Fractal geometry and dynamical systems

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ABSTRACT. These are (currently incomplete) lecture notes for the course *Fractal geometry and dynamical systems* taught at the University of Jyväskylä in the 3rd quarter of the 2025–2026 academic year.

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CONTENTS

1. INTRODUCTION AND PRELIMINARIES

Perhaps one of the oldest branches of mathematics is the study of *geometry*. What is geometry? The Merriam–Webster dictionary [Mer22] defines geometry as

> *a branch of mathematics that deals with the measurement, properties, and relationships of points, lines, angles, surfaces, and solids;*

and more broadly,

> *the study of properties of given elements that remain invariant under specified transformations.*

A classical setting for the study of geometry is the geometry of *smooth* objects, such as circles, lines, or graphs of smooth functions.

In contrast, classical examples of sets such as the middle-third Cantor set or the graph of the Weierstrass function are very far from being smooth. Such sets were originally considered to be aberrations that had to be handled in the development of (rigorous) mathematical analysis. However, especially in the past century, it has become clear that such irregular sets are surprisingly abundant.

The goal of these notes is to provide an introduction to the study of irregular sets, with a particular focus on those sets on the structured end of the spectrum. Particular attention will be given to sets exhibiting *self-similarity*. Self-similarity generally describes the phenomenon in which microscopic parts of a set have a similar structure to the set as a whole. We will also see that self-similarity ties in naturally with *invariance* under some smooth or continuous action. In particular, we will be able to draw on tools from ergodic theory and probability theory.

1.1. Context of the course. These notes are prepared for a course with about 24 lecture hours in total, and therefore the scope is limited heavily by time constraints. In particular, a vast amount of interesting material concerning fractal geometry and dynamical systems is omitted. I have attempted to go very deep into a particular topic (regularity of self-similar sets) to give a flavour for the subject. This depth has come at the substantial cost of having a quite narrow focus. If you are interested in a more comprehensive introduction, I would recommended the books [BSS23; BP17; Fal85; Fal97; Mat95].

1.2. Notational conventions. The natural numbers $\mathbb{N}$ begin at the index 1. Sometimes, I may use $\mathbb{N}_0$ to denote the natural numbers starting at 0. Usually, we will work in $\mathbb{R}$, or in $\mathbb{R}^d$ for some $d \in \mathbb{N}$.

In the worst case scenario we will work in a complete separable metric space (X, d). For $x \in X$ and $r > 0$, $B(x, r)$ denotes the *closed ball* of radius r centred at x. The open ball will be denoted by $B^\circ(x, r)$, but will be rarely used. The distance between a point and a set is denoted by

$$d(x, E) := \inf\{d(x, y) : y \in E\}.$$

The *diameter* of a set $E \subset X$ is the maximal distance between any pair of points:

$$\operatorname{diam} E = \sup\{d(x,y) : x, y \in E\}.$$

Note that for all $x \in E$, it holds that $E \subset B(x, \operatorname{diam} E)$. For $r > 0$, the *open r-neighbourhood* of a set E is the set

$$E^{(r)} = \{x \in X : d(x, E) < r\}.$$

Note that $E^{(r)}$ is a union of open balls, and therefore open.

1.3. Measures in metric spaces. The goal of this section is to provide an overview of the measure-theoretic concepts that will be important in this course, but it does assume you've already seen a good amount of abstract measure theory. You can find more detailed exposition of the concepts here in any good measure theory book. I personally recommend Folland's book [Fol99].

1.3.1. General measures. Let X be an arbitrary set.

Definition 1.1. A function $\mu\colon 2^X \to [0, \infty]$ is called an *outer measure* if $\mu^*(\varnothing) = 0$ and for arbitrary subsets $A, \{B_n\}_{n=1}^\infty$ of X with $A \subset \bigcup_{n=1}^\infty B_n$,

$$\mu(A) \leq \sum_{n=1}^\infty \mu(B_n).$$

If $\mathcal{M}$ is a σ-algebra on X, we say that the outer measure μ restricted to the sets in $\mathcal{M}$ is a *measure* if it is countably additive on disjoint sets in $\mathcal{M}$.

Every space X supports a trivial Σ-algebra on which an outer measure is a proper measure: namely, the σ-algebra $\Sigma = \{\varnothing, X\}$. However, this σ-algebra is not particularly useful since we cannot actually measure any interesting sets. Conversely, it turns in many cases because of set-theoretic obstructions, it is unreasonable to hope that Σ can consist of every subset of X.

If X comes with some additional structure, then we would like the σ-algebra to interact nicely with this additional structure. If X is a topological space, at the very least, we would like the σ-algebra to contain the open (equivalently, closed) sets. Such a σ-algebra is called a *Borel* σ-algebra.

1.3.2. Carathéodory's criterion. A powerful abstract criterion for proving that outer measures are measures on large σ-algebras is *Carathéodory's criterion*. If μ is an outer measure on X, we say that a set $F \subset X$ is *μ-measurable* if

$$\mu(E) = \mu(E \cap F) + \mu(E \cap F^c) \quad \text{for all} \quad E \subset X.$$

Since μ is an outer measure, the inequality $\mu(E) \leq \mu(E \cap F) + \mu(E \cap F^c)$, so we only have to prove the converse inequality. The converse inequality is also trivial if $\mu(E) = \infty$, so we need only worry about sets E with finite μ-outer measure.

Theorem 1.2. *If μ is an outer measure on X, then the collection $\mathcal{M}$ of μ-measurable sets is a σ-algebra and the restriction of μ to $\mathcal{M}$ is a (complete) measure.*

The proof is not too long but a bit outside the scope of this overview. You can find the details in [Fol99, §1.11].

1.3.3. Metric outer measures. Now suppose in addition that (X, d) is a complete metric space. Let ρ denote the separation between subsets of X: that is, for sets $E, F \subset X$,

$$\rho(E, F) = \inf\{d(x, y) : x \in E, y \in F\}.$$

An important class of measures in metric spaces is the following.

Definition 1.3. An outer measure μ on X is a *metric outer measure* if for all sets E, F with $\rho(E, F) > 0$,

$$\mu(E \cup F) = \mu(E) \cup \mu(F).$$

This property is very useful because, when combined with Carathéodory's criterion, it provides an easily checkable condition for a general outer measure to be a *bona fide* Borel measure.

Proposition 1.4. *If μ is a metric outer measure on X, then every Borel subset of X is μ-measurable.*

Proof. By Theorem 1.2, it suffices to show that every closed set is μ-measurable. Let F be an arbitrary closed set. Recall that we must show for $E \subset X$ with $\mu(E) < \infty$ that

$$\mu(E) \geq \mu(E \cap F) + \mu(E \setminus F).$$

Let $A_n = \{x \in E \setminus F : d(x, F) \geq n^{-1}\}$ denote those points which are far from F. Since F is closed, $E \setminus F = \bigcup_{n=1}^{\infty} A_n$. Also, $\rho(F, A_n) \geq n^{-1}$. Therefore, $\mu(E) \geq \mu(E \cap F) + \mu(A_n)$.

To complete the proof, it suffices to show that $\mu(E \setminus F) = \lim_{n\to\infty} \mu(A_n)$. **TODO:** write details □

1.4. Constructing measures by repeated subdivision. A particularly useful method for constructing measures, especially in this course, is by *repeated subdivision*.

Begin with a compact metric space (X, d) and let $\{J_n\}_{n=1}^{\infty}$ be a sequence of finite index sets. Let

$$\mathcal{J} = \bigcup_{k=0}^{\infty} \prod_{n=1}^{k} J_1 \times \cdots J_k.$$

Now, suppose we are given a hierarchy of non-empty compact subsets of $X_{\mathtt{i}}$ indexed by sequences $\mathtt{i} = (i_1, \ldots, i_k)$ where $k \in \mathbb{N}_0$ and $i_n \in J_n$ for all n (in other words, $\mathtt{i} \in J_1 \times \cdots \times J_k$), satisfying the following conditions:

(i) $X_{\varnothing} = X$,

(ii) $X_{\mathtt{i}j} \subset X_{\mathtt{i}}$ for all $k \in \mathbb{N}_0$, $\mathtt{i} \in J_1 \times \cdots \times J_k$, and $j \in J_{k+1}$.
(iii) $\lim\limits_{k\to\infty} \sup\limits_{\mathtt{i}\in J_1\times\cdots\times J_k} \operatorname{diam} X_{\mathtt{i}} = 0.$

Note that the sets $X_{\mathtt{i}j}$ need not be disjoint. Write

$$X_k = \bigcup_{\mathtt{i}\in J_1\times\cdots\times J_k} X_{\mathtt{i}}.$$

Note that $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ is a nested sequence of non-empty compact sets, so $K = \bigcap_{k=0}^{\infty} X_k$ is itself a non-empty compact set.

Next, consider an assignment μ, initially defined on the sets $X_{\mathtt{i}}$, with the following additional properties:
(a) $\mu(X_\varnothing) < \infty$.
(b) $\mu(X_{\mathtt{i}}) = \sum_{j\in J_{k+1}} \mu(X_{\mathtt{i}j})$ for all $k \in \mathbb{N}_0$ and $\mathtt{i} \in J_1 \times \cdots \times J_k$.
(c) $\lim\limits_{k\to\infty} \sup\limits_{\mathtt{i}\in J_1\times\cdots\times J_k} \mu(X_{\mathtt{i}}) = 0.$

In words, the second condition says that all of the mass of μ is divided equally among the "children" of $X_{\mathtt{i}}$. The third condition is just a non-degeneracy condition analogous to the condition on $\operatorname{diam} X_{\mathtt{i}}$.

In order to extend μ to a genuine measure, we first extend it as an outer measure by the rule

$$\mu(E) = \inf\left\{\sum_{\mathtt{i}\in\mathcal{E}} \mu(X_{\mathtt{i}}) : E\cap K \subset \bigcup_{\mathtt{i}\in\mathcal{E}} X_{\mathtt{i}}\right\}.$$

It is not too difficult to verify that $\mu(X_{\mathtt{i}})$ is the pre-assigned value for $\mathtt{i} \in \mathcal{J}$. Moreover, since the measures $\mu(X_{\mathtt{i}})$ and diameters $\operatorname{diam} X_{\mathtt{i}}$ converge to 0, one can additionally check that μ is a metric outer measure. Therefore μ defines a Borel measure on the metric space X.

Example 1.5. This gives a way to define Lebesgue measure on the interval $X = [0, 1]$. Let $J_1 = J_2 = \cdots = \{0, 1\}$ so that $\{X_{\mathtt{i}} : \mathtt{i} \in J_1 \times \cdots \times J_k\}$ is the set of dyadic intervals in $[0, 1]$ of width 2^{-k}. Take $\mu(J_k) = 2^{-k}$, and one can check that various conditions are satisfied.

2. SELF-SIMILARITY AND DIMENSION THEORY

We begin these notes by rigorously introducing the notion of a self-similar set, and with it the basic aspects of dimension theory (of sets) that will accompany us throughout these notes.

The *middle-thirds Cantor set* is almost always the first example that one sees of a fractal set. It will accompany us throughout this section as a very basic, yet fundamental example.

Example 2.1. The most concrete construction of the middle-thirds Cantor set is as an inductive construction using a sequence of nested intervals.

We will construct $C = \bigcap_{n=0}^{\infty} C_n$ where C_n is a disjoint union of 2^n compact intervals each of width 3^{-n}.

Begin with $C_0 = [0,1]$. Now, suppose we have constructed $C_n = \bigcup_{i=1}^{2^n}[a_i,b_i]$ for pairwise disjoint compact intervals $[a_i,b_i]$ with $b_i - a_i = 3^{-n}$. We subdivide this interval into two sub-intervals of length 3^{n-1}, the first of which has left-endpoint a_i, and the second of which has right-endpoint b_i. Specifically, these two intervals are given by

$$[a_i, a_i + 3^{-n+1}] \qquad \text{and} \qquad [b_i - 3^{-n+1}, b_i].$$

We then set

$$C_{n+1} = \bigcup_{i=1}^{2_n} [a_i, a_i + 3^{-n+1}] \cup [b_i - 3^{-n+1}, b_i].$$

Explicitly:
- $C_0 = [0,1]$,
- $C_1 = [0, 1/3] \cup [1/3, 1]$,
- $C_2 = [0,1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$,
- etc.

Note that the sets C_n are nested: $C_0 \supset C_1 \supset C_2 \supset \cdots$. Therefore $C = \bigcap_{n=0}^{\infty} C_n$ is a legitimate non-empty compact set.

We can already see a few basic properties. By a diagonalization argument (indeed, the one attributed to Cantor himself!), the Cantor set is uncountable. However, it also has length 0: this is since $C \subset C_n$ and C_n has length $(2/3)^n$.

2.1. Iterated function systems and attractors. Hutchinson introduced an elegant general framework for self-similar sets in [Hut81], which we now introduce.

Definition 2.2. Let (X,d) be a metric space. We say that a function $f\colon X \to X$ is a *contraction map* if there is a number $\lambda \in [0,1)$ so that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda d(x,y).$$

We call the number λ the *contraction ratio* of f.

We moreover say that f is a *contractive similarity* if $\lambda > 0$ and the above inequality is an equality.

Contraction maps are just those Lipschitz maps which move all pairs of points closer together.

Recall that any contraction map has a unique fixed point.

Proposition 2.3 (Banach contraction mapping). *Let (X,d) be a non-empty complete metric space and suppose $f\colon X \to X$ is a contraction map. Then there is a unique $x_* \in X$ such that $f(x_*) = x_*$, and moreover $x_* = \lim_{n\to\infty} f^n(x)$[1] for any starting point $x \in X$.*

If you do not recall the proof, it is a good exercise to prove it. Here is a brief sketch. The sequence of n-fold compositions $f^n(x)$ is Cauchy (here, we use contraction), and so has a limit, say x_*. By continuity of f, we certainly see that x_* is a fixed

[1] Superscripts will always denote n-fold composition, and never differentiation.

point. Finally, there can be no other fixed points for if y were a fixed point, then $d(x_*, y) = d(f(x_*), f(y)) \leq \lambda d(x_*, y)$ which can only happen if $y = x_*$.

Contraction maps act naturally on sets as well. Recall the *Hausdorff distance* on sets.

Definition 2.4. Let $A, B \subset X$ be non-empty sets. The *Hausdorff distance* between A and B is defined as

$$d_{\mathcal{H}}(A, B) := \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}.$$

Here, $d(x, E) := \inf_{y \in E} d(x, y)$ is the distance between the point x and the set E.

If A and B have $d_{\mathcal{H}}(A, B) < r$, then the open r-neighbourhood of A contains B, and the r-neighbourhood of B contains A. In fact, it is not too much more difficult to show that the Hausdorff distance is the smallest such value of r for which this is occurs.

We will be primarily interested in compact sets. Here, the theory is particularly elegant.

First, let's check that $d_{\mathcal{H}}$ actually defines a metric. That $d_{\mathcal{H}}(A, A) = 0$ and $d_{\mathcal{H}}(A, B) = d_{\mathcal{H}}(B, A)$ is immediate from the definition. Compactness implies that if $d_{\mathcal{H}}(A, B) = 0$, then in fact $A = b$. To see the triangle inequality, let A, B, C be non-empty and compact. Using the triangle inequality in X, we observe that $A \subset B^{(r_1)}$ and $B \subset C^{(r_2)}$, then $A \subset C^{(r_1+r_2)}$. Now if $d_{\mathcal{H}}(A, B) = s$, $d_{\mathcal{H}}(B, C) = t$, and $\varepsilon > 0$, then $A \subset B^{(s+\varepsilon)}$ and $B \subset C^{(t+\varepsilon)}$ so $A \subset C^{(s+t+2\varepsilon)}$. The other inclusion is analogous, so $d_{\mathcal{H}}(A, C) \leq s+t+2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $d_{\mathcal{H}}(A, C) \leq d_{\mathcal{H}}(A, B)+d_{\mathcal{H}}(B, C)$.

More importantly, if X is itself a complete metric space, then $\mathcal{K}(X)$ is also complete.

Proposition 2.5. *Let (X, d) be a complete metric space and let $\mathcal{K}(X)$ denote the set of all non-empty compact subsets of X. Then the space $(\mathcal{K}(X), d_{\mathcal{H}})$ is itself a complete metric space.*

Proof. Finally, assuming that X is complete, we prove that $\mathcal{K}(X)$ is complete. Let $(A_n)_{n=1}^\infty$ be a Cauchy sequence of compact sets.

We first reduce to the case that the sets A_n are nested. Define

$$K_n = \overline{\bigcup_{k=n}^{\infty} A_k}.$$

Observe that $\lim_{n\to\infty} d_{\mathcal{H}}(A_n, K_n) = 0$. Certainly $A_n \subset K_n$. The other inequality in the definition of the Hausdorff distance follows since A_n is itself Cauchy: for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ so that for all $N \leq n \leq k$, $A_k \subset A_n^{(\varepsilon)}$, and therefore $K_n \subset A_n^{(\varepsilon)}$.

Now, let us check that the sets K_n are compact. Since K_n is closed, it remains to show that K_n is totally bounded. Let $\varepsilon > 0$ be arbitrary. Let N be sufficiently large so that $d_{\mathcal{H}}(A_N, A_k) < \varepsilon/2$ for all $k \geq N$. Then, consider the family of balls $\{B^\circ(x, \varepsilon/2) : x \in A_N\}$. This is an open cover for A_N, and therefore has a finite sub-cover $\{B^\circ(x_j), \varepsilon/2)\}_{j=1}^m$ by compactness of A_N. Moreover, since $\mathcal{U} :=$

$\{B^\circ(x_j), \varepsilon)\}_{j=1}^m$ is a cover for $A_N^{(\varepsilon/2)}$, it follows that $\mathcal{U}$ is also a cover for A_k for all $k \geq N$, by choice of N. Therefore, K_N is totally bounded. But K_n is the union of K_N with finitely many compact sets A_k, and therefore K_n is totally bounded as well.

We now have a nested sequence of compact sets $(K_n)_{n=1}^\infty$. Let $K = \bigcap_{n=1}^\infty K_n$, which is itself non-empty and compact. It remains to see that $\lim_{n\to\infty} d_{\mathcal{H}}(K_n, K) = 0$ in the Hausdorff metric. Suppose for contradiction that there is a $\varepsilon > 0$ and a subsequence $(n_k)_{k=1}^\infty$ such that

$$E_k := K_{n_k} \setminus K^{(\varepsilon)} \neq \varnothing.$$

Since the sets K_{n_k} are compact and nested, the sets E_k are also compact and nested. Therefore, there exists a point $x \in E_k$ for all $k \in \mathbb{N}$. Moreover, by definition of E_k, $d(x, K) \geq \varepsilon$. But $x \in K_{n_k}$ for all $k \in \mathbb{N}$ and the sets K_{n_k} are nested, so in fact $x \in K$. This is a contradiction.

In particular, we conclude that $K_n \subset K^{(\varepsilon)}$ for all n sufficiently large. Since $K \subset K_n$ for all n, we conclude that $\lim_{n\to\infty} d_{\mathcal{H}}(K_n, K) = 0$ as required. □

One can check that contraction maps also act as contractions on the space $\mathcal{K}(X)$. However, the "limiting" theory is still not too interesting: for a compact set $K \subset X$, $\lim_{n\to\infty} f^n(K) = \{x_*\}$, where x_* is the unique fixed point of f.

Instead of considering a single contraction map, we will instead consider finite families of maps.

Definition 2.6. A *contracting iterated function system* (or *IFS* for short) on a non-empty complete metric space (X, d) is a collection of maps $(f_i)_{i\in\mathcal{I}}$, where $\mathcal{I}$ is a non-empty finite index set, such that each f_i is a contraction map on X.

An IFS $\Phi = \{f_i\}_{i\in\mathcal{I}}$ no longer acts on X naturally (since single points, in general, map to many points). However, the action on $\mathcal{K}(X)$ generalizes correctly when acting on subsets of X. In particular, if $K \subset X$ is compact, then

$$\Phi(K) := \bigcup_{i\in\mathcal{I}} f_i(K).$$

is a finite union of compact sets, and therefore still compact.

Hutchinson's observation is that the action of Φ is again a contraction.

Theorem 2.7. *Let (X, d) be a complete metric space and let $\Phi = \{f_i\}_{i\in\mathcal{I}}$ be an IFS. Suppose f_i has contraction r_i. Then $\Phi\colon \mathcal{K}(X) \to \mathcal{K}(X)$ is a contraction with contraction ratio $\max_{i\in\mathcal{I}} r_i$.*

Proof. Let $A, B \subset X$ be arbitrary non-empty compact sets with $d_{\mathcal{H}}(A, B) = t$.

Let $i \in \mathcal{I}$ be arbitrary. We first observe that $d_{\mathcal{H}}(f_i(A), f_i(B)) \leq r_i \cdot t$. Indeed, suppose $x \in f_i(A)$ and write $x = f_i(a)$. By compactness, get $b \in B$ such that $d(a, b) = d(a, B)$. Then

$$d(x, f_i(B)) \leq d(x, f_i(b)) = d(f_i(a), f_i(b)) \leq r_i d(a, b) = r_i d(a, B) \leq r_i t.$$

The inequality with A and B swapped holds by the symmetric argument.

In particular,

$$\sup_{x\in\Phi(A)} d(x,\Phi(B)) = \max_{i\in\mathcal{I}} \sup_{x\in f_i(A)} d(x,\Phi(B)) \leq \max_{i\in\mathcal{I}} r_i \cdot t.$$

Again, the inequality holds with A and B swapped by the symmetric argument. This completes the proof of the claim. □

Rephrasing the conclusion of this theorem, we obtain the following key corollary.

Corollary 2.8. *Let (X,d) be a non-empty complete metric space and let $\{f_i\}_{i\in\mathcal{I}}$ be an IFS. Then there exists a unique non-empty compact set $K \subset X$ such that*

$$K = \bigcup_{i\in\mathcal{I}} f_i(K).$$

This unique non-empty invariant compact set K is called the *attractor* of the IFS $\{f_i\}_{i\in\mathcal{I}}$.

Example 2.9. Let us return again to the Cantor set example. Recall the notation from Theorem 2.1. We can realize the middle-thirds Cantor set as the attractor of an IFS.

Fix the index set $\mathcal{I} = \{1,2\}$ and maps $f_1(x) = x/3$ and $f_2(x) = x/3 = 2/3$. Observe that $f([0,1]) = [0,1/3]$ and $f_2([0,1]) = [2/3,1]$. In other words, $\Phi(C_0) = C_1$.

In fact, we will see that $\Phi(C_n) = C_{n+1}$. With our current notation, it is a bit hard to understand what is going on, so let us introduce some better book-keeping to have a better idea of what is happening.

Instead of speaking of the intervals $[a_i,b_i]$, let us instead speak of sequences $\mathtt{i} \in \mathcal{I}^n$. Given a sequence $\mathtt{i} = (i_1,\ldots,i_n) \in \mathcal{I}^n$, we write $f_{\mathtt{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$. Then, we write $I_{\mathtt{i}} = f_{i_1} \circ \cdots \circ f_{i_n}([0,1])$.

A short computation shows that if $I_{\mathtt{i}} = [a,b]$, then $I_{\mathtt{i}1} = [a, a+3^{-(n+1)}]$ and $I_{\mathtt{i}2} = [b-3^{-(n+1)}, b]$. (One way to think about right-hand composition is to imagine $I_{\mathtt{i}} \cong [0,1]$, and then $I_{\mathtt{i}j} \cong I_j$, so the "placement relative to the parent" is the same as the "absolute placement relative to the original interval".) In particular, this shows for $n \geq 0$ that

$$C_n = \bigcup_{\mathtt{i}\in\mathcal{I}^n} I_{\mathtt{i}}.$$

Now the action of Φ is totally transparent:

$$\begin{aligned}\Phi(C_n) &= \bigcup_{\mathtt{i}\in\mathcal{I}^n} f_1 \circ f_{\mathtt{i}}([0,1]) \cup f_2 \circ f_{\mathtt{i}}([0,1]) \\ &= \bigcup_{\mathtt{j}\in\mathcal{I}^{n+1}} I_{\mathtt{j}} = C_{n+1}.\end{aligned}$$

This in particular shows that $\lim_{n\to\infty} C_n = C$ in the Hausdorff metric (fill in the details!). Recalling the contraction mapping principle, this means that C must be the attractor of the IFS Φ.

2.2. Hausdorff dimension. Informally, one would hope that the dimension of a geometric object gives some meaningful notion of "size", but one which does not concern itself too precisely with the exact appearance of the object itself but rather some more general, global property. For linear objects, such as real vector spaces, a notion of dimension arises naturally from the algebraic structure as the 'number of coordinates' required to uniquely identify a point in space. Smooth objects, which are locally linear, then inherit such a notion of dimension 'for free'.

However, the sets that we have constructed in the previous section are very far from being smooth. The goal in this section is to introduce some of the fundamental ways to define dimension in a purely geometric way.

One way to think about dimension is through *scaling*. It is quite natural to ask for the area of a (mathematically ideal) sheet of paper. But asking for the volume or the length of a sheet of paper doesn't make sense: the sheet has has zero volume, and 'infinite' length. So, a sheet of paper is 2-dimensional in the sense the natural measurement scale is 2-dimensional.

How can we define concepts like 'length' or 'area' in a universal way? The principle of scaling gives one answer: a 2-dimensional object, like a disc, has area which scales by the square of the diameter; whereas a 3-dimensional object, like a solid sphere, has volume which scales by the cube of the diameter. So, if we know nothing else, if we are given a general set E and we are told that it is s-dimensional, then we would expect that scaling it by some factor λ would in turn scale its 'size' by a factor of λ^s.

In order to choose the ideal exponent s, we need to determine the 'size' (like length, or area, or volume) of some abstract geometric object from an s-dimensional reference point. This is the role of *Hausdorff measure*.

Definition 2.10. Let $s \geq 0$. Then the *s-dimensional Hausdorff (outer) measure* of a set E is defined by

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(U_j)^s : E \subset \bigcup_{j=1}^{\infty} U_j \text{ and } \operatorname{diam} U_j \leq \delta \right\}.$$

We call any family of sets $\{U_j\}$ with $\operatorname{diam} U_j \leq \delta$ and $E \subset \bigcup_{j=1}^\infty U_j$ a *δ-cover for E.*

The limit $\lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$ exists by monotonicity, though it may of course take value $+\infty$.

Note that $\mathcal{H}^s_\delta$ for $\delta > 0$ is very far from being a measure. On the other hand, the outer measure $\mathcal{H}^s$ is in fact a proper Borel measure.

Proposition 2.11. *For $s \geq 0$, $\mathcal{H}^s$ is a metric outer measure. In particular, every Borel set is $\mathcal{H}^s$-measurable.*

Proof. Suppose $d(E, F) = \varepsilon > 0$ and let $0 < \delta < \varepsilon$. Then if $\{U_j\}$ is any cover of $E \cup F$ using sets of diameter at most δ, each U_j must intersect at most one of the two sets E and F. Therefore, we can partition $\{U_j\}$ into two parts based on whether $U_j \cap E = \varnothing$ or $U_j \cap F = \varnothing$. In particular,

$$\sum_j (\operatorname{diam} U_j)^s \geq \mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F).$$

Since $\{U_j\}$ was an arbitrary δ-cover, it follows that $\mathcal{H}^s_\delta(E \cup F) \geq \mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F)$. Taking the limit as δ goes to 0, combined with the fact that $\mathcal{H}^s$ is an outer measure (and therefore subadditive) yields the claim. □

Let us briefly note a few important properties of Hausdorff measure.

Lemma 2.12. *Suppose $f\colon X \to Y$ is a λ-Lipschitz for some $\lambda > 0$. Let $E \subset X$ be arbitrary. Then $\mathcal{H}^s(f(E)) \leq \lambda^s \mathcal{H}^s(E)$.*

In particular, if $f\colon X \to X$ is a similarity map, then $\mathcal{H}^s(f(E)) = \lambda^s \mathcal{H}^s(E)$.

Proof. Observe that if f is λ-Lipschitz and U is non-empty, then $\operatorname{diam} f(U) \leq \lambda \operatorname{diam} U$.

Now for the proof. If $\mathcal{H}^s(E) = \infty$, there is nothing to prove. Otherwise, suppose $\{U_i\}_{i=1}^\infty$ is an arbitrary $\lambda^{-1}\delta$-cover for E. Then $\{f(U_i)\}_{i=1}^\infty$ is a δ-cover for $f(E)$ and satisfies

$$\sum_{i=1}^\infty (\operatorname{diam} f(U_i))^s \leq \lambda^s \sum_{i=1}^\infty (\operatorname{diam} U_i)^s.$$

But $\{U_i\}_{i=1}^\infty$ was an arbitrary $\lambda^{-1}\delta$-cover, so

$$\mathcal{H}^s_\delta(f(E)) \leq \mathcal{H}^s_{\lambda^{-1}\delta}(E).$$

Taking the limit as δ goes to zero completes the proof. □

In the special case that $X = \mathbb{R}^d$, we see that Hausdorff s-measure is translation invariant. However, it is very much *not σ-finite*, except in the special case that $s = d$.

Hausdorff measure gives us a way to measure the size of some set from an s-dimensional vantage point. Hausdorff dimension is the exponent s from which that vantage point is the most natural.

Definition 2.13. The *Hausdorff dimension* of a set E is defined equivalently by

$$\dim_{\mathrm{H}} E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\}.$$

Before we continue, let us briefly justify why such an exponent exists.

Lemma 2.14. *Suppose $0 \leq s < t$.*
(i) If $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^t(E) = 0$.
(ii) If $\mathcal{H}^t(E) > 0$, then $\mathcal{H}^s(E) = \infty$.

Proof. If $\mathcal{H}^s(E) < \infty$, then for all $\delta > 0$, there exists a δ-cover $\{U_j\}$ for E with $\sum_{j=1}^{\infty}(\operatorname{diam} U_j)^s \leq \mathcal{H}^s(E) + 1$. Therefore

$$\mathcal{H}^(E) \leq \sum_{j=1}^{\infty}(\operatorname{diam} U_j)^t \leq \delta^{t-s}\sum_{j=1}^{\infty}(\operatorname{diam} U_j)^s \leq \delta^{t-s}(\mathcal{H}^s(E)+1).$$

Since $t > s$, δ^{t-s} converges to 0 so $\mathcal{H}^t(E) = 0$.

The second statement is just the contrapositive of the first. □

Considering again the example of a sheet of paper, the exponent 1 is too small (since the sheet has infinite length) whereas the exponent 3 is too large (since the sheet has zero volume). Coming up with visual intuition for fractional exponents is more difficult, but mathematically it works fine.

An interesting feature of Hausdorff dimension is that it is defined as an infimum over all possible covers. To give an upper bound on dimension, it suffices to come up with some explicit family of good covers.

Example 2.15. Recall the middle-thirds Cantor set C. At level n, there are 2^n construction intervals each of diameter 3^{-n}. Therefore,

$$\mathcal{H}^s(C) \leq 2^n 3^{-ns}.$$

Taking $s = \log 2/\log 3$, it follows that $\mathcal{H}^s(C) \leq 1$, and therefore $\dim_{\mathrm{H}} C \leq \frac{\log 2}{\log 3}$.

In contrast, proving a lower bound over all covers can be somewhat more difficult. To obtain lower bounds, an essential approach is to use measures which are well-distributed over the set E. We introduce this method in the next section.

2.3. The mass distribution principle and Frostman's lemma. In the next section, we prove an important converse to the mass distribution principle called *Frostman's lemma*.

In order to state Frostman's lemma, it is more natural to use *Hausdorff content* instead of Hausdorff measure. Hausdorff content is the set function $\mathcal{H}^s_\infty(E)$, and it is in a sense the opposite of Hausdorff s-measure in that there are no size constraints on the allowed covers of E. Hausdorff content is an outer measure, but unlike Hausdorff measure, it is very far from being a measure. In general, $\mathcal{H}^s_\infty(E) \leq \mathcal{H}^s(E)$, but it can be much smaller: for example, $\mathcal{H}^s_\infty(E) \leq (\operatorname{diam} E)^s < \infty$.

However, it can often be used in place of Hausdorff measure, because of the following lemma.

Lemma 2.16. *Let $s \geq 0$. Then $\mathcal{H}^s(E) = 0$ if and only if $\mathcal{H}^s_\infty(E) = 0$. In particular,*

$$\dim_{\mathrm{H}} E = \inf\{s \geq 0 : \mathcal{H}^s_\infty(E) = 0\}.$$

Proof. Of course, $\mathcal{H}^s_\infty(E) \leq \mathcal{H}^s(E)$ for all sets E. Thus, suppose E has $\mathcal{H}^s_\infty(E) = 0$. Let $\delta > 0$ be arbitrary. By definition of Hausdorff content, get a cover $\{U_i\}_{i=1}^{\infty}$ for E such that $\sum_{i=1}^{\infty}(\operatorname{diam} U_i)^s < \delta^s$. In particular, $\operatorname{diam} U_i < \delta$ for all i. Therefore,

$\{U_i\}_{i=1}^{\infty}$ is in fact a δ-cover for E, and therefore $\mathcal{H}^s_\delta(E) < \delta^s$ as well. Since $\delta > 0$ was arbitrary, it follows that $\mathcal{H}^s(E) = 0$. □

We now begin with the following lower bound for Hausdorff content.

Lemma 2.17 (Mass distribution principle). *Let $E \subset \mathbb{R}^d$ be compact and suppose μ is a finite Borel measure with $\operatorname{supp}\mu \subset E$ such that $\mu(A) \leq c(\operatorname{diam} A)^s$ for all bounded $A \subset \mathbb{R}^d$. Then*

$$\mathcal{H}^s_\infty(E) \geq \mu(E) \cdot c^{-1}.$$

In particular, if μ is not the zero measure, then $\dim_{\mathrm{H}} E \geq s$.

Proof. Let $\{U_j\}_{j=1}^{\infty}$ be an arbitrary cover of E, and let $x_j \in U_j$ for each j. Then

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(U_j) \leq \sum_{j=1}^{\infty} c(\operatorname{diam} U_j)^s.$$

Since $\{U_j\}$ was arbitrary, it follows that $\mathcal{H}^s(E) \geq c^{-1}$. □

An explanation in words for the mass distribution principle is as follows: since the measure cannot concentrate too much on any fixed ball $B(x, r)$, the only way is for the measure to be supported on many balls, i.e. for its support to have large dimension.

Conversely, it turns out that such measures must always exist. This result is due to Otto Frostman [Fro35].

Lemma 2.18 (Frostman's lemma). *For all $d \in \mathbb{N}$ there exists a constant $c_d > 0$ such that the following holds. Let $E \subset [0, 1]^d$ be compact and let $\alpha = \mathcal{H}^s_\infty(E)$. Then there exists a Borel measure μ with $\operatorname{supp}\mu \subset E$ and $\mu(E) \geq \alpha$ such that $\mu(A) \leq c_d(\operatorname{diam} A)^s$ for all bounded $A \subset \mathbb{R}^d$.*

Proof. We will only need two features of the Hausdorff content $\mathcal{H}^s_\infty(E)$: firstly, that it is an outer measure, and secondly that it satisfies $\mathcal{H}^s_\infty(E) \leq (\operatorname{diam} E)^s$ for all sets E.

The spirit of the proof is essentially to inductively normalize the Hausdorff content on the set E so that it becomes a measure. We construct the measure using the method of subdivision from §1.4, using the dyadic cubes as our indices.

Translating E if necessary, we may assume that E is contained in a single dyadic cube $[0, 2^j]^d$ for some $j \in \mathbb{Z}$. Call this cube Q_0.

We inductively assign weights $\mu(Q)$ to the dyadic cubes contain in Q_0 with the additional requirement that $\mu(Q) \leq \mathcal{H}^s_\infty(Q \cap E)$.

Assign Q_0 mass $\mu(Q_0) = \mathcal{H}^s_\infty(E)$. Now, suppose Q is an arbitrary dyadic cube to which we have assigned mass $\mu(Q) \leq \mathcal{H}^s_\infty(Q \cap E)$. Recall that Q has 2^d dyadic sub-cubes at the next level. Call them $\{P_1, \ldots, P_{2^d}\}$. Then, define $\mu(P_j)$ in such a way that:

(i) $\sum_{j=1}^{2^d} \mu(P_j) = \mu(Q)$
(ii) $\mu(P_j) \leq \mathcal{H}^s_\infty(P_j \cap E)$.

Such a choice is always possible since Hausdorff content is sub-additive, so

$$\mu(Q) \leq \mathcal{H}^s_\infty(Q \cap E) \leq \sum_{j=1}^{2^d} \mathcal{H}^s_\infty(P_j \cap E).$$

It is not too difficult to check that that this inductive assignment of mass satisfies the requirements for the method of subdivision. The resulting measure μ has, by definition, $\mu(E) = \alpha$. Moreover, if a dyadic cube has $Q \cap E = \varnothing$, then $\mu(Q) \leq \mathcal{H}^s_\infty(Q \cap E) = 0$. Therefore $\operatorname{supp} \mu \subset E$.

Finally, if Q is any dyadic cube, by construction of μ, $\mu(Q) \leq (\operatorname{diam} Q)^s$. There are two cases: if $Q \subset Q_0$, then $\mu(Q) \leq \mathcal{H}^s_\infty(Q \cap E) \leq (\operatorname{diam} Q)^s$; and if $Q_0 \subset Q$ then the inequality follows from the inequality for $Q = Q_0$ by monotonicity. But any set A with diameter satisfying $2^j \leq \operatorname{diam} A < 2^{j+1}$ for some $j \in \mathbb{Z}$ intersects c_d dyadic cubes at level j. Therefore, $\mu(A) \leq c_d (\operatorname{diam} Q)^s$, completing the proof. $\square$

Remark 2.19. This lemma is started for compact sets, but it is also true for general Borel sets (or even analytic sets). The difficulty is primarily of set-theoretic nature; the details can be found for example in [BP17, Appendix B].

2.4. Invariant measures for iterated function systems. We have now seen that the Hausdorff dimension of a general set can always be bounded below by a general measure. However, for the special sets which we are most interested in, we would to be able to say that the measures may take a particularly nice form.

For iterated function systems, there is a special family of candidate invariant measures, which we now introduce. Our next goal is to prove an analogue of the existence of measures.

Definition 2.20. A *weighted IFS* is an IFS $\{f_i\}_{i \in \mathcal{I}}$ combined with a family of weights $\boldsymbol{p} = \{p_i\}_{i \in \mathcal{I}}$ with the property that $p_i \geq 0$ and $\sum_{i \in \mathcal{I}} p_i = 1$.

Recall that every IFS has an associated attractor K satisfying $K = \bigcup_{i \in \mathcal{I}} f_i(K)$. We now show that every weighted IFS has an associated *invariant measure* $\mu_{\boldsymbol{p}}$ satisfying, for Borel sets E,

$$\mu_{\boldsymbol{p}}(E) = \sum_{i \in \mathcal{I}} p_i \mu_{\boldsymbol{p}}(f_i^{-1}(E)).$$

The general strategy of the proof is analogous to the existence of the attractor K: we will show that a weighted IFS acts as a contraction map on the space of probability measures on the metric space X.

As in the case of sets, we need to first introduce the space of measures. However, handling this in full rigour is a bit beyond the scope of these notes (certainly, it would take quite a long time), so we will just summarize the key required details.

The metric on the space of measures generates the weak topology. However, to make it a proper metric, instead of considering the action on continuous functions, we restrict the action to Lipschitz functions with constant at most 1.

Proposition 2.21. *Let (X, d) be a complete metric space and let $\mathcal{P}(X)$ denote the space of Borel probability measures on X. Define*

$$d_{\mathcal{P}}(\mu, \nu) = \sup\left\{\left|\int_X g(x)\,\mathrm{d}\mu(x) - \int_X g(x)\,\mathrm{d}\nu(x)\right| : g\colon X \to \mathbb{R} \text{ is 1-Lipschitz}\right\}.$$

Then $(\mathcal{P}(X), d_{\mathcal{H}})$ is a complete metric space. Moreover, if X is compact, then $(\mathcal{P}(X), d_{\mathcal{H}})$ is compact.

Proof. Perhaps I will include more details later, but here is a sketch for now.

The fact that $d_{\mathcal{P}}$ is symmetric, has $d_{\mathcal{P}}(\mu, \mu) = 0$, and satisfies the triangle inequality, is not too difficult.

Proving that $d_{\mathcal{P}}(\mu, \nu) \neq 0$ for $\mu \neq \nu$ is rather technical, since it requires the construction of a 1-Lipschitz function which witnesses the difference in mass.

Next, completeness follows by the Riesz representation theorem. Define a linear functional $L\colon C(X) \to \mathbb{R}$ by

$$L(f) := \lim_{n\to\infty} \int_X f(x)\,\mathrm{d}\mu_n(x).$$

The Riesz representation guarantees that L arises as integration against a measure:

$$L(f) = \int_X f(x)\,\mathrm{d}\mu(x).$$

Completing the proof requires proving convergence in norm of the linear functionals $L_n(f) := \int_X f(x)\,\mathrm{d}\mu_n(x)$. Here, one can simplify the argument by only considering norm convergence on the space of 1-Lipschitz functions, rather than worrying about all continuous functions.

Finally, total boundedness can be proven by considering measures of finite support. For $N \in \mathbb{N}$, let $P = \{x_1, \ldots, x_m\}$ be a set of points such that $\{B(x_j, 1/N)\}_{j=1}^m$ covers X, and then consider the set of measures μ supported on P and assigning mass j/N for some integer $0 \leq j \leq N$ to the points in P. The set of balls centred at this family of measures, for an appropriate radius ε, will be a cover for $\mathcal{P}(X)$. □

With these technical details out of the way let's prove the existence of invariant measures on iterated function systems.

Theorem 2.22. *Let $\{f_i\}_{i\in\mathcal{I}}$ be an IFS and $\boldsymbol{p}$ a set of weights. Then there exists a unique Borel probability measure $\mu_{\boldsymbol{p}}$ satisfying*

$$\mu_{\boldsymbol{p}}(E) = \sum_{i\in\mathcal{I}} p_i \mu_{\boldsymbol{p}}(f_i^{-1}(E)).$$

In particular, $\operatorname{supp}\mu_{\boldsymbol{p}} \subset K$.

Proof. For each i, let f_i have contraction ratio r_i.

Define a map $\Psi\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by the rule

$$\Psi(\mu)(A) = \sum_{i\in\mathcal{I}} p_i \mu(f_i^{-1}(A)).$$

Unpacking definitions, observe for measurable functions $g\colon X \to \mathbb{R}$ that

$$\int g \,\mathrm{d}\Psi(\mu) = \sum_{i\in\mathcal{I}} p_i \int (g \circ f_i)\,\mathrm{d}\mu.$$

For notational compactness, let $\mathcal{L}$ denote the 1-Lipschitz functions from X to $\mathbb{R}$. Note the following key observation: if g is 1-Lipschitz, then $r_i^{-1}(g \circ f_i)$ is also 1-Lipschitz. Then to see that Ψ is a contraction map,

$$\begin{aligned}
d_{\mathcal{P}}(\Psi(\mu), \Psi(\nu)) &= \sup\left\{\left|\int_X g\,\mathrm{d}\Psi(\mu) - \int_X g\,\mathrm{d}\Psi(\nu)\right| : g \in \mathcal{L}\right\}\\
&= \sup\left\{\left|\sum_{i\in\mathcal{I}} p_i \int_X (g\circ f_i)\,\mathrm{d}\mu - \int_X (g\circ f_i)\,\mathrm{d}\nu\right| : g \in \mathcal{L}\right\}\\
&\leq \sum_{i\in\mathcal{I}} p_i \sup\left\{\left|\sum_{i\in\mathcal{I}} \int_X (g\circ f_i)\,\mathrm{d}\mu - \int_X (g\circ f_i)\,\mathrm{d}\nu\right| : g \in \mathcal{L}\right\}\\
&\leq \sum_{i\in\mathcal{I}} p_i \sup\left\{r_i\left|\sum_{i\in\mathcal{I}} \int_X r_i^{-1}(g\circ f_i)\,\mathrm{d}\mu - \int_X r_i^{-1}(g\circ f_i)\,\mathrm{d}\nu\right| : g \in \mathcal{L}\right\}\\
&\leq \sum_{i\in\mathcal{I}} p_i r_i \sup\left\{r_i\left|\sum_{i\in\mathcal{I}} \int_X h\,\mathrm{d}\mu - \int_X h\,\mathrm{d}\nu\right| : h \in \mathcal{L}\right\}\\
&\leq \max_{i\in\mathcal{I}} r_i d_{\mathcal{P}}(\mu,\nu).
\end{aligned}$$

The result therefore follows by the Banach contraction mapping principle.

To see that $\operatorname{supp}\mu_{\boldsymbol{p}} \subset K$, observe that

$$\operatorname{supp}\mu_{\boldsymbol{p}} \subset \bigcup_{i\in\mathcal{I}} f_i(\operatorname{supp}\mu_{\boldsymbol{p}}).$$

Therefore, $K \cup \operatorname{supp}\mu_{\boldsymbol{p}} = K$, so $\operatorname{supp}\mu_{\boldsymbol{p}} \subset K$ as required. ☐

In general, the measures of the sets $f_{\mathtt{i}}(K)$ for $\mathtt{i} = (i_1, \ldots, i_n) \in \mathcal{I}^*$ satisfy the following bound:

$$\mu_{\boldsymbol{p}}(f_{\mathtt{i}}(K)) \geq p_{\mathtt{i}} \quad \text{where} \quad p_{\mathtt{i}} = p_{i_1} \cdots p_{i_n}.$$

However, since the images $f_{\mathtt{i}}(K)$ may overlap, the actual mass could be substantially larger. If the images are disjoint, then this cannot happen. This assumption is particularly important, and has a name:

Definition 2.23. We say that the IFS $\{f_i\}_{i\in\mathcal{I}}$ satisfies the *strong separation condition* if the attractor K satisfies $f_i(K) \cap f_j(K) = \varnothing$ for all $i \neq j$.

Under this assumption let's prove that $\mu_{\boldsymbol{p}}(f_{\mathtt{i}}(K)) = p_{\mathtt{i}}$ for all $\mathtt{i} \in \mathcal{I}^*$. We prove this by induction. The base case is immediate: $\mu_{\boldsymbol{p}}(K) = 1$. Then, for general $\mathtt{i} \in \mathcal{I}^* \setminus \{\varnothing\}$, applying the self-similarity relationship,

$$\mu_{\boldsymbol{p}}(f_{\mathtt{i}}(K)) = \sum_{i\in\mathcal{I}} p_j \mu_{\boldsymbol{p}}(f_j^{-1}(f_{\mathtt{i}}(K))).$$

If $\mathtt{i} = j\mathtt{k}$, then $f_j^{-1}(f_{\mathtt{i}}(K)) = f_{\mathtt{k}}(K)$ and we can apply the inductive step. Otherwise, if $\mathtt{i} = i\mathtt{k}$ where $i \neq j$, since $f_i(K) \cap f_j(K) = \varnothing$ and $f_{\mathtt{i}}(K) \subset K$,

$$f_j^{-1}(f_{\mathtt{i}}(K)) \cap K = \varnothing$$

so $\mu_{\boldsymbol{p}}(f_j^{-1}(f_{\mathtt{i}}(K))) = 0$.

2.5. Box dimension. In contrast to the Hausdorff dimension, the definition of box dimension is comparatively much simpler. For a *totally bounded* set E and $r > 0$, we let $N_r(E)$ denote the smallest number m for which there exist points $\{x_1, \ldots, x_m\} \subset E$ such that $E \subset \bigcup_{i=1}^m B(x_i, r)$.

The *upper box dimension* is the limit

$$\overline{\dim}_{\mathrm{B}} E = \limsup_{r \to 0} \frac{\log N_r(E)}{\log(1/r)}.$$

The *lower box dimension*, denoted by $\underline{\dim}_{\mathrm{B}} E$, is defined similarly albeit with a limit infimum in place of the limit supremum. When $\underline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{B}} E$, we say that the *box dimension exists* and write $\dim_{\mathrm{B}} E$ for the common value.

One way to think of the box dimension is as a modification of the Hausdorff dimension to only permit covers of a fixed radius. In particular, we have the following lemma:

Lemma 2.24. *Let $E \subset \mathbb{R}^d$ be a bounded set. Then*

$$\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}} E \leq \overline{\dim}_{\mathrm{B}} E.$$

Proof. The most interesting inequality is to show that $\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}} E$. Let $s = \underline{\dim}_{\mathrm{B}} E$, let $\varepsilon > 0$ and $\delta > 0$ be arbitrary. It suffices to show that $\mathcal{H}_\delta^{s+\varepsilon}(E) < \infty$, independently of δ.

By definition of the upper box dimension, get $0 < r < \delta$ and a family of balls $\{B(x_i, r)\}_{i=1}^m$ which covers E and with $m \leq r^{-(s+\varepsilon)}$. Therefore,

$$\mathcal{H}_\delta^{s+\varepsilon}(E) \leq \sum_{i=1}^m (2r)^{s+\varepsilon} \leq 2^{s+\varepsilon} < \infty$$

as required. □

2.6. Dimensions of self-similar sets. To conclude this section, we specialize our discussion of dimension to the case of self-similar sets.

3. ERGODIC THEORY INTERLUDE

3.1. Measure-preserving dynamics and ergodicity.

3.2. Birkoff's ergodic theorem.

3.3. Maker's ergodic theorem.

4. Applications of probability theory

4.1. Entropy and the Shannon–McMillan–Breiman theorem.

4.2. Local dimension and the law of large numbers.

4.3. Conditional expectation.

4.4. Rokhlin's theorem and disintegration of measures.

5. Exact dimensionality of self-similar measures

Acknowledgements

REPLACE

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