Exercise 5

DUE 12:15PM ON THURSDAY, FEBRUARY 19

Recall the notation from the previous assignment: if $\mathcal{I}$ is a finite index set and $\boldsymbol{p} \in \mathcal{P}(\mathcal{I})$, then $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \boldsymbol{p}^{\mathbb{N}}, \sigma)$ is a measure-preserving system where $\mathcal{B}$ is the Borel σ-algebra and σ is the left shift map.

The questions.

1. *(2 pt.)* Consider the space $([0,1]^2, \mathcal{B}([0,1]^2), m)$ where m is Lebesgue measure and $\mathcal{B}([0,1]^2)$ is the Borel σ-algebra on $[0,1]^2$. Let $\mathcal{A} \subset \mathcal{B}([0,1]^2)$ denote the sub-σ-algebra

$$\mathcal{A} = \mathcal{B}([0,1]) \times \{\varnothing, [0,1]\}$$

   where $\mathcal{B}([0,1])$ is the Borel σ-algebra on $[0,1]$.

   Let $f\colon [0,1]^2 \to \mathbb{R}$ be a Borel-measurable and integrable function. Prove that

$$\mathbb{E}(f \mid \mathcal{A})(x,y) = \int_0^1 f(x,z)\,\mathrm{d}z$$

   for Lebesgue a.e. $(x,y) \in [0,1]^2$.

   Hint: A system of conditional measures might be useful here.

2. *(3 pt.)* Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system.
   (i) Prove that the system is ergodic if and only if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B)$$

   (ii) Recall that a family $\mathcal{C}$ of sets is a semi-algebra of sets if the following hold:
   (a) $\varnothing \in \mathcal{C}$.
   (b) If $E \in \mathcal{C}$, then $X \setminus E$ is a finite union of elements in $\mathcal{C}$.
   (c) If $E, F \in \mathcal{C}$ then $E \cap F \in \mathcal{C}$.
   Let $\mathcal{C} \subset \mathcal{B}$ be a semi-algebra such that $\mathcal{B}$ is the σ-algebra generated by $\mathcal{C}$. Suppose for all $A, B \in \mathcal{C}$ that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B).$$

   Prove that the system is ergodic.

3. *(2 pt.)* Give an example of a finite index set $\mathcal{I}$, a probability vector $\boldsymbol{p} \in \mathcal{P}(\mathcal{I})$, and a continuous function $f\colon \mathcal{I}^{\mathbb{N}} \to \mathbb{R}$ such that Von Neumann's ergodic theorem (Theorem 3.26 in the notes) fails for $p = \infty$ in the measure-preserving space $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \boldsymbol{p}^{\mathbb{N}}, \sigma)$.

4. *(3 pt.)* Let $\mathcal{C}$ denote the set of cylinders in the measure-preserving space $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \boldsymbol{p}^{\mathbb{N}}, \sigma)$.
   (i) Prove for all cylinders $[\mathtt{i}], [\mathtt{j}] \in \mathcal{C}$ that
   $$\lim_{k\to\infty} \mu(\sigma^{-k}[\mathtt{i}] \cap [\mathtt{j}]) = \mu([\mathtt{i}])\mu([\mathtt{j}]).$$
   Conclude that the system is ergodic.
   This property is called strong mixing.
   (ii) Two measure-preserving systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{A}, \nu, S)$ are *isomorphic* if there are invariant subsets $X_0 \in \mathcal{B}$ and $Y_0 \in \mathcal{A}$ with full measure and a bijection $\pi\colon X_0 \to Y_0$ such that π and π^{-1} are measurable, $\mu \circ \pi^{-1} = \nu$, and $\pi \circ T = S \circ \pi$. Observe that ergodicity is an isomorphism invariant.
   Let $b \in \mathbb{N}$, $b \geq 2$. Recall from the notes that $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, T_b)$ is a measure-preserving system, where m is Lebesgue measure, $\mathcal{A}$ is the Borel σ-algebra, and T_b is multiplication by $b \pmod 1$. Show that $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \boldsymbol{p}^{\mathbb{N}}, \sigma)$ and $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, T_b)$ are isomorphic for an appropriate choice of $\mathcal{I}$ and $\boldsymbol{p}$.
   (iii) **(1 pt. bonus)** Prove that the shift space $(\mathcal{I}^{\mathbb{N}}, \mathcal{B}, \boldsymbol{p}^{\mathbb{N}}, \sigma)$ is *not isomorphic* to the space $(\mathbb{R}/\mathbb{Z}, \mathcal{A}, m, R_\alpha)$, where $\mathcal{A}$ is the Borel σ-algebra on $\mathbb{R}/\mathbb{Z}$, α is irrational, and $R_\alpha(x) = x + \alpha \pmod 1$ for any choice of $\mathcal{I}$ and $\boldsymbol{p}$.

5. **(1 pt. bonus)** Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system where μ is ergodic and does not contain any atoms. Prove that the σ-algebra of (exactly) T-invariant sets $\{E \in \mathcal{B} : T^{-1}(E) = E\}$ is not countably generated.