Exercise 2

Due 12:15pm on Thursday, January 22

The questions.

1. *(2 pt.)* Let (X, d) be a complete metric space and let $s \geq 0$.
   (i) Prove that the Hausdorff content is upper semi-continuous: if $(K_n)_{n=1}^{\infty}$ is a sequence of compact sets and $K = \lim_{n\to\infty} K_n$ in the Hausdorff metric, then
   $$\mathcal{H}^s_\infty(K) \geq \limsup_{n\to\infty} \mathcal{H}^s_\infty(K_n).$$
   (ii) Taking $X = [0, 1]$ and $s = 1/2$, give an example showing that Hausdorff content is not continuous.
   (iii) Taking $X = [0, 1]$ and $s = 1/2$, give an example showing that Hausdorff s-measure is not upper semi-continuous.

2. *(3 pt.)* Let $E \subset \mathbb{R}$ be a bounded set.
   (i) Let $E \subset [0, 1]$ and define a function $f_E\colon\ \mathbb{R} \to \mathbb{R}$ by
   $$f_E(u) = \frac{\log N_{2^{-u}}(E)}{\log 2}.$$
   Prove that there is a constant $M \geq 0$ so that for $v \leq u$,
   $$0 \leq f_E(u) - f_E(v) \leq u - v + M. \tag{1}$$
   Aside: this inequality essentially says that f_E is increasing and 1-Lipschitz.
   (ii) What is the analogue of equation (1) if instead $E \subset \mathbb{R}^d$ for a general $d \in \mathbb{N}$?
   (iii) Recall that a *dyadic interval* (of side length 2^{-n}) is an interval $[j2^{-n}, (j+1)2^{-n}]$ where $j \in \mathbb{Z}$. For $r > 0$, let $\Delta_r(E)$ denote the number of closed dyadic intervals of side-length 2^{-n} which intersect E, where $n \in \mathbb{Z}$ is such that $2^{-n} \leq r < 2^{-n+1}$. Prove that there is a constant $c \geq 1$ such that for all $r > 0$,
   $$c^{-1}\Delta_r(E) \leq N_r(E) \leq c\Delta_r(E).$$
   (iv) Let m denote Lebesgue measure. Prove that there is a constant $c \geq 1$ such that for all $r > 0$,
   $$c^{-1}\frac{m(E^{(r)})}{r} \leq N_r(E) \leq c\frac{m(E^{(r)})}{r}.$$

Aside: This means that the box dimension is the same as the Minkowski dimension, which is the exponential decay rate of the Lebesgue measure of the r-neighbourhood of the set E.

3. *(3 pt.)* Fix a binary sequence $(a_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$. Using this binary sequence, define a compact subset of $[0,1]$ inductively as follows. Begin with $K_0 = [0,1]$. Now, suppose we have constructed K_n as a non-empty union of dyadic intervals: that is,

$$K_n = \bigcup_{i=1}^{m_n} [a_j, a_j + 2^{-n}]$$

for numbers $a_j \in 2^{-n}\,\mathbb{Z}$.

We then define K_{n+1} as follows:

- If $a_{n+1} = 0$, we set $K_{n+1} = \bigcup_{j=1}^{m_n} [a_j, a_j + 2^{-n-1}]$.
- If $a_{n+1} = 1$, we set $K_{n+1} = \bigcup_{j=1}^{m_n} [a_j, a_j + 2^{-n-1}] \cup \bigcup_{j=1}^{m_n} [a_j + 2^{-n-1}, a_j + 2^{-n}]$.

In words, if $a_{n+1} = 0$, we replace each dyadic interval of width 2^{-n} with a single dyadic interval of width 2^{-n-1} with the same left endpoint; and if $a_{n+1} = 0$ we replace each dyadic interval with two dyadic intervals, one sharing the left endpoint and the other sharing the right endpoint.

By construction $K_0 \supset K_1 \supset K_2 \supset \cdots$. So, we may define $K = K(a_n)_{n=1}^{\infty} = \bigcap_{n=0}^{\infty} K_n$.

(i) For $r \in (0,1)$ and $k \in \mathbb{N}$ such that $2^{-k} \leq r < 2^{-k+1}$, show that there is a constant $c \geq 1$ so that

$$c^{-1} 2^{\sum_{n=1}^{k} a_n} \leq N_r(K) \leq c 2^{\sum_{n=1}^{k} a_n}.$$

Conclude that

$$\underline{\dim}_{\mathrm{B}} K = \liminf_{k\to\infty} \frac{\sum_{n=1}^{k} a_n}{k}$$
$$\overline{\dim}_{\mathrm{B}} K = \limsup_{k\to\infty} \frac{\sum_{n=1}^{k} a_n}{k}.$$

(ii) Prove that $\dim_{\mathrm{H}} K = \underline{\dim}_{\mathrm{B}} K$.

(iii) For any $0 \leq s \leq t \leq 1$, give an example of a set $K \subset [0,1]$ such that $\underline{\dim}_{\mathrm{B}} K = s$ and $\overline{\dim}_{\mathrm{B}} K = t$.

(iv) **(1 pt. bonus)** For any $0 \leq s \leq t \leq 1$, construct sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that

$$s = \max\{\underline{\dim}_{\mathrm{B}} K(a_n), \underline{\dim}_{\mathrm{B}} K(b_n)\}$$
$$t = \underline{\dim}_{\mathrm{B}}\big(K(a_n) \cup K(b_n)\big).$$

Hint: it may be easier to construct an increasing 1-Lipschitz function instead of a sequence. These two constructions are essentially equivalent by considering $f(k) = \sum_{n=1}^{k} a_n$.

4. *(2 pt.)* Let $\Phi = \{f_i(x) = r_i x + t_i\}_{i \in \mathcal{I}}$, with $r_i \in (0,1)$, be a self-similar IFS in $\mathbb{R}$. Denote the corresponding self-similar set by K'. We say that the IFS Φ has an *exact overlap* if there are words $\mathtt{j}, \mathtt{k} \in \mathcal{I}^*$ with $\mathtt{j} \neq \mathtt{k}$ such that $f_{\mathtt{j}} = f_{\mathtt{k}}$.

   (i) Suppose Φ has an exact overlap. Prove that an exact overlap occurs at a fixed level: namely, that that there exists an $n \in \mathbb{N}$ and $\mathtt{j}, \mathtt{k} \in \mathcal{I}^n$ with $\mathtt{j} \neq \mathtt{k}$ such that $f_{\mathtt{j}} = f_{\mathtt{k}}$.

   (ii) Fix $n \in \mathbb{N}$ and $\mathtt{j}, \mathtt{k} \in \mathcal{I}^n$ satisfying the conclusion of (i). Define a new IFS $\Phi' = \{f_{\mathtt{i}} : \mathtt{i} \in \mathcal{I}^n \setminus \{\mathtt{j}\}\}$ and let Φ' have attractor K'. Prove that $K = K'$.

   (iii) Prove that the similarity dimension of Φ' is strictly smaller than the similarity dimension Φ.