

Exercise 4

DUE 12:15PM ON THURSDAY, FEBRUARY 12

The questions.

1. (2 pt.) Let (X, \mathcal{B}, μ, T) be a measure-preserving system. Let $A \in \mathcal{B}$ have $\mu(A) > 0$. Prove that μ -a.e. $x \in A$ returns to A infinitely often (that is, $T^n x \in A$ for infinitely many n).
2. (3 pt.) Let (X, d) be a separable metric space, and let (X, \mathcal{B}, μ, T) be a measure-preserving system where \mathcal{B} is the Borel σ -algebra on X .
 - (i) Prove for μ -a.e. $x \in A$ that there is a sequence of times $n_k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} T^{n_k} x = x$.
 - (ii) Suppose moreover that the system is ergodic and every ball $B(x, r)$ has positive measure. Prove (without using the ergodic theorem) that μ -a.e. $x \in A$ has a dense orbit in X .
3. (1 pt.) Let (X, \mathcal{B}, μ) be a probability space and let $f: X \rightarrow X$ be a measurable function. Prove that the family of sets

$$\mathcal{D} = \{E \in \mathcal{B} : \mu(f^{-1}(E)) = \mu(E)\}$$

forms a *Dynkin class*. That is, it satisfies the following conditions:

- (a) $\emptyset \in \mathcal{D}$.
- (b) If $E \in \mathcal{D}$, then $X \setminus E \in \mathcal{D}$.
- (c) If $(E_n)_{n=1}^{\infty} \subset \mathcal{D}$ is a sequence of pairwise disjoint sets, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{D}.$$

4. (4 pt.) Let \mathcal{I} be a finite index set and consider the infinite product space $\mathcal{I}^{\mathbb{N}}$. Let $\mathbf{p}^{\mathbb{N}}$ denote the infinite product measure, where $\mathbf{p} \in \mathcal{P}(\mathcal{I})$, and let $\sigma: \mathcal{I}^{\mathbb{N}} \rightarrow \mathcal{I}^{\mathbb{N}}$ denote the left shift map. Let

$$\mathcal{C} := \{[i] : i \in \mathcal{I}^*\}$$

denote the set of cylinders, where we recall that

$$[i] = \{x \in \mathcal{I}^{\mathbb{N}} : i \prec x\}$$

and $i \prec x$ if $i = (i_1, \dots, i_n)$ and $x = (i_1, \dots, i_n, j_{n+1}, j_{n+2}, \dots)$ for some $j_{n+k} \in \mathcal{I}$.

(Recall that the infinite product measure is the unique probability measure defined on the σ -algebra generated by the cylinders \mathcal{C} by the rule $\mathbf{p}^{\mathbb{N}}([i]) = p_i$.)

- (i) Let $(r_i)_{i \in \mathcal{I}}$ be a set of numbers with $r_i \in (0, 1)$. Given $x, y \in \mathcal{I}^{\mathbb{N}}$, define

$$d(x, y) = \inf \{r_i : \{x, y\} \subset [i]\}.$$

Prove that d defines a metric on $\mathcal{I}^{\mathbb{N}}$, and in fact satisfies the stronger *ultrametric inequality*:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in \mathcal{I}^{\mathbb{N}}.$$

- (ii) Prove that $B^\circ(x, r) \in \mathcal{C}$ for all $x \in \mathcal{I}^{\mathbb{N}}$ and $r > 0$.
- (iii) Conclude that $\mathcal{I}^{\mathbb{N}}$ is a compact metric space.
Hint: You may use Tychonoff's theorem to save some work, or it's not so much work to do this manually in any case.
- (iv) Prove that if $[i] \in \mathcal{C}$, then $\mathbf{p}^{\mathbb{N}}(\sigma^{-1}([i])) = \mathbf{p}^{\mathbb{N}}([i])$.
- (v) Recall the π - λ theorem: If \mathcal{C} is a non-empty family of sets closed under finite intersection, \mathcal{D} is a Dynkin class of sets, and $\mathcal{C} \subseteq \mathcal{D}$, then the σ -algebra generated by \mathcal{C} is a subset of \mathcal{D} .
 Conclude that $\mathbf{p}^{\mathbb{N}}(\sigma^{-1}(E)) = \mathbf{p}^{\mathbb{N}}(E)$ for all Borel sets E .
- (vi) **(1 pt. bonus)** Assume $\#\mathcal{I} \geq 2$. Give an example of a shift-invariant Borel probability measure on $\mathcal{I}^{\mathbb{N}}$ which is *not* a convex combination of the product measures $\mathbf{p}^{\mathbb{N}}$ for $\mathbf{p} \in \mathcal{P}(\mathcal{I})$.