Attainable forms of Assouad spectra

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ABSTRACT. Let $d \in \mathbb{N}$ and let $\varphi\colon (0,1) \to [0,d]$. We prove that there exists a set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^\theta F = \varphi(\theta)$ for all $\theta \in (0,1)$ if and only if for every $0 < \lambda < \theta < \lambda^{1/2} < 1$,

$$0 \le (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \le (\theta - \lambda)\varphi\Big(\frac{\lambda}{\theta}\Big).$$

In particular, the following behaviours which have not previously been witnessed in any examples are possible: the Assouad spectrum can be non-monotonic on every open set, and can fail to be Hölder in a neighbourhood of 1.

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1. INTRODUCTION

The Assouad dimension is a particular notion of dimension which captures the scaling properties of the "thickest" part of a set. This in contrast to the more usual notions of box (or Hausdorff) dimension, which are in some sense a global measurement of scaling. The Assouad dimension was originally introduced in [1] to study the embedding theory of metric spaces. More recently, the Assouad dimension has received a significant amount of attention in the literature: see, for example, the books by Mackay and Tyson on conformal geometry [14], Robinson on embedding theory [15], and Fraser on Assouad dimension in fractal geometry [6]

If the box dimension and the Assouad dimension of a set agree, this implies that the set has a large amount of spatial regularity. For instance, this is the case for any Ahlfors-regular subset of $\mathbb{R}^d$. However, the box dimension and Assouad dimension can be distinct for many naturally-occurring sets, such as self-conformal sets with overlaps or self-affine sets. In order to obtain a more fine-grained understanding of the Assouad dimension in this situation, the *Assouad spectrum* was introduced by Fraser and Yu in [11]. This is a notion of dimension parametrized by a variable $\theta \in (0,1)$, which approaches the box dimension as θ approaches 0 and the (quasi-)Assouad dimension as θ approaches 1. We refer the reader to [6] for a general introduction to Assouad-type dimensions.

Besides being a useful bi-Lipschitz invariant and an important notion of fractal dimension in its own right, the Assouad spectrum provides quantitative information about the Assouad dimension. As a result, the Assouad spectrum has been explicitly studied for a wide range of examples (see, for example, [4, 8, 9, 10, 11]). This relationship has also been useful in applications outside of fractal geometry. For instance, the Assouad spectrum plays an important role in the work by Roos and Seeger [16] on L^p bounds for spherical maximal operators. The Assouad spectrum has also been used to obtain bounds for quasiconformal distortion in geometric mapping theory [12].

In this paper, rather than consider explicit examples and applications of the Assouad spectrum, we focus on the general question of classification: what constraints on a function $\varphi\colon (0,1) \to [0,d]$ guarantee that there is a set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^\theta F = \varphi(\theta)$ for all $\theta \in (0,1)$?

1.1. **Classifying Assouad spectra.**

We fix $d \in \mathbb{N}$ and work in $\mathbb{R}^d$ with the Euclidean norm. We write $B(x,\delta)$ to denote the open ball centred at x with radius δ. If F is a bounded subset of $\mathbb{R}^d$, for $\delta > 0$, we let $N_\delta(F)$ denote the least number of balls of radius δ required to cover F. Then, for $\theta \in (0,1)$, the *Assouad spectrum* of $F \subset \mathbb{R}^d$ is given by

$$\dim_{\mathrm{A}}^\theta F = \inf\Big\{\alpha : (\exists C > 0)(\forall 0 < \delta \le 1)(\forall x \in F)\ N_{\delta^{1/\theta}}(F \cap B(x,\delta)) \le C\Big(\frac{\delta}{\delta^{1/\theta}}\Big)^\alpha\Big\}.$$

In general, $\lim_{\theta\to 0} \dim_{\mathrm{A}}^\theta F = \overline{\dim}_{\mathrm{B}} F$, and $\lim_{\theta\to 1} \dim_{\mathrm{A}}^\theta F = \dim_{\mathrm{qA}} F$ [7], where $\dim_{\mathrm{qA}} F$ denotes the quasi-Assouad dimension of F as introduced by Lü and Xi [13]. Like the

Assouad dimension, the Assouad spectrum measures the worst-case local scaling of the set, but the Assouad spectrum specifies the relationship between the small and large scales.

The main result of this paper is to give a complete classification of possible forms of Assouad spectra.

Theorem A. *Let $d \in \mathbb{N}$ and let $\varphi\colon (0,1) \to [0,d]$ be a function. Then the following are equivalent:*

(a) There exists $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ for all $\theta \in (0,1)$.

(b) For every $0 < \lambda < \theta < \lambda^{1/2} < 1$,

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta - \lambda)\varphi\Big(\frac{\lambda}{\theta}\Big). \tag{1.1}$$

(c) φ is the supremum of functions of the form $\theta \mapsto h(\theta)/(1-\theta)$ where

$$h(\theta) = \begin{cases} \kappa(1 - \frac{c_1}{c_2} + c_2 - c_1) & : 0 < \theta < c_1 \\ \kappa(1 - \frac{c_1}{c_2} + c_2 - \theta) & : c_1 \leq \theta < c_2 \\ \kappa(1 - \frac{c_1}{c_2}) & : c_2 \leq \theta < c_1/c_2 \\ \kappa(1-\theta) & : c_1/c_2 < \theta < 1 \end{cases}$$

for $0 < c_1 \leq c_2 \leq c_1^{1/2} < 1$ and $\kappa \in [0,d]$.

The proof of this result is given in Section 3. For a geometric interpretation of the bound (b), we refer the reader to Section 2.1, and for a description of the class of functions in (c), see Section 3.3 (and in particular Figure 2). That (a) implies (b) is well-known (see, for example, [6, Theorem 3.3.1]), but all other implications are new.

We can interpret the first inequality in (1.1) as a growth rate constraint, and the second inequality as an oscillation constraint. In fact, the second inequality is always satisfied when φ is increasing (the short argument is given in Lemma 2.7), which yields the following corollary:

Corollary B. *Let $d \in \mathbb{N}$ and let $\varphi\colon (0,1) \to [0,d]$ be an increasing function. Then there exists a set $F \subset \mathbb{R}^d$ with $\dim_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ if and only if $\theta \mapsto (1-\theta)\varphi(\theta)$ is decreasing.*

We can also obtain results for the *upper Assouad spectrum*, which is defined by bounding the lower scale from above, rather than specifying the relationship precisely:

$$\overline{\dim}_{\mathrm{A}}^{\theta} F = \inf\Big\{\alpha : (\exists C > 0)(\forall 0 < \delta \leq 1)(\forall 0 < \delta' \leq \delta^{1/\theta})(\forall x \in F) \\ N_{\delta'}(B(x,\delta)) \leq C\Big(\frac{\delta}{\delta'}\Big)^{\alpha}\Big\}.$$

The upper Assouad spectrum is closely related to the Assouad spectrum: in fact $\overline{\dim}_{\mathrm{A}}^{\theta} F = \sup_{0<\theta'<\theta} \dim_{\mathrm{A}}^{\theta'} F$ by [7, Theorem 2.1]. Combining this with Corollary 2.8 gives a full characterization of the upper Assouad spectrum.

Corollary C. *Let $d \in \mathbb{N}$ and let $\overline{\varphi}\colon (0,1) \to [0,d]$ be an arbitrary function. Then the following are equivalent:*

(a) There exists a set $F \subset \mathbb{R}^d$ such that $\overline{\dim}_{\mathrm{A}}^{\theta} F = \overline{\varphi}(\theta)$ for all $\theta \in (0,1)$.
(b) $\overline{\varphi}(\theta)$ is increasing and $\theta \mapsto (1-\theta)\overline{\varphi}(\theta)$ is decreasing.
(c) $\overline{\varphi}$ is the supremum of functions of the form $\theta \mapsto f(\theta)/(1-\theta)$ where

$$f(\theta) = \begin{cases} \kappa(1-c) & : 0 < \theta \leq c \\ \kappa(1-\theta) & : c < \theta < 1 \end{cases}$$

for $c \in (0,1)$ and $\kappa \in [0,d]$.

Beyond giving a full classification, Theorem A also clarifies many of the properties of the Assouad spectrum: certain observations which might *a priori* depend on explicit properties of the Assouad spectrum in fact only require the bound (1.1). For instance, the observation that if $\overline{\dim}_{\mathrm{B}} F = 0$ then $\dim_{\mathrm{A}}^{\theta} F = 0$ only requires the fact that $\lim_{\theta \to 0} \dim_{\mathrm{A}}^{\theta} F = \overline{\dim}_{\mathrm{B}} F$ along with the general bound (see Proposition 2.4).

Having completed the classification, in Section 4 we construct some exceptional sets. Our first result concerns Hölder regularity.

Theorem D. *There is a compact set $F \subset \mathbb{R}$ such that $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ is not Hölder in any neighbourhood of 1.*

In fact, there is no lower control on the rate at which $\dim_{\mathrm{A}}^{\theta} F$ approaches $\dim_{\mathrm{qA}} F$. See Section 4.1 for the details. This result is sharp: in Proposition 2.5, we prove that $\dim_{\mathrm{A}}^{\theta} F$ is (uniformly) Lipschitz on $(0, 1-\delta)$ for all $\delta > 0$, with constants depending only on δ and the ambient dimension d. This observation, along with Theorem D, provides a complete answer to [11, Question 9.2].

Finally, we address the question of monotonicity. In [6, Question 17.7.1], Fraser conjectures that the Assouad spectrum must be monotonic in some neighbourhood of 1. We provide a strong negative answer to this question: we show that Assouad spectra that are non-monotonic on any open set are dense in the set of all possible Assouad spectra.

Theorem E. *For any $\epsilon > 0$ and function φ satisfying one of the equivalent constraints in Theorem A, there is a compact set $F \subset \mathbb{R}$ such that $\psi(\theta) = \dim_{\mathrm{A}}^{\theta} F$ is non-monotonic on any open subset of $(0,1)$ and $\|\psi - \varphi\|_{\infty} < \epsilon$.*

Since $\dim_{\mathrm{A}}^{\theta} F$ is Lipschitz on $(0, 1-\delta)$ for every $\delta > 0$, if φ is non-constant then by Rademacher's theorem φ must have strictly positive derivative on a set with positive Lebesgue measure. This is sharp: using similar techniques as used in the proof of Theorem E, one can construct examples of sets with quasi-Assouad dimension d, box dimension arbitrarily close to 0, and Assouad spectrum that is strictly decreasing on a dense open subset of $(0,1)$ with Lebesgue measure arbitrarily close to 1. We leave the details of such a construction to the interested reader.

1.2. **Rate constraints and the relationship with intermediate dimensions.** The intermediate dimensions are a different notion of dimension spectrum introduced in [5] that interpolate between the Hausdorff and box dimensions. In [2], the author and Banaji fully classify the possible forms of the intermediate dimensions. For simplicity, in the discussion that follows we assume that the intermediate dimensions exist and denote these dimensions by $\dim_\theta F$. We refer the reader to [2] for precise statements of the results in full generality.

Recall that the (upper right) Dini derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$ at x is given by

$$D^+f(x) = \limsup_{\epsilon \searrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}. \tag{1.2}$$

We then recall the following result:

Theorem 1.1 ([2]). *Let $g\colon (0,1) \to [0,d]$. Then there exists a non-empty bounded set $F \subset \mathbb{R}^d$ with $\dim_\theta F = g(\theta)$ if and only if*

$$0 \leq D^+g(\theta) \leq \frac{g(\theta)(d - g(\theta))}{d\theta} \tag{1.3}$$

for all $\theta \in (0,1)$.

On the other hand, if $\varphi\colon (0,1) \to [0,d]$, Corollary C gives that there exists $F \subset \mathbb{R}^d$ such that $\overline{\dim}_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ if and only if

$$0 \leq D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta} \tag{1.4}$$

for all $\theta \in (0,1)$. In particular, when $\dim_{\mathrm{B}} F < d$, $D^+g(\theta) \leq \frac{g(\theta)}{\theta} \cdot (d - \dim_{\mathrm{B}} F)/d$, so an arbitrary function which is the intermediate dimension of some set can be transformed to be the upper Assouad spectrum of a set through multiplication by a constant, reflection, and translation—and vice versa.

1.3. **Structure and outline of the paper.** In Section 2, we study the family of functions $\mathcal{A}_d$ (see Definition 2.1) which satisfy the bound (1.1) for some fixed $d \in \mathbb{N}$. This is the family which we will prove is the set of possible forms of Assouad spectra for subsets of $\mathbb{R}^d$. First, in Proposition 2.4, we establish a number of basic properties of such functions. The Assouad spectrum has been known to satisfy these properties, but here we only require the bound (1.1) and do not require any geometric properties of the Assouad spectrum itself. Then in Section 2.3, we establish the growth rate bounds and the corresponding Lipschitz constraints.

Now, in Section 3, we prove Theorem A. The general bound (a) implies (b) is standard, and follows by a straightforward covering argument: we give the details in Proposition 3.1. To see that (b) implies (a), we will construct a homogeneous Moran set with prescribed Assouad spectrum using the techniques from [2]. This result is encapsulated in Proposition 3.3, where for a function satisfying certain derivative constraints, there exists a homogeneous Moran set such that the Assouad spectrum is given by a certain convenient formula. It then remains to choose such a function carefully, which is done in Theorem 3.5. Next, the family considered in (c) is described fully in Section 3.3, and (c) implies (b) is given by proving in Proposition 3.6 that each function is an element of $\mathcal{A}_d$, and that $\mathcal{A}_d$ is closed under taking suprema (see Proposition 3.7). Finally, we relate the bound (1.1) with the family of functions in (c) in Lemma 3.9 and Lemma 3.10. These lemmas then combine to give the proof of (b) implies (c) in Theorem 3.11.

To conclude, we use the classification result, as well as the techniques used in the proof, to construct examples of sets with exceptional Assouad spectra. The result proving Hölder failure at 1 is described in Section 4.1. Then in Section 4.2 we use the general family of non-monotonic spectra from Section 3.3 along with ideas from the proof of Theorem 3.11 to construct a set with Assouad spectra which is not monotonic on any open subset of $(0, 1)$.

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2. Forms of the family of functions $\mathcal{A}_d$

We define the family of functions $\mathcal{A}_d$, which we will prove in Section 3 are the possible forms of the maps $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ for sets $F \subset \mathbb{R}^d$.

Definition 2.1. *Let $\mathcal{A}_d$ denote the set of functions $\varphi\colon (0,1) \to [0, d]$ where for any $0 < \lambda < 1$ and $\lambda < \theta < \lambda^{1/2}$,*

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta - \lambda)\varphi\Big(\frac{\lambda}{\theta}\Big). \tag{2.1}$$

In this section, we study properties of the family $\mathcal{A}_d$ directly: we emphasize that we do not require any geometric facts about the Assouad spectrum itself.

In Proposition 2.4, we will prove that functions in $\mathcal{A}_d$ are uniformly continuous. Thus, we will embed $\mathcal{A}_d$ in $C([0, 1])$ by defining $\varphi(0) = \lim_{\theta \to 0} \varphi(\theta)$ and $\varphi(1) = \lim_{\theta \to 1} \varphi(\theta)$. We will use this notation once we prove uniform continuity.

2.1. Rescaling and a geometric interpretation of the bound.

Given $\varphi \in \mathcal{A}_d$, define $\beta(\theta) = (1 - \theta)\varphi(\theta)$. In (2.1), the first inequality implies that $\beta(\theta)$ is decreasing, and the second states that for all $0 < \lambda < \theta < \lambda^{1/2} < 1$,

$$\frac{\beta(\lambda) - \beta(\theta)}{\theta - \lambda} \leq \frac{\beta\big(\frac{\lambda}{\theta}\big)}{1 - \frac{\lambda}{\theta}}. \tag{2.2}$$

The left hand side is the negative of the slope of the line passing through $(\lambda, \beta(\lambda))$ and $(\theta, \beta(\theta))$, and the right hand side is the negative of the slope of the line passing through $(\lambda/\theta, \beta(\lambda/\theta)$ and $(1, 0)$. This constraint is depicted in Figure 1.

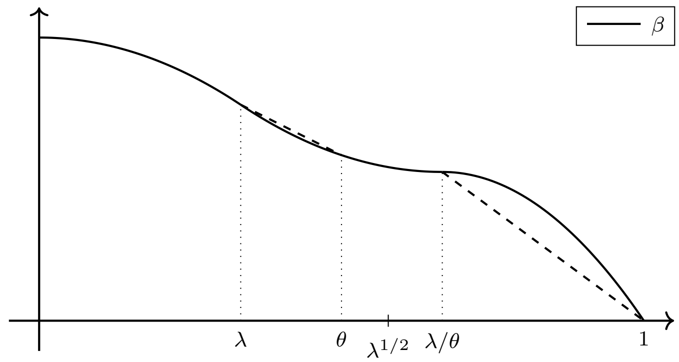

FIGURE 1. A plot of $\beta(\theta) = (1-\theta)\varphi(\theta)$ where $\varphi \in \mathcal{A}_d$, and the lines with slopes corresponding to (2.2).

2.2. **Basic properties.** In this section, we collect various properties of the family $\mathcal{A}_d$. We first note that (2.1) can be extended to all $0 < \lambda < \theta < 1$.

Lemma 2.2. *If $\varphi \in \mathcal{A}_d$, for any $0 < \lambda < \theta < 1$,*

$$0 \le (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \le (\theta - \lambda)\varphi\Big(\frac{\lambda}{\theta}\Big).$$

Proof. Write $\beta(\theta) = (1-\theta)\varphi(\theta)$. If $\lambda^{1/2} < \theta < 1$, then $\lambda/\theta < \lambda^{1/2} < \theta$ so $\beta(\theta) \le \beta(\lambda/\theta)$ and

$$\frac{\beta(\theta)}{1+\theta} \le \frac{\beta\big(\frac{\lambda}{\theta}\big)}{1+\frac{\lambda}{\theta}}.$$

Thus applying (2.2) to λ and λ/θ, by the above inequality

$$\beta(\theta) - \beta(\lambda) \le \theta\beta\Big(\frac{\lambda}{\theta}\Big) + \left(1 + \frac{\lambda}{\theta}\right)\beta(\theta) - (1+\theta)\beta\Big(\frac{\lambda}{\theta}\Big) \le \theta\beta\Big(\frac{\lambda}{\theta}\Big).$$

Then the result extends to $\theta = \lambda^{1/2}$ by continuity in λ and θ of the bound (2.1) □

Next, we observe the following useful lemma which was essentially proven in [11, Remark 3.8]. Here, we obtain it as a direct consequence of (2.1). Heuristically, this lemma states that the function $\varphi(\theta)$ is almost increasing, up to some possible local oscillations.

Lemma 2.3. *Let $\varphi \in \mathcal{A}_d$. Given $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$,*

$$\varphi(\theta_1) \le \max\left\{\varphi\Big(\frac{\theta_1}{\theta_2}\Big), \varphi\Big(\frac{\theta_2}{\theta_3}\Big), \ldots, \varphi\Big(\frac{\theta_{n-1}}{\theta_n}\Big), \varphi(\theta_n)\right\}.$$

In particular, for any $n \in \mathbb{N}$ and $\theta \in (0,1)$, $\varphi(\theta) \le \varphi(\theta^{1/n})$.

Proof. Let $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$. Applying Lemma 2.2 to each pair θ_i, θ_{i+1},

$$(1-\theta_1)\varphi(\theta_1) \leq (1-\theta_n)\varphi(\theta_n) + \sum_{k=2}^{n}(\theta_k - \theta_{k-1})\varphi\Big(\frac{\theta_{k-1}}{\theta_k}\Big)$$

from which the result follows. Taking $\theta_i = \theta^{\frac{n-i+1}{n}}$ for each $i = 1, \ldots, n$, observe that $\theta_{k-1}/\theta_k = \theta^{1/n}$ and $\theta_n = \theta^{1/n}$ so that $\varphi(\theta) \leq \varphi(\theta^{1/n})$. □

We now have the following essential properties of $\mathcal{A}_d$. All of these properties have been previously observed for the Assouad spectrum, but the main point here is that these properties only depend on the family $\mathcal{A}_d$ and not on other properties of the Assouad spectrum. Some of these properties will be used in the proof of Theorem 3.5, so we cannot formally depend on the corresponding results for the Assouad spectrum. We draw on ideas from [7, 11].

Proposition 2.4. *Let $\varphi \in \mathcal{A}_d$ be arbitrary. Then the following properties hold:*

(i) The limits $\varphi(0) := \lim_{\theta\to 0}\varphi(\theta)$ and $\varphi(1) := \lim_{\theta\to 1}\varphi(\theta)$ exist.
(ii) Each $\varphi \in \mathcal{A}_d$ is uniformly continuous.
(iii) $\varphi(0) = \inf_{\theta\in(0,1)}\varphi(\theta)$ and $\varphi(1) = \sup_{\theta\in(0,1)}\varphi(\theta)$.
(iv) For any $\theta_0 \in (0,1)$, if $\varphi(\theta_0) = \varphi(1)$, then $\varphi(\theta_0) = \varphi(\theta)$ for all $\theta_0 < \theta < 1$.
(v) If $\varphi(0) = 0$, then $\varphi(\theta) = 0$ for all θ.

Proof. First, we show that $\varphi(\theta)$ is continuous on $(0,1)$. For $0 < \theta_1 < \theta_2 < 1$ we have $\theta_1 < \theta_1/\theta_2 < 1$, so applying Lemma 2.2 we obtain

$$(1-\theta_2)\varphi(\theta_2) \leq (1-\theta_1)\varphi(\theta_1) \leq \frac{\theta_1}{\theta_2}(1-\theta_2)\varphi(\theta_2) + \Big(1 - \frac{\theta_1}{\theta_2}\Big)\varphi\Big(\frac{\theta_1}{\theta_2}\Big). \tag{2.3}$$

This implies that

$$|\varphi(\theta_1) - \varphi(\theta_2)| \leq \frac{\varphi(\theta_1/\theta_2)}{\theta_2(1-\theta_1)}|\theta_2 - \theta_1|.$$

Since $\varphi(\theta_1/\theta_2) \leq d$, it follows that $\varphi(\theta)$ is Lipschitz on any closed subinterval of $(0,1)$, and therefore continuous on $(0,1)$.

Now we see (i). Observe that $(1-\theta)\varphi(\theta)$ is a bounded decreasing function of θ, so $\lim_{\theta\to 0}(1-\theta)\varphi(\theta)$ exists so $\lim_{\theta\to 0}\varphi(\theta)$ exists as well. To see that $\lim_{\theta\to 1}\varphi(\theta)$ exists, we use the proof from [7, Section 3.2]. Set $L = \limsup_{\theta\to 1}\varphi(\theta)$ and let $\epsilon > 0$. Since $\varphi(\theta)$ is continuous, we can find $0 < u < v < 1$ such that $\varphi(\theta) > L - \epsilon$ for all $\theta \in [u, v]$. Thus by Lemma 2.3, with

$$X := \bigcup_{n=1}^{\infty}[u^{1/n}, v^{1/n}]$$

we have $\varphi(\theta) > L - \epsilon$ for all $\theta \in X$. But $v^{1/n} \geq u^{1/(n+1)}$ for all $n \geq n_0$ with $\frac{n_0}{n_0+1} \geq \frac{\log v}{\log u}$, so in fact $(u^{1/n_0}, 1) \subset X$. Thus $\lim_{\theta\to 1}\varphi(\theta)$ exists as well. In particular, combining the existence of endpoint limits with continuity of φ on $(0,1)$, (ii) also follows immediately.

To see (iii), if $\theta_1 \in (0,1)$, then $\theta_n = \theta_1^{1/n}$ is a sequence converging monotonically to 1 with $\varphi(\theta_n) \geq \varphi(\theta_1)$ by Lemma 2.3. Thus $\varphi(1) \geq \varphi(\theta_1)$. Similarly $\varphi(\theta_1^n) \leq \varphi(\theta_1)$ for any $n \in \mathbb{N}$, and $\lim_{n\to\infty} \theta_1^n = 0$. But θ_1 was arbitrary, giving (iii).

Now we see (iv). Suppose $\varphi(1) = \varphi(\theta_1)$ for some $0 < \theta_1 < 1$. By (2.3),

$$(1-\theta_1)\varphi(1) - (1-\theta_2)\varphi(\theta_2) \leq (\theta_2 - \theta_1)\varphi(\theta_1/\theta_2) \leq (\theta_2-\theta_1)\varphi(1)$$

since $\varphi(\theta_1/\theta_2) \leq \varphi(1)$ by (iii). This implies that $\varphi(1) \leq \varphi(\theta_2)$, so (iv) follows.

To see (v), if $\varphi(0) = 0$, then $\lim_{\theta\to 0}(1-\theta)\varphi(\theta) = 0$. But $(1-\theta)\varphi(\theta)$ is a decreasing function of θ, so $(1-\theta)\varphi(\theta) = 0$ for all $\theta \in (0,1)$, i.e. $\varphi(\theta) = 0$ for all $\theta \in (0,1)$. □

2.3. **Rate constraints and increasing functions.** Now, we obtain bounds on growth rates of functions in $\mathcal{A}_d$. These growth rate bounds are similar to the bounds obtained in [2].

We recall that the Dini derivative is defined in (1.2). We obtain the following regularity property for functions $\varphi \in \mathcal{A}_d$.

Proposition 2.5. *Let $\varphi \in \mathcal{A}_d$ be arbitrary and $\theta \in (0,1)$. Then*

$$-\varphi(1) \leq D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}$$

In particular, φ is Lipschitz on $[0, 1-\delta]$ for any $\delta > 0$.

Proof. The first inequality in (2.1) is equivalent to saying that $\beta(\theta) = (1-\theta)\varphi(\theta)$ is decreasing. Since φ is continuous by Proposition 2.4, by [3, Corollary 11.4.2] β is decreasing if and only if $D^+\beta \leq 0$, or equivalently

$$D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}.$$

This gives the upper bound.

To obtain the lower bound, let $0 < \lambda < \theta < \lambda^{1/2} < 1$ be arbitrary. Then rearranging (2.1), we obtain

$$\frac{\varphi(\theta)-\varphi(\lambda)}{\theta-\lambda} \geq \frac{\varphi(\theta) - \frac{\lambda}{\theta}\varphi(\lambda)}{1-\frac{\lambda}{\theta}} - \varphi\Big(\frac{\lambda}{\theta}\Big).$$

But $|\varphi(\theta) - (\lambda/\theta)\varphi(\lambda)| \leq d$ so that

$$\lim_{\theta\searrow\lambda} \frac{\varphi(\theta) - \frac{\lambda}{\theta}\varphi(\lambda)}{1-\frac{\lambda}{\theta}} = 0$$

and therefore $D^+\varphi(\lambda) \geq -\varphi(1)$. But this holds for all $\lambda \in (0,1)$, so in fact $D^+\varphi(\lambda) \geq -\varphi(1)$, as required. □

Remark 2.6. *In Section 4.1, we will see that, in general, elements of $\mathcal{A}_d$ need not be Lipschitz (in fact, not even Hölder) on the entire interval $[0,1]$.*

Now, we obtain the following result concerning increasing functions.

Lemma 2.7. *If $\varphi\colon (0,1) \to [0,d]$ is increasing, then $\varphi \in \mathcal{A}_d$ if and only if*

$$D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}. \tag{2.4}$$

Proof. The forward direction is just Proposition 2.5.

To obtain the reverse implication, let $0 < \lambda < \theta < \lambda^{1/2} < 1$. Since φ is increasing, $\varphi(\lambda) \leq \varphi(\theta) \leq \varphi(\lambda/\theta)$ and

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta-\lambda)\varphi(\theta) \leq (\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big)$$

which is (2.1). □

We obtain the following convenient application, which we use to characterize the upper Assouad spectra.

Corollary 2.8. *Let $\varphi \in \mathcal{A}_d$. Then $\overline{\varphi} \in \mathcal{A}_d$ where*

$$\overline{\varphi}(\theta) = \sup_{0<\theta'\leq\theta} \varphi(\theta').$$

Proof. As proven in Lemma 2.7, since $\overline{\varphi}(\theta)$ is increasing, we only need to verify that $D^+\overline{\varphi}(\theta) \leq \overline{\varphi}(\theta)/(1-\theta)$. Since $\varphi(\theta) \leq \overline{\varphi}(\theta)$, it suffices to show $D^+\overline{\varphi} \leq \max\{D^+\varphi, 0\}$.

Fix θ_0 and let $(\theta_n)_{n=1}^\infty \to \theta_0$ be strictly decreasing. Passing to a subsequence if necessary, we may assume $\overline{\varphi}(\theta_n) > \overline{\varphi}(\theta_0)$ for all n; otherwise $D^+\overline{\varphi}(\theta_0) \leq 0$. Thus for each n there is $\theta_0 < \theta_n' \leq \theta_n$ be such that $\varphi(\theta_n') = \overline{\varphi}(\theta_n)$. Thus

$$\frac{\overline{\varphi}(\theta_n) - \overline{\varphi}(\theta_0)}{\theta_n - \theta_0} \leq \frac{\varphi(\theta_n') - \varphi(\theta_0)}{\theta_n' - \theta_0} \leq D^+\varphi(\theta_0).$$

But $(\theta_n)_{n=1}^\infty$ was an arbitrary sequence, so the result follows. □

3. Classifying the forms of Assouad spectra

In this section, we prove Theorem A.

3.1. Bounding the Assouad spectrum.

We recall the following general bounds, which are given in [11, Proposition 3.4] and [6, Theorem 3.3.1]. We include the details here for completeness.

Proposition 3.1. *For any set $F \subset \mathbb{R}^d$, the function $\varphi(\theta) = \dim_{\mathrm{A}}^\theta F$ is in $\mathcal{A}_d$.*

Proof. Let $0 < \theta_1 < \theta_2 < 1$ and let $\epsilon > 0$ be arbitrary. For $\delta > 0$ sufficiently small, since $B(x,\delta^{\theta_2}) \subset B(x,\delta^{\theta_1})$ for all $x \in F$,

$$\begin{aligned}\sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_1})) &\geq \sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_2}))\\ &\geq \left(\frac{\delta^{\theta_2}}{\delta}\right)^{(\varphi(\theta_2)-\epsilon)}\\ &= \left(\delta^{\theta_1-1}\right)^{(\varphi(\theta_2)-\epsilon)\left(\frac{1-\theta_2}{1-\theta_1}\right)}\end{aligned}$$

which proves that $(1-\theta_1)\varphi(\theta_1) \geq (1-\theta_2)(\varphi(\theta_2)-\epsilon)$. This gives the lower inequality in (2.1).

To obtain the upper inequality, by covering $B(x,\delta^{\theta_1})$ by balls with radius δ^{θ_2},

$$\sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_1})) \leq \sup_{x\in F} N_{\delta^{\theta_2}}(F\cap B(x,\delta^{\theta_1})) \sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_2})).$$

This implies for all $\delta > 0$ sufficiently small

$$\begin{aligned}\sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_1})) &\leq \left(\frac{\delta^{\theta_1}}{\delta^{\theta_2}}\right)^{\varphi(\theta_1/\theta_2)+\epsilon} \left(\frac{\delta^{\theta_2}}{\delta}\right)^{\varphi(\theta_2)+\epsilon} \\ &= \left(\delta^{\theta_1-1}\right)^{(\varphi(\theta_1/\theta_2)+\epsilon)\left(\frac{\theta_2-\theta_1}{1-\theta_1}\right)+(\varphi(\theta_2)+\epsilon)\left(\frac{1-\theta_2}{1-\theta_1}\right)}\end{aligned}$$

which implies that

$$(1-\theta_1)\varphi(\theta_1) \leq (\theta_2-\theta_1)(\varphi(\theta_1/\theta_2)+\epsilon) + (1-\theta_2)(\varphi(\theta_2)+\epsilon)$$

as required. □

3.2. **Constructing sets with prescribed spectra.** In this section, for any $\varphi \in \mathcal{A}_d$ we construct a homogeneous Moran set C such that $\dim_{\mathrm{A}}^\theta C = \varphi(\theta)$ for all $\theta \in (0,1)$. The techniques here are based on ideas first introduced by the author and Banaji in [2]: we refer the reader to that paper for more details on this general technique. Since the construction in [2] is phrased using a convenient super-exponential rescaling, we adopt the same convention here.

We first recall the notion of homogeneous Moran sets from [2]. The construction is analogous to the usual 2^d-corner Cantor set, except that the subdivision ratios need not be the same at each level.

Let $\mathcal{I} = \{0,1\}^d$, set $\mathcal{I}^* = \bigcup_{n=0}^\infty \mathcal{I}^n$, and denote the word of length 0 by $\varnothing$. Let $\boldsymbol{r} = (r_n)_{n=1}^\infty \subset (0,1/2]$ and for each n and $\boldsymbol{i} \in \mathcal{I}$, define $S_{\boldsymbol{i}}^n \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$S_{\boldsymbol{i}}^n(x) \coloneqq r_n x + b_{\boldsymbol{i}}^n$$

where $b_{\boldsymbol{i}}^n \in \mathbb{R}^d$ has

$$(b_{\boldsymbol{i}}^n)^{(j)} = \begin{cases} 0 & : \boldsymbol{i}^{(j)} = 0 \\ 1-r_n & : \boldsymbol{i}^{(j)} = 1 \end{cases}.$$

Given $\sigma = (\boldsymbol{i}_1,\ldots,\boldsymbol{i}_n) \in \mathcal{I}^n$, we write $S_\sigma = S_{\boldsymbol{i}_1}^1 \circ \cdots \circ S_{\boldsymbol{i}_n}^n$. Then set

$$C = C(\boldsymbol{r}) \coloneqq \bigcap_{n=1}^\infty \bigcup_{\sigma\in\mathcal{I}^n} S_\sigma([0,1]^d).$$

We refer to the set C as a *homogeneous Moran set*.

Given $\delta > 0$, let $k = k(\delta)$ be such that $r_1\cdots r_k \leq \delta < r_1\cdots r_{k-1}$. We then define

$$s(\delta) = s_{\boldsymbol{r}}(\delta) \coloneqq \frac{k(\delta)\cdot d\log 2}{-\log\delta}.$$

Heuristically, $s(\delta)$ is the best candidate for the box dimension of C at scale δ.

Definition 3.2. *Let $0 \leq \lambda < \alpha \leq d$ and let $\mathcal{G}(\lambda, \alpha)$ denote the set of functions $g\colon (0, \infty) \to (\lambda, \alpha)$ satisfying*

$$\lambda - (\lambda - g(y)) \exp(-t) \leq g(y + t) \leq \alpha - (\alpha - g(y)) \exp(-t)$$

for any $y > 0$ and $t > 0$.

This family is the same as the family defined in [2, Definition 3.1] (see [2, Lemmma 3.2]).

We first establish a general construction result to prescribe Assouad spectra for homogeneous Moran sets.

Proposition 3.3. *Let $d \in \mathbb{N}$ and $g \in \mathcal{G}(0, d)$. Then there exists a homogeneous Moran set C such that*

$$\dim_{\mathrm{A}}^{\theta} C = \limsup_{x \to \infty} \frac{g\left(x + \log \frac{1}{\theta}\right) - \theta g(x)}{1 - \theta}. \tag{3.1}$$

Proof. If

$$\limsup_{x \to \infty} (D^{+} g(x) + g(x)) = 0,$$

then

$$\limsup_{x \to \infty} \frac{g\left(x + \log \frac{1}{\theta}\right) - \theta g(x)}{1 - \theta} = 0$$

and we can define the Moran set $C(\boldsymbol{r})$ where $\boldsymbol{r}$ is a sequence converging monotonically to 0. Otherwise, performing an appropriate translation to g which does not change (3.1), [2, Lemmma 3.4] provides a sequence $\boldsymbol{r} \subset (0, 1/2]$ so that

$$|s_{\boldsymbol{r}}(\exp(-\exp(x))) - g(x)| \leq d \log(2) \cdot \exp(-x). \tag{3.2}$$

To obtain (3.1), we first note that it follows directly from the definition that

$$\dim_{\mathrm{A}}^{\theta} C = \limsup_{\delta \to 0} \sup_{x \in C} \frac{\log N_{\delta^{1/\theta}}(C \cap B(x, \delta))}{(1 - 1/\theta) \log \delta}.$$

Observe that there is some constant $M > 0$ such that $B(x, \delta)$ intersects at most M cylinders in level $k(\delta)$. In particular, $C \cap B(x, \delta)$ can be covered by $M \cdot 2^{d(k(\delta^{1/\theta}) - k(\delta))}$ balls of radius $\delta^{1/\theta}$. On the other hand, $C \cap B(x, \delta)$ contains an interval in level $k(\delta)$, and therefore contains a δ-separated subset of size $2^{d(k(\delta^{1/\theta}) - 1 - k(\delta))}$. Thus there is a constant $M' > 0$ so that

$$M' \cdot 2^{k(\delta^{1/\theta}) - k(\delta)} \leq \sup_{x \in C} N_{\delta}^{1/\theta}(C \cap B(x, \delta)) \leq M \cdot 2^{k(\delta^{1/\theta}) - k(\delta)}$$

and therefore

$$\limsup_{\delta\to 0}\sup_{x\in C}\frac{\log N_{\delta^{1/\theta}}(C\cap B(x,\delta))}{(1-1/\theta)\log\delta} = \limsup_{\delta\to 0}\frac{\theta(k(\delta^{1/\theta})-k(\delta))\cdot d\log 2}{(1-\theta)\cdot(-\log\delta)}$$
$$= \limsup_{\delta\to 0}\frac{s(\delta^{1/\theta})-\theta\cdot s(\delta)}{1-\theta}.$$

Applying (3.2) then yields the desired formula. $\square$

Definition 3.4. *Given a sequence of continuous functions $(f_k)_{k=1}^{\infty}$ each defined on some interval $[0,a_k]$, the* concatenation *of $(f_k)_{k=1}^{\infty}$ is the function $f\colon (0,\sum_{k=1}^{\infty} a_k)\to\mathbb{R}$ given as follows: for each $x\in\mathbb{R}^+$ with $\sum_{j=0}^{k-1} a_j < x \le \sum_{j=0}^{k} a_j$ where $a_0=0$ we define*

$$f(x) = f_k\left(x-\sum_{j=0}^{k-1} a_j\right).$$

The concatenation of a finite tuple of functions is defined similarly.

Now, we can prove that (b) implies (a) in Theorem A. For the convenience of the reader, we also give an explicit description of the construction technique in $\mathbb{R}$. Note that, in the proof of Theorem 3.5, the precise choice of the contractions $(r_i)_{i=1}^{\infty}\subset(0,1/2]$ is concealed in the application of [2, Lemmma 3.4] in Proposition 3.3.

Let $\varphi\in\mathcal{A}_1$ be some fixed function. Fix some small constant δ_1. Then we will inductively choose constants $r_1^{(n)},\dots,r_{m_n}^{(n)}$ in $(0,1/2]$ for each $n\in\mathbb{N}$ so that for each $1\le i\le m_n$,

$$2^j \approx \left(\frac{1}{r_1^{(n)}\cdots r_j^{(n)}}\right)^{\varphi(\theta)} \tag{3.3}$$

where θ is such that $\delta_n^{1/\theta}\approx\delta_n\cdot r_1^{(n)}\cdots r_j^{(n)}$, and m_n satisfies $\delta_n\cdot r_1^{(n)}\cdots r_{m_n}^{(n)}\approx\delta_n^n$. Then take R_n very small, and set $\delta_{n+1}=\delta_n\cdot r_1^{(n)}\cdots r_{m_n}^{(n)}\cdot R_n$. Now let C denote the Moran set corresponding to the sequence

$$(\delta_1, r_1^{(1)},\dots,r_{m_1}^{(1)},R_1,r_1^{(2)},\dots,r_{m_2}^{(2)},R_2,\dots).$$

Now for $n\in\mathbb{N}$ and $x\in C$, $N_{\delta_n^{1/\theta}}(C\cap B(x,\delta_n))\approx 2^j$ where $\delta_n^{1/\theta}\approx\delta_n\cdot r_1^{(n)}\cdots r_j^{(n)}$. In particular, (3.3) guarantees that the Assouad spectrum of C with respect to θ at scale δ_n is precisely $\varphi(\theta)$, for all n sufficiently large so that $1/n\le\theta$.

The main details of the proof are to show (1) that such a choice of the constants r_i is possible, and (2) that for fixed θ and sufficiently small scales δ not of the form δ_n, the Assouad spectrum of C at scale δ is at most $\varphi(\theta)$.

Theorem 3.5. *Let $\varphi\in\mathcal{A}_d$ be arbitrary. Let α be such that $\varphi(1)\le\alpha\le d$. Then there exists a homogeneous Moran set $C\subseteq\mathbb{R}^d$ such that $\dim_{\mathrm{A}} C=\alpha$ and, for all $\theta\in(0,1)$,*

$$\dim_{\mathrm{A}}^{\theta} C = \varphi(\theta).$$

Proof. We may assume $\alpha > 0$, or the result is immediate. We will prove the result for the Assouad spectrum, and then explain how to modify the proof to accommodate the Assouad dimension as well.

First, we apply some convenient rescaling to $\varphi(\theta)$. Given $y \in (0, \infty)$, $\exp(-y) \in (0, 1)$ so we may define

$$\xi(y) = (1 - \exp(-y))\varphi(\exp(-y)).$$

In particular, given $0 < y_1 < y_2 < \infty$, it follows that $0 < \exp(-y_2) < \exp(-y_1) < 1$ so

$$\begin{aligned} 0 &\leq (1 - \exp(-y_2))\varphi(\exp(-y_2)) - (1 - \exp(-y_1))\varphi(\exp(-y_1)) \\ &\leq \exp(-y_1)\left(1 - \exp(-(y_2 - y_1))\right)\varphi(\exp(-(y_2 + y_1))) \end{aligned}$$

or equivalently

$$0 \leq \xi(y_2) - \xi(y_1) \leq \exp(-y_1)\xi(y_2 - y_1). \tag{3.4}$$

Moreover, observe that $\varphi(1) = \lim_{y\to 0} \varphi(\exp(-y))$ so $\lim_{y\to 0} \xi(y) = 0$, and similarly $\lim_{y\to\infty} \xi(y) = \varphi(0)$. In particular ξ is continuous, increasing, and bounded.

Now for $z \in (0, \alpha)$, let ξ_z denote the function

$$\xi_z(y) = \xi(y) + \exp(-y)z$$

and similarly $\Psi_z(y) = \exp(-y)z$. We note that $\xi_z(0) = \Psi_z(0) = z$.

Now, choose constants w_n, z_n such that the functions $f_n := \xi_{z_n}|_{[0,n]}$ and $e_n := \Psi_{w_n}|_{[0,n]}$ satisfy $f_n(n) = e_n(0)$ and $e_n(n) = f_{n+1}(0)$ for all $n \in \mathbb{N}$. Then, let g be the infinite concatenation of the sequence

$$\big(f_1, e_1, f_2, e_2, \ldots\big).$$

First, let us verify that $g \in \mathcal{G}(0, \varphi(1)) \subset \mathcal{G}(0, \alpha)$. Let $n \in \mathbb{N}$. Note that $e_n \in \mathcal{G}(\varphi(0), \varphi(1))$ since the e_n are differentiable with $e_n'(x) = \varphi(0) - e_n(x)$. Next let $0 < y < y + t < \infty$. First observe that

$$\xi(y + t) \leq \xi(t) + \xi(y)\exp(-t) \leq (1 - \exp(-t))\varphi(1) + \xi(y)\exp(-t)$$

by (3.4) and (iii) in Proposition 2.4. Thus

$$\begin{aligned} f_n(y + t) &= \xi(y + t) + \exp(-(y + t))z_n \\ &\leq (1 - \exp(-t))\varphi(1) + \xi(y)\exp(-t) + \exp(-(y + t))z_n \\ &= (1 - \exp(-t))\varphi(1) + f_n(y)\exp(-t) \end{aligned}$$

as required. To obtain the other bound, since ξ is increasing,

$$\begin{aligned} f_n(y + t) &= \xi(y + t) + \exp(-(y + t))z_n \\ &\geq \xi(y)\exp(-t) + \exp(-y)\exp(-t)z_n \\ &= f_n(y)\exp(-t). \end{aligned}$$

Now, let C denote the Moran set corresponding to the function g. Let $\theta \in (0,1)$: we must show that $\dim_{\mathrm{A}}^{\theta} C = \varphi(\theta)$. Let $\tau = \log(1/\theta)$. By Proposition 3.3, it suffices to show

$$\varphi(\theta) = \limsup_{x\to\infty} \frac{g(x+\tau) - \theta g(x)}{1-\theta}. \tag{3.5}$$

For $n \in \mathbb{N}$ set $x_n = 2\sum_{i=1}^{n-1} i$ and let $N \in \mathbb{N}$ be sufficiently large so that $N \geq \tau + 1$.

Now if $n \geq N$, $g(x_n + \tau) = f_n(\tau)$ and $g(x_n) = z_n$ so that

$$\frac{g(x_n+\tau) - \theta g(x_n)}{1-\theta} = \frac{(1-\theta)\varphi(\theta) + \theta z_n - \theta z_n}{1-\theta} = \varphi(\theta).$$

This gives the lower bound in (3.5).

It remains to see the upper bound. We first observe for all $y > 0$ and $z \in \mathbb{R}$ that

$$\frac{\xi_z(y+\tau) - \theta\xi_z(y)}{1-\theta} \leq \varphi(\theta).$$

Indeed, expanding the definition of ξ_z and applying (3.4),

$$\begin{aligned}
\xi_z(y+\tau) - \theta\xi_z(y) &= \xi(y+\tau) + \exp(-(y+\tau))z - \exp(-\tau)(\xi(y) + \exp(-y)z) \\
&= \xi(y+\tau) - \exp(-\tau)\xi(y) \\
&\leq \xi(\tau) = (1-\theta)\varphi(\theta).
\end{aligned}$$

Now let $x \geq x_N$ be arbitrary and let n be such that $x \in [x_n - (n-1), x_n + n]$. First note that for $y \in [x_n - (n-1), x_n + 2n]$, $g(y) = \exp(-(y - x_n))z_n + \phi(y)$ where

$$\phi(y) = \begin{cases} 0 & : x_n - (n-1) \leq y \leq x_n \\ \xi(y - x_n) & : x_n \leq y \leq x_n + n \\ \xi(n)\exp(-(y - x_n + n)) & : x_n + n \leq y \leq x_n + 2n \end{cases}$$

by choice of the constants w_n and z_n. If $x \in [x_n, x_n + n]$, since $x + \log(1/\theta) \leq x_n + 2n$ and $g(y) \leq \xi_{z_n}(y - x_n)$ for all $y \in [x_n, x_n + 2n]$, the prior computation shows that $g(x+\tau) - \theta g(x) \leq (1-\theta)\varphi(\theta)$. Otherwise, $x \in [x_n - (n-1), x_n]$. If $x + \tau \leq x_n$, then $g(x+\tau) - \theta g(x) = 0 \leq (1-\theta)\varphi(\theta)$, and if $x_n < x + \tau \leq x_n + n$, then

$$g(x+\tau) - \theta g(x) = \xi(x + \tau - x_n) \leq \xi(\tau)$$

since ξ is increasing. Thus (3.5) holds, finishing the proof.

In order to obtain the result for the Assouad dimension as well, we modify the construction as follows. Define functions $u_n \colon [0, 1/n] \to (0, \alpha)$ by the rule $u_n(x) = \alpha - (\alpha - q_n)\exp(-x)$. Choosing the constants q_n appropriately and modifying the constants w_n and z_n, the concatenation $\tilde{g}$ of the sequence

$$(f_1, e_1, u_1, f_2, e_2, u_2, \ldots)$$

is continuous and $\tilde{g} \in G(0,\alpha)$ since $\alpha \geq \varphi(1)$. Since the u_n are supported on intervals with lengths converging to 0, the same arguments as before yield the correct bounds for $\dim_{\mathrm{A}}^{\theta} C$ up to an error decaying to 0 as n goes to infinity. On the other hand, the same arguments as given in [2, Lemmma 3.7 and Theorem 3.9] give that $\dim_{\mathrm{A}} C = \alpha$. We leave the precise details to the reader. □

3.3. **Families of monotonic and non-monotonic spectra.** In this section, we define two general parametrized families of functions in $\mathcal{A}_d$. The first is a 2-parameter family composed of increasing functions, and the second is a 3-parameter family composed of functions which (outside of degenerate cases) are non-monotonic.

3.3.1. *Monotonic spectra.* Let

$$M_d = \{(\kappa, c) : 0 \leq \kappa \leq d, 0 < c < 1\}$$

and for $\boldsymbol{i} = (\kappa, c) \in M_d$, we may define

$$f_{\boldsymbol{i}}(\theta) = \begin{cases} \kappa(1-c) & : \theta \in [0,c] \\ \kappa(1-\theta) & : \theta \in [c,1] \end{cases}.$$

Then, let

$$\mathcal{M}_d := \Big\{\theta \mapsto \frac{f_{\boldsymbol{i}}(\theta)}{1-\theta} : \boldsymbol{i} \in M_d\Big\}. \tag{3.6}$$

A direct argument shows that each $\varphi \in \mathcal{M}_d$ is increasing and in $\mathcal{A}_d$.

3.3.2. *Non-monotonic spectra.* This family generalizes the example considered in [6, Theorem 3.4.16]. Let

$$C_d = \{(\kappa, c_1, c_2) : 0 \leq \kappa \leq d, 0 \leq c_1 \leq c_2 \leq c_1^{1/2} \leq 1\}.$$

Suppose $\boldsymbol{c} = (\kappa, c_1, c_2) \in C_d$. If $c_1 = 0$, let $h_{\boldsymbol{c}}(\theta) = \kappa(1-\theta)$ for $\theta \in [0,1]$. Otherwise, $c_2 \leq c_1/c_2$. Thus we may define $h = h_{\boldsymbol{c}} \colon [0,1] \to [0,d]$ to be the unique continuous function which has slope 0 on $[0,c_1] \cup [c_2, c_1/c_2]$, has slope $-\kappa$ on $[c_1,c_2] \cup [c_1/c_2, 1]$, and satisfies $h(1) = 0$. Now, let

$$\mathcal{C}_d = \Big\{\theta \mapsto \frac{h_{\boldsymbol{c}}(\theta)}{1-\theta} : \boldsymbol{c} \in C_d\Big\} \tag{3.7}$$

We note that $h_{\boldsymbol{c}}$ satisfies a certain rescaling invariance: for $c_2^2 \leq \theta \leq c_2$,

$$h_{\boldsymbol{c}}(\theta) - h_{\boldsymbol{c}}\Big(\frac{\theta}{c_2}\Big) = \kappa(c_2 - c_1).$$

In particular, $h_{\boldsymbol{c}}(c_2^2)/(1-c_2^2) = h_{\boldsymbol{c}}(c_2)/(1-c_2)$.

There are degenerate cases: if $c_2 = c_1^{1/2}$, then $h_{(\kappa,c_1,c_2)} = f_{(\kappa,c_1)}$ and if $c_1 = c_2$ or $\kappa = 0$, then $h_{\kappa,c_1,c_2} = 0$. Otherwise, $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ is increasing on $[0,c_1]$ and $[c_2,1]$, constant in $[c_1/c_2, 1]$, and strictly decreasing on $[c_1,c_2]$. A plot of the function $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ for non-degenerate parameters is given in Figure 2.

Proposition 3.6. *For any $d \in \mathbb{N}$, $\mathcal{C}_d \subset \mathcal{A}_d$.*

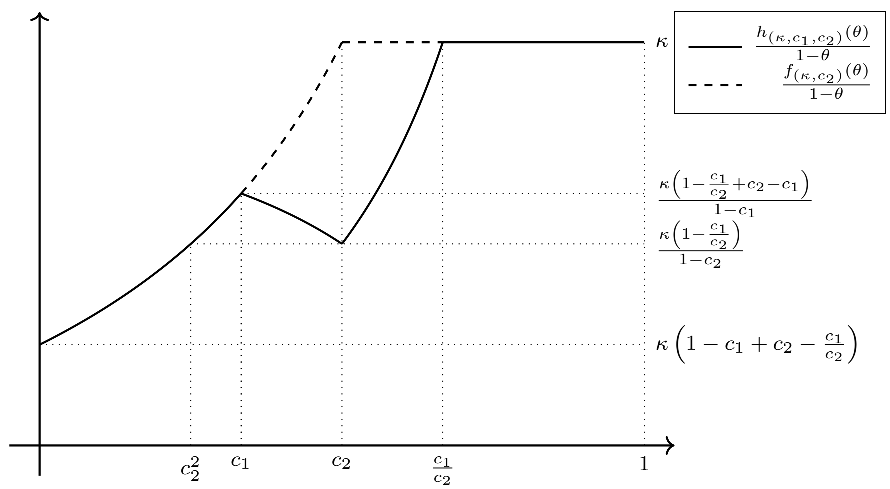

FIGURE 2. A plot of $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ and $f_{\boldsymbol{i}}(\theta)/(1-\theta)$ where $\boldsymbol{c} = (\kappa, c_1, c_2)$ and $\boldsymbol{i} = (\kappa, c_2)$.

Proof. Fix $0 < \theta_1, \theta_2 < 1$ and $\boldsymbol{c} = (\kappa, c_1, c_2) \in C_d$. We may assume $0 < c_1 < 1$. Since $h_{\boldsymbol{c}}$ is decreasing, it suffices to show that

$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} \leq h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big) \tag{3.8}$$

for all $0 < \lambda < \theta < \lambda^{1/2} < 1$. Since (3.8) is invariant under scaling by a positive factor, we may assume $\kappa = 1$. We prove this result in cases depending on the positions of λ and θ.

If $\lambda \in [0, c_2^2] \cup [c_2, 1]$, then $h_{\boldsymbol{c}}(\lambda)/(1-\lambda) \leq h_{\boldsymbol{c}}(\theta)/(1-\theta)$ for all $\lambda < \theta$. In particular, as argued in Lemma 2.7, (3.8) holds for all such λ. Thus we may assume $\lambda \in [c_2^2, c_2]$.

For convenience, write $I_1 = [c_2^2, c_1]$, $I_2 = [c_1, c_2]$, $I_3 = [c_2, c_1/c_2]$, and $I_4 = [c_1/c_2, 1]$. We complete the proof with casework on the positions of λ and θ.

First, suppose $\lambda \in I_1$ so $h_{\boldsymbol{c}}(\lambda) = (1 + c_2)(1 - c_1/c_2)$.

(1) If $\theta \in I_1$, then
$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} = 0 \leq h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big).$$

(2) If $\theta \in I_2$, then
$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} = 1 - \frac{c_1}{\theta} \leq h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big)$$
where the last inequality follows since $\lambda/\theta \geq c_2$.

(3) Suppose $\theta \in I_3$ so that $\lambda/\theta \leq c_1/c_2$ and $h_{\boldsymbol{c}}(\lambda/\theta) \geq 1 - c_1/c_2$. Thus
$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} = \frac{c_2 - c_1}{\theta} \leq 1 - \frac{c_1}{c_2} \leq h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big).$$

Now, suppose $\lambda \in I_2$.

(1) If $\theta \in I_2$, then $\lambda/\theta \geq c_1/c_2$ and
$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} = 1 - \frac{\lambda}{\theta} = h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big).$$
(2) If $\theta \in I_3$ then
$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} = \frac{c_2 - \lambda}{\theta} \leq 1 - \frac{\lambda}{\theta}.$$
If $\lambda/\theta \geq c_1/c_2$, then $h_{\boldsymbol{c}}(\lambda/\theta) = 1 - \lambda/\theta$ and the bound follows. Otherwise, $c_2 \leq \lambda/\theta \leq c_1/c_2$ so
$$h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big) = 1 - \frac{c_1}{c_2} \geq \frac{c_2 - c_1}{\theta} \geq \frac{c_2 - \lambda}{\theta}.$$
(3) If $\theta \in I_4$, then $c_1/c_2 \leq \lambda/\theta$ and
$$h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta) = \theta - \lambda - \Big(\frac{c_1}{c_2} - c_2\Big) \leq \theta - \lambda = \theta h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big).$$

This treats all the cases $0 < \lambda < \theta < \lambda^{1/2} < 1$, as required. □

3.4. **Closure under suprema.** In this section, we prove that $\mathcal{A}_d$ is closed under taking suprema. This essentially follows since $\mathcal{A}_d$ is uniformly Lipschitz on $[0, 1-\delta]$ for any $\delta > 0$.

Proposition 3.7. *Let $(\varphi_j)_{j \in \mathcal{J}}$ be some family of elements in $\mathcal{A}_d$. Then $\sup_{j \in \mathcal{J}} \varphi_j \in \mathcal{A}_d$.*

Proof. Let $f = \sup_{j \in \mathcal{J}} \varphi_j$. Get a sequence $J_1 \subset J_2 \subset \cdots \subset \mathcal{J}$ such that each J_n is finite and with
$$f_n \coloneqq \max\{\varphi_i : i \in J_n\}$$
that $f = \lim_{n \to \infty} f_n$ pointwise. An easy computation shows that if $\varphi_1, \varphi_2 \in \mathcal{A}_d$, then $\max\{\varphi_1, \varphi_2\} \in \mathcal{A}_d$; in particular, each $f_n \in \mathcal{A}_d$.

We first show that $f \in C([0,1])$. Since $(f_n)_{n=1}^\infty$ is monotonically increasing, by the Arzelà-Ascoli Theorem, it suffices to show that $(f_n)_{n=1}^\infty$ is uniformly bounded and uniformly equicontinuous. Uniform boundedness is immediate, so we must verify uniform equicontinuity.

Set $b = \lim_{n \to \infty} f_n(1)$ and let N be sufficiently large so that $f_N(1) > b - \epsilon/2$. Since f_N is continuous, get $\delta > 0$ so that $f_N(y) > f_N(1) - \epsilon/2$ for all $y \in [1-\delta, 1]$. Then $|f_n(x) - f_n(y)| \leq \epsilon$ whenever $x, y \in [1-\delta, 1]$. Finally, since each $f_n \in \mathcal{A}_d$, f_n is uniformly Lipschitz on $[0, 1-\delta]$ as proven in Proposition 2.5. It follows that $(f_n)_{n=1}^\infty$ is uniformly equicontinuous on $[0,1]$.

Thus $f \in C([0,1])$. To verify that $f \in \mathcal{A}_d$, let $0 < \lambda < \theta < \lambda^{1/2} < 1$ be arbitrary. Then for any $\epsilon > 0$, get n such that $\|f_n - f\|_\infty \leq \epsilon$ so that
$$\begin{aligned}(1-\lambda)f(\lambda) - (1-\theta)f(\theta) &\leq (1-\lambda)(f_n(\lambda) + \epsilon) - (1-\theta)(f_n(\theta) - \epsilon) \\ &\leq (\theta - \lambda) f_n(\lambda/\theta) + 2\epsilon \\ &\leq (\theta - \lambda) f(\lambda/\theta) + 3\epsilon\end{aligned}$$
for any $\epsilon > 0$, so the inequality holds. The lower inequality follows identically. □

Remark 3.8. *Note that $\mathcal{A}_d$ is not compact: for example, consider the functions $\varphi_n(\theta) = \min\{c_n/(1-\theta), 1\}$ with constants $c_n > 0$. If $\lim_{n\to\infty} c_n = 0$, then φ_n converges pointwise to the function which is 0 on $[0,1)$ and 1 at 1, and hence has no uniformly convergent subsequence. However, a simple modification of the above proof gives that for every $\delta > 0$, the restriction of $\mathcal{A}_d$ to $C([0, 1-\delta])$ is compact.*

3.5. **Assouad spectra as a supremum of functions.** Let $\varphi \in \mathcal{A}_d$ and write $\beta(\theta) = (1-\theta)\varphi(\theta)$. We first introduce a particular reparametrization of the family $h_{(\kappa,c_1,c_2)}$ for $(\kappa, c_1, c_2) \in \mathcal{C}_d$ by choosing c_2 so that $h_{(\kappa,c_1,c_2)}(c_1)$ has a prescribed value y. This construction, and its properties, will be used in the proof of Theorem 3.11, as well as in Theorem 4.4.

Let $0 < \lambda < 1$. Then for $\kappa \in [y/(1-\lambda), d]$, let

$$c(\kappa, \lambda, y) = \frac{\lambda + y/\kappa - 1 + \sqrt{(\lambda + y/\kappa - 1)^2 + 4\lambda}}{2}.$$

The constraints on κ guarantee that $(\kappa, \lambda, c(\kappa, \lambda, y)) \in \mathcal{C}_d$. Moreover, $c(\kappa, \lambda, y)$ is chosen precisely so that

$$h_{(\kappa,\lambda,c(\kappa,\lambda,y))}(\lambda) = y.$$

The main idea in the proof of Theorem 3.11 is, for fixed $\lambda \in (0,1)$, to choose the constant κ so that $h_{(\kappa,\lambda,c(\kappa,\lambda,\beta(\lambda)))} \le \beta(\lambda)$ on $(0,1)$. Our strategy is to choose κ so that the inequality is satisfied on the open interval $\big(\lambda, c(\kappa, \lambda, \beta(\lambda))\big)$. It will then follow from the two following lemmas that such a choice is in fact sufficient to guarantee that $h \le \beta$ on the interval $(\lambda/c(\kappa, \lambda, y), 1)$.

In fact, we will prove more: we will see that κ can be chosen as a continuous and increasing function of λ. The latter feature will be essential in the proof of Theorem 4.4.

Lemma 3.9. *Let $\varphi \in \mathcal{A}_d$ and $\lambda \in (0,1)$, and $\kappa \in [\varphi(\lambda), d]$. Then:*

(a) If $\lambda_1 \le \lambda_2$, $\kappa_1 \le \kappa_2$, and $y_1 \ge y_2$,

$$\frac{\lambda_1}{c(\kappa_1, \lambda_1, y_1)} \le \frac{\lambda_2}{c(\kappa_2, \lambda_2, y_2)}.$$

(b) If $\lambda < \theta < \lambda^{1/2}$ and $\kappa = \frac{\beta(\lambda)-\beta(\theta)}{\theta-\lambda} \ge \varphi(\lambda)$, then

$$h_{(\kappa,\lambda,c(\kappa,\lambda,\beta(\lambda)))}\Big(\frac{\lambda}{\theta}\Big) \le \beta\Big(\frac{\lambda}{\theta}\Big).$$

(c) Let $\varphi(\lambda) \le \kappa_1 \le \kappa_2 \le d$. Then

$$h_{(\kappa_1,\lambda,c(\kappa_1,\lambda,\beta(\lambda)))}(\theta) \le h_{(\kappa_2,\lambda,c(\kappa_2,\lambda,\beta(\lambda)))}(\theta)$$

for all $\theta \ge c(\kappa_1, \lambda, \beta(\lambda))$.

Proof. To see (a), it is clear that $c(\kappa, \lambda, y)$ is an increasing function of y and a decreasing function of κ. Moreover, $\kappa_2 \in [y_2/(1-\lambda_1), d]$. Thus

$$\frac{\lambda_1}{c(\kappa_1, \lambda_1, y_1)} \le \frac{\lambda_1}{c(\kappa_2, \lambda_1, y_2)}.$$

Then if $\psi(\lambda) = \lambda / c(\kappa_2, \lambda, y_2)$, we have for $\lambda_1 < \lambda < \lambda_2$,

$$\psi'(\lambda) = \frac{1}{2} + \frac{1 - \lambda - \frac{y}{\kappa}}{2\sqrt{\left(1 - \lambda - \frac{y}{\kappa}\right)^2 + 4\lambda}}.$$

But $1 - \lambda - y/\kappa \geq 0$ so $\lambda / c(\kappa_2, \lambda, y_2)$ is an increasing function of λ, and the result follows.

Next, to see (b), by (2.2),

$$h_{(\kappa,\lambda,c(\kappa,\lambda,\beta(\lambda)))}\Big(\frac{\lambda}{\theta}\Big) = \frac{\beta(\lambda) - \beta(\theta)}{\theta - \lambda} \cdot \Big(1 - \frac{\lambda}{\theta}\Big) \leq \frac{\beta\big(\frac{\lambda}{\theta}\big)}{1 - \frac{\lambda}{\theta}} \cdot \Big(1 - \frac{\lambda}{\theta}\Big) = \beta\Big(\frac{\lambda}{\theta}\Big)$$

as claimed.

We now see (c). Since $\kappa_1 < \kappa_2$, $\lambda / c(\kappa_1, \lambda) \leq \lambda / c(\kappa_2, \lambda)$, so it suffices to observe that

$$h_{(\kappa_1,\lambda,c(\kappa_1,\lambda,\beta(\lambda)))}(c(\kappa_1, \lambda, \beta(\lambda))) \leq h_{(\kappa_2,\lambda,c(\kappa_2,\lambda,\beta(\lambda)))}(c(\kappa_1, \lambda, \beta(\lambda))).$$

But this follows since

$$h_{(\kappa_i,\lambda,c(\kappa_i,\lambda,\beta(\lambda)))}\big(c(\kappa_1, \lambda, \beta(\lambda))\big) = \kappa_i \cdot (c(\kappa_i, \lambda, \beta(\lambda)) - \lambda)$$

for $i = 1, 2$ and $c(\kappa_1, \lambda, \beta(\lambda)) > c(\kappa_2, \lambda, \beta(\lambda))$. □

Now for $\lambda \in (0, 1)$, let

$$\tilde{\kappa}_\varphi(\lambda) = \max\Big\{\sup_{\lambda < \theta < \lambda^{1/2}} \frac{\beta(\lambda) - \beta(\theta)}{\theta - \lambda}, \varphi(\lambda)\Big\}$$

and set $\tilde{\kappa}_\varphi(0) = \varphi(0)$ and $\tilde{\kappa}_\varphi(1) = \varphi(1)$. Then let for $\lambda \in [0, 1]$

$$\begin{aligned} \kappa_\varphi(\lambda) &= \sup_{0 \leq \lambda' \leq \lambda} \tilde{\kappa}_\varphi(\lambda') \\ c_\varphi(\lambda) &= c(\kappa_\varphi(\lambda), \lambda, \beta(\lambda)) \\ \boldsymbol{c}_\varphi(\lambda) &= (\kappa_\varphi(\lambda), \lambda, c_\varphi(\lambda)) \end{aligned} \tag{3.9}$$

using the convention $c_\varphi(0) = 0$ and $c_\varphi(1) = 1$. Observe that $\boldsymbol{c}_\varphi(\lambda) \in \mathcal{C}_d$. A plot of the function $\kappa_\varphi(\lambda)$ is given in Figure 3.

Lemma 3.10. *Let $\varphi \in \mathcal{A}_d$. Then:*

(a) κ_φ is a continuous increasing function with $\kappa_\varphi(0) = 0$, $\kappa_\varphi(1) = \varphi(1)$, and $\varphi \leq \kappa_\varphi$.
(b) If φ is increasing, then $\kappa_\varphi = \varphi$.
(c) For any $0 \leq \lambda \leq 1$ and $0 \leq \theta \leq 1$,

$$h_{\boldsymbol{c}_\varphi(\lambda)}(\theta) \leq (1 - \theta)\varphi(\theta). \tag{3.10}$$

Proof. We first see (a). Of course, κ_φ is increasing and $\kappa_\varphi(0) = 0$ by definition. Moreover, it follows from the same proof as the lower bound in Proposition 2.5 that $\tilde{\kappa}_\varphi(\lambda) \leq \varphi(1)$ for all λ, so $\kappa_\varphi(1) = \varphi(1)$. Next, continuity of κ_φ follows since

$$\lim_{\theta \searrow \lambda} \tilde{\kappa}_\varphi(\theta) \leq \tilde{\kappa}_\varphi(\lambda),$$

which holds by continuity of φ.

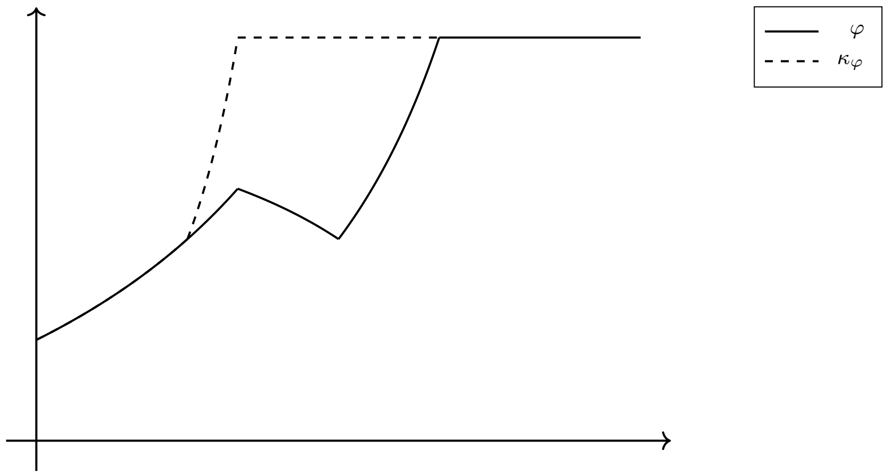

FIGURE 3. A plot of $\varphi \in \mathcal{A}_d$ and κ_φ.

To see (b), if φ is increasing, then $\tilde{\kappa}_\varphi = \varphi$, so $\tilde{\kappa}_\varphi$ is increasing and therefore $\tilde{\kappa}_\varphi = \kappa_\varphi$.

Finally, to verify (c), write $\beta(\theta) = (1-\theta)\varphi(\theta)$. We observed above that $h_{\boldsymbol{c}_\varphi(\lambda)}(\lambda) = (1-\lambda)\varphi(\lambda)$. Since $\varphi(0) \leq \varphi(\lambda)$ for all λ by Proposition 2.4, (3.10) holds for $\lambda = 0$. In addition, $h_{\boldsymbol{c}_\varphi(1)}(\theta) = 0$ for all θ, again giving (3.10).

Otherwise, $0 < \lambda < 1$. Write

$$q_\lambda = h_{(\tilde{\kappa}_\varphi(\lambda),\lambda,c(\tilde{\kappa}_\varphi(\lambda),\lambda,\beta(\lambda)))}.$$

We first show that $q_\lambda(\theta) \leq \beta(\theta)$ for all $\theta \in (0,1)$. Since q_λ is constant on $[0,\lambda] \cup [c(\tilde{\kappa}_\varphi(\lambda),\lambda,\beta(\lambda)),\lambda/c_\varphi(\lambda)]$, β is decreasing, and $q_\lambda(1) = 0$, we may assume $\theta \in (\lambda, c_\varphi(\lambda)) \cup (\lambda/c_\varphi(\lambda),1)$. Moreover, $\tilde{\kappa}_\varphi(\lambda)$ is chosen precisely so that $q_\lambda(\theta) \leq \beta(\theta)$ for all $\theta \in (c, c(\tilde{\kappa}_\varphi(\lambda),\lambda,\beta(\lambda)))$.

It remains to verify the inequality for $\theta \in (\lambda/c(\tilde{\kappa}_\varphi(\lambda),\lambda,\beta(\lambda)),1)$. Let $\theta_0 = \lambda/\theta$. If $\varphi(\lambda) \leq \varphi(\theta_0)$, rearranging and applying (2.2),

$$\frac{\beta(\lambda)}{1-\lambda} \leq \frac{\beta(\lambda)-\beta(\theta_0)}{\theta_0 - \lambda} \leq \frac{\beta(\theta)}{1-\theta}.$$

Since $\tilde{\kappa}_\varphi(\lambda) \geq \varphi(\lambda)$, by Lemma 3.9 (c),

$$q_\lambda(\theta) \leq h_{(\varphi(\lambda),\lambda,c(\varphi(\lambda),\lambda,\beta(\lambda)))}(\theta) = \frac{\beta(\lambda)}{1-\lambda}(1-\theta) \leq \beta(\theta).$$

Otherwise, $\kappa_0 = \frac{\beta(\lambda)-\beta(\theta_0)}{\theta_0-\lambda} \in [\varphi(\lambda),\tilde{\kappa}_\varphi(\lambda)]$ so by Lemma 3.9 (b) and (c),

$$q_\lambda(\theta) \leq h_{(\kappa_0,\lambda,c(\kappa_0,\lambda,\beta(\lambda)))}(\theta) \leq \beta(\theta)$$

as claimed.

We now verify (3.10). As before, since $\kappa_\varphi(\lambda) \geq \tilde{\kappa}_\varphi(\lambda)$, it suffices to verify (3.10) for $\theta \in (\lambda/c_\varphi(\lambda),1)$. Suppose $\lambda' \leq \lambda$ is such that $\kappa_\varphi(\lambda) = \kappa_{\tilde{\kappa}_\varphi}(\lambda')$. But $\lambda/c_\varphi(\lambda) \geq$

$\lambda'/c(\tilde{\kappa}_\varphi(\lambda'), \lambda', \beta(\lambda'))$ by Lemma 3.9 (a), so

$$h_{\kappa_\varphi(\theta)} = q_{\lambda'}(\theta) \leq \beta(\theta).$$

But all relevant quantities are continuous in their parameters, and the result follows. □

We now complete the remaining equivalence in the classification result. We recall that the families $\mathcal{C}_d \supset \mathcal{M}_d$ are defined in Section 3.3.

Theorem 3.11. *Let $\varphi\colon [0,1] \to [0,d]$. Then $\varphi \in \mathcal{A}_d$ if and only if there is a family $(\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{C}_d$ such that $\varphi = \sup_{\lambda\in\Lambda}\varphi_\lambda$. Moreover, if φ is increasing, the family can be taken so that $(\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{M}_d$.*

Proof. First let $\varphi \in \mathcal{A}_d$. In the context of Proposition 3.7 and Proposition 3.6, it suffices to construct a family of functions in $(\varphi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{C}_d$ satisfying the required properties. Let $\Lambda \subset (0,1)$ be a countable dense subset and for each $\lambda \in \Lambda$, let $\varphi_\lambda(\theta) = h_{\boldsymbol{c}_\varphi(\lambda)}(\theta)/(1-\theta)$. Then $\varphi_\lambda(\lambda) = (1-\lambda)\varphi(\lambda)$ and $\varphi_\lambda(\theta) \leq (1-\theta)\varphi(\theta)$ for all $\theta \in [0,1]$ by Lemma 3.10, so $\varphi = \sup_{\lambda\in\Lambda}\varphi_\lambda$ by continuity of φ.

If φ is increasing then $\kappa_\varphi = \varphi$ so $h_{\boldsymbol{c}_\varphi(\lambda)} = f_{(\varphi(\lambda),\lambda)} \in \mathcal{M}_d$ for all $\lambda \in \Lambda$. □

4. Examples with exceptional Assouad spectra

4.1. **Hölder failure at 1.** Here, we prove the following result which states that there is no control of the rate at which Assouad spectrum approaches the quasi-Assouad dimension.

Proposition 4.1. *Let $f : [0,1] \to [0,d]$ be an increasing function with $f(0) > 0$. Then there exists a set $F \subset \mathbb{R}$ such that $\dim_{\mathrm{A}}^\theta F \leq f(\theta)$ for all $\theta \in (0,1)$ and $\lim_{\theta\to 1}\dim_{\mathrm{A}}^\theta F = f(1)$.*

Proof. Let $h(\theta) = \inf_{0<\theta'\leq\theta}(1-\theta')f(\theta')$. By definition, $h(\theta)$ is decreasing, $h(\theta) \leq (1-\theta)f(\theta)$, and since $f(\theta) \leq d$, $\lim_{\theta\to 1} h(\theta) = 0$. Observe also that $\varphi(\theta) = h(\theta)/(1-\theta)$ is increasing, so $\varphi \in \mathcal{A}_d$ and $\varphi(\theta) \leq f(\theta)$.

To see that $\lim_{\theta\to 1}\varphi(\theta) = f(1)$, note that $(1-\theta)f(\theta) \geq (1-\theta)(f(1)-\epsilon)$ for any $\epsilon > 0$ and θ sufficiently close to 1. Since $\lim_{\theta\to 1}(1-\theta)f(\theta) = 0$ and $(1-\theta)f(\theta) > 0$ for all $\theta \in [0,1)$ since $f(0) > 0$, it follows that $h(\theta)/(1-\theta) \geq f(1) - \epsilon$ for some θ. But $\varphi(\theta) \geq h(\theta)/(1-\theta)$, so the claim follows.

Finally, apply Theorem 3.5. □

We also consider an explicit example. Consider the function $f(\theta) = 1 + \frac{1}{\log(1-\theta)}$; note that f is not Hölder at 1. A direct computation shows that there is some minimal $\theta_0 \in (0,1)$ so that $(1-\theta)f(\theta)$ is decreasing on $[\theta_0, 1]$. Thus if we define

$$\sigma(\theta) = \begin{cases} \frac{(1-\theta_0)f(\theta_0)}{1-\theta} & : 0 < \theta \leq \theta_0 \\ f(\theta) & : \theta_0 < \theta < 1 \end{cases}$$

then σ is a continuous increasing function of θ with $(1-\theta)\sigma(\theta)$ decreasing. Thus by Lemma 2.7, $\sigma \in \mathcal{A}_1$. A plot of $\sigma(\theta)$ and the upper bound $\min\{(1-\theta_0)f(\theta_0)/(1-\theta), 1\}$ is given in Figure 4.

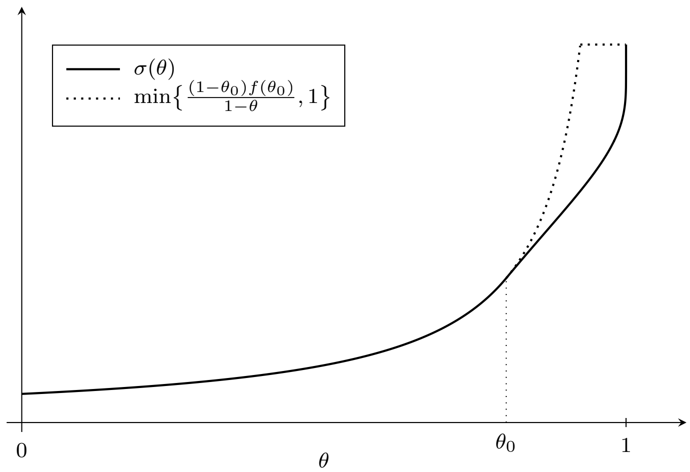

FIGURE 4. Plot of a spectrum which is not Hölder at 1, along with the general upper bound.

4.2. **Non-monotonicity on any open set.** In this section, we now prove that Assouad spectra which are non-monotonic on every open subset of $(0,1)$ are dense in the set of all Assouad spectra. Recall for $\varphi \in \mathcal{A}_d$ that κ_φ is defined in (3.9).

The idea behind the proof is to begin with the parametrized family of functions $h_{\boldsymbol{c}_\varphi(\lambda)}$ for $\lambda \in (0,1)$, which satisfy $\varphi(\theta) = (1-\theta)\sup_{\lambda\in(0,1)} h_{\boldsymbol{c}_\varphi(\lambda)}(\theta)$ for $\theta \in (0,1)$. We will then choose new constants $\boldsymbol{c}'_\varphi(\lambda) = (\kappa_\varphi(\lambda), \lambda, y_\lambda)$ in such a way that the function

$$f(\theta) = h_{\boldsymbol{c}'_\varphi(\theta)}(\theta)/(1-\theta) = y_\theta/(1-\theta)$$

is non-monotonic on every open set. The following lemma guarantees that such a choice can be done so that f is in fact the supremum of the functions $\theta \mapsto h_{\boldsymbol{c}'_\varphi(\lambda)}(\theta)/(1-\theta)$, as long as the constants y_λ are sufficiently close to $\beta(\lambda)$.

Lemma 4.2. *Let $\varphi \in \mathcal{A}_d$. Then for every $\epsilon > 0$, there is $\phi = \phi_\epsilon \in \mathcal{A}_d$ such that $\|\varphi - \phi\|_\infty < \epsilon$, $\kappa_\varphi \leq \kappa_\phi$, and $\omega(\lambda) < \phi(\lambda)(1-\lambda)$ for all $\lambda \in (0,1)$ where*

$$\begin{aligned} \Delta(\theta) &= c(\kappa_\varphi(\theta), \theta, (1-\theta)\phi(\theta)) \\ \omega\Big(\frac{\theta}{\Delta(\theta)}\Big) &= h_{(\kappa_\varphi(\theta),\theta,\Delta(\theta))}\Big(\frac{\theta}{\Delta(\theta)}\Big). \end{aligned} \tag{4.1}$$

Proof. We recall that the same argument as Lemma 3.9 gives that $\lambda \mapsto \lambda/\Delta(\lambda)$ is a strictly increasing function of λ with $\lim_{\lambda\to 0} \lambda/\Delta(\lambda) = 0$ and $\lim_{\lambda\to 1} \lambda/\Delta(\lambda) = 1$. Thus ω is well-defined.

If $\varphi(0) = 0$, then $\varphi(\theta) = 0$ for all θ and the result follows immediately. Thus we may assume $\varphi(\theta)(1-\theta) > 0$ for all $\theta \neq 1$. Write $\beta(\lambda) = (1-\lambda)\varphi(\lambda)$ and $\delta(\lambda) = \epsilon(1-\lambda)$ for $\lambda \in (0,1)$.

Let $\epsilon > 0$ be arbitrary and let $\lambda_0 \in (0,1)$ be sufficiently small so that $\beta(0) - \beta(\lambda_0) < \delta(\lambda_0)$. Then choose y_0 so that

$$\max\{0, \beta(0) - \delta(\lambda_0)\} < y_0 < \beta(\lambda_0).$$

and for $\theta \in [0, \lambda_0]$ let $\zeta(\theta) = y_0$.

We now inductively construct ζ as follows. Observe that, having defined ζ on some interval $[0, \lambda_n]$, Δ is well-defined on $[0, \lambda_n]$ so we may set $\lambda_{n+1} = \lambda_n / \Delta(\lambda_n)$, so that ω is well-defined on $[0, \lambda_{n+1}]$. Moreover, if $\zeta(\theta) < \beta(\theta)$ for all $\theta \in [0, \lambda_n]$, then $\omega(\theta) < \beta(\theta)$ for all $\theta \in [0, \lambda_{n+1}]$. Thus we may choose $\zeta\colon (\lambda_n, \lambda_{n+1}] \to [0, d]$ so that

(i) ζ is continuous and decreasing on $[0, \lambda_{n+1}]$,
(ii) $\max\{\beta(\theta) - \delta(\theta), \omega(\theta), 0\} < \zeta(\theta) < \beta(\theta)$ for $\theta \in [\lambda_{n+1}, \lambda_{n+2}]$, and
(iii) for $0 < \theta_1 < \theta_2 \leq \lambda_{n+1}$,

$$\zeta(\theta_2) \geq h_{(\kappa_\varphi(\theta_1), \theta_1, \Delta(\theta_1))}(\theta_2).$$

Moreover, since $\lambda_{n+1} \geq \lambda_n^{1/2}$, $\lim_{n\to\infty} \lambda_n = 1$ so ζ is defined on $[0,1]$ by setting $\zeta(1) = 0$. The choice of ζ to satisfy (i), (ii), and (iii) guarantees that

$$h_{(\kappa_\varphi(\lambda), \lambda, \Delta(\lambda))}(\lambda) = \zeta(\lambda) \geq \sup_{\theta \in (0,1)} h_{(\kappa_\varphi(\theta), \theta, \Delta(\theta))}(\lambda).$$

In particular, if $\mathcal{Q} \subset (0,1)$ is any dense subset,

$$\zeta = \sup_{q \in \mathcal{Q}} h_{(\kappa_\varphi(q), q, \Delta(q))}.$$

Thus $\phi \in \mathcal{A}_d$ by Proposition 3.7, and satisfies the required properties. □

Remark 4.3. *Note that if $\varphi \in \mathcal{A}_d$ is strictly increasing, we can simply take $\phi = \varphi$ in Lemma 4.2.*

Theorem 4.4. *Let $\varphi \in \mathcal{A}_d$. Then for any $\epsilon > 0$, there exists $F = F_\epsilon \subset \mathbb{R}^d$ such that $f(\theta) = \dim_{\mathrm{A}}^\theta F$ is non-monotonic on every open subset of $(0,1)$ and $\|f - \varphi\|_\infty \leq \epsilon$.*

Proof. As usual, define $\beta(\theta) = (1-\theta)\varphi(\theta)$. By Theorem 3.5, it suffices to construct a function $f \in \mathcal{A}_d$ satisfying the desired properties.

By Lemma 4.2, we may assume $\varphi > 0$ for all $\theta \in [0,1]$ and that there is an increasing function $\kappa\colon [0,1] \to [0,d]$ with $\kappa(0) > 0$ and $\kappa \leq \kappa_\varphi$ such that with

$$\Delta(\theta) = c(\kappa(\theta), \theta, \beta(\theta))$$
$$\omega\Big(\frac{\theta}{\Delta(\theta)}\Big) = h_{(\kappa(\theta), \theta, \Delta(\theta))}\Big(\frac{\theta}{\Delta(\theta)}\Big),$$

ω is defined on $[0,1]$ and $\omega(\theta) < \beta(\theta)$ for $\theta \in (0,1)$. Observe that for each $0 < \lambda_1 < \Delta(\lambda_1) \leq \lambda_2 < 1$,

$$h_{(\kappa(\lambda_1), \lambda_1, \Delta(\lambda_1))}(\lambda_2) \leq \omega(\lambda_2).$$

Since β is decreasing, there is a decreasing function $\psi\colon [0,1] \to [0,d]$ so that for all $\theta \in (0,1)$

$$\max\{\omega(\theta), \beta(\theta) - \epsilon(1-\theta)\} \leq \psi(\theta) < \beta(\theta). \tag{4.2}$$

Now for $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{D}_n = \{j \cdot 2^{-n} : j \in \mathbb{Z}\} \cap (0,1)$ so that $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$ is a dense subset of $(0,1)$. We will inductively choose constants $\psi(\lambda) < y_\lambda < \beta(\lambda)$ for $\lambda \in \mathcal{D}_n$ and $n \in \mathbb{N}$. When $n = 1$, $\mathcal{D}_1 = \{1/2\}$ and choose $\psi(\lambda_{1/2}) < y_{1/2} < \beta(\lambda_{1/2})$.

Now assume we have chosen constants y_λ for all $\lambda \in \mathcal{D}_n$. Set

$$\beta_n = \max\{h_{(\kappa(\lambda),\lambda,y_\lambda)} : \lambda \in \mathcal{D}_n\}.$$

First suppose $\ell_1 < \lambda < \ell_2$ where $\ell_1, \ell_2 \in \mathcal{D}_n$. We will successively constrain y_λ as follows. First, let y_λ be so that

$$\max\{\beta_n(\lambda), \psi(\lambda), y_{\ell_2}\} < y_\lambda < \min\Big\{y_{\ell_1}, \beta(\lambda), \frac{y_{\ell_1} + y_{\ell_2}}{2}\Big\}. \tag{4.3}$$

Now, since $y_{\ell_1} > \psi(\ell_1)$, there is some maximal $\ell_1 < \ell_0 < \ell_2$ such that for all $\theta \in [\ell_1, \ell_0]$,

$$\max\{\beta_n(\theta), \psi(\theta)\} = h_{(\kappa(\ell_1),\ell_1,\Delta(\ell_1))}(\theta).$$

Let m be minimal so that $\ell_1 + 2^{-n-m} \le \ell_0$ and let y_λ be so that

$$\frac{y_{\ell_1} - y_\lambda}{2^{-n-m}} \le \kappa(\ell_1) + \frac{1}{n+m}. \tag{4.4}$$

Similarly, since $y_{\ell_2} > \psi(\ell_2)$, there is some minimal $\ell_0 \le \ell_0' < \ell_2$ such that for all $\theta \in [\ell_0', \ell_2]$,

$$\max\{\beta_n(\theta), \psi(\theta)\} = y_{\ell_2}.$$

Let k be minimal so that $\ell_2 - 2^{-n-m} \ge \ell_0'$. If $k = 1$, let y_λ be so that

$$\frac{y_\lambda - y_{\ell_2}}{2^{-n-k}} \le \frac{1}{n+k}$$

and if $k > 1$, let y_λ be so that

$$\frac{h_{(\kappa(y_\lambda),\lambda,y_\lambda)}(\ell_0') - y_{\ell_2}}{2^{-n-k}} \le \frac{1}{n+k}.$$

Such a choice is possible since κ is increasing.

If instead $\lambda = 1 - 2^{-n-1}$, simply choose y_λ so that

$$\max\{\beta_n(\lambda), \psi(\lambda), y_{\ell_2}\} < y_\lambda < \beta(\lambda). \tag{4.5}$$

and if $\lambda = 2^{-n-1}$, choose y_λ so that

$$\max\{\beta_n(\lambda), \psi(\lambda)\} < y_\lambda < \min\Big\{y_{\ell_1}, \beta(\lambda)\Big\}. \tag{4.6}$$

Finally, set $\nu = \lim_{n\to\infty} \beta_n$. Then

$$\psi(\theta) < \nu(\theta) < \beta(\theta)$$

and with $f(\theta) = \nu(\theta)/(1-\theta)$, $f \in \mathcal{A}_d$ by Proposition 3.7.

Fix $\lambda \in \mathcal{D}$. We first note that (4.3), (4.5), and (4.6) guarantee that

$$(1-\lambda)f(\lambda) = h_{(\kappa(\lambda),\lambda,y_\lambda)}(\lambda).$$

In particular, $0 \le \varphi(\lambda) - (1-\lambda)f(\lambda) \le \epsilon$, so $\|f - \varphi\|_\infty \le \epsilon$.

It remains to show that f is non-monotonic on every open set: since $y_\lambda < \beta(\lambda)$,

$$\theta \mapsto h_{(\kappa(\lambda),\lambda,y_\lambda)}(\theta)/(1-\theta) \in \mathcal{C}_d \setminus \mathcal{M}_d$$

so it suffices to show that $D^{-}\nu(\lambda) = 0$ and $D^{+}\nu(\lambda) = \kappa(\lambda)$, where D^{-} denotes the upper left Dini derivative.

We prove that $D^{+}\nu(\lambda) = \kappa(\lambda)$: the argument that $D^{-}\nu(\lambda) = 0$ follows similarly. We first note that $D^{+}\nu(\lambda) \geq \kappa(\lambda)$ since $D^{+}\beta_n(\lambda) = \kappa(\lambda)$ for all n sufficiently large. To obtain the converse bound, since $y_\lambda < (y_{\ell_1} + y_{\ell_2})/2$ from (4.3), it suffices to prove that there is a monotonic sequence $(\lambda_k)_{k=1}^{\infty}$ converging to λ from the right such that $\lim_{k\to\infty}(y_\lambda - y_{\lambda_k})/(\lambda_k - \lambda) = \kappa(\lambda)$. But such a sequence is guaranteed by (4.4) (take a sequence of λ_k for which $m = 1$). □

Remark 4.5. *To obtain a concrete example with non-monotonic spectra, one can take $d = 1$ and $\varphi(\theta) = (1+\theta)/2$. Then, in the notation of the proof of Theorem 4.4, $\kappa(\theta) = \varphi(\theta)$, $\Delta(\theta) = \theta^{1/2}$ and $\omega(\theta) = (1+\theta^2)(1-\theta)/2$.*

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