside an exceptional set of dimension ε. Of course, $\widehat{\mu}$ is a continuous function, so for us the sensible notion of dimension is that an upper box dimension estimate with exponent ε holds when "zooming out".

In this note, we study the relationship between Fourier decay outside sparse frequencies and L^2 norms of measures. We emphasize that the results of this note are not new. The goal of this note is to give an exposition of recent results due to Khalil [Kha23+], with extensions due to Banaji & Yu [BY24+]. In particular, many details will follow the general lines of [Kha23+, Section 11]. The main difference in the approach here is to use Shmerkin's inverse theorem for L^q norms to obtain a (morally) special case of Khalil's result, which by an observation of [BY24+] is sufficiently general to apply to all self-similar measures, independent of separation conditions.

1.2. Affine non-concentration. The canonical example of a measure μ without Fourier decay outside sparse frequencies is one which is supported on a hyperplane.

Example 1.2. Let W be m-dimensional affine subspace of $\mathbb{R}^d$, where $0 \leq m \leq d-1$, and suppose $\operatorname{supp}\mu \subset W$. Then if $\xi \in W^\perp$, $x \cdot \xi = 0$ so

$$\widehat{\mu}(\xi) = \int e^{-2\pi i x\cdot\xi}\,\mathrm{d}\mu(x) = \mu(\mathbb{R}^d).$$

In particular, $\widehat{\mu}$ has no Fourier decay on a subspace of dimension $d - m$.

In a sense which we make precise below, it turns out that this is the *only* type of obstruction to Fourier decay outside sparse frequencies. Let us introduce a small amount of notation. We denote the Grassmanians of linear and affine k-dimensional planes in $\mathbb{R}^d$ by $\mathbb{G}(d,k)$ and $\mathbb{A}(d,k)$ respectively. We let π_W denote the orthogonal projection onto a subspace $W \leq \mathbb{R}^d$. Given a subset $E \subset \mathbb{R}^d$, we write $E(\delta)$ to denote the open δ-neighbourhood of E.

We consider the following key hypothesis, which can be considered as a certain quantitative non-concentration on proper affine subspaces.

Definition 1.3. We say that ν is *affinely non-concentrated* if there is a $C > 0$, $c > 0$, and $\alpha > 0$ such that for all $x \in \operatorname{supp}\nu$, $0 < r < 1$, $\eta \in (0,1)$, and all $W \in \mathbb{A}(d, d-1)$,

$$\nu\big(W^{(\eta r)} \cap B(x,r)\big) \leq C\eta^\alpha \nu\big(B(x,cr)\big).$$

To make the dependence on the parameters precise, we will also say that ν is (C,c,α)-affinely non-concentrated.

This definition is essentially the non-concentration assumption introduced in [Kha23+, Definition 11.1], though with the extra flexibility that the outer ball $B(x,cr)$ can be larger than the original ball $B(x,r)$ to avoid boundary effects. (It is possible that this version already falls under [Kha23+, Definition 11.1], though this was not entirely clear to the author.) Even though this modification causes no additional technical difficulty, it turns out that it is very useful since it ensures that the non-concentration assumption is satisfied by self-similar measures which are not supported on proper hyperplanes; this is an observation due to Banaji & Yu [BY24+]. The details are given in Proposition 3.2.

2. FOURIER DECAY FROM AFFINE NON-CONCENTRATION

In this section, our main goal is to show that affine non-concentration implies polynomial Fourier decay outside sparse frequencies. At a heuristic level, this fact has certainly been well-known for a while, though the first concrete manifestation of this can be found in [Kha23+, Section 11]. In this document, we will not directly use Khalil's results concerning L^2-flattening in higher dimensions; instead, we will appeal to a (conceptually) more general result due to Shmerkin, which is the inverse theorem for L^1-norms.

The inverse theorem has motivations in additive combinatorics. A classical result of Freiman states that if $A \subset \mathbb{Z}$ and $|A + A| \leq K|A|$, then A must in fact have quite a special structure: it is a subset of a generalized arithmetic progressions with dimension and size bounded only as a function of K. Unfortunately, in a fractal setting, the situation one is often faced with is a much weaker type of sumset control: namely of the form $|A + A| \leq |A|^{1+\delta}$ for an extremely small (but absolute) constant $\delta > 0$. In this regime, Freiman's theorem—or even the strongest possible quantitative bounds, such as the conjectured polynomial bounds (see for instance [GGM+23+] for more background)—essentially gives nothing useful.

In [Bou03; Bou10], Bourgain introduced a fundamental and wide-reaching insight: if $|A + A| \leq |A|^{1+\delta}$ and moreover A satisfies a weak non-concentration approximately analogous to having positive Hausdorff dimension, then A must admit a very special decomposition into "good" scales where either A is essentially full dimensional, or A is essentially zero dimensional, and a sparse collection of "bad" scales. In other words, sets with "intermediate" fractal dimension (in a uniform sense) must exhibit some type of growth under sumset.

In Bourgain's work, $A \subset \mathbb{Z}$ is essentially a one-dimensional set. In higher dimensions, there is a new type of obstruction: at fixed scales, a given set can be saturated on non-trivial proper affine subspaces. However, it turns out that this is the only type of obstruction: in [Hoc15+], Hochman proved an inverse theorem for entropy (here, entropy replaces the sumset operation) which says that either the entropy grows under convolution, or the measures must have a special structure essentially of this form.

Building on this work, as well as earlier work [Shm19], Shmerkin recently established an analogous inverse theorem except for L^2 norms instead of entropy. As we will see, controlling the L^2 norms can be much more useful, since L^2 control gives much finer control over local oscillation than entropy (which is, at least heuristically, a global property). For Fourier analytic applications, L^2 norms are also particularly useful because of the duality inherent in the L^2 theory of the Fourier transform.

2.1. Shmerkin's inverse theorem for L^2-norms. In this section, we state our key technical tool, which is Shmerkin's inverse theorem for L^q-norms [Shm23+, Theorem 1.2]. The proof of this result is quite technical, and relies on deep results in additive combinatorics, such as the Asymmetric Balog–Szemerédi–Gowers Theorem as proven by Tao–Vu [TV06, Theorem 3.2], and also on Hochman's theorem on saturation of self-convolutions [Hoc15+, Theorem 4.2].

As a result, we will not provide a proof here. Instead, the goal in this section is to state this result precisely and explain why this result is essential for our application.

We begin by introducing some notation and terminology for discretizations of measures. Let $\mathcal{D}_k$ denote the set of half-open standard dyadic cubes of side-length 2^{-k} in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, we let $\mathcal{D}_k(x)$ denote the unique dyadic cube in $\mathcal{D}_k$ containing x. More generally, if $A \subset \mathbb{R}^d$, we let $\mathcal{D}_k(A)$ denote the set of dyadic cubes which intersect A. We also write $|A|_{2^{-k}} = |\mathcal{D}_k(A)|$. For a non-negative integer S, we write $[S] = \{0, \ldots, S-1\}$; more generally, if S is not an integer, then $[S] \coloneqq [\lfloor S \rfloor]$.

Definition 2.1. For $m \in \mathbb{N}$, a 2^{-m}*-set* is a subset of $2^{-m}\mathbb{Z} \cap [0,1)^d$, and a 2^{-m}*-measure* is a probability measure supported on a 2^{-m}-set.

A 2^{-SL}-set is called (L, S)-uniform if there is a sequence $(R_s)_{s\in[S]}$ such that for each s and each $I \in \mathcal{D}_{sL}$,

$$|A \cap I|_{2^{-(s+1)L}} = R_s.$$

We refer to the numbers R_s as the *branching numbers* of A and we will also say that $(L, (R_s)_{s\in[S]})$-uniform to make the branching numbers explicit.

The idea of a uniform set is that the dyadic tree representing A has the property that, with step size L, every node in the tree has the same number of nodes L levels down.

Given a general Borel probability μ supported on $[0,1)^d$, we let μ_k denote the scale-2^{-k} discretization of μ:

$$\mu_k = \sum_{\lambda \in 2^{-k}\mathbb{Z}^d} \nu(\mathcal{D}_k(\lambda))\delta_\lambda.$$

In other words, μ_k is formed by concentrating the mass of μ on the bottom left corner of each dyadic cube in $\mathcal{D}_k$. Of course, μ_k is a 2^{-m}-measure.

Theorem 2.2 ([Shm23+]). *For each $q \in (1, \infty)$ and $\delta > 0$ the following holds if $L \geq L(\delta) \in \mathbb{N}$ and $\varepsilon < \varepsilon(L)$, and $S = S(\delta, L, \varepsilon)$ is sufficiently large:*

Let $m = SL$. Suppose μ and ν are 2^{-m}-measures on $[0,1)^d$ such that

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m} \|\mu\|_q.$$

Then there exist 2^{-m}-sets $A \subset \operatorname{supp}\mu$ and $B \subset \operatorname{supp}\nu$ and a sequence $(k_s)_{s\in[S]}$ taking values in $\{0, \ldots, d\}$ such that:

(A1) $\|\mu|_A\|_q \geq 2^{-\delta m}\|\mu\|_q$.

(A2) $\mu(x) \leq 2\mu(y)$ *for all* $x, y \in A$.

(A3) A *is* $(L, (R_s)_{s\in[S]})$*-uniform for some sequence* $(R_s)_{s\in[S]}$.

(A4) For each $s \in [S]$ *and each* $I \in \mathcal{D}_{sL}(A)$*, there is a* $W_I \in \mathbb{G}(d, d-k_s)$ *such that*

$$R_s \geq 2^{(k_s-\delta)L}|\pi_{W_I}(A \cap I)|_{2^{-(s+1)L}}.$$

(B1) $\|\nu|_B\|_1 = \nu(B) \geq 2^{-\delta m}$.

(B2) $\nu(x) \leq 2\nu(y)$ *for all* $x, y \in B$.

(B3) *B is (L, S)-uniform.*
(B4) *For each $s \in [S]$ and each $I \in \mathcal{D}_{sL}(B)$, there is a $V_I \in \mathbb{A}(d, k_s)$ such that*

$$I \cap B \subset \mathcal{D}_{(s+1)L}(V_I).$$

Heuristically, Theorem 2.2 states that at least one of the following must happen: either the L^q norm of $\mu * \nu$ decreases substantially (i.e. the measure becomes more spread out), or there are subsets $A \subset \operatorname{supp} \mu$ and $B \subset \operatorname{supp} \nu$ such that:

1. A captures a large proportion of the L^q-norm of μ and B captures a large proportion of the mass of ν;
2. μ is roughly uniform on A and ν is roughly uniform on B;
3. A and B are uniformly branching with respect to the same scale sequence;
4. $A \cap I$ is densely contained in a product of a k_s-dimensional subspace with an arbitrary measure, and $B \cap I$ is contained in a k_s-dimensional plane.

In particular, we can already see why affine non-concentration is very useful to prove L^2-smoothing. At a heuristic level, if ν is affinely non-concentrated and $\mu * \nu$ is not much smoother than μ, since condition (B4) can only happen when $k_s = d$, condition (A4) implies that A essentially has d-dimensional branching at "all" levels. In particular, the L^2-norm of μ is already as small as possible.

In the next section, we make this heuristic precise.

2.2. Affine non-concentration implies L^2-smoothing. The goal of this section is to establish the following special case of Theorem 2.2: if ν is an affinely non-concentrated measure and μ is an arbitrary measure, then the L^2 norm of $\mu * \nu$ must be quantitatively smaller than the L^2 norm of μ, unless the L^2 norm of μ was already quite small to begin with. In particular, this will imply that every affinely non-concentrated measure is arbitrarily L^2-smooth under convolution.

This result is essentially [Kha23+, Proposition 11.10] (but with slightly different assumptions, see the discussion following Definition 1.3) and follows by the same proof; see [BY24+] for more discussion. That this also follows from Theorem 2.2 is certainly well-known; see for instance the discussion in [Shm23+, Section 4.2]. We emphasize that [Kha23+] and [Shm23+] appeared independently.

Corollary 2.3 ([Kha23+; Shm23+]). *Suppose $\alpha > 0$, $c \geq 1$, and $C > 0$. Then for every $\gamma \in (0, 1)$, there exists $\varepsilon > 0$ and $S_0, L \in \mathbb{N}$ depending on γ, α, c, and C such that for all $S \geq S_0$, the following holds:*

Let $m = SL$ and suppose μ and ν are 2^{-m}-measures on $[0, 1)^d$ such that
(i) $\|\mu\|_2^2 > 2^{-md(1-\gamma)}$, *and*
(ii) *For all $W \in \mathbb{A}(d, d-1)$ and $0 \leq n \leq m$,*

$$\nu\big(W(\eta 2^{-n}) \cap B(x, 2^{-n})\big) \leq C\eta^{\alpha} \nu\big(B(x, c2^{-n})\big).$$

Then

$$\|\nu * \mu\|_2 < 2^{-\varepsilon m} \|\mu\|_2 .$$

Proof. Let $\alpha > 0$, $\gamma \in (0,1)$, $c \geq 1$, and $C > 0$ be fixed as in the hypothesis. We begin by choosing the appropriate constants. Let $\delta_0 > 0$ and $\kappa > 0$ be sufficiently small so that for all $0 < \delta \leq \delta_0$,

$$(d-\delta)(1-\kappa) - 3\delta > d(1-\gamma). \tag{2.1}$$

Then let $\delta > 0$ be sufficiently small and $L_0 \in \mathbb{N}$ sufficiently large so that for all $L \geq L_0$,

$$\frac{\delta - \frac{\log(16c^2)}{L}}{\alpha\left(1 - \frac{\log C}{L}\right)} \leq \kappa. \tag{2.2}$$

Finally, get parameters $L = L(\delta) \in \mathbb{N}$ with $L \geq L_0$, $\varepsilon < \varepsilon(L)$, and threshold $S(\delta, L)$ for which the hypotheses of Theorem 2.2 is satisfied for all $S \geq S_0$.

Now suppose for contradiction that for some $S \geq S_0$ and $m = SL$ that

$$\|\nu * \mu\|_2 \geq 2^{-\varepsilon m} \|\mu\|_2 .$$

Then by Theorem 2.2, get subsets $A \subset \operatorname{supp}\mu$ and $B \subset \operatorname{supp}\nu$ and sequence $(k_s)_{s\in[S]} \subset \{0,\dots,d\}$ satisfying Items (A1) to (A4) and Items (B1) to (B4) respectively.

For each $s \in [S]$ and $I \in \mathcal{D}_{sL}(B)$, fix some $y_I \in I \cap B$. Since each ball $B(y_I, c2^{-sL})$ intersects at most $16c^2$ distinct dyadic cubes in level sL, if $\mathcal{A} \subset \mathcal{D}_{sL}(B)$ is arbitrary, then

$$\sum_{I\in\mathcal{A}} \nu\big(B(y_I, c2^{-sL})\big) \leq 16c^2 \nu\left(\bigcup_{I\in\mathcal{A}} B(y_I, c2^{-sL})\right). \tag{2.3}$$

Now, let $\mathcal{S} \subset [S]$ denote the set of levels where $k_s < d$. Suppose $s \in \mathcal{S}$, and let $I \in \mathcal{D}_{sL}(B)$. Since $k_s \leq d-1$, by conclusion (B4) there is a hyperplane $V_I \in \mathbb{A}(d, d-1)$ so that $I \cap B \subset \mathcal{D}_{(s+1)L}(V_I)$. Applying the affine non-concentration hypothesis,

$$\nu\big(V_I(c2^{-L}\cdot 2^{-sL}) \cap B(y_I, 2^{-sL})\big) \leq C(c2^{-L})^\alpha \nu\big(B(y_I, c2^{-sL})\big)$$

so by (2.3)

$$\sum_{J\in\mathcal{D}_{(s+1)L}(B)} \nu\big(B(y_J, c2^{-(s+1)L})\big) \leq 16c^2 \cdot C(c2^{-L})^\alpha \sum_{I\in\mathcal{D}_{sL}(B)} \nu\big(B(y_I, c2^{-sL})\big).$$

For $s \notin \mathcal{S}$, assuming c is sufficiently large, again by (2.3) we can apply the trivial bound

$$\sum_{J\in\mathcal{D}_{(s+1)L}(B)} \nu\big(B(y_J, c2^{-(s+1)L})\big) \leq 16c^2 \sum_{I\in\mathcal{D}_{sL}(B)} \nu\big(B(y_I, c2^{-sL})\big).$$

Therefore, applying the relevant bound for each $s \in [S]$,

$$2^{-\delta SL} \leq \nu(B) \leq \sum_{J\in\mathcal{D}_m(B)} \nu\big(B(y_J, c2^{-(s+1)L})\big) \leq (16c^2)^S (c2^{-L})^{\alpha|\mathcal{S}|}.$$

Rearranging, the choice of δ and L_0 in (2.2) implies that $|\mathcal{S}| \leq \kappa S$.

Moreover, by conclusion (A4), for each $s \in [S] \setminus \mathcal{S}$, $R_s \geq 2^{(d-\delta)L}$. Therefore applying the trivial bound $R_s \geq 1$ for $s \in \mathcal{S}$,

$$|A| \geq \prod_{s \in [S]} R_s \geq 2^{-m(d-\delta)(1-\kappa)}.$$

Moreover, by (A2), if $t = \max_{a \in A} \mu(x)$, since $t|A|/2 \leq \sum_{x \in A} \mu(x) \leq 1$ it follows that $\mu(x) \leq 2|A|^{-1}$ for all $x \in A$. Therefore by (A1) and the choice of κ in (2.1),

$$\|\mu\|_2^2 \leq 2^{2\delta m} \|\mu|_A\|_2^2 \leq 2 \cdot 2^{2\delta m}|A|^{-1} \leq 2 \cdot 2^{2\delta m - m(d-\delta)(1-\kappa)} \leq 2 \cdot 2^{-\delta m} \cdot 2^{-md(1-\gamma)}.$$

This contradicts the assumption on μ for m sufficiently small, as required. □

Of course, we can also apply Corollary 2.3 with $\mu = \nu$ where ν is affinely non-concentrated. It then follows that repeated self-convolutions of ν have L^2 norms essentially as small as possible. This deduction is standard; see for instance [Kha23+, Section 11.5].

Corollary 2.4. *Let $c \geq 1$, $\alpha > 0$, and $C > 0$ be arbitrary. Then for every $\varepsilon > 0$, there exists a constant $C' > 0$ and natural numbers n, m_0 such that for every (C, c, α)-affinely non-concentrated measure ν and $m \geq m_0$,*

$$\|\nu_m^{*n}\|_2^2 \leq C' 2^{-(d-\varepsilon)m}.$$

Proof. Fix $\gamma = \varepsilon/d$, and let $\varepsilon > 0$, and $S_0, L \in \mathbb{N}$ be the constants provided by Corollary 2.3 and set $m_0 = S_0 L$. Let $n \in \mathbb{N}$ be such that $n \geq d/\varepsilon$.

We first note that it suffices to verify the inequality for multiples of L since, for $0 \leq \ell < L$ and any probability measure μ,

$$\sum_{I \in \mathcal{D}_{SL+\ell}} \mu(I)^2 \leq 2^{d\ell} \sum_{I \in \mathcal{D}_{SL}} \mu(I)^2.$$

Thus let $m \geq m_0$ be of the form $m = SL$ for some integer $S \geq S_0$. Suppose $\left\|\nu_m^{*n'}\right\|_2^2 \leq 2^{-d(1-\gamma)m} = 2^{-(d-\varepsilon)m}$ for some $1 \leq n' \leq n$. Then by Young's inequality,

$$\|\nu_m^{*n}\|_2^2 = \left\|\nu_m^{*n'} * \nu_m^{*(n-n')}\right\|_2^2 \leq \left\|\mu_m^{*(n-n')}\right\|_1^2 \|\mu_m^{*n}\|_2^2 \leq 2^{-(d-\varepsilon)m}.$$

Otherwise, $\left\|\nu_m^{*n'}\right\|_2^2 > 2^{-d(1-\gamma)m}$ for all $1 \leq n' \leq n$. Applying Corollary 2.3 $(n-1)$ times,

$$\|\nu_m^{*n}\|_2 \leq 2^{-(n-1)\varepsilon m} \|\nu_m\|_2 \leq 2^{-(n-1)\varepsilon m}.$$

Therefore $2^{-(d-\varepsilon)m} < \|\nu_m^{*n}\|_2^2 \leq 2^{-(n-1)\varepsilon m}$, so $n < d/\varepsilon$, contradicting the choice of n. □

2.3. L^2-smoothing implies Fourier decay outside sparse frequencies. Now, let us recall the L^2 dimension, or the (lower) correlation dimension, of a measure μ:

$$\dim_2 \mu = \liminf_{m\to\infty} \frac{-\log \sum_{I\in\mathcal{D}_m} \mu(I)^2}{m} \in [0,d].$$

We can now formally introduce a notion of L^2 smoothing.

Definition 2.5. Let μ be a Borel probability measure. We say that μ is *arbitrarily L^2-smooth under convolution* if for all $\varepsilon > 0$, there is an $n \in \mathbb{N}$ so that $\dim_2 \mu^{*n} \geq d-\varepsilon$.

In this terminology, Corollary 2.4 states that an affinely non-concentrated measure ν is arbitrarily L^2-smooth under convolution, with dependence on the parameters of non-concentration.

The particular importance of the L^2 norm is the relationship with the Fourier transform.

Definition 2.6. Let $1 \leq p < \infty$. Then the *Fourier ℓ^p-dimension* of a measure μ is the number

$$\kappa_p := \sup\left\{ s \geq 0 : \int_{|\xi|\leq R} |\widehat{\mu}(\xi)|^p \,\mathrm{d}\xi \lesssim R^{d-s} \right\}.$$

In fact, Fourier ℓ^2-dimension and the L^2 dimension of a measure μ coincide. This proof can be found, for instance, in [FNW02]; for completeness, we give the details here.

Proposition 2.7. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with Fourier ℓ^2-dimension κ_2. Then $\dim_2 \mu = \kappa_2$.*

Proof. It essentially suffices to show for all $r > 0$, with $m \in \mathbb{N}$ chosen so that $2^{-(m+1)} < r \leq 2^{-m}$, that

$$\int_{|\xi|\leq r^{-1}} |\widehat{\mu}(\xi)|^2 \,\mathrm{d}\xi \approx r^{-2d} \int \mu\big(B(x,r)\big)^2 \,\mathrm{d}x \approx 2^{dm} \sum_{I\in\mathcal{D}_m} \mu(I)^2.$$

Fix $r > 0$ and define $f(x) = \mu\big(B(x,r)\big)$. Since μ is a probability measure, by Fubini's theorem,

$$\widehat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} \int_{|x-y|\leq r} \mathrm{d}\mu(y)\,\mathrm{d}x = \int \widehat{\chi_{B(y,r)}}(\xi)\,\mathrm{d}\mu(y) = r^d \widehat{\chi_{B(0,1)}}(r\xi)\cdot\widehat{\mu}(\xi).$$

Therefore, by Plancherel's formula,

$$\frac{1}{r^{2d}} \int \mu\big(B(x,r)\big)^2 \,\mathrm{d}x = \int |\widehat{\mu}(\xi)|^2 \widehat{\chi_{B(0,1)}}(r\xi)^2 \,\mathrm{d}\mu(\xi).$$

But by assumption, $|r\xi| \leq 1$, and since the magnitude of the Fourier transform of the indicator $\chi_{B(0,1)}$ is uniformly bounded away from 0 on a neighbourhood of 0 with constants only depending on the ambient dimension, the claim follows. This proves the first equality.

The second equality follows since the choice of m ensures that each ball $B(x,r)$ can be covered by $O_d(1)$ dyadic cubes in $\mathcal{D}_m$; and conversely each ball $B(x,r)$ contains at least one dyadic cube in some level $m+\ell$, where ℓ depends only on d.

Now, by definition of $s = \dim_2 \mu$, for all $\varepsilon > 0$ and m sufficiently large, $\sum_{I\in\mathcal{D}_m} \mu(I)^2 \leq 2^{-m(s-\varepsilon)}$. Therefore, writing $R = r^{-1}$,

$$\int_{|\xi|\leq R} |\widehat{\mu}(\xi)|^2 \,\mathrm{d}\xi \lesssim 2^{m(d-s+\varepsilon)} \lesssim R^{d-(s-\varepsilon)}.$$

Therefore $\kappa_2 \geq \dim_2 \mu - \varepsilon$; but $\varepsilon > 0$ was arbitrary, so $\kappa_2 \geq \dim_2 \mu$. The converse direction is proven analogously, as claimed. □

Finally, we recall *Chebyshev's inequality*: if $f \in L^p(\mathbb{R}^d)$ and m denotes Lebesgue measure, then

$$m\left(\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\right) \leq \frac{\int_{\{x:|f(x)|>\lambda\}} |f(x)|^p}{\lambda^p} \leq \frac{\|f\|_p^p}{\lambda^p}.$$

We can now show that good L^2-smoothing implies Fourier decay outside sparse frequencies. This is an observation due to Khalil, and we essentially follow the proof in [Kha23+, Section 11.6]

Proposition 2.8 ([Kha23+]). *Suppose μ is arbitrarily L^2-smooth under convolution. Then μ has Fourier decay outside sparse frequencies.*

Proof. Let $\varepsilon > 0$ be arbitrary. By Proposition 2.7 and the assumption, get some $n \in \mathbb{N}$ so that the convolution μ^{*n} has Fourier ℓ^2-dimension strictly larger than $d - \varepsilon/2$. Equivalently, for R sufficiently large,

$$\int_{|\xi|\leq R} |\widehat{\mu}(\xi)|^{2n} \,\mathrm{d}\xi \lesssim R^{\varepsilon/2}.$$

By Chebychev's inequality, we can convert the L^2 estimate to an exceptional set estimate:

$$m\left(\{\xi \in \mathbb{R}^d : |\xi| \leq 2R \text{ and } |\widehat{\mu}(\xi)| > R^{-\delta}/2\}\right) \lesssim R^{\varepsilon/2+2\delta n}.$$

But we recall that $\widehat{\mu}$ is Lipschitz continuous, so for R sufficiently large, if $|\widehat{\mu}(\xi)| > R^{-\delta}$ for some ξ, the $|\widehat{\mu}(\xi)| \geq R^{-\delta/2}$ on a ball of radius $R^{-2\delta}$. It therefore follows that

$$\{\xi \in \mathbb{R}^d : |\xi| \leq R \text{ and } |\widehat{\mu}(\xi)| \geq R^{-\delta}\}$$

can be covered by $O(R^{\varepsilon/2+\delta(2n+2d)})$ balls of radius 1. Taking δ sufficiently small so that $\delta(2n+2d) \leq \varepsilon/2$ yields the claim. □

3. APPLICATIONS FOR SELF-SIMILAR MEASURES

In the previous sections, we saw general conditions under which the Fourier transform has good decay properties almost everywhere. To conclude this note, we give a concrete application: we show that all self-similar measures in $\mathbb{R}^d$ satisfy the affine non-concentration hypothesis, regardless of separation conditions. The proof we give here is due to [BY24+]. We note that this result is similar to an analogous result due to Feng & Lau for measures of balls, namely that every self-similar measure has finite Frostman exponent: see [FL09, Proposition 2.2].

3.1. Affine non-concentration of self-similar measures. The proof is split into two parts. First, we prove an initial non-concentration estimate at a fixed scale, which holds for arbitrary measures.

Lemma 3.1. *Let ν be a finite compactly supported Borel measure on $\mathbb{R}^d$ which is not supported on an affine hyperplane. Then there exists $\delta, \gamma \in (0,1)$ such that for all $W \in \mathbb{A}(d, d-1)$,*

$$\nu\big(W(\gamma)\big) \leq (1-\delta)\mu(\mathbb{R}^d).$$

Proof. Suppose for contradiction that this is false, and get a sequence $(W_n)_{n=1}^\infty \subset \mathbb{A}(d, d-1)$ such that $\mu(W_n(1/n)) \geq \mu(\mathbb{R}^d) - 1/n$ for all $n \in \mathbb{N}$. By compactness of $\mathbb{A}(d, d-1)$, let $W = \lim_{n\to\infty} W_n$. Let B be a compact ball containing the support of ν, so that $W_n \cap B$ converges to $W \cap B$ in the Hausdorff metric. In particular, for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that for all $n \geq N$, $W_n(1/n) \cap B \subset W(\varepsilon) \cap B$. Therefore $\mu(W(\varepsilon)) = \mu(\mathbb{R}^d)$ for all $\varepsilon > 0$, so $\mu(W) = \mu(\mathbb{R}^d)$, which is a contradiction. ☐

Then we use self-similarity to repeatedly apply the initial non-concentration estimate to obtain exponential decay of the measures of neighbourhoods of hyperplanes.

Proposition 3.2 ([BY24+]). *Let ν be a self-similar measure in $\mathbb{R}^d$ and which is not supported on a hyperplane. Then there exists an $\alpha > 0$ so that for all $c > 1$, there is a constant $C > 0$ (depending on c) such that for all $0 < r < 1$, $W \in \mathbb{A}(d, d-1)$, and $\eta \in (0,1)$,*

$$\nu\big(W(\eta r) \cap B(x,r)\big) \leq C\eta^\alpha \nu\big(B(x, cr)\big).$$

Proof. Let ν be associated with the IFS $\{S_i\}_{i\in\mathcal{I}}$ and let $r_{\min} = \min\{r_i : i \in \mathcal{I}\}$. Write $K = \operatorname{supp}\nu$; without loss of generality, we may assume that K is contained in a ball of diameter 1. Let $\delta, \gamma \in (0,1)$ be as in Lemma 3.1. Fix $c > 1$, and define $\eta_n = (c-1)(\gamma r_{\min}/4)^{n-1}$.

Now let $0 < r < 1$, $W \in \mathbb{A}(d, d-1)$ and $x \in \operatorname{supp}\nu$ be arbitrary. We prove by induction in n that

$$\nu\big(W(r\eta_n) \cap B(x, r(1+\eta_n))\big) \leq (1-\delta)^{n-1}\nu\big(B(x, cr)\big). \tag{3.1}$$

For notational simplicity, write $r_n := r\eta_n$. First, the claim holds when $n = 1$ since by definition of r_1, $B(x, r + r_1) \subset B(x, cr)$ for all x. Now assume that the claim holds for $n \in \mathbb{N}$. Write $\nu = \sum_{\mathtt{i}\in\Lambda} p_{\mathtt{i}}\nu \circ S_{\mathtt{i}}^{-1}$, where Λ is a stopping time with the property that $2r_{\mathtt{i}}(r_n)^{-1} \in [r_{\min}, 1]$. Fix $x \in K$ and $W \in \mathbb{A}(d, d-1)$, and set

$$\Lambda' := \big\{\mathtt{i} \in \Lambda : S_{\mathtt{i}}(K) \cap W(r_{n+1}) \cap B(x, r + r_{n+1}) \neq \varnothing\big\}.$$

The choice of r_n and the definition of Λ ensures that $S_{\mathtt{i}}(K) \subset W(r_n) \cap B(x, r + r_n)$ for all $\mathtt{i} \in \Lambda'$, and the choice of η_n ensures that $S_{\mathtt{i}}^{-1}(W(r_{n+1})) \subset V(\gamma)$ for some $V \in \mathbb{A}(d, d-1)$. (Here, it is essential that $S_{\mathtt{i}}$ is a similarity). Therefore, applying Lemma 3.1,

$$\begin{aligned} \nu\big(W(r_{n+1}) \cap B(x, r + r_{n+1})\big) &\leq \sum_{\mathtt{i} \in \Lambda'} p_{\mathtt{i}} \nu\big(S_{\mathtt{i}}^{-1}(W(r_{n+1}) \cap B(x, r + r_{n+1}))\big) \\ &\leq \sum_{\mathtt{i} \in \Lambda'} p_{\mathtt{i}} (1 - \delta) \\ &\leq (1 - \delta) \nu\big(W(r_n) \cap B(x, r + r_n)\big) \\ &\leq (1 - \delta)^n \nu\big(B(x, cr)\big). \end{aligned}$$

In the last inequality, we have used the inductive hypothesis.

To complete the proof, let $\alpha > 0$ be sufficiently small so that $(\gamma r_{\min}/4)^\alpha \geq 1 - \delta$. Then by (3.1), for all $n \in \mathbb{N}$,

$$\nu\big(W(r\eta_n) \cap B(x, r)\big) \leq \frac{1}{(c-1)^\alpha} \eta_n^\alpha \nu\big(B(x, cr)\big).$$

Finally, if $\eta \in (0, 1)$ is arbitrary, if $\eta > c - 1$, then the claim is trivial, and otherwise let $n \in \mathbb{N}$ be maximal so that $\eta \leq \eta_n$ and

$$\begin{aligned} \nu\big(W(r\eta) \cap B(x, r)\big) &\leq \nu\big(W(r\eta_n) \cap B(x, r)\big) \\ &\leq \frac{1}{(c-1)^\alpha} \eta_n^\alpha \nu\big(B(x, cr)\big) \\ &\leq \left(\frac{4}{\gamma r_{\min}(c-1)}\right)^\alpha \eta^\alpha \nu\big(B(x, cr)\big) \end{aligned}$$

as required. □

Combining this result with Corollary 2.4 due to Khalil–Shmerkin yields the following application.

Corollary 3.3 ([BY24+]). *Let μ be a self-similar measure not supported on a proper affine subspace. Then μ is arbitrarily L^2-smooth under convolution.*

Further applying Proposition 2.8 yields the following.

Corollary 3.4 ([BY24+]). *Let μ be a self-similar measure. Then μ has Fourier decay outside sparse frequencies if and only if μ is not supported on a proper affine subspace.*

ACKNOWLEDGEMENTS

The author thanks Amlan Banaji for interesting conversations about this topic, and for sharing a draft version of his preprint [BY24+] during the preparation of this note. He also thanks Vilma Orgoványi for many useful comments on a draft of this document.

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