

# Exercise 5

DUE 12:15PM ON THURSDAY, FEBRUARY 19

Recall the notation from the previous assignment: if  $\mathcal{I}$  is a finite index set and  $\mathbf{p} \in \mathcal{P}(\mathcal{I})$ , then  $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$  is a measure-preserving system where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra and  $\sigma$  is the left shift map.

## The questions.

1. (2 pt.) Consider the space  $([0, 1]^2, \mathcal{B}([0, 1]^2), m)$  where  $m$  is Lebesgue measure and  $\mathcal{B}([0, 1]^2)$  is the Borel  $\sigma$ -algebra on  $[0, 1]^2$ . Let  $\mathcal{A} \subset \mathcal{B}([0, 1]^2)$  denote the sub- $\sigma$ -algebra

$$\mathcal{A} = \mathcal{B}([0, 1]) \times \{\emptyset, [0, 1]\}$$

where  $\mathcal{B}([0, 1])$  is the Borel  $\sigma$ -algebra on  $[0, 1]$ .

Let  $f: [0, 1]^2 \rightarrow \mathbb{R}$  be a Borel-measurable and integrable function. Prove that

$$\mathbb{E}(f | \mathcal{A})(x, y) = \int_0^1 f(x, z) dz$$

for Lebesgue a.e.  $(x, y) \in [0, 1]^2$ .

*Hint: A system of conditional measures might be useful here.*

2. (3 pt.) Let  $(X, \mathcal{B}, \mu, T)$  be a measure-preserving system.

- (i) Prove that the system is ergodic if and only if for all  $A, B \in \mathcal{B}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k} A \cap B) = \mu(A)\mu(B)$$

- (ii) Recall that a family  $\mathcal{C}$  of sets is a *semi-algebra* of sets if the following hold:

- (a)  $\emptyset \in \mathcal{C}$ .
- (b) If  $E \in \mathcal{C}$ , then  $X \setminus E$  is a finite disjoint union of elements in  $\mathcal{C}$ .
- (c) If  $E, F \in \mathcal{C}$  then  $E \cap F \in \mathcal{C}$ .

Let  $\mathcal{C} \subset \mathcal{B}$  be a semi-algebra such that  $\mathcal{B}$  is the  $\sigma$ -algebra generated by  $\mathcal{C}$ . Suppose for all  $A, B \in \mathcal{C}$  that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k} A \cap B) = \mu(A)\mu(B).$$

Prove that the system is ergodic.

3. (2 pt.) Give an example of a finite index set  $\mathcal{I}$ , a probability vector  $\mathbf{p} \in \mathcal{P}(\mathcal{I})$ , and a continuous function  $f: \mathcal{I}^{\mathbb{N}} \rightarrow \mathbb{R}$  such that Von Neumann's ergodic theorem (Theorem 3.26 in the notes) fails for  $p = \infty$  in the measure-preserving space  $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$ .
4. (3 pt.) Let  $\mathcal{C}$  denote the set of cylinders in the measure-preserving space  $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$ .
  - (i) Prove for all cylinders  $[i], [j] \in \mathcal{C}$  that

$$\lim_{k \rightarrow \infty} \mu(\sigma^{-k}[i] \cap [j]) = \mu([i])\mu([j]).$$

Conclude that the system is ergodic.

*This property is called strong mixing.*

- (ii) Two measure-preserving systems  $(X, \mathcal{B}, \mu, T)$  and  $(Y, \mathcal{A}, \nu, S)$  are *isomorphic* if there are invariant subsets  $X_0 \in \mathcal{B}$  and  $Y_0 \in \mathcal{A}$  with full measure and a bijection  $\pi: X_0 \rightarrow Y_0$  such that  $\pi$  and  $\pi^{-1}$  are measurable,  $\mu \circ \pi^{-1} = \nu$ , and  $\pi \circ T = S \circ \pi$ . Observe that ergodicity is an isomorphism invariant.  
Let  $b \in \mathbb{N}$ ,  $b \geq 2$ . Recall from the notes that  $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, T_b)$  is a measure-preserving system, where  $m$  is Lebesgue measure,  $\mathcal{A}$  is the Borel  $\sigma$ -algebra, and  $T_b$  is multiplication by  $b$  (mod 1). Show that  $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$  and  $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, T_b)$  are isomorphic for an appropriate choice of  $\mathcal{I}$  and  $\mathbf{p}$ .
  - (iii) **(1 pt. bonus)** Prove that the shift space  $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \mathbf{p}^{\mathbb{N}}, \sigma)$  is *not isomorphic* to the space  $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, R_{\alpha})$ , where  $\mathcal{A}$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}/\mathbb{Z}$ ,  $\alpha$  is irrational, and  $R_{\alpha}(x) = x + \alpha \pmod{1}$  for any choice of  $\mathcal{I}$  and  $\mathbf{p}$ .
5. **(1 pt. bonus)** Let  $(X, \mathcal{B}, \mu, T)$  be a measure-preserving system where  $\mu$  is ergodic and does not contain any atoms. Prove that the  $\sigma$ -algebra of (exactly)  $T$ -invariant sets  $\{E \in \mathcal{B} : T^{-1}(E) = E\}$  is not countably generated.