Exercise 4 Solutions

THURSDAY, FEBRUARY 5

1. Write
$$E = A \setminus \bigcup_{n=1}^{\infty} T^{-n} A.$$
This is the set of points in A which never return to A. By the Poincaré recurrence theorem, $\mu(E) = 0$. Moreover, if $x \in A$ only returns to A finitely many times, then there is an index $m \in \mathbb{N} \cup \{0\}$ such that x returns to A for the final time, and therefore $T^m x \in E$. Thus
$$\{x \in A : x \text{ returns to } A \text{ finitely many times}\} \subset \bigcup_{m=1}^{\infty} T^{-m} E.$$
But T is measure preserving, so $\mu(T^{-m}E) = 0$ so that μ-a.e. $x \in A$ returns to A infinitely many times.

2. (i) Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X and consider the family of balls
$$\mathcal{B} := \{B(x_n, 1/k) : (n,k) \in \mathbb{N} \times \mathbb{N}\}.$$
Observe that each $x \in X$ is contained in balls $B \in \mathcal{B}$ of arbitrarily small radius. By Q1, for each $B \in \mathcal{B}$ there is a subset $A_B \subset B$ of full measure (but possibly 0, if $\mu(B) = 0$) such that each $x \in A_B$ returns to B infinitely many times. Then, set
$$Y = X \setminus \bigcup_{B \in \mathcal{B}}^{\infty} (B \setminus A_B).$$
Since $\mu(B \setminus A_B) = 0$, certainly certainly, $\mu(Y) = \mu(X)$.
Moreover, fix $x \in Y$ and let $\varepsilon > 0$ be arbitrary. By definition, there is a $B \in \mathcal{B}$ with $\operatorname{diam} B \leq \varepsilon$ such that $x \in A_B$, and therefore $T^n x \in B$ for infinitely many $n \in \mathbb{N}$, and in particular $d(T^n x, x) \leq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Since this is true for all $\varepsilon > 0$, the claim follows.

   (ii) Let $\mathcal{B}$ be the same family of balls from Q2(i). For $B \in \mathcal{B}$ and consider the set
$$M_B = \{x \in X : T^n x \in B \text{ for infinitely many } n\} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}(B).$$

Since $\mu(B) > 0$ by assumption, by Proposition 3.7 in the notes, $\mu(M_B) = 1$. In particular, if we set

$$Y = \bigcap_{B \in \mathcal{B}} M_B$$

then $\mu(Y) = \mu(X)$ and if $x \in Y$, for all $B \in \mathcal{B}$, there are infinitely many $n \in \mathbb{N}$ such that $T^n x \in B$. By definition of $\mathcal{B}$, this means that the orbit of x is dense in X.

3. Clearly $\varnothing \in \mathcal{D}$. Next, if $E \in \mathcal{D}$, then

$$\mu(f^{-1}(X \setminus E)) = \mu(X \setminus f^{-1}(E)) = \mu(X) - \mu(f^{-1}(E)) = \mu(X \setminus E).$$

Finally, if $(E_n)_{n=1}^{\infty} \subset \mathcal{D}$ are pairwise disjoint, then $(f^{-1}(E_n))_{n=1}^{\infty}$ are also pairwise disjoint sets so

$$\mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = \sum_{n=1}^{\infty} \mu(f^{-1}(E_n)) = \sum_{n=1}^{\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

as required.

4. Write $x\rceil_m$ to denote the unique $\mathtt{i} \in \mathcal{I}^m$ with $\mathtt{i} \prec x$.

(i) Certainly $d(x,y) = d(y,x)$ and $d(x,x) = 0$ since $\{x,x\} \subset [x\rceil_m]$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} r_{x\rceil_m} = 0$. Conversely, if $d(x,y) = 0$, then $\{x,y\} \subset [\mathtt{i}]$ for infinitely many $\mathtt{i}$, and in particular for $\mathtt{i}$ with $r_{\mathtt{i}}$ arbitrarily small. Therefore $x = y$.
It remains to verify the ultrametric inequality. Let $x,y,z \in \mathcal{I}^{\mathbb{N}}$ be arbitrary. If $x = y$ or $y = z$, then the inequality is immediate. Otherwise, let $\mathtt{i}$ be the maximal common prefix of x and y, and let $\mathtt{j}$ be the maximal common prefix of y and z. Since $\mathtt{i}$ and $\mathtt{j}$ are both prefixes of y, either $\mathtt{i} \prec \mathtt{j}$ or $\mathtt{j} \prec \mathtt{i}$. Without loss of generality, $\mathtt{i} \prec \mathtt{j}$ so in fact $\mathtt{i} \prec z$. Then $d(x,y) = r_{\mathtt{i}}$ and $d(y,z) \leq r_{\mathtt{i}}$ so

$$d(x,z) \leq r_{\mathtt{i}} = \max\{d(x,y), d(y,z)\}$$

as required.

(ii) Let $x = (i_n)_{n=1}^{\infty} \in \mathcal{I}^{\mathbb{N}}$ and $r > 0$ be arbitrary. Let $\mathtt{i} \prec x$ be the prefix of minimal length with the property that $r_{\mathtt{i}} < r$.
If $y \in [\mathtt{i}]$, then $\mathtt{i}$ is a common prefix of x and y so $d(x,y) \leq r_{\mathtt{i}} < r$ and therefore $y \in B^{\circ}(x,r)$. Conversely, suppose $d(x,y) < r$. If $x = y$ there is nothing to prove. Otherwise, let $\mathtt{j}$ be the maximal common prefix of x and y so that $d(x,y) = r_{\mathtt{j}} < r$. But $\mathtt{j}$ is therefore a prefix of x with $r_{\mathtt{j}} < r$, where as $\mathtt{i}$ was the prefix of minimal length with this property. Therefore $\mathtt{i}$ is a prefix of $\mathtt{j}$ and $\{x,y\} \subset [\mathtt{i}]$.
We have therefore shown that $B^{\circ}(x,r) = [\mathtt{i}] \in \mathcal{C}$, as required.

(iii) By Tychonoff's theorem, $\mathcal{I}^{\mathbb{N}}$ is compact with the product topology. But the product topology and the metric topology coincide by Q4(ii) (they are both generated by $\mathcal{C}$), and therefore $\mathcal{I}^{\mathbb{N}}$ is in fact a compact metric space.

(iv) Let $[\mathtt{i}] \in \mathcal{C}$ be arbitrary. Observe that

$$\sigma^{-1}([\mathtt{i}]) = \bigcup_{j \in \mathcal{I}} [j\mathtt{i}]$$

disjointly, so

$$p_{\mathtt{i}} = \mu_{\boldsymbol{p}}(\sigma^{-1}([\mathtt{i}])) = \sum_{j \in \mathcal{I}} \mu_{\boldsymbol{p}}([j\mathtt{i}]) = \sum_{j \in \mathcal{I}} p_j p_{\mathtt{i}} = p_{\mathtt{i}}$$

as claimed.

(v) Certainly $\mathcal{C} \cup \{\varnothing\}$ is non-empty and closed under finite intersection, and by Q3 the measurable sets $\{E : \mu(E) = \mu(\sigma^{-1}(E))\}$ form a Dynkin class. Therefore by the π-λ theorem, $\mu(E) = \mu(\sigma^{-1}(E))$ for all E in the σ-algebra generated by $\mathcal{C}$. But Q4(ii) implies that this is precisely the Borel σ-algebra, as required.

(vi) Since $\#\mathcal{I} \geq 2$, say $\{0, 1\} \subset \mathcal{I}$. Let $x = 010101\ldots$ and consider the measure

$$\mu = \frac{1}{2}\left(\delta_x + \delta_{\sigma x}\right).$$

Observe that $\operatorname{supp} \mu = \mathcal{O}_\sigma(x) = \{x, \sigma x\}$ and σ acts by permutation on $\operatorname{supp} \mu$. Therefore μ is σ-invariant.

To see that μ is not a convex combination of measures $\boldsymbol{p}^{\mathbb{N}}$, observe that $\boldsymbol{p}^{\mathbb{N}}$ is atomic if and only if $p_i = 1$ for some $i \in \mathcal{I}$, in which case $\boldsymbol{p}^{\mathbb{N}} = \delta_{z_i}$ where z_i is the sequence obtained by repeating i. So, the only convex combinations of measures $\boldsymbol{p}^{\mathbb{N}}$ which are also a countable sums of atoms are of the form

$$\sum_{i=1}^{\infty} \lambda_i \delta_{z_i} \quad \text{where} \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i = 1.$$

But μ is a finite sum of atoms, and is certainly not of this form.