

Exercise 2

DUE 12:15PM ON THURSDAY, JANUARY 22

The questions.

1. (3 pt.) Let (X, d) be a complete metric space and let $s \geq 0$.
 - (i) Prove that the Hausdorff content is upper semi-continuous: if $(K_n)_{n=1}^\infty$ is a sequence of compact sets and $K = \lim_{n \rightarrow \infty} K_n$ in the Hausdorff metric, then

$$\mathcal{H}_\infty^s(K) \geq \limsup_{n \rightarrow \infty} \mathcal{H}_\infty^s(K_n).$$

- (ii) Taking $X = [0, 1]$ and $s = 1/2$, give an example showing that Hausdorff content is not continuous.
 - (iii) Taking $X = [0, 1]$ and $s = 1/2$, give an example showing that Hausdorff s -measure is not upper semi-continuous.
2. (3 pt.) Let $E \subset \mathbb{R}$ be a bounded set.
 - (i) Let $E \subset [0, 1]$ and define a function $f_E: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_E(u) = \frac{\log N_{2^{-u}}(E)}{\log 2}.$$

Prove that there is a constant $M \geq 0$ so that for $v \leq u$,

$$(1) \quad 0 \leq f_E(u) - f_E(v) \leq u - v + M.$$

Aside: this inequality essentially says that f_E is increasing and 1-Lipschitz.

- (ii) What is the analogue of equation (1) if instead $E \subset \mathbb{R}^d$ for a general $d \in \mathbb{N}$?
 - (iii) Recall that a *dyadic interval* (of side length 2^{-n}) is an interval $[j2^{-n}, (j+1)2^{-n}]$ where $j \in \mathbb{Z}$. For $r > 0$, let $\Delta_r(E)$ denote the number of closed dyadic intervals of side-length 2^{-n} which intersect E , where $n \in \mathbb{Z}$ is such that $2^{-n} \leq r < 2^{-n+1}$. Prove that there is a constant $c \geq 1$ such that for all $r > 0$,

$$c^{-1} \Delta_r(E) \leq N_r(E) \leq c \Delta_r(E).$$

- (iv) Let m denote Lebesgue measure. Prove that there is a constant $c \geq 1$ such that for all $r > 0$,

$$c^{-1} \frac{m(E^{(r)})}{r} \leq N_r(E) \leq c \frac{m(E^{(r)})}{r}.$$

Aside: This means that the box dimension is the same as the Minkowski dimension, which is the exponential decay rate of the Lebesgue measure of the r -neighbourhood of the set E .

3. (3 pt.) Fix a binary sequence $(a_n)_{n=1}^\infty \in \{0, 1\}^\mathbb{N}$. Using this binary sequence, define a compact subset of $[0, 1]$ inductively as follows. Begin with $K_0 = [0, 1]$. Now, suppose we have constructed K_n as a non-empty union of dyadic intervals: that is,

$$K_n = \bigcup_{i=1}^{m_n} [a_i, a_i + 2^{-n}]$$

for numbers $a_j \in 2^{-n} \mathbb{Z}$.

We then define K_{n+1} as follows:

- If $a_{n+1} = 0$, we set $K_{n+1} = \bigcup_{j=1}^{m_n} [a_j, a_j + 2^{-n-1}]$.
- If $a_{n+1} = 1$, we set $K_{n+1} = \bigcup_{j=1}^{m_n} [a_j, a_j + 2^{-n-1}] \cup \bigcup_{j=1}^{m_n} [a_j + 2^{-n-1}, a_j + 2^{-n}]$.

In words, if $a_{n+1} = 0$, we replace each dyadic interval of width 2^{-n} with a single dyadic interval of width 2^{-n-1} with the same left endpoint; and if $a_{n+1} = 1$ we replace each dyadic interval with two dyadic intervals, one sharing the left endpoint and the other sharing the right endpoint.

By construction $K_0 \supset K_1 \supset K_2 \supset \dots$. So, we may define $K = K(a_n)_{n=1}^\infty = \bigcap_{n=0}^\infty K_n$.

- (i) For $r \in (0, 1)$ and $k \in \mathbb{N}$ such that $2^{-k} \leq r < 2^{-k+1}$, show that there is a constant $c \geq 1$ so that

$$c^{-1} 2^{\sum_{n=1}^k a_n} \leq N_r(K) \leq c 2^{\sum_{n=1}^k a_n}.$$

Conclude that

$$\begin{aligned} \underline{\dim}_B K &= \liminf_{k \rightarrow \infty} \frac{\sum_{n=1}^k a_n}{k} \\ \overline{\dim}_B K &= \limsup_{k \rightarrow \infty} \frac{\sum_{n=1}^k a_n}{k}. \end{aligned}$$

- (ii) Prove that $\dim_H K = \underline{\dim}_B K$.
 (iii) For any $0 \leq s \leq t \leq 1$, give an example of a set $K \subset [0, 1]$ such that $\underline{\dim}_B K = s$ and $\overline{\dim}_B K = t$.
 (iv) **(1 pt. bonus)** For any $0 \leq s \leq t \leq 1$, construct sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ such that

$$\begin{aligned} s &= \max \{ \underline{\dim}_B K(a_n), \underline{\dim}_B K(b_n) \} \\ t &= \underline{\dim}_B (K(a_n) \cup K(b_n)). \end{aligned}$$

Hint: it may be easier to construct an increasing 1-Lipschitz function instead of a sequence. These two constructions are essentially equivalent by considering $f(k) = \sum_{n=1}^k a_n$.

4. (1 pt.) Let $\Phi = \{f_i(x) = r_i x + t_i\}_{i \in \mathcal{I}}$, with $r_i \in (0, 1)$, be a self-similar IFS in \mathbb{R} . Denote the corresponding self-similar set by K' . We say that the IFS Φ has an *exact overlap* if there are words $j, k \in \mathcal{I}^*$ with $j \neq k$ such that $f_j = f_k$.
- (i) Suppose Φ has an exact overlap. Prove that an exact overlap occurs at a fixed level: namely, that there exists an $n \in \mathbb{N}$ and $j, k \in \mathcal{I}^n$ with $j \neq k$ such that $f_j = f_k$.
 - (ii) Fix $n \in \mathbb{N}$ and $j, k \in \mathcal{I}^n$ satisfying the conclusion of (i). Define a new IFS $\Phi' = \{f_i : i \in \mathcal{I}^n \setminus \{j\}\}$ and let Φ' have attractor K' . Prove that $K = K'$.
 - (iii) Prove that the similarity dimension of Φ' is strictly smaller than the similarity dimension Φ .