

# Exercise 2

DUE 12:15PM ON THURSDAY, JANUARY 22

## The questions.

1. (3 pt.) Let  $(X, d)$  be a complete metric space and let  $s \geq 0$ .
  - (i) Prove that the Hausdorff content is upper semi-continuous: if  $(K_n)_{n=1}^\infty$  is a sequence of compact sets and  $K = \lim_{n \rightarrow \infty} K_n$  in the Hausdorff metric, then

$$\mathcal{H}_\infty^s(K) \geq \limsup_{n \rightarrow \infty} \mathcal{H}_\infty^s(K_n).$$

- (ii) Taking  $X = [0, 1]$  and  $s = 1/2$ , give an example showing that Hausdorff content is not continuous.
  - (iii) Taking  $X = [0, 1]$  and  $s = 1/2$ , give an example showing that Hausdorff  $s$ -measure is not upper semi-continuous.

2. (3 pt.) Let  $E \subset \mathbb{R}$  be a bounded set.

- (i) Let  $E \subset [0, 1]$  and define a function  $f_E: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f_E(u) = \frac{\log N_{2^{-u}}(E)}{\log 2}.$$

Prove that there is a constant  $M \geq 0$  so that for  $v \leq u$ ,

$$(1) \quad 0 \leq f_E(u) - f_E(v) \leq u - v + M.$$

*Aside: this inequality essentially says that  $f_E$  is increasing and 1-Lipschitz.*

- (ii) What is the analogue of equation (1) if instead  $E \subset \mathbb{R}^d$  for a general  $d \in \mathbb{N}$ ?
  - (iii) Recall that a *dyadic interval* (of side length  $2^{-n}$ ) is an interval  $[j2^{-n}, (j+1)2^{-n}]$  where  $j \in \mathbb{Z}$ . For  $r > 0$ , let  $\Delta_r(E)$  denote the number of closed dyadic intervals of side-length  $2^{-n}$  which intersect  $E$ , where  $n \in \mathbb{Z}$  is such that  $2^{-n} \leq r < 2^{-n+1}$ . Prove that there is a constant  $c \geq 1$  such that for all  $r > 0$ ,

$$c^{-1}\Delta_r(E) \leq N_r(E) \leq c\Delta_r(E).$$

- (iv) Let  $m$  denote Lebesgue measure. Prove that there is a constant  $c \geq 1$  such that for all  $r > 0$ ,

$$c^{-1}\frac{m(E^{(r)})}{r} \leq N_r(E) \leq c\frac{m(E^{(r)})}{r}.$$

*Aside: This means that the box dimension is the same as the Minkowski dimension, which is the exponential decay rate of the Lebesgue measure of the  $r$ -neighbourhood of the set  $E$ .*

3. (3 pt.) Fix a binary sequence  $(a_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ . Using this binary sequence, define a compact subset of  $[0, 1]$  inductively as follows. Begin with  $K_0 = [0, 1]$ . Now, suppose we have constructed  $K_n$  as a non-empty union of dyadic intervals: that is,

$$K_n = \bigcup_{i=1}^{m_n} [a_j, a_j + 2^{-n}]$$

for numbers  $a_j \in 2^{-n} \mathbb{Z}$ .

We then define  $K_{n+1}$  as follows:

- If  $a_{n+1} = 0$ , we set  $K_{n+1} = \bigcup_{j=1}^{m_n} [a_j, a_j + 2^{-n-1}]$ .
- If  $a_{n+1} = 1$ , we set  $K_{n+1} = \bigcup_{j=1}^{m_n} [a_j, a_j + 2^{-n-1}] \cup \bigcup_{j=1}^{m_n} [a_j + 2^{-n-1}, a_j + 2^{-n}]$ .

In words, if  $a_{n+1} = 0$ , we replace each dyadic interval of width  $2^{-n}$  with a single dyadic interval of width  $2^{-n-1}$  with the same left endpoint; and if  $a_{n+1} = 1$  we replace each dyadic interval with two dyadic intervals, one sharing the left endpoint and the other sharing the right endpoint.

By construction  $K_0 \supset K_1 \supset K_2 \supset \dots$ . So, we may define  $K = K(a_n)_{n=1}^{\infty} = \bigcap_{n=0}^{\infty} K_n$ .

- (i) For  $r \in (0, 1)$  and  $k \in \mathbb{N}$  such that  $2^{-k} \leq r < 2^{-k+1}$ , show that there is a constant  $c \geq 1$  so that

$$c^{-1} 2^{\sum_{n=1}^k a_n} \leq N_r(K) \leq c 2^{\sum_{n=1}^k a_n}.$$

Conclude that

$$\begin{aligned} \underline{\dim}_B K &= \liminf_{k \rightarrow \infty} \frac{\sum_{n=1}^k a_n}{k} \\ \overline{\dim}_B K &= \limsup_{k \rightarrow \infty} \frac{\sum_{n=1}^k a_n}{k}. \end{aligned}$$

- (ii) Prove that  $\dim_H K = \underline{\dim}_B K$ .  
 (iii) For any  $0 \leq s \leq t \leq 1$ , give an example of a set  $K \subset [0, 1]$  such that  $\underline{\dim}_B K = s$  and  $\overline{\dim}_B K = t$ .  
 (iv) (**1 pt. bonus**) For any  $0 \leq s \leq t \leq 1$ , construct sequences  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  such that

$$\begin{aligned} s &= \max \{\underline{\dim}_B K(a_n), \underline{\dim}_B K(b_n)\} \\ t &= \underline{\dim}_B (K(a_n) \cup K(b_n)). \end{aligned}$$

*Hint: it may be easier to construct an increasing 1-Lipschitz function instead of a sequence. These two constructions are essentially equivalent by considering  $f(k) = \sum_{n=1}^k a_n$ .*

4. (1 pt.) Let  $\Phi = \{f_i(x) = r_i x + t_i\}_{i \in \mathcal{I}}$ , with  $r_i \in (0, 1)$ , be a self-similar IFS in  $\mathbb{R}$ . Denote the corresponding self-similar set by  $K'$ . We say that the IFS  $\Phi$  has an *exact overlap* if there are words  $j, k \in \mathcal{I}^*$  with  $j \neq k$  such that  $f_j = f_k$ .
- (i) Suppose  $\Phi$  has an exact overlap. Prove that an exact overlap occurs at a fixed level: namely, that there exists an  $n \in \mathbb{N}$  and  $j, k \in \mathcal{I}^n$  with  $j \neq k$  such that  $f_j = f_k$ .
  - (ii) Fix  $n \in \mathbb{N}$  and  $j, k \in \mathcal{I}^n$  satisfying the conclusion of (i). Define a new IFS  $\Phi' = \{f_i : i \in \mathcal{I}^n \setminus \{j\}\}$  and let  $\Phi'$  have attractor  $K'$ . Prove that  $K = K'$ .
  - (iii) Prove that the similarity dimension of  $\Phi'$  is strictly smaller than the similarity dimension  $\Phi$ .