Extended Stochastic Hybrid Systems and Their Reachability Problem*

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Abstract. In this paper we generalize a model for stochastic hybrid systems. First, we prove that this model is a right Markov process and it satisfies some mathematical properties. Second, we propose a method based on the theory of Dirichlet forms to study the reachability problem associated with these systems.

Keywords: stochastic hybrid systems, reachability problem, extended automata, Markov processes, Dirichlet forms.

1 Introduction

In this paper we extend the stochastic hybrid system (SHS) model introduced by Lygeros et. al. in [15]. We call this new model *extended stochastic hybrid systems* (ESHS) (see section 2). In the first step we prove that this model is a Borel right process with the CADLAG property (section 3). In a second step we mainly study the reachability problem for ESHS (section 4). In a probabilistic framework, the reachability problem consists in determining the probability that the system trajectories enter some prespecified set starting from a certain set of initial conditions with a given probability distribution. Our investigation begins with a simple observation, namely, an ESHS is interleaving between a jump process and some diffusion processes. Therefore, studying the reachability problem for this model requires two reachability problems to be solved: one for the jump process and another one for the diffusion processes. Dealing with the standard apparatus of Markov processes (hitting times, hitting probability, harmonic measure), solving the reachability problem seems to be quite difficult.

This work promotes a new method, based on the theory of Dirichlet forms [14, 17], for the reachability problem. It has already been proved in the literature that Dirichlet forms constitute a powerful tool for studying Markov processes (see, for example, [1,17] and the references therein). Dirichlet form techniques have found striking applications in the study of stochastic partial differential equations [1, 8]. This is mainly due to the fact that they allow to develop a highly nontrivial stochastic analysis under some minimal regularity hypothesis, for instance, on

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very irregular spaces without differentiable structure like fractals, or on infinite dimensional spaces like path spaces or spaces of measures.

For Dirichlet forms, a lot of work was carried out on axiomatizations and representation results. This provides a mathematical vehicle for zooming in and out at different levels of abstraction in a consistent way. For example, in the most abstract view, the Dirichlet spaces can be seen as mixing a linear space structure with a partial order structure, by providing simple compatibility axioms. In more concrete applications a Dirichlet space defines a logical type of functions with an inner product given explicitly by a logical expression. The advantage of Dirichlet forms which derives from this is that they can be easily implemented. There are two main streams: one is symbolic (like using a model-checker or a theorem prover or their combination like PVS [13]) and another one is numerical [10]. For the reachability problem the symbolic approach has been intensively applied (see e.g. the papers in [18]), especially because the accessible states can be generated. In the case of PVS, we can link these techniques with the huge mathematical libraries made available by the theorem provers.

The basic idea of the reachability method proposed here is to employ the correspondence between some 'nice' Markov processes (like our process) and some quadratic forms, called *Dirichlet forms*, defined using the process generator. Each (quasi-regular) Dirichlet form can be expressed as the sum of its parts: continuous, jumping and killing corresponding to the same parts of the Markov process considered. A Dirichlet form comes with the so-called notion of *capacity*, which is, roughly speaking, a nonlinear extension of a measure. The capacity associated with a Dirichlet form is in a very close connection with the hitting times of the corresponding Markov process. We investigate the possible benefits of applying a Dirichlet form based method to study the reachability problem of ESHS. Developing a reachability analysis methodology for this model will involve dealing with its two characteristics: forced jumps and diffusion segments between two consecutive jumps. In what follows, we will work with the corresponding jumping and continuous Dirichlet forms. Usually a target set E in the state space is a level set for a given function F, i.e. $E = \{\alpha | F(\alpha) > l\}$ (F can be chosen as the Euclidean norm or as the distance to the boundary of E). The probability of the set of trajectories which hit E until time horizon $T > 0$ can be expressed as $P\{\sup_{t\in[0,T]} F(\alpha_t) > l\}$. An upper estimation for this probability will be given in terms of the Dirichlet form induced by F on $\mathbb{R}$. This form corresponds to the process $F(\alpha_t)$. One might, for instance, use the small induced processes rather then the huge original process to deal with the reachability problem. The induced Dirichlet form capacity (of $E^* = (l, \infty)$) plays an essential role in obtaining the reach event probability estimation. Intuitively, this capacity is the Laplace transform of the hitting time of the target set. If the model H is discretized then the induced process is a one-dimensional jump process and therefore the computation of Laplace transform and the mean level-crossing time is feasible. It is interesting to note that the capacity of the target set is subadditive. So even if the target set were very complex, then the capacity of target set is at most the sum of capacities of its parts.

2 Stochastic Hybrid Model

Stochastic Hybrid Systems (SHS) introduced in [15] are a class of non-linear, continuous-time stochastic hybrid processes. In this section we give a generalization of SHS called extended stochastic hybrid systems (ESHS). ESHS will be object for the reachability problem studied in section 4. ESHS can be considered, as well, a generalization of the model used in [4].

2.1 Model Description

ESHS involve a hybrid state space, with both continuous and discrete states. The continuous and the discrete parts of the state variable have their own natural dynamics, but the main point is to capture the interaction between them.

The time t is measured continuously. The state of the system is represented by a continuous variable x and a discrete variable i. The continuous variable evolves in some 'cells' X_i (open sets in the Euclidean space) and the discrete variable belongs to a countable set Q. The intrinsic difference between the discrete and continuous variables, consists of the way that they evolve through time. The continuous state is governed by an SDE that depends on the hybrid state. The discrete dynamics produces transitions in both (continuous and discrete) state variables x, i. Transitions occur when the continuous state hits a predefined set of the state space (forced transitions). Whenever a transition occurs the hybrid state is reset instantly to a new value. The value of the discrete state after the transition is determined uniquely by the hybrid state before the transition. On the other hand, the new value of the continuous state obeys a probability law which depends on the last hybrid state. Thus, a sample trajectory has the form $(q(t), x(t), t \geq 0)$, where $(x(t), t \geq 0)$ is piecewise continuous and $q(t) \in Q$ is piecewise constant. Let $(0 \leq T_1 \leq T_2 \leq ... \leq T_i \leq T_{i+1} \leq ...)$ be the sequence of jump times at which the continuous and the discrete part of the system interact. This time sequence is generated when the state of the system passes through a set of 'marked states' called set-interface.

2.2 Mathematical Model

Definition 1 (Extended Stochastic Hybrid System). *An ESHS is an extended automaton $H = (Q, X, Dom, D, b, \sigma, Init, G, R)$ where*

- *Q is a countable set of discrete variables*[1]*;*
- *$X = \mathbb{R}^n$ is the continuous state space and $\mathcal{B}(\mathbb{R}^n)$ its σ-algebra Borel;*
- *$Dom : Q \to 2^X$, $Q \ni i \mapsto X_i \subset X$, with X_i a relatively compact open set;*
- *$D : Q \to 2^X$ assigns to each $i \in Q$ a measurable subset (an interface set) D_i of X such that $\partial X_i \subset D_i$.*
- *$b : Q \times X \to \mathbb{R}^n$, $\sigma : Q \times X \to \mathbb{R}^{n \times n}$;*
- *$Init : \mathcal{B}(Q \times X) \to [0, 1]$ is an initial probability measure on $(Q \times X, \mathcal{B}(Q \times X))$ concentrated on $\cup_{i \in Q} \{i\} \times X_i$;*

[1] Q can be taken as the set of natural numbers.

– $G : Q \times Q \to 2^X$ *maps each* $(i, j) \in Q \times Q$ *into* $G(i, j) \subset X$ *such that* $(\forall)$ $(i, j) \in Q \times Q$, $G(i, j)$ *is a measurable subset of* D_i *(possibly empty);* $(\forall)$ $i \in Q$, *the family* $\{G(i, j) \mid j \in Q\}$ *is a disjoint partition of* D_i;
– $R : Q \times Q \times X \to \mathcal{P}(X)$ *assigns to each* $(i, j) \in Q \times Q$ *and* $x \in G(i, j)$ *a reset probability kernel on* X *concentrated on* X_j.

To describe the dynamics, we need to consider an n-dimensional standard Wiener process $(W(t), t \geq 0)$ in a complete probability space $(\Omega, \mathcal{F}, P)$.

Assumption 1 *The functions* $b(i, x)$ *and* $\sigma(i, x)$ *are bounded and Lipschitz continuous in* x. *For all* $i, j \in Q$ *and for any measurable set* $A \subset X_j$, $R(i, j, x)(A)$ *is a measurable function in* x.

The first part of the Assumption 1 ensures, for any $i \in Q$, the existence and uniqueness (Theorem 6.2.2. in [2]) of the solution for the following SDE

$$dx(t) = b(i, x(t))dt + \sigma(i, x(t))dW_t,$$

where W_t is a n-dimensional standard Wiener process. Moreover, the assumption on R ensures that events we encounter later are measurable w.r.t. the underlying σ-field, hence their probabilities make sense.

We can introduce the ESHS execution.

Definition 2 (ESHS Execution). *A stochastic process* $\alpha_t = (q(t), x(t))$ *is called a ESHS execution if there exists a sequence of stopping times* $T_0 = 0 \leq T_1 \leq T_2 \leq \ldots$ *such that for each* $j \in \mathbb{N}$,

– $\alpha_0 = (q(0), x(0))$ *is a* $Q \times X$*-valued random variable extracted according to the probability measure* $Init$;
– *For* $t \in [T_j, T_{j+1})$, $q(t) = q(T_j)$ *is constant and* $x(t)$ *is a (continuous) solution of the SDE:*

$$dx(t) = b(q(T_j), x(t))dt + \sigma(q(T_j), x(t))dW_t \tag{1}$$

where W_t *is a* n*-dimensional standard Wiener process;*
– $T_{j+1} = \inf\left\{t \geq T_j : x(t-) \in D_{q(T_j)}\right\}$ *(which is equal to* $+\infty$ *if the process never hits the target* $D_{q(T_j)}$*);*
– $x(T_{j+1}^-) \in G(q(T_j), q(T_{j+1}))$, *where* $x(T_{j+1}^-)$ *denotes* $\lim_{t \uparrow T_{j+1}} x(t)$;
– *The probability distribution of* $x(T_{j+1})$ *is governed by the law* $R\left(q(T_j), q(T_{j+1}), x(T_{j+1}^-)\right)$.

Remark 1. The continuous post jump location $x(T_{j+1})$ is given by R and the hybrid state before jump, but the discrete post jump location $q(T_{j+1})$ depends only on the hybrid state before jump. In the stochastic model from [4], both continuous and discrete post jump locations depend on the hybrid state before jump and some i.i.d. random variables, which are independent of W_t.

3 Model Summary

3.1 Hybrid State Space

Let us define the hybrid state space and its 'boundary' as follows:

$$\mathbb{S} = \bigcup_{i \in Q} \{i\} \times X_i; \ \overline{\mathbb{S}} = \mathbb{S} \cup \overline{\partial} \mathbb{S};$$

$$\overline{\partial} \mathbb{S} = \bigcup_{i \in Q} \{i\} \times D_i = \bigcup_{i \in Q} \left(\{i\} \times \bigcup_{j \in Q, j \neq i} G(i,j) \right).$$

Let us denote $\mathbb{S}_i = \{i\} \times X_i$, $\overline{\partial}\mathbb{S}_i = \{i\} \times D_i$ for every $i \in Q$. We can refine the last notation as follows: $\overline{\partial}\mathbb{S}_i = \bigcup_{j \in Q, j \neq i} \overline{\partial}\mathbb{S}_{ij}$ where $\overline{\partial}\mathbb{S}_{ij} = \{i\} \times G(i,j)$ for $i \neq j$. Define $\mathcal{B}(\widetilde{\mathbb{S}})$ as the σ-algebra on the set $\widetilde{\mathbb{S}} = Q \times \mathbb{R}^n$ generated by the sets $\{2^Q \times \mathcal{B}(\mathbb{R}^n)\}$. Let $\mathcal{B}(\overline{\mathbb{S}})$ be the σ-algebra on $\overline{\mathbb{S}}$ induced by $\mathcal{B}(\widetilde{\mathbb{S}})$. It is possible to define a metric ρ on $\overline{\mathbb{S}}$ in such a way that $\rho(\alpha_n, \alpha) \to 0$ as $n \to \infty$ with $\alpha_n = (i_n, x_n)$, $\alpha = (i, x)$ if and only if there exists m such that $i_n = i$ for all $n \geq m$ and $x_{m+k} \to x$ as $k \to \infty$. The metric ρ restricted to any component X_i is equivalent to the usual Euclidean metric [12]. Then $(\overline{\mathbb{S}}, \mathcal{B}(\overline{\mathbb{S}}))$ is a Borel space[2]. A cemetery point $\Delta \notin \overline{\mathbb{S}}$ is adjoined to $\overline{\mathbb{S}}$ as an isolated point, $\overline{\mathbb{S}}_\Delta \equiv \overline{\mathbb{S}} \cup \{\Delta\}$. Using the Lebesgue measure λ on $\mathbb{R}^n$, we define a measure m on $\mathcal{B}(\overline{\mathbb{S}})$ such that for each $i \in Q$ the projection of m to X_i is exactly $\lambda|_{X_i}$.

$$m(Q' \times A) := card Q' \cdot \lambda(A), \ Q' \subset Q, \ A \in \mathcal{B}(\mathbb{R}^n).$$

One can then consider the reset probability kernel of an ESHS as a transition measure $R : \overline{\mathbb{S}} \times \mathcal{B}(\overline{\mathbb{S}}) \to [0,1]$ such that: (i) $R((i,x), \cdot) = 0$, for all $(i,x) \in \overline{\mathbb{S}} \backslash \overline{\partial}\mathbb{S}$; (ii) for all $A \in \mathcal{B}(\overline{\mathbb{S}})$, $R(\cdot, A)$ is measurable; (iii) for all $(i,x) \in \overline{\partial}\mathbb{S}$ the function $R((i,x), \cdot)$ is a probability measure concentrated on $\{j\} \times X_j$ where j is the unique value of the discrete state such that $x \in G(i,j)$.

In short we have, the stochastic execution is given by

$$dx(t) = \{b(q(t), x(t)) + \sum_{j=0}^{\infty} [x(T_j) - x(T_j^-)]\delta_{(t-T_i)}\}dt + \sigma(q(t), x(t))dW_t;$$

$$q(t) = \sum_{j=0}^{\infty} [q(T_j) - q(T_j^-)] 1_{(T_j \leq t)}$$

where δ is the Dirac measure and

$$x(T_j)(\omega) = \Psi(\omega, (q(T_j^-), x(T_j^-)(\omega)); \ q(T_j)(\omega) = \sum_{i \in Q} i 1_{G(q(T_j^-),i)}(x(T_j^-)(\omega))$$

[2] Recall that a Borel space is a topological space which is homeomorphic to a Borel subset of a complete separable metric space.

3.2 Markov Property

In this subsection we prove that any ESHS is a Borel right process (i.e. a right Markov process whose semigroup maps $\mathcal{B}(\overline{\mathbb{S}})$[3] into itself). A *right process* was originally defined by P. A. Meyer [19] as a process satisfying two *hypothèses droites* HD1 and HD2. A right process is a strong Markov process with the following properties: (i) The state space is Lusin[4] (i.e. it is isomorphic to a Borel subset of a compact metrizable space); (ii) It is right continuous; (iii) The p-excessive functions ($p > 0$) of the process are almost surely right continuous (If P_t is a Markov semigroup then a function f is p-excessive if it non-negative and $e^{-pt}P_t f \leq f$ for all $t \geq 0$ and $e^{-pt}P_t f \nearrow f$ as $t \searrow 0$).

Let H be an ESHS. We use the same notations as in the subsection 3.1.

Proposition 1. *Under the standard Assumption 1, any Extended Stochastic Hybrid System, H is a Borel right process.*

Proof. First we prove that H is a right Markov process. Clear, the state space $\overline{\mathbb{S}}$ is Lusin space (it is a Borel space and its closure can be embedded in a compact space). We have a countable number of state spaces $(\overline{\mathbb{S}}_i)_{i\in Q}$ provided with the *Feller semigroups*[5] $(P_t^i)_{i\in Q}$ associated with the diffusion processes defined by (1). Δ can be considered adjoined to all these spaces. The spaces Ω, Ω_i are taken as the canonical realizations with values in $\overline{\mathbb{S}}$, $\overline{\mathbb{S}}_i$ (see the remark below). We provide Ω with the measures P^α, $\alpha \in \overline{\mathbb{S}}$ such that if $\alpha \in \overline{\mathbb{S}}_i$ the measure P^α is equal to P^α corresponding to Ω_i (P^α is a measure on Ω supported by Ω_i). Since $\overline{\mathbb{S}}$ is a Borel space, then $\overline{\mathbb{S}}$ is homeomorphic to a subset of the Hilbert cube, $\mathcal{H}$[6] (Urysohn's theorem, Prop. 7.2 [6]). Moreover, X is a homeomorphic with a Borel subset of a compact metric space (Lusin space) because it is a locally compact Hausdorff space with countable base (see [12] and the references therein). The ESHP model is obtained by 'melange' operation [20] of the diffusion processes $(i, x(t))_{i\in Q}$, each one defined on $\overline{\mathbb{S}}_i$. Since each diffusion is, in particular, a right process, then the whole process is a right Markov process (Th.1 in [20]). The 'renaissance' kernel Ψ [20] used to mix the diffusion processes is given by

$$\Psi(\omega^i, A) = R((i, x(T_i^-)), A),\ A \in \mathcal{B}(\mathbb{S})$$

where ω^i is a diffusion trajectory and $x(T_i^-)$ is its boundary hitting point.

Secondly, we prove that H is a Borel right process. Let (P_t) be the transition semigroup of (α_t) i.e. $P_t f(\alpha) = \mathbf{E}^\alpha[f(\alpha_t)]$, $\alpha \in \overline{\mathbb{S}}$, where $\mathbf{E}^\alpha$ is the expectation w.r.t. P^α, for all functions f for which the right-hand sides make sense.
Let be f a bounded Borel function on $\overline{\mathbb{S}}$ and f_j the restriction of f to $\overline{\mathbb{S}}_i$. Let $t > 0$. Then $t \in [T_i, T_{i+1})$ where $(T_j)_{j\in\mathbb{N}}$ is the sequence of stopping times from definition 2. The construction of H implies the following equality

$$P_t f(\alpha) = \mathbf{E}^\alpha[f(\alpha_t)] = \mathbf{E}^\alpha[f(\alpha_t^{q(T_i)})|T_i \leq t < T_{i+1}] = P_t^{q(T_i)} f_{q(T_i)}(\alpha)$$

[3] here $\mathcal{B}(\overline{\mathbb{S}})$ is understood as the set of all real Borel functions defined on $\overline{\mathbb{S}}$.

[4] This condition is missing in other Markov process monographs, see e.g. [7].

[5] i.e. the function $\alpha \to P_t^i f(\alpha)$ is continuous for each $t \in \mathbb{R}_+$ if f is a bounded continuous function.

[6] $\mathcal{H}$ is the product of countable many copies of $[0, 1]$.

where $(\alpha_t^{q(T_i)})$ is the restriction of (α_t) to $[T_i, T_{i+1})$. Note that if $\alpha \notin \overline{\mathbb{S}}_{q(T_i)}$ then $P_t f(\alpha) = 0$. Therefore $P_t f$ is supported by $\overline{\mathbb{S}}_{q(T_i)}$ where it is a Borel function ($P_t^{q(T_i)}$ is Feller). Then $P_t f$ is a Borel function.

Remark 2. It is clear from the construction, that an ESHS has the CADLAG property (i.e. its trajectories are right continuous with left limits). Then the underlying probability space Ω can be taken, in a canonical way, equal to $D_{[0,\infty)}(\overline{\mathbb{S}})$[7].

3.3 Process Generator

We denote by $\mathcal{B}_b(\overline{\mathbb{S}})$ the set of all bounded measurable functions $f : \overline{\mathbb{S}} \to \mathbb{R}$. This is a Banach space under the norm $\|f\| = \sup_{\alpha \in \overline{\mathbb{S}}} |f(\alpha)|$. Associated with the semigroup (P_t) is its *strong generator* which, loosely speaking, is the derivative of P_t at $t = 0$. Let $D(L) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists

$$\lim_{t \searrow 0} \frac{1}{t}(P_t f - f) \tag{2}$$

and denote this limit Lf. The limit refers to convergence in the norm $\|\cdot\|$, i.e. for $f \in D(L)$ we have

$$\lim_{t \searrow 0} || \frac{1}{t}(P_t f - f) - Lf || = 0.$$

The results from [3] give us the possibility to write the generator of any process as the sum of its corresponding continuous and jump parts. For $f \in C^2(Q \times \mathbb{R}^n) = \{f : Q \times \mathbb{R}^n \to \mathbb{R} |\ f(i, \cdot) \in C^2(\mathbb{R}^n) \text{ for all } i \in Q\}$ the expression of the ESHS generator [21] is

$$Lf(\alpha) = L^c f(\alpha) + L^j f(\alpha)$$

where

$$L^c f(\alpha) = \frac{1}{2} \sum_{k,m,l=1}^{n} \sigma_{kl}(\alpha)\sigma_{ml}(\alpha) \frac{\partial^2 f}{\partial \alpha_k \partial \alpha_m} + \sum_{k=1}^{n} b_k(\alpha) \frac{\partial f}{\partial \alpha_k}$$

and L^j is the *Lévy operator*, given by

$$L^j f(\alpha) = \int_{\partial \mathbb{S}} (f(\beta) - f(\alpha)) R(\alpha, d\beta).$$

Since any ESHS is a right Markov process, according to [5], there exists a Lévy system associated to the process. Recall that a Lévy system (n, dH_t) for a process (α_t) is a kernel $n(\alpha, d\beta)$ and a perfect continuous additive functional H_t (see the definition in [7]) such that such that for all α in the state space $\overline{\mathbb{S}}$, for

[7] $D_{[0,\infty)}(\overline{\mathbb{S}})$ is the space of all right continuous functions on $[0, \infty)$ with left limits taking values in $\overline{\mathbb{S}}$.

all stopping times T, and for all positive Borel measurable functions f on $\overline{\mathbb{S}} \times \overline{\mathbb{S}}$ that are 0 on the diagonal, the *Lévy system identity* holds:

$$\mathbf{E}^{\alpha} \sum_{0<t\leq T} f(\alpha_{t-}, \alpha_t) = \mathbf{E}^{\alpha} \int_0^T \int f(\alpha_t, \beta) n(\alpha_t, d\beta) dH_t \tag{3}$$

where both sides may be infinite. We will assume without loss of generality that $H_t(\omega) = t$ for all t and all ω, since one can always perform a time change on α_t [3]. For our process the Lévy kernel n can be chosen to be equal with R.

Remark 3. [17] Let $D[L]$ be the domain of the generator L. Under the assumptions 1, since (α_t) is a right Markov process[8], there exists a *quasi-regular Dirichlet form*[9] $(\mathcal{E}, D[\mathcal{E}])$ on $L^2(\overline{\mathbb{S}}, m)$ associated with the process, given by

$$\begin{cases} D[L] \subset D[\mathcal{E}] \\ \mathcal{E}(u,v) = (-Lu, v), \ u \in D[L], \ v \in D[\mathcal{E}]. \end{cases}$$

We can think of a Dirichlet form $\mathcal{E}$ as a recipe for a Markov process $(\alpha_t)_{t\geq 0}$, in the sense that $\mathcal{E}$ describes the behavior of the composed process $u(\alpha_t)$ for every u in the domain of $\mathcal{E}$. There is no guarantee that the 'coordinates' $(u(\alpha_t))_u$ can be put together in a consistent way to form a process with reasonable sample paths.

3.4 Jump Measures

Assumption 2 *For any $\varepsilon > 0$, $x \in \mathbb{S}$, $j(\cdot, \mathbb{S} \backslash B_{\varepsilon}(x))$ is a locally integrable function w.r.t. m.*

For each $\mathcal{B}(\overline{\mathbb{S}})$-measurable function u, let us define $Ru(\alpha) := \int_{\overline{S}} u(\beta) R(\alpha, d\beta)$, $\alpha \in \overline{\mathbb{S}}$. It is clear that Ru satisfies some properties as below: (i) Ru is a $\mathcal{B}(\overline{\mathbb{S}})$-measurable function. (ii) If $\alpha \in \overline{\partial \mathbb{S}}$ (there exist $i \neq j$ such that $\alpha = (i,x)$ and $x \in G(i,j)$) then $Ru(i,x) = \int_{\mathbb{S}_j} u(j,y) R((i,x),(j,dy))$. Moreover, if $supp(u) \cap \mathbb{S}_j = \emptyset$ then $Ru(i,x) = 0$. If $u = 1_A$, $A \in \mathcal{B}(\mathbb{S})$ then $R1_A(i,x) = R((i,x), A \cap \{j\} \times X_j)$. (iii) $Ru(\alpha) = 0$ if $\alpha \in \overline{\mathbb{S}} \backslash \overline{\partial \mathbb{S}}$ or $supp(u) \cap \mathbb{S} = \emptyset$.

Let us define for any $u, v \in \mathcal{B}(\overline{\mathbb{S}})$, the following energies

$$\overrightarrow{\mathcal{E}}_R(u,v) = \int_{\overline{S}} u(\alpha) Rv(\alpha) m(d\alpha) \tag{4}$$

$$\overleftarrow{\mathcal{E}}_R(u,v) = \int_{\overline{S}} Ru(\alpha) v(\alpha) m(d\alpha) \tag{5}$$

Some simple computations give

$$\overrightarrow{\mathcal{E}}_R(u,v) = \int_{\overline{S}} u(\alpha) \{ \int_{\overline{S}} v(\beta) R(\alpha, d\beta) \} m(d\alpha) = \int_{\overline{\partial \mathbb{S}}} u(\alpha) \{ \int_{\mathbb{S}} v(\beta) R(\alpha, d\beta) \} m(d\alpha)$$

[8] In Meyer's sense, i.e. the state space $\overline{\mathbb{S}}$ is Lusin.

[9] See the definition 3.1 from [17]

$$
\begin{aligned}
&= \sum_{i\in Q} \int_{\overline{\partial}\mathbb{S}_i} u(i,x)\{\sum_{j\in Q} \int_{\mathbb{S}_j} v(j,y)R((i,x),(j,dy))\}(i,\lambda(dx)) \\
&= \sum_{i\in Q} \sum_{l\in Q, l\neq i} \int_{\overline{\partial}\mathbb{S}_{il}} u(i,x)\{\sum_{j\in Q} \int_{\mathbb{S}_j} v(j,y)R((i,x),(j,dy))\}(i,\lambda(dx)) \\
&= \sum_{i\in Q} \sum_{j\in Q, j\neq i} \int_{\overline{\partial}\mathbb{S}_{ij}} u(i,x) \int_{\mathbb{S}_j} v(j,y)R((i,x),(j,dy))(i,\lambda(dx)) \\
&= \sum_{i\in Q} \sum_{j\in Q, j\neq i} \int_{\overline{\partial}\mathbb{S}_{ij}} \int_{\mathbb{S}_j} u(i,x)v(j,y)R((i,x),(j,dy))(i,\lambda(dx)). \qquad (6)
\end{aligned}
$$

$$
\overleftarrow{\mathcal{E}}_R(u,v) = \sum_{i\in Q} \sum_{j\in Q, j\neq i} \int_{\overline{\partial}\mathbb{S}_{ij}} \int_{\mathbb{S}_j} v(i,y)u(j,x)R((i,y),(j,dx))(i,\lambda(dy)). \qquad (7)
$$

Letting in (6) $u = 1_{\overline{\partial}\mathbb{S}_{ij}}$, $v = 1_{\mathbb{S}_j}$ for $i \neq j$, we get $\int_{\overline{S}} u(\alpha)Rv(\alpha)m(d\alpha) = \lambda(G(i,j))$. Therefore, if $u = 1_{\overline{\partial}\mathbb{S}_i}$, $v = 1_{\mathbb{S}}$ then $\int_{\overline{S}} u(\alpha)Rv(\alpha)m(d\alpha) = \sum_{j\neq i} \lambda(G(i,j)) = \lambda(D_i)$. In the last, if $u = 1_{\overline{\partial}\mathbb{S}}$, $v = 1_{\mathbb{S}}$ then $\int_{\overline{S}} u(\alpha)Rv(\alpha)m(d\alpha) = \sum_{i\in Q} \lambda(D_i)$. Analogously, letting in (7) $u = 1_{\overline{\partial}\mathbb{S}}$, $v = 1_{\mathbb{S}}$ then $\int_{\overline{S}} Ru(\alpha)v(\alpha)m(d\alpha) = 0$.

Remark 4. If the energy $\overrightarrow{\mathcal{E}}_R(u,v)$ (resp. $\overleftarrow{\mathcal{E}}_R(u,v)$) is nonzero then $supp(u) \cap \overline{\partial}\mathbb{S} \neq \emptyset$ and $supp(v) \cap \mathbb{S} \neq \emptyset$ (resp. $supp(v) \cap \overline{\partial}\mathbb{S} \neq \emptyset$ and $supp(u) \cap \mathbb{S} \neq \emptyset$).

R induces a *positive jump measure* J and a *positive symmetric jump measure* $\widetilde{J}$[10] on $\overline{\mathbb{S}}\times\overline{\mathbb{S}}\backslash d$ (d is the diagonal set) by

$$
\int_{\overline{\mathbb{S}}\times\overline{\mathbb{S}}\backslash d} f(\alpha,\beta)J(d\alpha,d\beta) = \int_{\overline{\mathbb{S}}}\int_{\overline{\mathbb{S}}} f(\alpha,\beta)R(\alpha,d\beta)m(d\alpha);\ f \in C_0(\overline{\mathbb{S}}\times\overline{\mathbb{S}}\backslash d); \quad (8)
$$

$$
\int_{\overline{\mathbb{S}}\times\overline{\mathbb{S}}\backslash d} f(\alpha,\beta)\widetilde{J}(d\alpha,d\beta) = \frac{1}{2}\left\{ \int_{\overline{\mathbb{S}}}\int_{\overline{\mathbb{S}}} f(\alpha,\beta)R(\alpha,d\beta)m(d\alpha) + \int_{\overline{\mathbb{S}}}\int_{\overline{\mathbb{S}}} f(\alpha,\beta)R(\beta,d\alpha)m(d\beta) \right\}.
$$

Remark 5. In the theory of stochastic processes the most frequently used jump measures are defined by means of symmetric transition kernels R (i.e. the energies defined by (4) and (5) are equal). In our case the computations (6) and (7) show

[10] The symmetric *Dirichlet form* associated with $\widetilde{J}$ on $L^2(\overline{\mathbb{S}},m)$ is given by the following formula

$$
\mathcal{E}(u,v) = \int_{\overline{\mathbb{S}}\times\overline{\mathbb{S}}\backslash d} (u(\alpha)-u(\beta))(v(\alpha)-v(\beta))\widetilde{J}(d\alpha,d\beta)
$$
$$
\mathcal{D}[\mathcal{E}] = \{u \in L^2(\mathbb{S},m),\ \mathcal{E}(u,u) < \infty\}.
$$

where $\mathcal{D}[\mathcal{E}]$ is called the domain of $\mathcal{E}$.

that we can have equality between the two above energies only in some special cases. Necessarily this kind of equality implies that the Lesbesgue measure of the interface sets must be zero. But, this will sequentially imply that the energy should be zero, which is a very trivial case.

Remark 6. A simple computation gives the following:

$$\begin{aligned}\int_{\overline{\mathbb{S}}\times\overline{\mathbb{S}}\setminus d} f(\alpha,\beta)\widetilde{J}(d\alpha,d\beta) &= \frac{1}{2}\left\{\int_{\overline{\partial\mathbb{S}}}\int_{\overline{\mathbb{S}}} f(\alpha,\beta)R(\alpha,d\beta)m(d\alpha)\right.\\ &\quad\left.+\int_{\overline{\partial\mathbb{S}}}\int_{\overline{\mathbb{S}}} f(\alpha,\beta)R(\beta,d\alpha)m(d\beta)\right\}\\ &= \frac{1}{2}\left\{\int_{\overline{\partial\mathbb{S}}}\int_{\mathbb{S}} f(\alpha,\beta)R(\alpha,d\beta)m(d\alpha)\right.\\ &\quad\left.+\int_{\overline{\partial\mathbb{S}}}\int_{\overline{\mathbb{S}}} f(\alpha,\beta)R(\beta,d\alpha)m(d\beta)\right\}\\ &= \int_{\overline{\partial\mathbb{S}}}\int_{\mathbb{S}} \frac{f(\alpha,\beta)+f(\beta,\alpha)}{2}R(\alpha,d\beta)m(d\alpha).\end{aligned}$$

4 Reachability Problem

In this section we study the reachability problem for ESHS. Recall from our paper [9] the reachability definitions in the stochastic framework. Let E be a Borel set of the state space $\overline{\mathbb{S}}$. Define the reachable "events" associated to E:

$$\begin{aligned}Reach_T(E) &= \{\omega\in\Omega \mid \exists t\in[0,T] : \alpha_t(\omega)\in E\}\\ Reach_\infty(E) &= \{\omega\in\Omega \mid \exists t\geq 0 : \alpha_t(\omega)\in E\}.\end{aligned}$$

where $T>0$ is a finite positive time horizon. The measurability of these events can be easily obtained using the CADLAG property of the process and the fact that the underlying probability space is a Borel space. We deal in this section with the issue of the computation of reach set probabilities.

To compute the reach set probabilities, we propose a Dirichlet form based approach, which takes into consideration the two main features of the hybrid executions: (1) forced jumps; (2) diffusion segments between consecutive jumps.

First we summarize the main ideas of the computation method of the reach event probabilities. Let B, E be two Borel sets, where B is the initial condition set and E is the reachability target set. We can suppose that $E\subset\mathbb{S}_i$, $i\in Q$ (otherwise E can be written as a partition $\bigcup_{i\in Q}(E\cap\mathbb{S}_i)$). Then the reachable event associated to E is the intersection of two other events:

1. the set of all trajectories which jump in $\mathbb{S}_i$, i.e. the 'jump' reachable event, denoted by $Reach_T^j(\mathbb{S}_i)$;
2. the set of all diffusion paths (corresponding to $\mathbb{S}_i$) which reach E, i.e. the 'continuous' reachable event, denoted by $Reach_T^c(E)$.

Let be $\mathcal{E}$ the quasi-regular Dirichlet form associated to (α_t) (see remark 3). Each quasi-regular Dirichlet form[11] can be expressed as the sum of three parts: diffusion $\mathcal{E}^c$, jumping $\mathcal{E}^j$ and killing $\mathcal{E}^k$: $\mathcal{E} = \mathcal{E}^c + \mathcal{E}^j + \mathcal{E}^k$ (Beurling-Deny Formulae). The continuous part is used to study the continuous part of the process (i.e. it corresponds to the continuous pieces of process trajectories). The jumping part corresponds to the trajectory discontinuities of the process and finally, the killing part is connected with those trajectories which go the cemetery point.

We employ the first two parts of $\mathcal{E}$ to solve the reachability problem:

(i) the jump part $\mathcal{E}^j$ given by the jump measure (8) is used to compute $P[Reach^j_T(\mathbb{S}_i)]$.
(ii) the diffusion part $\mathcal{E}^c$ is employed to compute $P[Reach^c_T(E)]$.

Intuitively, the probability of the reachable event $P(Reach_T(E))$, $E \subset \mathbb{S}_i$, is a conditional probability, which is equal to the probability of $Reach^c_T(E)$ conditioned by the event $Reach^j_T(\mathbb{S}_i)$. It is clear that the probability to jump in $\mathbb{S}_i$ is given by the reset probability kernel of ESHS, R. Since R also defines the jump measure $\widetilde{J}$, then the above jump probability should be related with the the jump component of the associated Dirichlet form. To compute $P[Reach^c_T(E)]$ we use the function F (which gives E as a level set) to obtain the induced Dirichlet form of $\mathcal{E}^c|_{\mathbb{S}_i}$. The induced Dirichlet forms are easy to deal with because their state space is $\mathbb{R}$. The Dirichlet form expression associated to different kind of diffusion is well known in the literature [14,17]. Therefore, the induced Dirichlet forms have nice representation formula [16], which allow to translate the computation problem of the 'continuous' reachable event probability from the initial state space to the 'induced' state space.

Suppose that B and $\overline{\mathbb{S}}_i$ are disjoint. More exactly, the computation steps are:

(A) compute the probability that the process, started in B, jumps in $\mathbb{S}_i$, using the estimation (11) of the Prop. 3 from below;
(B) give an upper bound for the probability that the process, arrived in $\mathbb{S}_i$, hits E, using the estimation (13) of Prop. 4 from below.

4.1 Computation of 'Jump' Reachable Event Probabilities

For $A \in \mathcal{B}(\overline{\mathbb{S}})$ define processes p, p^* and $\widetilde{p}$ as follows:

$$p(t,A) = \sum_{k=1}^{\infty} I_{(t \geq T_k)} I_{(\alpha_{T_k} \in A)};\ p^*(t) = \sum_{k=1}^{\infty} I_{(t \geq T_k)};$$
$$\widetilde{p}(t,A) = \int_0^t R(\alpha_{s-}, A) dp^*(s) = \sum_{T_k \leq t} R(\alpha_{T_k -}, A).$$

Note that p, p^* are counting processes and $p^*(t) = k$ if $t \in [T_k, T_{k+1})$, $k = 0, 1, 2, ...$ where $T_0 = 0$. $\widetilde{p}(t,A)$ is the compensator of $p(t,A)$ (see [12] for more explanations). The following process is a local martingale.

$$q(t,A) = p(t,A) - \widetilde{p}(t,A). \tag{9}$$

[11] See [14,17] for the theory of symmetric and non-symmetric Dirichlet forms.

Proposition 2. *For each $\alpha \in \overline{\mathbb{S}}$ we have*

$$P^{\alpha}(Reach_T^j(\mathbb{S}_i)) = \mathbf{E}^{\alpha}\widetilde{p}(T, \mathbb{S}_i)$$

Proof. If in the Lévy system identity (3) we take $f(\alpha, \beta) = 1_{\mathbb{S}_i^c}(\alpha) \cdot 1_{\mathbb{S}_i}(\beta)$ then the first member will give the expectation of the set of all trajectories which jump in $\mathbb{S}_i$ in the time interval $[0, T]$, i.e. $P^{\alpha}(Reach_T^j(\mathbb{S}_i))$

$$\begin{aligned} P^{\alpha}(Reach_T^j(\mathbb{S}_i)) &= \mathbf{E}^{\alpha}\int_0^T \int f(\alpha_t, \beta)R(\alpha_t, d\beta)dt \\ &= \mathbf{E}^{\alpha}\int_0^T \int_{\mathbb{S}_i} 1_{\mathbb{S}_i^c}(\alpha_t)R(\alpha_t, d\beta)dt = \mathbf{E}^{\alpha}\int_0^T 1_{\mathbb{S}_i^c}(\alpha_t)R(\alpha_t, \mathbb{S}_i)dt \\ &= \mathbf{E}^{\alpha}\sum_{T_i \leq T} R(\alpha_{T_i^-}, \mathbb{S}_i) = \mathbf{E}^{\alpha}\widetilde{p}(T, \mathbb{S}_i). \end{aligned}$$

Remark 7. The proof of the Prop. 2 can be derived using the martingale property of q defined by (9).

For a Borel set G of state space we define the first hitting time $T_G := \inf\{t > 0 | \alpha_t \in G\}$. Let τ_G denote the first leaving time from G, i.e. $\tau_G = T_{\overline{\mathbb{S}} \backslash G}$ is the first hitting time of $\overline{\mathbb{S}} \backslash G$. Let p_t^G be the transition function of the restriction of the process (α_t) to G. Dealing with the jump times we use the following result:

Lemma 1. *[14] Let G be a relatively compact open set. For any h, f, g bounded positive Borel functions on $\overline{\mathbb{S}}$ such that $supp[h] \subset G$, $supp[f] \subset G$ and $supp[g] \subset \overline{\mathbb{S}} \backslash \overline{G}$ we have*

$$\mathbf{E}^{h \cdot m}(f(\alpha_{\tau_G^-})g(\alpha_{\tau_G});\ \tau_G \leq T) = 2\int_0^T \left[\int\int p_t^G h(\alpha)f(\alpha)g(\beta)\widetilde{J}(d\alpha, d\beta)\right] dt \quad (10)$$

where , $\mathbf{E}^{h \cdot m}$ is the expectation given by the probability $P^{h \cdot m}(A) = \int_{\overline{\mathbb{S}}} P^{\alpha}(A)h(\alpha)m(d\alpha)$.

Proposition 3. *Let $\overline{\partial}\mathbb{S}_{\to i}$ be the subset of $\overline{\partial}\mathbb{S}$ from where the process (α_t) can jump in $\mathbb{S}_i$ ($i \in Q$). For any Borel set $B \subset \mathbb{S}_i$ (resp. $B \subset \overline{\mathbb{S}}_i^c$), $h = 1_B$, we have*

$$\begin{aligned} P^{h \cdot m}(\tau_{\mathbb{S}_i} \leq T) &= \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} \int_{\mathbb{S}_i} p(\beta, t, B)R(\alpha, d\beta)m(d\alpha)\right] dt \\ \textit{(resp. } P^{h \cdot m}(T_{\mathbb{S}_i} \leq T) &= \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} p(\alpha, t, B)m(d\alpha)\right] dt\textit{).} \end{aligned} \quad (11)$$

Proof. By setting $f = 1_{\mathbb{S}_i}$, $h = 1_B$, $B \subset \mathbb{S}_i$, $g = 1_{\overline{\mathbb{S}} \backslash \mathbb{S}_i}$ in (10) we get the probability to remain in $\mathbb{S}_i$ until time horizon T, if the process has started in $\mathbb{S}_i$

$$\begin{aligned}
P^{h\cdot m}(\tau_{\mathbb{S}_i} \leq T) &= \int_0^T \Bigg[\int_{\overline{\partial}\mathbb{S}} \int_{\mathbb{S}} \{ p_t^{\mathbb{S}_i} h(\alpha) f(\alpha) g(\beta) \\
&\quad + p_t^{\mathbb{S}_i} h(\beta) f(\beta) g(\alpha) \} R(\alpha, d\beta) m(d\alpha) \Bigg] dt \\
&= \int_0^T \left[\int_{\overline{\partial}\mathbb{S} \setminus \overline{\partial}\mathbb{S}_i} \int_{\mathbb{S}_i} p_t^{\mathbb{S}_i} h(\beta) R(\alpha, d\beta) m(d\alpha) \right] dt \\
&= \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} \int_{\mathbb{S}_i} p_t^{\mathbb{S}_i} h(\beta) R(\alpha, d\beta) m(d\alpha) \right] dt \\
&= \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} \int_{\mathbb{S}_i} p(\beta, t, B) R(\alpha, d\beta) m(d\alpha) \right] dt.
\end{aligned}$$

Letting $f = 1_{\overline{\mathbb{S}}_i^c}$, $h = 1_B$, $B \subset \overline{\mathbb{S}}_i^c$, $g = 1_{\overline{\mathbb{S}}_i}$ the probability to hit $\mathbb{S}_i$ until T is

$$\begin{aligned}
P^{h\cdot m}(T_{\mathbb{S}_i} \leq T) &= \int_0^T \Bigg[\int_{\overline{\partial}\mathbb{S}} \int_{\mathbb{S}} \{ p_t^{\overline{\mathbb{S}}_i^c} h(\alpha) f(\alpha) g(\beta) \\
&\quad + p_t^{\overline{\mathbb{S}}_i^c} h(\beta) f(\beta) g(\alpha) \} R(\alpha, d\beta) m(d\alpha) \Bigg] dt \\
&= \int_0^T \left[\int_{\overline{\partial}\mathbb{S} \setminus \overline{\partial}\mathbb{S}_i} \int_{\overline{\mathbb{S}}_i} p_t^{\overline{\mathbb{S}}_i^c} h(\alpha) R(\alpha, d\beta) m(d\alpha) \right] dt \\
&= \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} p_t^{\overline{\mathbb{S}}_i^c} h(\alpha) R(\alpha, \overline{\mathbb{S}}_i) m(d\alpha) \right] dt \\
&= \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} p_t^{\overline{\mathbb{S}}_i^c} h(\alpha) m(d\alpha) \right] dt \\
&= \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} p(\alpha, t, B) m(d\alpha) \right] dt.
\end{aligned}$$

Remark 8. (11) gives the probability of hitting $\mathbb{S}_i$ through the Borel set B. Similar arguments as those used in the Prop. 3 lead to $P^{h\cdot m}(\tau_G \leq T) = \int_0^T \left[\int_{\overline{\partial}\mathbb{S}_{\to i}} \int_G p(\beta, t, B) R(\alpha, d\beta) m(d\alpha) \right] dt$ for any open set $G \subset \mathbb{S}_i$, $h = 1_B$, $B \subset \mathbb{S}_i$. This estimation can be useful when G is known as a safety set of the state space. An interesting problem would be to find h or B maximizing the above probability. It is known that the transition probabilities $p(\beta, t, \cdot)$ for the diffusion[12] are the solutions of Kolmogorov backward equations [2].

4.2 Computation of 'Continuous' Reachable Event Probabilities

Suppose that the given Borel set $E \subset \mathbb{S}_i$ can be written as a level set $\{\alpha \in \mathbb{S}_i | F(\alpha) > l\}$, where $F : \mathbb{S}_i \to \mathbb{R}$ is a given function. Note that, in general,

[12] The restriction of the process to G is a diffusion process.

$F(\alpha_t)$ is not itself a Markov process (see [22]). The computation of probability of $Reach^c_T(E)$ means, in fact, the computation of $P^m(\sup_{t\in[0,T]} F(\alpha_t) > l)$.

We consider the restriction (α^i_t) of our process to $\overline{\mathbb{S}}_i$. This is a diffusion process and the properties of the corresponding Dirichlet form $\mathcal{E}_i$ are well known [14,17]. Moreover, we can suppose that $\mathcal{E}_i$ is a regular symmetric Dirichlet form [11]. Let $\mathcal{B}^*$ denote the σ-algebra Borel of $\mathbb{R}$ and let m^* denote the image of m under F. We construct a form $\mathcal{E}^*_i$ on $D[\mathcal{E}^*_i]\times D[\mathcal{E}^*_i] \subset L^2(\mathbb{R},m^*)\times L^2(\mathbb{R},m^*)$ by

$$\mathcal{E}^*_i(u^*,v^*) \ : \ = \mathcal{E}_i(u^*\circ F, v^*\circ F),\ u^*,v^*\in D[\mathcal{E}^*_i],$$
$$D[\mathcal{E}^*_i] = \{u^*\in L^2(\mathbb{R},m^*)|u^*\circ F\in D[\mathcal{E}_i].$$

Suppose that F is chosen such that $\mathcal{E}^*_i$ is a regular Dirichlet form [16]. Then there exists a certain $\mathbb{R}$-valued Markov process, (α^{i*}_t) associated with $\mathcal{E}^*_i$. Note if E^* is open in $\mathbb{R}$ and $E = F^{-1}(E^*)$ then we can consider the two first hitting times T_E and T_{E^*}. We define for $p > 0$, the p-capacity of E^*

$$Cap^*_p(E^*) = \inf\{\mathcal{E}^*_i(u^*,u^*) + p(u^*,u^*)_{m^*}|u^*\in D[\mathcal{E}^*_i],\ u^*\geq 1\ m^* - a.e. \text{ on } E^*\}$$

where $(u^*,u^*)_{m^*}$ is the inner product of $L^2(\mathbb{R},m^*)$. To give an upper estimate on $P^m(T_E\leq T)$ we need the following assumption:

Assumption 3 *We suppose that $m(\overline{\mathbb{S}}_i) < \infty$, $1\in D[\mathcal{E}_i]$ and $E^*\subset\mathbb{R}$ is an open subset of finite Cap^*-capacity such that $E = F^{-1}(E^*)$.*

The Prop. 1.24 from [16] becomes in our case as follows:

Proposition 4. *For all $p > 0$, we have:*

$$P^m(T_E\leq T)\leq e^p\{\mathbf{E}^{m^*}e^{-pT^*_{E^*}/T} + Tp^{-1}\int_{\mathbb{R}}\mathbf{E}^r e^{-pT^*_{E^*}/T}k^*(dr)\}; \tag{12}$$
$$P^m(T_E\leq T)\leq p^{-1}e^p\min\{T\mathcal{E}^*_i(u^*,u^*) + p(u^*,u^*)_{m^*}|u^*\in D[\mathcal{E}^*_i], \quad u^*\geq 1,\ m^* - a.e.\ on\ E^*\}. \tag{13}$$

where k^* is the killing measure associated with the killing part of $\mathcal{E}^*_i$.

Since we deal with the restriction of the process to $\overline{\mathbb{S}}_i$, the estimations (12), (13) still remain true if we replace P^m with $P^{g\cdot m}$, where $g = 1_{\mathbb{S}_i}$.

5 Conclusions and Further Work

In this paper we have extended and developed a stochastic hybrid system model introduced in [15]. The contributions of the paper are twofold: to prove correctness of this model (existence of the solution process which is a Borel right process), and to build mathematical tools for computing the probability of reaching a given set. The first contribution is necessary for any kind of mathematical exploration and serves as foundations for the development of the results. The second contribution can be viewed as a stochastic version of reachability. The method suggested is based on computation of so-called Dirichlet forms (already used for this purpose in stochastic analysis).

The possibility of implementation of the Dirichlet forms in a theorem prover leads to new directions in the stochastic hybrid system reachability study. On the other hand, for a stochastic hybrid system H with the Markov property, endowed with the corresponding Dirichlet form $\mathcal{E}$, using an appropriate proper map Φ from the state space $\mathbb{S}$ onto a 'smaller' space $\mathbb{S}^*$, one can construct the induced Dirichlet form $\mathcal{E}^*$ associated to a simpler system H^* (the image of H through Φ). The reachability problem for H and the target set $E \subset \mathbb{S}$ can be reduced to the reachability problem for H^* and the target set $\Phi(E)$. Therefore, it is possible to introduce a notion of *bisimulation of stochastic hybrid systems via Dirichlet forms,* which is a relatively new issue in the probabilistic framework.

One of the possible approaches which derives from this work is to study the reachability problem for ESHS with control. To solve this problem, it could be possible to use a cross-fertilization method which combines the Dirichlet form theory with the dynamic programming. For example, we intend to combine the results connected with the 'jumping' Dirichlet forms with the representations of reachable sets for the diffusion paths by means of viscosity solutions for some partial differential equations [23].

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