We've already familiar with the definition of rotation matrix in the following form:

$$R = R(\psi)R(\theta)R(\phi)$$

When apply R on some vector $\vec{v}$, the effect is rotating the vector first around the ϕ-axis (x-axis in world frame) then around θ-axix(y-axis) then ψ-axis(z-axis). The final vector $R\vec{v}$ is world frame.

We can interpret rotation matrix in another way: change of basis. What that mean in linear algebra is, changing the representation of vector in one frame to representation in another frame.

The rotation matrix we used actually change basis from body frame to world frame, we can noted as ${}^W_B R$, where the subscript means the source frame (here the body frame), and the superscript means to destination frame(here the world frame).

The construction for any ${}^A_B M$ is simple. Supposed we have basis i_A, j_A, k_A in frame A and basis i_B, j_B, k_B in frame B, and we also supposed they are all unit vectors. Then ${}^A_B M$ can be defined as:

$${}^A_B M = \begin{bmatrix} i_B \cdot i_A & j_B \cdot i_A & k_B \cdot i_A \\ i_B \cdot j_A & j_B \cdot j_A & k_B \cdot j_A \\ i_B \cdot k_A & j_B \cdot k_A & k_B \cdot k_A \end{bmatrix}$$

So the our rotation matrix ${}^W_B R$ will have the form:

$${}^W_B R = \begin{bmatrix} i_B \cdot i_W & j_B \cdot i_W & k_B \cdot i_W \\ i_B \cdot j_W & j_B \cdot j_W & k_B \cdot j_W \\ i_B \cdot k_W & j_B \cdot k_W & k_B \cdot k_W \end{bmatrix}$$

We can easily find ${}^W_B R_{13} = k_B \cdot i_W$ and ${}^W_B R_{23} = k_B \cdot j_W$. But what does that mean? It means ${}^W_B R_{13}$ is the dot product of the z-axis of the body frame and the x-axis of the world frame, similar for ${}^W_B R_{23}$:

$$\begin{aligned} {}^W_B R_{13} &= |k_B||i_W| \cos(\alpha) \\ {}^W_B R_{23} &= |k_B||j_W| \cos(\beta) \end{aligned}$$

With the fact that the (negated) thrust generated is always aligned with the z-axis (k_B in the above equations), if we constrain ${}^W_B R_{13}$ and ${}^W_B R_{23}$, we're constraining the α and β angles.

Notice that α and β is not the same as ϕ and θ. While ϕ and θ are angles about attitude, α and β are angles between frame axes. A more intuitive way of thinking might be, α and β are angles between the thrust (negated thrust in NED frame) and x-axis and y-axis in world frame.

Let's just ignore β and think about α. α is the angle between the body frame z-axis (or the thrust vector, since thrust vector is always aligned with the negated body frame z-axis) and the world frame x-axis. If we fix α, we will actually get something like the follow:

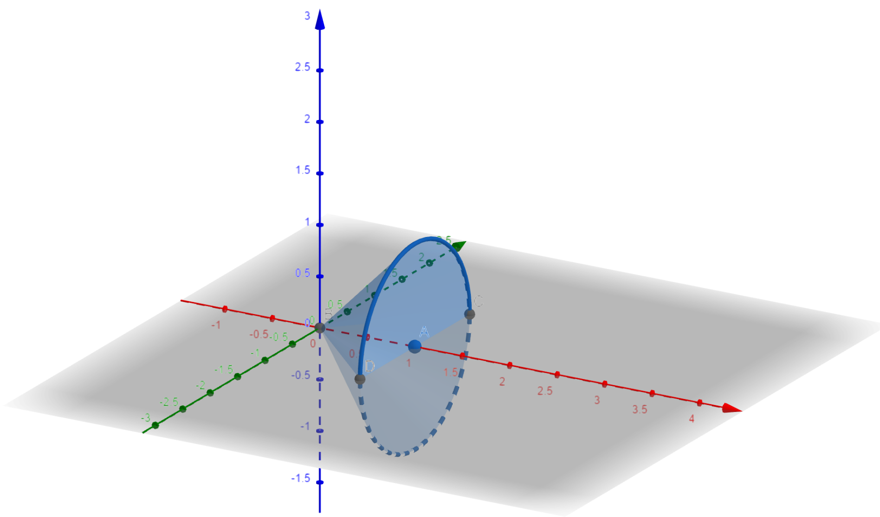

Figure 1: Fixing α

It's a cone whose coning angle is $2 \cdot \alpha$. In general, if we constrain R_{13} which is $\cos(\alpha)$ to be within some specific range $[-v, v]$, that will ends up two cones symmetrical to the z-axis of the world frame, whose coning angle is greater than or equal to $2 \cdot \arccos(v)$. Any vectors outside those two cones will meet the constrain.

The same principal applies to β. That will finally give us the valid space of thrust vector as follow for example, by constraining both R_{13} and R_{23} to $[-\sqrt{2}/2, \sqrt{2}/2]$ (or in other words, constraining both α and β to $[45°, 135°]$):

Notice the outcome thrust vector is not within the cones, but within the space delimited by the surface of the four cones.

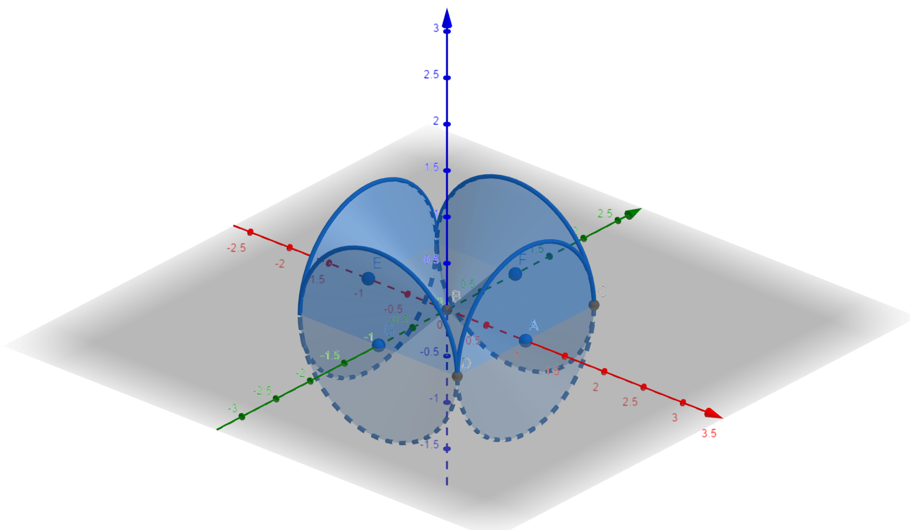

Figure 2: Constraining with $-\sqrt{2}/2 \leq R_{13}, R_{23} \leq -\sqrt{2}/2$

One thing we can see is, by constraining R_{13}, R_{23} to be within $[-\sqrt{2}/2, \sqrt{2}/2]$, the resulting thrust vector may still possible be perpendicular to the z-axis of the world frame and that's what we really want to avoid.

Don't believe that? Try calculate the rotation matrix when $\phi = -\pi/4, \theta = \pi/2, \psi = 0$ (a reminder: we have $\theta = \pi/2$!) and see what R_{13} and R_{23} will be.

So if we want the drone to maintain some attitudes that're safe by constraining with R_{13} and R_{23} ONLY, we have to make R_{13}, R_{23} smaller. Here's what the space look like if they're within $[-0.5, 0.5]$ (corresponding to $60° \leq \alpha, \beta \leq 120°$):

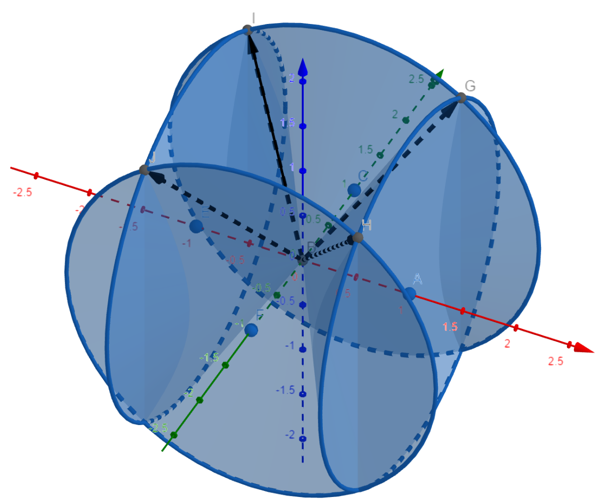

Figure 3: Constraining with $-0.5 \leq R_{13}, R_{23} \leq 0.5$