Matrix Norms

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We consider matrix norms on $(\mathbb{C}^{m,n}, \mathbb{C})$. All results holds for $(\mathbb{R}^{m,n}, \mathbb{R})$.

Definition 1 (Matrix Norms)**.** *A function* $\|\cdot\| : \mathbb{C}^{m,n} \longrightarrow \mathbb{C}$ *is called a* **matrix norm** *on* $\mathbb{C}^{m,n}$ *if for all* $A, B \in \mathbb{C}^{m,n}$ *and all* $\alpha \in \mathbb{C}$

1. $\|A\| \geq 0$ *with equality if and only if* $A = 0$. *(positivity)*
2. $\|\alpha A\| = |\alpha|\,\|A\|$. *(homogeneity)*
3. $\|A + B\| \leq \|A\| + \|B\|$. *(subadditivity)*

A matrix norm is simply a vector norm on the finite dimensional vector spaces $(\mathbb{C}^{m,n}, \mathbb{C})$ of $m \times n$ matrices.

Equivalent norms

Adapting some general results on vector norms to matrix norms give

Theorem 2.

1. *All matrix norms are equivalent. Thus, if* $\|\cdot\|$ *and* $\|\cdot\|'$ *are two matrix norms on* $\mathbb{C}^{m,n}$ *then there are positive constants* μ *and* M *such that* $\mu\|A\| \leq \|A\|' \leq M\|A\|$ *holds for all* $A \in \mathbb{C}^{m,n}$.
2. *A matrix norm is a continuous function* $\|\cdot\| : \mathbb{C}^{m,n} \longrightarrow \mathbb{R}$.

Submultiplicativity

- For matrix norms we usually require that the norm of a product is bounded by the product of the norms. Thus for square matrices $A, B \in \mathbb{C}^{n,n}$ and a matrix norm we most often have the additional property

   4. $\|AB\| \leq \|A\|\|B\|$ (submultiplicativity).

- For a square matrix A and a submultiplicative matrix norm $\|\cdot\|$ we have

$$\|A^k\| \leq \|A\|^k \text{ for } k \in \mathbb{N}. \tag{1}$$

Consistent Matrix norms

When m and n vary we have a family of norms which are formally different for each m and n since they are defined in different spaces. However, the most common matrix norms are defined by the same formula for all m, n and we consider mainly such norms.

Definition 3 (Consistent Matrix Norms)**.** *A submultiplicative matrix norm which is defined for all $m, n \in \mathbb{N}$, is said to be a consistent matrix norm.*

The Frobenius Matrix Norm

- For $A \in \mathbb{C}^{m,n}$ we define the Frobenius norm by

$$\|A\|_F := \Big(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\Big)^{1/2}.$$

- $\|\boldsymbol{A}\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$ (singular values of $\boldsymbol{A}$.)
- The Frobenius norm is a consistent matrix norm which is subordinate to the Euclidian vector norm.

Subordinate Matrix Norm

A matrix norm $\| \;\|$ on $\mathbb{C}^{m,n}$ is subordinate to the vector norms $\| \;\|_\alpha$ on $\mathbb{C}^n$ and $\| \;\|_\beta$ on $\mathbb{C}^m$ if

$$\|A\boldsymbol{x}\|_\beta \leq \|A\|\|\boldsymbol{x}\|_\alpha \text{ for all } A \in \mathbb{C}^{m,n} \text{ and } \boldsymbol{x} \in \mathbb{C}^n.$$

Operator Norm

Definition 4. *Suppose* $m, n \in \mathbb{N}$ *are given and let* $\|\cdot\|_\alpha$ *be a vector norm on* $\mathbb{C}^n$ *and* $\|\cdot\|_\beta$ *a vector norm on* $\mathbb{C}^m$*. For* $A \in \mathbb{C}^{m,n}$ *we define*

$$\|A\| := \|A\|_{\alpha,\beta} := \max_{\boldsymbol{x} \neq 0} \frac{\|A\boldsymbol{x}\|_\beta}{\|\boldsymbol{x}\|_\alpha}. \tag{2}$$

We call this the (α, β) **operator norm***, the* (α, β)*-norm, or simply the* α*-norm if* $\alpha = \beta$*.*

Operator norm properties

The operator norm has the following properties:

- It is a matrix norm
- It is subordinate to the vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$.
- It is consistent if the vector norms $\|\cdot\|_\alpha = \|\cdot\|_\beta$ and they are defined for all m, n.
- There is some $\boldsymbol{x}^* \in \mathbb{C}^n$ with $\|\boldsymbol{x}^*\|_\alpha = 1$ such that

$$\|A\| = \max_{\|\boldsymbol{x}\|_\alpha = 1} \|A\boldsymbol{x}\|_\beta = \|A\boldsymbol{x}^*\|_\beta.$$

The p matrix norm

- The operator norms $\|\cdot\|_p$ defined from the p-vector norms are of special interest.
- We define

$$\|A\|_p := \max_{\boldsymbol{x}\neq 0} \frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p} = \max_{\|\boldsymbol{y}\|_p=1} \|A\boldsymbol{y}\|_p. \tag{3}$$

- The p-norms are consistent matrix norms which are subordinate to the p-vector norm.

Explicit expressions

- For $A \in \mathbb{C}^{m,n}$ we have:
- $\|A\|_1 = \max_{1 \le j \le n} \sum_{k=1}^{m} |a_{k,j}|$
- $\|A\|_2 = \sigma_1$, the largest singular value of A
- $\|A\|_\infty = \max_{1 \le k \le m} \sum_{j=1}^{m} |a_{k,j}|$
- If $A \in \mathbb{C}^{n,n}$ is nonsingular then $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$, the smallest singular value of A.
- Proof:

Unitary Transformations

- An important property of the 2-norm is that it is invariant with respect to unitary transformations.
- Let $k, m, n \in \mathbb{N}$, $V \in \mathbb{C}^{k,m}$, $U \in \mathbb{C}^{n,n}$, $A \in \mathbb{C}^{m,n}$, $V^H V = I$ and $U^H U = I$. Then
- 1. $\|VA\|_2 = \|A\|_2$ and $\|V\|_2 = 1$,

  2. $\|AU\|_2 = \|A\|_2$.
- Proof:

Example

- $A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- $\|A\|_1 = 6$
- $\|A\|_2 = 5.465$
- $\|A\|_\infty = 7.$
- $\|A\|_F = 5.4772$

Perturbation of linear systems

- Consider the system of two linear equations

$$\begin{aligned} x_1 \quad &+x_2 &= \quad 20 \\ x_1 \quad &+0.999x_2 &= \quad 19.99 \end{aligned}$$

- The exact solution is $x_1 = x_2 = 10$.
- Suppose we replace the second equation by

$$x_1 + 1.001x_2 = 19.99,$$

- the exact solution changes to $x_1 = 30$, $x_2 = -10$.
- A small change in one of the coefficients, from 0.999 to 1.001, changed the exact solution by a large amount.

Ill Conditioning

- A mathematical problem in which the solution is very sensitive to changes in the data is called **ill-conditioned** or sometimes **ill-posed.**
- Such problems are difficult to solve on a computer.
- If at all possible, the mathematical model should be changed to obtain a more well-conditioned or properly-posed problem.

Perturbations

- We consider what effect a small change (perturbation) in the data A,$\boldsymbol{b}$ has on the solution $\boldsymbol{x}$ of a linear system $A\boldsymbol{x} = \boldsymbol{b}$.
- Suppose $\boldsymbol{y}$ solves $(A + E)\boldsymbol{y} = \boldsymbol{b}+\boldsymbol{e}$ where E is a (small) $n \times n$ matrix and $\boldsymbol{e}$ a (small) vector.
- How large can $\boldsymbol{y}-\boldsymbol{x}$ be?
- To measure this we use vector and matrix norms.

Conditions on the norms

- $\|\cdot\|$ will denote a vector norm on $\mathbb{C}^n$ and also a submultiplicative matrix norm on $\mathbb{C}^{n,n}$ which in addition is subordinate to the vector norm.
- Thus for any $A, B \in \mathbb{C}^{n,n}$ and any $\boldsymbol{x} \in \mathbb{C}^n$ we have

$$\|AB\| \leq \|A\| \, \|B\| \text{ and } \|A\boldsymbol{x}\| \leq \|A\| \, \|\boldsymbol{x}\|.$$

- This is satisfied if the matrix norm is the operator norm corresponding to the given vector norm or the Frobenius norm.

Absolute and relative error

- The difference $\|y - x\|$ measures the absolute error in y as an approximation to x,
- $\|y - x\|/\|x\|$ or $\|y - x\|/\|y\|$ is a measure for the relative error.

Perturbation in the right hand side

Theorem 5. *Suppose* $A \in \mathbb{C}^{n,n}$ *is invertible,* $\boldsymbol{b}, \boldsymbol{e} \in \mathbb{C}^n$*,* $\boldsymbol{b} \neq \boldsymbol{0}$ *and* $A\boldsymbol{x} = \boldsymbol{b}$*,* $A\boldsymbol{y} = \boldsymbol{b} + \boldsymbol{e}$*. Then*

$$\frac{1}{K(A)} \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{b}\|} \leq \frac{\|\boldsymbol{y} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq K(A) \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{b}\|}, \qquad K(A) = \|A\| \|A^{-1}\|. \tag{4}$$

- Proof:
- Consider (4). $\|\boldsymbol{e}\| / \|\boldsymbol{b}\|$ is a measure for the size of the perturbation $\boldsymbol{e}$ relative to the size of $\boldsymbol{b}$. $\|\boldsymbol{y} - \boldsymbol{x}\| / \|\boldsymbol{x}\|$ can in the worst case be

$$K(A) = \|A\| \|A^{-1}\|$$

times as large as $\|\boldsymbol{e}\| / \|\boldsymbol{b}\|$.

Condition number

- $K(A)$ is called the **condition number with respect to inversion of a matrix**, or just the condition number, if it is clear from the context that we are talking about solving linear systems.
- The condition number depends on the matrix A and on the norm used. If $K(A)$ is large, A is called **ill-conditioned** (with respect to inversion).
- If $K(A)$ is small, A is called **well-conditioned** (with respect to inversion).

Condition number properties

- Since $\|A\|\|A^{-1}\| \geq \|AA^{-1}\| = \|I\| \geq 1$ we always have $K(A) \geq 1$.
- Since all matrix norms are equivalent, the dependence of $K(A)$ on the norm chosen is less important than the dependence on A.
- Usually one chooses the spectral norm when discussing properties of the condition number, and the l_1 and l_∞ norm when one wishes to compute it or estimate it.

The 2-norm

- Suppose $\boldsymbol{A}$ has singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ if $\boldsymbol{A}$ is square.
- $K_2(\boldsymbol{A}) = \|\boldsymbol{A}\|_2 \|\boldsymbol{A}^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$
- $K_2(\boldsymbol{A}) = \|\boldsymbol{A}\|_2 \|\boldsymbol{A}^{-1}\|_2 = \frac{|\lambda_1|}{|\lambda_n|}, \quad \boldsymbol{A}$ normal.
- It follows that $\boldsymbol{A}$ is ill-conditioned with respect to inversion if and only if σ_1/σ_n is large, or $|\lambda_1|/|\lambda_n|$ is large when $\boldsymbol{A}$ is normal.
- $K_2(\boldsymbol{A}) = \|\boldsymbol{A}\|_2 \|\boldsymbol{A}^{-1}\|_2 = \frac{\lambda_1}{\lambda_n}, \quad \boldsymbol{A}$ positive definite.

The residual

Suppose we have computed an approximate solution $\boldsymbol{y}$ to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. The vector $\boldsymbol{r}(\boldsymbol{y} :) = \boldsymbol{A}\boldsymbol{y} - \boldsymbol{b}$ is called the residual vector , or just the residual. We can bound $\boldsymbol{x} - \boldsymbol{y}$ in term of $\boldsymbol{r}(\boldsymbol{y})$.

Theorem 6. *Suppose $\boldsymbol{A} \in \mathbb{C}^{n,n}$, $\boldsymbol{b} \in \mathbb{C}^n$, $\boldsymbol{A}$ is nonsingular and $\boldsymbol{b} \neq \boldsymbol{0}$. Let $\boldsymbol{r}(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{y} - \boldsymbol{b}$ for each $\boldsymbol{y} \in \mathbb{C}^n$. If $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ then*

$$\frac{1}{K(\boldsymbol{A})} \frac{\|\boldsymbol{r}(\boldsymbol{y})\|}{\|\boldsymbol{b}\|} \leq \frac{\|\boldsymbol{y} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq K(\boldsymbol{A}) \frac{\|\boldsymbol{r}(\boldsymbol{y})\|}{\|\boldsymbol{b}\|}. \tag{5}$$

Discussion

- If $\boldsymbol{A}$ is well-conditioned, (5) says that $\|\boldsymbol{y}-\boldsymbol{x}\|/\|\boldsymbol{x}\| \approx \|\boldsymbol{r}(\boldsymbol{y})\|/\|\boldsymbol{b}\|$.
- In other words, the accuracy in $\boldsymbol{y}$ is about the same order of magnitude as the residual as long as $\|\boldsymbol{b}\| \approx 1$.
- If $\boldsymbol{A}$ is ill-conditioned, anything can happen.
- The solution can be inaccurate even if the residual is small
- We can have an accurate solution even if the residual is large.

Perturbation in A

We consider next a perturbation in $\boldsymbol{A}$.

Theorem 7. *Suppose* $\boldsymbol{A}, \boldsymbol{E} \in \mathbb{C}^{n,n}$, $\boldsymbol{b} \in \mathbb{C}^n$ *with* $\boldsymbol{A}$ *invertible and* $\boldsymbol{b} \neq \boldsymbol{0}$. *If* $\|\boldsymbol{A}^{-1}\boldsymbol{E}\| < 1$ *for some operator norm then* $\boldsymbol{A}+\boldsymbol{E}$ *is invertible. If* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *and* $(\boldsymbol{A}+\boldsymbol{E})\boldsymbol{y} = \boldsymbol{b}$ *then*

$$\frac{\|\boldsymbol{y}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \frac{\|\boldsymbol{A}^{-1}\boldsymbol{E}\|}{1-\|\boldsymbol{A}^{-1}\boldsymbol{E}\|} \leq \frac{K(\boldsymbol{A})}{1-\|\boldsymbol{A}^{-1}\boldsymbol{E}\|}\frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|}. \tag{6}$$

- $\|\boldsymbol{E}\|/\|\boldsymbol{A}\|$ is a measure of the size of the perturbation $\boldsymbol{E}$ in $\boldsymbol{A}$ relative to the size of $\boldsymbol{A}$.
- The condition number again plays a crucial role.

The Spectral Radius

- We define the **spectral radius** of a matrix $\boldsymbol{A} \in \mathbb{C}^{n,n}$ as the maximum absolute values of the eigenvalues.

$$\rho(\boldsymbol{A}) = \max_{\lambda \in \sigma(\boldsymbol{A})} |\lambda|. \tag{7}$$

- For any submultiplicative matrix norm $\|\cdot\|$ on $\mathbb{C}^{n,n}$ and any $\boldsymbol{A} \in \mathbb{C}^{n,n}$ we have $\rho(\boldsymbol{A}) \le \|\boldsymbol{A}\|$.
- Proof:
- Let $\boldsymbol{A} \in \mathbb{C}^{n,n}$ and $\epsilon > 0$ be given. There is a submultiplicative matrix norm $\|\cdot\|'$ on $\mathbb{C}^{n,n}$ such that $\rho(\boldsymbol{A}) \le \|\boldsymbol{A}\|' \le \rho(\boldsymbol{A}) + \epsilon$.
- Proof:

Limits

- For any $\boldsymbol{A} \in \mathbb{C}^{n,n}$ we have

$$\lim_{k\to\infty} \boldsymbol{A}^k = 0 \iff \rho(\boldsymbol{A}) < 1.$$

- Convergence can be slow:
- $$\boldsymbol{A} = \begin{bmatrix} 0.99 & 1 & 0 \\ 0 & 0.99 & 1 \\ 0 & 0 & 0.99 \end{bmatrix}, \quad \boldsymbol{A}^{100} = \begin{bmatrix} 0.4 & 9.37 & 1849 \\ 0 & 0.4 & 37 \\ 0 & 0 & 0.4 \end{bmatrix},$$
  $$\boldsymbol{A}^{2000} = \begin{bmatrix} 10^{-9} & \epsilon & 0.004 \\ 0 & 10^{-9} & \epsilon \\ 0 & 0 & 10^{-9} \end{bmatrix}$$