

Exercise 5

DUE 12:15PM ON THURSDAY, FEBRUARY 19

Recall the notation from the previous assignment: if \mathcal{I} is a finite index set and $\mathbf{p} \in \mathcal{P}(\mathcal{I})$, then $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$ is a measure-preserving system where \mathcal{B} is the Borel σ -algebra and σ is the left shift map.

The questions.

1. (2 pt.) Consider the space $([0, 1]^2, \mathcal{B}([0, 1]^2), m)$ where m is Lebesgue measure and $\mathcal{B}([0, 1]^2)$ is the Borel σ -algebra on $[0, 1]^2$. Let $\mathcal{A} \subset \mathcal{B}([0, 1]^2)$ denote the sub- σ -algebra

$$\mathcal{A} = \mathcal{B}([0, 1]) \times \{\emptyset, [0, 1]\}$$

where $\mathcal{B}([0, 1])$ is the Borel σ -algebra on $[0, 1]$.

Let $f: [0, 1]^2 \rightarrow \mathbb{R}$ be a Borel-measurable and integrable function. Prove that

$$\mathbb{E}(f \mid \mathcal{A})(x, y) = \int_0^1 f(x, z) \, dz$$

for Lebesgue a.e. $(x, y) \in [0, 1]^2$.

Hint: A system of conditional measures might be useful here.

2. (3 pt.) Let (X, \mathcal{B}, μ, T) be a measure-preserving system.

(i) Prove that the system is ergodic if and only if for all $A, B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B)$$

- (ii) Recall that a family \mathcal{C} of sets is a *semi-algebra* of sets if the following hold:

- (a) $\emptyset \in \mathcal{C}$.
- (b) If $E \in \mathcal{C}$, then $X \setminus E$ is a finite disjoint union of elements in \mathcal{C} .
- (c) If $E, F \in \mathcal{C}$ then $E \cap F \in \mathcal{C}$.

Let $\mathcal{C} \subset \mathcal{B}$ be a semi-algebra such that \mathcal{B} is the σ -algebra generated by \mathcal{C} . Suppose for all $A, B \in \mathcal{C}$ that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B).$$

Prove that the system is ergodic.

3. (2 pt.) Give an example of a finite index set \mathcal{I} , a probability vector $\mathbf{p} \in \mathcal{P}(\mathcal{I})$, and a continuous function $f: \mathcal{I}^{\mathbb{N}} \rightarrow \mathbb{R}$ such that Von Neumann's ergodic theorem (Theorem 3.26 in the notes) fails for $p = \infty$ in the measure-preserving space $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$.
4. (3 pt.) Let \mathcal{C} denote the set of cylinders in the measure-preserving space $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$.

(i) Prove for all cylinders $[i], [j] \in \mathcal{C}$ that

$$\lim_{k \rightarrow \infty} \mu(\sigma^{-k}[i] \cap [j]) = \mu([i])\mu([j]).$$

Conclude that the system is ergodic.

This property is called strong mixing.

- (ii) Two measure-preserving systems (X, \mathcal{B}, μ, T) and (Y, \mathcal{A}, ν, S) are *isomorphic* if there are invariant subsets $X_0 \in \mathcal{B}$ and $Y_0 \in \mathcal{A}$ with full measure and a bijection $\pi: X_0 \rightarrow Y_0$ such that π and π^{-1} are measurable, $\mu \circ \pi^{-1} = \nu$, and $\pi \circ T = S \circ \pi$. Observe that ergodicity is an isomorphism invariant.

Let $b \in \mathbb{N}$, $b \geq 2$. Recall from the notes that $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, T_b)$ is a measure-preserving system, where m is Lebesgue measure, \mathcal{A} is the Borel σ -algebra, and T_b is multiplication by $b \pmod{1}$. Show that $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$ and $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, T_b)$ are isomorphic for an appropriate choice of \mathcal{I} and \mathbf{p} .

- (iii) **(1 pt. bonus)** Prove that the shift space $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$ is *not isomorphic* to the space $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, R_\alpha)$, where \mathcal{A} is the Borel σ -algebra on \mathbb{R}/\mathbb{Z} , α is irrational, and $R_\alpha(x) = x + \alpha \pmod{1}$ for any choice of \mathcal{I} and \mathbf{p} .
5. **(1 pt. bonus)** Let (X, \mathcal{B}, μ, T) be a measure-preserving system where μ is ergodic and does not contain any atoms. Prove that the σ -algebra of (exactly) T -invariant sets $\{E \in \mathcal{B} : T^{-1}(E) = E\}$ is not countably generated.