2.2

Newton's 2^{nd} law :

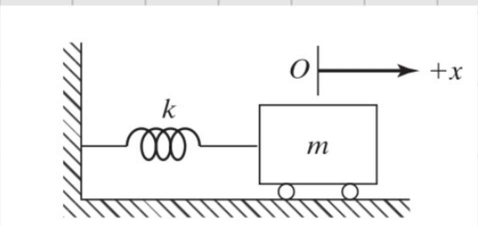
$$\vec{F}(t) = \frac{d}{dt}\left(m\frac{d\vec{x}}{dt}\right)$$

if $m = const$

$$\vec{F} = m\ddot{\vec{x}}$$

for rotational motion

$$\vec{M}(t) = J\ddot{\vec{\theta}}$$



: $F(t) = -kx = m\ddot{x}$

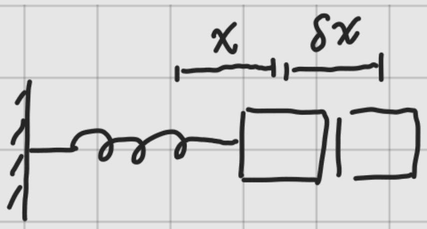
D'Alembert's Principle :

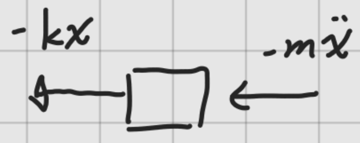
$$\vec{F}(t) - \underbrace{m\ddot{\vec{x}}}_{} = 0$$

視其為假想力 (inertia force)
(慣性力)

Principle of Virtual displacement

假設其受到一個虛位移 δx, 則總虛功 $= 0$





$$\delta W_s = -(kx)\,\delta x \quad ; \quad \delta W_i = -(m\ddot{x})\,\delta x$$

$$\delta W_s + \delta W_i = 0$$

$$\Rightarrow \quad -m\ddot{x}\,\delta x - kx\,\delta x = 0 \qquad \Rightarrow \quad +m\ddot{x} + kx = 0$$

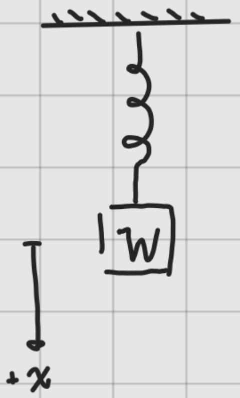
〃 of Conservation of energy

$$T + U = \text{const} \quad \Rightarrow \quad \frac{d}{dt}(T+U) = 0$$

$$T = \frac{1}{2}m\dot{x}^2 \quad , \quad U = \frac{1}{2}kx^2$$

$$\Rightarrow \quad m\ddot{x} + kx = 0$$

m



$$W = mg = k\delta_{st}$$

$$\Rightarrow \quad m\ddot{x} = -k(x+\delta_{st}) + W$$

$$\Rightarrow \quad m\ddot{x} + kx = 0$$

solution :

assuming $x(t) = Ce^{st} \rightarrow (ms^2 + k)C = 0$

$$\Rightarrow \quad s = \pm\left(\frac{-k}{m}\right)^{\frac{1}{2}} = \pm i\omega_n$$

$$\Rightarrow \quad x(t) = A_1\cos\omega_n t + A_2\sin\omega_n t$$

I.C.

$$x(t=0) = A_1 = x_0 \quad , \quad \dot{x}(t=0) = \omega_n A_2 = \dot{x}_0$$

$$\Rightarrow x(t) = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$

set $A_1 = A\cos\phi = x_0 \quad ; \quad A_2 = A\sin\phi = \frac{\dot{x}_0}{\omega_n}$

$$\Rightarrow \quad x(t) = A\cos(\omega_n t - \phi)$$

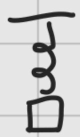
已知

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \quad , \quad W = k\,\delta_{st} \rightarrow k = \frac{mg}{\delta_{st}}$$

$$\Rightarrow \quad \omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\left(\frac{g}{\delta_{st}}\right)^{1/2} \quad ; \quad T_n = \frac{1}{f_n}$$

natural frequency



$$x(t) = A\cos(\omega_n t - \phi) \qquad \text{— ①}$$

lead

$$\dot{x}(t) = \frac{d}{dt}x(t) = \omega_n A\cos\left(\omega_n t - \phi + \frac{\pi}{2}\right) \qquad \text{— ②}$$

$$\ddot{x}(t) = \frac{d^2}{dt^2}(x(t)) = \omega_n^2 A\cos\left(\omega_n t - \phi + \pi\right)$$

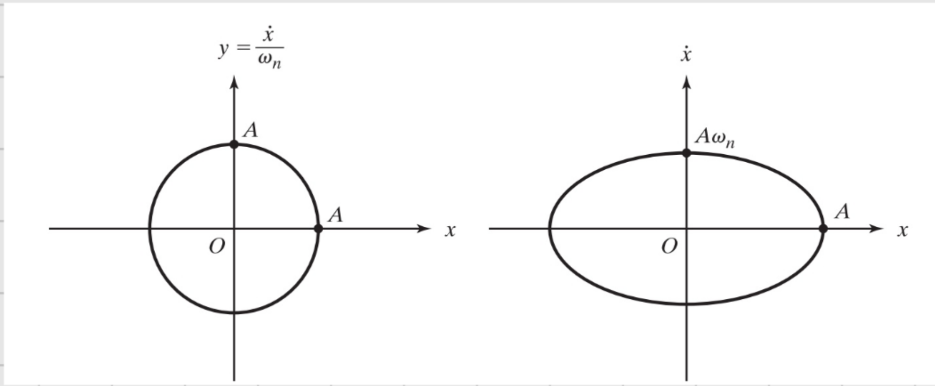
state space (phase plane) :

$$① \rightarrow \cos(\omega_n t - \phi) = \frac{x}{A} \quad ; \quad ② \rightarrow \sin(\omega_n t - \phi) = -\frac{\dot{x}}{\omega_n A} \equiv -\frac{y}{A}$$

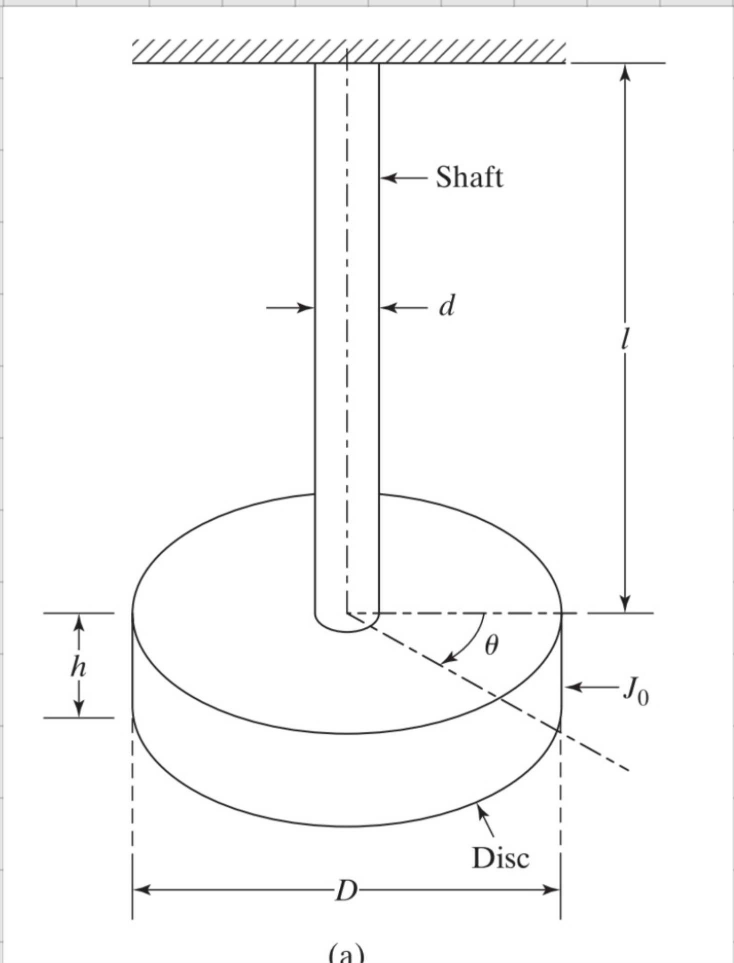
$$\because \quad \sin^2 + \cos^2 = 1 \qquad \therefore \quad \frac{x^2}{A^2} + \frac{y^2}{A^2} = 1$$

where $y = \dfrac{\dot{x}}{\omega_n}$

undamped



2.3 Undamped Torsional System



shear modulus

$$M_t = \frac{G I_o}{l}\theta$$

$$I_o = \frac{\pi}{32} d^4$$

torsional spring constant

$$k_t = \frac{M_t}{\theta} = \frac{G I_o}{l}$$

$$-M_t = J_o\ddot{\theta} \quad \Rightarrow \quad J_o\ddot{\theta} + k_t\theta = 0$$

polar

$$\Rightarrow \quad J_o s^2 + k_t = 0 \qquad s = \left(\frac{-k_t}{J_o}\right)^{\frac{1}{2}} = i\omega_n$$

$$\omega_n = \left(\frac{k_t}{J_0}\right)^{1/2} \quad , \quad f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\left(\frac{k_t}{J_0}\right)^{1/2}$$

$$\tau_n = \frac{1}{f_n} = 2\pi\left(\frac{J_0}{k_t}\right)^{1/2}$$

$$\Rightarrow \theta(t) = A_1 \cos\omega_n t + A_2 \sin\omega_n t$$

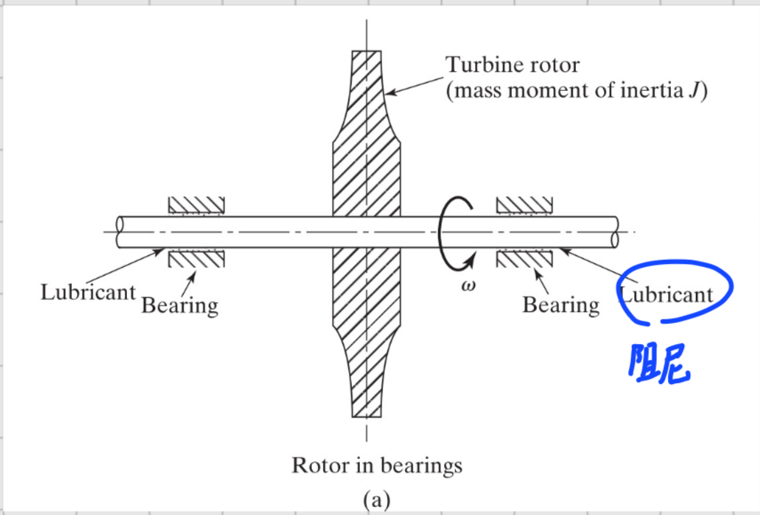
I.C.

$$\theta(t=0) = \theta_0 \quad , \quad \dot{\theta}(t=0) = \dot{\theta}_0$$

$$\Rightarrow A_1 = \theta_0 \quad , \quad A_2 = \dot{\theta}_0 / \omega_n$$

ex.

2.4 Response



$\Rightarrow J\dot{\omega} + c_t \omega = 0$

rotational damping const

sol. $\Rightarrow \omega(t) = A e^{st}$

I.C. $\omega(t=0) = \omega_0 \Rightarrow \omega(t) = \omega_0 e^{st}$ 代回

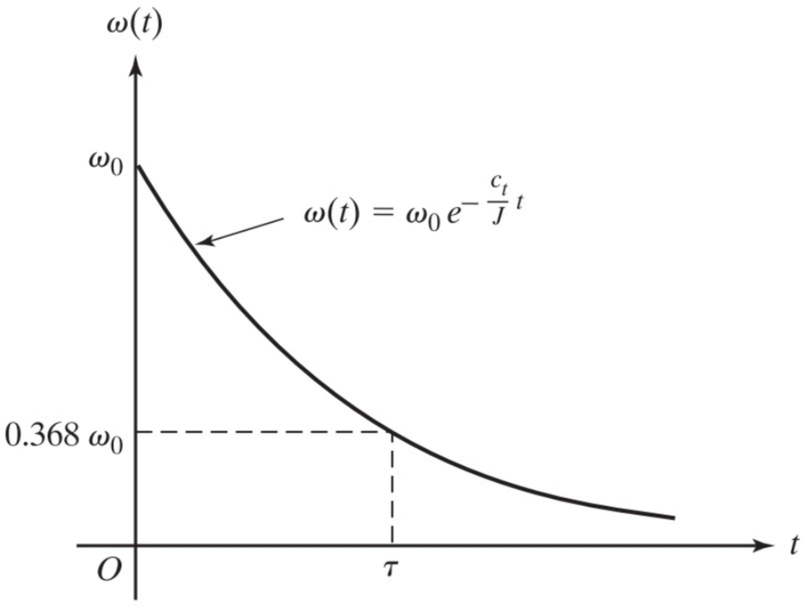
$$J\omega_0 s e^{st} + C_t \omega_0 e^{st} = 0$$

$$\Rightarrow \omega_0 e^{st} \left(Js + C_t \right) = 0$$

assume $\omega_0 \neq 0$

$$Js + C_t = 0 \Rightarrow S = -\frac{C_t}{J}$$

$$\Rightarrow \omega(t) = \omega_0 e^{-\frac{C_t}{J} t}$$



Variation of angular velocity
(b)

time constant (τ)

$$\tau = \frac{J}{c_t}$$

$$\omega(\tau) = \omega_0 e^{-1}$$

$$= 0.368\,\omega_0$$

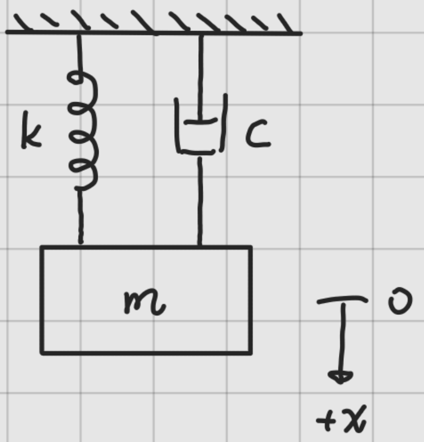
2.5

用能量法找 natural frequencies

ex.

2.6 Viscous Damping

viscous damping force : $F = \underbrace{-}_{\text{與速度方向相反}} c\dot{x}$



$$m\ddot{x} = -c\dot{x} - kx$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = 0$$

$$\rightarrow ms^2 + cs + k = 0$$

$$\rightarrow S_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$\Rightarrow x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

Critical Damping Constant C_c

$$\left(\frac{C_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\Rightarrow C_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

Damping ratio ζ zeta

$$\zeta \equiv \frac{c}{c_c}$$

$$\Rightarrow \frac{c}{2m} = \zeta \omega_n \quad , \quad \omega_n = \left(\frac{k}{m}\right)^{\frac{1}{2}}$$

$$\Rightarrow S_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$\Rightarrow x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2-1})\omega_n t}$$

Case 1. Underdamped system ($\zeta < 1$)

$$\rightarrow \zeta^2 - 1 < 0$$

$$\Rightarrow S_1 = \left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n$$

$$S_2 = \left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n$$

$$\Rightarrow x(t) = e^{-\zeta\omega_n t}\left\{C_1' \cos\sqrt{1-\zeta^2}\,\omega_n t + C_2' \sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

$$= X e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right)$$

$$X = \sqrt{C_1'^2 + C_2'^2} \quad , \quad \phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right)$$

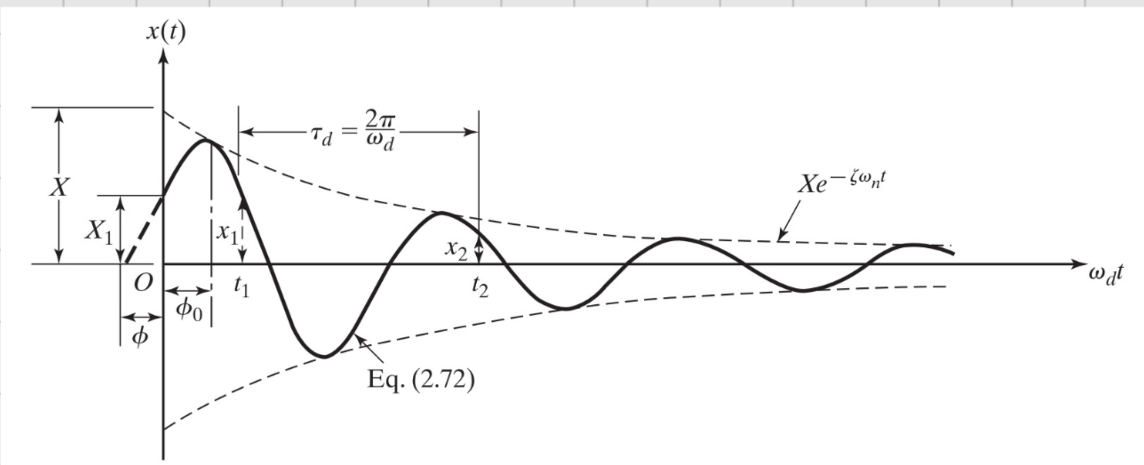
I.C. $\quad x(t=0) = x_0 \quad , \quad \dot{x}(t=0) = \dot{x}_0$

$$\Rightarrow \quad x_0 = C_1' \quad , \quad \dot{x}_0 = -\zeta\omega_n C_1' + \sqrt{1-\zeta^2}\,\omega_n$$

$$\rightarrow C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$\Rightarrow x(t) = e^{-\zeta\omega_n t}\left\{ x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin\sqrt{1-\zeta^2}\,\omega_n t \right\}$$

衰減



frequency of damped vibration : ω_d

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

Case 2. Critically damped system ($\zeta = 1$)

$c = c_c$

$$S_1 = S_2 = -\frac{c_c}{2m} = -\omega_n$$

$$\Rightarrow \quad x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

I.C.

$$x(t=0) = x_0 \quad , \quad \dot{x}(t=0) = \dot{x}_0$$

$$C_1 = x_0 \quad , \quad C_2 = \dot{x}_0 + \omega_n x_0$$

$$\Rightarrow \quad x(t) = [\, x_0 + (\dot{x}_0 + \omega_n x_0)t \,]e^{-\omega_n t}$$

※ aperiodic
(非週期)

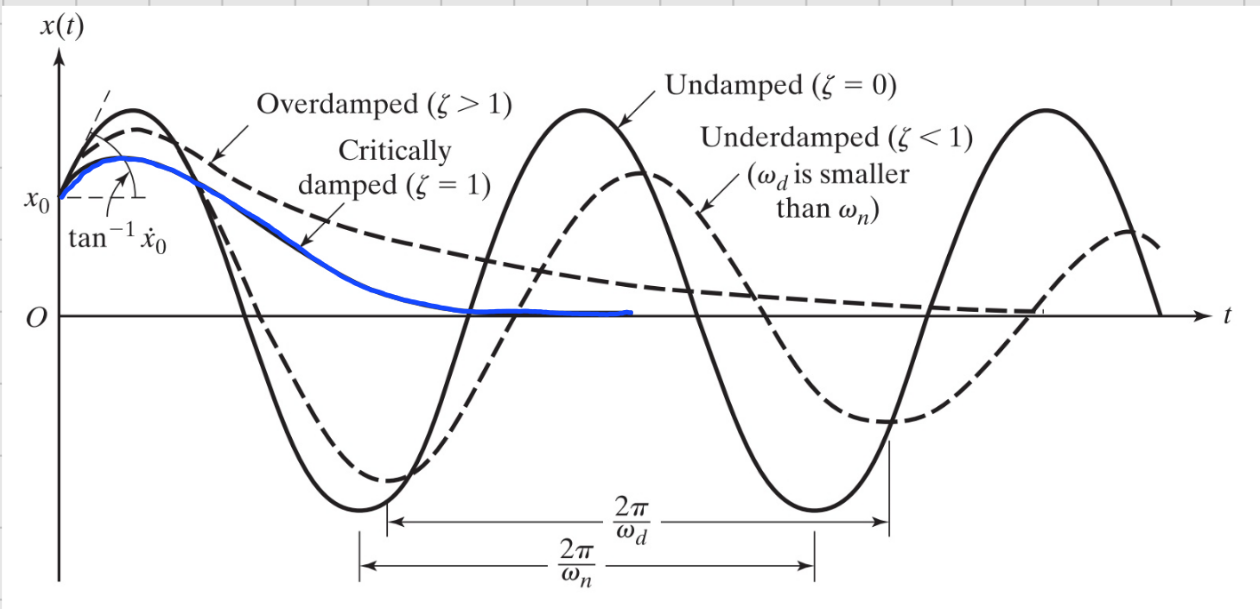


FIGURE 2.24 Comparison of motions with different types of damping.

Case 3. Overdamped system $(\zeta > 1)$

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

$s_2 \ll s_1$

I.C.

$$\Rightarrow \quad C_1 = \frac{x_0\omega_n(\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0\omega_n(\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

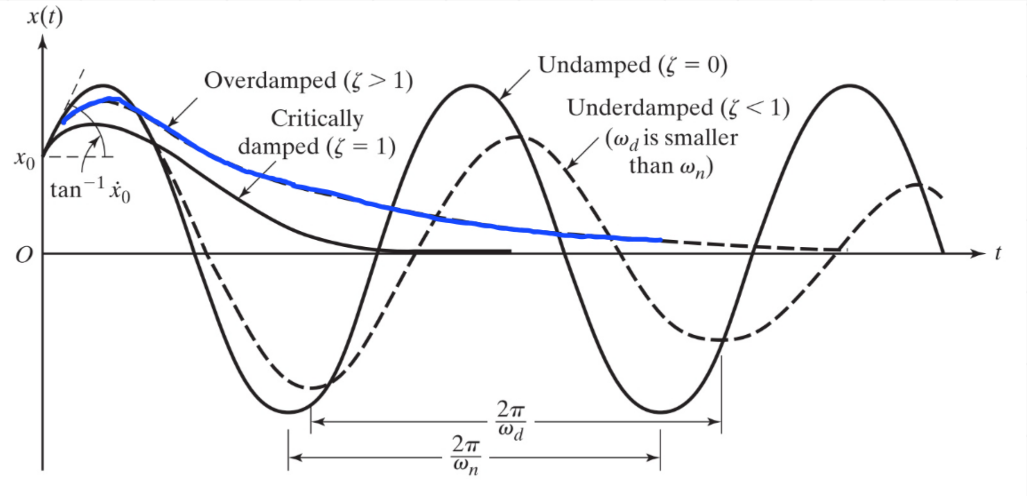


FIGURE 2.24 Comparison of motions with different types of damping.

∴ 臨界阻尼能在最短的時間，返回到靜止

2.6.3 對數遞減

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1} \cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2} \cos(\omega_d t_2 - \phi_0)}$$

where $t_2 = t_1 + T_d$, $T_d = \frac{2\pi}{\omega_d}$

$$\cos(\omega_d t_1 + \underbrace{\omega_d T_d}_{=2\pi} - \phi_0) = \cos(\omega_d t_1 - \phi_0)$$

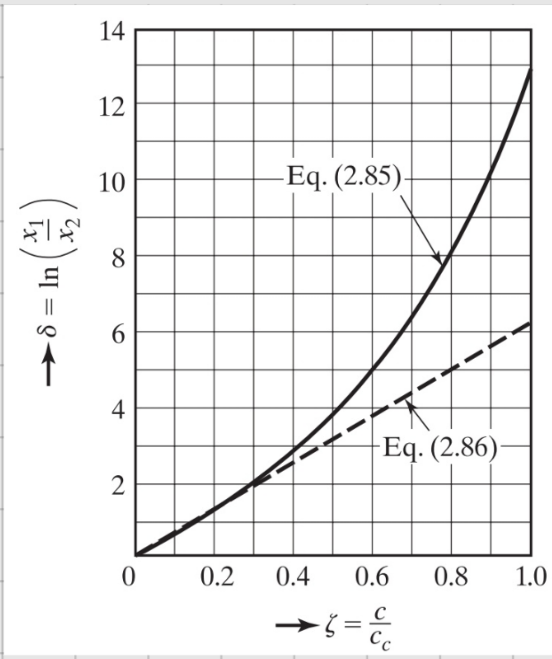
$$\Rightarrow \frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + T_d)}} = e^{\zeta\omega_n T_d}$$

logarithmic decrement :

$$\delta \equiv \ln \frac{x_1}{x_2} = \zeta\omega_n T_d$$

<cf>

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} \Rightarrow \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$



small damping $\zeta \ll 1$

$$\delta \simeq 2\pi\zeta$$

$$\zeta \simeq \frac{\delta}{2\pi}$$

if δ 已知

$$\Rightarrow \quad \zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

∴ 通过測定任意兩個連續位移 ⇒ 阻尼比

也可以測定任意兩個相隔完整周期的位移

set $t_1 \sim t_{m+1}$

$\qquad\qquad\quad = t_1 + mT_d$

$$\frac{x_1 x_2 \cdots x_m}{x_2 \cdots x_{m+1}} = \frac{x_1}{x_{m+1}} = \left(e^{\zeta\omega_n T_d}\right)^m$$

$$\Rightarrow \quad \delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right)$$

$\Rightarrow \zeta$

In a viscously damped system

$$\frac{dW}{dt} = F_V = -cv^2 = -c\left(\frac{dx}{dt}\right)^2$$

$$\Rightarrow \Delta W = \int_{t=0}^{\tau_d = \frac{2\pi}{\omega_d}} c\left(\frac{dx}{dt}\right)^2 dt$$

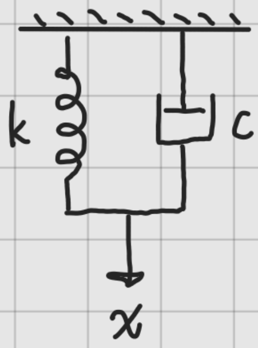
assume $x(t) = X \sin\omega_d t$

$$\Rightarrow \Delta W = \int c(X\omega_d \cos\omega_d t)^2\, dt = \int_0^{2\pi} cX^2 \omega_d \cos^2\omega_d t \cdot d(t\omega_d)$$

$$= \pi c \omega_d X^2$$

能量損耗(ΔW) $\propto$ 振幅^2(X^2)

In



$$F = -kx - c\dot{x}$$

assume : $x(t) = X\sin\omega_d t$

$$F = -kX\sin\omega_d t - c\omega_d X\cos\omega_d t$$

$$\Delta W = \int_{t=0}^{\tau_d = \frac{2\pi}{\omega_d}} F v\, dt = \pi c \omega_d X^2$$

∵ 彈簧力在一完整週期淨功 = 0

specific damping capacity

$$\frac{\Delta W}{\underbrace{W}_{\text{total energy} \,=\, \text{max 動能}}} = \frac{\pi c \omega_d X^2}{\frac{1}{2} m V_{max}^2} = \frac{\pi c \cancel{\omega_d} \cancel{X^2}}{\frac{1}{2} m \omega_d^{\cancel{3}} \cancel{X^2}} = 2\left(\frac{2\pi}{\omega_d}\right)\left(\frac{c}{2m}\right)$$

$$= 2\delta \simeq 4\pi\zeta$$

$$= \text{const}$$

<cf> set $x(t) = X \sin\omega_d t$,

$$V(t) = \underbrace{X\omega_d}_{V_{max}} \cos\omega_d t$$

loss coefficient $= \dfrac{\left(\Delta W / 2\pi\right)}{W}$ 每弧度的耗損

同理於 torsional

viscous damping torque : $T = -c_t\dot{\theta}$

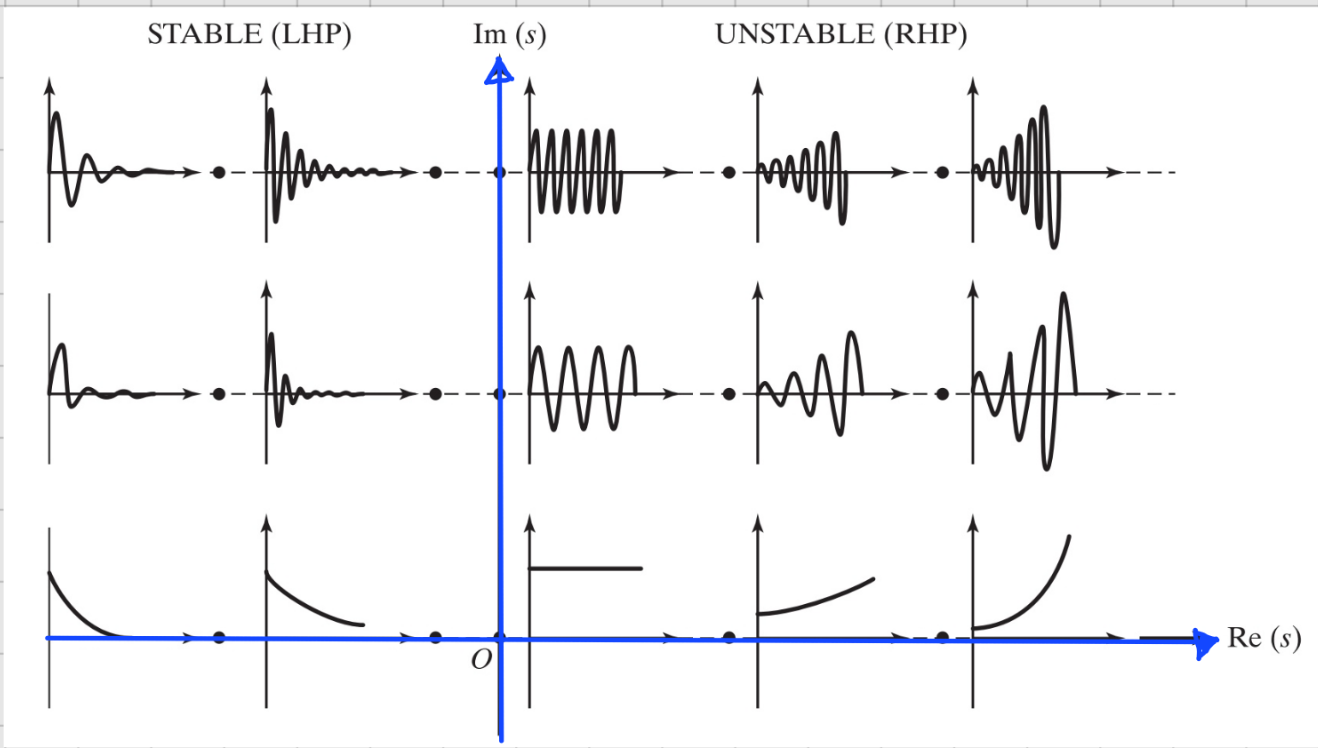
$$J_0\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0$$

underdamped case : $\omega_d = \sqrt{1-\zeta^2}\ \omega_n$

$$\omega_n = \sqrt{\frac{k_t}{J_o}}$$

$$\zeta = \frac{C_t}{C_{tc}} = \frac{C_t}{2 J_o \omega_n}$$

2.7



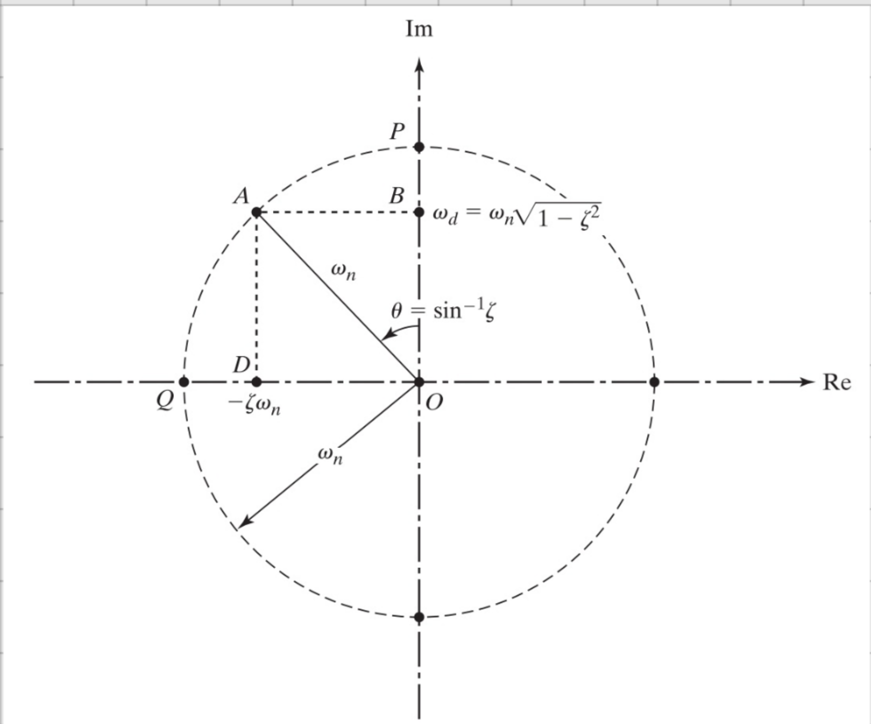
S越左,衰減越快 ；右邊則會增長

當根處在虛部≠0時,也會振盪

2.8

$$S_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

Case 1 : $S_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$



$$\sin\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

$$\theta = \sin^{-1}\zeta$$

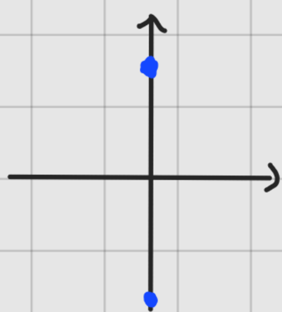
time constant : $T \equiv \dfrac{1}{\zeta\omega_n} \Rightarrow \zeta\omega_n = \dfrac{1}{T}$

Variation of the damping ratio :

$c < 0 \rightarrow$ unstable sys

$c = 0$:

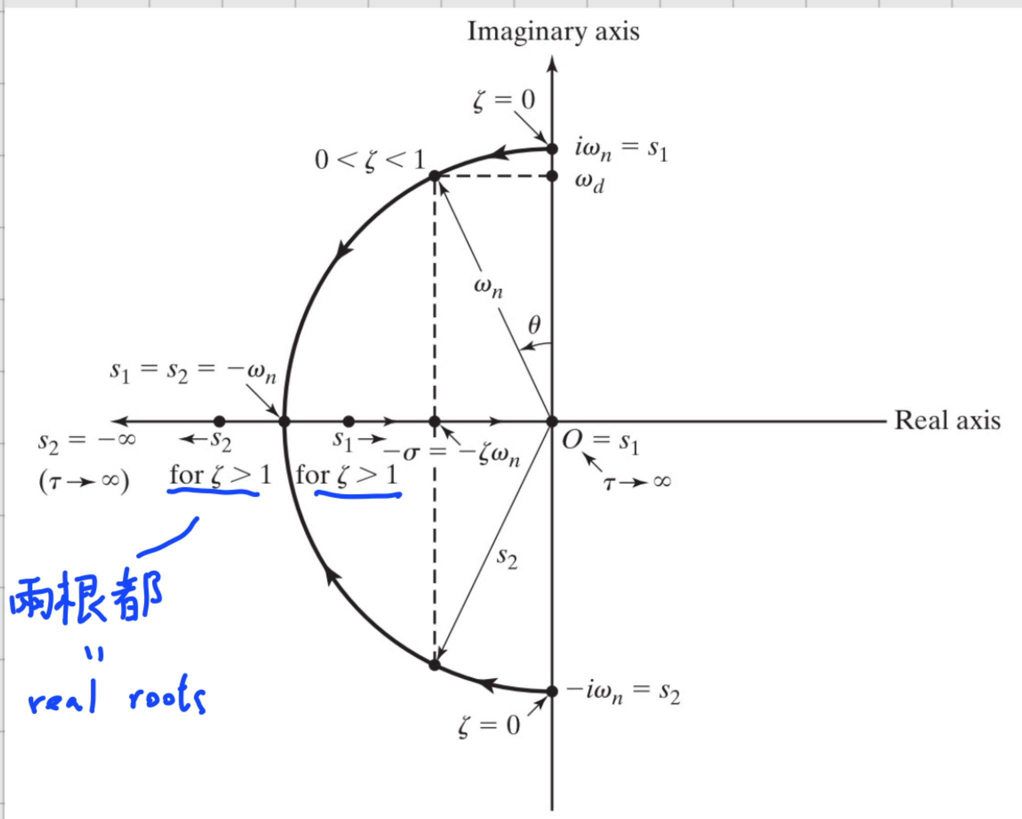
$$S_{1,2} = 0 \pm i\omega_n = \pm i\sqrt{\frac{k}{m}}$$



$c>0$: $(0<\zeta<1)$

$$s_{1,2} = -\underbrace{\zeta\omega_n}_{=\,\sigma} \pm i\underbrace{\omega_n\sqrt{1-\zeta^2}}_{=\,\omega_d} = -\sigma \pm i\omega_d$$

$$\Rightarrow \sigma^2 + \omega_d^2 = \omega_n^2$$



$\zeta \to \infty$, $s_1 \to -\infty$, $s_2 \to 0$

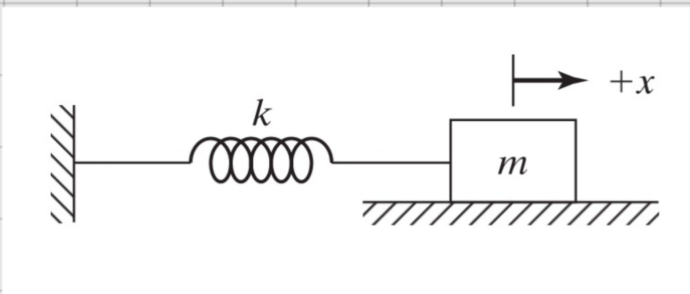
ex.

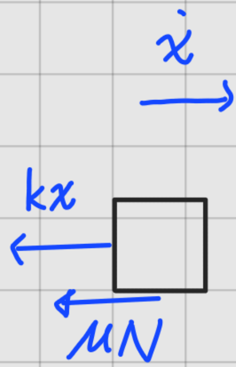
$k \cdot m \sim s$

2.9 Coulomb Damping

= dry-friction damping

friction force : $F = \mu N$





Case 1.

$$m\ddot{x} = -kx - \mu N \quad \rightarrow \quad m\ddot{x} + kx = \underbrace{-\mu N}_{\because \text{const}}$$

$$ms^2 + k = 0 \quad \rightarrow \quad s = \sqrt{-\frac{k}{m}} = i\omega_n$$

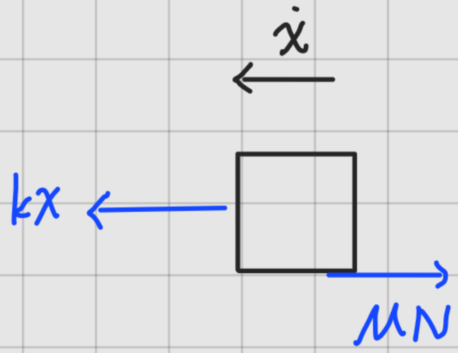
$$\Rightarrow x_h = A_1 \cos\omega_n t + A_2 \sin\omega_n t$$

$$x_p = B_1$$

$$\rightarrow 0 + kB_1 = -\mu N \quad \rightarrow \quad B_1 = \frac{-\mu N}{k} = x_p$$

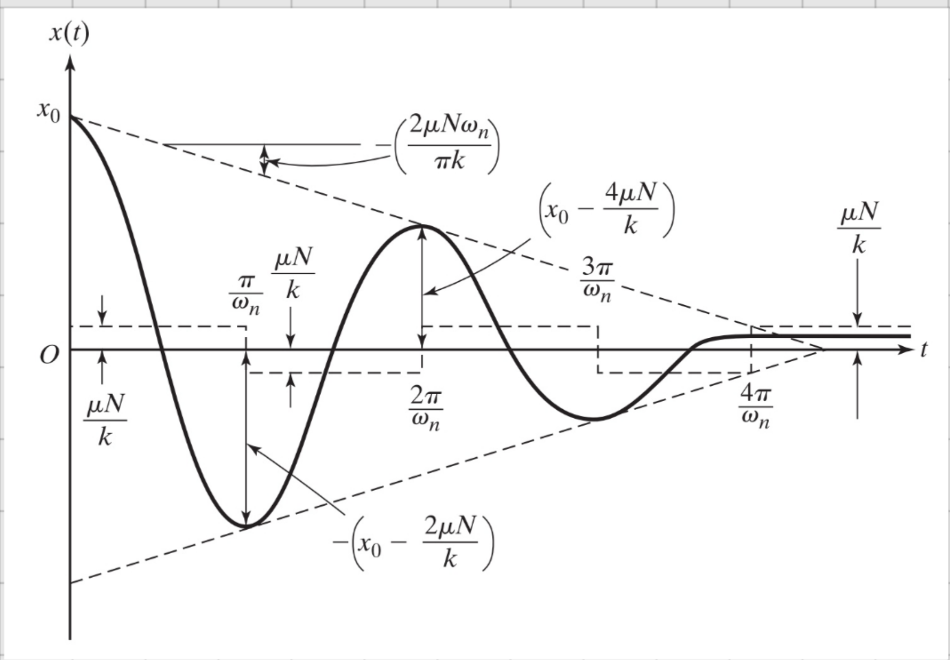
$$\Rightarrow x(t) = A_1 \cos\omega_n t + A_2 \sin\omega_n t - \frac{\mu N}{k}$$

Case 2.



$$m\ddot{x} + kx = \mu N$$

$$\Rightarrow x(t) = A_3 \cos\omega_n t + A_4 \sin\omega_n t + \frac{\mu N}{k}$$



$$\Rightarrow \quad m\ddot{x} + \mu mg \, \mathrm{sgn}(\dot{x}) + kx = 0$$ 非線性

<cf>

sgn(y) 表 $y > 0$ 時, $= 1$

$y = 0$, $= 0$

$y < 0$, $= -1$

assume I.C. $x(t=0)=x_0$, $\dot{x}(t=0)=0$

1. half cycle

$$x(t) = A_3\cos\omega_n t + A_4\sin\omega_n t + \frac{\mu N}{k}$$

$$x_0 = A_3 + \frac{\mu N}{k} \Rightarrow A_3 = x_0 - \frac{\mu N}{k}$$

$$A_4 = 0$$

$$\Rightarrow x(t) = \left(x - \frac{\mu N}{k}\right)\cos\omega_n t + \frac{\mu N}{k}$$

$$0 \le t \le \frac{\pi}{\omega_n}$$

When $t = \pi/\omega_n$

$$-x_1 = -\left(x_0 - \frac{2\mu N}{k}\right)$$

$$\dot{x}\left(t=\frac{\pi}{\omega_n}\right) = -\left(x - \frac{\mu N}{k}\right)\omega_n \cancel{\sin\omega_n t}^{\,0} = 0$$

2以上結果做為I.C

2 half cycle

$$x(t) = A_1 \cos\omega_n t + A_2 \sin\omega_n t - \frac{\mu N}{k}$$

I.C

$$-A_1 = -x_0 + \frac{3\mu N}{k} \quad , \quad A_2 = 0$$

$$\Rightarrow x(t) = \left(x_0 - \frac{3\mu N}{k}\right)\cos\omega_n t - \frac{\mu N}{k}$$

$$\pi/\omega_n \le t \le 2\pi/\omega_n$$

$$x_2 = x_0 - \frac{4\mu N}{k} \qquad \dot{x} = 0$$

3 half cycle

⋮

當 $kx < \mu N$

$$x_0 - r\frac{2\mu N}{k} \le \frac{\mu N}{k}$$

半週期數

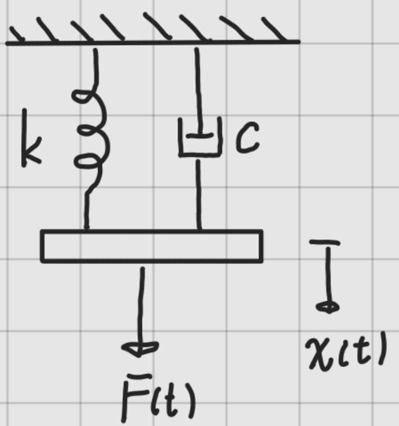
Torsional system 同理

$$J_o\ddot{\theta} + k_t\theta = -T$$

$$J_o\ddot{\theta} + k_t\theta = T$$

2.10 Hysteretic Damping
(遲滯)

由於材料變形所引起的阻尼



位移 $x(t)$ 所需要的 force F

$$F = kx + c\dot{x}$$

set $x(t) = X\sin\omega t$

$$\Rightarrow F(t) = kX\sin\omega t + cX\omega\cos\omega t$$

$$= kX\sin\omega t \pm c\omega X(1-\sin^2\omega t)^{\frac{1}{2}}$$

$$= kx \pm c\omega\sqrt{X^2 - X^2\sin^2\omega t}$$

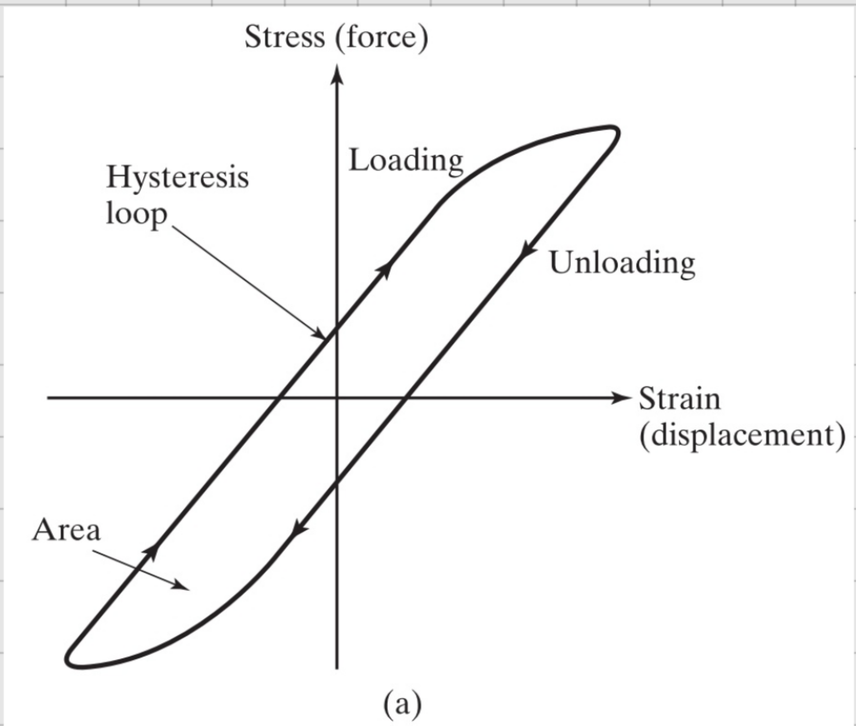
$$= kx \pm c\omega\sqrt{X^2 - x^2}$$

energy dissipated :

$$\Delta W = \oint F\,dx \qquad \frac{dX}{dt} = X\omega\cos\omega t$$

$$= \int_0^{2\pi/\omega} (kX\sin\omega t + cX\omega\cos\omega t)(\omega X\cos\omega t)\,dt$$

$$= \pi\omega c X^2$$



(a)

∵ 由實驗得知，ΔW 與 ω 無關

∴ assume

damping coefficient : $c = \dfrac{h}{\omega}$ hysteresis damping const

$$\Rightarrow \Delta W = \pi h X^2$$

---

if set $x = X e^{i\omega t}$

$$F = (k + i\omega c) x$$

$$= (k + ih) x = k\big(1 + i \underbrace{\tfrac{h}{k}}_{\beta}\big)$$

called : complex stiffness of the system

$$\rightarrow \Delta W = \pi k \beta X^2 \qquad / \text{per cycle}$$

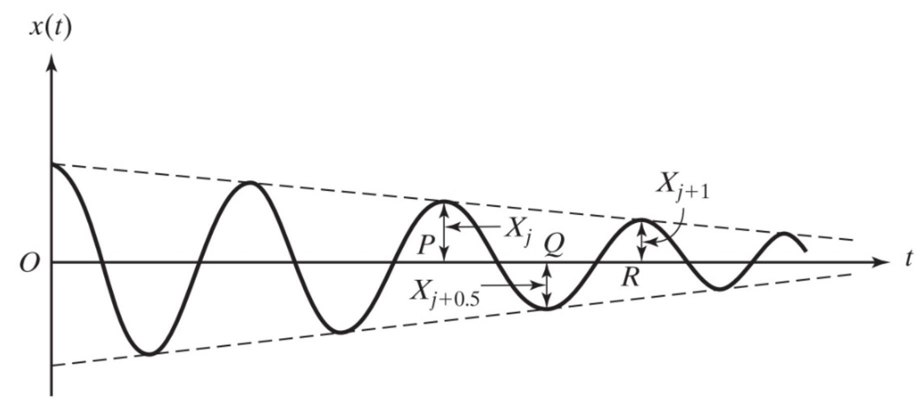


FIGURE 2.46 Response of a hysteretically damped system.

$P \rightarrow Q$

$$\frac{kX_j^2}{2} - \frac{\pi k \beta X_j^2}{4} - \frac{\pi k \beta X_{j+0.5}^2}{4} = \frac{kX_{j+0.5}^2}{2}$$

$$\frac{X_j}{X_{j+0.5}} = \sqrt{\frac{2+\pi\beta}{2-\pi\beta}}$$

$$\Rightarrow \quad \frac{X_j}{X_{j+1}} = \frac{2+\pi\beta}{2-\pi\beta} \simeq 1+\pi\beta$$

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \simeq \pi\beta$$

assume "近似 harmonic"

$$\delta \simeq 2\pi\zeta_{eq} \simeq \pi\beta = \frac{\pi h}{k}$$

$$\Rightarrow \quad \zeta_{eq} = \frac{h}{2k}$$

$$C_{eq} = C_c \zeta_{eq} = \frac{h}{\omega}$$

2.11 Stability

define 1.

if a system 的自由振动在 $t \to \infty$，响应 $\to 0$

called： asymptotically stable
(渐近)

特征根在 LHP

if 〃 $t \to \infty$，响应 $\to \infty$

called： unstable

〃 RHP

if 〃 $t \to \infty$，不衰減也不增長

called： stable

〃 虛軸上