Complex numbers

The complex unit i is defined to have the property that $i^2 = -1$. A complex number z is introduced to have the form

$$z = a + b\,i$$

in which a and b are real numbers: $a, b \in \mathbb{R}$. Here, the quantity a is called the *real part* of z, denoted by $a = \mathrm{Re}(z)$. The quantity b is the *imaginary part* of z, denoted by $b = \mathrm{Im}(z)$. If $\mathrm{Re}(z) = a = 0$, then $z = b\,i$ is called *purely imaginary.* If $\mathrm{Im}(z) = b = 0$, then $z = a$ and z is real. The real numbers are therefore a subset of the complex numbers. The set of all complex numbers is denoted by $\mathbb{C}$. To visualize the complex numbers, one can plot $z = a + b\,i$ as a point with coordinates (a, b) in a two-dimensional graph with a horizontal real axis and a vertical imaginary axis. This graph is called the *complex plane.*

Here are some elementary algebraic rules for computation with complex numbers.
Addition of two complex numbers $z = a + b\,i$ and $w = c + d\,i$ is defined to give $z + w = (a + c) + (b + d)\,i$. In the graph, this corresponds to adding up the vectors (a, b) and (c, d) in the usual way for vectors in $\mathbb{R}^2$.
Multiplication of two complex numbers $z = a + b\,i$ and $w = c + d\,i$ is defined to give $zw = (ac - bd) + (ad + bc)\,i$. This corresponds to working out the four terms of the product $(a + b\,i)(c + d\,i)$ as usual, and then replacing i^2 by -1.

For the exponential function with a purely imaginary exponent, we have by definition that

$$e^{\theta\,i} = \cos(\theta) + \sin(\theta)\,i.$$

This means that $e^{\theta\,i}$ is the number that appears in the complex plane on the unit circle having the angle θ. (The unit circle has the point 0 as its center and radius 1.) Note that with this definition it is not hard to prove, using the standard trigonometric identities, that:

$$e^{\theta\,i} e^{\phi\,i} = e^{(\theta+\phi)\,i}.$$

For $z = a + b\,i$ we write: $e^z = e^{a+b\,i} = e^a e^{b\,i} = e^a \cos(b) + e^a \sin(b)\,i$. Every complex number $z = a + b\,i$ can be written in '*polar form*' as follows:

$$z = re^{\theta\,i},$$

with $r = \sqrt{a^2 + b^2}$ and θ the angle of (a, b) in the complex plane (so that $\tan(\theta) = \frac{b}{a}$). The radius r is called the *modulus* of z and denoted by $r = |z|$. The angle θ is called the *argument* of z and denoted by $\theta = \arg(z)$. Note that, with such notation, multiplication of $z = re^{\theta\,i}$ and $w = se^{\phi\,i}$ gives:

$$zw = re^{\theta\,i} se^{\phi\,i} = (rs)e^{(\theta+\phi)\,i}.$$

This shows that the modulus of zw satisfies $|zw| = |z||w|$ and the argument satisfies $\arg(zw) = \arg(z) + \arg(w)$.

A useful concept is that of the *complex conjugate* of $z = a + b\,i$. It is denoted by $\overline{z}$ and defined by $\overline{z} = a - b\,i$. Note that $z\overline{z} = (a + b\,i)(a - b\,i) = a^2 + b^2 = |z|^2$, which is real and non-negative. This property can be used to perform division by a complex number and bring the result into standard form:

$$\frac{a + b\,i}{c + d\,i} = \frac{a + b\,i}{c + d\,i} \cdot \frac{c - d\,i}{c - d\,i} = \frac{(a + b\,i)(c - d\,i)}{c^2 + d^2},$$

in which the numerator can be worked out as usual and the denominator is real (making division easy).

Note also that $z = re^{\theta\,i}$ has $\overline{z} = re^{-\theta\,i}$. For addition, multiplication, and exponentiation we have the convenient rules: $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{e^z} = e^{\overline{z}}$. Also: $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2\,i\,\mathrm{Im}(z)$.

Finally, when trying to solve a polynomial equation $z^n + a_1 z^{n-1} + \ldots + a_n = 0$ of degree n (with possibly complex coefficients $a_1, \ldots, a_n$), the *fundamental theorem of algebra* that states that this can be rewritten as $(z-p_1)(z-p_2)\cdots(z-p_n) = 0$ for a unique set of n complex numbers $p_1, p_2, \ldots, p_n$. These numbers $p_1, p_2, \ldots, p_n$ are the zeros of the polynomial on the left-hand side (also called the roots of the equation); they need not all be distinct. If the coefficients of the equation are all real, then the roots of the equation are real or they come in complex conjugate pairs.