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Fundamental study

Towards a unified view of bisimulation: a comparative study

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Abstract

The realm of approaches to operational descriptions and equivalences for concurrent systems in the literature lead to a series of different attempts to give a uniform characterization of what should be considered a bisimulation, mostly in an algebraic and/or categorical framework. Meanwhile the realm of such approaches calls itself for comparison and/or unification. We investigate how different abstract characterizations of bisimulations are related. In particular, we consider the coalgebraic approach of Aczel and Mendler, the observation structures (Kripke structures) of Degano, De Nicola and Montanari, the algebraic approach of Malacaria, the domain theoretic view of Abramsky and the categorical setting of Joyal, Nielsen and Winskel. The framework of Aczel and Mendler turns out to be the most general one in the sense that the other approaches can be translated into it. These translations, where the relation between the categorical setting of Joyal, Nielsen and Winskel with the coalgebraic approach is the most complicated one, enhance the understanding of the different approaches and contribute to a unified view of bisimulation.

Keywords: Bisimulation; Concurrency; Semantics; Event structures

Contents

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1. Introduction

Bisimulation was introduced by Milner and Park [27, 28, 31] in order to identify processes that cannot be distinguished by an external agent. Since then a large variety of notions of "bisimulation" have been studied, e.g. on labelled transition systems [14, 29, 7, 16], on event structures [18, 17, 21, 11, 34] and on Petri nets [20, 3, 6, 13]. Abramsky [2] extends the notion of bisimulation to transition systems with divergence.

> Degano et al. [12] remark that "the realm of approaches to operational descriptions and equivalences for concurrent systems in the literature calls for unification".
>
> Joyal et al. [22] write: "There are confusingly many models for concurrency and all too many equivalences on them. To an extent their representation as categories of models has helped explain and unify the apparent differences. But hitherto this category-theoretic approach has lacked any convincing way to adjoin abstract equivalences to these categories of models."

By now, a series of different attempts have been made to give a uniform characterization of what should be considered as a bisimulation, mostly in algebraic and/or categorical framework [5, 2, 12, 22, 23]. Meanwhile this realm of approaches to abstract characterization in the literature calls itself for comparison and/or unification. The purpose of this paper is to investigate how these abstract characterizations can be classified, how they are related and how suitable they are to encompass the concrete notions of bisimulation.

In Section 2 we recall briefly Milner's definition of strong (resp. weak) bisimulation. Then we deal with the coalgebraic approach of Aczel and Mendler [5] in Section 3, where we introduce the concept of (backward–forward) AM-bisimulation (Definition 3.2). Taking this notion as a point of reference we obtain as main results:

Observation structures of Degano et al. [12]: In Section 4 we transform the observation structures (Kripke structures) of Degano et al. [12] into transition systems (for the general case in Definition 4.3 and with special treatment of the τ-action in Definition 4.5). Conversely, we give an example of a very simple transition system that cannot be turned into an observation structure while preserving the graph structure.

Based on the above-mentioned transformations strong bisimulation (resp. weak bisimulation) on observation structures turns out to be a special case of strong bisimulation (resp. weak bisimulation) on transition systems (Lemma 4.4 resp. Lemmas 4.7 and 4.10). As the coalgebraic framework of Aczel and Mendler [5] covers strong and weak bisimulation on transition systems, and as observation structures arise to be a weaker concept than transition systems, we argue that the approach of Degano et al. [12] can be subsumed by the concept of AM-bisimulation.

Algebraic approach of Malacaria [23]. The *algebraic* approach of Malacaria [23], which we review in Section 5, is dual to the *coalgebraic* approach of Aczel and Mendler [5] in its interpretation of a transition system in the following sense: the coalgebraic view gives for each state the information about its immediate successors, the algebraic view yields for each state the information about its predecessors. In principle, both approaches are equivalent. In the algebraic view bisimulation can be characterized in terms of common subalgebras which adds an interesting perspective to the understanding of bisimulation.

Domain theoretic view of abramsky [2]. Abramsky [2] studies bisimulation on transition systems with divergence (Definition 6.1). We discuss this approach in Section 6. The coalgebras for AM-bisimulation can be embedded into this slightly broader model (Remark 6.2), and Abramsky's partial bisimulation and AM-bisimulation coincide on the subclass of transition systems with empty divergence set (Remark 6.5). These results carry over to the corresponding categories (Lemma 6.13). One obtains also that, under weak restrictions, Abramsky's domain equation (Definition 6.8) is suitable for describing AM-bisimulation. Further we point out that there is an interesting analogy between the settings of Aczel and Mendler [5] and Abramsky [2]: In both approaches bisimulation on a transition system is characterized by equality in a final object of a suitable category (Remark 6.10).

Categorical setting of Joyal et al. [22]. It is not difficult to see that concept of AM-bisimulation can be viewed as an instance of the concepts of **P**-bisimulation (Definition 7.1) and of path-**P**-bisimulation (Definition 7.3) of Joyal et al. [22], which we review in Section 7. For this result we choose a suitable category of transition systems together with a suitable subcategory (Remarks 7.2 and 7.4). In a general context the relation between AM-bisimulation and path-**P**-bisimulation turns out to be more complex. We consider the questions:

(1) Given a category M of models with a notion of bisimulation described in terms of path-**P**-bisimulation, is it possible to characterize this bisimulation in terms of coalgebras? This question has a positive answer (Theorem 7.6). However, the transition systems obtained are rather abstract.
(2) Conversely, given a category M of models with a notion of bisimulation which can be modelled as AM-Bisimulation, can we model this bisimulation as path-**P**-bisimulation for some subcategory **P**? In (Theorem 7.9) we establish conditions under which this is possible.

In addition, we describe the interplay of these results in Corollary 7.12. One might argue that the positive result of Theorem 7.9 together with the rather strong and

complicated conditions we need in Section 7.2 for the translation of an AM-bisimulation into a path-**P**-bisimulation might be a hint that AM-bisimulation is the more promising characterization.
It should be noted that by relating all approaches to the coalgebraic one we establish implicitly a relation between any two of the considered formalisms.

Apart from translating approaches into another an alternative way of getting information about the expressiveness of an approach consists of considering concrete bisimulations and trying to find a representation within the model. We consider in Section 8 a variety of bisimulations on event structures and discuss if and how these bisimulations can be modelled as AM-bisimulation and in the categorical setting of Joyal et al. [22]. In the introduction of Section 8 we present a table displaying the results obtained sofar. Finally, we give some hints at the limitations of these two models (Section 8.5).

In part these results have been presented in [25, 32].

2. Transition systems and Milner's bisimulations

We make frequent use of the following category of transition systems.

Definition 2.1. Let L be a set of labels.

(1) A *transition system* over L is a triple $\mathscr{T} = (S, \longrightarrow, i_S)$, where
   S is a set of states,
   $\longrightarrow \subseteq S \times L \times S$ is the transition relation and
   i_S is the initial state.
   Occasionally, we are not interested in the initial state, we then consider transition systems $\mathscr{T} = (S, \longrightarrow)$ without initial state.

(2) The *category* $\mathbf{T}_L$ has as objects transition systems $\mathscr{T} = (S, \longrightarrow, i_S)$ over L. Let $\mathscr{T}_0 = (S_0, \longrightarrow, i_{S_0})$ and $\mathscr{T}_1 = (S_1, \longrightarrow, i_{S_1})$ be transition systems over L. A map $\sigma : S_0 \rightarrow S_1$ is a morphism iff
   (i) $\sigma(i_{S_0}) = i_{S_1}$ and
   (ii) for all $s, s' \in S_0$, $l \in L : s \xrightarrow{l} s'$ implies $\sigma(s) \xrightarrow{l} \sigma(s')$.

(3) Let $\tau \in L$ denote the *silent action*. Let $\hat{\ } : L \rightarrow L^*$ be the function

$$\hat{l} := \begin{cases} l; & l \neq \tau \\ \varepsilon; & l = \tau, \end{cases}$$

   where ε denotes the empty word.

(4) On a transition system $\mathscr{T} = (S, \longrightarrow, i_S)$ over L an additional transition relation $\Longrightarrow \subseteq S \times L^* \times S$ is defined as follows:

$$s \overset{\hat{l}}{\Longrightarrow} s' : \iff \begin{cases} s\,(\xrightarrow{\tau})^* \xrightarrow{l} (\xrightarrow{\tau})^* s', & l \in L \backslash \{\tau\}, \\ s\,(\xrightarrow{\tau})^* s', & l = \tau. \end{cases}$$

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ FA \arrow[r, "F\sigma"] & FB \end{tikzcd}$$

Fig. 1. Definition of homomorphism.

Definition 2.2. Let $\mathscr{T}_0 = (S_0, \longrightarrow, i_{S_0})$ and $\mathscr{T}_1 = (S_1, \longrightarrow, i_{S_1})$ be transition systems over some set of labels L. A relation $R \subseteq S_0 \times S_1$ is a

strong bisimulation, iff for all $(s,t) \in R$, $l \in L$:

(i) if $s \xrightarrow{l} s'$ in $\mathscr{T}_0$ then $t \xrightarrow{l} t'$ in $\mathscr{T}_1$ and $(s',t') \in R$ for some $t' \in S_1$, and

(ii) if $t \xrightarrow{l} t'$ in $\mathscr{T}_1$ then $s \xrightarrow{l} s'$ in $\mathscr{T}_0$ and $(s',t') \in R$ for some $s' \in S_0$.

weak bisimulation, iff for all $(s,t) \in R$, $l \in L$:

(i) if $s \xrightarrow{l} s'$ in $\mathscr{T}_0$ then $t \xRightarrow{\hat{l}} t'$ in $\mathscr{T}_1$ and $(s',t') \in R$ for some $t' \in S_1$, and

(ii) if $t \xrightarrow{l} t'$ in $\mathscr{T}_1$ then $s \xRightarrow{\hat{l}} s'$ in $\mathscr{T}_0$ and $(s',t') \in R$ for some $s' \in S_0$.

These definitions carry over to transition systems without initial states.

3. The view of Aczel and Mendler [5]

Aczel and Mendler [5] prove that "every set-based functor on the category of classes has a final coalgebra". To establish this result they introduce the general notion of F-bisimulation, where F is an endofunctor on **Class**. We transfer this definition to the category **Set**, call it AM-bisimulation and define in addition a notion of backward–forward AM-bisimulation. As we will show in this paper AM-bisimulation (seen in a slightly broader sense) is adequate to capture a great variety of concrete instances of bisimulation and seems to be the most promising abstract characterization.

A *coalgebra* for an endofunctor F on a category $\mathbf{C}$ is a pair (A, α) consisting of an object A and a morphism $\alpha : A \to F(A)$ of $\mathbf{C}$. A morphism $\sigma : A \to B$ in $\mathbf{C}$ is a *homomorphism* between coalgebras (A, α) and (B, β) iff $\beta \circ \sigma = (F\sigma) \circ \alpha$ (see Fig. 1). Coalgebras and homomorphisms constitute a category, denoted by $\mathbf{C_F}$.

Example 3.1. Let L be a set of labels. Let $F := \mathscr{P}(L \times \_)$ be an endofunctor on **Set**, where $\mathscr{P}$ denotes the powerset operator.

(1) Any coalgebra (A, α) in $\mathbf{Set}_F$ can be seen as a transition system $\mathscr{T}_{(A,\alpha)} = (A, \longrightarrow)$ without initial state and vice versa, where $x \xrightarrow{l} x'$ in $\mathscr{T}_{(A,\alpha)}$ iff $(l, x') \in \alpha(x)$.

(2) With each coalgebra (A, α) in $\mathbf{Set}_F$ one may associate its "inverse coalgebra" (A, α^-), where $\alpha^- : A \to \mathscr{P}(L \times A)$ and $(l, x) \in \alpha^-(x') :\iff (l, x') \in \alpha(x)$.

Definition 3.2. (1) Let F be an endofunctor on **Set**. A coalgebra (R, γ) is an F-*bisimulation* between coalgebras (A, α) and (B, β), iff $R \subseteq A \times B$ and the projection

$$\begin{tikzcd} A \arrow[d, "\alpha"] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & B \arrow[d, "\beta"] \\ FA & FR \arrow[l, "F\pi_1"'] \arrow[r, "F\pi_2"] & FB \end{tikzcd}$$

Fig. 2. Definition of AM-bisimulation.

$\pi_1 : (R, \gamma) \to (A, \alpha)$ of R on A and the projection $\pi_2 : (R, \gamma) \to (B, \beta)$ of R on B are homomorphisms, i.e. the diagram in Fig. 2 is commutative.

(2) Let $F := \mathscr{P}(L \times \_)$ be the endofunctor on **Set** from Example 3.1.

(a) An *AM-bisimulation* is a F-bisimulation for this special functor.

(b) A *backward–forward AM-bisimulation* is an AM-bisimulation (R, γ) between coalgebras (A, α) and (B, β), such that (R, γ^-) is an AM-bisimulation between (A, α^-) and (B, β^-).

The translation of coalgebras into transition systems and vice versa carries over to the morphisms of the categories. Here we obtain:

Lemma 3.3. *Let L be a set of labels, let $F = \mathscr{P}(L \times \_)$ be the endofunctor on* **Set** *from Example* 3.1. *A map $\sigma : A \to B$ is a homomorphism between coalgebras (A, α) and (B, β) iff for the transition systems $T_{(A,\alpha)}$ and $T_{(B,\beta)}$ holds*

(i) *if $x \xrightarrow{l} x'$ in $T_{(A,\alpha)}$ then $\sigma(x) \xrightarrow{l} \sigma(x')$ in $T_{(B,\beta)}$ and*

(ii) *if $y \xrightarrow{l} y'$ in $T_{(B,\beta)}$ and there exists $x \in A$ with $y = \sigma(x)$, then there exists some $x' \in A$ with $y' = \sigma(x')$ such that $x \xrightarrow{l} x'$ in $T_{(A,\alpha)}$.*

Proof. Straightforward. □

Lemma 3.4. *Let (A, α) and (B, β) be coalgebras to $F = \mathscr{P}(L \times \_)$ on* **Set.**

(1) *Let $R \subseteq A \times B$, define $\gamma : R \to FR$, where $\forall (x, y),\ (x', y') \in R,\ l \in L$:*

$$(l, x', y') \in \gamma(x, y) :\iff (l, x') \in \alpha(x),\ (l, y') \in \beta(y).$$

Then for all $(x, y) \in R$:

$$(F\pi_1 \circ \gamma)(x, y) \subseteq (\alpha \circ \pi_1)(x, y),\ (F\pi_2 \circ \gamma)(x, y) \subseteq (\beta \circ \pi_2)(x, y).$$

(2) *Let (R, γ) be an AM-bisimulation between (A, α) and (B, β). Then for all $(x', y') \in R$:*

$$(F\pi_1 \circ \gamma^-)(x', y') \subseteq (\alpha^- \circ \pi_1)(x', y') \text{ and } (F\pi_2 \circ \gamma^-)(x', y') \subseteq (\beta^- \circ \pi_2)(x', y').$$

Proof. Straightforward. □

$$\begin{tikzcd}
A^n \arrow[d, "f_A"] & R^n \arrow[l, "(\pi_1,\ldots,\pi_1)"'] \arrow[r, "(\pi_2,\ldots,\pi_2)"] \arrow[d, "f_R"] & B^n \arrow[d, "f_B"] \\
A & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}$$

Fig. 3. Compatible relation R.

Let (A,α) and (B,β) be coalgebras for the functor $F=\mathscr{P}(L\times \_)$ on **Set**. Then

- by Lemma 3.3 $R\subseteq A\times B$ is a strong bisimulation between $T_{(A,\alpha)}$ and $T_{(B,\beta)}$ iff R can be *turned* into a *coalgebra* (R,γ), such that the diagram in Fig. 2 commutes, i.e. (R,γ) is an AM-bisimulation between (A,α) and (B,β).
- let the sets A and B consist of terms of some (process) language with a set of operators, e.g. $\Sigma=\{stop, a., +, ||\}$, such that A and B may also be viewed as Σ-algebras. In this situation, one may ask when a strong bisimulation R between $T_{(A,\alpha)}$ and itself that is an equivalence is a congruence. More general, the question is when a strong bisimulation R between $T_{(A,\alpha)}$ and $T_{(B,\beta)}$ is "compatible" with Σ. Here we call $R\subseteq A\times B$ *compatible* with Σ if $(a_i,b_i)\in R$, $i=1,2,\ldots,n$, implies $(f_A(a_1,a_2,\ldots,a_n),\ f_B(b_1,b_2,\ldots,b_n)\in R$ for every n-ary operator symbol $f\in\Sigma$. One can prove that $R\subseteq A\times B$ is compatible with Σ iff R can be *turned* into a Σ-algebra, such that for every n-ary operator symbol $f\in\Sigma$ the diagram in Fig. 3 commutes.

Thus a relation $R\subseteq A\times B$ is:

a strong bisimulation iff it can be turned into a coalgebra that displays the same behaviour as (A,α) and (B,β) and

compatible with Σ iff it can be turned into a Σ-algebra that displays the same behaviour as (A,Σ) and (B,Σ).

4. The view of Degano et al. [12]

Degano et al. [12] remark that "*the realm of approaches to operational descriptions and equivalences for concurrent systems in the literature calls for unification* $[\cdots]$. *At an appropriate level of abstraction many of the semantics proposed so far can be recast within a common framework* $[\cdots]$.

As this common framework Degano et al. [12] propose the concept of an observation structure and introduce four types of bisimulation of decreasing distinguishing power for observation structures to capture the essence of "bisimulation": strong bisimulation, branching bisimulation, weak bisimulation and jumping bisimulation. These observation structures are closely related with Kripke structures: Every Kripke structure can be viewed as an observation structure and vice versa. Various equivalences and bisimulations have been studied on Kripke structures, e.g. in [8, 15].

Observation structures differ from transition systems with labels in some set D by the fact that labels are attached to nodes instead of edges.

Definition 4.1. Given a set D of observations, an *observation structure* is a triple $\mathcal{O}=(S,\rightarrow,o)$, where

S is a set of nodes,
$\rightarrow\ \subseteq S\times S$ is the transition relation and
$o:S\rightarrow D$ is an observation function mapping nodes into observations.

An *observation structure with start state* is a quadruple $\mathcal{O}=(S,\rightarrow,o,i_S)$, where $(S,\rightarrow,o)$ is an observation structure and $i_S\in S$ is a state such that any node can be reached from i_S. i_S is called start state.

Given an observation structure $\mathcal{O}=(S,\rightarrow,o,i_S)$ with start state we often denote the underlying observation structure $(S,\rightarrow,o)$ also by $\mathcal{O}$.

Definition 4.2. Given an observation structure $(S,\rightarrow,o)$, a symmetric relation R on S, such that $r\,R\,s$ implies $o(r)=o(s)$, is a *strong bisimulation* if $r\,R\,s$ and $r\rightarrow r'$ implies that there exists s', with $s\rightarrow s'$ and $r'\,R\,s'$.
branching bisimulation if $r\,R\,s$ and $r\rightarrow r'$ implies that there exist $s_0,s_1,\ldots,s_n, n\geqslant 0$, with $s=s_0\rightarrow\cdots\rightarrow s_n$ and $r\,R\,s_i$ for $i<n$ and $r'\,R\,s_n$.
weak bisimulation if $r\,R\,s$ and $r\rightarrow r'$ implies that there exist $s_0,s_1,\cdots,s_n$, with $s=s_0\rightarrow\cdots\rightarrow s_k\rightarrow\cdots\rightarrow s_n$, $0<k\leqslant n$, and $o(s_0)=o(s_i)$ for $0<i\leqslant k$, $o(s_i)=o(s_n)$ for $k<i<n$ and $r'\,R\,s_n$.
jumping bisimulation if $r\,R\,s$ and $r\rightarrow r'$ implies that there exists s', with $s\rightarrow^{*}s'$ and $r'\,R\,s'$.

The question arises, how the observation structure approach is related to the coalgebraic setting of [5]. Degano et al. [12] argue that

(1) the observation structure is more flexible and general than the transition system as the labelling of a node can be the observation of a whole computation and
(2) consequently e.g. strong and branching bisimulation on observation structures are generalizations of the terms introduced on transition systems.

However, the framework of transition systems has been extended very early to allow for arbitrary labelling of transitions and in [5] the labelling can be taken from some arbitrary set. As we show in the following an observation structure can be easily transformed into a transition system and based on this transformation bisimulation on observation structures turns out to be a special case of bisimulation on transition systems.

Definition 4.3. Let $\mathcal{O}=(S,\rightarrow,o,i_S)$ be an observation structure over D with start state. Choose $\hat{s}\notin S$ and put

$$S':=S\cup\{\hat{s}\}\text{ and}$$
$$\rightharpoonup\ \subseteq S\times D\times S,\text{ where } s\overset{d}{\rightharpoonup}s' \text{ iff } \begin{array}{l}(s=\hat{s}\text{ and } s'=i_S\text{ and } d=o(i_S))\text{ or}\\ (s\neq\hat{s}\text{ and } s\rightarrow s'\text{ and } o(s')=d.)\end{array}$$

We call $TS(\mathcal{O})=(S',\rightharpoonup,\hat{s})$ the transition system associated with $\mathcal{O}$. Fig. 4 shows an observation structure $\mathcal{O}$ with its associated transition system $TS(\mathcal{O})$. Please note that

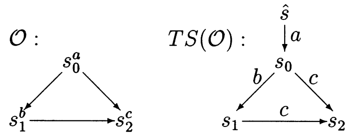

Fig. 4. An observation structure $\mathcal{O}$ and its associated transition system $TS(\mathcal{O})$.

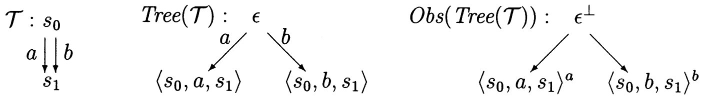

Fig. 5. A transition system which cannot be turned into an observation structure.

the graph structure is basically preserved by the transformation and that one can obtain $\mathcal{O}$ from $TS(\mathcal{O})$.

By the above, it is clear that observation structures (with start state) can be considered as coalgebras for the functor $F(X)=\mathcal{P}(D\times X)$ over the category **Set**, i.e. in the coalgebraic setting of [5]. Conversely there are very simple transition systems which cannot be turned into an observation structure while preserving the graph structure, see the transition system $\mathcal{T}$ in Fig. 5. However, one may transform the reachable part of a transition system with initial state into a tree ($Tree(\mathcal{T})$ in Fig. 5) which can then be turned into an observation structure ($Obs(Tree(\mathcal{T}))$ in Fig. 5) by moving a label from an edge to the node it points to and by introducing some dummy observation at the start state.

Degano et al. [12] write "strong and branching equivalences are straightforward generalizations of the corresponding notions over labelled transition systems". From the above point of view, however, one obtains the following results:

Lemma 4.4. *Let $\mathcal{O}=(S,\rightarrow,o,i_S)$ be an observation structure over D with start state.*
(1) *If $R\subseteq S\times S$ is a strong bisimulation on $\mathcal{O}$ then R is a strong bisimulation on $TS(\mathcal{O})$.*
(2) *If $R\subseteq S\times S$ is a strong bisimulation on $TS(\mathcal{O})$ and $r\,R\,s$ implies $o(r)=o(s)$ then $R\cup R^{-1}$ is a strong bisimulation on $\mathcal{O}$.*
(3) *Let $r,s\in S$ with $o(r)=o(s)$.*
There is a strong bisimulation R on $\mathcal{O}$ with $r\,R\,s$ iff there is a strong bisimulation $\hat{R}\subseteq S\times S$ on $TS(\mathcal{O})$ with $r\,\hat{R}\,s$.

Proof. (1), (2) and (3) "$\Rightarrow$" are obvious.

Let $\hat{R}$ be a strong bisimulation on $TS(\mathcal{O})$ with $r\,\hat{R}\,s$. Remove from $\hat{R}$ all pairs (r_1,s_1) with $o(r_1)\neq o(s_1)$. The resulting relation $\bar{R}$ is nonempty. $R:=\bar{R}\cup\bar{R}^{-1}$ is a strong bisimulation on $\mathcal{O}$: let $r_1\,R\,s_1$ and $r_1\rightarrow r_2$ with $o(r_2)=d$. Hence $r_1 \xrightarrow{d} r_2$ in $TS(\mathcal{O})$. As $r_1\,\hat{R}\,s_1$ or $s_1\,\hat{R}\,r_1$ we get $s_1 \xrightarrow{d} s_2$ in $TS(\mathcal{O})$ for some s_2, and $r_2\,\hat{R}\,s_2$ or $s_2\,\hat{R}\,r_2$. Hence $o(s_2)=d$ and $r_2\,R\,s_2$. $\square$

We will now turn to the concept of weak bisimulation on observation structures and show that it can also be subsumed in the coalgebraic setting.

Definition 4.5. Let $\mathcal{O}=(S,\rightarrow,o,i_S)$ be an observation structure over D with start state i_S. Consider the transition system $TS(\mathcal{O})=(S',\rightharpoonup,\hat{s})$ from Definition 4.3. For all observations $d\in D$ let $Path_d$ denote the set of all simple[1] directed paths in $TS(\mathcal{O})$ where all transitions are labelled with d. Each set $Path_d$ is partially ordered by the subpath relation. Let $s \overset{d}{\rightharpoonup} s'$ be a transition in $TS(\mathcal{O})$, that is located on two maximal paths p_1 and p_2 in $Path_d$. Then $s \overset{d}{\rightharpoonup} s'$ is either the first transition in both p_1 and p_2 or neither the first transition in p_1 nor the first transition on p_2. Hence we may define a transition system $TS_\tau(\mathcal{O}):=(S',\rightharpoondown,\hat{s})$ with labels in $D\cup\{\tau\}$, where

$$s \overset{\tau}{\rightharpoondown} s' \quad \text{iff } s \overset{d}{\rightharpoonup} s' \text{ is not the first transition in a maximal path of } Path_d \text{ and}$$
$$s \overset{d}{\rightharpoondown} s' \quad \text{iff } s \overset{d}{\rightharpoonup} s' \text{ is the first transition in a maximal path of } Path_d.$$

Fig. 6 shows an observation structure $\mathcal{O}$ with its associated transition systems $TS(\mathcal{O})$ and $TS_\tau(\mathcal{O})$.

Remark 4.6. De Nicola and Vaandrager [15] introduce doubly labelled transition systems, i.e. transition systems where nodes *and* edges are labelled. A doubly labelled transitions systems $\mathcal{D}$ models a Kripke structure $KS(\mathcal{D})$ and a transition systems $LTS(\mathcal{D})$ at the same time. De Nicola and Vaandrager [15] give a construction how to obtain from a Kripke structure $\mathcal{O}$ a doubly labelled transition system $DLT(\mathcal{O})$. The underlying transition system $LTS(DLT(\mathcal{O}))$ is similar to our $LT_\tau(\mathcal{O})$, where we view $\mathcal{O}$ as observation structure. The main difference lies in our introduction of a new initial state, which allows us to recover all information contained in the labels of $\mathcal{O}$ whereas $LTS(DLT(\mathcal{O})$ looses the label of the original initial state of the Kripke structure.

Lemma 4.7. *Let $\mathcal{O}=(S,\rightarrow,o,i_S)$ be an observation structure over D. If $R\subseteq S\times S$ is a weak bisimulation on $\mathcal{O}$ then R is a weak bisimulation on $TS_\tau(\mathcal{O})$.*

Proof. Let $r\,R\,s$ and $r \overset{a}{\rightharpoondown} r'$ in $TS_\tau(\mathcal{O})$.

Case 1: $a\neq\tau, a=d'$. Hence $r\rightarrow r'$ in $\mathcal{O}$ and $o(r')=d'$. As R is a weak bisimulation on $\mathcal{O}$ there exist $s_0,s_1,\ldots,s_n$ with

$$s=s_0\rightarrow\cdots\rightarrow s_k\rightarrow\cdots\rightarrow s_n, \qquad 0<k\leqslant n,$$

and $o(s_0)=o(s_i)$ for $0<i\leqslant k$ and $o(s_i)=o(s_n)$ for $k<i<n$ and $r'\,R\,s_n$. Hence $o(r)=o(s)=o(s_i)$ for $0<i\leqslant k$ and $d'=o(r')=o(s_n)$ for $k<i<n$., i.e. in $TS_\tau(\mathcal{O})$ we have

$$s=s_0 \overset{\tau}{\rightharpoondown} s_1 \overset{\tau}{\rightharpoondown}\cdots\overset{\tau}{\rightharpoondown} s_k \overset{d'}{\rightharpoondown} s_{k+1} \overset{\tau}{\rightharpoondown}\cdots\overset{\tau}{\rightharpoondown} s_n$$

and obtain therefore $s \overset{\hat{d}'}{\Longrightarrow} s_n$ and $r'\,R\,s_n$.

[1] A path is simple iff every edge occurs at most once.

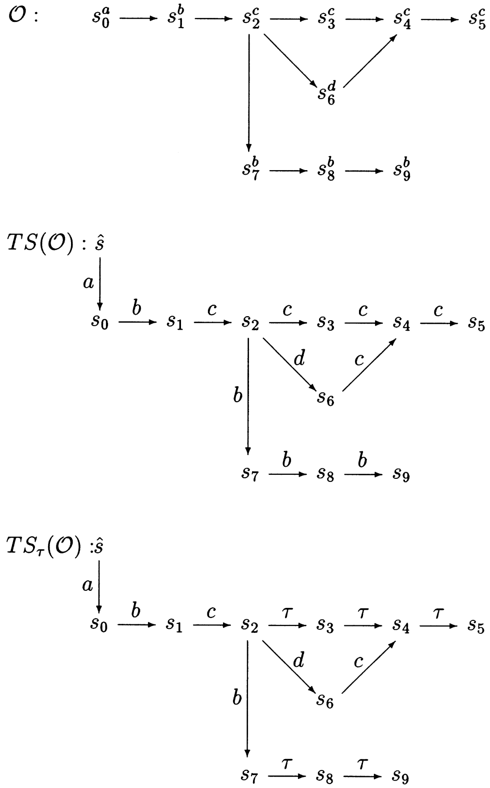

Fig. 6. An observation structure $\mathcal{O}$ and its associated transition systems.

Case 2: $a=\tau$. Hence there must be $d\in D$ with $o(r)=o(s)=o(r')=d$. As R is a weak bisimulation on $\mathcal{O}$ there exist $s_0, s_1, \ldots, s_n$ with

$$s=s_0 \to \cdots \to s_k \to \cdots \to s_n, \; 0<k\leqslant n,$$

and $o(s)=o(s_0)=o(s_n)=o(s_i)$ for $0<i<n$ and $r' R s_n$., i.e. in $TS_\tau(\mathcal{O})$ we have

$$s=s_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_k \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_n$$

and obtain therefore $s \stackrel{\varepsilon}{\Longrightarrow} s_n$ and $r' R s_n$. $\square$

The definition of weak bisimulation on observation structures from [12] requires that for related states $(r,s)\in R$ holds: if there is a transition $r\to r'$ then there is at least one transition starting in s.[2] This is not required for Milner's weak bisimulation on transition systems if the transition is labelled with τ. Therefore in general a weak bisimulation $\hat{R}$ on the transition system $TS_\tau(\mathcal{O})$ of an observation structure $\mathcal{O}$ does not induce a weak bisimulation on $\mathcal{O}$ including the pairs of $\hat{R}$ (see Example 4.8).

[2] This is due to the requirement $0<k$ in Definition 4.2.

Example 4.8. Consider the observation structure $\mathcal{O}=(\{i_S,r_1,r_2,s_1\},\{i_s\to r_1,\ i_s\to s_1,\ r_1\to r_2\},\{o(i_s)=e,\ o(r_1)=o(r_2)=o(s_1)=d\})$. Then $\hat{R}=\{(i_S,i_S),(r_1,s_1),(r_2,s_1)\}$ is a weak bisimulation on $TS_\tau(\mathcal{O})$. But there is no weak bisimulation R on $\mathcal{O}$ with $(r_1,s_1)\in R$: $\mathcal{O}$ includes the transition $r_1\to r_2$, but there is no transition starting at s_1.

However Degano et al. [12] write: "Our version of weak equivalence requires the same sequence of observations (possibly with stuttering) along the corresponding paths." In this sense the states r_1 and s_1 in the observation structure $\mathcal{O}$ of Example 4.8 should be weakly equivalent as they have – up to stuttering – the same sequence of observations. So we propose to change the definition of weak bisimulation on observation structures in order to adjust it to the verbal description. It turns out that then the equivalence to Milner's definition can be established.

Given an observation structure $(S,\to,o)$, a symmetric relation R on S, such that $r\,R\,s$ implies $o(r)=o(s)$, is a *w-bisimulation* if $r\,R\,s$ and $r\to r'$ implies that there exists $s_0,s_1,\dots,s_n$, with $s=s_0\to\cdots\to s_k\to\cdots\to s_n$, $0\leqslant k\leqslant n$, and $o(s_0)=o(s_i)$ for $0\leqslant i\leqslant k$, $o(s_i)=o(s_n)$ for $k<i<n$ and $r'\,R\,s_n$.

Remark 4.9. Please note that our definition of w-bisimulation is still different from jumping bisimulation, as e.g. in case of jumping bisimulation a transition $r\to r'$ with observations $o(r)=d$ and $o(r')=d'$ may be matched with transitions $s\to s_1\to s'$ with observations $o(s)=d$, $o(s')=d'$ and $o(s_1)=e\notin\{d,d'\}$, which is not possible with w-bisimulation. Please note that w-bisimilarity implies jumping bisimilarity.

Lemma 4.10. *Let $\mathcal{O}=(S,\to,o,i_S)$ be an observation structure over D with start state.*

(1) *If $R\subseteq S\times S$ is a w-bisimulation on $\mathcal{O}$ then R is a weak bisimulation on $TS_\tau(\mathcal{O})$.*
(2) *If $R\subseteq S\times S$ is a weak bisimulation on $TS_\tau(\mathcal{O})$ and $r\,R\,s$ implies $o(r)=o(s)$ then $R\cup R^{-1}$ is a w-bisimulation on $\mathcal{O}$.*
(3) *Let $r,s\in S$ with $o(r)=o(s)$.*

There is a w-bisimulation R on $\mathcal{O}$ with $r\,R\,s$ iff there is a weak bisimulation $\hat{R}\subseteq S\times S$ on $TS_\tau(\mathcal{O})$ with $r\,\hat{R}\,s$.

Proof. (1) By Lemma 4.7.

(2) Let w.o.l.g. $r\,R\,s$. Let $r\to r'$ in $\mathcal{O}$ with $o(r)=d$ and $o(r')=d'$.

Case 1: $d=d'$. Then $r\xrightarrow{\tau}r'$ in $TS_\tau(\mathcal{O})$. As R is a weak bisimulation for some s' we have $s\stackrel{\varepsilon}{\Longrightarrow}s'$ in $TS_\tau(\mathcal{O})$ and $r'\,R\,s'$. i.e. $s(\xrightarrow{\tau})^*s'$. Hence $o(s')=o(s)=d$ and there exist $s_0,s_1,\dots,s_n: d=o(s)=o(s_n)=o(s_i)$, $i=1\dots n$, and $s_n=s'$, $n\geqslant 0$.

Case 2: $d\neq d'$. Then $r\xrightarrow{d'}r'$ in $TS_\tau(\mathcal{O})$. As R is a weak bisimulation for some s' we have $s\stackrel{\hat{d'}}{\Rightarrow}s'$ in $TS_\tau(\mathcal{O})$ and $r'\,R\,s'$. I.e. $s(\xrightarrow{\tau})^*\xrightarrow{d}(\xrightarrow{\tau})^*s'$. Hence there exist $s_0,s_1,\dots,s_n: s_0=s$ and $s_n=s'$ with

$$s=s_0\xrightarrow{\tau}s_1\xrightarrow{\tau}\cdots\xrightarrow{\tau}s_k\xrightarrow{d'}s_{k+1}\xrightarrow{\tau}\cdots\xrightarrow{\tau}s_n,\quad k\geqslant 0,$$

in $TS_\tau(\mathcal{O})$. Hence $d = o(s) = o(s_i)$ for $0 < i \leqslant k$ and $d' = o(r') = o(s_n) = o(s_i)$ for $k < i < n$.

(3) Analogous to the Proof of (3). in Lemma 4.4. □

As the coalgebra framework also covers the case of weak bisimulation on transition systems (see [4]) it follows that observation structures with w-bisimulation can be modelled in the coalgebraic setting of [5].

Degario et al. [12] sketch how event structures can be turned into different observation trees by varying the observation function. It is an open question which bisimulations on event structures precisely can be modelled with these observation structures and the proposed bisimulations on observation structures. To our knowledge this question is also open for other models of concurrency.

5. The view of Malacaria [23]

Malacaria [23] studies *simulation* and *strong bisimulation* as observational equivalences on transition systems in an algebraic context. The aim of his approach is to get rid of the "syntactical nature" of the definition of observational equivalences and to give abstract algebraic tools "to characterize these equivalences as mathematically as possible".

On the one hand, Malacaria [23] introduces a category of transition systems $\mathbf{T}_{Malacaria}$, that has as objects transition systems $\mathcal{T} = (S, \rightarrow)$ over some set of labels L without an initial state. A morphism from $\mathcal{T}_0 = (S_0, \rightarrow)$ to $\mathcal{T}_1 = (S_1, \rightarrow)$ is a mapping $\sigma : S_0 \rightarrow S_1$ with $s \xrightarrow{l} s'$ in $\mathcal{T}_0$ implies $\sigma(s) \xrightarrow{l} \sigma(s')$ in $\mathcal{T}_1$, $s, s' \in S_0$, $l \in L$.

On the other hand, Malacaria [23] defines a category **A-CBA** of actions over complete atomic Boolean algebras and shows that there are (contravariant) functors between $\mathbf{T}_{Malacaria}$ and **A-CBA** that define a (contravariant) equivalence between these categories.

Definition 5.1. (1) A *complete atomic Boolean algebra* $\mathcal{A}$ is a Boolean algebra $\mathcal{A} = (A, \wedge, \vee)$ which is *complete*, i.e. each subset $V \subseteq A$ has an *inf* and a *sup*, and is *atomic,* i.e. there exists a nonempty subset $At(A)$ of A such that the following properties hold:

(a) $\forall v \in A$, $a \in At(A)$: $a \not\leqslant v \Rightarrow (a \wedge v = 0)$.

(b) $\forall v \neq 0 \in A$ $\exists a \in At(A)$: $a \leqslant v$.

(2) Let $\mathcal{A} = (A, \wedge, \vee)$ be a complete atomic Boolean algebra, let L be a set. An*action over* $\mathcal{A}$ is a pair $(\mathcal{A}, \alpha)$ such that $\alpha : L \times A \rightarrow A$ is a map with

(i) $\alpha(l, 0) = 0$ for all $l \in L$ and

(ii) $\alpha(l, \vee V) = \bigvee_{v \in V} \alpha(l, v)$ for all $l \in L, V \subseteq A$.

(3) Let $\mathcal{T} = (S, \rightarrow)$ be a transition system over L without an initial state. With $\mathcal{T}$ [23] associates an algebra $Ac(\mathcal{T}) := (\mathcal{P}(\mathcal{S}), \alpha)$, where $\mathcal{P}(S)$ is the powerset of S considered as complete atomic Boolean algebra with $\cap$ and $\cup$ as meet resp. join and

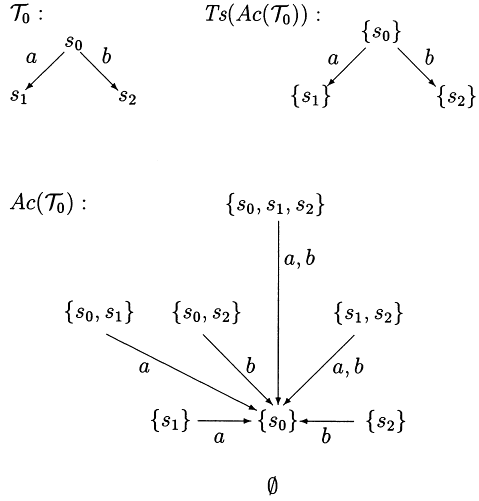

Fig. 7. Transformation of a transition system into an algebra and vice versa.

$\alpha : L \times \mathscr{P}(S) \to \mathscr{P}(S)$ is a map with

$$\alpha(l, V) := \{s \in S \mid \exists s' \in V : s \xrightarrow{l} s'\}, \quad l \in L,\ V \subseteq S.$$

A *subalgebra* $\mathscr{A}'$ of $Ac(\mathscr{T})$ is a set $A' \subseteq \mathscr{P}(S)$ such that: for any $v \in V \subseteq A'$ and for any $l \in L$ the elements $\emptyset$, S, $\cup V$, $\cap V$, $\neg V$, and $\alpha(l, v)$ are in A'.

(4) With an action $(\mathscr{A}, \alpha)$ over a complete atomic Boolean algebra $\mathscr{A}$ [2] associates a transition system $Ts(\mathscr{A}, \alpha) := (At(A), \to)$, where

$$s \xrightarrow{l} s' : \Longleftrightarrow s \leqslant \alpha(l, s').$$

Consequently one may interpret a transition system $\mathscr{T}_0 = (S_0, \longrightarrow_0)$ as an algebra and obtain from this algebra a transition system which is isomorphic to $\mathscr{T}_0$. The transition system $\mathscr{T}_1 = (S_1, \longrightarrow_1)$ resulting from $Alg(\mathscr{T}_0)$ is

$S_1 := At(\mathscr{P}(S_0)) = \{\{s\} \mid s \in S_0\}$ as states and
$\{s\} \xrightarrow{l}_1 \{s'\} : \Longleftrightarrow \{s\} \subseteq \alpha(l, \{s'\})$ as transition relation.

Fig. 7 illustrates these two transformations. In the above representation of a transition system as an algebra $(\mathscr{P}(S), \alpha)$ the map α yields for a state s' all immediate predecessors, i.e. all states from which s' can be reached via a single transition. This construction is dual to the coalgebraic view of [5] where the coalgebra gives for each state the information on the immediate successors.

In order to be able to give an algebraic characterization of bisimulation Malacaria [23] considers a restricted notion of strong bisimulation. For a strong bisimulation R between transition systems $\mathscr{T}_0 = (S_0, \longrightarrow_0)$ and $\mathscr{T}_1 = (S_1, \longrightarrow_1)$ it is requested that for every state $s_0 \in S_0$ there must exist a bisimilar state $s_1 \in S_1$, i.e. a state such that $(s_0, s_1) \in R$ and vice versa. This restriction is not strong, as we are usually interested in transition systems with an initial state i and may ignore states that cannot be reached from i. We will call this bisimulation *Mal-bisimulation*. Using the translation from transition systems into algebras [23] gives a characterization of bisimulation:

Theorem 5.2. *Transition systems $\mathcal{T}_0$, $\mathcal{T}_1$ are in Mal-bisimulation iff $Ac(\mathcal{T}_0)$ and $Ac(\mathcal{T}_1)$ have an isomorphic subalgebra.*

Example 5.3. Consider the transition system $\mathcal{T}_0$ in Fig. 7 and the transition system $\mathcal{T}_1 := (\{t_0, t_1, t_2, t_3\}, \{t_0 \xrightarrow{a} t_1, t_0 \xrightarrow{a} t_2, t_0 \xrightarrow{b} t_3, \}, t_0)$. $\mathcal{T}_0$ and $\mathcal{T}_1$ are Mal-bisimilar. The sets

$$T_0' := \{\emptyset,\ \{s_0\},\ \{s_1\},\quad \{s_2\},\ \{s_0, s_1\},\quad \{s_0, s_2\},\ \{s_1, s_2\},\quad \{s_0, s_1, s_2, s_3\}\},$$

$$T_1' := \{\emptyset,\ \{t_0\},\ \{t_1, t_2\},\ \{t_3\},\ \{t_0, t_1, t_2\},\ \{t_0, t_3\},\ \{t_1, t_2, t_3\},\ \{t_1, t_2, t_3, t_4\}\}$$

are isomorphic subalgebras of $Ac(\mathcal{T}_0)$ resp. $Ac(\mathcal{T}_1)$.

The above view adds an interesting perspective to the understanding of the nature of bisimulation. Clearly every notion of bisimulation in some model **M** that can be described in the coalgebra framework and yields a Mal-bisimulation can be characterized via the isomorphic subalgebra paradigm.

6. The view of Abramsky [2]

As part of a general program "domain theory in logical form" Abramsky [2] provides a general relationship between domain theory and operational notions of observability. In particular, Abramsky [2] defines a domain $\mathcal{D}$ that allows for a (fully abstract) characterization of partial (resp. finitary) bisimulation on transition systems with divergence. We consider the question how this view of bisimulation is related to the coalgebraic approach of [5].

Definition 6.1. (1) A *transition system with divergence* is a structure
$\mathcal{T} = (S, Act, \longrightarrow, \uparrow)$ where
S is a set of processes or agents,
Act is a set of atomic actions,
$\longrightarrow\ \subseteq S \times Act \times S$ is the transition relation and
$\uparrow \subseteq S$ is a predicate.
Write $s\uparrow$ iff $s \in \uparrow$ and $s \downarrow$ iff $s \notin \uparrow$. $s\uparrow$ means "s may diverge" while $s\downarrow$ is read as "s definitely converges". Call a transition system $\mathcal{T}$ *terminating* iff $\uparrow = \emptyset$.

(2) A (*finite*) *synchronization tree* is a transition system $\mathcal{T} = (S, Act, \longrightarrow, \uparrow)$, where
- $(S, \longrightarrow)$ is a directed tree with a root $r \in S$ (in the graph theoretical sense) and
- the set S is finite.

(3) Let States be some countable set. $Synch(Act)$ denotes the set of all finite synchronization trees $\mathcal{T} = (S, Act, \longrightarrow, \uparrow)$ with $S \subseteq$ States.

Remark 6.2. A transition system with divergence can be seen as an object in $\mathbf{Set}_F$ and vice versa.

Definition 6.3. Let $\mathscr{T}_0 = (S_0, Act, \longrightarrow, \uparrow)$ and $\mathscr{T}_1 = (S_1, Act, \longrightarrow, \uparrow)$ be transition systems with divergence over the same set of actions Act.

(1) A *partial bisimulation* is a relation $R \subseteq S_0 \times S_1$, such that for all $(s,t) \in R$, $a \in Act$:
(i) if $s \xrightarrow{a} s'$ in $\mathscr{T}_0$ then $t \xrightarrow{a} t'$ in $\mathscr{T}_1$ and $(s',t') \in R$ for some $t' \in S_1$, and
(ii) if $s\downarrow$ then ($t\downarrow$ and $t \xrightarrow{a} t'$ in $\mathscr{T}_1$ implies $s \xrightarrow{a} s'$ in $\mathscr{T}_0$ and $(s',t') \in R$ for some $s' \in S_0$.)

(2) For $s \in S_0, t \in S_1$

$s \sqsubseteq^{pb} t$ iff there exists a partial bisimulation R with $s\,R\,t$.
$s \sqsubseteq^{fb} t$ iff for all $\mathscr{S} \in Synch(Act)$ holds: $r \sqsubseteq^{pb} s \Rightarrow r \sqsubseteq^{pb} t$,
where r is the root of $\mathscr{S}$.
$\sqsubseteq^{fb}$ is called finitary bisimulation.

Both relations, i.e. partial and finitary bisimulation, are reflexive and transitive but not symmetrical. Partial bisimulation implies finitary bisimulation, but not vice versa.

Example 6.4. Let

$$\mathscr{T}_0 := (\{s_i \mid i \in \mathbb{N}\}, Act, \{s_0 \xrightarrow{a_i} s_i \mid i \geqslant 1\}, \emptyset) \text{ and}$$

$$\mathscr{T}_1 := (\{t_i \mid i \in \mathbb{N}\} \cup \{u\}, Act, \{t_0 \xrightarrow{a_i} t_i \mid i \geqslant 1\} \cup \{t_0 \xrightarrow{b} u\}, \emptyset)$$

be transition systems with divergence. Here $s_0 \sqsubseteq^{fb} t_0$, $s_0 \not\sqsubseteq^{pb} t_0$, and $t_0 \not\sqsubseteq^{fb} s_0$.

Remark 6.5. Partial bisimulation and Milner's strong bisimulation coincide on terminating transition systems and can hence be viewed as AM-bisimulation.

The notion of partial bisimulation is used in [2] to define a category of transition systems with divergence:

Definition 6.6. Let Act be a countable set of actions.

The objects of $\mathbf{T}_{Abramsky}$ are the transition systems with divergence over Act. Let $\mathscr{T}_0 = (S_0,\ Act,\ \longrightarrow, \uparrow)$ and $\mathscr{T}_1 = (S_1,\ Act,\ \longrightarrow, \uparrow)$ be objects of $\mathbf{T}_{Abramsky}$. A map $\sigma : S_0 \rightarrow S_1$ is a morphisms between $\mathscr{T}_0$ and $\mathscr{T}_1$, iff

$$\forall s \in S_0\colon\ s \sqsubseteq^{fb} \sigma(s) \wedge \sigma(s) \sqsubseteq^{fb} s.$$

Abramsky defines in [2] a class of so-called *finitary transition systems with divergence*, which are transition systems with divergence that satisfy the two axiom schemes

(BN) $\Box \bigvee_{i \in I} \Phi_i \leqslant \bigvee_{J \in \mathbf{Fin}(I)} \Box \bigvee_{j \in J} \Phi_j$ $(\Phi_i \in \mathscr{L}_\omega)$ (bounded non-determinancy) and
(FA) $\bigwedge_{J \in \mathbf{Fin}(I)} \Diamond \bigwedge_{j \in J} \Phi_j \leqslant \Diamond \bigwedge_{i \in I} \Phi_i$ $\quad(\Phi_i \in \mathscr{L}_\omega)$ (finite approximability),

where I is some index set, $\mathbf{Fin}(I)$ is the set of finite subsets of I and $\mathscr{L}_\omega$ is a finitary subset of a domain logic $\mathscr{L}_\infty$ in the sense of [1]. In [2] it is shown that partial and finitary bisimulation coincide on finitary transition systems with divergence.

Remark 6.7. Let (A_i, α_i), $i = 0, 1$, be coalgebras in $\mathbf{Set}_F$ such that their related transition systems are finitary. Then for $s_i \in A_i$, $i = 0, 1$:

$$s_0 \sqsubseteq^{fb} s_1 \text{ iff there is an AM-bisimulation } (R, \gamma) \text{ between}$$
$$(A_0, \alpha_0) \text{ and } (A_1, \alpha_1) \text{ with } (s_0, s_1) \in R.$$

Definition 6.8. Let Act be a countable set of actions. Let $\mathscr{D}$ be defined as the initial solution (in **SFP**) of the domain equation

$$\mathscr{D} = \mathscr{P}^0 \left(\sum_{a \in Act} \mathscr{D} \right),$$

where $\mathscr{P}^0$ is Plotkin's powerdomain with empty set.

Abramsky shows in [2] that for any transition system with divergence $\mathscr{T}$ over a countable set Act there is a mapping $[\![\ ]\!]$: $\mathscr{T} \to \mathscr{D}$ such that for all states s, t of $\mathscr{T}$:

$$s \sqsubseteq^{fb} t \iff [\![s]\!] \sqsubseteq_{\mathscr{D}} [\![t]\!].$$

Remark 6.9. Let (A_i, α_i), $i = 0, 1$ be coalgebras in $\mathbf{Set}_F$ such that their related transition systems are finitary. Then AM-bisimulation can be characterized by $\mathscr{D}$ in the following sense: for $s_i \in A_i$, $i = 0, 1$,

$$[\![s_0]\!] \sqsubseteq_{\mathscr{D}} [\![s_1]\!] \text{ iff there is an AM-bisimulation } (R, \gamma) \text{ between}$$
$$(A_0, \alpha_0) \text{ and } (A_1, \alpha_1) \text{ with } (s_0, s_1) \in R.$$

There is yet another aspect that makes the comparison between these two approaches interesting. In [2] the object $\mathscr{D}$ is also considered as transition system with divergence $(\mathscr{D}, Act, \longrightarrow, \uparrow)$ defined by

$$s \xrightarrow{a} s' :\iff \langle a, s' \rangle \in s \text{ and}$$
$$s\uparrow :\iff \bot \in s.$$

This transition system $\mathscr{D}$ is a final object in $\mathbf{T}_{Abramsky}$ and for transition systems $\mathscr{T}_i$, $i = 0, 1$, in $\mathbf{T}_{Abramsky}$ holds: for all states s_i of $\mathscr{T}_i$

$$s_0 \sqsubseteq^{fb} s_1 \iff fin_0(s_0) \sqsubseteq_{\mathscr{D}} fin_1(s_1).$$

where $fin_i : \mathscr{T}_i \to \mathscr{D}$ are the unique morphisms in $\mathbf{T}_{Abramsky}$. Combining this with Remark 6.7 one obtains:

Remark 6.10. Let (A_i, α_i), $i=0,1$ be coalgebras in $\mathbf{Set}_F$ such that their related transition systems are finitary. Then for $s_i \in A_i$, $i = 0, 1$:

$$\begin{aligned} \mathit{fin}_0(s_0) = \mathit{fin}_1(s_1) \text{ iff there is an AM-bisimulation } (R, \gamma) \text{ between}\\ (A_0, \alpha_0) \text{ and } (A_1, \alpha_1) \text{ with } (s_0, s_1) \in R. \end{aligned}$$

Here equality holds because of Remark 6.5.

Analogously, it can be shown, see e.g. [5, 4], that $\mathbf{Class}_F$, where $F = \mathscr{P}(Act \times \_)$, has a final object $\mathcal{O}$ and that for two coalgebras (A_i, α_i) and $s_i \in A_i$, $i = 0, 1$,

$$\begin{aligned} \overline{\mathit{fin}_0}(s_0) = \overline{\mathit{fin}_1}(s_1) \text{ iff there is an } F\text{-bisimulation } (R, \gamma) \text{ between}\\ (A_0, \alpha_0) \text{ and } (A_1, \alpha_1) \text{ with } (s_0, s_1) \in R, \end{aligned}$$

where $\overline{\mathit{fin}_i} : (A_i, \alpha_i) \to \mathcal{O}$ is the unique morphism in $\mathbf{Class}_F$, hence

Remark 6.11. For coalgebras (A_i, α_i), $i=0,1$, in $\mathbf{Set}_F$ with associated finitary transition systems and $s_i \in A_i$, $i = 0, 1$:

$$\mathit{fin}_0(s_0) = \mathit{fin}_1(s_1) \iff \overline{\mathit{fin}_0}(s_0) = \overline{\mathit{fin}_1}(s_1).$$

If we conversely consider terminating transition systems $\mathscr{T}_i$ and states s_i of $\mathscr{T}_i$, $i = 0, 1$, then we may summarize as follows:

$$s_0 \sqsubseteq^{fb} s_1 \iff \mathit{fin}_0(s_0) \sqsubseteq_{\mathscr{D}} \mathit{fin}_1(s_1)$$

and if interpreted as coalgebras

$$s_0 \sqsubseteq^{pb} s_1 \iff \overline{\mathit{fin}_0}(s_0) = \overline{\mathit{fin}_1}(s_1).$$

For terminating finitary transition systems we obtain

$$s_0 \sqsubseteq^{fb} s_1 \iff \overline{\mathit{fin}_0}(s_0) = \overline{\mathit{fin}_1}(s_1). \tag{$*$}$$

In the above, we freely interpreted coalgebras as (terminating) transition systems and vice versa. Both approaches, Acel and Mendler [5] and Abramsky [2], work in a categorical framework. So the question arises if this switching of view can be captured also on the categorical level such that the results about the characterization of bisimulation are maintained.

One can prove that the mapping from $\mathbf{Set}_F$ to $\mathbf{T}_{Abramsky}$ that associates a terminating transition system with a coalgebra and is the identity mapping on morphisms is a functor under which Remark 6.7 remains valid.

To go from $\mathbf{T}_{Abramsky}$ to $\mathbf{Set}_F$ one cannot use the simple interpretation of a terminating transition system as a coalgebra as can be seen by example:

Example 6.12. Consider the (finitary) transition systems $\mathscr{T}_0$ and $\mathscr{T}_1$ from Fig. 8, where we assume that all states converge. In the category $\mathbf{T}_{Abramsky}$ exists a morphism σ from

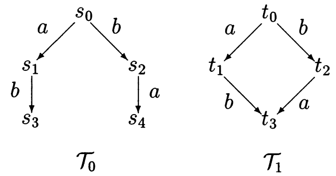

Fig. 8. Transition systems $\mathscr{T}_0$ and $\mathscr{T}_1$.

T_1 to T_0, take for example $\sigma(t_i) := s_i$, $0 \leqslant i \leqslant 3$. But there is no morphism from $\mathscr{T}_1$ to $\mathscr{T}_0$ in $\mathbf{Set}_F$.

Hence to establish a functor from $\mathbf{T}_{Abramsky}$ to $\mathbf{Set}_F$ we proceed as follows. Let **TermFinT** be the full subcategory of $\mathbf{T}_{Abramsky}$ which consists of terminating finitary transition systems. Let $\mathscr{T} = (S, Act, \longrightarrow, \emptyset)$ be an object of **TermFinT** and put

$$\hat{S} := \{[s]_{fb} \mid s \in S\}, \text{ where } [s]_{fb} \text{ denotes the equivalence class of } s \text{ with respect to } \sqsubseteq^{fb}, \text{ and}$$

$$[s]_{fb} \xrightarrow{a} [t]_{fb} :\iff \exists s' \in [s]_{fb}, \quad t' \in [t]_{fb} : s' \xrightarrow{a} t' \text{ in } \mathscr{T}.$$

Lemma 6.13. *Let $\mathscr{T}_i = (S_i, Act, \longrightarrow_i, \emptyset)$, $i = 0, 1$, be objects of* **TermFinTS**, *let $\sigma : S_0 \to S_1$ be a morphism from $\mathscr{T}_0$ to $\mathscr{T}_1$. Then G defined as*

$$G(\mathscr{T}_0) := (\hat{S}_0, \longrightarrow_0) \text{ and}$$

$$G(\sigma)[p]_{fb} := [f(p)]_{fb}$$

is a functor from **TermFinTS** *to* $\mathbf{Set}_F$. *For $s_i \in S_i$, $i = 0, 1$,*

$$s_0 \sqsubseteq^{fb} s_1 \text{ iff there is an AM-bisimulation } (R, \gamma) \text{ between } G(\mathscr{T}_0) \text{ and } G(\mathscr{T}_1), \text{ such that } ([s_0]_{fb}, [s_1]_{fb}) \in R.$$

Proof. We prove first that $G(\sigma)$ is a morphism in $\mathbf{Set}_F$ using the characterization of Lemma 3.3.

To show condition (i) let $[x]_{fb} \xrightarrow{a}_0 [x']_{fb}$ be a transition in $G(\mathscr{T}_0)$. Then there exist some $\hat{x} \in [x]_{fb}$, $\hat{x}' \in [x']_{fb}$ with $\hat{x} \xrightarrow{a}_0 \hat{x}'$ in $\mathscr{T}_0$. As σ is a morphism in **TermFinTS** we obtain $\hat{x} \sqsubseteq^{pb} \sigma(\hat{x})$. Therefore there exists some $y' \in S_1$ such that $\sigma(\hat{x}) \xrightarrow{a}_1 y'$ in $G(\mathscr{T}_1)$ and $\hat{x}' \sqsubseteq^{pb} y'$. Using again that σ is a morphism we get $\hat{x}' \sqsubseteq^{pb} \sigma(\hat{x}')$. Thus $\sigma(\hat{x}') \sqsubseteq^{pb} y'$ and therefore $[\sigma(x)]_{fb} = [\sigma(\hat{x})]_{fb} \xrightarrow{a}_1 [y']_{fb} = [\sigma(\hat{x}')]_{fb} = [\sigma(x')]_{fb}$.

Now let $[y]_{fb} \xrightarrow{a}_1 [y']_{fb}$ be a transition in $G(\mathscr{T}_1)$, where $[y]_{fb} = G(\sigma)[x]_{fb}$ for some $[x]_{fb} \in \hat{S}_0$. Then there exist some $\hat{y} \in [y]_{fb}$, $\hat{y}' \in [y']_{fb}$ with $\hat{y} \xrightarrow{a}_1 \hat{y}'$. As $x \sqsubseteq^{pb} \sigma(x)$ and $[y]_{fb} = [\sigma(x)]_{fb}$ we obtain $\hat{y} \sqsubseteq^{pb} x$. Thus there exists some $x' \in S_0$ with $x \xrightarrow{a}_0 x'$ and $\hat{y}' \sqsubseteq^{pb} x'$, i.e. we have $[x]_{fb} \xrightarrow{a}_0 [x']_{fb}$. As $x' \sqsubseteq^{pb} \sigma(x')$ we obtain further $[\sigma(x')]_{fb} = [\hat{y}']_{fb}$.

If $R \subseteq S_0 \times S_1$ is a partial bisimulation with $(s,t) \in R$ then $(\hat{R}, \hat{\gamma})$, where

$\hat{R} := \{([p]_{fb}, [q]_{fb}) \mid (p,q) \in R\}$ and
$(a, [p']_{fb}, [q']_{fb}) \in \hat{\gamma}([p]_{fb}, [q]_{fb}) :\iff [p]_{fb} \xrightarrow{a}_0 [p']_{fb}, [q]_{fb} \xrightarrow{a}_1 [q']_{fb}$,
where $a \in Act$ and $([p]_{fb}, [q]_{fb}), ([p']_{fb}, [q']_{fb}) \in \hat{R}$,

is an AM-bisimulation between $G(\mathscr{T}_0)$ and $G(\mathscr{T}_1)$ with $([s]_{fb}, [t]_{fb}) \in \hat{R}$.

If (R, γ) is an AM-bisimulation between $G(\mathscr{T}_0)$ and $G(\mathscr{T}_1)$ with $([s]_{fb}, [t]_{fb}) \in R$. Then

$$\hat{R} := \{(p', q') \mid p' \in [p]_{fb},\ q' \in [q]_{fb}, ([p]_{fb}, [q]_{fb}) \in R\}.$$

is a partial bisimulation with $(s,t) \in R$. $\square$

Now, we obtain a result analogous to $(*)$ in Remark 6.11:

Corollary 6.14. *Let* $\mathscr{T}_i = (S_i, Act, \longrightarrow_i, \emptyset)$ *be objects of* **TermFinT,** $s_i \in S_i$, $i = 0, 1$. *Then*

$$s_0 \sqsubseteq^{fb} s_1 \iff \overline{fin}_0([s_0]_{fb}) = \overline{fin}_1([s_1]_{fb}).$$

7. The view of Joyal et al. [22]

Joyal et al. [22] write: “There are confusingly many models for concurrency and all too many equivalences on them. To an extent their representation as categories of models has helped explain and unify the apparent differences. But hitherto this category-theoretic approach has lacked any convincing way to adjoin abstract equivalences to these categories of models.” [22] then propose to characterize bisimulation in a category **M** of models via a subcategory **P** of **M** of “path objects”. Such a path object represents “a particular run or history of a process”.

Definition 7.1. Let **M** be a category of models, let **P** be a category of path objects, where **P** is a subcategory of **M**.

(1) A *path* is a morphism $p : P \to X$ from an object P in **P** to an object X in **M**.

(2) In **M** a morphism $f : X \to Y$ is called **P**-*open*, iff whenever there are objects P, Q and a morphism $m : P \to Q$ in **P** and paths $p : P \to X$, $q : Q \to Y$, such that $f \circ p = q \circ m$, then there exists a path $r : Q \to X$ with $r \circ m = p$ and $f \circ r = q$.

Fig. 9 illustrates this “path lifting condition”. **P**-open morphisms include all the identity morphisms and are closed under composition.

(3) Two objects X_1 and X_2 of **M** are called **P**-*bisimilar*, iff there exists an object X in **M** and **P**-open morphisms $f_1 : X \to X_1$ and $f_2 : X \to X_2$.

In categories **M** with pullbacks the relation **P**-bisimilarity is transitive and therefore it is an equivalence relation. One can find categories with pullbacks for transition

$$
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "m"] & X \arrow[d, "f"] \\
Q \arrow[r, "q"] \arrow[ur, "r"] & Y
\end{tikzcd}
$$

Fig. 9. Path lifting condition.

systems, synchronization trees, event structures, transition systems with independence and Petri nets e.g. in [22, 23].

Let **Bran** be the full subcategory of $\mathbf{T}_L$ (Definition 2.1), which has finite synchronisation trees with at most one maximal branch as objects. Joyal et al. [22] show that **Bran**-bisimulation models precisely Milner's strong bisimulation. Modifying the category of transition systems [10] captures Milner's weak bisimulation, trace equivalence, testing equivalence, barbed bisimulation and probabilistic bisimulation as **P**-bisimulation. On event structures, Petri nets and transition systems with independence [22, 30] introduce a new notion of bisimulation the so-called strong history preserving bisimulation and characterize it in terms of **P**-bisimulation.

Remark 7.2. As **Bran**-bisimulation and Milner's strong bisimulation coincide on the category $\mathbf{T}_L$ AM-bisimulation can be viewed as an instance of **P**-bisimulation. [3]

To obtain a logic characteristic of **P**-bisimulation Joyal, Nielsen, and Winskel propose in [22] a second characterization of bisimulation in terms of category theory.

Definition 7.3. Let **M** be a category of models, let **P** be a small category of path objects, where **P** is a subcategory of **M**, let I be a common initial object of **M** and **P**.

(1) Two objects X_1 and X_2 of **M** are called *path*-**P**-*bisimilar* iff there is a set R of pairs of paths (p_1, p_2) with common domain P, so $p_1 : P \to X_1$ is a path in X_1 and $p_2 : P \to X_2$ is a path in X_2, such that
   (o) $(\imath_1, \imath_2) \in R$, where $\imath_1 : I \to X_1$ and $\imath_2 : I \to X_2$ are the unique paths starting in the initial object,

   and for all $(p_1, p_2) \in R$ and for all $m : P \to Q$, where m is in **P**, holds
   (i) if there exists $q_1 : Q \to X_1$ with $q_1 \circ m = p_1$ then there exists $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ and $(q_1, q_2) \in R$ (see Fig. 10) and
   (ii) if there exists $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ then there exists $q_1 : Q \to X_1$ with $q_1 \circ m = p_1$ and $(q_1, q_2) \in R$.

(2) Two objects X_1 and X_2 are *strong path*-**P**-bisimilar iff they are path-**P**-bisimilar and the set R further satisfies:
   (iii) If $(q_1, q_2) \in R$, with $q_1 : Q \to X_1$ and $q_2 : Q \to X_2$ and $m : P \to Q$, where m is in **P**, then $(q_1 \circ m, q_2 \circ m) \in R$, see Fig. 11.

---

[3] In [24] we discuss some subtle differences between **Bran**-bisimulation and AM-bisimulation.

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[d, "m"] \arrow[dr, "p_2"] & \\ X_1 & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] & X_2 \end{tikzcd}$$

Fig. 10. Path-**P**-bisimulation, illustration for condition (i).

$$\begin{tikzcd} & P \arrow[dl, "q_1 \circ m"'] \arrow[d, "m"] \arrow[dr, "q_2 \circ m"] & \\ X_1 & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] & X_2 \end{tikzcd}$$

Fig. 11. The new condition for strong Path-**P**-bisimulation.

Sometimes the set R is called a (strong) path-**P**-bisimulation between the objects X_1 and X_2.

On transition systems strong bisimulation can be modelled as (strong) path-**Bran**-bisimulation [22]. For event structures (strong) history preserving bisimulation can be captured by (strong) path-**Pos**-bisimulation [4] [22].

Remark 7.4. As (strong) path-**Bran**-bisimulation and Milner's strong bisimulation coincide on the category $\mathbf{T}_L$ AM-bisimulation can be viewed as an instance of (strong) path-**P**-bisimulation.

Joyal et al. [22] give the following relations between **P**-bisimulation and path-**P**-bisimulation:

Theorem 7.5. (1) *Let* **M** *be a category of models, let* **P** *be a small category of path objects, where* **P** *is a subcategory of* **M**, *let I be a common initial object of* **M** *and* **P**.
If two objects X_1 *and* X_2 *of* **M** *are* **P**-*bisimilar, then* X_1 *and* X_2 *are strong path-***P** *bisimilar.*
(2) *Let* **M** *be the subcategory of rooted presheaves in* $[\mathbf{P}^{op}, \mathbf{Set}]$. *Rooted presheaves* X_1, X_2 *are strong path-***P***-bisimilar iff they are* **P**-*bisimilar.*

As the relation between **P**-bisimulation and path-**P**-bisimulation is well understood we concentrate in this paper on the weaker concept of path-**P**-bisimulation. The translation of the path-**P**-bisimulation to AM-bisimulation also covers **P**-bisimulation. As we already need rather strong conditions to go from AM-bisimulation to path-**P**-bisimulation the chances to obtain a **P**-bisimulation in the general case are rather low.

[4] For the definition of the category **Pos** see Section 8.

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[d, "m"] \\ X & Q \arrow[l, "q"] \end{tikzcd}$$

Fig. 12. Defining the transitions of $T_{\mathbf{P},\mathbf{M}}$.

7.1. From path-$\mathbf{P}$-bisimulation to AM-bisimulation

In this section we study the following question: Start in a setting suitable for path-$\mathbf{P}$-bisimulation, i.e.

- let $\mathbf{M}$ be a category of models,
- let $\mathbf{P}$ be a small subcategory of $\mathbf{M}$ of path objects,
  such that $\mathbf{P}$ and $\mathbf{M}$ have a common initial object I, and
- let X_1 and X_2 be objects in $\mathbf{M}$.

Is there a way to associate coalgebras (A_i, α_i) with X_i, $i=1,2$, such that X_1 and X_2 are path-$\mathbf{P}$-bisimilar iff (A_1, α_1) and (A_2, α_2) are AM-bisimilar?

We show in the following that indeed we can define an operator T from $\mathbf{M}$ to the coalgebras in $\mathbf{Set}_F$ such that two objects in $\mathbf{M}$ are path-$\mathbf{P}$-bisimilar iff the corresponding coalgebras are AM-bisimilar. This result shows that AM-bisimulation is a least as powerful as path-$\mathbf{P}$-bisimulation.

Theorem 7.6. *Let* $\mathbf{M}$ *be a category of models, let* $\mathbf{P}$ *be a small subcategory of* $\mathbf{M}$ *of path objects, such that* $\mathbf{P}$ *and* $\mathbf{M}$ *have a common initial object* I. *There exists an operator* $T:\mathbf{M}\to\mathbf{Set}_F$ *such that*:

Objects X_1 *and* X_2 *of* $\mathbf{M}$ *are* (*strong*) *path-*$\mathbf{P}$*-bisimilar iff there exists a* (*backward-forward*) *AM-bisimulation* (R,γ) *between* $(A,\alpha):=T(X_1)$ *and* $(B,\beta):=T(X_2)$ *with* $(\iota_1, \iota_2)\in R$, *where* $\iota_1: I\to X_1$ (*resp.* $\iota_2: I\to X_2$) *is the unique path from* I *to* X_1 (*resp.* X_2).

Proof. We define for each object X of $\mathbf{M}$ a labelled transition system $T_{\mathbf{P},\mathbf{M}}(X)=(S,\sigma)$ in $\mathbf{Set}_F$ over the set of labels $\bigcup_{P,Q\in\mathbf{P}}\{(m,P,Q)\mid m\in Hom_{\mathbf{M}}(P,Q)\}$:

$S:=\{p:P\to X \mid P\in\mathbf{P},\ p\in Hom_{\mathbf{M}}(P,X)\}$.
$(m,P,Q,q)\in\sigma(p):\iff q\circ m=p$, see Fig. 12.

Let X_1 and X_2 be path-$\mathbf{P}$-bisimilar. Then there exists a set R consisting of pairs of paths (p_1,p_2) with common domain P. We define a map $\gamma: R\to FR$ and show that (R,γ) is an AM-bisimulation between (A,α) and (B,β). Let for all $(p_1,p_2),(q_1,q_2)\in R$, $p_i:P\to X_i$, $q_i:Q\to X_i$, $i=1,2, m\in Hom_{\mathbf{M}}(P,Q)$

$$(m,P,Q,q_1,q_2)\in\gamma(p_1,p_2):\iff q_1\circ m=p_1\wedge q_2\circ m=p_2.$$

Let $(m,P,Q,q_1)\in(\alpha\circ\pi_1)(p_1,p_2)$. Then $(m,P,Q,q_1)\in\alpha(p_1)$ and therefore $q_1\circ m=p_1$. As $(p_1,p_2)\in R$ this implies by condition (i) of the definition of path-$\mathbf{P}$-bisimulation that there is some $q_2:Q\to X_2$ with $q_2\circ m=p_2$ and $(q_1,q_2)\in R$. Thus, we have $(m,P,Q,q_1,q_2)\in\gamma(p_1,p_2)$ and hence $(m,P,Q,q_1)\in(F\pi_1\circ\gamma)(p_1,p_2)$.

Let $(m,P,Q,q_1)\in(F\pi_1\circ\gamma)(p_1,p_2)$. Then there exists some $q_2:Q\to X_2$ such that $(m,P,Q,q_1,q_2)\in\gamma(p_1,p_2)$. By the above definition of γ this implies $q_1\circ m=p_1$. By definition of $T_{\mathbf{P},\mathbf{M}}(X_1)$ we get $(m,P,Q,q_1)\in\alpha(p_1)$ and therefore $(m,P,Q,q_1)\in(\alpha\circ\pi_1)(p_1,p_2)$.

Assume further that the set R is a strong path-**P**-bisimulation between X_1 and X_2. In order to prove that the constructed AM-bisimulation (R,γ) is backward–forward by Lemma 3.4 it is enough to show $(\alpha^-\circ\pi_1)\subseteq(F\pi_1\circ\gamma^-)$.

Let $(m,P,Q,p_1)\in(\alpha^-\circ\pi_1)(q_1,q_2)$. Then we have $(m,P,Q,p_1)\in\alpha^-(q_1)$ and therefore $(m,P,Q,q_1)\in\alpha(p_1)$. Thus by definition of (A,α) we get the equation $q_1\circ m=p_1$. As $(q_1,q_2)\in R$ we get by (iii) that $(q_1\circ m,q_2\circ m)\in R$. By definition of γ we obtain $(m,P,Q,q_1,q_2)\in\gamma(q_1\circ m,q_2\circ m)$. This implies $(m,P,Q,q_1\circ m,q_2\circ m)\in\gamma^-(q_1,q_2)$ and we get finally by the equation $q_1\circ m=p_1$ that $(m,P,Q,p_1)\in(F\pi_1\circ\gamma^-)(q_1,q_2)$.

Now let (R,γ) be an AM-bisimulation between (A,α) and (B,β), such that $(\imath_1,\imath_2)\in R$. As R may relate paths p_1 and p_2 with different domains we define a subset of R to establish the path-**P**-bisimulation:

$$R':=\{(p_1,p_2)\in R\mid\exists P\in\mathbf{P}\colon\ p_1\in Hom_{\mathbf{M}}(P,X_1),\ p_2\in Hom_{\mathbf{M}}(P,X_2)\}.$$

We have $(\imath_1,\imath_2)\in R'$. Now let $(p_1,p_2)\in R'$, $m\in Hom_{\mathbf{M}}(P,Q)$ for some object Q in **P** and $q_1:Q\to X_1$ a path, such that $q_1\circ m=p_1$. This implies $(p_1,p_2)\in R$ and $(m,P,Q,q_1)\in(\alpha\circ\pi_1)(p_1,p_2)$. As (R,γ) is an AM-bisimulation there exists some $q_2:Q\to X_2$ with $(m,P,Q,q_1,q_2)\in\gamma(p_1,p_2)$. Therefore, we get $(m,P,Q,q_2)\in\beta(p_2)$ and thus by definition of (B,β) we have $q_2\circ m=p_2$. As q_1 and q_2 have the same domain and $(q_1,q_2)\in R$ we conclude $(q_1,q_2)\in R'$ and thus R' fullfills condition (i).

Assume further that the AM-bisimulation (R,γ) is backward–forward. To show condition (iii) let $(q_1,q_2)\in R'$, i.e. q_1 and q_2 are paths with the same domain Q, let $m\in Hom_{\mathbf{M}}(P,Q)$. Then $q_1\circ m\in Hom_{\mathbf{M}}(P,X_1)$. By definition of the operator $T_{\mathbf{P},\mathbf{M}}$ we get $(m,P,Q,q_1)\in\alpha(q_1\circ m)$. This implies

$$(m,P,Q,q_1\circ m)\in\alpha^-(q_1)=(\alpha^-\circ\pi_1)(q_1,q_2)=(F\pi_1\circ\gamma^-)(q_1,q_2).$$

Thus there exists some $p_2:P\to X_2$ such that $(m,P,Q,q_1\circ m,p_2)\in\gamma^-(q_1,q_2)$. As R is a backward–forward AM-bisimulation we get $(m,P,Q,p_2)\in\beta^-(q_2)$ and therefore $(m,P,Q,q_2)\in\beta(p_2)$. With the definition of $T_{\mathbf{P},\mathbf{M}}$ we conclude $q_2\circ m=p_2$. Thus $(q_1\circ m,q_2\circ m)\in R'$. □

Consequently, any concrete notion of bisimulation on some model **M** for concurrent processes that can be captured by the framework of [22], i.e. for which two objects are bisimilar iff there is a path-**P**-bisimulation between them in the corresponding category, can be given a characterization in terms of coalgebras and hence transition systems. However, the transition systems obtained by the above construction are rather abstract and not related directly to the intuitive understanding of the given bisimulation. For a notion of bisimulation on some model there are often some quite natural ways of defining an operator T that associates a transition system with an object in some model **M** such that two objects O_1, O_2 are bisimilar iff the corresponding transition systems

$T(O_1)$ and $T(O_2)$ are bisimilar, see e.g. [26]. We deal with such "natural" operators in the next section.

7.2. *From AM-bisimulation to path-P-bisimulation*

We now consider the question: let B be a concrete notion of bisimulation in some category $\mathbf{M}$ of models, that can be modelled as AM-bisimulation, i.e. there is an operator $T:\mathbf{M} \to \mathbf{Set_F}$, where F is the functor $F(X) = \mathscr{P}(L \times X)$ for some set of labels L, such that objects X_1 and X_2 of $\mathbf{M}$ are B-bisimilar iff $T(X_1)$ and $T(X_2)$ are AM-bisimilar. Under which conditions can we model B as path-$\mathbf{P}$-bisimulation for some path category $\mathbf{P}$? The AM-bisimulation is a path-**Bran**-bisimulation in the category $\mathbf{T}_L$ (see Remark 7.4) but the question is to find a subcategory $\mathbf{P}$ of $\mathbf{M}$ that enables us to give a characterization of B as path-$\mathbf{P}$-bisimulation in the category $\mathbf{M}$.

The following result suggests to take as objects of the category $\mathbf{P}$ those objects X which have a "final reachable" state in $T(X)$. If it is then possible to select morphisms for $\mathbf{P}$ such that the operator T is "connecting" to the category $\mathbf{P}$ then the desired characterization can be concluded.

Let $\mathbf{M}$ be a category of models, let $\mathbf{P}$ be a small subcategory of $\mathbf{M}$ of path objects, such that $\mathbf{P}$ and $\mathbf{M}$ have a common initial object I. Let L be a set of labels, T an operator which associates to each object X from $\mathbf{M}$ a transition system $T(X) = (S, \longrightarrow, i_S)$ in $\mathbf{T}_L$. We call the operator T *connecting to* $\mathbf{P}$ iff the following conditions C1–C5 hold:

C1: T evolves into a functor from $\mathbf{M}$ to $\mathbf{T}_L$.

C2: For all $P \in \mathbf{P}$ holds: there exists a state f in the transition system $T(P) = (S, \longrightarrow, i_S)$ such that $\forall x \in S : x \longrightarrow^* f$. We choose one of these states and call it the *final reachable state* f of $T(P)$.

C3: Let X be an object of $\mathbf{M}$ and $s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} s_n$, $n \geqslant 1$, be a derivation in $T(X)$, such that s_1 is the initial state of $T(X)$. Then there exists an object P in $\mathbf{P}$, such that $T(P)$ has a derivation $t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} t_n$, where t_1 is the initial and t_n the final reachable state of $T(P)$. Further on for any object Y of $\mathbf{M}$ with a derivation $u_1 \xrightarrow{a_1} u_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} u_n$ in $T(Y)$, where u_1 is the initial state of $T(Y)$, there exists a morphism $p:P \to Y$ in $\mathbf{M}$ such that $T(p)(t_i) = u_i$, $i = 1, 2, \ldots, n$.

C4: For derivations of length $n = 1$ the initial object I can be chosen as object P of $\mathbf{P}$ in condition C3.

C5: Let P and Q be objects of $\mathbf{P}$, X an object of $\mathbf{M}$, $p:P \to X$, $q:Q \to X$ morphisms in $\mathbf{M}$, $m:P \to Q$ a morphisms in $\mathbf{P}$. Let $t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} t_n$ be a derivation of $T(P)$, where t_1 is the initial state and t_n the final reachable state of $T(P)$. Then holds:

$$q \circ m = p \iff \forall 1 \leqslant i \leqslant n\colon\ T(q \circ m)(t_i) = T(p)(t_i). \tag{2}$$

Lemma 7.7. *Let* $\mathbf{M}$ *be a category of models, let* $\mathbf{P}$ *be a small subcategory of* $\mathbf{M}$ *of path objects, such that* $\mathbf{P}$ *and* $\mathbf{M}$ *have a common initial object I. Let X be an object*

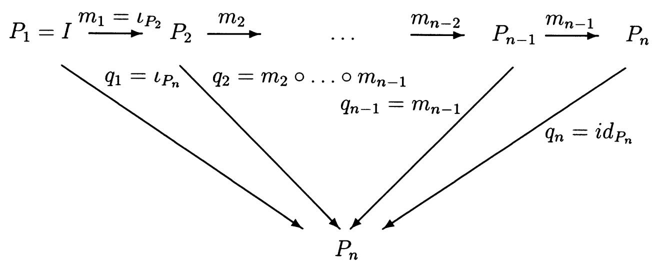

Fig. 13. Illustration for the proof of Lemma 7.7.

in $\mathbf{M}$. *We define*

$$T_{\mathbf{P},\mathbf{M}}(X) := (S, \longrightarrow, \iota_X)$$

as the transition system over $L := \bigcup_{P,Q\in\mathbf{P}} \{(m,P,Q) \mid m \in Hom_{\mathbf{P}}(P,Q)\}$, *where*

$S := \{p : P \to X \mid P \in \mathbf{P},\ \mathbf{p} \in Hom_{\mathbf{M}}(P,X)\}$.
$p \overset{(m,P,Q)}{\longrightarrow} q :\Longleftrightarrow q \circ m = p$, *see Fig.* 12.
ι_X *is the morphism from* I *to* X.

The operator $T_{\mathbf{P},\mathbf{M}}$ *is connecting to* $\mathbf{P}$.

Proof. Let $f : X_1 \to X_2$ be a morphism in $\mathbf{M}$. Choosing $T_{\mathbf{P},\mathbf{M}}(f)(p) := f \circ p$, where $p : P \to X_1$ is a state of $T_{\mathbf{P},\mathbf{M}}(X_1)$ and P is an object in $\mathbf{P}$, turns the operator $T_{\mathbf{P},\mathbf{M}}$ into a functor. As final reachable state of the transition system $T_{\mathbf{P},\mathbf{M}}(P)$ take the identity of P, i.e. id_P.

Let X be an object of $\mathbf{M}$. For $n=1$ condition C3 holds for the initial object. For $n>1$ consider a derivation $s_1 \overset{a_1}{\longrightarrow} s_2 \overset{a_2}{\longrightarrow} \cdots \overset{a_{n-1}}{\longrightarrow} s_n$ in $T_{\mathbf{P},\mathbf{M}}(X)$, where s_1 is the initial state. By the above definition of the operator $T_{\mathbf{P},\mathbf{M}}$ there exist path objects P_i, morphisms $p_i : P_i \to X$, $1 \leqslant i \leqslant n$, and morphisms $m_j : P_j \to P_{j+1}$, $1 \leqslant j \leqslant n-1$, such that
(1) $a_j = (m_j, P_j, P_{j+1})$, $1 \leqslant j \leqslant n-1$,
(2) $p_{j+1} \circ m_j = p_j$, $1 \leqslant j \leqslant n-1$ and
(3) $P_1 = I$, $m_1 = \iota_{P_2} : I \to P_2$.
Choose as path object $P = P_n$. Let $q_i := \prod_{k=i}^{n-1} m_k : P_i \to P_n$ for $1 \leqslant i \leqslant n$. Then $q_1 = \iota_{P_n}$ and $p_n = id_{P_n}$. Thus in $T_{\mathbf{P},\mathbf{M}}(P_n)$ we find the derivation $q_1 = \iota_{P_n} \overset{a_1}{\longrightarrow} q_2 \overset{a_2}{\longrightarrow} \cdots \overset{a_{n-1}}{\longrightarrow} q_n$ (see Fig. 13).

Let Y be an object of $\mathbf{M}$ with a derivation

$$u_1 \overset{(\iota_{P_2}, I, P_2)}{\longrightarrow} u_2 \overset{(m_2,P_2,P_3)}{\longrightarrow} \cdots \overset{(m_{n-1},P_{n-1},P_n)}{\longrightarrow} u_n$$

in $T_{\mathbf{P},\mathbf{M}}(Y)$, where u_1 is the initial state of $T_{\mathbf{P},\mathbf{M}}(Y)$. We obtain:
(1) $u_i \in Hom_{\mathbf{M}}(P_i, Y)$, $1 \leqslant i \leqslant n$,
(2) $u_1 = \iota_X : I \to Y$ and
(3) $u_i = u_{i+1} \circ m_i$, $1 \leqslant i \leqslant n-1$.
For the morphism $u_n : P_n = P \to Y$ holds $T_{\mathbf{P},\mathbf{M}}(u_n)(p_i) = u_i$, $1 \leqslant i \leqslant n$.

Let P, Q be objects of $\mathbf{P}$, X an object of $\mathbf{M}$, $p: P \to X$, $q: Q \to X$ morphisms in $\mathbf{M}$, $m: P \to Q$ a morphism in $\mathbf{P}$. Let $p_1 \xrightarrow{a_1} p_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} p_n$ be a derivation in $T_{\mathbf{P},\mathbf{M}}(P)$, where p_1 is the initial state and p_n is the final reachable state of $T_{\mathbf{P},\mathbf{M}}(P)$. As in the proof of condition 3 we have some information on the structure of $T_{\mathbf{P},\mathbf{M}}(P)$:

(1) $a_j = (m_j, P_j, P_{j+1})$, where $m_j \in Hom_{\mathbf{P}}(P_j, P_{j+1})$, $1 \leqslant j \leqslant n-1$, for objects $P_i \in \mathbf{P}$, $1 \leqslant i \leqslant n$,
(2) $p_i \in Hom_{\mathbf{M}}(P_i, P)$, $1 \leqslant i \leqslant n$,
(3) $P_1 = I$ and $m_1 = \iota_{P_2}: I \to P_2$,
(4) $P_n = P$ and $p_n = id_P$, and
(5) $p_j = p_{j+1} \circ m_j$, $1 \leqslant j \leqslant n-1$.

Let $T_{\mathbf{P},\mathbf{M}}(q \circ m)(p_i) = T_{\mathbf{P},\mathbf{M}}(p)(p_i)$ for $1 \leqslant i \leqslant n$. Choosing $i = n$ we have $p_n = id_P$, thus we obtain:

$$\begin{aligned} q \circ m &= q \circ m \circ id_P \\ &= q \circ m \circ p_n \\ &= T_{\mathbf{P},\mathbf{M}}(q \circ m)(p_n) \\ &= T_{\mathbf{P},\mathbf{M}}(p)(p_n) \\ &= p \circ p_n \\ &= p. \end{aligned}$$

As we need initial states and a rich structure of morphisms for connecting operators we use the category $\mathbf{T}_L$ as a link between the category of models $\mathbf{M}$, where we study a concrete notion of bisimulation, and the category $\mathbf{Set}_F$, where the concept of AM-bisimulation was introduced. $\square$

Definition 7.8. Let $\mathscr{T}_1 = (S, \longrightarrow_1, s_1)$ and $\mathscr{T}_2 = (T, \longrightarrow_2, t_1)$ be transition systems in $\mathbf{T}_L$, (A, α) the coalgebra with $\mathscr{T}_{(A,\alpha)} = (S, \longrightarrow_1)$ and (B, β) the coalgebra with $\mathscr{T}_{(B,\beta)} = (T, \longrightarrow_2)$.

- $\mathscr{T}_1$ and $\mathscr{T}_2$ are *AM-bisimilar* iff there exists an AM-bisimulation (R, γ) between (A, α) and (B, β) with $(s_1, t_1) \in R$.
- $\mathscr{T}_1$ and $\mathscr{T}_2$ are *backward–forward AM-bisimilar* iff there exists an AM-bisimulation (R, γ) between (A, α) and (B, β) with $(s_1, t_1) \in R$ and (R, γ^-) is an AM-bisimulation between (A, α^-) and (B, β^-).

Theorem 7.9. *Let* $\mathbf{M}$ *be a category of models. Let* B *be a bisimulation on* $\mathbf{M}$, *which an operator* $T: \mathbf{M} \to \mathbf{T_L}$ *models as AM-bisimulation.*

If there exists a small subcategory $\mathbf{P}$ *of* $\mathbf{M}$, *such that* $\mathbf{P}$ *and* $\mathbf{M}$ *have a common initial object* I *and the operator* T *is connecting to* $\mathbf{P}$, *then objects* X_1 *and* X_2 *of* $\mathbf{M}$ *are path-*$\mathbf{P}$*-bisimilar iff* $T(X_1) = (S, \longrightarrow, s_1)$ *and* $T(X_2) = (T, \longrightarrow, t_1)$ *are AM-bisimilar (iff* X_1, X_2 *are B-bisimilar).*

Proof. Let (R, γ) be an AM-bisimulation between $T(X_1) = (S, \longrightarrow, s_1)$ and $T(X_2) = (T, \longrightarrow, t_1)$ with $(s_1, t_1) \in R$. To obtain a path-$\mathbf{P}$-bisimulation R' between X_1 and X_2 we consider a state (s,t) in (R,γ) which is reachable from (s_1,t_1). Let

$$(s_1,t_1)\xrightarrow{a_1}(s_2,t_2)\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}(s_n,t_n)=(s,t)$$

be a derivation of (s,t). With the projections π_1 and π_2 we obtain derivations $s_1\xrightarrow{a_1}s_2\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}s_n$ and $t_1\xrightarrow{a_1}t_2\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}t_n$ in $T(X_1)$ resp. $T(X_2)$. By condition C3 there exists an object P of $\mathbf{P}$, such that $T(P)$ has a derivation $u_1\xrightarrow{a_1}u_2\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}u_n$. Further on there exist morphisms $p_i : P\to X_i,\ i=1,2$, such that $T(p_1)(u_j)=s_j$ and $T(p_2)(u_j)=t_j,\ j=1,2,\dots,n$.

Let $M(s,t)$ be the set of all pairs of morphisms (p_1,p_2), which can be obtained from a reachable state (s,t) in (R,γ) in the way described above. I.e. first consider all derivations of (s,t), second all objects P of $\mathbf{P}$ corresponding to a derivation, and finally any pair of morphisms (p_1,p_2), which maps $T(P)$ on $T(X_1)$ (resp. $T(X_2)$) in the way described above. We claim that the set

$$R' := \bigcup_{(s,t)\in R,\,(s,t)\text{ reachable}} M(s,t)$$

is a path-$\mathbf{P}$-bisimulation between X_1 and X_2. Condition C4 implies $(\iota_1,\iota_2)\in R'$, where $\iota_i : I\to X_i,\ i=1,2$.

Let $(p_1,p_2)\in R'$ with $p_i : P\to X_i,\ i=1,2$, for some P in $\mathbf{P}$. Let $m : P\to Q$ be some morphism in $\mathbf{P}$, $q_1 : Q\to X_1$ be a path in $\mathbf{M}$ such that $q_1\circ m=p_1$. Using the definition of R' we obtain the following derivations:

in (R,γ): $(s_1,t_1)\xrightarrow{a_1}(s_2,t_2)\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}(s_n,t_n)$,

in $T(X_1)$: $s_1\xrightarrow{a_1}s_2\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}s_n$,

in $T(X_2)$: $t_1\xrightarrow{a_1}t_2\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}t_n$ and

in $T(P)$: $u_1\xrightarrow{a_1}u_2\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}u_n$.

By definition of R' holds $T(p_1)(u_j)=s_j,\ j=1,2,\dots,n$, and $T(p_2)(u_j)=t_j,\ j=1,2,\dots,n$. As $T(m)$ is a morphism in $\mathbf{T}_L$, there exists a derivation

in $T(Q)$: $T(m)(u_1)\xrightarrow{a_1}T(m)(u_2)\xrightarrow{a_2}\cdots\xrightarrow{a_{n-1}}T(m)(u_n)$.

Condition C2 implies that there exists a final reachable state f in $T(Q)$. Therefore we obtain a derivation

in $T(Q)$: $T(m)(u_n)\xrightarrow{a_n}v_{n+1}\xrightarrow{a_{n+1}}\cdots\xrightarrow{a_{n+k-1}}v_{n+k}=f$.

Combining these derivations of $T(Q)$ we obtain – using the morphism $T(q_1)$ and $p_1=q_1\circ m$ – a derivation

in $T(X_1)$: $s_1\xrightarrow{a_1}\cdots\xrightarrow{a_{n-1}}s_n\xrightarrow{a_n}T(q_1)(v_{n+1})\xrightarrow{a_{n+1}}\cdots\xrightarrow{a_{n+k-1}}T(q_1)(v_{n+k})$.

As (R,γ) is an AM-bisimulation, there exist derivations

in (R,γ): $(s_n,t_n)\xrightarrow{a_n}(T(q_1)(v_{n+1}),t_{n+1})\xrightarrow{a_{n+1}}\cdots\xrightarrow{a_{n+k-1}}(T(q_1)(v_{n+k}),t_{n+k})$

and

$$\text{in } T(X_2)\text{: } t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} t_n \xrightarrow{a_n} t_{n+1} \xrightarrow{a_{n+1}} \cdots \xrightarrow{a_{n+k-1}} t_{n+k}$$

for states $t_{n+1},\ldots,t_{n+k} \in T(X_2)$. Thus by condition C3 there exists a morphism $q_2 : Q \to X_2$ such that $T(q_2) \circ T(m)(u_j) = t_j$, $j = 1,2,\ldots,n$, and $T(q_2)(v_{n+j}) = t_{n+j}$, $j = 1,2,\ldots,k$. This implies by condition C5: $q_2 \circ m = p_2$. By construction we have $(q_1,q_2) \in R'$.

Let R' be a path-**P**-bisimulation between X_1 and X_2, let $T(X_1) = (S, \longrightarrow_1, s_1)$ and $T(X_2) = (T, \longrightarrow_2, t_1)$, let (A,α) and (B,β) be the coalgebras with $\mathscr{T}_{(A,\alpha)} = (S, \longrightarrow_1)$ and $\mathscr{T}_{(B,\beta)} = (T, \longrightarrow_2)$.

Let P be an object of **P**, f be the final reachable state of $T(P)$, X be an object of **M** and $p : P \to X$ a path. $\mathit{Reach}(p,P,X) := T(p)(f)$ denotes the image of the final reachable state f in the transition system $T(P)$ under the morphism $T(p)$. Let

$$\begin{aligned} R := \{(s,t) \mid\ & \exists P \in \mathbf{P}, (p_1,p_2) \in R': \\ & p_1 : P \to X_1,\ p_2 : P \to X_2, \\ & s = \mathit{Reach}(p_1,P,X_1),\ t = \mathit{Reach}(p_2,P,X_2)\}. \end{aligned}$$

Let $(s,t),(s',t') \in R$, let P,Q be objects of **P**, let $(p_1,p_2),(q_1,q_2) \in R'$, such that $s = \mathit{Reach}(p_1,P,X_1)$, $t = \mathit{Reach}(p_2,P,X_2)$, $s' = \mathit{Reach}(q_1,Q,X_1)$, $t' = \mathit{Reach}(q_2,Q,X_2)$. Define

$$(a,s',t') \in \gamma(s,t)$$

iff there exists a morphism $m : P \to Q$, such that

$p_1 = q_1 \circ m$,
$p_2 = q_2 \circ m$ and
$T(m)(f) \xrightarrow{a} g$ is a transition in $T(Q)$, where f is the final reachable state of $T(P)$ and g is the final reachable state of $T(Q)$.

We claim that (R,γ) is an AM-bisimulation between (A,α) and (B,β) with $(s_1,t_1) \in R$.

Due to condition C4 we have $(s_1,t_1) \in R$. Let $(a,s') \in (\alpha \circ \pi_1)(s,t)$. As $(s,t) \in R$ there exists an object $P \in \mathbf{P}$ and morphisms $p_1 : P \to X_1$, $p_2 : P \to X_2$ such that $s = \mathit{Reach}(p_1, P,X_1), t = \mathit{Reach}(p_2,P,X_2)$ and $(p_1,p_2) \in R'$. Let

$$\text{in } T(P) : u_1 \xrightarrow{a_1} u_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} u_n$$

be a derivation of the final reachable state u_n from the initial state u_1. Then we obtain

$$\text{in } (A,\alpha) : T(p_1)(u_1) \xrightarrow{a_1} T(p_1)(u_2) \xrightarrow{a_2} \ldots \xrightarrow{a_{n-1}} T(p_1)(u_n) = s$$

a derivation for s. As $(a,s') \in \alpha(s)$ we get

$$\text{in } (A,\alpha) : T(p_1)(u_1) \xrightarrow{a_1} T(p_1)(u_2) \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} s \xrightarrow{a} s'.$$

By condition C3 there exists an object Q in **P** such that we find a derivation

$$\text{in } T(Q) : v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} v_n \xrightarrow{a} v_{n+1},$$

where v_1 is the initial state and v_{n+1} is the final reachable state of $T(Q)$. Further there exist morphisms $m: P \to Q$ with $T(m)(u_j) = v_j$, $j = 1, 2, \dots, n$, and $q_1 : Q \to X_1$ with $T(q_1)(v_j) = T(p_1)(u_j)$, $j = 1, 2, \dots, n$, and $T(q_1)(v_{n+1}) = s'$. This implies with condition C5 that $q_1 \circ m = p_1$. As R' is a path-**P**-bisimulation, there exists a morphism $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ and $(q_1, q_2) \in R'$. Thus $(Reach(q_1, P, X_1), Reach(q_2, Q, X_2)) \in R$, where $s' = Reach(q_1, Q, X_1)$ and $(a, s', Reach(q_2, Q, X_2)) \in \gamma(s, t)$. Therefore $(a, s') \in (F\pi_1 \circ \gamma)(s, t)$.

Let $(a, s') \in (F\pi_1 \circ \gamma)(s, t)$. Then there exists some $t' \in B$ with $(a, s', t') \in \gamma(s, t)$. By definition of R and γ we obtain: there exist objects P and Q in **P**, morphisms $p_1 : P \to X_1, q_1 : Q \to X_1$ and a morphism $m: P \to Q$ such that holds: $s = Reach(p_1, P, X_1)$, $s' = Reach(q_1, Q, X_1)$, $p_1 = q_1 \circ m$, $T(m)(f) \xrightarrow{a} g$ is a transition in $T(Q)$, where f is the final reachable state of $T(P)$ and g is the final reachable state of $T(Q)$. This implies $s = T(p_1)(f) = T(q_1 \circ m)(f) \xrightarrow{a} T(q_1)(g) = s'$ in (A, α) and thus $(a, s') \in (\alpha \circ \pi_1)(s, t)$. □

For an operator T the property "connecting to **P**" is not sufficient to ensure the equivalence between backward–forward AM-bisimulation and strong path-**P**-bisimulation, as the following example shows:

Example 7.10. Consider the category $\mathbf{T}_L$ with the path category **Bran**, defined in Section 7. Choose as operator T the identity Id on $\mathbf{T}_L$. T is connecting to **Bran**. For the transition systems $\mathscr{T}_0$ and $\mathscr{T}_1$ from Fig. 8 holds: $\mathscr{T}_0$ and $\mathscr{T}_1$ are strong path-**Bran**-bisimilar by Theorem 7.6, as the transition systems $T_{\mathbf{Bran},\mathbf{T}_L}(\mathscr{T}_0)$ and $T_{\mathbf{Bran},\mathbf{T}_L}(\mathscr{T}_1)$ are the same. But there is no backward–forward AM-Bisimulation (R, γ) between $\mathscr{T}_0$ and $\mathscr{T}_1$ with $(s_0, t_0) \in R$. I.e. strong path-**P**-bisimulation does not imply backward–forward AM-bisimulation in general.

Remark 7.11. It is an open problem whether for an operator T which is connecting to some path category **P** backward–forward AM-bisimulation implies strong path-**P**-bisimulation in general.

By Lemma 7.7 there always exists a connecting operator for any category **M** of models with subcategory **P**. $T_{\mathbf{P},\mathbf{M}}$ and any other operator T which is connecting to **P** yield the same bisimulation in the following sense.

Corollary 7.12. *Let* **M** *be a category of models, let* **P** *be a small subcategory of* **M** *of path objects, such that* **P** *and* **M** *have a common initial object I. Let T be a connecting operator to* **P**, *let* X_1 *and* X_2 *be objects of* **M**.

$T(X_1)$ and $T(X_2)$ are AM-bisimilar iff $T_{\mathbf{P},\mathbf{M}}(X_1)$ and $T_{\mathbf{P},\mathbf{M}}(X_2)$ are AM-bisimilar.

Fig. 14 summarizes how **P**-bisimulation, path-**P**-bisimulation, and AM-bisimulation are related:[5] Under certain restrictions **P**-bisimulation and strong path-**P**-bisimulation

[5] For simplicity in this diagram we do *not* mention the conditions which are (sometimes) necessary to establish an equivalence.

$$\begin{array}{|ccc|}
\mathbf{P}\text{-bisimulation} & & \\
\Updownarrow \text{(Theorem 7.5)} & & \\
\text{strong path-}\mathbf{P}\text{-bisimulation} & \Rightarrow & \text{path-}\mathbf{P}\text{-bisimulation} \\
\Updownarrow \text{(Theorem 7.6)} & & \Updownarrow \text{(Theorem 7.6)} \\
\text{bf AM-bisimulation with } T_{\mathbf{P},\mathbf{M}} & \Rightarrow & \text{AM-bisimulation with } T_{\mathbf{P},\mathbf{M}} \\
\not\Downarrow \text{(Example 7.10)} & & \Updownarrow \text{(Theorem 7.9)} \\
\text{bf AM-bisimulation with } T \neq T_{\mathbf{P},\mathbf{M}} & \Rightarrow & \text{AM-bisimulation with } T \neq T_{\mathbf{P},\mathbf{M}}
\end{array}$$

Fig. 14. Relations between the different bisimulation concepts.

are equivalent (Theorem 7.5). For path-**P**-bisimulation (resp. strong path-**P**-bisimulation) and AM-bisimulation (resp. backward–forward AM-bisimulation) holds: If we are in a setting suitable for path-**P**-bisimulation (resp. strong path-**P**-bisimulation), there exists an operator such that these concepts coincide, take e.g. the operator $T_{\mathbf{P},\mathbf{M}}$ (Theorem 7.6). If the operator T is different from $T_{\mathbf{P},\mathbf{M}}$, the situation becomes more complex: for an operator T that is connecting to **P**, AM-bisimulation and path-**P**-bisimulation describe the same equivalence (Theorem 7.9), and AM-bisimulation with T is the same as AM-bisimulation with $T_{\mathbf{P},\mathbf{M}}$ (Corollary 7.12). In general strong path-**P**-bisimulation does not imply backward–forward AM-bisimulation even for an operator connecting to **P** (Example 7.10). It is an open question if the converse holds (Remark 7.11).

The next section on bisimulations on event structures includes different instantiations of the general relations displayed in Fig. 14: Taking the category of event structures as category of models, i.e. $\mathbf{M}=\mathbf{E}_{Act}$, choosing the path category [6] **P** as

Lin, and taking the operator $T \neq T_{\mathbf{P},\mathbf{M}}$ as T_{int}, all concepts of Fig. 14 except "bf AM-bisimulation with $T \neq T_{\mathbf{P},\mathbf{M}}$" are equivalent (Corollary 8.9). Backward–forward AM-bisimulation with T_{int} implies these concepts, and – although T_{int} is connecting – strong path-**Lin**-bisimulation does not imply backward–forward AM-bisimulation with T_{int}.

Step, and taking the operator $T \neq T_{\mathbf{P},\mathbf{M}}$ as T_{step}, all the concepts on the right-hand side are equivalent, i.e. path-**Step**-bisimulation, AM-bisimulation with $T_{\mathbf{Step},\mathbf{E}_{Act}}$ and AM-bisimulation with T_{step} (Corollary 8.9) coincide, while all the concepts on the left-hand side describe equivalences different from path-**Step**-bisimulation.

Pos, and taking the operator $T \neq T_{\mathbf{P},\mathbf{M}}$ as T_{pos}, some concepts on the right-hand side are different: path-**Pos**-bisimulation implies AM-bisimulation with T_{pos}, but the converse does not hold. Consequently, the operator T_{pos} is not connecting to **Pos** (Lemma 8.10). For the left-hand side holds: **PosC**-bisimulation and strong path-**PosC**-bisimulation coincide [7] (Theorem 8.11), but are different from path-**Pos**-bisimulation. Backward–forward AM-bisimulation with T_{pos} differs from AM-bisimulation with T_{pos}. It is open how strong path-**Pos**-bisimulation and backward–forward AM-bisimulation with T_{pos} are related.

[6] The above-mentioned categories will be defined in Section 8.1, the operators T_* are introduced in Section 8.3.

[7] These results are obtained for a slightly broader category of event structures.

bisimulation	AM I	AM II	path-**P**	**P**
interleaving	T_{int}	$T_{\mathbf{Lin},\mathbf{E}_{Act}}$ (+bf)	**Lin** (+ s)	**Lin**
bf	T_{int} + bf			
step	T_{step}	$T_{\mathbf{Step},\mathbf{E}_{Act}}$	**Step**	
pomset	T_{pom}			
weak history preserving	T_{whp}			
history preserving	T_{hp}	$T_{\mathbf{Pos},\mathbf{E}_{Act}}$	**Pos**	
strong history preserving	T_{hp} + bf	$T_{\mathbf{Pos},\mathbf{E}_{Act}}$ + bf	**Pos** + s	**PosC**

Fig. 15. Modelling bisimulations on event structures.

8. An application: bisimulations on event structures

In the previous sections we studied the relation between the various characterizations of bisimulation abstractly. In order to get still more insight into the power and the limitations of the methods we consider here a variety of *concrete notions* of bisimulation on event structures which we try to model in terms of the abstract concepts. For this we focus here on the coalgebraic approach of Aczel and Mendler [5], i.e. on AM-bisimulation, and the categorical setting of Joyal et al. [22], i.e. on path-**P**-bisimulation and **P**-bisimulation.

Fig. 15 summarizes our results:

Column "bisimulation" lists the concrete notions of bisimulation on event structures we study in this section. We define these bisimulations in Section 8.2.

Column "AM I" shows that we are able to model all these bisimulations directly in the coalgebraic framework of Aczel and Mendler [5] by suitable operators. Its entries are the names of the operators T_*, which we use to model a concrete notion of bisimulation as AM-bisimulation – see Section 8.3. We put "+ bf" to indicate that we use backward–forward AM-bisimulation. The transition systems obtained by the operators in this columns have the configurations (resp. derivations) as states.

Column "AM II" displays further possibilities to model a bisimulation in the coalgebraic framework of Aczel and Mendler [5]. These results are achieved by applying Theorem 7.6 on the modelling of a concrete bisimulation on event structures as a path-**P**-bisimulation. We do this for interleaving bisimulation in Corollary 8.4, for step bisimulation in Corollary 8.9, and for history preserving bisimulation (resp. strong history preserving bisimulation) in Corollary 8.12. Again we put "+ bf" to indicate that we take backward–forward AM-bisimulation. Putting "+ bf" in brackets expresses that AM-bisimulation and backward–forward AM-bisimulation coincide for this particular operator. The transition systems obtained by the operators $T_{*,\mathbf{E}_{Act}}$ have morphisms of $\mathbf{E}_{Act}$ as states.

Column "path-P" shows the successful modelling in the categorical setting of Joyal et al. [22]. We model interleaving bisimulation as path-**Lin**-bisimulation (Corollary 8.4), step bisimulation as path-**Step**-bisimulation (Theorem 8.6) and history preserving bisimulation (resp. strong history preserving bisimulation) as path-**Pos**-bisimulation (resp. strong path-**Pos**-bisimulation) (Corollary 8.12). See Section 8.1 for the definition of these path categories. We put "+s" to indicate that we take strong path-**P**-bisimulation. "(s)" expresses that path-**P**-bisimulation and strong path-**P**-bisimulation coincide. The results concerning (strong) history preserving bisimulation are obtained from analogous results for event structures with consistency relation given in [22].

Column "P" deals with the concept of **P**-bisimulation of the categorical setting of Joyal et al. [22]. Here we give as a new result that interleaving bisimulation and **Lin**-bisimulation coincide (Theorem 8.1) and recall from [22] that for event structures with consistency relation [8] strong history preserving bisimulation is the same as **PosC**-bisimulation.

Besides the positive results for the categorical setting of [22] we also obtain some kind of negative results in the sense that for a concrete notion of bisimulation a "natural choice" of the path category **P** does not model this bisimulation as path-**P**-bisimulation and/or as **P**-bisimulation. In particular we obtain for the path categories

Lin: Strong path-**Lin**-bisimulation does not coincide with bf-bisimulation (Remark 8.5).

Step: Step bisimulation is different from **Step**-bisimulation (Corollary 8.8).

Pos: Pos-bisimulation and path-**Pos**-bisimulation are stronger concepts than pomset bisimulation (Corollary 8.12).

For the coalgebraic approach of Aczel and Mendler [5] we address in Section 8.5 the question if there are concrete notions of bisimulations which do not fit into this framework. As candidates we study generalized pomset bisimulation and partial word bisimulation.

8.1. Event structures

Let Act be a set of actions. A (*prime*) *event structure*

$$\mathscr{E} = (E, \leqslant, \sharp, l)$$

over the set of actions Act consists of

E, a set of events,
$\leqslant \subseteq E \times E$, a causal dependency relation, which is a partial order,
$\sharp \subseteq E \times E$, an irreflexive and symmetric conflict relation, and
$l : E \to Act$, a labelling function,

[8] A slight modification of the prime event structures that we use throughout this section.

which together satisfy:
(1) For all $e \in E$ the set $\downarrow(e) := \{e' \in E \mid e' \leqslant e\}$ is finite, and
(2) for all $d, e, f \in E$ holds: if $d \leqslant e$ and $d \sharp f$ then $e \sharp f$.
In an event structure two events $e_1, e_2 \in E$ are called *concurrent*, $e_1 co\, e_2$, iff they are not related by $\leqslant$ or $\sharp$. An event structure is called *finite* if its set of events is finite. An event structure is called *conflict-free* if its conflict relation is the empty set.

A set $X \subseteq E$ is called a *configuration* of the event structure $\mathscr{E}$ iff

X is a finite set,
X is leftclosed in E, and
for all $e, f \in X$ holds: $\neg e \sharp f$.

Sometimes we consider a configuration X itself as an event structure $(X, \leqslant \cap (X \times X), \emptyset, l_{|X})$. $Conf(\mathscr{E})$ denotes the set of all configurations of an event structure $\mathscr{E}$.

The *category* $\mathbf{E}_{Act}$ has as *objects* the prime event structures $\mathscr{E} = (E, \leqslant, \sharp, l)$ over Act, where $E \subseteq Ev$ for some "universal" set Ev of events. Let $\mathscr{E} = (E, \leqslant_E, \sharp_E, l_E)$ and $\mathscr{F} = (F, \leqslant_F, \sharp_F, l_F)$ be objects of $\mathbf{E}_{Act}$. A total map $\eta : E \to F$ is a *morphism* from $\mathscr{E}$ to $\mathscr{F}$ iff

$$\forall e \in E : l_E(e) = l_F(\eta(e)),$$

$$\forall X \in Conf(\mathscr{E}) : \eta(X) \in Conf(\mathscr{F}), \text{ and}$$

$$\forall X \in Conf(\mathscr{E})\, \forall e, e' \in X : \eta(e) = \eta(e') \Rightarrow e = e'.$$

To model bisimulations on event structures in the categorical setting of [22] we define subcategories of $\mathbf{E}_{Act}$:

Lin denotes the full subcategory of $\mathbf{E}_{Act}$ that consists of finite, conflict free event structures $(E, \leqslant, \emptyset, l)$, where the dependency relation is a total order.

Step is the full subcategory of $\mathbf{E}_{Act}$ that consists of steps as objects. Here a step is defined as follows:

Let $\mathscr{E} = (E, \leqslant_E, \emptyset, l_E)$, $\mathscr{M} = (M, \leqslant_M, \emptyset, l_M)$ be finite event structures with $E \cap M = \emptyset$ and $\leqslant_M = \{(m, m) \mid m \in M\}$. Then $\mathscr{F} := E; M$ denotes the event structure $(E \cup M, \leqslant_F, \emptyset, l_E \cup l_M)$, where $e \leqslant_F f$ iff $e = f$ or ($e \in E$ and $f \in M$) or $e \leqslant_E f$. Call an event structure

$$\mathscr{S} := \mathscr{M}_1; \mathscr{M}_2; \ldots; \mathscr{M}_n, \quad n \geqslant 0,$$

a *step*, where $\mathscr{M}_i = (M_i, \leqslant_{M_i}, \emptyset, l_i)$ are event structures, M_i are finite sets, M_i are pairwise disjoint and $\leqslant_{M_i} = \{(m, m) \mid m \in M_i\}$. For an event e of an event structure $\mathscr{E}$ let

$$depth_{\mathscr{E}}(e) := \begin{cases} 1 & \downarrow\{e\} = \{e\} \\ 1 + \max\{depth_{\mathscr{E}}(f) \mid f \in \downarrow\{e\}, f \neq e\} & \text{otherwise.} \end{cases}$$

Let $\mathscr{S} := \mathscr{M}_1; \mathscr{M}_2; \ldots; \mathscr{M}_n$, be a step, where all $\mathscr{M}_i$ are different from the empty event structure, let e be an event of $\mathscr{S}$. Then $e \in \mathscr{M}_i \Leftrightarrow depth_{\mathscr{S}}(e) = i$, $i \in \{1, 2, \ldots, n\}$.

Thus the representation of a step by nonempty event structures $\mathscr{M}_i$ is uniquely determined.

Pos is the full subcategory of $\mathbf{E}_{Act}$ that has as objects the finite, conflict free event structures $(E, \leqslant, \emptyset, l)$.

Further we need the following structures as labels on transition systems:

Pomsets: A *pomset* is a isomorphism class $[\mathscr{E}]$, where $\mathscr{E}$ is a finite, conflict-free event structure of $\mathbf{E}_{Act}$, i.e. $\mathscr{E} \in \mathbf{Pos}$. Pom_{Act} denotes the set of all pomsets.

Derivations: Let $\mathscr{E}$ be an event structure, $X = \{e_1, e_2, \ldots, e_n\} \in Conf(\mathscr{E})$ a configuration of $\mathscr{E}$. We call the sequence $e_1 e_2 \ldots e_n$ a *derivation of* X, iff there exist configurations $X_0, X_1, \ldots, X_n \in Conf(\mathscr{E})$ with

$$X_0 = \emptyset,$$

$$X_n = X, \text{ and}$$

$$X_i \backslash X_{i-1} = \{e_i\}, \; i = 1, 2, \ldots, n.$$

Let $e_1 e_2 \ldots e_n$ be a derivation of X, $f_1 f_2 \ldots f_n$ be a derivation of Y. These derivations are equal,

$$e_1 e_2 \cdots e_n \sim f_1 f_2 \cdots f_n,$$

iff there exists an isomorphism $\eta : X \to Y$ of $\mathbf{E}_{Act}$ with $\eta(e_1 e_2 \cdots e_n) := \eta(e_1)\eta(e_2) \cdots \eta(e_n) = f_1 f_2 \cdots f_n$. $Der(X)$ denotes the set of all *equivalence classes* $[e_1 e_2 \cdots e_n]$ *of derivations* of a configuration X, $Der_{Act} := \bigcup_{X \in Conf(\mathscr{E}), \mathscr{E} \in \mathbf{E}_{Act}} Der(X)$.

8.2. Concrete bisimulations on event structures

The various notions of bisimulation on event structures are usually defined in terms of transition relations on the configurations of an event structure. Let $\mathscr{E} = (E, \leqslant, \sharp, l)$ be an event structure over Act, let $X, X' \in Conf(\mathscr{E})$ be configurations of $\mathscr{E}$.

$$X \to X', \text{ iff } X \subseteq X'.$$

$$X \xrightarrow{a} X', \text{ iff } a \in Act, \; X \subseteq X', \; X' \setminus X = \{e\}, \; l(e) = a.$$

$$X \xrightarrow{M} X', \text{ iff } M \in \mathbb{N}^{Act}, \; X \subseteq X', \; \forall e, f \in X' \setminus X : e \neq f \Rightarrow e co f \text{ and}$$

$$\forall a \in Act : M(a) = |\{e \in X' \setminus X \mid l(e) = a\}|.$$

$$X \xrightarrow{p} X', \text{ iff}$$

$$p \in Pom_{Act}, \; X \subseteq X' \text{ and}$$

$$p = [X' \setminus X].$$

Let $\mathscr{E}, \mathscr{F}$ be event structures. A relation $R \subseteq Conf(\mathscr{E}) \times Conf(\mathscr{F})$ with $(\emptyset, \emptyset) \in R$ is called

interleaving bisimulation iff $\forall (X,Y)\in R, a\in Act$:
(i) $X \xrightarrow{a} X' \Rightarrow \exists Y' \in Conf(\mathscr{F}): Y \xrightarrow{a} Y', (X',Y')\in R$, and
(ii) $Y \xrightarrow{a} Y' \Rightarrow \exists X' \in Conf(\mathscr{E}): X \xrightarrow{a} X', (X',Y')\in R$.
bf-bisimulation (this definition is due to [21], where it is called backward–forward bisimulation) iff it is an interleaving bisimulation and
$\forall (X',Y')\in R,\ a\in Act$:
(i) $X \xrightarrow{a} X' \Rightarrow \exists Y \in Conf(\mathscr{F}): Y \xrightarrow{a} Y',\ (X,Y)\in R$, and
(ii) $Y \xrightarrow{a} Y' \Rightarrow \exists X \in Conf(\mathscr{E}): X \xrightarrow{a} X',\ (X,Y)\in R$.
step bisimulation iff $\forall (X,Y)\in R, M\in \mathbb{N}^{Act}$:
(i) $X \xrightarrow{M} X' \Rightarrow \exists Y' \in Conf(\mathscr{F}): Y \xrightarrow{M} Y',\ (X',Y')\in R$, and
(ii) $Y \xrightarrow{M} Y' \Rightarrow \exists X' \in Conf(\mathscr{E}): X \xrightarrow{M} X',\ (X',Y')\in R$.
pomset bisimulation $\forall (X,Y)\in R, p\in Pom_{Act}$:
(i) $X \xrightarrow{p} X' \Rightarrow \exists Y' \in Conf(\mathscr{F}): Y \xrightarrow{p} Y',\ (X',Y')\in R$, and
(ii) $Y \xrightarrow{p} Y' \Rightarrow \exists X' \in Conf(\mathscr{E}): X \xrightarrow{p} X',\ (X',Y')\in R$.
weak history preserving bisimulation [19] iff $\forall (X,Y)\in R$:
(o) there exists an isomorphism between
$(X, \leqslant_E \cap (X\times X), \emptyset, l_{E|X})$ and $(Y, \leqslant_F \cap (Y\times Y), \emptyset, l_{F|Y})$,
(i) $X \to X' \Rightarrow \exists Y' \in Conf(\mathscr{F}): Y \to Y',\ (X',Y')\in R$, and
(ii) $Y \to Y' \Rightarrow \exists X' \in Conf(\mathscr{E}): X \to X',\ (X',Y')\in R$.
A set R of triples (X,Y,η) with $(\emptyset,\emptyset,\emptyset)\in R$, where $X\in Conf(\mathscr{E})$, $Y\in Conf(\mathscr{F})$ and $\eta: X\to Y$ is an isomorphism in $\mathbf{E}_{\mathrm{Act}}$, is called
history preserving bisimulation iff $\forall (X,Y,\eta)\in R$
(i) $X \to X' \Rightarrow \exists Y' \in Conf(\mathscr{F}), \eta': Y\to Y',\ \eta'_{|X}=\eta,\ (X',Y',\eta')\in R$, and
(ii) $Y \to Y' \Rightarrow \exists X' \in Conf(\mathscr{E}), \eta': X\to X',\ \eta'_{|X}=\eta,\ (X',Y',\eta')\in R$.
strong history preserving bisimulation [22]
iff it is a history preserving bisimulation and $\forall (X',Y')\in R,\ a\in Act$:
(i) $X \to X' \Rightarrow \exists Y \in Conf(\mathscr{F}),\ \eta': Y\to Y',\ \eta'_{|X}=\eta,\ (X,Y,\eta)\in R$, and
(ii) $Y \to Y' \Rightarrow \exists X \in Conf(\mathscr{E}),\ \eta': X\to X',\ \eta'_{|X}=\eta,\ (X,Y,\eta)\in R$.

8.3. Modelling with AM-bisimulation

The above summarized notions of bisimulation can be viewed as AM-bisimulation in the following sense: For each notion B of bisimulation we give an operator T_B from the category $\mathbf{E}_{Act}$ of event structures to a suitable category $\mathbf{T}_B$ of transition systems with initial states such that two event structures $\mathscr{E}_1$, $\mathscr{E}_2$ are B-bisimilar iff $T_B(\mathscr{E}_1)$ and $T_B(\mathscr{E}_2)$ are AM-bisimilar.

$T_{int}(\mathscr{E}) := (Conf(\mathscr{E}), \to_{int}, \emptyset)$ is a transition system over $L_{int} := Act$, where $X \xrightarrow{a}_{int} X'$ iff $X \xrightarrow{a} X'$.

$T_{step}(\mathscr{E}) := (Conf(\mathscr{E}), \to_{step}, \emptyset)$ is a transition system over $L_{step} := \mathbb{N}^{Act}$, where $X \xrightarrow{M}_{step} X'$ iff $X \xrightarrow{M} X'$.

$T_{pom}(\mathscr{E}) := (Conf(\mathscr{E}), \to_{pom}, \emptyset)$ is a transition system over $L_{pom} := Pom_{Act}$, where $X \xrightarrow{p}_{pom} X'$ iff $X \xrightarrow{p} X'$.

$T_{whp}(\mathscr{E}) := (Conf(\mathscr{E}), \rightarrow_{whp}, \emptyset)$ is a transition system over $L_{whp} := Pom_{Act}$ where $X \xrightarrow{p}_{whp} X'$ iff $X \subseteq X'$ and $p = [X']$.

$T_{hp}(\mathscr{E}) := (\{Der(X) \mid X \in Conf(\mathscr{E})\}, \rightarrow_{hp}, \varepsilon)$ is a transition system over $L_{hp} := Der_{Act}$, where

$$e_1 e_2 \cdots e_n \xrightarrow{[e_1 e_2 \cdots e_n e_{n+1}]}_{hp} e_1 e_2 \cdots e_n e_{n+1}$$

iff $X' \backslash X = \{e_{n+1}\}$, where $X = \{e_1, e_2, \ldots, e_n\}$, $X' = \{e_1, e_2, \ldots, e_n, e_{n+1}\}$.

AM-bisimulation and backward–forward AM-bisimulation do not coincide for the transition systems $T_*(\mathscr{E})$, where $* \in \{int, step, pom, hp\}$. It is an open problem whether AM-bisimulation and backward-forward AM-bisimulation coincide in the case of the operator T_{whp}.

Event structures $\mathscr{E}$ and $\mathscr{F}$ are (interleaving, step, pomset)-bisimilar, iff $T_*(\mathscr{E})$ and $T_*(\mathscr{F})$ are AM-bisimilar for $* \in (int, step, pom)$. Moreover $\mathscr{E}$ and $\mathscr{F}$ are bf-bisimilar iff $T_{int}(\mathscr{E})$ and $T_{int}(\mathscr{F})$ are backward–forward AM-bisimilar. In [26] we showed: event structures $\mathscr{E}$ and $\mathscr{F}$ are weak history preserving bisimilar (history preserving bisimilar) iff $T_{whp}(\mathscr{E})$ and $T_{whp}(\mathscr{F})$ ($T_{hp}(\mathscr{E})$ and $T_{hp}(\mathscr{F})$) are AM-bisimilar. Moreover $\mathscr{E}$ and $\mathscr{F}$ are strong history preserving bisimilar iff $T_{hp}(\mathscr{E})$ and $T_{hp}(\mathscr{F})$ are backward–forward AM-bisimilar.

8.4. *Modelling with* **P**-*bisimulation and path*-**P**-*bisimulation*

Joyal, Nielsen, and Winskel consider in [22] (strong) history preserving bisimulation on event structures with consistency relation and give a modelling as path-**P**-bisimulation (resp. **P**-bisimulation). By a lengthy transformation this result can be carried over to event strucures. In addition, we give here a modelling of interleaving and step bisimulation in this setting and discuss also pomset, bf- and weak history preserving bisimulation.

There are two different ways to model a concrete notion of bisimulation on event structures as **P**-bisimulation (resp. path-**P**-bisimulation): On the one hand, we can choose a category **P** of path objects and try to show directly that the concrete notion of bisimulation and **P**-bisimulation (resp. path-**P**-bisimulation) coincide. On the other, we can take the modelling of a concrete bisimulation as AM-bisimulation by an operator T from Section 8.3, choose some category **P** of path objects and try to show that the operator T is connecting to **P**. In the following we will demonstrate both approaches.

Theorem 8.1. *Event structures are* **Lin**-*bisimilar iff they are interleaving bisimilar.*

Proof. Let $\mathscr{E}_1 = (E_1, \leqslant_1, \sharp_1, l_1)$, $\mathscr{E}_2 = (E, \leqslant_2, \sharp_2, l_2)$ be **Lin**-bisimilar. Then there exists an event structure $\mathscr{E} = (E, \leqslant, \sharp, l)$ and **Lin**-open morphisms $p_i : \mathscr{E} \rightarrow \mathscr{E}_i$, $i = 1, 2$. We claim that

$$R := \{(p_1(X), p_2(X)) \mid X \in Conf(\mathscr{E})\}$$

is an interleaving bisimulation between $\mathscr{E}_1$ and $\mathscr{E}_2$. As $\emptyset \in Conf(\mathscr{E})$ we obtain $(\emptyset, \emptyset) \in R$.

Let $(p_1(X), p_2(X)) \in R$ for some configuration $X \in Conf(\mathscr{E})$, let $p_1(X) \xrightarrow{a} Y'$ be a transition in $\mathscr{E}_1$. From $p_1(X) \in Conf(\mathscr{E}_1)$ we construct an event structure $\mathscr{P} = (P, \leqslant_P, \sharp_P, l_P)$, where $P := X$, $\leqslant_P$ is a linearization of $\leqslant_1 \cap (X \times X)$, $\sharp_P := \emptyset$ and $l_p := l_{1|X}$. Let $\hat{e}$ be the event in $\{\hat{e}\} = Y' \setminus p_1(X)$. Let $\mathscr{Q} := (Q, \leqslant_Q, \emptyset, l_Q)$ be an event structure, where $Q := P \cup \{\hat{e}\}$, $\forall e \in Q$: $e \leqslant_Q \hat{e}$ and $\forall e, f \in P : e \leqslant_Q f :\Longleftrightarrow e \leqslant_P f$, $\sharp_Q := \emptyset$ and $\forall e \in P : l_Q(e) := l_P(e)$ and $l_Q(\hat{e}) := a$. Both $\mathscr{P}$ and $\mathscr{Q}$ are objects in **Lin**.

Let $p : \mathscr{P} \to \mathscr{E}, m : \mathscr{P} \to \mathscr{Q}$ and $q : \mathscr{Q} \to \mathscr{E}_1$ the morphisms with

- $\forall e \in P : p(e) := e$,
- $\forall e \in P : m(e) := e$ and
- $\forall e \in P : q(e) := p_1(e)$, $q(\hat{e}) = \hat{e}$.

Then we have $p_1 \circ p = q \circ m$. As p_1 is **Lin**-open, there exists a morphism $r : \mathscr{Q} \to \mathscr{E}$ with $r \circ m = p$ and $p_1 \circ r = q$. Thus $Y := r(Q) = X \cup \{r(\hat{e})\} \in Conf(\mathscr{E})$, $p_1(Y) = Y'$, and $X \xrightarrow{a} Y$ is a transition between configurations in $\mathscr{E}$. As p_2 is a morphism, $p_2(X) \xrightarrow{a} p_2(Y)$ is a transition in $\mathscr{E}_2$. By definition of R holds $(p_1(Y), p_2(Y)) = (Y', p_2(Y)) \in R$.

Let now $R \subseteq Conf(\mathscr{E}) \times Conf(\mathscr{F})$ be an interleaving bisimulation between $\mathscr{E}_1$ and $\mathscr{E}_2$. Let $T_{int}(\mathscr{E}) = (Conf(\mathscr{E}), \alpha)$ and $T_{int}(\mathscr{F}) = (Conf(\mathscr{F}), \beta)$ be the related coalgebras. Let for all (X, Y), $(X', Y') \in R$

$$(a, X', Y') \in \gamma(X, Y) :\Longleftrightarrow (a, X') \in \alpha(X),\ (a, Y') \in \beta(Y).$$

We claim that unfolding this coalgebra (R, γ) into a tree $\mathscr{S}$ and constructing from $\mathscr{S}$ an event structure $\mathscr{E}$ with morphism $p_i : \mathscr{E} \to \mathscr{E}_i$, $i = 1, 2$, makes a $\mathscr{E}_1$ and $\mathscr{E}_2$ **Lin**-bisimilar.

The synchronization tree $\mathscr{S} = (S, \to, s)$ of (R, γ) is defined as follows:

$$\langle (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n) \rangle$$

is a state of $\mathscr{S}$ iff $(X_1, Y_1) = (\emptyset, \emptyset) \xrightarrow{a_1} (X_2, Y_2) \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} (X_n, Y_n)$ is a derivation in (R, γ). There is a transition

$$\langle (X_1, Y_1), \ldots, (X_n, Y_n) \rangle \xrightarrow{a} \langle (X_1, Y_1), \ldots, (X_n, Y_n), (X_{n+1}, Y_{n+1}) \rangle,$$

in $\mathscr{S}$ iff $(X_n, Y_n) \xrightarrow{a} (X_{n+1}, Y_{n+1})$ is a transition in (R, γ). $\langle (\emptyset, \emptyset) \rangle$ is the initial state of $\mathscr{S}$. The event structure $\mathscr{E} = (E, \leqslant, \sharp, l)$ associated with $\mathscr{S} = (S, \to, s)$ is constructed as

$E := S \setminus \{s\}$,
$e \leqslant f :\Longleftrightarrow (e, f) \in Tran^*$, where $Tran^*$ is the reflexive and transitive closure of $Tran := \{(e, f) \mid e \xrightarrow{a} f$ for an action $a\}$.
$e \sharp f :\Longleftrightarrow \neg(e \leqslant f \vee f \leqslant e)$ and
$l(e) = a :\Longleftrightarrow e = \langle (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n) \rangle \wedge (X_{n-1}, Y_{n-1}) \xrightarrow{a} (X_n, Y_n)$.

Let $p_1 : \mathscr{E} \to \mathscr{E}_1$, $p_2 : \mathscr{E} \to \mathscr{E}_2$ be the maps with

- $p_1(\langle (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n) \rangle) := e$ iff $\{e\} = X_n \setminus X_{n-1}$, and
- $p_2(\langle (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n) \rangle) := e$ iff$\{e\} = Y_n \setminus Y_{n-1}$.

We claim that p_1 and p_2 are **Lin**-open.

We first show that p_1 is a morphism in $\mathbf{E}_{Act}$. By construction of (R,γ) we have: $(X,Y) \xrightarrow{a} (X',Y')$ implies $X \xrightarrow{a} X'$. Thus p_1 preserves labels. A configuration C in $\mathscr{E}$ with $n \geqslant 1$ elements is a set

$$\begin{aligned} C = \{&\langle(\emptyset,\emptyset),(X_2,Y_2)\rangle,\\ &\langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3)\rangle,\\ &\ldots\\ &\langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3),\ldots,(X_{n+1},Y_{n+1})\rangle\}. \end{aligned}$$

and

$$p_1(C) = \bigcup_{i=2}^{n+1} X_i \backslash X_{i-1} = X_{n+1} \in Conf(\mathscr{E}_1).$$

Let e,e' be events of $C \neq \emptyset \in Conf(\mathscr{E})$ with $p_1(e) = p_1(e')$. Then

$$e = \langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3),\ldots,(X_i,Y_i)\rangle \text{ and}$$
$$e' = \langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3),\ldots,(X_j,Y_j)\rangle,$$

$2 \leqslant i,j \leqslant |C|+1$. Assume $i \neq j$. Let w.o.l.g. $i<j$. Then on the one hand $X_i \subseteq X_{j-1}$ and therefore $p_1(e) = p_1(e') \in X_i$. On the other hand $X_j \backslash X_{j-1} = \{p_1(e)\} = \{p_1(e')\}$ – contradiction. Therefore we have $i=j$ and thus $e=e'$.

Finally, we prove that p_1 is **Lin**-open. Let $\mathscr{P} = (P, \leqslant_P, \emptyset, l_P)$ and $\mathscr{Q} = (Q, \leqslant_Q, \emptyset, l_Q)$ be objects in **Lin**, let $p: \mathscr{P} \to \mathscr{E}$, $m: \mathscr{P} \to \mathscr{Q}$, $q: \mathscr{Q} \to \mathscr{E}_1$ be morphisms with $q \circ m = p_1 \circ p$. We show the existence of a morphisms $r: \mathscr{Q} \to \mathscr{E}$ with $p = r \circ m$ and $q = p_1 \circ r$ by induction on $n := |Q| - |P|$.

In case of $n=0$ the morphism m is bijective: m is injective, because $P \in Conf(\mathscr{P})$. As $|P| = |Q|$ we know that m is surjective. As the map m^{-1} preserves labels and maps configurations of $\mathscr{Q}$ on configurations of $\mathscr{P}$ and is injective on Q, m^{-1} is a morphism in $\mathbf{E}_{Act}$. We choose $r := p \circ m^{-1}$ and obtain: $r \circ m = p \circ m^{-1} \circ m = p$ and $p_1 \circ r = p_1 \circ p \circ m^{-1} = q$, because $q \circ m = p_1 \circ p$.

Now let $|Q| - |P| = n+1$. Let $\hat{e}$ be the maximal event in Q, let $\mathscr{Q}' := (Q', \leqslant', \emptyset, l')$, where $Q' := Q \backslash \{\hat{e}\}$, $\leqslant' := \leqslant_Q \cap (Q' \times Q')$, $l' := l_{Q|Q'}$. Let $m': \mathscr{P} \to \mathscr{Q}'$ be the morphism with $m'(e) := m(e)$ for all $e \in P$ and $q': \mathscr{Q}' \to \mathscr{E}_1$ be the morphism with $q'(e) := q(e)$ for all $e \in Q'$. Then $q' \circ m' = p_1 \circ p$, and by the induction hypothesis there exists a morphism $r': \mathscr{Q}' \to \mathscr{E}$ with $p = r' \circ m'$ and $q' = p_1 \circ r'$. The morphism r' maps Q' to a configuration $C \in Conf(\mathscr{E})$, where

$$\begin{aligned} C = \{&\langle(\emptyset,\emptyset),(X_2,Y_2)\rangle,\\ &\langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3)\rangle,\\ &\ldots\\ &\langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3),\ldots,(X_{k+1},Y_{k+1})\rangle\}, \end{aligned}$$

$k = |Q'|$, $p_1(C) = X_{k+1}$ and $q'(Q) = p_1(r'(Q)) = X_{k+1}$.

As there is a transition $Q' \xrightarrow{a} Q$ in $T_{int}(\mathscr{Q})$ there is a transition $q(Q') = X_{k+1} \xrightarrow{a} q(Q)$ in $T_{int}(\mathscr{E}_1)$. R is an interleaving bisimulation, $(X_{k+1}, Y_{k+1}) \in R$, hence there exists a configuration $Y' \in C(\mathscr{E}_2)$ with $(q(Q), Y') \in R$, where $Y_{k+1} \xrightarrow{a} Y'$ is a transition in $T_{int}(\mathscr{E}_2)$. By definition of γ there is a transition $(X_{k+1}, Y_{k+1}) \xrightarrow{a} (q(Q), Y')$ in (R, γ) and thus an event

$$f := \langle (\emptyset, \emptyset), (X_2, Y_2), (X_3, Y_3), \ldots, (X_{k+1}, Y_{k+1}), ((q(Q), Y') \rangle$$

in the event structure $\mathscr{E}$. Let $\forall e \in Q'$: $r(e) := r'(e)$ and $r(\hat{e}) := f$. This map r is the desired morphism. $\square$

Remark 8.2. To prove Theorem 8.1 one could use the results of [22] concerning open maps and the coreflection between the category $\mathbf{S}_{Act}$ of synchronization trees and $\mathbf{E}_{Act}$. In this setting one obtains easily that there exists a span of **Bran**-open maps in $\mathbf{S}_{Act}$ iff there exists a span of **Lin**-open maps in $\mathbf{E}_{Act}$ – but it remains to prove that synchronisation trees $\mathscr{S}_1$ and $\mathscr{S}_2$ associated with event structures $\mathscr{E}_1$ and $\mathscr{E}_2$ are **Bran**-bisimilar, i.e. strong bisimilar, iff the transition systems $T_{int}(\mathscr{E}_1)$ and $T_{int}(\mathscr{E}_2)$ are strong bisimilar. This involves again the technique of unfolding transition systems into synchronization trees.

Lemma 8.3. *The operator T_{int} is connecting to* **Lin**.

Proof. C1: Let $\mathscr{E}$ and $\mathscr{F}$ be event structures, $\eta : \mathscr{E} \to \mathscr{F}$ be a morphism in $\mathbf{E}_{Act}$. Defining $T_{int}(\eta)(X) := \eta(X)$, where $X \in Conf(\mathscr{E})$, turns T_{int} into a functor from $\mathbf{E}_{Act}$ to $\mathbf{T}_{Act}$.

C2: Let $\mathscr{P} = (P, \leqslant, \emptyset, l)$ be an object of **Lin**. The configuration P is reachable from all states of $T_{int}(\mathscr{P})$.

C3: Let $s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} s_n$, $n \geqslant 1$, be a derivation of a transition system. Let $\mathscr{P} = (P, \leqslant, \emptyset, l)$ be a path object in **Lin,** where

$$P := \{\langle s_1, s_2 \rangle, \langle s_1, s_2, s_3 \rangle, \ldots, \langle s_1, s_2, s_3, \ldots, s_n \rangle\},$$

$$\langle s_1, s_2, \ldots, s_i \rangle \leqslant \langle s_1, s_2, \ldots, s_j \rangle :\Longleftrightarrow i \leqslant j \text{ und}$$

$$l(\langle s_1, s_2, \ldots, s_i \rangle) := a_{i-1}, \ 2 \leqslant i \leqslant n.$$

Let $\mathscr{E}$ be an event structure with derivation $u_1 \xrightarrow{a_1} u_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} u_n$ in $T_{int}(\mathscr{E})$, where $u_1 = \emptyset$ is the initial state of $T_{int}(\mathscr{E})$. The map $p : P \to X$ with $p(\langle s_1, s_2, \ldots, s_i \rangle) := e_i$ is the desired morphism, where $\{e_i\} = u_i \setminus u_{i-1}$, $i = 2, \ldots, n$.

C4: The empty event structure fullfills condition C3.

C5: Let $\mathscr{P}$ and $\mathscr{Q}$ be objects of **Lin,** $\mathscr{E}$ be an event structure, $p : \mathscr{P} \to \mathscr{E}$, $q : \mathscr{Q} \to \mathscr{E}$ morphisms in $\mathbf{E}_{Act}$, $m : \mathscr{P} \to \mathscr{Q}$ a morphism in **Lin.** Let $\emptyset \xrightarrow{a_1} \{e_1\} \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} \{e_1, e_2, \ldots, e_{n-1}\}$ be a derivation in $T_{int}(\mathscr{P})$, where $\{e_1, e_2, \ldots, e_{n-1}\}$ is the final reachable state of $T_{int}(\mathscr{P})$. Let for all configurations $\{e_1, e_2, \ldots, e_i\} \in Conf(\mathscr{P}), (T_{int}(q) \circ T_{int}(m))\{e_1, e_2, \ldots, e_i\} = T_{int}(p)\{e_1, e_2, \ldots, e_i\}$, $0 \leqslant i \leqslant n-1$. Then we have $(q \circ m)(e_i) = p(e_i)$ for all $1 \leqslant i \leqslant n-1$ and thus $q \circ m = p$. $\square$

Corollary 8.4. *Let $\mathscr{E}_1$, $\mathscr{E}_2$ be event structures in $\mathbf{E}_{Act}$. The following are equivalent*:
(1) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are interleaving-bisimilar.*
(2) *$T_{int}(\mathscr{E}_1)$ and $T_{int}(\mathscr{E}_2)$ are AM-bisimilar.*
(3) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are* **Lin**-*bisimilar.*
(4) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are path*-**Lin**-*bisimilar.*
(5) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are strong path*-**Lin**-*bisimilar.*
(6) *$T_{\mathbf{Lin},\mathbf{E}_{Act}}(\mathscr{E}_1)$ and $T_{\mathbf{Lin},\mathbf{E}_{Act}}(\mathscr{E}_2)$ are AM-bisimilar.*
(7) *$T_{\mathbf{Lin},\mathbf{E}_{Act}}(\mathscr{E}_1)$ and $T_{\mathbf{Lin},\mathbf{E}_{Act}}(\mathscr{E}_2)$ are backward–forward AM-bisimilar.*

Proof. $1 \Leftrightarrow 2$: See Section 8.3.
$1 \Leftrightarrow 3$: Theorem 8.1.
$3 \Rightarrow 5$: Theorem 7.5.
$5 \Rightarrow 4$: By definition.
$4 \Leftrightarrow 2$: Theorem 7.9, T_{int} is connecting to **Bran** by Lemma 8.3.
$4 \Leftrightarrow 6$: Theorem 7.6.
$5 \Leftrightarrow 7$: Theorem 7.6. $\square$

Remark 8.5. Neither **Lin**-bisimulation nor (strong) path-**Lin**-bisimulation coincide with bf-bisimulation.

Theorem 8.6. *Event structures in $\mathbf{E}_{Act}$ are step bisimilar iff they are path*-**Step**-*bisimilar.*

Proof. We use the characterization of step bisimulation as AM-bisimulation and apply Theorem 7.9 in order to obtain a path-**Step**-bisimulation. We have to show that T_{step} fullfills all five conditions, where $\mathbf{M}=\mathbf{E}_{Act}$, $\mathbf{P}=\mathbf{Step}$ and $L=\mathbb{N}^{Act}$.

C1: Let $\mathscr{E}$, $\mathscr{F}$ be event structures, $\eta:\mathscr{E}\to\mathscr{F}$ a morphism in $\mathbf{E}_{Act}$. Defining $T_{step}(\eta)(X) := \eta(X)$, where $X\in Conf(\mathscr{E})$, turns T_{step} into a functor from $\mathbf{E}_{Act}$ to $\mathbf{T}_{\mathbb{N}^{Act}}$.

C2: Let $\mathscr{S}=(S,\leqslant,\sharp,l)=\mathscr{M}_1;\mathscr{M}_2;\ldots;\mathscr{M}_n$, $n\geqslant 0$, be a step, where $\mathscr{M}_i=(M_i,\leqslant_{M_i},\emptyset,l_i)$. Choose S as final reachable state. Let $X\in Conf(\mathscr{S})$. Then $S\backslash X=R\cup\bigcup_{i=k+1}^{n}M_i$ for some set $R\subseteq M_k$, where $k\in\{1,2,\ldots,n\}$. Let $A(a):=|\{e\in R\mid l(e)=a\}|$, and $A_i(a):=|\{e\in M_{i+1}\mid l(e)=a\}|$, $i=k,\ k+1,\ldots,n-1$, $a\in Act$. Then

$$X \xrightarrow{A} \bigcup_{i=1}^{k} M_i \xrightarrow{A_k} \bigcup_{i=1}^{k+1} M_i \xrightarrow{A_k} \cdots \xrightarrow{A_n} S$$

is a derivation from X to S in $T_{step}(\mathscr{S})$.

C3: Let $s_1 \xrightarrow{A_1} s_2 \xrightarrow{A_2} \cdots \xrightarrow{A_{n-1}} s_n$, $n\geqslant 1$, be a derivation in a transition system of $\mathbf{T}_{\mathbb{N}^{Act}}$. Let $\mathscr{S}=(S,\leqslant_S,\sharp_S,l_S)=\mathscr{M}_1;\mathscr{M}_2;\ldots;\mathscr{M}_{n-1}$, where $\mathscr{M}_i=(M_i,\leqslant_{M_i},\emptyset,l_i)$, $\leqslant_{M_i}=\{(m,m)\mid m\in M_i\}$, M_i pairwise disjoint, $\forall a\in Act$, $\forall 1\leqslant i\leqslant n-1: A_i(a)=|\{e\in M_i\mid l_i(e)=a\}|$ be a step. In $T_{step}(\mathscr{S})$ we find a derivation

$$\emptyset \xrightarrow{A_1} M_1 \xrightarrow{A_2} M_1\cup M_2 \xrightarrow{A_3} \cdots \xrightarrow{A_{n-1}} S,$$

where $\emptyset$ is the initial state and S the final reachable state of $T_{step}(\mathscr{S})$.

Let $\mathscr{E}=(E,\leqslant_E,\#_E,l_E)$ be an event structure with a derivation $X_1=\emptyset \xrightarrow{A_1} X_2 \xrightarrow{A_2} X_3 \xrightarrow{A_3} \cdots \xrightarrow{A_{n-1}} X_n$ in $T_{step}(\mathscr{E})$. For the sets M_i and $X_{i+1}\backslash X_i$ we obtain $\forall a\in Act$: $A_i(a)=|\{e\in M_i \mid l_i(e)=a\}|=|\{e\in X_{i+1}\backslash X_i \mid l_E(e)=a\}|$, $1\leqslant i\leqslant n-1$. Thus there exists bijective maps $p_i : M_i\rightarrow X_{i+1}\backslash X_i$ with $l_E(p_i(e))=l_i(e)$ for all events $e\in M_i$, $i=1,2,\ldots,n-1$. We claim that $p:=\bigcup_{i=1}^{n-1} p_i$ is the desired morphism from $\mathscr{S}$ to $\mathscr{E}$.

p preserves labels and is injective on configurations of $\mathscr{S}$. As the sets X_n are conflict free, this holds for $p(Y)\subseteq X_n$, where $Y\in Conf(\mathscr{S})$. Thus it remains to show that the image of a configuration $Y\in Conf(\mathscr{S})$ is leftclosed in E.

Let $e\in p(Y)$ for a configuration $Y\in Conf(\mathscr{S})$, let $e'\leqslant_E e$ and $e'\neq e$. As X_n is leftclosed we have $e'\in X_n$. As $e'\leqslant_E e$, there exists $j\in\{1,2,\ldots,n-1\}$ with $e'\in X_j, e\notin X_j$. Thus we obtain for the events $f,\ f'\in S$ with $p(f)=e,\ p(f')=e'$ that $f'\leqslant_S f$. As Y is a configuration, $f'\in Y$ and $p(f')=e'\in p(Y)$. This implies $\forall 1\leqslant i\leqslant n: T_{step}(p)(\bigcup_{j<i} M_i)=\bigcup_{j<i} p_i(M_i)=X_{i+1}$.

C4: The empty event structure fullfills condition C3.

C5: Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be steps, let $\mathscr{E}$ be an event structure, $m:\mathscr{S}_1\rightarrow\mathscr{S}_2$, $p:\mathscr{S}_1\rightarrow\mathscr{E}$ and $q:\mathscr{S}_2\rightarrow\mathscr{E}$ morphisms. Let $\emptyset=X_0 \xrightarrow{A_1} X_1 \xrightarrow{A_2} X_2 \xrightarrow{A_3} \cdots \xrightarrow{X_{n-1}} X_n$ be a derivation in $\mathscr{S}_1$, where X_n is the final reachable state of $\mathscr{S}_1$.

Let $\forall 0\leqslant i\leqslant n: (T_{step}(q)\circ T_{step}(m))(X_i)=T_{step}(p)(X_i)$. Then we have for $i=n$: $(T_{step}(q)\circ T_{step}(m))(X_n)=T_{step}(p)(X_n)$. As p, q and m are injective on configurations we obtain for all $e\in X_n: (q\circ m)(e)=p(e)$, i.e. $q\circ m=p$. $\square$

Example 8.7. Path-**Step**-bisimulation and strong path-**Step**-bisimulation do not coincide. Consider the event structures $\mathscr{E}$ and $\mathscr{F}$ from Fig. 16. The dotted lines between the circles around the events mean that all events inside one circle are in conflict with all events inside the other circle.

$\mathscr{E}$ and $\mathscr{F}$ are step-bisimilar and thus path-**Step**-bisimilar by Theorem 8.6. But there exists no strong path-**Step** bisimulation between $\mathscr{E}$ and $\mathscr{F}$. Assume that R is a strong path-**Step** bisimulation between $\mathscr{E}$ and $\mathscr{F}$. Then for R holds:

$(o_1,o_2)\in R$: Consider the event structure $\mathscr{O}:=(\{g_1,g_2\},\emptyset,\emptyset,l_O)$, which consists of two concurrent events g_1 and g_2, where $l_O(g_1):=a,\ l_O(g_2):=b$. $\mathscr{O}$ is a step. $o_1:\mathscr{O}\rightarrow\mathscr{E}$, where $o_1(g_1):=e_1, o_2(g_2):=e_2$, and $o_2:\mathscr{O}\rightarrow\mathscr{F}$, where $o_2(g_1):=f_1, o_2(g_2):=f_2$ are morphisms in $\mathbf{E}_{Act}$. Hence $(o_1,o_2)\in R$.

$(o_1\circ m_1, o_2\circ m_1)\in R$: Let $\mathscr{P}:=(\{g'\},\{g'\leqslant_P g'\},\emptyset,l_P(g'):=a)$. $m_1:\mathscr{P}\rightarrow\mathscr{O}$, where $m_1(g'):=g_1$, is a morphisms in **Step.** As R is a strong path-**Step** bisimulation, we obtain $(o_1\circ m_1,o_2\circ m_1)\in R$.

$q_1\circ m_2=(o_1\circ m_1)$ gives the contradiction: Let $\mathscr{Q}:=(\{g_1'',g_2''\},\leqslant_Q,\emptyset,l_Q)$ be the event structure, where $l_Q(g_1''):=a,\ l_Q(g_2''):=c$ and $g_1''\leqslant_Q g_2''$. We define morphisms

$$m_2:\mathscr{P}\rightarrow\mathscr{Q},\ m_2(g'):=g_1'',$$

$$q_1:\mathscr{Q}\rightarrow\mathscr{E} \text{ mit } q_1(g_1''):=e_1 \text{ and } q_1(g_2''):=e_3.$$

Then $q_1\circ m_2=(o_1\circ m_1)$, but there is no morphism $q_2:\mathscr{Q}\rightarrow\mathscr{F}$ with $q_2(g_1'')=f_1$.

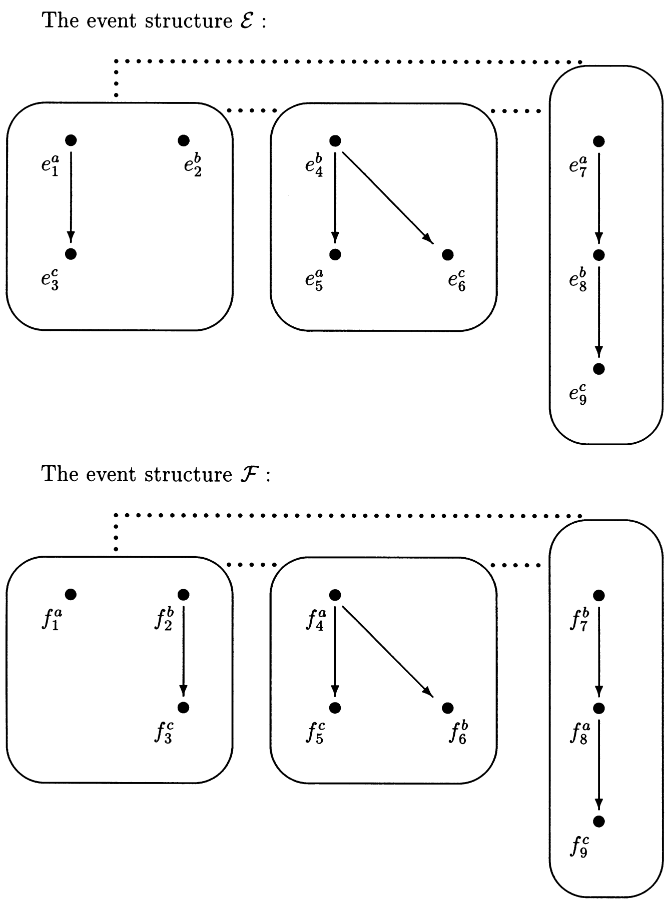

Fig. 16. Step-bisimilar event structures $\mathscr{E}$ and $\mathscr{F}$.

Corollary 8.8. **Step**-*bisimulation and step bisimulation do not coincide.*

Proof. Assume that **Step**-bisimulation and step bisimulation coincide. As the event structures $\mathscr{E}$ and $\mathscr{F}$ of Example 8.7 are step bisimilar, they are **Step**-bisimilar. Hence by Theorem 7.5 they are strong path-**Step**-bisimilar. □

Corollary 8.9. *Let $\mathscr{E}_1$, $\mathscr{E}_2$ be event structures in $\mathbf{E}_{Act}$. The following are equivalent*:
(1) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are step-bisimilar.*
(2) *$T_{step}(\mathscr{E}_1)$ and $T_{step}(\mathscr{E}_2)$ are AM-bisimilar.*
(3) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are path*-**Step**-*bisimilar.*
(4) *$T_{\mathbf{Step},\mathbf{E}_{Act}}(\mathscr{E}_1)$ and $T_{\mathbf{Step},\mathbf{E}_{Act}}(\mathscr{E}_2)$ are AM-bisimilar.*

Proof. $1 \Leftrightarrow 2$: See Section 8.3.
$1 \Leftrightarrow 3$: Theorem 8.6.
$3 \Leftrightarrow 4$: Theorem 7.6. □

$\mathcal{G}$: $g_1^a \longrightarrow g_2^b$ $\qquad$ $\mathcal{H}$: $h_1^a \quad h_2^b$

Fig. 17. T_{pom}, T_{whp} and T_{hp} cannot evolve into functors.

Lemma 8.10. *The operators T_{pom}, T_{whp} and T_{hp} (introduced in Section 8.3) are not connecting to any subcategory* $\mathbf{P}$ *of* $\mathbf{E}_{Act}$.

Proof. Let $\mathscr{G}$ and $\mathscr{H}$ be the event structures of Fig. 17. $\eta:\mathscr{G}\to\mathscr{H}$, where $\eta(g_1)=h_1$ and $\eta(g_2)=h_2$, is a morphism in $\mathbf{E}_{Act}$. In $\mathbf{T}_L$ there exists no morphism from $T_*(\mathscr{G})$ to $T_*(\mathscr{H})$, where $*\in\{pom, whp, hp\}$, $L\in\{[\mathbf{Pos}], Der_{Act}\}$. Hence the operators T_{pom}, T_{whp} and T_{hp} do not yield functors. $\square$

This result means in particular that we cannot make use of T_{hp} and Theorem 7.9 to obtain a characterization of history preserving bisimulation as path-**P**-bisimulation. However, Joyal, Nielsen, and Winskel characterize in [22] (strong) history preserving bisimulation for the category $\mathbf{EC}_{Act}$ of event structures with consistency relation with the path category **PosC**, which consists of all finite event structures without any conflict:

Theorem 8.11. (1) *Event structures in* $\mathbf{EC}_{Act}$ *are strong history preserving bisimilar iff they are* **PosC**-*bisimilar.*

(2) *Event structures in* $\mathbf{EC}_{Act}$ *are (strong) history preserving bisimilar iff they are (strong) path-***PosC***-bisimilar.*

One can translate the second result of Theorem 8.11 for the category $\mathbf{E}_{Act}$ to obtain the following:

Corollary 8.12. *For event structures $\mathscr{E}_1$ and $\mathscr{E}_2$ in $\mathbf{E}_{Act}$ are equivalent:*

(1) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are (strong) history preserving bisimilar.*

(2) *$\mathscr{E}_1$ and $\mathscr{E}_2$ are (strong) path-***Pos***-bisimilar.*

(3) *$T_{\mathbf{Pos},\mathbf{E}_{Act}}(\mathscr{E}_1)$ and $T_{\mathbf{Pos},\mathbf{E}_{Act}}(\mathscr{E}_2)$ are (backward–forward) AM-bisimilar.*

Remark 8.13. It is an open question whether it is possible to model step, pomset, weak history preserving and bf-bisimulation in the open map approach of [22].

8.5. Beyond the Aczel/Mendler approach?

In this section we give two examples of concrete bisimulations which hint at possible limitations of the Aczel/Mendler approach. In our attempts to view generalized pomset bisimulation and partial word bisimulation as coalgebras we encountered some obstacles as shown below. It is an open problem if these bisimulations can be modelled in a satisfactory way as AM-bisimulation.

Generalized pomset bisimulation was introduced in [20] as a notion of equivalence for Petri nets. In [18], Example 7.4, this kind of bisimulation was studied for event structures, without a formal definition. Here we transfer the definition from Petri nets to prime event structures.

Let $\mathscr{E}$, $\mathscr{F}$ be event structures. A relation $R \subseteq Conf(\mathscr{E}) \times Conf(\mathscr{F})$ is called

gpomset bisimulation iff $(\emptyset, \emptyset) \in R$ and for all $(X, Y) \in R$ holds:

(i) if $X \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} X_n$ in $T_{int}(\mathscr{E})$ then $Y \xrightarrow{a_1} Y_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} Y_n$ in $T_{int}(\mathscr{F})$ with $[X_n \backslash X] = [Y_n \backslash Y]$ and $(X_i, Y_i) \in R$ for all $1 \leqslant i \leqslant n$ and

(ii) if $Y \xrightarrow{a_1} Y_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} Y_n$ in $T_{int}(\mathscr{F})$ then $X \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} X_n$ in $T_{int}(\mathscr{E})$ with $[X_n \backslash X] = [Y_n \backslash Y]$ and $(X_i, Y_i) \in R$ for all $1 \leqslant i \leqslant n$.

$[\mathscr{E}]_{gpom}$ denotes the equivalence class of an event structure $\mathscr{E}$ to gpomset bisimulation in the category $\mathbf{E}_{Act}$. $GPom_{Act}$ is the set of all these equivalence classes.

Let $\mathscr{E}$ be an event structure. For $X \in Conf(\mathscr{E})$ let $\sharp_{\mathscr{E}}(X) := \{ f \in E \mid \exists e \in X \colon e \sharp f \}$, define $E' := E \backslash (X \cup \sharp_{\mathscr{E}}(X))$. $\mathscr{E} \backslash X := (E', \leqslant \cap (E' \times E'), \sharp \cap (E' \times E'), l_{|E'})$ denotes the "sub-event structure" of $\mathscr{E}$ including all events from which a finite subset may be added to X in order to get a larger configuration. For configurations and "sub-event structures" of $\mathscr{E}$ holds, see [26]:

(1) Let $\mathscr{E}' := \mathscr{E} \backslash X$ for some configuration $X \in Conf(\mathscr{E})$, $X' \in Conf(\mathscr{E}')$. Then $X \cup X'$ is a configuration of $\mathscr{E}$.

(2) Let $X', X'' \in Conf(\mathscr{E})$ with $X' \subseteq X''$. Define $\mathscr{E}' := \mathscr{E} \backslash X'$ and $X := X'' \backslash X'$. Then X is a configuration of $\mathscr{E}'$.

In order to model gpomset bisimulation in the coalgebraic framework of [5] one has to find an operator T_{gpom} which associates with an event structure $\mathscr{E} = (E, \leqslant, \sharp, l)$ a transition system $T_{gpom}(\mathscr{E})$ such that $\mathscr{E}_1$ and $\mathscr{E}_2$ are gpomset bisimilar iff $T_{gpom}(\mathscr{E}_1)$ and $T_{gpom}(\mathscr{E}_2)$ are AM-bisimilar. In the following, we present an operator T_{gpom} that satisfies these requirements. $T_{gpom}(\mathscr{E}) = (Conf(\mathscr{E}), \longrightarrow, \emptyset)$ is the transition system over $L := Pom_{Act} \times Act^{+} \times GPom_{Act}^{*}$, where

$$
\begin{aligned}
X \overset{(p, a_1 a_2 \ldots a_n, G)}{\longrightarrow} X' \colon \iff\ & [X' \backslash X] = p, \\
& \exists n \geqslant 1,\ \exists X_1, X_2, \ldots, X_{n-1} \in Conf(\mathscr{E}) \colon \\
& X \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots X_{n-1} \xrightarrow{a_n} X' \text{ in } T_{int}(\mathscr{E}), \\
& G = ([\mathscr{E} \backslash X_i]_{gpom})_{i=1}^{n-1}.
\end{aligned}
$$

Theorem 8.14. *Event structures $\mathscr{E}$ and $\mathscr{F}$ are gpom-bisimilar iff $T_{gpom}(\mathscr{E})$ and $T_{gpom}(\mathscr{F})$ are AM-bisimilar.*

Proof. Let $\mathscr{E}$ and $\mathscr{F}$ be prime event structures. Let $T_{gpom}(\mathscr{E}) = (Conf(\mathscr{E}), \longrightarrow_1, \emptyset)$ and $T_{gpom}(\mathscr{F}) = (Conf(\mathscr{F}), \longrightarrow_2, \emptyset)$, (A, α) the coalgebra with $\mathscr{T}_{(A,\alpha)} = (Conf(\mathscr{E}), \longrightarrow_1)$ and (B, β) the coalgebra with $\mathscr{T}_{(B,\beta)} = (Conf(\mathscr{F}), \longrightarrow_2)$.

Let R be a gpomset bisimulation between $\mathscr{E}$ and $\mathscr{F}$. Let for $(X, Y), (X', Y') \in R$

$$
\begin{aligned}
(p, a_1 a_2 \ldots a_n, G, X', Y') \in \gamma(X, Y) \colon \iff\ & (p, a_1 a_2 \ldots a_n, G, X') \in \alpha(X), \\
& (p, a_1 a_2 \ldots a_n, G, Y') \in \beta(Y).
\end{aligned}
$$

Let $(p, a_1a_2\ldots a_n, G, X') \in (\alpha\circ\pi_1)\ (X, Y)$. Then $(p, a_1a_2\ldots a_n, T, X') \in \alpha(X)$ and thus by definition of T_{gpom} we obtain a derivation $X \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots X_{n-1} \xrightarrow{a_n} X_n = X'$ in $T_{int}(\mathscr{E})$. Further holds: $[X'\backslash X] = p$ and $G = ([\mathscr{E}\backslash X_i]_{gpom})_{i=1}^{n-1}$. As R is a gpomset bisimulation there exists a derivation $Y \xrightarrow{a_1} Y_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} Y_n$ in $T_{int}(\mathscr{F})$ with $[X_n\backslash X] = [Y_n\backslash Y]$ and $(X_i, Y_i) \in R$ for all $1 \leqslant i \leqslant n$. Thus $p = [Y_n\backslash Y]$ and $(X', Y') \in R$. For each $1 \leqslant i \leqslant n-1$ let $\hat{R}_i := \{(\hat{X}, \hat{Y}) \mid \exists (X, Y) \in R\colon X_i \subseteq X,\ Y_i \subseteq Y,\ \hat{X} = X\backslash X_i,\ \hat{Y} = Y\backslash Y_i\}$. As R is a gpomset bisimulation and $(X_i, Y_i) \in R$ the sets $\hat{R}$ are gpomset bisimulations between $\mathscr{E}\backslash X_i$ and $\mathscr{F}\backslash Y_i$. Hence $[\mathscr{E}\backslash X_i]_{gpom} = [\mathscr{F}\backslash Y_i]_{gpom}$ for $1 \leqslant i \leqslant n-1$. This implies $(p, a_1a_2\ldots a_n, G, Y') \in \beta(Y)$ and we get $(p, a_1a_2\ldots a_n, G, X', Y') \in \gamma(X, Y)$. Hence $(p, a_1a_2\ldots a_n, G, X') \in (F\pi_1 \circ \gamma)\ (X, Y)$. Lemma 3.4 gives the other inclusion.

Now let (R, γ) be an AM-bisimulation between (A, α) and (B, β) with $(\emptyset, \emptyset) \in R$. Let $(p, a_1a_2\ldots a_n, G, X', Y') \in \gamma(X, Y)$ be a transition in (R, γ). Then there are transitions $(p, a_1a_2\ldots a_n, G, X') \in \alpha(X)$ and $(p, a_1a_2\ldots a_n, G, Y') \in \beta(Y)$, i.e. there exist derivations $X \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots X_{n-1} \xrightarrow{a_n} X_n = X'$ in $T_{int}(\mathscr{E})$ and $Y \xrightarrow{a_1} Y_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} Y_n = Y'$ in $T_{int}(\mathscr{F})$, where $[\mathscr{E}\backslash X_i]_{gpom} = [\mathscr{F}\backslash Y_i]_{gpom}$ for $1 \leqslant i \leqslant n-1$, as both transitions have the same "G" as label. Let R_i be a gpomset bisimulation which establishes $[\mathscr{E}\backslash X_i]_{gpom} = [\mathscr{F}\backslash Y_i]_{gpom}$, $1 \leqslant i \leqslant n-1$. Let $R_{(p, a_1a_2\ldots a_n, G, X', Y', X, Y)} := \bigcup_{i=1}^{n-1} \{(\bar{X} \cup X_i,\ \bar{Y} \cup Y_i) \mid (\bar{X}, \bar{Y}) \in R_i\}$ the union of all these relations, where we add the events of X_i (resp. Y_i) to obtain configurations of $\mathscr{E}$ (resp. $\mathscr{F}$). We claim that

$$\hat{R} := R \cup \bigcup_{\substack{(p, a_1a_2\ldots a_n, G, X', Y') \in \gamma(X, Y),\\ (X, Y), (X', Y') \in R,\\ (p, a_1a_2\ldots a_n, G) \in L.}} R_{(p, a_1a_2\ldots a_n, G, X', Y', X, Y)}$$

is a gpomset bisimulation between $\mathscr{E}$ and $\mathscr{F}$.

As $(\emptyset, \emptyset) \in R$ we obtain $(\emptyset, \emptyset) \in \hat{R}$. Now let $(X, Y) \in \hat{R}$.

First, we deal with the case that $(X, Y) \in R$. Let $X \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} X_n$ be a derivation in $T_{int}(\mathscr{E})$. Then $(p, a_1a_2\ldots a_n, G, X') \in \alpha(X)$, where $p = [X'\backslash X]$ and $G = ([\mathscr{E}\backslash X_i]_{gpom})_{i=1}^{n-1}$. As (R, γ) is an AM-bisimulation there exists some configuration $Y' \in Conf(\mathscr{F})$ with $(p, a_1a_2\ldots a_n, G, Y') \in \beta(Y)$ and $(X', Y') \in R$. Thus be definition of T_{gpom} there exists a derivation $Y \xrightarrow{a_1} Y_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} Y'$ in $T_{int}(\mathscr{F})$ with $[Y'\backslash Y] = p$ and $G = ([\mathscr{F}\backslash Y_i]_{gpom})_{i=1}^{n-1}$. By construction of $\hat{R}$ we have $(X_i, Y_i) \in \hat{R}$ for all $1 \leqslant i \leqslant n-1$.

If $(X, Y) \notin R$ then there exists some relation of type $R_{(p, a_1a_2\ldots a_n, G, X', Y', X, Y)}$ (see above) with $(X, Y) \in R_{(p, a_1a_2\ldots a_n, G, X', Y', X, Y)}$. As the corresponding set R_i is a gpomset bisimulation conditions (i) and (ii) of gpomset bisimulation hold for R_i and thus for $\hat{R}$. □

However, the definition of $T_{gpom}(\mathscr{E})$ exhibits the following drawback: in order to define the transitions $X \overset{(p, a_1a_2\ldots a_n, G)}{\longrightarrow} X'$ we make explicit use of the gpomset bisimulation by referring to $[\mathscr{E}\backslash X_i]_{gpom}$ in G. While this might be considered not important in the case of finite event structures, the construction may become awkward in the infinite case, as can be seen in the following example, where we need the "global" information $[\mathscr{E}]_{gpom}$ in order to obtain the transition relation for $T_{gpom}(\mathscr{E})$:

Example 8.15. Let $\mathscr{E}=(E,\leqslant,\emptyset,l)$ be the event structure with $E:=\{e_i \mid i\geqslant 1\}$, $e_i\leqslant e_j :\Longleftrightarrow i\leqslant j$, $l(e_i)=a$ for all $i\geqslant 1$. Let $X_i:=\{e_j\in E \mid j\leqslant i\}$, $i\geqslant 0$. There is e.g. a transition in $T_{gpom}(\mathscr{E})$ from X_i to X_{i+2}. The label of such a transition is (p,a^2,G), where $p=[X_2]$ and $G=([\mathscr{E}]_{gpom})$. Hence in order to define $T_{gpom}(\mathscr{E})$ we make use of $[\mathscr{E}]_{gpom}$. In particular, the labelling of a transition from X_i to X_{i+2} contains the infinite object $[\mathscr{E}]_{gpom}$.

It is open if it is possible to find a characterization of generalized pomset bisimulation that does not have this drawback.

Lemma 8.16. *The operator T_{gpom} is not connecting to any subcategory $\mathbf{P}$ of $\mathbf{E}_{Act}$.*

Proof. Analogous to the proof of Lemma 8.10: take again the event structures $\mathscr{G}$ and $\mathscr{H}$ of Fig. 17. □

Remark 8.17. It is an open question whether AM-bisimulation and backward–forward AM-bisimulation for the transition systems $T_{gpom}(\mathscr{E})$ coincide.

We now consider partial word bisimulation in the Aczel/Mendler approach:

Let $p,q\in Pom_{Act}$ be pomsets. p is *less sequential* than q, denoted by $p\leqslant q$, iff there exist event structures $\mathscr{E}=(E,\leqslant_E,\emptyset,l_E)\in p$, $\mathscr{F}=(F,\leqslant_F,\emptyset,l_F)\in q$ and a bijective map $f:E\to F$ such that $\forall e\in E: l_E(e)=l_F(f(e))$ and $\forall e,e'\in E: e\leqslant_E e' \Rightarrow f(e)\leqslant_F f(e')$. Let $\mathscr{E}$, $\mathscr{F}$ be event structures. A relation $R\subseteq Conf(\mathscr{E})\times Conf(\mathscr{E})$ with $(\emptyset,\emptyset)\in R$ is called

partial word bisimulation [33] iff for all $(X,Y)\in R$, $p\in Pom_{Act}$ holds:
(i) $X\xrightarrow{p}X'\Rightarrow\exists Y'\in Conf(\mathscr{F})$, $q\in Pom_{Act}: Y\xrightarrow{q}Y'$, $(X',Y')\in R$, $q\leqslant p$ and
(ii) $Y\xrightarrow{p}Y'\Rightarrow\exists X'\in Conf(\mathscr{F})$, $q\in Pom_{Act}: X\xrightarrow{q}X'$, $(X',Y')\in R$, $q\leqslant p$.

Theorem 8.18. *Let $\mathscr{E}$ and $\mathscr{F}$ be event structures.*

Let $T_{pom}(\mathscr{E})=(Conf(\mathscr{E}),\longrightarrow_1,\emptyset)$ and $T_{pom}(\mathscr{F})=(Conf(\mathscr{F}),\longrightarrow_2,\emptyset)$, let (A,α) be the coalgebra with $\mathscr{T}_{(A,\alpha)}=(C(\mathscr{E}),\longrightarrow_1)$ and (B,β) be the coalgebra with $\mathscr{T}_{(B,\beta)}=(Conf(\mathscr{F}),\longrightarrow_2)$.

$\mathscr{E}$ and $\mathscr{F}$ are partial word bisimilar iff there exists a coalgebra (R,γ) with $(\emptyset,\emptyset)\in R$, such that for (A,α) and (B,β) holds:
(i) $(\alpha\circ\pi_1)\subseteq(F\pi_1\circ\gamma)$,
(ii) *if* $(p,X',Y')\in\gamma(X,Y)$ *and* $(p,X')\in(\alpha\circ\pi_1)\ (X,Y)$ *then* $(q,Y')\in(\beta\circ\pi_2)\ (X,Y)$ *for some* $q\leqslant p$,
(iii) $(\beta\circ\pi_2)\subseteq(F\pi_2\circ\gamma)$ *and*
(iv) *if* $(p,X',Y')\in\gamma(X,Y)$ *and* $(p,Y')\in(\beta\circ\pi_1)\ (X,Y)$ *then* $(q,X')\in(\alpha\circ\pi_1)\ (X,Y)$ *for some* $q\leqslant p$.

Proof. Let R be a partial word bisimulation between $\mathscr{E}$ and $\mathscr{F}$. Let for all (X, Y), $(X', Y') \in R$, $r \in Pom_{Act}$

$$(r, X', Y') \in \gamma(X, Y) :\iff (p, X') \in \alpha(X),\ (q, Y') \in \beta(Y),\\ p \leqslant q \vee q \leqslant p, \quad r = \max\{p, q\}.$$

Then (R, γ) is the desired coalgebra. The proof of the other implication is straightforward. □

Conditions (i) and (iii) are weaker than the ones required by AM-bisimulation; however (ii) and (iv) are stronger than those of AM-bisimulation. It remains open if partial word bisimulation can be captured by the Aczel/Mendler approach in the strict sense.

Remark 8.19. It is an open question whether it is possible to model gpomset and partial word bisimulation in the open map approach of [22].

9. Conclusion

We have shown how the various approaches to an abstract characterization of bisimulation relate to each other. It turns out that AM-bisimulation is the most flexible abstract characterization. The results obtained for event structures can be easily transferred to Petri nets and other models of computation.

The notion AM-bisimulation gives a new perspective on the phenomen "bisimulation": While Milner introduces bisimulation as a relation which he interprets as "a kind of invariant holding between a pair of dynamic systems" [27], AM-bisimulation itself is a dynamic system.

Apart from serving as an abstraction the coalgebraic setting allows to compare – via bisimulation – objects that stem from different models of computation in the following sense. Let $\mathbf{M}_1$ (e.g. event structures) and $\mathbf{M}_2$ (e.g. Petri nets) be models of computation, each with a notion of bisimulation, say B_1 for $\mathbf{M}_1$ (resp. B_2 for $\mathbf{M}_2$). Let us further assume the existence of mappings $T_i : \mathbf{M}_i \to \mathbf{Set}_F$, $i = 1, 2$, for some $F(X) = \mathscr{P}(L \times X)$, such that for X_i, $Y_i \in \mathbf{M}_i$ holds:

$$X_i \sim_{B_i} Y_i \quad \text{iff } T(X_i) \text{ and } T(Y_i) \text{ are (backward–forward) AM-bisimilar.}$$

(I.e. the mappings T_i "model" the bisimulations B_i, as e.g. the operator T_{step} models step bisimulation on event structures.) This justifies that we may now use T_i, $i = 1, 2$, to compare an object X_1 from $\mathbf{M}_1$ with an object X_2 from $\mathbf{M}_2$ by constructing the transition systems $T_1(X_1)$ and $T_2(X_2)$ and investigating their relationship in terms of (backward–forward) AM-bisimulation.

When dealing with a concrete notion B of bisimulation in a context of a process calculus with a set Σ of operators, the question arises under which conditions B is compatible with the operators of Σ. Hence, it is interesting to know which abstract settings are suitable to handle this question. We briefly sketched the issue for the coalgebraic setting. It is not difficult to see that the question can be easily handled in the algebraic view of [23]. Recently there are attempts to treat the problem in the open map approach [9], where it is requested that the operators can be turned into functors preserving open maps.

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