# Convexity and Subadditivity

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ABSTRACT. This work in progress discusses various properties of functions which satisfy some form of convexity or subadditivity, with a focus on functions satisfying both.

CONTENTS



## 1. SUBADDITIVITY

**Definition 1.1.** *Let $A, B$ be abelian semigroups. We say that a function $f : A \to B$ is* subadditive *if*

$$f(x+y) \leq f(x) + f(y)$$

*for all $x, y \in A$.*

A natural first example of subadditivity is for a sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ satisfying $a_{n+m} \leq a_n + a_m$. As a fundamental illustration of the nice properties of subadditivity, we have the following result due to Fekete [1]:

**Lemma 1.2** (Subadditivity). *If $(a_i)_{i=1}^{\infty}$ is subadditive, then $\lim_{n\to\infty} a_n/n$ exists and is equal to its infimum $L := \inf_{n\geq 1} a_n/n$.*

*Proof.* For any $\epsilon > 0$, let $n$ be such that $a_n/n < L + \epsilon$ and $b = \max\{a_i : 1 \leq i \leq n\}$. For $m \geq n$, write $m = qn + r$ with $0 \leq r < n$. Then from the subadditivity property, we have

$$a_{qn+r} \leq qa_n + a_r \leq qa_n + b$$

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so that

$$\begin{aligned}\frac{a_m}{m} &\leq \frac{qa_n}{m} + \frac{b}{m} \\ &< \frac{qn(L+\epsilon)}{m} + \frac{b}{m} \xrightarrow{m\to\infty} L+\epsilon\end{aligned}$$

since $qn/m \to 1$ as $m \to \infty$. □

1.1. **Subadditivity for functions on the positive reals.** Here we establish some conditions which guaratee that a function $f : (0, \infty) \to \mathbb{R}$ is subadditive:

**Proposition 1.3.** *(i) If $f(t)/t$ is decreasing on $(0, \infty)$, then $f(t)$ is subadditive.*
*(ii) If $f : (0, \infty) \to \mathbb{R}$ is concave with $\limsup_{t\to 0} f(t) \geq 0$, then $f$ is subadditive.*

*Proof.* To see (i), we have

$$f(t_1+t_2) = t_1\frac{f(t_1+t_2)}{t_1+t_2} + t_2\frac{f(t_1+t_2)}{t_1+t_2} \leq t_1\frac{f(t_1)}{t_1} + t_2\frac{f(t_2)}{t_2} = f(t_1) + f(t_2)$$

as claimed.

To see (ii), if $f(t)$ is concave, for $0 < a < b$, let $0 < t < a$ be arbitrary and let $\alpha$ be such that $\alpha t + (1-\alpha)b = a$. Then by concavity, we have

$$f(a) \geq \alpha f(t) + (1-\alpha)f(b) = \alpha f(t) + \frac{a-\alpha t}{b} f(b).$$

Thus

$$f(a) \geq \alpha \limsup_{t\to 0} f(t) + f(b) \limsup_{t\to 0} \frac{a-\alpha t}{b} \geq \frac{a}{b} f(b)$$

so that $f(t)/t$ is decreasing. Then apply (i). □

We can also establish the equivalent statement of Lemma 1.2 for functions $f : (0, \infty) \to \mathbb{R}$. The key technical detail is to establish a continuous equivalence of the maximum $\max\{a_i : 1 \leq i \leq n\}$ in the proof of Lemma 1.2.

The proofs of the following lemma and theorem are due to Hille [3]:

**Lemma 1.4.** *Let $f : (0, \infty) \to \mathbb{R}$ be measurable and subadditive. Then $f$ is bounded on any compact subset of $(0, \infty)$.*

*Proof.* Let $a \in (0, \infty)$ be arbitrary. If $t_1, t_2 \in (0, \infty)$ satisfy $t_1 + t_2 = a$, then $f(a) \leq f(t_1) + f(t_2)$. It follows that, with

$$E_a := \{t : f(t) \geq f(a)/2, 0 < t < a\},$$

we have $(0, a) = E_a \cup (a - E_a)$ and therefore $m(E_a) \geq a/2$. Suppose for contradiction $f$ is unbounded on some interval $(\alpha, \beta)$ with $0 < \alpha < \beta < \infty$.

If $f$ is not bounded above on $(\alpha, \beta)$, then there exists a sequence $(t_n)_{n=1}^\infty$ where each $f(t_n) \geq 2n$ and $(t_n)_{n=1}^\infty \to t_0 \in [\alpha, \beta]$. But now each $E_{t_n} = \{t : f(t) \geq n, 0 < t < t_n\} \subset [0, \beta]$ has $m(E_{t_n}) \geq t_n/2 \geq \alpha/2$, a contradiction. Thus $f$ is bounded above on any interval $(\alpha, \beta)$.

If $f$ is not bounded below on $(\alpha, \beta)$, then there exists a sequence $(t_n)_{n=1}^{\infty}$ where each $f(t_n) \le -n$ and $(t_n)_{n=1}^{\infty} \to t_0 \in [\alpha, \beta]$. Let $M = \sup\{f(t) : 2 < t < 5\} < \infty$. Now if $t' \in (2,5)$, we have $f(t' + t_n) \le f(t') + f(t_n) \le M - n$. For sufficiently large $n$, $(t_0 + 3, t_0 + 4) \subset (t_n + 2, t_n + 5)$ so for each $t \in (t_0 + 3, t_0 + 4)$, we have $f(t) \le M - n$, a contradiction. Thus $f$ is bounded below on any interval $(\alpha, \beta)$, and hence bounded below on any compact subset of $(0, \infty)$. □

The previous lemma is the key technical result for the following theorem; the remaining details of the proof are similar to Lemma 1.2.

**Theorem 1.5.** *Let $f : (0, \infty) \to \mathbb{R}$ be measurable and subadditive. Then*

$$\lim_{n \to \infty} \frac{f(t)}{t} = \inf_{t > 0} \frac{f(t)}{t} < \infty.$$

*Proof.* We first assume $L := \inf_{t>0} \frac{f(t)}{t} > -\infty$; the case $L = -\infty$ follows analgously. For any $\epsilon > 0$, let $b > 0$ be such that $f(b)/b < L + \epsilon$. Now for any $t \ge 2b$, let $n \in \mathbb{N}$ and $b \le r < 2b$ such that $t = nb + r$. Then

$$\begin{aligned} L \le \frac{f(t)}{t} &= \frac{f(nb + r)}{t} \le \frac{nf(b) + f(r)}{t} \\ &\le \frac{n}{t} \cdot \frac{f(b)}{b} + \frac{f(r)}{t}. \end{aligned}$$

But $r \in [b, 2b]$ and since $\sup\{f(t) : t \in [b, 2b]\} < \infty$ by Lemma 1.4, we have $\lim_{t \to \infty} \frac{f(t)}{t} \le L + \epsilon$. But $\epsilon > 0$ was arbitrary, so the desired result holds. □

**Remark 1.6.** *Of course, subadditivity is preserved under isomorphism. Let $A, B, C$ be abelian semigroups and $f : A \to C$ a subadditive function. If $T : A \to B$ is an isomorphism of semigroups, then $g = T \circ f \circ T^{-1}$ is also subadditive. For example, submultiplicativity is equivalent to subadditivity by using the map $T(x) = -\log(x)$ as a function from $(0, 1)$ (with multiplication) to $(0, \infty)$ (with addition).*

1.2. **Approximate subadditivity and other variants.** Sometimes, it is useful to consider an approximate form of subadditivity.

**Definition 1.7.** *We say that $f : (0, \infty) \to \mathbb{R}$ is* approximately subadditive *if there exist constants $c \in \mathbb{R}$ and $r \in (0, \infty)$ such that*

$$f(x + y + r) \le f(x) + f(y) + c$$

For example, the following result holds, and the proof is essentially same as Theorem 1.5:

**Theorem 1.8.** *Let $f : (0, \infty) \to \mathbb{R}$ be approximately subadditive. Then $\lim_{t \to \infty} f(t)/t$ exists and is equal to $\inf_{t>0} f(t)/t$.*

We can also consider types of subadditivity for functions of two variables. This result is motivated by the technique used in [2, Prop. 3.1]:

**Theorem 1.9.** *Let $f : (0,\infty) \times (0,1) \to \mathbb{R}$ and suppose for any $\epsilon > 0$ sufficiently small we have*

*(i) there exists some $\delta > 0$ such that whenever $s, t \in (0,\infty)$ have $s/t < \delta$,*
$$f(t+s, 2\epsilon) \geq f(t,\epsilon),$$
*and*

*(ii) there exists constants $r \in (0,\infty)$ and $D > 0$ such that for any $\epsilon \in (0, 1/2)$ and $p \in \mathbb{N}$, there exists $N(\epsilon) > 0$ so that*
$$f(p(t+r), 2\epsilon) \geq D^p (f(t,\epsilon))^p$$
*for any $t \geq N(\epsilon)$.*

*Then*
$$\lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{\log f(t,\epsilon)}{t} = \lim_{\epsilon\to 0} \liminf_{t\to\infty} \frac{\log f(t,\epsilon)}{t}.$$

*Proof.* Let $\epsilon > 0$ be arbitrary and sufficiently small, and set
$$L := \limsup_{t\to\infty} \frac{\log f(t,\epsilon)}{t} \qquad M := \liminf_{t\to\infty} \frac{\log f(t,4\epsilon)}{t}.$$
It suffices to show that $L \leq M$. Let $t_0 \geq N(\epsilon)$ be arbitrary and let $t_0 + r \leq s_1 \leq s_2 \leq \cdots$ be a sequence tending to infinity such that
$$\lim_{n\to\infty} \frac{\log f(s_n, 4\epsilon)}{s_n} = M.$$
Now for $n \in \mathbb{N}$ sufficiently large, there exists $p_n \in \mathbb{N}$ and $0 < s \leq t_0 + r$ such that $s_n = p_n(t_0 + r) + s$ and $s/(p_n(t_0+r)) < \delta$. Applying (i) and then (ii), we have
$$f(s_n, 4\epsilon) = f(p(t_0+r)+s, 4\epsilon) \geq f(p_n(t_0+r), 2\epsilon) \geq D^{p_n} f(t_0,\epsilon)^{p_n}$$
so that
$$\frac{\log f(s_n,4\epsilon)}{s_n} \geq \frac{\log(D) + \log f(t_0,\epsilon)}{s_n/p_n}.$$
Now, observe that $\lim_{n\to\infty} s_n/p_n = t_0 + r$ so that
$$M \geq \frac{\log(D) + \log f(t_0,\epsilon)}{t_0 + r} \tag{1.1}$$
where $t_0 > 0$ is arbitrary.

Moreover, we observe that $\lim_{t\to\infty} f(t,\epsilon) = \infty$ as a consequence of (ii). Let $(t_n)_{n=1}^\infty$ be a sequence tending to infinity with $\lim_{n\to\infty} \frac{\log f(t_n,\epsilon)}{t_n} = L$. Then for each $n \in \mathbb{N}$ with $t_n \geq N(\epsilon)$, we have by (1.1)
$$M \geq \lim_{n\to\infty} \frac{\log D + \log f(t_n,\epsilon)}{t_n + r} = L$$
as required. □

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