Properties of Positive Definite Symmetric Matrices

- Suppose $A \in \mathbb{R}^n$ is a symmetric positive definite matrix, i.e., $A = A^T$ and

$$\forall x \in \mathbb{R}^n \setminus \{0\}. \, x^T A x > 0. \tag{3}$$

- Then we can easily show the following properties of A.
- All diagonal elements are positive: In (3), put x with $x_j = 1$ for $j = i$ and $x_j = 0$ for $j \neq i$, to get $A_{ii} > 0$.
- The largest element in magnitude in the entire matrix occurs in the diagonal: Fix $i \neq j$ between 1 and n. In (3), put x with $x_k = 1$ for $k = i$, $x_k = \pm 1$ for $k = j$ and $x_k = 0$ for $j \neq k \neq i$, to get $|A_{ij}| < \max(A_{ii}, A_{jj})$.
- All leading principle minors (i.e., the 1×1, 2×2, 3×3, $\ldots, m \times m$ matrices in the upper left corner) are positive definite: In (3), put x with $x_k = 0$ for $k > m$ to prove that the top left $m \times m$ matrix is positive definite.

Eigen/diagonal Decomposition

- Let $A \in \mathbb{R}^{nXn}$ be a **square** matrix with $\boldsymbol{n}$ **linearly independent eigenvectors**
- **Theorem**: Exists an **eigen decomposition**

$$\mathrm{A} = \mathbf{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$$

diagonal

Unique for distinct eigen-values

the diagonal matrix $\boldsymbol{\Lambda}$ Is formed using the eigenvalues $\mathbf{S}$

$$\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m), \quad \lambda_i \geq \lambda_{i+1}$$

Diagonal Decomposition: why/how

Let ***U*** have the eigenvectors as columns: $U = \quad v_1 \quad \ldots \quad v_n$

Then, ***AU*** can be written

$$AU = A \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \ldots & \lambda_n v_n \end{bmatrix} = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ldots & \\ & & \lambda_n \end{bmatrix}$$

Thus $\boldsymbol{AU=U\Lambda}$, or $\boldsymbol{U^{-1}AU=\Lambda}$

Diagonal decomposition - example

Recall $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; so, \lambda_1 = 1, \lambda_2 = 3.$

The eigenvectors $\begin{matrix} 1 \\ 1 \end{matrix} \div$ and $\begin{matrix} 1 \\ 1 \end{matrix} \div$ form $U = \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$

Inverting, we have $U^{1} = \begin{matrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{matrix}$

Recall $UU^{-1} = 1$.

$$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \quad \begin{matrix} 1 & 0 \\ 0 & 3 \end{matrix} \quad \begin{matrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{matrix}$$

Example Continued

Let's divide $\boldsymbol{U}$ (and multiply $\boldsymbol{U}^{-1}$) by $\sqrt{2}$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\boldsymbol{Q}$ $\qquad$ $\boldsymbol{\Lambda}$ $\qquad$ $(\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{T})$

Symmetric Eigen Decomposition

- If $\mathbf{S} \in \mathbb{R}^{m \times m}$ is a **symmetric** matrix:
- **Theorem**: There exists a (unique) **eigen decomposition** $S = Q \quad Q^T$
- where $\boldsymbol{Q}$ is **orthogonal:**
  - $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$
  - Columns of $\boldsymbol{Q}$ are normalized eigenvectors
  - Columns are orthogonal
  - (everything is real)

Spectral Decomposition

- We have seen in the previous pages and in lecture notes that if $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix then it has an orthonormal set of eigenvectors $u_1, u_2, \ldots, u_n$ corresponding to (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we have:
- The **spectral decomposition**: $Q^T AQ = \Lambda$ where
- $Q = [u_1, u_2, \ldots, u_n]$ is an orthogonal matrix with $Q^{-1} = Q^T$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is diagonal.
- Assume $A \in \mathbb{R}^n$ reprsents the linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ in the standard basis of $\mathbb{R}^n$.
- Then, the matrix $S := Q^{-1} \in \mathbb{R}^{n\times n}$ is the matrix for the change of basis into one in which f is reprsented by $B := \Lambda$. More generally, $B = SAS^{-1}$:

$$\begin{tikzcd}
\text{old coordinates} \arrow[r, "A"] \arrow[d, "S"'] & \text{old coordinates} \arrow[d, "S"] \\
\text{new coordinates} \arrow[r, "B"] & \text{new coordinates}
\end{tikzcd}$$

Example Continued

Let's divide $\boldsymbol{U}$ (and multiply $\boldsymbol{U}^{-1}$) by $\sqrt{2}$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\boldsymbol{Q}$ (S^{-1}) $\boldsymbol{\Lambda}$ $(\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{T}$ (S)

Mapping of the Identity Matrix

- $AS * I = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$

- $SB * I = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$

A = $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Properties of Product of A and A^t

- Let $A \in \mathbb{R}^{mXn}$; $AA^t \in \mathbb{R}^{mXm}$; $A^t\mathrm{A} \in \mathbb{R}^{nXn}$
- Both AA^t and $A^t\mathrm{A}$ are symmetric matrices
- They have the same eigenvalues:
- Let

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$$

- $A^tA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$; eigenvalues are

$$\lambda_1 = 3 \,;\, \lambda_2 = 1$$

Eigenvalues of AA^t

- $AA^t = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

- Eigenvalues are: $\lambda_1 = 3$; $\lambda_2 = 1$; $\lambda_3 = 0$
- If $AA^t x = \lambda x$, then $A^t AA^t x = \lambda A^t x$, that is $A^t Ay = \lambda y$ where $A^t x = y$
- So they have the same eigenvalues; the eigenvectors are different; x and $A^t x$

Singular Value Decomposition (SVD)

- Let $A \in \mathbb{R}^{m \times n}$ be an arbitrary matrix.
- Then $A^T A \in \mathbb{R}^{n \times n}$ and $AA^T \in \mathbb{R}^{m \times m}$ are symmetric matrices.
- They are also positive semi-definite since for example $x^T A^T A x = (Ax)^T (Ax) = (\|Ax\|_2)^2 \geq 0$.
- We will show that $A = USV^T$, called the SVD of A, where $V \in \mathbb{R}^{n \times n}$ and $U \in \mathbb{R}^{m \times m}$ are orthogonal matrices whereas the matrix $S = U^T AV \in \mathbb{R}^{m \times n}$ is diagonal with $S = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p)$ where $p = \min(m, n)$ and the non-negative numbers $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_p \geq 0$ are the **singular values** of A.
- If r is the rank of A then A has exactly r positive singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_r > 0$ with $\sigma_{r+1} = \sigma_{r+2} = \ldots = \sigma_p = 0$.

SVD

- Note that if the SVD for A as above exists then, since $U^T U = I_m$, we have $A^T A = VS^T U^T USV^T = VS^T SV^T$, where $S^T S = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots, \sigma_p^2) \in \mathbb{R}^{n \times n}$ is a diagonal matrix, thus giving the spectral decomposition of the positive semi-definite matrix $A^T A$.
- This gives us a method to find the SVD of A.
- Obtain the eigenvalues $\sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \geq \ldots \geq \sigma_p^2 \geq 0$ and the corresponding eigenvectors $v_1, \cdots, v_p$ of $A^T A$. If $p < n$, the other eigenvalues of A are zero with corresponding eigenvectors $v_{p+1}, \cdots, v_n$ which make the orthogonal matrix $V = [v_1, \cdots, v_n]$.
- From the SVD we have $AV = US$, thus when $\sigma_i > 0$, i.e., for $1 \leq i \leq r$, we get $\frac{1}{\sigma_i} A v_i = u_i$.
- Extend the set $u_1, \cdots, u_r$ to an orthonormal basis $u_1, \cdots, u_r, \cdots u_m$ of $\mathbb{R}^m$ which gives the orthogonal matrix $U = [u_1, \cdots, u_m]$.

SVD

- Note the following useful facts.
- For $1 \leq i \leq r$, the vector u_i is an eigenvector of AA^T with eigenvalue σ_i^2. Check!
- AA^T is similar to SS^T (with identical eigenvalues) and A^TA is similar to S^TS (with identical eigenvalues).
- The diagonal elements of the diagonal matrices $S^TS \in \mathbb{R}^{n\times n}$ and $SS^T \in \mathbb{R}^{m\times m}$ are $\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots, \sigma_p^2$ followed by $n-p$ zeros and $m-p$ zeros respectively.
- The singular values of A are the positive square roots of the eigenvalues of AA^T or A^TA.

Similarity → Clustering

- We can compute the similarity between two document vector representations x_i and x_j by $x_i x_j^T$
- Let $X = [x_1 \ldots x_N]$
- Then XX^T is a matrix of similarities
- XX^T is symmetric
- So $XX^T = Q\Lambda Q^T$
- So we can decompose this similarity space into a set of orthonormal basis vectors (given in Q) scaled by the eigenvalues in Λ
  - This leads to PCA (Principal Components Analysis)

Singular Value Decomposition

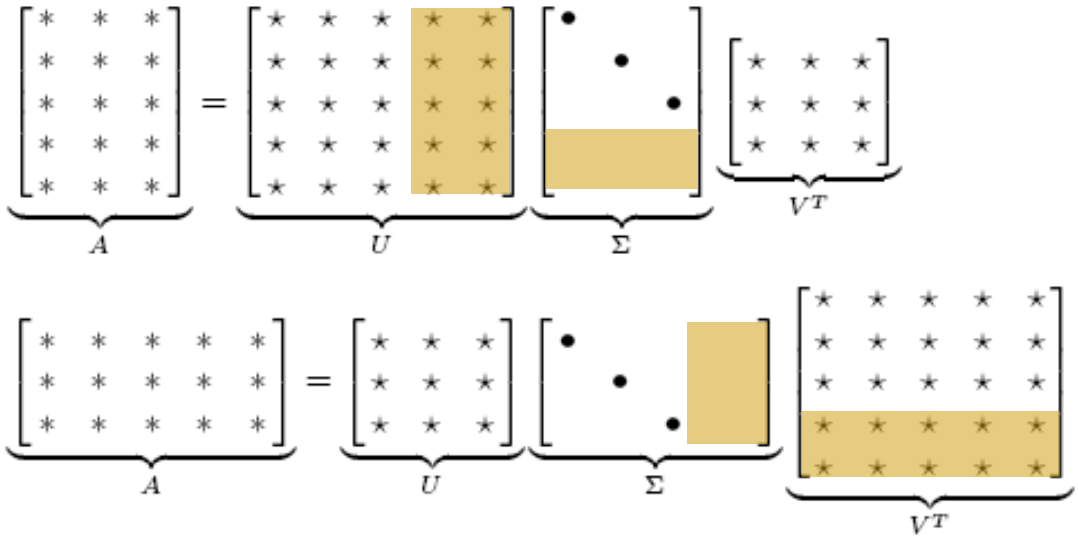
For an $M \times N$ matrix **A** of rank r there exists a factorization (Singular Value Decomposition = **SVD**) as follows:

$$A = U \quad V^T$$

$M \times M$ $\quad$ $M \times N$ $\quad$ V is $N \times N$

Singular Value Decomposition

- Illustration of SVD dimensions and sparseness



SVD Example

Let $A = \begin{matrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{matrix}$

Thus M=3, N=2. Its SVD is

$$\begin{matrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{matrix} \begin{matrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{matrix} \begin{matrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{matrix}$$

Lecture 4. The Singular Value Decomposition

The singular value decomposition (SVD) is a matrix factorization whose computation is a step in many algorithms. Equally important is the use of the SVD for conceptual purposes. Many problems of linear algebra can be better understood if we first ask the question, what if we take the SVD?

A Geometric Observation

The SVD is motivated by the following geometric fact:

The image of the unit sphere under any $m \times n$ matrix is a hyperellipse.

The SVD is applicable to both real and complex matrices. However, in describing the geometric interpretation, we assume as usual that the matrix is real.

The term "hyperellipse" may be unfamiliar, but this is just the m-dimensional generalization of an ellipse. We may define a hyperellipse in $\mathbb{R}^m$ as the surface obtained by stretching the unit sphere in $\mathbb{R}^m$ by some factors $\sigma_1, \ldots, \sigma_m$, possibly zero, in some orthogonal directions $u_1, \ldots, u_m \in \mathbb{R}^m$. For convenience, let us take the u_i to be unit vectors. The vectors $\{\sigma_i u_i\}$ are the *principal semiaxes* of the hyperellipse, with lengths $\sigma_1, \ldots, \sigma_m$. If A has rank r, exactly r of the lengths σ_i will turn out to be nonzero, and in particular, if $m > n$, at most n of them will be nonzero.

The italicized statement above has the following meaning. By the unit sphere, we mean the usual Euclidean sphere in n-space, i.e., the unit sphere in the 2-norm; let us denote it by S. Then AS, the image of S under the mapping A, is a hyperellipse as just defined.

This geometric fact is not obvious. We shall restate it in the language of linear algebra and prove it later. For the moment, assume it is true.

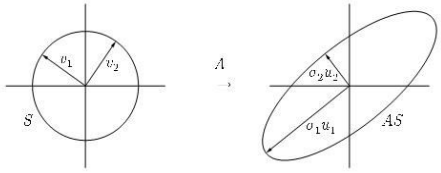


Figure 4.1. *SVD of a 2×2 matrix.*

Let S be the unit sphere in $\mathbb{R}^n$, and take any $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. For simplicity, suppose for the moment that A has full rank n. The image AS is a hyperellipse in $\mathbb{R}^m$. We now define some properties of A in terms of the shape of AS. The key ideas are indicated in Figure 4.1.

First, we define the n *singular values* of A. These are the lengths of the n principal semiaxes of AS, written $\sigma_1, \sigma_2, \ldots, \sigma_n$. It is conventional to assume that the singular values are numbered in descending order, $\sigma_1 > \sigma_2 > \cdots > \sigma_n \geq 0$.

Next, we define the n *left singular vectors* of A. These are the unit vectors $\{u_1, u_2, \ldots, u_n\}$ oriented in the directions of the principal semiaxes of AS, numbered to correspond with the singular values. Thus the vector $\sigma_i u_i$ is the ith largest principal semiaxis of AS.

Finally, we define the n *right singular vectors* of A. These are the unit vectors $\{v_1, v_2, \ldots, v_n\} \in S$ that are the preimages of the principal semiaxes of AS, numbered so that $Av_j = \sigma_j u_j$.

The terms "left" and "right" in the definitions above are decidedly awkward. They come from the positions of the factors U and V in (4.2) and (4.3), below. What is awkward is that in a sketch like Figure 4.1, the left singular vectors correspond to the space on the right and the right singular vectors correspond to the space on the left! One could resolve this problem by interchanging the two halves of the figure, with the map A pointing from right to left, but that would go against deeply-ingrained habits.

Reduced SVD

We have just mentioned that the equations relating right singular vectors $\{v_j\}$ and left singular vectors $\{u_j\}$ can be written

$$Av_j = \sigma_j u_j, \qquad 1 \le j \le n. \tag{4.1}$$

This collection of vector equations can be expressed as a matrix equation,

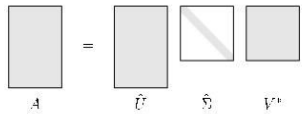
$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix},$$

or, more compactly, $AV = \hat{U}\hat{\Sigma}$. In this matrix equation, $\hat{\Sigma}$ is an $n \times n$ diagonal matrix with positive real entries (since A was assumed to have full rank n), $\hat{U}$ is an $m \times n$ matrix with orthonormal columns, and V is an $n \times n$ matrix with orthonormal columns. Thus V is unitary, and we can multiply on the right by its inverse V^* to obtain

$$A = \hat{U}\hat{\Sigma}V^*. \tag{4.2}$$

This factorization of A is called a *reduced singular value decomposition*, or *reduced SVD*, of A. Schematically, it looks like this:

Reduced SVD $\{m \ge n\}$



Full SVD

In most applications, the SVD is used in exactly the form just described. However, this is not the standard way in which the idea of an SVD is usually formulated. We have introduced the awkward term "reduced" and the unsightly hats on U and Σ in order to distinguish the factorization (4.2) from the more standard "full" SVD. This "reduced" vs. "full" terminology and hatted notation will be maintained throughout the book, and we shall make a similar distinction between reduced and full QR factorizations.

The idea is as follows. The columns of $\hat{U}$ are n orthonormal vectors in the m-dimensional space $\mathbb{C}^m$. Unless $m = n$, they do not form a basis of

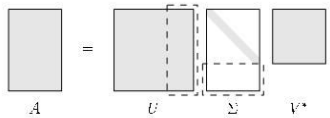
$\mathbb{C}^m$, nor is $\hat{U}$ a unitary matrix. However, by adjoining an additional $m - n$ orthonormal columns, $\hat{U}$ can be extended to a unitary matrix. Let us do this in an arbitrary fashion, and call the result U.

If $\hat{U}$ is replaced by U in (4.2), then $\hat{\Sigma}$ will have to change too. For the product to remain unaltered, the last $m-n$ columns of U should be multiplied by zero. Accordingly, let Σ be the $m \times n$ matrix consisting of $\hat{\Sigma}$ in the upper $n \times n$ block together with $m - n$ rows of zeros below. We now have a new factorization, the *full SVD* of A:

$$A = U\Sigma V^*. \tag{4.3}$$

Here U is $m \times m$ and unitary, V is $n \times n$ and unitary, and Σ is $m \times n$ and diagonal with positive real entries. Schematically:

Full SVD ($m \geq n$)



The dashed lines indicate the "silent" columns of U and rows of Σ that are discarded in passing from (4.3) to (4.2).

Having described the full SVD, we can now discard the simplifying assumption that A has full rank. If A is rank deficient, the factorization (4.3) is still appropriate. All that changes is that now not n but only r of the left singular vectors of A are determined by the geometry of the hyperellipse. To construct the unitary matrix U, we introduce $m - r$ instead of just $m - n$ additional arbitrary orthonormal columns. The matrix V will also need $n - r$ arbitrary orthonormal columns to extend the r columns determined by the geometry. The matrix Σ will now have r positive diagonal entries, with the remaining $n - r$ equal to zero.

By the same token, the reduced SVD (4.2) also makes sense for matrices A of less than full rank. One can take $\hat{U}$ to be $m \times n$, with $\hat{\Sigma}$ of dimensions $n \times n$ with some zeros on the diagonal, or further compress the representation so that $\hat{U}$ is $m \times r$ and $\hat{\Sigma}$ is $r \times r$ and strictly positive on the diagonal.

Formal Definition

Let m and n be arbitrary; we do not require $m \geq n$. Given $A \in \mathbb{C}^{m\times n}$, not necessarily of full rank, a *singular value decomposition* (SVD) of A is a factorization

$$A = U\Sigma V^* \tag{4.4}$$

where

$$\begin{aligned} U &\in \mathbb{C}^{m\times m} && \text{is unitary,} \\ V &\in \mathbb{C}^{n\times n} && \text{is unitary,} \\ \Sigma &\in \mathbb{R}^{m\times n} && \text{is diagonal.} \end{aligned}$$

In addition, Σ is assumed to have its diagonal entries σ_j nonnegative and in nonincreasing order; that is, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, where $p = \min(m, n)$.

Note that the diagonal matrix Σ has the same shape as A even when A is not square, but U and V are always square unitary matrices.

It is clear that the image of the unit sphere in $\mathbb{R}^n$ under a map $A = U\Sigma V^*$ must be a hyperellipse in $\mathbb{R}^m$. The unitary map V^* preserves the sphere, the diagonal matrix Σ stretches the sphere into a hyperellipse aligned with the canonical basis, and the final unitary map U rotates or reflects the hyperellipse without changing its shape. Thus, if we can prove that every matrix has an SVD, we shall have proved that the image of the unit sphere under any linear map is a hyperellipse, as claimed at the outset of this lecture.

Existence and Uniqueness

Theorem 4.1. *Every matrix $A \in \mathbb{C}^{m\times n}$ has a singular value decomposition (4.4). Furthermore, the singular values $\{\sigma_j\}$ are uniquely determined, and, if A is square and the σ_j are distinct, the left and right singular vectors $\{u_j\}$ and $\{v_j\}$ are uniquely determined up to complex signs (i.e., complex scalar factors of absolute value 1).*

Proof. To prove existence of the SVD, we isolate the direction of the largest action of A, and then proceed by induction on the dimension of A.

Set $\sigma_1 = \|A\|_2$. By a compactness argument, there must be a vector $v_1 \in \mathbb{C}^n$ with $\|v_1\|_2 = 1$ and $\|u_1\|_2 = \sigma_1$, where $u_1 = Av_1$. Consider any extensions of v_1 to an orthonormal basis $\{v_j\}$ of $\mathbb{C}^n$ and of u_1 to an orthonormal basis $\{u_j\}$ of $\mathbb{C}^m$, and let U_1 and V_1 denote the unitary matrices with columns u_j and v_j, respectively. Then we have

$$U_1^* A V_1 = S = \begin{bmatrix} \sigma_1 & w^* \\ 0 & B \end{bmatrix}, \tag{4.5}$$

where 0 is a column vector of dimension $m-1$, w^* is a row vector of dimension $n-1$, and B has dimensions $(m-1)\times(n-1)$. Furthermore,

$$\left\| \begin{bmatrix} \sigma_1 & w^* \\ 0 & B \end{bmatrix} \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2 \geq \sigma_1^2 + w^*w = (\sigma_1^2 + w^*w)^{1/2} \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2,$$

implying $\|S\|_2 \geq (\sigma_1^2 + w^*w)^{1/2}$. Since U_1 and V_1 are unitary, we know that $\|S\|_2 = \|A\|_2 = \sigma_1$, so this implies $w = 0$.

If $n = 1$ or $m = 1$, we are done. Otherwise, the submatrix B describes the action of A on the subspace orthogonal to v_1. By the induction hypothesis, B has an SVD $B = U_2\Sigma_2V_2^*$. Now it is easily verified that

$$A = U_1 \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V_2 \end{bmatrix}^* V_1^*$$

is an SVD of A, completing the proof of existence.

For the uniqueness claim, the geometric justification is straightforward: if the semiaxis lengths of a hyperellipse are distinct, then the semiaxes themselves are determined by the geometry, up to signs. Algebraically, we can argue as follows. First we note that σ_1 is uniquely determined by the condition that it is equal to $\|A\|_2$, as follows from (4.4). Now suppose that in addition to v_1, there is another linearly independent vector w with $\|w\|_2 = 1$ and $\|Aw\|_2 = \sigma_1$. Define a unit vector v_2, orthogonal to v_1, as a linear combination of v_1 and w,

$$v_2 = \frac{w - (v_1^*w)v_1}{\|w - (v_1^*w)v_1\|_2}.$$

Since $\|A\|_2 = \sigma_1$, $\|Av_2\|_2 \le \sigma_1$; but this must be an equality, for otherwise, since $w = v_1c + v_2s$ for some constants c and s with $|c|^2 + |s|^2 = 1$, we would have $\|Aw\|_2 < \sigma_1$. This vector v_2 is a second right singular vector of A corresponding to the singular value σ_1; it will lead to the appearance of a vector y (equal to the last $n-1$ components of $V_1^*v_2$) with $\|y\|_2 = 1$ and $\|By\|_2 = \sigma_1$. We conclude that, if the singular vector v_1 is not unique, then the corresponding singular value σ_1 is not simple. To complete the uniqueness proof we note that as indicated above, once σ_1 and v_1 and u_1 are determined, the remainder of the SVD is determined by the action of A on the space orthogonal to v_1. Since v_1 is unique up to sign, this orthogonal space is uniquely defined, and the uniqueness of the remaining singular values and vectors now follows by induction. □

Exercises

1. Determine SVDs of the following matrices (by hand calculation):

$$\text{(a)} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{(d)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{(e)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2. Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper (not exactly a standard mathematical

Theorem 3. *If λ_i is a repeated root with multiplicity $m \geqq 2$, then there exist m orthonormal eigenvectors corresponding to λ_i.*

Proof. First, we note that corresponding to λ_i there will be at least one eigenvector x_i. For any arbitrary nonzero vector x_i one can always find an additional $n-1$ vectors y_j, $j = 2, \ldots, n$, so that x_i, together with the $n-1$ y-vectors forms an orthonormal basis. Collect the y vectors in a matrix Y, i.e.,

$$Y = [y_2, \ldots, y_n],$$

and define

$$B = [x_i \quad Y]. \tag{8}$$

Then

$$B'AB = \begin{bmatrix} \lambda_i x_i' x_i & x_i' AY \\ \lambda_i Y' x_i & Y'AY \end{bmatrix} = \begin{bmatrix} \lambda_i & 0 \\ 0 & Y'AY \end{bmatrix} \tag{9}$$

since (1) the products in the first column under the $_{11}$ element are products of orthogonal vectors, and (2) replacing in the first row (other than in the $_{11}$ element) the terms $x_i'A$ by $\lambda_i x_i'$ also leads to products of orthogonal vectors. B is an orthogonal matrix, hence its transpose is also its inverse. Therefore A and $B'AB$ are similar matrices (see Definition 4) and they have the same eigenvalues. From (9), the characteristic polynomial of $B'AB$ can be written as

$$\det(B'AB - \lambda I_n) = (\lambda_i - \lambda)\det(Y'AY - \lambda I_{n-1}). \tag{10}$$

If a root, say λ_i, has multiplicity $m \geqq 2$, then in the factored form of the polynomial the term $(\lambda_i - \lambda)$ occurs m times; hence if $m \geqq 2$, $\det(Y'AY - \lambda_i I_{n-1}) = 0$, and the null space of $(B'AB - \lambda_i I_n)$ has dimension greater than or equal to 2. In particular, if $m = 2$, the null space has dimension 2, and there are two linearly independent and orthogonal eigenvectors in this nullspace.[1] If the multiplicity is greater, say 3, then there are at least two orthogonal eigenvectors x_{i_1} and x_{i_2} and we can find another $n-2$ vectors y_j such that $[x_{i_1}, x_{i_2}, y_3, \ldots, y_n]$ is an orthonormal basis and repeat the argument above.

It also follows that if a root has multiplicity m, there cannot be more than m orthogonal eigenvectors corresponding to that eigenvalue, for that would lead to the conclusion that we could find more than n orthoganl eigenvectors, which is not possible. ∎

[1] Note that any set of n linearly independent vectors in n-space can be transformed into an orthonormal basis the Schmidt orthogonalization process.

2.0.5 Existence and Uniqueness Theorem

Theorem 2.1 *Let A be a complex $m \times n$ matrix. A has a singular value decomposition of the form*

$$A = U \Sigma V^*,$$

where Σ is a uniquely determined $m \times n$ (real) diagonal matrix, U is an $m \times m$ unitary matrix, and V is an $n \times n$ unitary matrix.

Proof We prove only existence. The uniqueness part of the proof follows directly from the geometric interpretation. A (more rigorous?) algebraic argument can be found, e.g., in [Trefethen/Bau].

We use induction on the dimensions of A. All of the following arguments assume $m \geq n$ (the case $m < n$ can be obtained by transposing the arguments).

For $n = 1$ (and any m) the matrix A is a column vector. We take $V = 1$, $\hat{\Sigma} = \|A\|_2$ and $\hat{U} = \frac{A}{\|A\|_2}$. Then, clearly, we have found a reduced SVD, i.e., $A = \hat{U}\hat{\Sigma}V^*$. The full SVD is obtained by extending $\hat{U}$ to U by the Gram-Schmidt algorithm and adding the necessary zeros to $\hat{\Sigma}$.

We now assume an SVD exists for the case $(m-1, n-1)$ and show it also exists for (m, n). To this end we pick $\boldsymbol{v}_1 \in \mathbb{C}^n$ such that $\|\boldsymbol{v}_1\|_2 = 1$ and

$$\|A\|_2 = \sup_{\substack{\boldsymbol{v}_1 \in \mathbb{C}^n \\ \|\boldsymbol{v}_1\|_2 = 1}} \|A\boldsymbol{v}_1\|_2 > 0.$$

Now we take

$$\boldsymbol{u}_1 = \frac{A\boldsymbol{v}_1}{\|A\boldsymbol{v}_1\|_2}. \tag{7}$$

Next, we use the Gram-Schmidt algorithm to arbitrarily extend $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ to unitary matrices by adding columns $\widetilde{U}_1$ and $\widetilde{V}_1$, i.e.,

$$U_1 = \begin{bmatrix} \boldsymbol{u}_1 & \widetilde{U}_1 \end{bmatrix} \quad V_1 = \begin{bmatrix} \boldsymbol{v}_1 & \widetilde{V}_1 \end{bmatrix}.$$

This results in

$$\begin{aligned} U_1^* A V_1 &= \begin{bmatrix} \boldsymbol{u}_1^* \\ \widetilde{U}_1^* \end{bmatrix} A \begin{bmatrix} \boldsymbol{v}_1 & \widetilde{V}_1 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{u}_1^* A \boldsymbol{v}_1 & \boldsymbol{u}_1^* A \widetilde{V}_1 \\ \widetilde{U}_1^* A \boldsymbol{v}_1 & \widetilde{U}_1^* A \widetilde{V}_1 \end{bmatrix}. \end{aligned}$$

We now look at three of these four blocks:

- Using (7) and the specific choice of $\boldsymbol{v}_1$ we have

$$\begin{aligned} \boldsymbol{u}_1^* A \boldsymbol{v}_1 &= \frac{(A\boldsymbol{v}_1)^*}{\|A\boldsymbol{v}_1\|_2} A\boldsymbol{v}_1 \\ &= \frac{\|A\boldsymbol{v}_1\|_2^2}{\|A\boldsymbol{v}_1\|_2} = \|A\boldsymbol{v}_1\|_2 \\ &= \|A\|_2. \end{aligned}$$

  For this quantity we introduce the abbreviation $\sigma_1 = \|A\|_2$.

- Again, using (7) we get
$$\widetilde{U}_1^* A\boldsymbol{v}_1 = \widetilde{U}_1^* \boldsymbol{u}_1 \|A\boldsymbol{v}_1\|_2.$$
This, however, is a zero vector since U_1 has orthonormal columns, i.e., $\widetilde{U}_1^* \boldsymbol{u}_1 = \boldsymbol{0}$.

- We show that $\boldsymbol{u}_1^* A\widetilde{V}_1 = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$ by contradiction. If it were nonzero then we could look at the first row of the block matrix $U_1^* AV_1$ and see that
$$U_1^* AV_1(1,:) = \begin{bmatrix} \sigma_1 & \boldsymbol{u}_1^* A\widetilde{V}_1 \end{bmatrix}$$
with $\| \begin{bmatrix} \sigma_1 & \boldsymbol{u}_1^* A\widetilde{V}_1 \end{bmatrix} \|_2 > \sigma_1$. On the other hand, we know that unitary matrices leave the 2-norm invariant, i.e.,
$$\|U_1^* AV_1\|_2 = \|A\|_2 = \sigma_1.$$
Since the norm of the first row of the block matrix $U_1^* AV_1$ cannot exceed that of the entire matrix we have reached a contradiction.

We now abbreviate the fourth block with $\widetilde{A} = \widetilde{U}_1^* A\widetilde{V}_1$ and can write the block matrix as
$$U_1^* AV_1 = \begin{bmatrix} \sigma_1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \widetilde{A} \end{bmatrix}.$$
To complete the proof we apply the induction hypothesis to $\widetilde{A}$, i.e., we use the SVD $\widetilde{A} = U_2 \Sigma_2 V_2^*$. Then
$$\begin{aligned} U_1^* AV_1 &= \begin{bmatrix} \sigma_1 & \boldsymbol{0}^T \\ \boldsymbol{0} & U_2 \Sigma_2 V_2^* \end{bmatrix} \\ &= \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \Sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & V_2 \end{bmatrix}^* \end{aligned}$$
or
$$A = U_1 \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \Sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & V_2 \end{bmatrix}^* V_1^*,$$
another SVD (since the product of unitary matrices is unitary). ■

2.1 SVD as a Change of Basis

We now discuss the use of the SVD to diagonalize systems of linear equations. Consider the linear system
$$A\boldsymbol{x} = \boldsymbol{b}$$
with $A \in \mathbb{C}^{m \times n}$. Using the SVD we can write
$$A\boldsymbol{x} = U\Sigma V^* \boldsymbol{x} \quad \Longleftrightarrow \quad \boldsymbol{b} = U\boldsymbol{b}'.$$
Thus, we can express $\boldsymbol{b} \in \text{range}(A)$ in terms of $\text{range}(U)$:
$$U\boldsymbol{b}' = \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{b}' = U^* \boldsymbol{b}$$

where we have used the columns of U as an orthonormal basis for range(A).

Similarly, any $\boldsymbol{x} \in \mathbb{C}^n$ (the domain of A) can be written in terms of range(V):

$$\boldsymbol{x}' = V^* \boldsymbol{x}.$$

Now

$$\begin{aligned} A\boldsymbol{x} = \boldsymbol{b} \quad &\iff \quad U^* A\boldsymbol{x} = U^* \boldsymbol{b} \\ &\iff \quad U^* U \Sigma V^* \boldsymbol{x} = U^* \boldsymbol{b} \\ &\iff \quad I \Sigma V^* \boldsymbol{x} = U^* \boldsymbol{b} \\ &\iff \quad \Sigma \boldsymbol{x}' = \boldsymbol{b}', \end{aligned}$$

and we have diagonalized the linear system.

In summary, expressing the range space of A in terms of the columns of U and the domain space of A in terms of the columns of V converts $A\boldsymbol{x} = \boldsymbol{b}$ to a diagonal system.

2.1.1 Connection to Eigenvalues

If $A \in \mathbb{C}^{m \times m}$ is square with a linearly independent set of eigenvectors (i.e., nondefective), then

$$AX = \Lambda X \quad \iff \quad X^{-1} A X = \Lambda,$$

where X contains the eigenvectors of A as its columns and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$ is a diagonal matrix of the eigenvalues of A.

If we compare this eigen-decomposition of A to the SVD we see that the SVD is a generalization: A need not be square, and the SVD always exists (whereas even a square matrix need not have an eigen-decomposition). The price we pay is that we require two unitary matrices U and V instead of only X (which is in general not unitary).

2.1.2 Theoretical Information via SVD

A number of theoretical facts about the matrix A can be obtained via the SVD. They are summarized in

Theorem 2.2 *Assume $A \in \mathbb{C}^{m \times n}$, $p = \min(m, n)$, and $r \leq p$ denotes the number of positive singular values of A. Then*

1. $\text{rank}(A) = r$
2. $\text{range}(A) = \text{range}(U(:, 1:r))$
   $\text{null}(A) = \text{range}(V(:, r+1, :n))$
3. $\|A\|_2 = \sigma_1$
   $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2}$
4. *The eigenvalues of A^*A are the σ_i^2 and the $\boldsymbol{v}_i$ are the corresponding (orthonormalized) eigenvectors. The eigenvalues of AA^* are the σ_i^2 and possibly $m - n$ zeros. The corresponding orthonormalized eigenvectors are given by the $\boldsymbol{u}_i$.*

5. *If $A = A^*$ (Hermitian or real symmetric), then the eigen-decomposition $A = X\Lambda X^*$ and the SVD $A = U\Sigma V^*$ are almost identical. We have $U = X$, $\sigma_i = |\lambda_i|$, and $\boldsymbol{v}_i = \mathrm{sign}(\lambda_i)\boldsymbol{u}_i$.*

6. *If $A \in \mathbb{C}^{m\times m}$ then $|\det(A)| = \prod_{i=1}^m \sigma_i$.*

Proof We discuss items 1–3 and 6.

1. Since U and V are unitary matrices of full rank and $\mathrm{rank}(\Sigma) = r$ the statement follows from the SVD $A = U\Sigma V^*$.

2. Both statements follow from the fact that the range of Σ is spanned by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r$ and that U and V are full-rank unitary matrices whose ranges are $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively.

3. The invariance of the 2-norm and Frobenius norm under unitary transformations imply $\|A\|_2 = \|\Sigma\|_2$ and $\|A\|_F = \|\Sigma\|_F$. Since Σ is diagonal we clearly have $\|\Sigma\|_2 = \max_{\substack{\boldsymbol{x}\in\mathbb{C}^n \\ \|\boldsymbol{x}\|_2=1}} \|\Sigma\boldsymbol{x}\|_2 = \max_{1\le i\le r} \sigma_i = \sigma_1$. The formula for $\|\Sigma\|_F$ follows directly from the definition of the Frobenius norm.

6. We know that the determinant of a unitary matrix is either plus or minus 1, and that of a diagonal matrix is the product of the diagonal entries. Finally, the determinant of a product of matrices is given by the product of their determinant. Thus, the SVD yields the stated result.

■

2.1.3 Low-rank Approximation

Theorem 2.3 *The $m \times n$ matrix A can be decomposed into a sum of r rank-one matrices:*

$$A = \sum_{j=1}^{r} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^*. \tag{8}$$

Moreover, the best 2-norm approximation of rank ν ($0 \le \nu \le r$) to A is given by

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^*.$$

In fact,

$$\|A - A_\nu\|_2 = \sigma_{\nu+1}. \tag{9}$$

Proof The representation (8) of the SVD follows immediately from the full SVD (6) by splitting Σ into a sum of diagonal matrices $\Sigma_j = \mathrm{diag}(0, \ldots, 0, \sigma_j, 0, \ldots, 0)$.

Formula (9) for the approximation error follows from the fact that $U^*AV = \Sigma$ and the expansion for A_ν so that $U^*(A - A_\nu)V = \mathrm{diag}(0, \ldots, 0, \sigma_{\nu+1}, \ldots)$ and $\|A - A_\nu\|_2 = \sigma_{\nu+1}$ by the invariance of the 2-norm under unitary transformations and item 3 of the previous theorem.

The claim regarding the best approximation property is a little more involved, and omitted. ■

Remark There are many possible rank-ν decompositions of A (e.g., by taking partial sums of the LU or QR factorization). Theorem 2.3, however, says that the ν-th partial sum of the SVD captures as much of the *energy* of A (measured in the 2-norm) as possible. This fact gives rise to many applications in image processing, data compression, data mining, and other fields. See, e.g., the Matlab scripts `svd_compression.m` and `qr_compression.m`.

A geometric interpretation of Theorem 2.3 is given by the best approximation of a hyperellipsoid by lower-dimensional ellipsoids. For example, the best approximation of a given hyperellipsoid by a line segment is given by the line segment corresponding to the hyperellipsoids longest axis. Similarly, the best approximation by an ellipse is given by that ellipse whose axes are the longest and second-longest axis of the hyperellipsoid.

2.1.4 Computing the SVD by Hand

We now list a simplistic algorithm for computing the SVD of a matrix A. It can be used fairly easily for manual computation of small examples. For a given $m \times n$ matrix A the procedure is as follows:

1. Form A^*A.
2. Find the eigenvalues and orthonormalized eigenvectors of A^*A, i.e.,
$$A^*A = V\Lambda V^*.$$
3. Sort the eigenvalues according to their magnitude, and let $\sigma_j = \sqrt{\lambda_j}$, $j = 1, \ldots, n$.
4. Find the first r columns of U via
$$\boldsymbol{u}_j = \sigma_j^{-1} A\boldsymbol{v}_j, \qquad j = 1, \ldots, r.$$
Pick the remaining columns such that U is unitary.

Example Find the SVD for
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{bmatrix}.$$

1. $$A^*A = \begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix}.$$
2. The eigenvalues (in order of decreasing magnitude) are $\lambda_1 = 17$ and $\lambda_2 = 1$, and the corresponding eigenvectors
$$\widetilde{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \widetilde{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
so that (after normalization)
$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Low-rank Approximation

- SVD can be used to compute optimal **low-rank approximations**.
- Approximation problem: Find $\boldsymbol{A_k}$ of rank $\boldsymbol{k}$ such that

$$A_k = \min_{X:\mathrm{rank}(X)=k} \|A \quad X\|_F$$

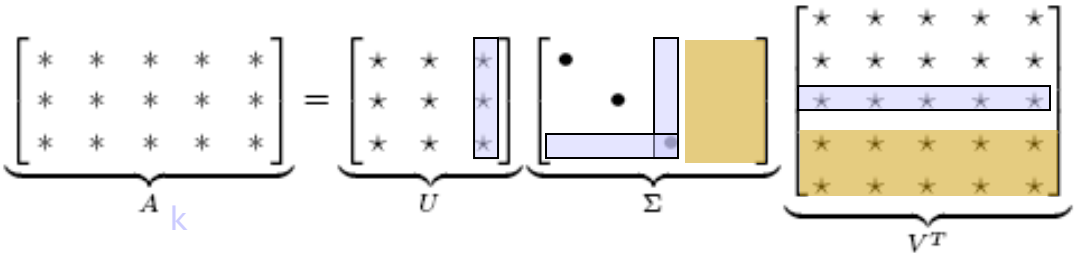
Frobenius norm

$$\|\mathrm{A}\|_F \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}\,.$$

Low-rank Approximation

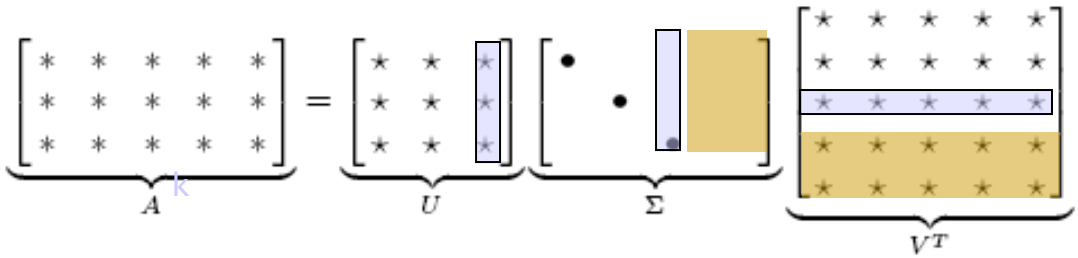
- Solution via SVD

$$A_k = U\,\mathrm{diag}(\ _1, \ldots,\ _k, \underbrace{0, \ldots, 0}_{\text{set smallest } r\text{-}k \text{ singular values to zero}})\,V^T$$



Reduced SVD

- If we retain only k singular values, and set the rest to 0, then we don't need the matrix parts in color
- Then Σ is $k \times k$, U is $M \times k$, V^T is $k \times N$, and A_k is $M \times N$
-
-



al

Approximation Error

- How good (bad) is this approximation?
- It's the best possible, measured by the Frobenius norm of the error:

$$\min_{X:\mathrm{rank}(X)=k} \|A \quad X\|_F = \|A \quad A_k\|_F = \quad {}_{k+1}$$

where the σ_i are ordered such that $\sigma_i \geq \sigma_{i+1}$.

Suggests why Frobenius error drops as *k* increases.

SVD Low-rank Approximation

- Whereas the term-doc matrix A may have M=50000, N=10 million (and rank close to 50000)
- We can construct an approximation A_{100} with rank 100.
  - <u>Of all rank 100 matrices, it would have the lowest Frobenius error.</u>
- Great … but why would we??
- Answer: *Latent Semantic Indexing*

2.0.5 Existence and Uniqueness Theorem

Theorem 2.1 *Let A be a complex $m \times n$ matrix. A has a singular value decomposition of the form*

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where Σ is a uniquely determined $m \times n$ (real) diagonal matrix, U is an $m \times m$ unitary matrix, and V is an $n \times n$ unitary matrix.

Proof We prove only existence. The uniqueness part of the proof follows directly from the geometric interpretation. A (more rigorous?) algebraic argument can be found, e.g., in [Trefethen/Bau].

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Next, we use the Gram-Schmidt algorithm to arbitrarily extend $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ to unitary matrices by adding columns $\widetilde{U}_1$ and $\widetilde{V}_1$, i.e.,

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This results in

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We now look at three of these four blocks:

- Using (7) and the specific choice of $\boldsymbol{v}_1$ we have

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- Again, using (7) we get
$$\widetilde{U}_1^* A\boldsymbol{v}_1 = \widetilde{U}_1^* \boldsymbol{u}_1 \|A\boldsymbol{v}_1\|_2.$$
This, however, is a zero vector since U_1 has orthonormal columns, i.e., $\widetilde{U}_1^* \boldsymbol{u}_1 = \boldsymbol{0}$.

- We show that $\boldsymbol{u}_1^* A\widetilde{V}_1 = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$ by contradiction. If it were nonzero then we could look at the first row of the block matrix $U_1^* AV_1$ and see that
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We now abbreviate the fourth block with $\widetilde{A} = \widetilde{U}_1^* A\widetilde{V}_1$ and can write the block matrix as
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To complete the proof we apply the induction hypothesis to $\widetilde{A}$, i.e., we use the SVD $\widetilde{A} = U_2 \Sigma_2 V_2^*$. Then
$$\begin{aligned} U_1^* AV_1 &= \begin{bmatrix} \sigma_1 & \boldsymbol{0}^T \\ \boldsymbol{0} & U_2 \Sigma_2 V_2^* \end{bmatrix} \\ &= \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \Sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & V_2 \end{bmatrix}^* \end{aligned}$$
or
$$A = U_1 \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \Sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & V_2 \end{bmatrix}^* V_1^*,$$
another SVD (since the product of unitary matrices is unitary). ■

2.1 SVD as a Change of Basis

We now discuss the use of the SVD to diagonalize systems of linear equations. Consider the linear system
$$A\boldsymbol{x} = \boldsymbol{b}$$
with $A \in \mathbb{C}^{m \times n}$. Using the SVD we can write
$$A\boldsymbol{x} = U\Sigma V^* \boldsymbol{x} \quad \Longleftrightarrow \quad \boldsymbol{b} = U\boldsymbol{b}'.$$
Thus, we can express $\boldsymbol{b} \in \text{range}(A)$ in terms of $\text{range}(U)$:
$$U\boldsymbol{b}' = \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{b}' = U^* \boldsymbol{b}$$

Latent Semantic Indexing via the SVD

What it is

- From term-doc matrix A, we compute the approximation $A_{k.}$
- There is a row for each term and a column for each doc in A_k
- Thus docs live in a space of $k<<r$ dimensions
  - These dimensions are not the original axes
- But why?

Vector Space Model: Pros

- **Automatic** selection of index terms
- **Partial matching** of queries and documents *(dealing with the case where no document contains all search terms)*
- **Ranking** according to **similarity score** *(dealing with large result sets)*
- **Term weighting** schemes *(improves retrieval performance)*
- Various extensions

Problems with Lexical Semantics

- Ambiguity and association in natural language
  - **Polysemy**: Words often have a **multitude of meanings** and different types of usage *(more severe in very heterogeneous collections).*
  - The vector space model is unable to discriminate between different meanings of the same word.

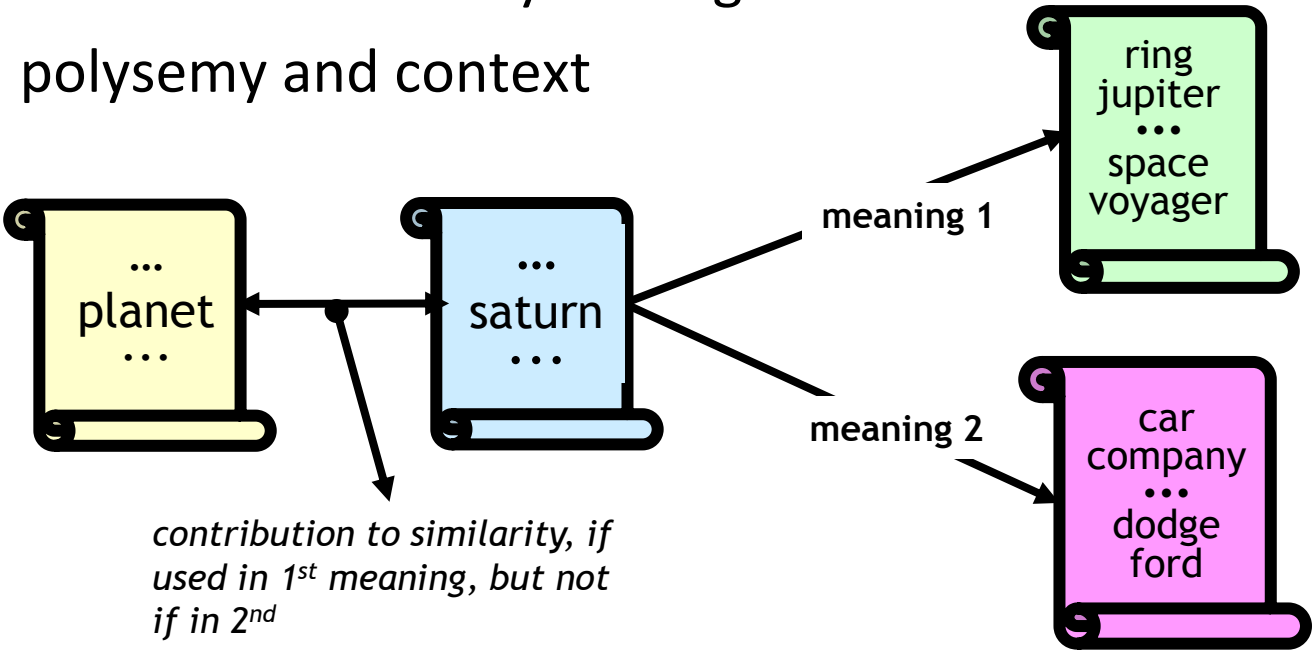
Problems with Lexical Semantics

- **Synonymy**: Different terms may have an **identical or a similar meaning** (weaker: words indicating the same topic).
- No associations between words are made in the vector space representation.

$$\mathrm{sim}_{\mathrm{true}}(d, q) > \cos(\angle(\vec{d}, \vec{q}))$$

Polysemy and Context

- Document similarity on single word level: polysemy and context



Latent Semantic Indexing (LSI)

- Perform a **low-rank approximation** of **document-term matrix** (typical rank **100–300**)
- General idea
  - Map documents (*and* terms) to a **low-dimensional** representation.
  - Design a mapping such that the low-dimensional space reflects **semantic associations** (latent semantic space).
  - Compute document similarity based on the **inner**

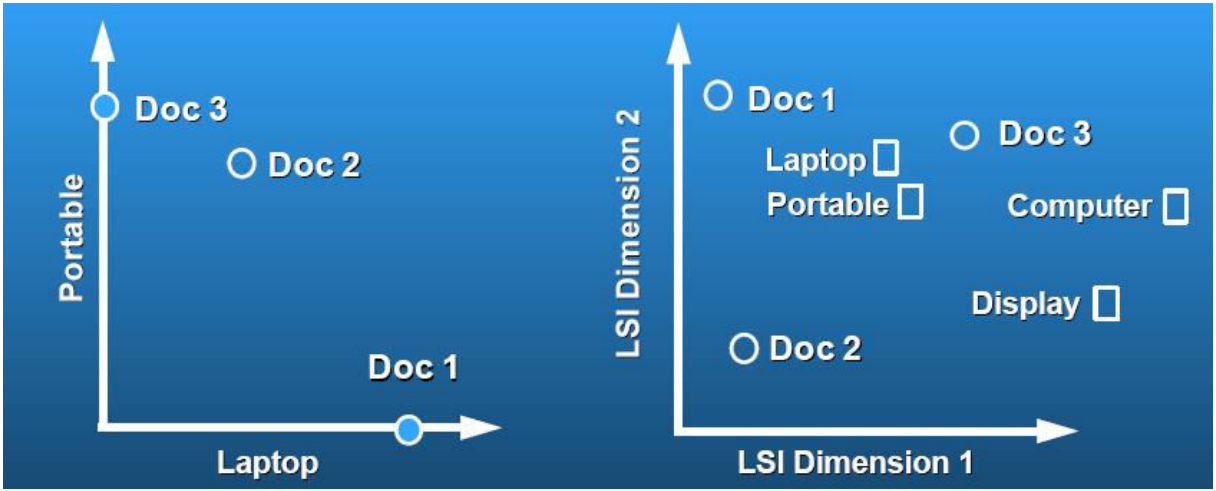
Goals of LSI

- LSI takes documents that are semantically similar (= talk about the same topics), but are not similar in the vector space (because they use different words) and re-represents them in a reduced vector space in which they have higher similarity.

- Similar terms map to similar location in low

Latent Semantic Analysis

- **Latent semantic space**: illustrating example



courtesy of Susan Dumais

Performing the maps

- Each row and column of *A* gets mapped into the *k*-dimensional LSI space, by the SVD.
- Claim – this is not only the mapping with the best (Frobenius error) approximation to *A*, but in fact *improves* retrieval.
- A query *q* is also mapped into this space, by

$$q_k = q^T U_k \Sigma_k^{-1}$$

LSA Example

- A simple example term-document matrix

C	d_1	d_2	d_3	d_4	d_5	d_6
ship	1	0	1	0	0	0
boat	0	1	0	0	0	0
ocean	1	1	0	0	0	0
wood	1	0	0	1	1	0
tree	0	0	0	1	0	1

LSA Example

- Example of C = UΣVT: The matrix U

U	1	2	3	4	5
ship	−0.44	−0.30	0.57	0.58	0.25
boat	−0.13	−0.33	−0.59	0.00	0.73
ocean	−0.48	−0.51	−0.37	0.00	−0.61
wood	−0.70	0.35	0.15	−0.58	0.16
tree	−0.26	0.65	−0.41	0.58	−0.09

LSA Example

- Example of C = UΣVT: The matrix Σ

Σ	1	2	3	4	5
1	2.16	0.00	0.00	0.00	0.00
2	0.00	1.59	0.00	0.00	0.00
3	0.00	0.00	1.28	0.00	0.00
4	0.00	0.00	0.00	1.00	0.00
5	0.00	0.00	0.00	0.00	0.39

LSA Example

- Example of $C = U\Sigma V^T$: The matrix V^T

V^T	d_1	d_2	d_3	d_4	d_5	d_6
1	−0.75	−0.28	−0.20	−0.45	−0.33	−0.12
2	−0.29	−0.53	−0.19	0.63	0.22	0.41
3	0.28	−0.75	0.45	−0.20	0.12	−0.33
4	0.00	0.00	0.58	0.00	−0.58	0.58
5	−0.53	0.29	0.63	0.19	0.41	−0.22

LSA Example: Reducing the dimension

U	1	2	3	4	5
ship	−0.44	−0.30	0.00	0.00	0.00
boat	−0.13	−0.33	0.00	0.00	0.00
ocean	−0.48	−0.51	0.00	0.00	0.00
wood	−0.70	0.35	0.00	0.00	0.00
tree	−0.26	0.65	0.00	0.00	0.00

Σ_2	1	2	3	4	5
1	2.16	0.00	0.00	0.00	0.00
2	0.00	1.59	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00

V^T	d_1	d_2	d_3	d_4	d_5	d_6
1	−0.75	−0.28	−0.20	−0.45	−0.33	−0.12
2	−0.29	−0.53	−0.19	0.63	0.22	0.41
3	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00

Original matrix C vs. reduced $C_2 = U\Sigma_2 V^T$

C	d_1	d_2	d_3	d_4	d_5	d_6
ship	1	0	1	0	0	0
boat	0	1	0	0	0	0
ocean	1	1	0	0	0	0
wood	1	0	0	1	1	0
tree	0	0	0	1	0	1

C_2	d_1	d_2	d_3	d_4	d_5	d_6
ship	0.85	0.52	0.28	0.13	0.21	−0.08
boat	0.36	0.36	0.16	−0.20	−0.02	−0.18
ocean	1.01	0.72	0.36	−0.04	0.16	−0.21
wood	0.97	0.12	0.20	1.03	0.62	0.41
tree	0.12	−0.39	−0.08	0.90	0.41	0.49

Why the reduced dimension matrix is better

- Similarity of d2 and d3 in the original space: 0.
- Similarity of d2 and d3 in the reduced space: $0.52 * 0.28 + 0.36 * 0.16 + 0.72 * 0.36 + 0.12 * 0.20 + -0.39 * -0.08 \approx 0.52$
- Typically, LSA increases recall and hurts precision

Empirical evidence

- Experiments on TREC 1/2/3 – Dumais
- Lanczos SVD code (available on netlib) due to Berry used in these experiments
  - Running times of ~ one day on tens of thousands of docs [still an obstacle to use!]
- Dimensions – various values 250-350 reported. Reducing *k* improves recall.
  - (Under 200 reported unsatisfactory)
- Generally expect recall to improve – what about precision?

Empirical evidence

- Precision at or above median TREC precision
  - Top scorer on almost 20% of TREC topics
- Slightly better on average than straight vector spaces
- Effect of dimensionality:

Dimensions	Precision
250	0.367
300	0.371
346	0.374

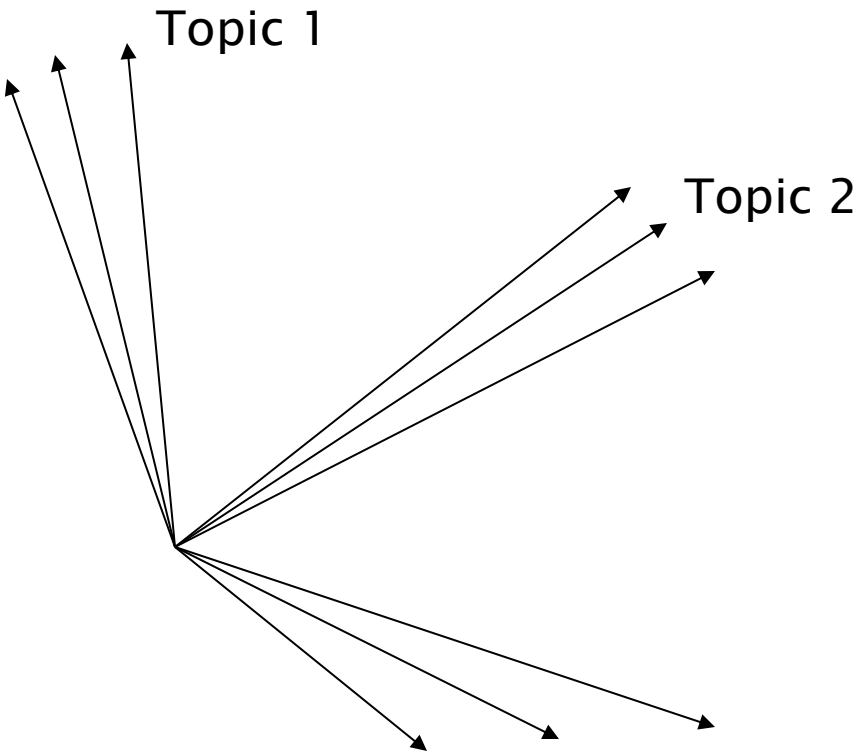
Failure modes

- Negated phrases
  - TREC topics sometimes negate certain query/terms phrases – precludes simple automatic conversion of topics to latent semantic space.
- Boolean queries
  - As usual, freetext/vector space syntax of LSI queries precludes (say) “Find any doc having to do with the following 5 companies”

But why is this clustering?

- We've talked about docs, queries, retrieval and precision here.
- What does this have to do with clustering?
- Intuition: Dimension reduction through LSI brings together "related" axes in the vector space.

Simplistic picture



Some wild extrapolation

- The “dimensionality” of a corpus is the number of distinct topics represented in it.
- More mathematical wild extrapolation:
  - if A has a rank k approximation of low Frobenius error, then there are no more than k distinct topics in the corpus.

LSI has many other applications

- In many settings in pattern recognition and retrieval, we have a feature-object matrix.
  - For text, the terms are features and the docs are objects.
  - Could be opinions and users …
  - This matrix may be redundant in dimensionality.
  - Can work with low-rank approximation.
  - If entries are missing (e.g., users' opinions), can recover if dimensionality is low.

Principal Components

- Most Popular Feature Extraction Scheme in
  - Signal Processing
  - Pattern Recognition
  - Machine Learning
- Each new feature is a linear combination of the original features
- Let $X_1, X_2, \ldots, X_n$ be n D-dimensional patterns.
- Any such D-dimensional vector is represented as

$$X = \sum_{j=1}^{D} \beta_j P_j \qquad\qquad Y = \sum_{j=1}^{d} \beta_j P_j$$

Where d < D ; β_j and P_j are the weight and basis vectors;
X and Y are the actual and approximate representations.

Characterization of Error

- Error $= E\{(X-Y)^t(X-Y)\}$

  $= E\{\sum_{j=d+1}^{D} \beta_j^2\}$ $\qquad P_i, P_j$ are orthonormal

- Error = $E\{\sum_{d+1}^{D} \beta_j \beta_j^t\} = E\{\sum_{d+1}^{D} P_j^t X X^t P_j\}$

This is because $\beta_j = \beta_j^t = X^t P_j$

$$P_j^t X = P_j^t \sum_{i=1}^{D} \beta_i P_i = \beta_j$$

- So, Error = $\sum_{j=d+1}^{D} P_j^t \Omega P_j$

Eigenvectors of the Covariance Matrix

The covariance matrix $\Omega = E\{XX^t\}$

By looking at the eigenvectors of the covariance matrix, we have

$$\boldsymbol{\Omega P_j} = \lambda_j \boldsymbol{P_j}$$

So, error = $\sum_{j=d+1}^{D} P_j^t \Omega P_j = \sum_{j=d+1}^{D} P_j^t P_j \lambda_j = \sum_{d+1}^{D} \lambda_j$

So, error can be minimized by choosing the least eigenvalues for the entries in the sum above. So, the eigenvalues are

Example

Consider six 2-dimensional patterns:

(1,2), (2,1), (2,2), (6,7), (7,6), (6,6)

$$\mu = \frac{1}{6}\sum_{i=1}^{6} X_i = \frac{1}{6}(24,24) = (4,4)$$

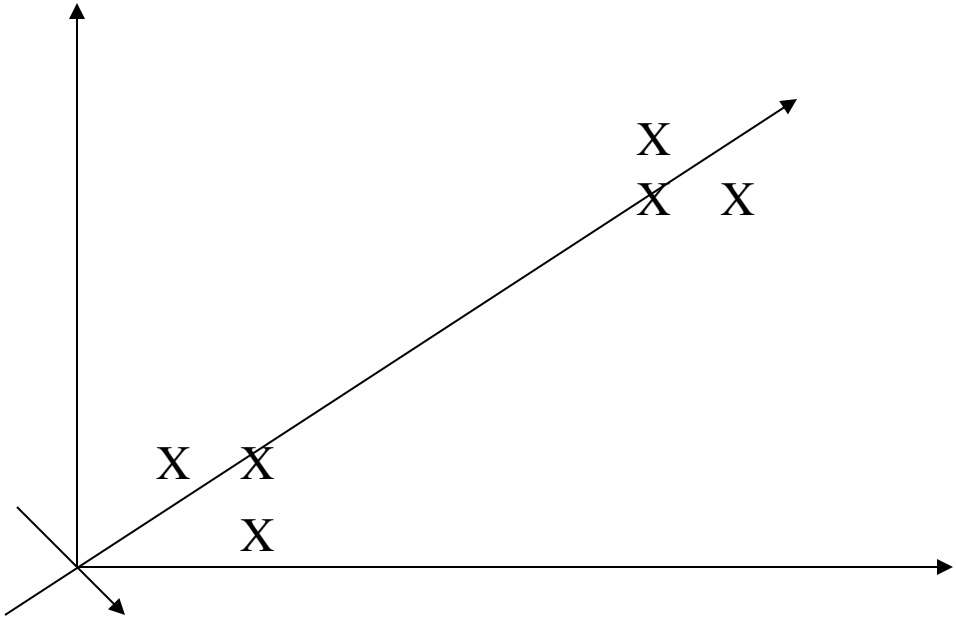
$$\Omega = \frac{1}{6}\sum_{1}^{6}(X_i - \mu)(X_i - \mu)^t = \begin{pmatrix} \frac{17}{3} & \frac{16}{3} \\ \frac{16}{3} & \frac{17}{3} \end{pmatrix}$$

The eigenvalues and eigenvectors are:

$$\lambda_1 = 11; \lambda_2 = \frac{1}{3}$$

$$P_1 = (1,1); P_2 = (1,-1)$$

The two eigenvector directions are shown below:



Singular Value Decomposition

- For an m x n matrix A of rank r there exists a factorization as follows:

$$\underset{m \times n}{A} = \underset{m \times m}{U} \underset{m \times n}{\Sigma} \underset{n \times n}{V^t}$$

- The columns of U are orthonormal eigenvectors of AA^t
- The columns of V are orthonormal eigenvectors of $A^t A$
- Eigenvalues $\lambda_1, \cdots, \lambda_r$ of AA^t are eigenvalues of $A^t A$
- Singular values $\sigma_i = \sqrt{\lambda_i}$ and

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad A^t A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\lambda_1 = 3; \lambda_2 = 1$$

Thus *M*=3, *N*=2. Its SVD is

$$\begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Let D = $\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}6\\7\end{pmatrix}\begin{pmatrix}7\\6\end{pmatrix}\begin{pmatrix}6\\6\end{pmatrix}$

$$\mu = \begin{pmatrix}4\\4\end{pmatrix}$$

$$X = \begin{bmatrix} -3 & -2 \\ -2 & -3 \\ -2 & -2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 2 \end{bmatrix} \qquad X^t X = \begin{bmatrix} 34 & 32 \\ 32 & 34 \end{bmatrix}$$

$$\lambda_1 = 66; \lambda_2 = 2$$

If normalized then the eigenvalues are 11 and 1/3

And the eigenvectors are $\begin{pmatrix}1\\1\end{pmatrix}$ $\begin{pmatrix}1\\-1\end{pmatrix}$

$$\begin{bmatrix} 34 & 32 \\ 32 & 34 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 66 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$Ax = \lambda x$$

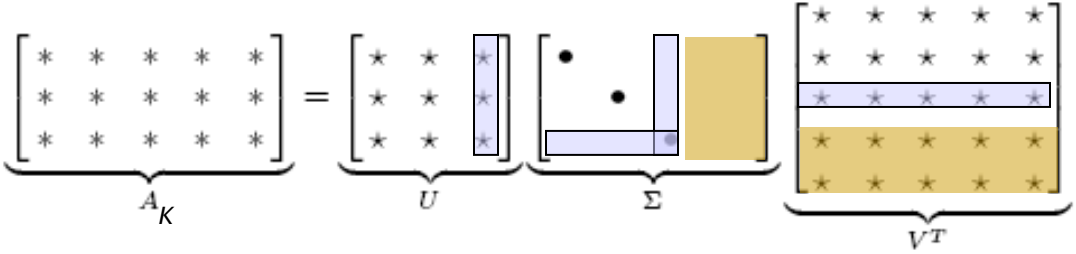
$$A^2 x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$$

SVD may be used in PCA.

Low-rank Approximation

- Solution via SVD

set smallest r-k singular values to zero



Problems with Lexical Semantics

- Ambiguity and association in natural language
  - **Polysemy**: Words often have a **multitude of meanings** and different types of usage *(more severe in very heterogeneous collections).*
  - The vector space model is unable to discriminate between different meanings of the same word.

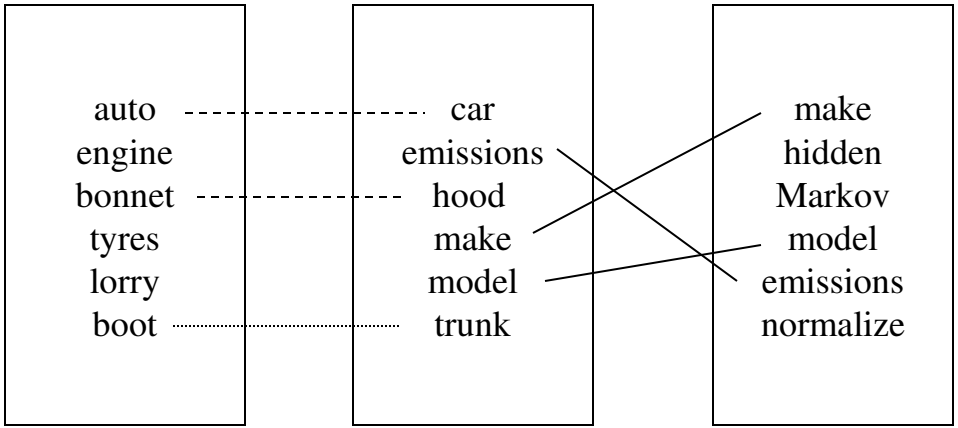
$$\mathrm{sim}_{\mathrm{true}}(d, q) < \cos(\angle(\vec{d}, \vec{q}))$$

- **Synonymy**: Different terms may have an **identical or a similar meaning** (weaker: words indicating the same topic).
- No associations between words are made in the vector space representation.

$$\mathrm{sim}_{\mathrm{true}}(d, q) > \cos(\angle(\vec{d}, \vec{q}))$$

The Problem

- Example: Vector Space Model



Synonymy

Will have small cosine

but are related

Polysemy

Will have large cosine

but not truly related

- Perform a ***low-rank approximation*** of ***document-term matrix*** (typical rank **100-300**)
- General idea
  - Map documents (*and* terms) to a **low-dimensional** representation.
  - Design a mapping such that the low-dimensional space reflects **semantic associations** (latent semantic space).
  - Compute document similarity based on the **inner product** in this **latent semantic space**

Latent Semantic Analysis

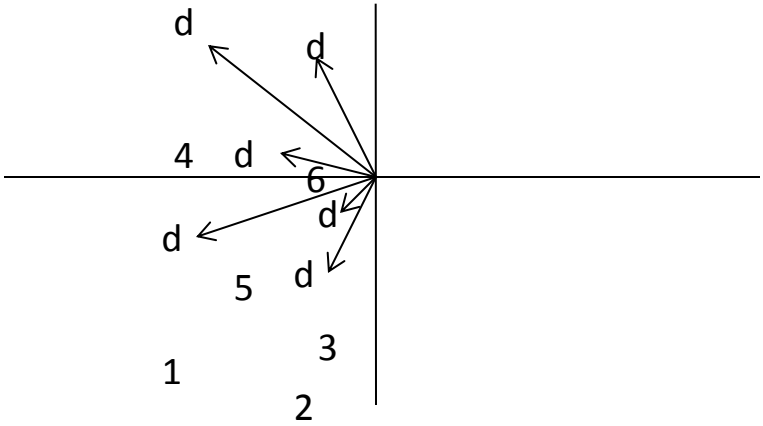
- **Latent semantic space**: illustrating example

	d1	d2	d3	d4	d5	d6
Ship	1	0	1	0	0	0
Boat	0	1	0	0	0	0
Ocean	1	1	0	0	0	0
Voyage	1	0	0	1	1	0
Trip	0	0	0	1	0	1

- The reduced dimensional data in 2-d is given below

	d1	d2	d3	d4	d5	d6
SV1	-1.62	-0.6	-0.44	-0.97	-0.7	-0.26
SV2	-0.46	-0.84	-0.3	1.0	0.35	0.65

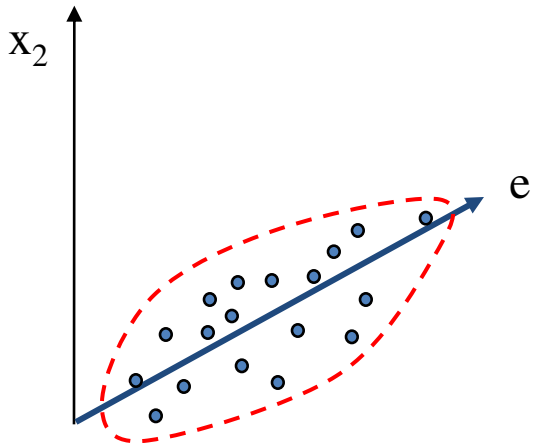
LSA



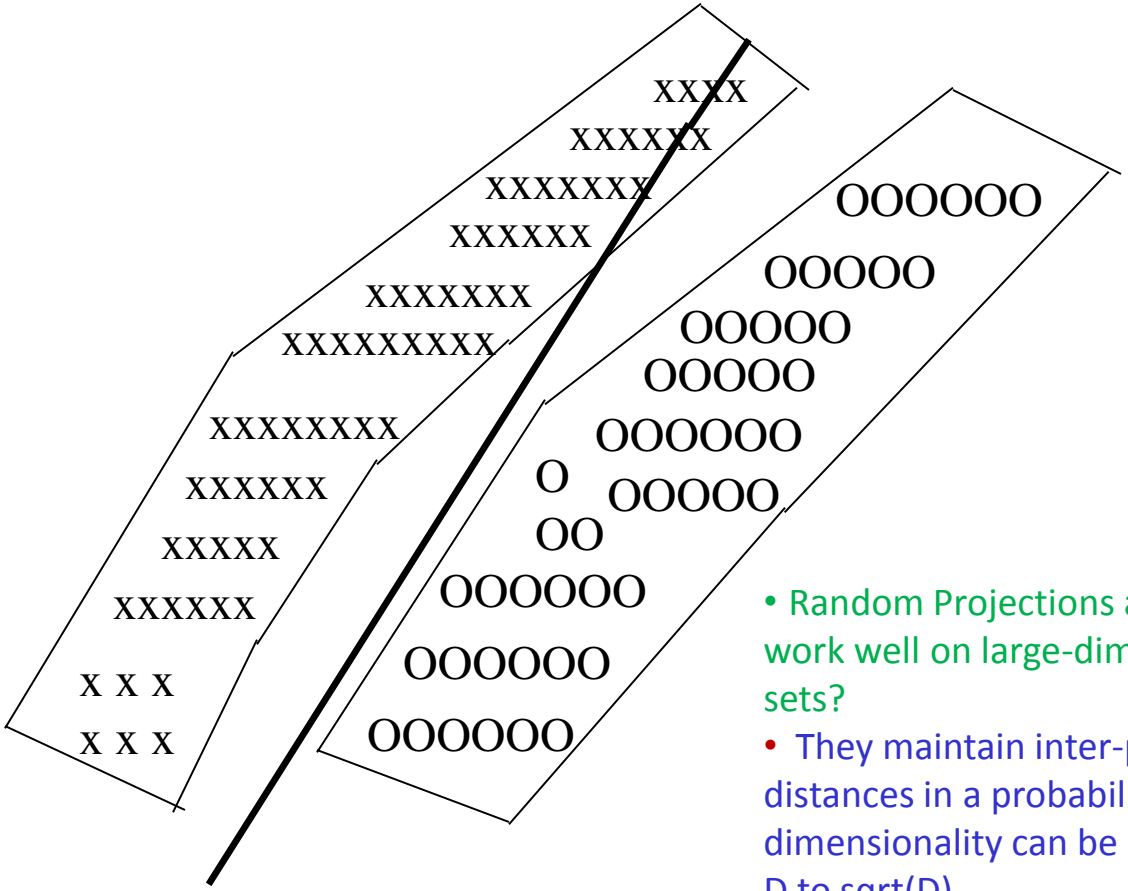
- d2 is similar to d3
  - d2: boat, ocean; d3: ship
- d4 is similar to d5 and d6 ($\cos\theta$)
  - d4: voyage, trip; d5: voyage; d6: trip

Dimensionality Reduction: PCA

- Find the eigenvectors of the covariance matrix
- The eigenvectors define the new space



Dimensions = 206



- Random Projections are believed to work well on large-dimensional data sets?
- They maintain inter-point distances in a probabilistic sense; the dimensionality can be reduced from D to sqrt(D).

Non-negative Matrix Factorization (NMF)

$$\mathbf{V} \approx \mathbf{WH}$$

$$\begin{cases} \mathbf{V} \in R^{n \times m} \\ \mathbf{W} \in R^{n \times r} \\ \mathbf{H} \in R^{r \times m} \end{cases}$$

$$\min \left\| \mathbf{V} - \mathbf{WH} \right\|^2$$

V: Data matrix.
W: Mixing matrix
H: Encoding matrix

The only constraint is **NON-NEGATIVIY** of the matrices

Only **ADDITIVE**,not subtractive, combinations are allowed

- W links documents to topics.
- H links topics to words.

Equivalance of NMF and K-means Clustering

- Objective function of K-means clustering is equivalent to:

$$\min J_{K-means} = \sum_{k=1}^{K} \sum_{i \in C_k} \left\| \mathbf{x}_i - \mathbf{w}_k \right\|^2$$

$\mathbf{x}_i$:ith sample, $\mathbf{w}_k$:kth cluster centroid, K: # of clusters

- The cth row of the encoding matrix **H** would have non-zero elements for the samples which belong to cth cluster.
- Columns of the mixing matrix **W** can be considered as cluster mean vectors

# of 1s is *t*

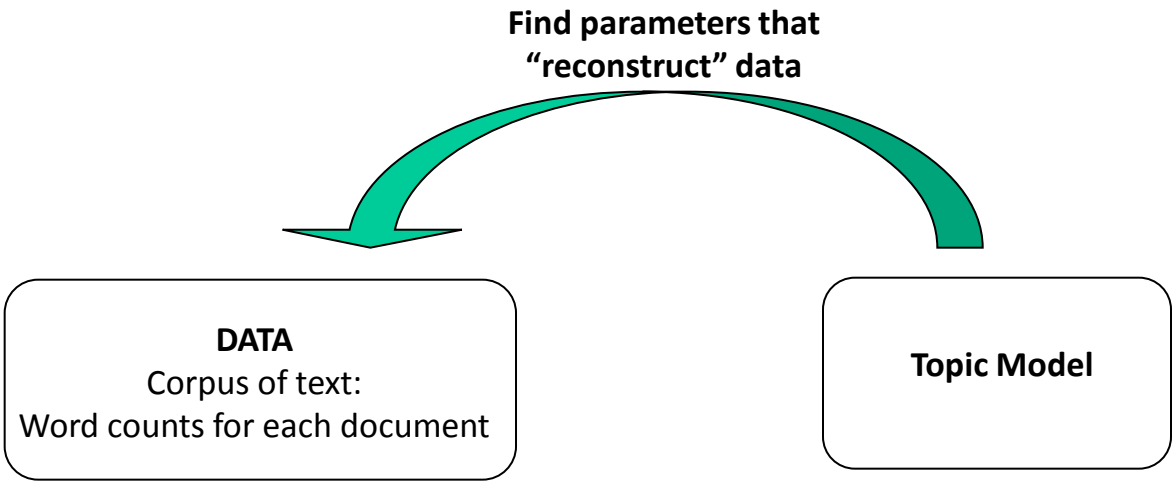
$$\mathbf{h}_c = [000...00\underbrace{11.....111}100...000]/t$$

$$\mathbf{W} = \mathbf{V}\mathbf{H}^T$$

$$\begin{bmatrix} 4 & 19 \\ 4 & 19 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 6 & 6 & 7 \\ 1 & 2 & 1 & 6 & 7 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Approximation

Model is Generative



Probabilistic Topic Models

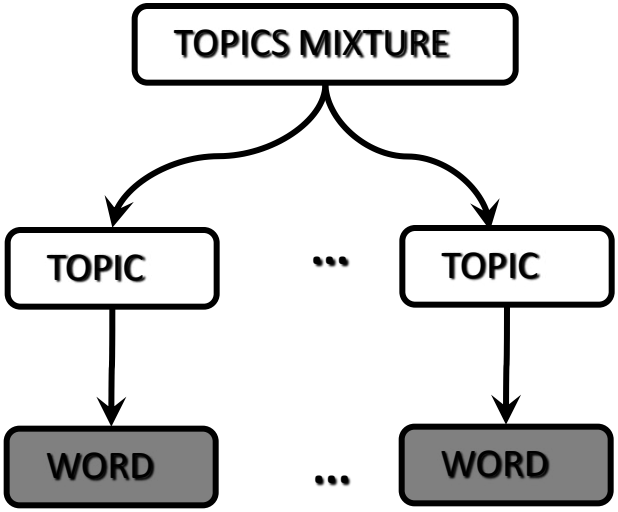
- Each document is a probability distribution over topics (distribution over topics = gist)

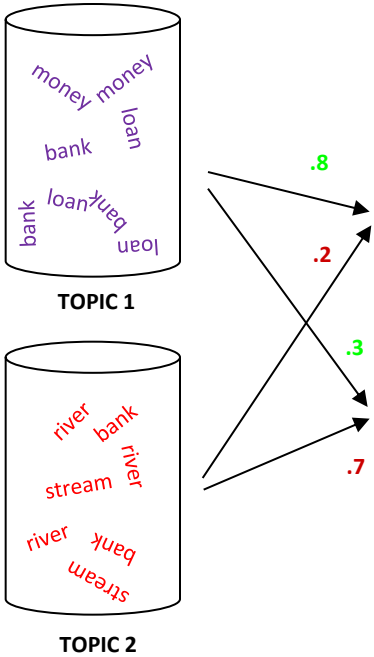
- Each topic is a probability distribution over words

1. for each document, choose
   a mixture of topics

2. For every word slot, sample a topic [1..T] from the mixture

3. sample a word from the topic





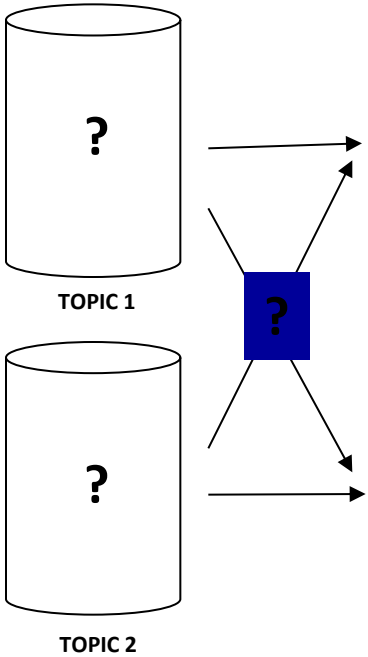
DOCUMENT 1: money[1] bank[1] bank[1] loan[1] river[2] stream[2] bank[1] money[1] river[2] bank[1] money[1] bank[1] loan[1] money[1] stream[2] bank[1] money[1] bank[1] bank[1] loan[1] river[2] stream[2] bank[1] money[1] river[2] bank[1] money[1] bank[1] loan[1] bank[1] money[1] stream[2]

DOCUMENT 2: river[2] stream[2] bank[2] stream[2] bank[2] money[1] loan[1] river[2] stream[2] loan[1] bank[2] river[2] bank[2] bank[1] stream[2] river[2] loan[1] bank[2] stream[2] bank[2] money[1] loan[1] river[2] stream[2] bank[2] stream[2] bank[2] money[1] river[2] stream[2] loan[1] bank[2] river[2] bank[2] money[1] bank[1] stream[2] river[2] bank[2] stream[2] bank[2] money[1]

Mixture components

Mixture weights

Bayesian approach: use priors



DOCUMENT 1: money? bank? bank? loan? river? stream? bank? money? river? bank? money? bank? loan? money? stream? bank? money? bank? bank? loan? river? stream? bank? money? river? bank? money? bank? loan? bank? money? stream?

DOCUMENT 2: river? stream? bank? stream? bank? money? loan? river? stream? loan? bank? river? bank? bank? stream? river? loan? bank? stream? bank? money? loan? river? stream? bank? stream? bank? money? river? stream? loan? bank? river? bank? money? bank? stream? river? bank? stream? bank? money?

Example: Topics From an Educational Corpus

(TASA: (Touchstone Applied Science Associates) Corpus- Lang., Arts, Social Studies, Health, Sciences)

- 37K docs, 26K words
- 1700 topics, e.g.:

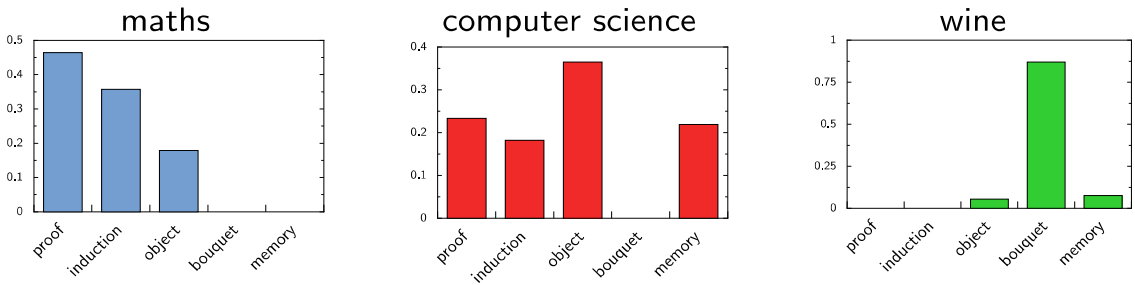
PRINTING	PLAY	TEAM	JUDGE	HYPOTHESIS	STUDY
PAPER	PLAYS	GAME	TRIAL	EXPERIMENT	TEST
PRINT	STAGE	BASKETBALL	COURT	SCIENTIFIC	STUDYING
PRINTED	AUDIENCE	PLAYERS	CASE	OBSERVATIONS	HOMEWORK
TYPE	THEATER	PLAYER	JURY	SCIENTISTS	NEED
PROCESS	ACTORS	PLAY	ACCUSED	EXPERIMENTS	CLASS
INK	DRAMA	PLAYING	GUILTY	SCIENTIST	MATH
PRESS	SHAKESPEARE	SOCCER	DEFENDANT	EXPERIMENTAL	TRY
IMAGE	ACTOR	PLAYED	JUSTICE	TEST	TEACHER
PRINTER	THEATRE	BALL	EVIDENCE	METHOD	WRITE
PRINTS	PLAYWRIGHT	TEAMS	WITNESSES	HYPOTHESES	PLAN
PRINTERS	PERFORMANCE	BASKET	CRIME	TESTED	ARITHMETIC
COPY	DRAMATIC	FOOTBALL	LAWYER	EVIDENCE	ASSIGNMENT
COPIES	COSTUMES	SCORE	WITNESS	BASED	PLACE
FORM	COMEDY	COURT	ATTORNEY	OBSERVATION	STUDIED
OFFSET	TRAGEDY	GAMES	HEARING	SCIENCE	CAREFULLY
GRAPHIC	CHARACTERS	TRY	INNOCENT	FACTS	DECIDE
SURFACE	SCENES	COACH	DEFENSE	DATA	IMPORTANT
PRODUCED	OPERA	GYM	CHARGE	RESULTS	NOTEBOOK
CHARACTERS	PERFORMED	SHOT	CRIMINAL	EXPLANATION	REVIEW

PLSI – a Probabilistic Approach

- Models each word in a document as a sample from a mixture model.

$$p(w) = \sum_{i=1}^{k} p(z_i)p(w|z_i) \qquad \text{s.t.} \quad \sum_{i=1}^{k} p(z_i) \stackrel{!}{=} 1$$

- Introduces the concept of a *topic*.



- Each word generated from one topic, different words can be generated by different topics.
- Problems: (1) *not well-defined* on documents level, (2) overfitting ($kV + kM$ parameters).

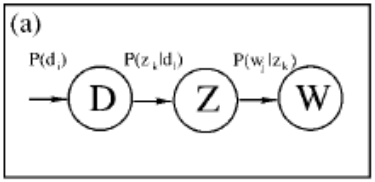
PLSI

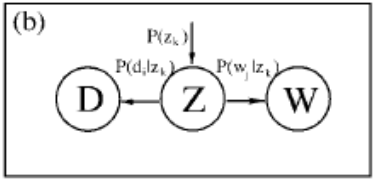
- Assume a multinomial distribution of topics (z)

$$mult(m_1, m_2, \cdots, m_k \mid \mu, n) = \binom{n}{m_1, m_2, \cdots, m_k} \Pi_{k=1}^{K} \mu_k^{m_k}$$

$$P(d_i, w_j) = P(d_i)P(w_j \mid d_i), \quad P(w_j \mid d_i) = \sum_{k=1}^{K} P(w_j \mid z_k)P(z_k \mid d_i).$$

$$P(d_i, w_j) = \sum_{k=1} P(z_k)P(d_i \mid z_k)P(w_j \mid z_k),$$





Question: How to determine z ?

Relation with LSI

- Relation

$$\mathbf{P} = \hat{\mathbf{U}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}^T$$

$$P(d,w) = \sum_{z \in Z} P(z)P(d \mid z)P(w \mid z)$$

$$\hat{\mathbf{U}} = (P(d_i \mid z_k))_{i,k} \qquad \hat{\mathbf{\Sigma}} = \mathbf{diag}(P(z_k))_{k} \qquad \hat{\mathbf{V}} = (P(w_j \mid z_k))_{j,k}$$

- Difference:
  - LSI: minimize L-2 norm
  - pLSI: maximize log-likelihood of training data

Problem of PLSI

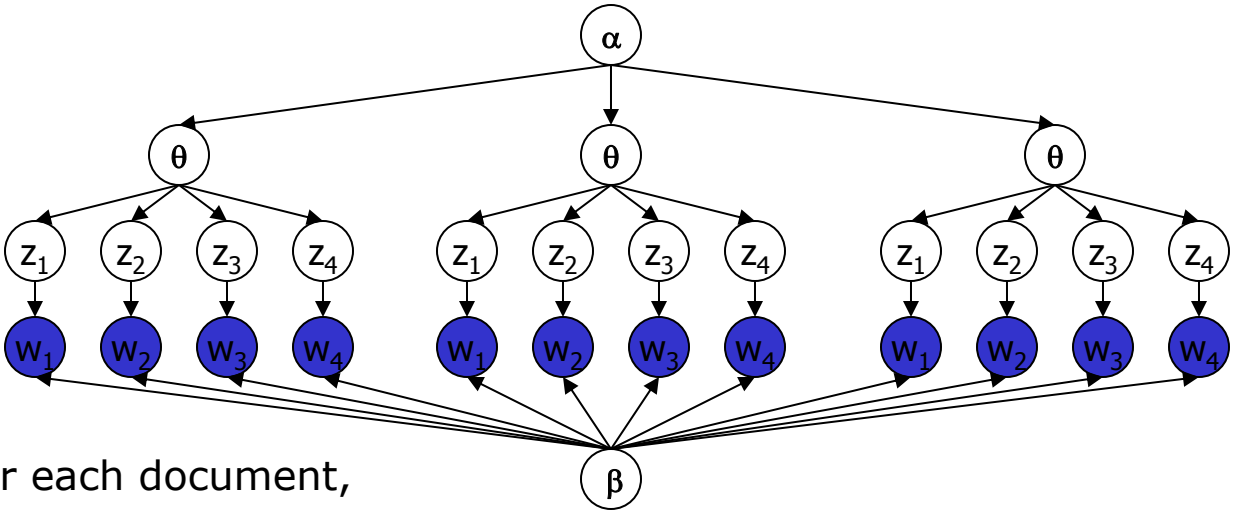
- It is not a proper generative model for document:
  - Document is generated from a mixture of topics
- The number of topics may grow linearly with the size of the corpus
- Difficult to generate a new document

Dirichlet Distributions

$$p(\theta|\alpha) = \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

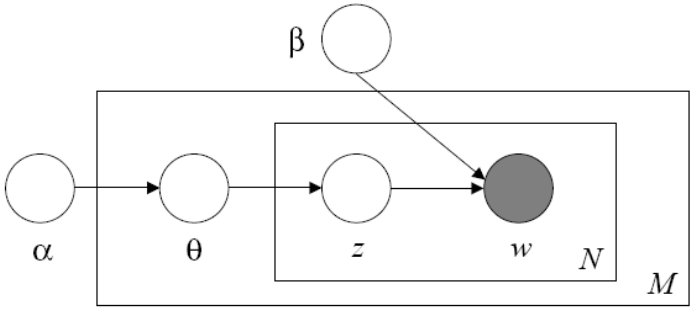
- Useful Facts:
  - This distribution is defined over a (k-1)-simplex. That is, it takes k non-negative arguments which sum to one. Consequently it is a natural distribution to use over multinomial distributions.
  - In fact, the Dirichlet distribution is the conjugate prior to the multinomial distribution. (This means that if our likelihood is multinomial with a Dirichlet prior, then the posterior is also Dirichlet!)
  - The Dirichlet parameter α_i can be thought of as a prior count of the ith class.

The LDA Model



- For each document,
- Choose $\theta \sim \text{Dirichlet}(\alpha)$
- For each of the N words wn:
  - Choose a topic z_n» Multinomial(θ)
  - Choose a word w_n from $p(w_n|z_n,\beta)$, a multinomial probability conditioned on the topic z_n.

The LDA Model

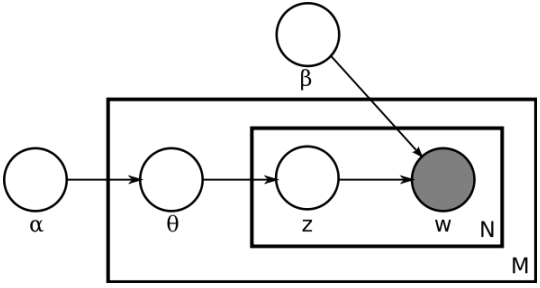


For each document,

- Choose θ» Dirichlet(α)
- For each of the N words w_n:
  - Choose a topic z_n» Multinomial(θ)
  - Choose a word w_n from $p(w_n|z_n,\beta)$, a multinomial probability conditioned on the topic z_n.

LDA (Latent Dirichlet Allocation)

- Document = mixture of topics (as in pLSI), but according to a Dirichlet prior
  - When we use a uniform Dirichlet prior, pLSI=LDA
- A word is also generated according to another variable β: $\beta_{ij} = p(w^j = 1 \mid z^i = 1)$



$$p(\mathbf{w} \mid \alpha, \beta) = \int p(\theta \mid \alpha) \left(\prod_{n=1}^{N} \sum_{z_n} p(z_n \mid \theta) p(w_n \mid z_n, \beta) \right) d\theta.$$

$$p(D \mid \alpha, \beta) = \prod_{d=1}^{M} \int p(\theta_d \mid \alpha) \left(\prod_{n=1}^{N_d} \sum_{z_{dn}} p(z_{dn} \mid \theta_d) p(w_{dn} \mid z_{dn}, \beta) \right) d\theta_d.$$

Performance of LDA on Reuters Data

About the experiments

- 100-topic LDA trained on a 16'000 documents corpus of news articles by Reuters (the news agency).
- Some standard stop words removed.

Top words from some of the $p(w|z)$

"Arts"	"Budgets"	"Children"	"Education"
new	million	children	school
film	tax	women	students
show	program	people	schools
music	budget	child	education
movie	billion	years	teachers
play	federal	families	high
musical	year	work	public

Knowledge is used in

- Refining query
- Collecting the documents
- Representing query and documents
- Indexing documents
- *Finding similarity and in ranking*
- *Grouping and Classification*

(B1 "00573"...) (B2 "00631"...) (B3 "005113"...) (B4 "004"...) (B5 "'00514"...) (B6 "005133"...)
(B7 "00513"...) (B8 "005131"...) (B9 "00435"...) (B10 "'005133"...) (Bll "005133"...) (B12 "0051".
(B13 "004"...) (B14 "006454"...) (B15 "00654"...) (B16 "00637"...) (B17 "0066"...) (B18 "00642".
(B19 "0063"...) (B20 "0063"...)(B21 "00637"...) (B22 "0063“…) (B23 "0063"...) (B24 "00435"...
(B25 "00633"...)(B26 "0063"...)(B27 "00633“…) (B28 "005133"...) (B29 "004125"...) (B30 "00637
(B31 "005265"...) (B32 "0046“…)(B33 "005133"...)(B34 "00516"...)(B35 "01M35"...) (B36 "00435"
(B37 "0051“…)(B38 "005133"...)(B39 "005133"...)(B40 "00574"...)(B41 "00435" . .)(B42 "005756
(B43 "00422"...)(B44 "004"...)(B45 "004"...)(B46"0063“. ..)(B47 "00435“…)(B48 "0046", . .)
(B49 "00543"...)(B50 "00513"...)(B51 "0046"..) (B52 "00435“…)(B53 "0051"...)(B54 "005131"..
(B55 "00543"...)(B56 "0043Y' . .)(B57 "004165“...) (B58 "005"...) (B59 "0057"...) (B60 "0046"..
(B61 "0051"..) (B62 "00516". ,) (B63 "00511"...) (B64 "0063"...) (B65 "004"...)(B66 "004"..)
(B67 "00575"...) (B68 "00424"...) (B69 "00436",..) (B70 "00421"...)(B71 "00574" . ,) (B72 "004".
(B73 "00633"...)(B74 "00512028"...)(B75 "005133"...)(B76 "00435" . .)(B77 "00435"...)(B78 "0042
(B79 "005131"...) (B80 "004"...)(B81 "00574"' . .) (B82 "00543".. ,) (B83 "00543"...) (B84 "0066'
(B85 "00435")(B86 "00513Y'...) (B87 "00526"...) (B88 "005265"...) (B89 "0063"., .) (B90 "00435'
(B91 "00512"'...) (B92 "00468".. ,) (B93 "00436"...) (B94 "0051",..) (B95 "0066"...)(B96 "00411".
(B97 "0051"...) (B98 "00574".. ,) (B99 "0046"...) (B100 "00513"...)(BI01 "00574"...) (BI02 "00435
(B103 "00435"...)(B104 "005453"...)(B105 "0057"'...)(B106 "'00435"...)(B107 "00411". , .)
(B108 "00512"...)

• (rank library1 library2 library3 “PROGRAMMING”)
Rank 1 Library LIBRARY1 with 17 books
Rank 2 Library LIBRARY2 with 15 books
Rank 3 Library LIBRARY3 with 9 books

• (rank library1 library2 library3 “ARTIFICIAL INTELLIGENCE”)
Rank 1 Library LIBRARY1 with 12 books
Rank 2 Library LIBRARY3 with 9 books
Rank 3 Library LIBRARY2 with 4 books

• (rank library1 library2 library3 "DATA PROCESSING")
Rank 1 Library LIBRARY2 with 31 books
Rank 2 Library LIBRARYI with 19 books
Rank 3 Library LIBRARY3 with 16 books

• Instead of simply taking counts, now we can use the number of *citations to these books using Google Scholar* to get a better characterization of the impact of the collection in a library in a given area.
• Further, one can use Amazon.com to get significant related books for strengthening an area in a given library.
• One can use the Table of Contents to get books that complement each other or at different levels.

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http://en.wikipedia.org/wiki/Pattern_recognition

pattern recognition

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Pattern recognition

From Wikipedia, the free encyclopedia

For other uses, see Pattern recognition (disambiguation).

In machine learning, **pattern recognition** is the assignment of some sort of output value (or *label*) to a given input value (or *instance*), according to some specific algorithm. An example of pattern recognition is classification, which attempts to assign each input value to one of a given set o *classes* (for example, determine whether a given email is "spam" or "non-spam"). However, pattern recognition is a more general problem that encompasses other types of output as well. Other examples are regression, which assigns a real-valued output to each input; sequence labeling, assigns a class to each member of a sequence of values (for example, part of speech tagging, which assigns a part of speech to each word in an input sentence); and parsing, which assigns a parse tree to an input sentence, describing the syntactic structure of the sentence.

Pattern recognition algorithms generally aim to provide a reasonable answer for all possible inputs and to do "fuzzy" matching of inputs. This is opposed to *pattern matching* algorithms, which look for exact matches in the input with pre-existing patterns. A common example of a pattern-ma algorithm is regular expression matching, which looks for patterns of a given sort in textual data and is included in the search capabilities of many text editors and word processors. In contrast to pattern recognition, pattern matching is generally not considered a type of machine learning, a pattern-matching algorithms (especially with fairly general, carefully tailored patterns) can sometimes succeed in providing similar-quality output to the sort provided by pattern-recognition algorithms.

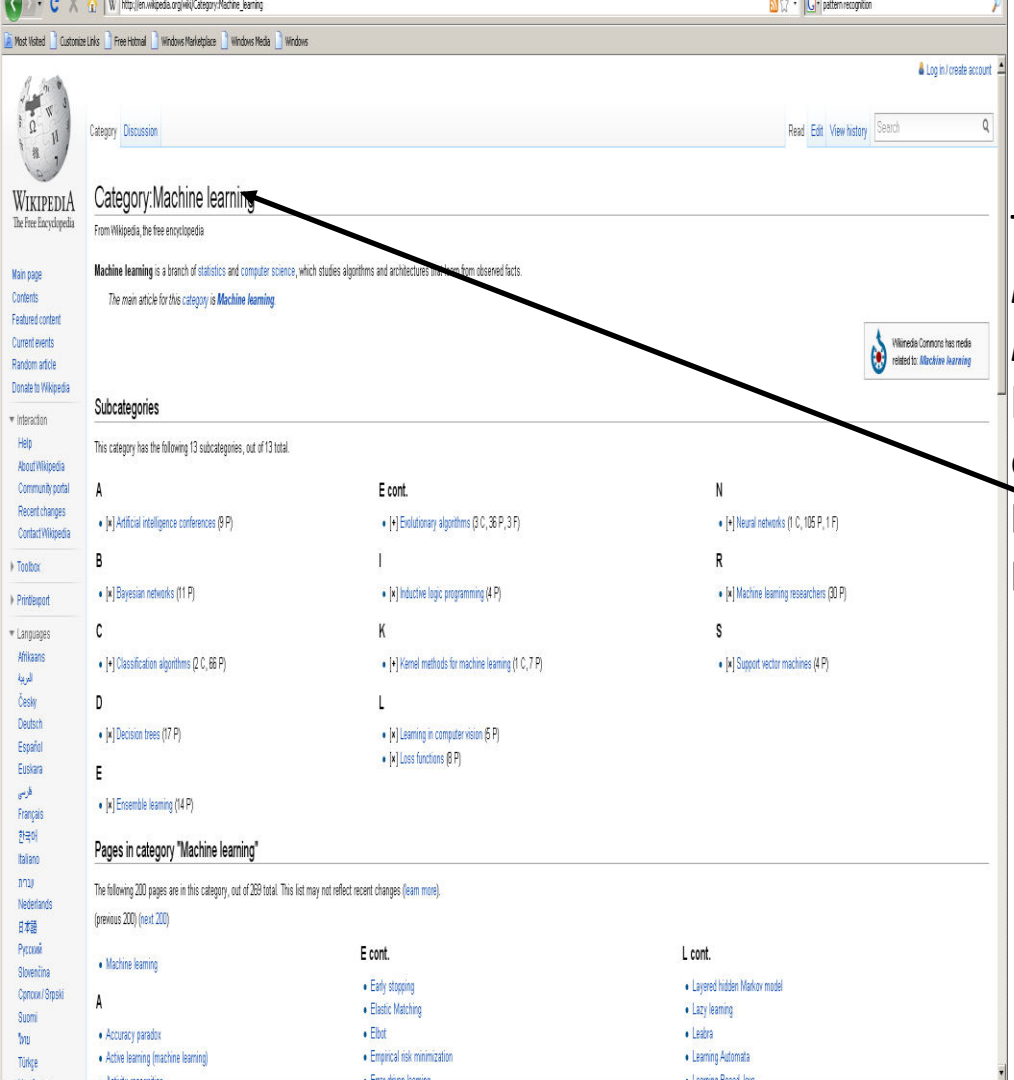
Pattern recognition is studied in many fields, including psychology, psychiatry, ethology, cognitive science, traffic flow and computer science.

Contents [hide]

1 Overview
2 Problem statement (supervised version)
3 Uses
4 Algorithms
4.1 Classification algorithms (supervised algorithms predicting categorical labels)
4.2 Clustering algorithms (unsupervised algorithms predicting categorical labels)
4.3 Regression algorithms (predicting real-valued labels)
4.4 Categorical sequence labeling algorithms (predicting sequences of categorical labels)
4.5 Real-valued sequence labeling algorithms (predicting sequences of real-valued labels)
4.6 Parsing algorithms (predicting tree structured labels)
4.7 General algorithms for predicting arbitrarily-structured labels
4.8 Ensemble learning algorithms (supervised meta-algorithms for combining multiple learning algorithms together)
5 See also
6 References
7 Further reading
8 External links

Overview

Pattern recognition is generally categorized according to the type of learning procedure used to generate the output value. *Supervised learning* assumes that a set of *training data* (the *training set*) has been provided, consisting of a set of instances that have been properly labeled by hand v



Category:Machine learning

From Wikipedia, the free encyclopedia

Machine learning is a branch of statistics and computer science, which studies algorithms and architectures that learn from observed facts.

The main article for this category is ***Machine learning***

Subcategories

This category has the following 13 subcategories, out of 13 total.

A
- [+] Artificial intelligence conferences (9 P)

B
- [+] Bayesian networks (11 P)

C
- [+] Classification algorithms (2 C, 66 P)

D
- [+] Decision trees (17 P)

E
- [+] Ensemble learning (14 P)

E cont.
- [+] Evolutionary algorithms (3 C, 36 P, 3 F)

I
- [+] Inductive logic programming (4 P)

K
- [+] Kernel methods for machine learning (1 C, 7 P)

L
- [+] Learning in computer vision (5 P)
- [+] Loss functions (8 P)

N
- [+] Neural networks (1 C, 105 P, 1 F)

R
- [+] Machine learning researchers (30 P)

S
- [+] Support vector machines (4 P)

Pages in category "Machine learning"

The following 200 pages are in this category, out of 369 total. This list may not reflect recent changes (learn more).

(previous 200) (next 200)

- Machine learning

A
- Accuracy paradox
- Active learning (machine learning)

E cont.
- Early stopping
- Elastic Matching
- Elbot
- Empirical risk minimization

L cont.
- Layered hidden Markov model
- Lazy learning
- Leabra
- Learning Automata

The concept ***Pattern Recognition*** belongs to two categories: **Machine Learning**, **Formal sciences**

The different meanings that **Pattern Recognition** may refer to are listed in its disambiguation page.

Pattern recognition (disambiguation) - Wikipedia, the free encyclopedia - Mozilla Firefox

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http://en.wikipedia.org/wiki/Pattern_recognition_(disambiguation)

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Pattern recognition (disambiguation)

From Wikipedia, the free encyclopedia

Pattern recognition is a field about machine learning.

Pattern recognition may also refer to:

- Pattern recognition (psychology)
- *Pattern Recognition* (novel), a novel by William Gibson
- *Pattern Recognition*, an album by Sea Scouts
- "Pattern Recognition", a song by Sonic Youth from *Sonic Nurse*

This disambiguation page lists articles associated with the same title.
If an internal link led you here, you may wish to change the link to point directly to the intended article.

Categories: Disambiguation pages

This page was last modified on 12 April 2011 at 11:50.

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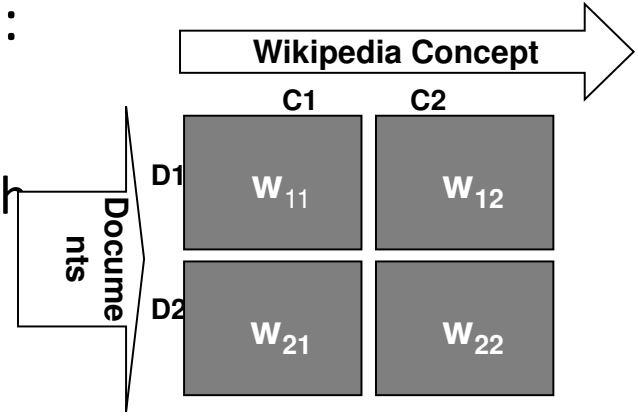
Concept Mapping

- Match documents to Wikipedia concepts.
- Result: A document-concept Matrix
- Matching Schemes:
  - Exact Match
  - Relatedness Match



d1

Pattern Recognition, or **PR** for short, is a combination of **computer science**, **classification**, and **philosophy**. …

Exact match

Wikipedia concept dictionary

Pattern Recognition, PR
…
computer science
…
classification
…
philosophy
…

document-concept Vector

	Wiki Concepts C							
	Pattern Recognition	**…**	**computer science**	**…**	**classification**	**…**	**philosophy**	**…**
d1	2	…	1	…	1	…	1	…

Concept Mapping Schemes: Exact Match

- Each document is scanned to find Wikipedia concepts (article titles).
- The searched Wikipedia concepts are used to comprise the concept vector of the corresponding document.
- Synonymous phrases to the same concept are grouped together through the redirect links in Wikipedia.

Concept Mapping Schemes: Exact Match

- **A dictionary** is constructed, with each entry corresponding to a topic covered by Wikipedia.
- Each entry includes not only the **preferred Wikipedia concept** which is used as the title of the article, but also all the **redirected concepts** representing the same topic.
- Based on the dictionary, both preferred concepts and redirected concepts are retrieved from documents.
- Only preferred concepts are used to build the **concept vector for each document**. The **weight** of each preferred concept equals to the frequency of itself plus the frequencies of all the redirected concepts appearing in a document.
- The document-concept TFIDF matrix is further calculated based on the document-concept frequency matrix.

Pattern Recognition, or **PR** for short, is a combination of **computer science**, **classification**, and **philosophy**. …

term vector of document d1

terms *W*

	w_1	w_2	…	w_N
d1	tfidf11	tfidf12		tfidf1N

×

Wiki article terms *W*

Wikipedia term concept matrix

Wiki concepts C

	c_1	c_2	…	c_M
w_1	$tfidf_{11}$	$tfidf_{12}$	…	$tfidf_{1N}$
w_2	$tfidf_{21}$	$tfidf_{22}$	…	$tfidf_{2N}$
…	…	…	…	…
w_N	$tfidf_{N1}$	$tfidf_{N2}$	…	$tfidf_{NM}$

document-concept Vector

Wiki concepts C

	c_1	c_2	…	c_M
d1	r_{11}	r_{12}		r_{1N}

Concept Mapping: Relatedness Match

Step 1: A **Wikipedia term-concept matrix** is constructed from Wikipedia article collection.

		Wiki concepts C			
		c_1	c_2	...	c_N
Wiki article terms *W*	w_1	val_{11}	val_{12}	...	val_{1N}
	w_2	val_{21}	val_{22}	...	val_{2N}
	...	...	...	...	...
	w_N	val_{N1}	val_{N2}	...	val_{NN}

- The values in the matrix are TFIDF scores, which denote the relatedness of each term to each Wikipedia concept article.
- For each word, only top *k* concepts with highest *TFIDF* scores are selected. (For example, k = 5)

Concept Mapping: Relatedness Match

Step 2: The word-concept matrix is used as a bridge to build **Document-Wikipedia concept Matrix**. The relatedness of a Wikipedia concept c_k to a given document d_j is calculated as:

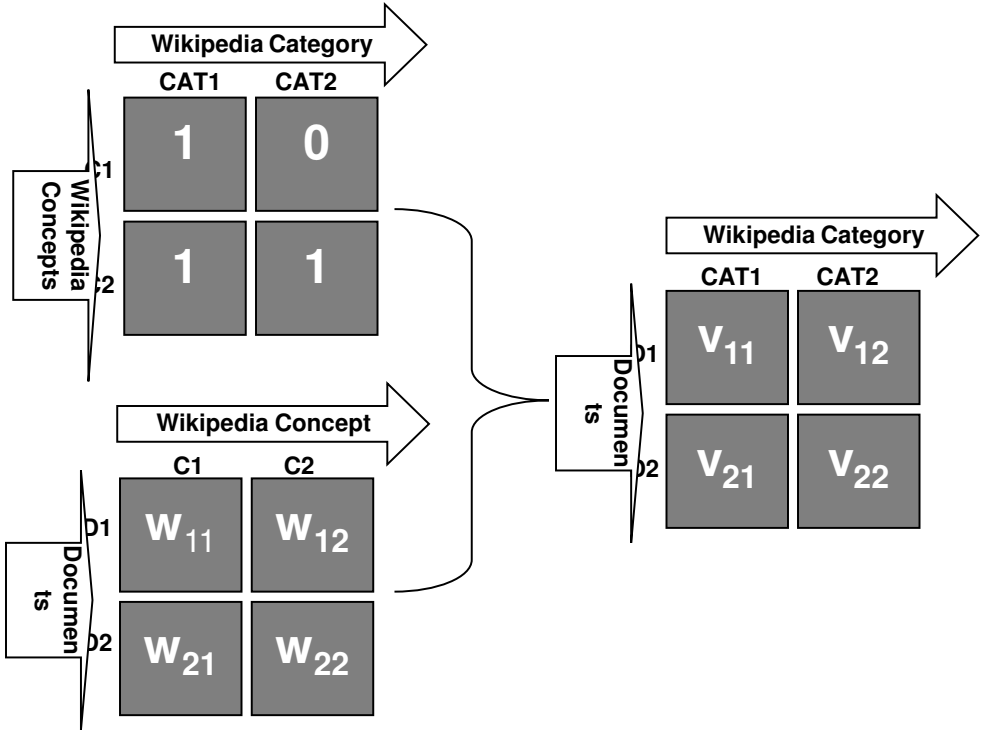
$$r_k^{d_j} = \sum tfidf_{d_j}^{w_i} \, tfidf_{c_k}^{w_i}$$

$tfidf_{d_j}^{w_i}$: the TFIDF score of word w_i in document d_j ; representing the importance of w_i to d_j

$tfidf_{c_k}^{w_i}$: the TFIDF score of word w_i in concept c_k ; representing the importance of w_i to c_k

- For each document, only the top *M* concepts with highest relatedness score are selected. (For example, M is 200)

Category Mapping



Concept Mapping
(Exact Match or Relatedness Match)

The hierarchical categorization system of Wikipedia

document-concept Vector

Wiki concepts C

	c_1	c_2	…	c_M
d1	r_{11}	r_{12}		r_{1N}

Wiki Concept-Category Matrix

Wiki categories *CAT*

Wiki Concepts C	cat_1	cat_2	…	cat_k
c_1	1	0	…	0
c_2	0	0	…	1
…	…	…	…	…
c_N	0	1	…	0

document-category Vector

Wiki categories *CAT*

	cat_1	cat_2	…	cat_k
d1	v_{11}	v_{12}		v_{1k}

Category Mapping

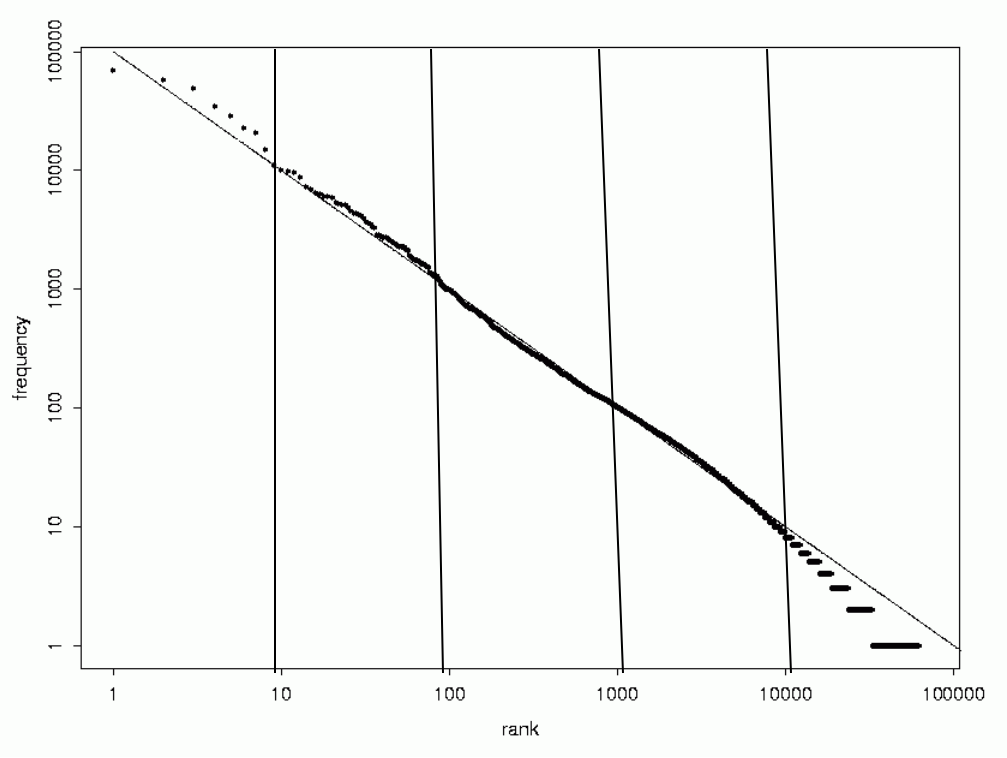
- Exact match:
  - A **document-category frequency matrix** is first derived from the **document-concept frequency matrix** by replacing each concept with its corresponding categories.
  - If a category is mapped to a document through more than one concept, the sum of the frequencies of these concepts is the category's frequency.
  - Based on the generated document-category frequency matrix, we further derive the **document-category TFIDF matrix**

Zipf's Law

- Look at the frequency of English words:
  - "the" is the most common, followed by "of", "to", etc.
  - claim: frequency of the n-th most common ~ 1/n (power law, $\alpha = 1$)
- General theme:
  - *rank* events by their *frequency of occurrence*
  - resulting distribution often is a power law!
- Other examples:
  - North America city sizes
  - personal income
  - file sizes
  - genus sizes (number of species)
- People seem to dither over exact form of these distributions (e.g. value of α), but not heavy tails

- Grows (but more slowly) with corpus size
- <u>Heap's Law</u>: The vocabulary size is
  - $m = kT^b$
- where $b \approx 0.5$, $k \approx 30–100$; T = # tokens
- m is decreased by case-folding, stemming
- Indexing all numbers could make it extremely large (so usually don't)
- Spelling errors contribute a fair bit of size
- <u>Zipf's Law:</u> The kth most frequent term has frequency proportional to $1/k$.

Zipf's law log-log plot



Rough analysis

- The *i* th most frequent term has frequency proportional to 1/*i*
- Let this frequency be *c/i.*
- Then $\sum c/i = c \log c$
- The number of terms is *c.*
- Also, the frequency of the most frequent term is *c.*
- *Remove the frequently occurring terms; how about Very Low Freq. terms?*
- We have grouped into log *c* groups; *c* words in each group.
- For dimensionality reduction, only a small number of groups are adequate.

Results on Cade2 Data

- # Documents: 1504 # Terms: 25893

Method	SVM	KNNC	DT
Unsupervised + MI (90 terms)	**79.29**	75.39	53.51
LSI (100 topics)	61.71	69.33	**75.19**
LDA (5 topics)	79.07	**86.32**	72.66
NMF (10 topics)	71.87	85.35	61.32
Random Projections	68.75	67.38	49.69

Results on NG2 Data

- # Documents: 1908 # Terms: 15566

Method	SVM	KNNC	DT
Unsupervised + MI (90 terms)	**97.77**	95.02	94.49
LSI (2 topics)	52.29	92.39	83.87
LDA (5 topics)	97.64	**97.64**	**95.28**
NMF (100 topics)	79.55	91.74	52.03
Random Projections	85.84	79.55	61.07

Results on R2 Data

- # Documents: 6215 # Terms: 14296

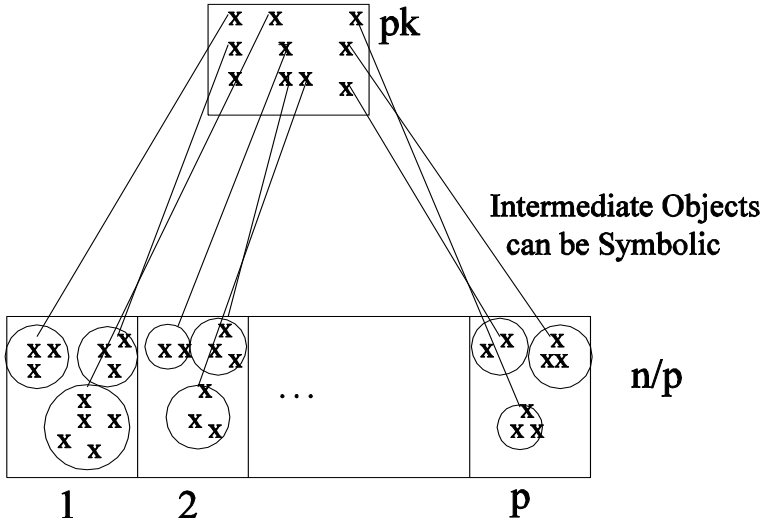
Method	SVM	KNNC	DT
Unsupervised + MI (90 terms)	**98.59**	**97.13**	93.98
LSI (100 topics)	72.51	95.39	**94.15**
LDA (100 topics)	97.86	94.32	92.07
NMF (5 topics)	94.15	71.72	59.61
Random Projections	94.88	88.92	81.90

Results on WebKB2 Data

- # Documents: 2739 # Terms: 7434

Method	SVM	KNNC	DT
Unsupervised + MI (90 terms)	**91.42**	75.82	**75.60**
LSI (100 topics)	69.56	**81.20**	69.89
LDA (100topics)	81.77	78.48	67.72
NMF (2 topics)	60.13	60.35	59.12
Random Projections	81.09	72.19	61.53

Method	Cade2	NG2	R2	WebKB2
Unsupervised + MI	2.7178	2.8964	5.5748	3.4527
LSI	17.6605	12.4350	18.6004	8.1939
LDA	252.192	447.332	593.821	517.159
NMF	1070	697	2765	523
Random Projections	4.2768	2.9449	8.1740	1.4567



1. Divide the set into p blocks, each of size n/p patterns.
2. Cluster patterns in each block into K clusters.
3. Collect the pK representatives and cluster them into K clusters.
4. Relabel the clusters obtained in step 2.

- What is the optimal number of blocks (P)?

# Patterns	# Blocks	# Clusters	# Distances (1-level)	# Distns. (2-level)
100	2	5	4950	2495
1000	10	5	499,500	50,525
10000	100	5	4,99,95,000	619,750

  - Select based on the memory available
  - Formally solve the associated optimization problem

- It is possible to show that the two-level algorithm gives a partition that is a constant factor approximation of the partition of one-level algorithm in terms of quality (fitness or accuracy).

(References: Murty and Krishna, Efficient Technique for Data Clustering, Pattern Recognition, 1980; Guha, Rastogi, Motwani, and O'Callaghan, Clustering Data Streams, FOCS 2000.)

What is MapReduce?

- Restricted parallel programming model meant for large clusters
  - User implements Map() and Reduce()
- Parallel computing framework
  - Libraries take care of EVERYTHING else
    - Parallelization
    - Fault Tolerance
    - Data Distribution
    - Load Balancing
- Useful model for many practical tasks

Map and Reduce

- Functions borrowed from functional programming languages (eg. Lisp)
- Map()
  - Process a key/value pair to generate intermediate key/value pairs
- Reduce()
  - Merge all intermediate values associated with the same key

Map

```
(define (square n)
(* n n))
 (map square '(1 2 3 4 5)) => (1 4 9 16 25)
```

Reduce

```
(define (plus a b)
   (+ a b))
(foldl plus 0 '(1 2 3 4 5)) => 15
   (5+...(2+(1+0)))
```

Example 1

- Suppose we want to count the number of occurrences of each word in a bunch of documents. For example,
- Doc 1: the weather is good
- Doc 2: today is good
- Doc 3: good weather is good.
- Map processes many pairs of the form

  <filename, contents>
- It generates many pairs of the form <word, 1> (Or <word, count> if you prefer...)

Word Index

- Map: emit <word, url> for each word in a given web document
- Reduce: combine all the <word, url> pairs associated with a given word into a single <word, url_list>. We can also sort the urls in the list, e.g., by word frequency

Example: Counting Words

- Map()
  - Input <filename, file text>
  - Parses file and emits <word, count> pairs
    - eg. <”hello”, 1>
- Reduce()
  - Sums all values for the same key and emits <word, TotalCount>
    - eg. <”hello”, (3 5 2 7)> => <”hello”, 17>

- Reduce processes all of the <word, count> pairs generated by the Map process for each word
  - Then it generates a single <word, total> for that word

```
void map (string url, string contents)
{
    foreach (word w in contents) {
        EmitIntermediate (word, 1);
    }
}
```

Example: Reduce

```
void reduce (string word, list values) {
    int total = 0;
    foreach (v in values) {
        total += (int) v;
    }
    Emit (word, total);
}
```

MapReduce

- Initially, the system selects M and R, the number of tasks to run map and reduce, respectively
- Then the input file(s) are split into M pieces, one for each map task
- One master task is in charge of submitting the M map and R reduce jobs to the server farm

How MapReduce Works

- User to do list:
  - indicate:
    - Input/output files
    - **M**: number of map tasks
    - **R**: number of reduce tasks
    - **W**: number of machines
  - Write *map* and *reduce* functions
  - Submit the job
- This requires no knowledge of parallel/distributed systems!!!

The Map Workers

- A map worker processes its portion of the input file and hands it to the user-defined map function
- As the user-defined map function generates <key,value> pairs, they are buffered in memory
- Periodically, the pairs are written out to disk into R intermediate datafiles (chosen by hash(key)%R)

The Map Workers

- When a map worker is finished, it will have produced R output files
- Then it notifies the master task that it is finished
- Note: the output files are kept in the worker machine
- If there are any more map tasks, the worker will be assigned another such task by the master

Map output

- Doc 1: the weather is good
- Doc 2: today is good
- Doc 3: good weather is good.
- Worker 1:
  - (the 1), (weather 1), (is 1), (good 1).
- Worker 2:
  - (today 1), (is 1), (good 1).
- Worker 3:
  - (good 1), (weather 1), (is 1), (good 1).

The Reduce Workers

- A reduce worker is notified when an intermediate data file is ready
- It requests the file from the map worker, and the file is transferred using a special protocol
- When all M intermediate files for a given reduce task are finished, the reduce task combines them into a <key, value_list> structure (by sorting on the keys first)

Reduce Input

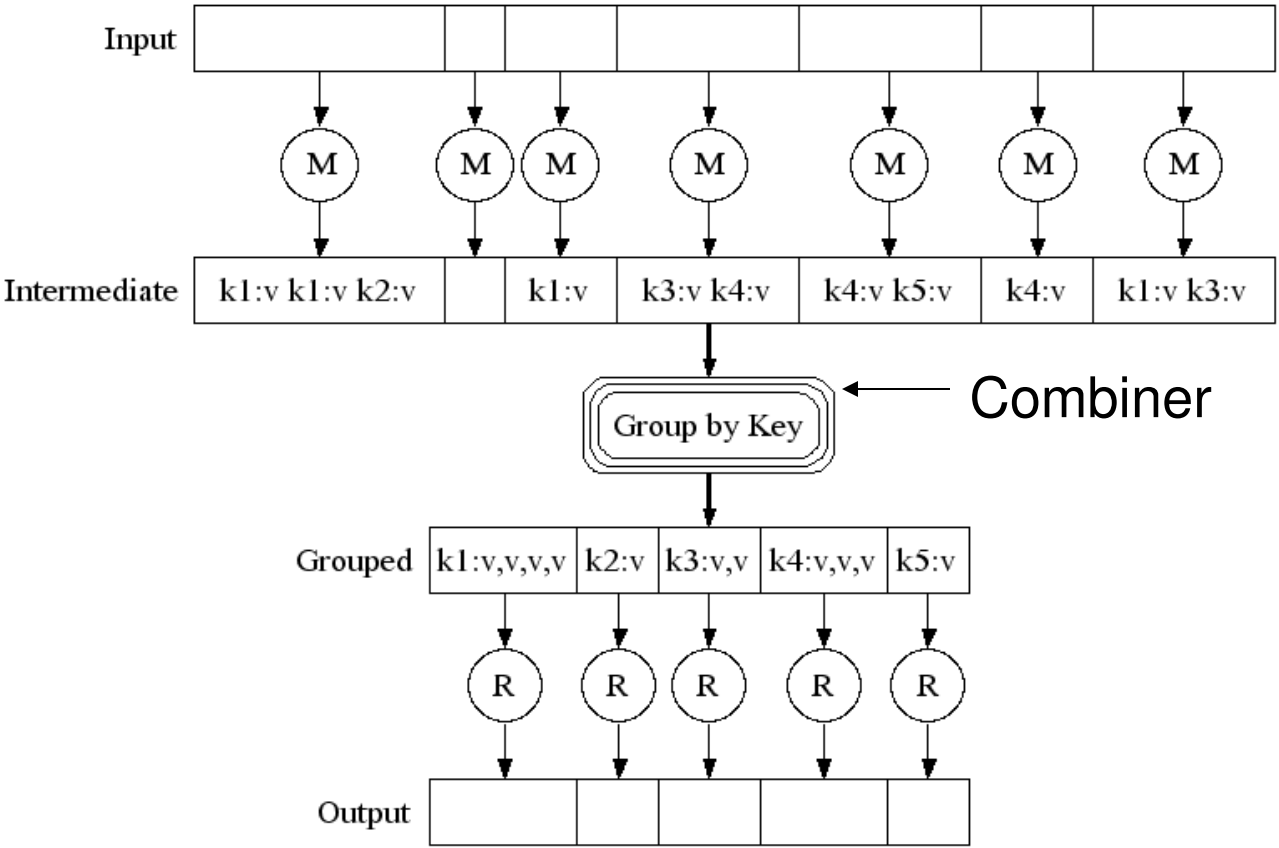
- Worker 1:
  - (the 1)
- Worker 2:
  - (is 1), (is 1), (is 1)
- Worker 3:
  - (weather 1), (weather 1)
- Worker 4:
  - (today 1)
- Worker 5:
  - (good 1), (good 1), (good 1), (good 1)

The Reduce Workers

- Once the <key, value_list> pairs are generated, the user-defined reduce function is called on each
- The workers append the output generated by the user-defined reduce function into a result file
- When the worker is finished, it notifies the master (which may give it another reduce task)

Reduce Output

- Worker 1:
  - (the 1)
- Worker 2:
  - (is 3)
- Worker 3:
  - (weather 2)
- Worker 4:
  - (today 1)
- Worker 5:
  - (good 4)


Input
M
M
M
M
M
M
M
Intermediate
k1:v k1:v k2:v
k1:v
k3:v k4:v
k4:v k5:v
k4:v
k1:v k3:v
Group by Key
Combiner
Grouped
k1:v,v,v,v
k2:v
k3:v,v
k4:v,v,v
k5:v
R
R
R
R
R
Output

Other Applications: grep

- Map: emit a <file, line> record whenever the line matches the pattern
- Reduce: do nothing! Pass through the intermediate <file, line> records

Web Link Graph

- Map: emit record <source, target> for each link found in the source (going to target)
- Reduce: collect all the target links from a given source into a single <source, target_list> record

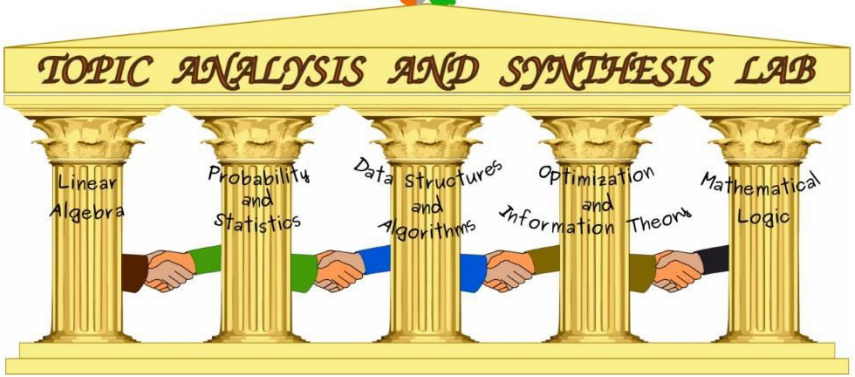
Reverse Link Graph

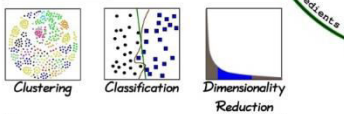
- Map: emit record <target, source> for each link found in the source (going to target)
- Reduce: collect all the sources to a given target into a single <target, source_list> record
  - Note: This is the same reduce as before...

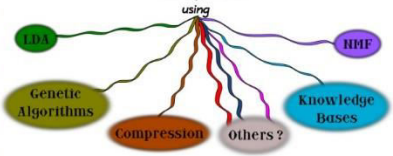
Research Directions

- Computing the eigenvalues and eigenvectors of a covariance matrix of size 100,000 x 100,000 may not be practical.
- Dimensionality reduction is important; can we ignore stop words and rare words, both?
- Relation between NMF and frequent patterns
- Topics are clusters; can we use efficient clustering schemes?
- Naïve Bayes classifier is popular. How far can we stretch it?
- How to improve the speed of GAs?
- Are heavy-tail or power-law distributions important in classification?
- What are efficient ways of summarizing documents? LDA + HMM?
- Wikipedia + LDA worked well on academic researchers data set. How do we make LDA more semantic?

TOPIC ANALYSIS AND SYNTHESIS LAB







References

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(some of the slides are taken from the lecture notes of Dr. P. Raghavan)

- Han and Kamber, *Data Mining*, Elsevier, 2008.
- Jing, Ng, and Huang, Knowledge-based vector space model for text clustering, *Knowl Inf Syst*, 25:35-55, 2010.
- Hu *et al*, Exploiting Wikipedia as External Knowledge for Document Clustering, KDD 2009.(some slides are from the presentation file)