Reachability Questions in Piecewise Deterministic Markov Processes

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Abstract. We formulate a stochastic hybrid system model that allows us to capture a class of Markov processes known as piecewise deterministic Markov processes (PDMPs). For this class of stochastic processes we formulate a probabilistic reachability problem. Basic properties of PDMPs are reviewed and used to show that the reachability question is indeed well defined. Possible methods for computing the reach probability are then concerned.

Keywords: Hybrid systems, reachability, Markov processes, hitting times.

1 Introduction

Hybrid systems generalise both discrete state-transition systems and continuous dynamical systems. Hybrid systems posses continuous dynamics defined within regions of the state space and discrete transitions among the regions. Although the deterministic models for hybrid systems capture many characteristics of real systems, in practice stochastic features arise because of the inherent uncertainty in the environment of most real world applications. Few stochastic hybrid system models have been proposed in the literature (e.g. [8,12]). Different researchers have tried to propose different models from their own perspectives. The most important difference lies in where the randomness is introduced: in the continuous evolution, discrete transition times and discrete transition destinations.

In this paper a class of stochastic processes, called piecewise-deterministic Markov processes, is proposed as a model for studying stochastic hybrid systems. The piecewise-deterministic Markov processes (PDMPs) are an important class of non-linear continuous-time stochastic hybrid dynamical systems which admit theoretical analysis (see [4,5]). PDMPs are stochastic models in which the randomness appears as point events, i.e., there is a sequence of random occurrences at fixed or random times $T_1 < T_2 < T_3 < ...$ but there is no additional component of uncertainty between these times. A PDMP consists of a mixture of deterministic motion along a vector field and random jumps governed by some prescribed probabilistic law. The main properties of PDMPs are that they are

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strong Markov processes, they have an exact characterisation of the extended generator and they are right processes. The PDMP viewed as a solution of a martingale problem and the generator-based computations of process distributions and process functional expectations have been well studied in [5].

In this paper, we sketch some approaches for studying the reachability problem for PDMP. In a probabilistic framework, the reachability problem consists of determining the probability that the system trajectories enter some prespecified set starting from a certain set of initial conditions with a given probability distribution. Developing a methodology for the reachability analysis of stochastic hybrid systems will involve dealing with two aspects:

1. the theoretical aspect of the measurability of the reachability sets; and
2. the computational aspect regarding how to estimate the probability of the reachability events and how to quantify the level of approximation introduced.

The PDMP reachability problem approach presented in this paper is based on hitting time expectations. We prove that the reachable sets are events (measurable sets) in the underlying probability space, so we can deal with probabilistic notions in order to estimate their "measure". Also, we develop a formal link between the hitting distributions and some potential theory notions, namely reduced functions and excessive functions.

2 Definitions and Notation

2.1 Background

Evolution will take place in a hybrid state space consisting of discrete and continuous components. The discrete component may take one of countably many values. The continuous component may take values in Euclidean space, $\mathbb{R}^n$, whose dimension may be different for different values of the discrete state. More formally, consider a countable set Q, a map $d : Q \to \mathbb{N}$, and a map $X : Q \to 2^{\mathbb{R}^{d(\cdot)}}$ assigning to each value of $q \in Q$ a subset $X(q) \subseteq \mathbb{R}^{d(q)}$. We will use $q \in Q$ to denote the discrete part of the state and $x \in X(q)$ to denote the continuous part[1]. We will refer to the set

$$\mathbb{S} = \mathbb{S}(Q, d, X) = \bigcup_{q \in Q} \{q\} \times X(q)$$

as the *state space* of the system and its elements $\alpha = (q, x) \in \mathbb{S}(Q, d, X)$ as the *states*. Notice that in order to define the state space one needs to specify three elements, (Q, d, X). We define the *boundary* of $\mathbb{S}(Q, d, X)$ by

[1] If the set Q is finite then we can dispense with the additional complication of having the dimension of the continuous state depending on the discrete state. We can embed everything into a single Euclidean space of high enough dimension $\mathbb{R}^m$ where $m = \max_{q \in Q}\{d(q)\}$. This is likely to be the case for most applications.

$$\partial\mathbb{S}(Q,d,X)=\bigcup_{q\in Q}\{q\}\times\partial X(q),$$

where $\partial X(q)$ denotes the boundary of $X(q)$ in the Euclidean topology of $\mathbb{R}^{d(q)}$.

A *vector field*, f, on the hybrid state space, $\mathbb{S}(Q,d,X)$, can be thought as a function $f:\mathbb{S}(Q,d,X)\to\mathbb{R}^{d(\cdot)}$ assigning to each $(q,x)\in\mathbb{S}$ a vector (direction) $f(q,x)\in\mathbb{R}^{d(q)}$. We define the *flow* of the vector field f as a function $\phi:\mathbb{S}(Q,d,X)\times\mathbb{R}\to\mathbb{S}(Q,d,X)$ with $\phi(\alpha,t)=(\phi_q(\alpha,t),\phi_x(\alpha,t))$, $\phi_q(\alpha,t)\in Q$, $\alpha=(q,x)$ and $\phi_x(q,x,t)\in X(q)$, given by

$$\begin{aligned}\phi_x(q,x,0)&=x\\ \phi_q(q,x,t)&=q \text{ for all } t\in\mathbb{R}\\ \frac{d}{dt}\phi_x(\alpha,t)&=f(\phi(\alpha,t)) \text{ for all } t\in\mathbb{R}.\end{aligned}$$

Notice that, depending on the properties of f and the sets $X(q)$, the flow may not be defined for all $(\alpha,t)\in\mathbb{S}(Q,d,X)\times\mathbb{R}$. This technical point will be ignored for the time being, but will be addressed by appropriate assumptions later on.

Consider now a hybrid state space, $\mathbb{S}(Q,d,X)$ and a vector field f on that state space. If $\partial\mathbb{S}(Q,d,X)\not\subseteq\mathbb{S}(Q,d,X)$ (for example, if $\mathbb{S}(Q,d,X)$ is an open set in the product topology), then certain parts of $\partial\mathbb{S}(Q,d,X)$ may be reachable from points in $\mathbb{S}(Q,d,X)$ under the flow of f and others not. We denote the *former* by $\Gamma((Q,d,X),f)$. In other words,

$$\Gamma((Q,d,X),f)=\{\alpha\in\partial\mathbb{S}(Q,d,X)\mid\exists(\alpha',t)\in\mathbb{S}(Q,d,X)\times\mathbb{R}_+,\alpha=\phi(\alpha',t)\}\,.$$

Clearly $\partial\mathbb{S}(Q,d,X)\cap\mathbb{S}(Q,d,X)\subseteq\Gamma((Q,d,X),f)$ for all f, and the two sets are equal if $\mathbb{S}(Q,d,X)$ is closed.

Given (Q,d,X), for each $q\in Q$ let $\mathcal{B}(q)$ denote the Borel σ-algebra of $X(q)$ (where $X(q)$ is provided with the trace topology with respect to the Euclidean topology of $\mathbb{R}^{d(q)}$). Define $\mathcal{B}(\overline{\mathbb{S}})=\sigma(\bigcup_{q\in Q}\{q\}\times\mathcal{B}(q))$, where $\overline{\mathbb{S}}=Q\times\mathbb{R}^{\infty}$ and $(\overline{\mathbb{S}},\mathcal{B}(\overline{\mathbb{S}}))$ will be a measurable space.

For simplicity we will drop dependencies on Q, d, X whenever they are clear from the context.

2.2 Piecewise Deterministic Markov Processes

Definition 1. *A Piecewise Deterministic Markov Process (PDMP) H is a collection $H=((Q,d,X),\ f,\ Init,\ \lambda,\ R)$, where*

- *Q is a countable set.*
- *$d:Q\to\mathbb{N}$ is a map giving the dimensions of the continuous state spaces.*
- *$X:Q\to 2^{\mathbb{R}^{d(\cdot)}}$ maps each $q\in Q$ to a subset of Euclidean space of appropriate dimension, $X(q)\subseteq\mathbb{R}^{d(q)}$.*
- *$f:\mathbb{S}(Q,d,X)\to\mathbb{R}^{d(\cdot)}$ is a vector field.*
- *$Init:\mathcal{B}(\overline{\mathbb{S}})\to[0,1]$ is an initial probability measure on $(\overline{\mathbb{S}},\mathcal{B}(\overline{\mathbb{S}}))$, with $Init(\mathbb{S}^c)=0$.*

– $\lambda : \mathbb{S}(Q, d, X) \to \mathbb{R}_+$ *is a transition rate function.*
– $R : \mathcal{B}(\overline{\mathbb{S}}) \times (\mathbb{S}(Q, d, X) \cup \Gamma((Q, d, X), f)) \to [0, 1]$ *is a transition probability measure, with* $R(\mathbb{S}^c, \cdot) = 0$.

To avoid technical pitfalls we introduce the following standing assumption.

Assumption 1 *H satisfies the following well-posedness assumptions:*

1. *For all $q \in Q$, $f(q, \cdot)$ is a globally Lipschitz continuous function.*
2. *$X(q) \subset \mathbb{R}^{d(q)}$ is an open set in the Euclidean topology for all $q \in Q$.*
3. *$\lambda : \mathbb{S} \to \mathbb{R}_+$ is measurable.*
4. *For all $\alpha \in \mathbb{S}$ there exists $\varepsilon > 0$ such that the functions $t \mapsto \lambda(\phi(\alpha, t))$ is integrable for all $t \in [0, \varepsilon)$.*

One can show that under Assumption 1 the space $(\overline{\mathbb{S}}, \mathcal{B}(\overline{\mathbb{S}}))$ is a Borel space (i.e. is homeomorphic to a Borel subset of a complete separable metric space) and $\mathcal{B}(\overline{\mathbb{S}})$ is a sub-σ-algebra of its Borel σ-algebra.

To define an *execution* of the PDMP, consider the *exit time*, $t^* : \mathbb{S} \to \mathbb{R}_+ \cup \infty$ defined by

$$t^*(\alpha) = \inf\{t \geq 0 \ : \ \phi(\alpha, t) \notin \mathbb{S}\}.$$

$t^*(\alpha)$ is the first time when the process hits the boundary. Notice that we may have $t^*(\alpha) = \infty$ for some $\alpha \in \mathbb{S}$. We define the *survivor function*, $F : \mathbb{S} \times \mathbb{R}_+ \to [0, 1]$ by

$$F(\alpha, t) = \begin{cases} \exp(-\int_0^t \lambda(\phi(\alpha, \tau))d\tau) & \text{if } t < t^*(\alpha) \\ 0 & \text{if } t \geq t^*(\alpha) \end{cases}$$

The survivor function is used to define the jump times as random variables. Note that a PDMP allows predictable jumps. The executions of H can now be "generated" using the Algorithm 1. All random extractions in Algorithm 1 are assumed to be independent one of another.

Algorithm 1 (Generation of Executions of H)
  set $T = 0$
  select $\mathbb{S}$-valued "random variable", $\widehat{\alpha}$, according to $Init$
  repeat
    select $\mathbb{R}_+$-valued random variable $\widehat{T}$ such that $P[\widehat{T} > t] = F(\widehat{\alpha}, t)$
    set $\alpha_t = \phi(\widehat{\alpha}, t - T)$ for all $t \in [T, T + \widehat{T})$
    select $\mathbb{S}-$valued "random variable" $\widehat{\alpha}$ according to $R(\cdot, \phi(\widehat{\alpha}, \widehat{T}))$
    set $T = T + \widehat{T}$
  until true

To ensure that α is defined on the entire $\mathbb{R}_+$ one needs to exclude the possibility of infinite number of discrete transitions taking place in a finite amount of time (Zeno executions). Let N_t denote the number of discrete transitions in the interval $[0, t]$. The following assumption is introduced in [5].

Assumption 2 *For all* $t \in \mathbb{R}_+$, $\mathbb{E}[N_t] < \infty$.

Other nasties (escape to infinity in finite time, etc.) are excluded by Assumption 1. Under Assumptions 1 and 2, Algorithm 1 defines a function $\alpha : \mathbb{R}_+ \to \mathbb{S}$, which is right continuous with left limits. A *stochastic execution* of H with the starting point $\alpha = (q, x) \in \mathbb{S}$ is a sample path starting with α of the piecewise deterministic Markov process H. Let $\mathcal{H}$ denote the family of all stochastic executions of H. Let $\mathcal{H}_\alpha$ denote the set of all stochastic executions which have $\alpha \in \mathbb{S}$ as starting point. The random events associated with a sample path are:

1. The random extraction of the initial condition;
2. The random extraction of the time of each transition;
3. The random extraction of the destination of each transition.

Overall, if the sample path takes N discrete transitions (possibly with $N = \infty$) one has to deal with N random variables taking values in $\mathbb{R}_+$ and $N + 1$ "random variables" taking values in $\mathbb{S}$. Davis [5] provides a construction for relating all of these random phenomena to a single random extraction from the Hilbert cube. $(\Omega, \mathcal{A}, P)$ will therefore be assumed to be the underlying probability space, where $\Omega = [0, 1]^\infty$ is the Hilbert cube, $\mathcal{A}$ is the product σ-field of the Lebesgue sets of $[0, 1]$ and P is the product of the Lebesgue measures. Recall that a property holds P-*almost sure* $(P - a.s)$ if there exists a set $\Omega' \subset \Omega$ such that $P(\Omega') = 0$ and the property holds everywhere on $\Omega \backslash \Omega'$.

The following fact is shown in [5,4].

Theorem 1 (Strong Markov Property). *Under Assumptions 1 and 2, Algorithm 1 defines a strong Markov process.*

2.3 An Application to Air Traffic Control

To illustrate the possible practical applications of PDMP we provide a high level discussion of how they can be used to model sector transitions in Air Traffic Control (ATC); other possible applications to finance, insurance, etc. are discussed in [5].

Under current ATC procedures, air space is divided into sectors. Different air traffic controllers are responsible for managing air traffic in the different sectors. Therefore, whenever aircraft move between sectors they have to be handed of from one air traffic controller to another. This involves communication between air traffic controllers (and possibly physical movement of air strips from one ATC workstation to another!) Since the hand-off process distracts the attention of the ATC from other tasks (such as maintaining aircraft separation) sector transitions can become safety critical especially in areas where the aircraft density is high, for example the Terminal Maneuvering Areas (TMA).

Piecewise deterministic Markov processes can be used to capture the ATC sector transition process. Sector transitions are characterised by the following types of dynamics:

Continuous Dynamics. Accurate models of continuous dynamics are not particularly important for sector transitions. In certain cases it may also be possible to separate the vertical and horizontal components of the continuous model. Specifically for transitions at the TMA gates the horizontal and vertical may interact, since aircraft may be climbing when leaving the TMA or descending when entering it.

Discrete Dynamics. Sector transitions are intrinsically discrete phenomena. The state of the system undergoes a transition when an aircraft changes sector and is handed of from one ATC to another. One needs to ensure that the aircraft will find itself in a safe configuration in the new sector before the transition is allowed to go ahead. If not (for example, if the traffic density in the new sector is already too high) the transition should be delayed. In this case, one has to ensure that it is safe for the aircraft to remain in its current sector.

Currently all these tasks are performed by ATC based on established procedures. Some flexibility is allowed depending on traffic conditions; for example an air traffic controller may delay accepting a flight coming into a sector if there are more urgent problems to deal with. Introducing automation safely requires full understanding of these procedures. One would have to ensure, for example, that the system does not give rise to situations where an aircraft can neither be handed on nor remain in the current sector safely.

Stochastic Phenomena. There is some uncertainty in the time at which a hand-off from one sector to the other will take place. The exact timing of the transition may depend on the traffic in both sectors. Hand-off may take place earlier if traffic in the current sector is heavy, or may be delayed if traffic in the receiving sector is heavy. The effect of this uncertainty tends to be small in current conditions, but may become more pronounced as traffic densities increase.

The above discussion suggests that it is possible to model sector transitions using a PDMP. Let $X(q)$ with $q \in Q$ denote the area covered by each ATC sector, where Q is the total number of the ATC sectors. Since precise information about continuous motion of the aircraf is not crucial, we can assume that the continuous motion of the aircraft in each sector is deterministic, governed by a given vector field f. The transition rate function λ will have support in a neighbourhood of the sector boundary. The magnitude of λ will depend on the traffic density in the current sector, ρ_c, and the traffic density in the new sector, ρ_n; more specifically λ increases as ρ_c increases or ρ_n decreases. Since the next sector is known, R can be modelled by a simple function.

A central problem in the air traffic context is determining the probability of conflict, i.e. the probability two aircraft come closer than a minimum allowed distance. If this probability can be computed, an alert can be issued when it exceeds a certain threshold. In the context of PDMP, the computation of the conflict probability reduces to a reachability problem: computing the probability that the PDMP modelling the aircraft motion reaches an unsafe part of the state space (one where two aircraft come closer than the minimum allowed distance). Motivated by this observation, problems of reachability for PDMP will be studied in Section 4.

3 Properties of PDMPs

3.1 PDMPs as a Markov Family

Under our assumptions the PDMP H can be thought of as a process defined on Ω or as a Markov family defined on $D_{\mathbb{S}} = D_{\mathbb{S}}[0,\infty)$, where $D_{\mathbb{S}}$ denote the set of right-continuous functions z on $\mathbb{R}_+$ with values in $\mathbb{S}$ and with left limit $\lim_{s \nearrow t} z(s)$ for all $t \in (0,\infty)$ (see [5]). More concretely, let $\widetilde{\alpha}_t(z) = z(t)$ for $z \in D_{\mathbb{S}}$. Let $\mathcal{F}_t^0$ denote the natural filtration, $\mathcal{F}_t^0 = \sigma\{\widetilde{\alpha}_s, s \leq t\}$ and $\mathcal{F}^0 = \vee_t \mathcal{F}_t^0$. The construction of a PDMP defines for each starting point $\alpha = (q,x) \in \mathbb{S}$ a measurable mapping $\psi_\alpha : \Omega \to D_{\mathbb{S}}$ such that $\widetilde{\alpha}_t(\psi_\alpha(\omega)) = \alpha_t(\omega)$. Let P_α denote the image measure, $P_\alpha = P\psi_\alpha^{-1}$. This defines a family of measures $\{P_\alpha, \alpha \in \mathbb{S}\}$. Thus, indeed, a PDMP can be thought of as a Markov family defined on $D_{\mathbb{S}}$. Moreover, every elementary event $\omega \in \Omega$ can be thought of as a stochastic execution of H. So $\mathcal{H}_\alpha$ can be identified with Ω.

Let $\Im$ be the family of all finite stopping times associated to H and $\mathcal{P}(\mathbb{S})$ the lattice of probabilities on $\mathbb{S}$. For all $\mu \in \mathcal{P}(\mathbb{S})$ we define a measure P^μ on $(D_{\mathbb{S}}, \mathcal{F}^0)$ by $P^\mu(E) = \int P_\alpha(E)\mu(d\alpha)$. We then denote by $\mathcal{F}_\infty^\mu$ (resp. $\mathcal{F}_t^\mu$) the completion of $\mathcal{F}_\infty^0$ (resp. of $\mathcal{F}_t^0$) with respect to P^μ. We also set

$$\mathcal{F}_\infty := \bigcap_{\mu \in \mathcal{P}(\mathbb{S})} \mathcal{F}_\infty^\mu \text{ and } \mathcal{F}_t := \bigcap_{\mu \in \mathcal{P}(\mathbb{S})} \mathcal{F}_t^\mu$$

Obviously, $\{\mathcal{F}_{t+}^0\}$, $\{\mathcal{F}_t^\mu\}$, $\{\mathcal{F}_t\}$ are *admissible families* [1]. We call $\{\mathcal{F}_t\}_{t \geq 0}$ the *minimum completed admissible family.* For a PDMP we have

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$$

(see [5]). Usually, we denote the $p(t,\alpha,E)$ the transition function for H defined for $t \in \mathbb{R}_+, \alpha \in \mathbb{S}$ and $E \in \mathcal{B}(\mathbb{S})$. The measure P_α and the transition function p are related by $P_\alpha[\alpha_t \in E] = p(t,\alpha,E)$, $(t,\alpha,E) \in \mathbb{R}_+ \times \mathbb{S} \times \mathcal{B}(\mathbb{S})$.

3.2 Hitting Times

$\overline{\mathbb{S}}$ is a locally compact separable metric space, and we assume that Δ is adjoined to $\overline{\mathbb{S}}$ as the point at infinity. Let $\mathcal{B}$ (resp. $\mathcal{B}_\Delta$) be the Borel σ-algebra of $\overline{\mathbb{S}}$ (resp. $\overline{\mathbb{S}}_\Delta$). We define $\zeta(\omega) = \inf\{t : \alpha_t(\omega) = \Delta\}$ provided the set in braces is not empty and $\zeta(\omega) = \infty$ if it is empty. ζ is a numerical random variable which is called the *lifetime* of the process H. For $A \subset \mathbb{S}_\Delta$ we define three functions:

$$D_A(\omega) = \inf\{t \geq 0 : \alpha_t(\omega) \in A\}$$
$$T_A(\omega) = \inf\{t > 0 : \alpha_t(\omega) \in A\}$$
$$\widehat{T}_A(\omega) = \inf\{t > 0 : \alpha_{t-}(\omega) \in A\}$$

where in all cases the infimum of the empty set is understood to be $+\infty$. We call D_A (resp. T_A) the *first entry* (resp. *hitting*) *time* of A.

Assumption 3 *The given Markov process H has the state space $(\overline{\mathbb{S}}, \mathcal{B}^*)$, where*

$$\mathcal{B}^* = \mathcal{B}^*(\overline{\mathbb{S}}) = \bigcap_{\mu \in \mathcal{P}(\overline{S})} \mathcal{B}^\mu(\overline{\mathbb{S}})$$

is the σ-algebra of all universally measurable sets of $\overline{\mathbb{S}}$, and is quasi-left continuous on $[0, \zeta)$ (i.e. whenever (T_n) is an increasing sequence from $\Im$ with limit T, then almost surely $\alpha_{T_n} \to \alpha_T$ on $[T < \zeta]$).

Assumption 3 is not very restrictive, see [1] for further discussions. At least, if $t^*(\alpha) = \infty$ for all $\alpha \in \mathbb{S}$, $\lambda \in C_b(\mathbb{S})$ and the function $\alpha \to Rf(\alpha)$ is continuous for $f \in C_b(\mathbb{S})$ (where $C_b(\mathbb{S})$ is the of bounded continuous functions on $\mathbb{S}$) then H has a Feller semi-group (see [5]). The Feller property implies the quasi-left continuity on $[0, \infty)$ (see [7]).

A normal Markov process is called *standard process* if its state space is a locally compact space with a countable base, its minimum completed admissible family is right continuous, its paths are Càdlàg and it is a quasi-left continuous strong Markov process.

Proposition 1. *Under Assumptions 1, 2 and 3 the PDMP H is a standard process.*

Proof. The proof follows from Assumption 3 and the construction of a PDMP, all the standard process requirements have been actually stated in [5].

Lemma 1. *Under Assumptions 1, 2 and 3 the hitting times for a PDMP, H, have the following properties:*

1. *If G is open (in $\mathbb{S}_\Delta$) then $D_G = T_G$ and D_G is an $\{\mathcal{F}^0_{t+}\}$ stopping time.*
2. *If $A \subset \mathbb{S}$ we define $d_A = \min(D_A, \zeta) = D_{A \cup \{\Delta\}}$. D_A is an $\{\mathcal{F}_t\}$ stopping time if and only if d_A is.*

Proof. Part 1 follows from [1] Chapter I, Lemma 10.8. Part 2 follows from [1] Chapter I, Lemma 10.9.

For any $A \subset \mathbb{S}$ and $t \geq 0$ let

$$R_t(A) = \{\omega : \alpha_s(\omega) \in A \cup \{\Delta\}, \text{ for some } s, 0 \leq s \leq t\},$$
$$R^*_t(A) = \{\omega : \alpha_s(\omega) \in A, \text{ for some } s, 0 \leq s \leq t\}.$$

Notice that $R_t(G)$ and $R^*_t(G)$ are in $\mathcal{F}^0_t$ if G is open.

Lemma 2. *For any set A, d_A is an $\{\mathcal{F}_t\}$ stopping time if and only if $R_t(A) \in \mathcal{F}_t$ for all t.*

Proof. This result is an immediate consequence of the definitions of d_A and $R_t(A)$.

Lemma 3. *[1] If $K \subset \mathbb{S}$ is compact then*

1. *$R_t(K) \in \mathcal{F}_t$ for all t.*
2. *If $\{G_n\}$ is a decreasing sequence of open subsets of $\mathbb{S}$ with $G_n \supset \overline{G}_{n+1} \supset K$ for all n and such that $K = \cap G_n = \cap \overline{G}_n$, then $d_{G_n} \uparrow d_K$ a.s. and $P^{\mu}(R_t(G_n)) \downarrow P^{\mu}(R_t(K))$ for all μ and t.*

Proposition 2. *For any Borel set B, $R_t(B) \in \mathcal{F}_t$.*

Proof. Since every Borel set is the union of a countable increasing sequence of compact sets, then the result is follows directly from Lemma 3.

Corollary 1. *D_A and T_A are $\{\mathcal{F}_t\}$ stopping times for all Borel subsets A of $\mathbb{S}_{\Delta}$.*

3.3 Hitting Distributions

For a Borel set $E \in \mathcal{B}(\mathbb{S})$, let us define the $p-$order *hitting distributions* H_p^E by

$$H_p^E(\alpha, f) = \mathbb{E}_{\alpha}(e^{-pT_E} f(\alpha_{T_E})), p > 0, \alpha \in \mathbb{S}$$

for every universally measurable function f. If $p = 0$, we write H^E instead of H_0^E, this kernel being defined by $H^E(\alpha, f) = \mathbb{E}_{\alpha}(f(\alpha_{T_E}) I_{[T_E < \infty]})$. H_p^E is a Markovian kernel on $(\mathbb{S}, \mathcal{B}^*(\mathbb{S}))$ (see [13]).

3.4 Excessive Functions

Definition 2. *A universally measurable function f on $\mathbb{S}$ is said to be p-*excessive *$(p > 0)$ w.r.t. the kernel semi-group of PDMP H if*

$$f(\alpha) \geq 0, e^{-pt} p_t f(\alpha) \nearrow f(\alpha) \ P - a.s. \ t \searrow 0, \alpha \in \mathbb{S}, (p > 0).$$

A 0-excessive function is simply called excessive.

A PDMP satisfying the standard conditions is a Borel right process (see [5]). We denote by $B(\mathbb{S})$ the set of all bounded measurable functions $f : \mathbb{S} \to \mathbb{R}$. This is a Banach space under the norm

$$\|f\| = \sup_{\alpha \in S} |f(\alpha)| .$$

The operator semi-group (P_t) associated to a PDMP, H, maps $B(\mathbb{S})$ into $B(\mathbb{S})$. Let $B_0(\mathbb{S})$ be the subset of $B(\mathbb{S})$ consisting of excessive functions. The semi-group (P_t) is said to be *strongly continuous* on $B_0(\mathbb{S})$ and $B_0(\mathbb{S})$ is a closed linear subspace of $B(\mathbb{S})$ (see [5]).

If f is a $p-excessive$ function then $H_p^E(\cdot, f)$ is called the *p-reduced function* (or the reduced function if $p = 0$) of f on E. The p-reduced function of 1 on E

is denoted by e_E^p (or e_E if $p = 0$) and is called the *p-equilibrium potential* of E (or the *equilibrium potential* if $p = 0$). If f is *p-excessive*, then so its p-harmonic average $H_p^E(\cdot, f)$ relative to a Borel set E (see [13]).

Given E a Borel set and f a p-excessive function ($p > 0$), let $L_{f,E} = \{g \in \mathcal{B}_0, g \geq f$ on $E\}$. The inferior envelope of $L_{f,E}$ is equal to $H_p^E f$ with except of at most some irregular points for E which belong to E. The same conclusion is true if $p = 0$ if $\mathbb{S}$ is the union of an increasing sequence (S_n) of measurable sets with bounded potentials $U(S_n)$ (XV T18 [13]). Since $L_{f,E}$ is a closed convex set the computation of $H_p^E(\cdot, f)$ can be approach using convex analysis techniques (see [9]).

4 Reachability

4.1 Reachability Definitions

To address the reachability questions assume that we have a given PDMP, H, and a set $E \in \mathcal{B}(\mathbb{S})$. Let us to define:

$$Reach_T(E) = \{\omega \in \Omega \mid \exists t \in [0, T] : \alpha_t(\omega) \text{ or } \alpha_{t-}(\omega) \in E\}$$
$$Reach_\infty(E) = \{\omega \in \Omega \mid \exists t \geq 0 : \alpha_t(\omega) \text{ or } \alpha_{t-}(\omega) \in E\}$$

Problem 1. Are $Reach_T(E)$ and $Reach_\infty(E)$ really events? In other words, can we define the measure $P[Reach_T(E)]$ and $P[Reach_\infty(E)]$ of these sets in the underlying probability space?

Notice that if $Reach_T(E)$ is an event then so is $Reach_\infty(E)$, since

$$Reach_\infty(E) = \bigcup_{n=0}^{\infty} Reach_n.$$

Problem 2. If it turns out that we can assign a probability to $Reach_T(E)$ and $Reach_\infty(E)$, can we compute these probabilities?

4.2 Reachability Computations

Under the above assumptions, the reachability events for a given PDMP, H, and a set $E \in \mathcal{B}(\mathbb{S})$ become:

$$Reach_T(E) = \{\omega \in \mathcal{H} : \exists t \in [0, T] \; \alpha_t(\omega) \text{ or } \alpha_{t-}(\omega) \in E \cup \{\Delta\}\}$$
$$Reach_\infty(E) = \{\omega \in \mathcal{H} : \exists t \geq 0 \; \alpha_t(\omega) \text{ or } \alpha_{t-}(\omega) \in E \cup \{\Delta\}\}.$$

where $\mathcal{H}$ denote the family of all stochastic executions of H. If the process is supposed to be quasi-left continuous on $[0, \infty)$, we can remove Δ from the above definitions.

Proposition 3. $Reach_T(E) \in \mathcal{F}_T$, $Reach_\infty(E) \in \mathcal{F}_\infty$.

Proof. We have $R_T(E) \in \mathcal{F}_T, R_\infty(E) \in \mathcal{F}_\infty$ (cf. Propostion 2). Since the process H is standard (cf. Proposition 1) then $P^\mu[Reach_T(E) \backslash R_T(E)] = 0$ for all $\mu \in \mathcal{P}(\mathbb{S})$ (see [1]). Therefore, since $Reach_T(E) = [Reach_T(E) \backslash R_T(E)] \cup R_T(E)$ then $Reach_T(E) \in \mathcal{F}_T$. The second part of the proposition follows since

$$Reach_\infty(E) = \bigcup_{n=0}^{\infty} Reach_n.$$

Let α be a given state in $\mathbb{S}$ and E a Borel set in $\mathbb{S}$. We are interested to compute the probability of the set of stochastic executions from $\mathcal{H}_\alpha$ which reach E at least one moment $t \in [0, T]$ or $t \in [0, \infty)$, i.e.,

$$P_\alpha(Reach_T(E)) = P_\alpha(D_E \leq T, \widehat{T}_E \leq T)$$
$$P_\alpha(Reach_\infty(E)) = P_\alpha(D_E < \infty, \widehat{T}_E < \infty)$$

Because the process is standard, the probabilities we should compute become

$$P_\alpha(R_T(E)) = P_\alpha(D_E \leq T) \text{ and } P_\alpha(R_\infty(E)) = P_\alpha(D_E < \infty). \tag{1}$$

To compute the probabilities from (1) one can start with the generator of the process H. Theoretically, it is possible, giving the generator of the process H, to determine the entire probabilities measure P_α. Technical conditions are required make this calculation feasible even in theory, and no one would think of carrying it out in practice except for artificial examples.

Alternatively, one can define the *hitting probabilities* relative to a Borel set E as below:

$$\varphi_E^p(\alpha) = H_p^E(\alpha, 1) = \mathbb{E}_\alpha(e^{-pD_E}) \text{ and } \varphi_E(\alpha) = H_{0+}^E(\alpha, 1) = P_\alpha(D_E < \infty). \tag{2}$$

φ_E^p (resp.φ_E) is a *p-excessive* (resp. *excessive*) function (see [7]).

A possible approach to compute the hitting probabilities of equation (2) is to treat them as excessive functions and invoke approximation results for excessive functions. A very important observation here is that these hitting probabilities relative to a Borel set E are exactly the equilibrium potentials associated to E, so they are equal to 1 for every regular point for E. Let us recall, that a point α is *regular* for E provided it is regular for T_E (i.e. $P_\alpha(T_E = 0) = 1$) and it is *irregular* for E provided it is irregular for T_E (i.e. $P_\alpha(T_E = 0) = 0$). We denote by E^r the set of all points which are regular for the Borel set E ($E^o \subset E^r \subset \overline{E}$). In this context we have to study some exceptional sets of the state space (without regular points) and the case when the process is starting from a point belonging to $\mathbb{S} \backslash E$.

4.3 Approximation of the Hitting Times

In this section, reachability questions are characterised as hitting time problems. Let us consider a fixed start point $\alpha_0 \in \mathbb{S}$ and the sequence $T_1 < T_2 < T_3 < \ldots$

of jump times associated to α_0. Let E a Borel subset of $\mathbb{S}$. Let A be the generator of the process H and let $D(A)$ denote the domain of the generator. It is known (see [5]) that the process $(C_t^u)_{t\in\mathbb{R}_+}$ defined by $C_t^u = u(\alpha_t) - u(\alpha_0) - \int_0^t Au(\alpha_s)ds$ is a martingale for each u in the domain $D(A)$. This fact implies that for each $t > 0$ we have $\mathbb{E}[u(\alpha_{t\wedge T_E}) - u(\alpha_0) - \int_0^{t\wedge T_E} Au(\alpha_s)ds] = 0$. Since we supposed that $T_E < \infty$ a.s., letting $t \to \infty$ gives

$$\mathbb{E}[u(\alpha_{T_E})] - \mathbb{E}[u(\alpha_0)] - \mathbb{E}[\int_0^{T_E} Au(\alpha_s)ds] = 0. \tag{3}$$

As in [10], we define the *occupation measure* μ_0 and *hitting distribution* μ_1 by

$$\mu_0(B) = \mathbb{E}[\int_0^{T_E} I_B(\alpha_s)ds] \text{ and } \mu_1(B) = P(\alpha_{T_E} \in B)$$

for all $B \in \mathcal{B}(\mathbb{S})$.

It is clear that if $B \subset E$ then $\mu_0(B) = 0$ and if $B \subset E^c$ then $\mu_1(B) = 0$. Therefore, μ_0 is concentrated in E^c and μ_1 is concentrated in E. With an integrability argument one can obtain from (3) the following equation, so-called *adjoint equation*:

$$\int_{E^c} Au(\alpha)\mu_0(d\alpha) + u(\alpha_0) - \int_E u(\alpha)\mu_1(d\alpha) = 0. \tag{4}$$

It is clear that

$$\mu_0(E^c) = \mathbb{E}(T_E) \textit{ and } \mu_1(E^c) = 0. \tag{5}$$

If we take $u = I_{E^c}$ and we suppose that $\alpha_0 \notin E$ then the adjoint equation becomes $\int_{E^c} AI_{E^c}(\alpha)\mu_0(d\alpha) + 1 = 0$. It is known that for every $u \in D(A)$, Au is given by

$$Au(\alpha) = \mathcal{L}_f u(\alpha) + \lambda(\alpha)\int_{\mathbb{S}}(u(\beta) - u(\alpha))R(d\beta, \alpha).$$

Thus, we get $\int_{E^c}\{\lambda(\alpha)\int_{\mathbb{S}}(I_{E^c}(\beta) - I_{E^c}(\alpha))R(d\beta,\alpha)\}\mu_0(d\alpha) + 1 = 0$ and, if we denote $\Lambda_E(\alpha) = \lambda(\alpha)\int_E R(d\beta,\alpha) = \lambda(\alpha)R(E,\alpha)$ then

$$\int_{E^c} \Lambda_E(\alpha)\mu_0(d\alpha) = 1.$$

Thus, once the measure μ_0 has been determined (using for e.g. linear programming methods), the hitting time mean can be obtained from (5).

The following lemma is extremely useful to estimate the expectations of the moments of hitting times.

Lemma 4. *[6] Let E be a Borel subset of $\mathbb{S}$, $T = T_E$ and let $m(t) = \sup\{P_\alpha[T > t], \alpha \in \mathbb{S}\}$. Then, for all $t \geq 0, \alpha \in \mathbb{S}$ we have $\mathbb{E}_\alpha T \leq \frac{t}{1-m(t)}$. Moreover, if $m < 1$, for $0 < p < -\frac{1}{t}\ln m$ then $\mathbb{E}_\alpha e^{pT}$ is an analytic function of p in a neighbourhood of the origin, i.e., we have*

$$\mathbb{E}_\alpha e^{pT} = \sum_{k=1} \infty \frac{p^k}{k!}\mathbb{E}_\alpha T^k.$$

If we suppose that T_E is finite then there exists T_k such that $T_E \in [T_k, T_{k+1})$. Using lemma 4 and the survivor function from Algorithm 1 we obtain the following estimation for all $k \geq 1$

$$\mathbb{E}_{\alpha_0} T_E \leq \frac{T_{k+1}}{1 - P_{\alpha_{T_k}}[T_E > T_{k+1}]} \leq \frac{T_{k+1}}{\exp(-\int_{T_k}^{T_E} \lambda(\phi(\alpha_{T_k}, \tau))d\tau)}$$

Then we compute the quantities $\tau_k = \frac{T_k}{\mathbb{E}_{\alpha_0} T_E}$, $k = 0, 1, 2, 3, \ldots$ (with the convention $T_0 = 0$) and let τ_{k_0} be the biggest one. Then $T_E \in [T_{k_0}, T_{k_0+1})$, i.e. the number of steps after the set E is reached is k_0.

5 Conclusions and Further Work

In this paper we set up PDMPs, as models for stochastic hybrid systems, and the corresponding reachability problem. An important result is that, under some technical assumptions, a PDMP is a standard Markov process. This fact has as a consequence that the "reach" events are real events in the underlying probability space, for which we can compute the probability measures. This result can be considered as the main result of this paper. It allows us to treat the reachability problem with some well-known tools like hitting times, hitting distributions or hitting time moments. We obtained also a method which allows us to estimate the number of steps after a certain set is reached. In an ongoing work we will develop some randomised algorithms for evaluating the system performance in terms of quantities such as the average time to reach a certain set and we will apply these algorithms to aircraft conflict detection.

Current work focuses on two methods for computing the probabilities of the reach events. In the remaining of this section we summarise the main points of these two approaches.

5.1 Via Capacity Theory

An other approach to study some important properties of "reach" probabilities is via capacity theory. These probabilities can be thought of as Choquet capacities on the state space $\mathbb{S}$ (see [1] for more details about Choquet capacities).

Let $\mathbb{S}$ be a locally compact separable metric space and let $\mathcal{K}$ be the class of all compact subsets of $\mathbb{S}$. A function $\varphi : \mathcal{K} \to \mathbb{R}$ is called a *Choquet capacity* if

1. if $A, B \in \mathcal{K}$ and $A \subset B$, then $\varphi(A) \leq \varphi(B)$;
2. given $A \in \mathcal{K}$ and $\varepsilon > 0$ there exists an open set $G \supset A$ such that for every $B \in \mathcal{K}$ with $A \subset B \subset G$ one has $\varphi(B) - \varphi(A) < \varepsilon$;
3. $\varphi(A \cup B) + \varphi(A \cap B) \leq \varphi(A) + \varphi(B)$ for all $A, B \in \mathcal{K}$ (strongly subaddtive).

Given a Choquet capacity φ one defines the *inner capacity*, $\varphi_*(A)$, of an arbitrary set $A \subset \mathbb{S}$ by $\varphi_*(A) = \sup_{K \subset A} \varphi(K)$ where the supremum is taken over all compact subsets of A. One next defines the outer capacity $\varphi^*(A)$, of an arbitrary set by $\varphi^*(A) = \inf_{G \supset A} \varphi_*(G)$ where the infimum is taken over all open sets containing A.

Let μ (a probability measure on $\mathbb{S}$) and t be fixed; then $\varphi(K) = P^\mu[R_t(K)]$ is a Choquet capacity on the compact subsets of $\mathbb{S}$ (see [1]). In this case it is enough to compute these probabilities only for the compact sets.

A good presentation of the operations on Choquet capacities can be found in [2], Chapter V. In this context very important are the approximation results for the hitting times. For the computation of Choquet capacities we can use similar methods as in the fractal theory (see [14], [15]), where these capacities are very useful. The capacity theory offers us sufficient tools to study the reachability problem for PDMP. It is clear that there is an overlapping between the capacity theory and the theory of hitting times, but the capacity theory is very well studied, richer then the second one and it could be a suitable candidate for our problem.

5.2 Via Optimal Control

An alternative, indirect approach to reachability questions is using optimal control. In this direction we consider the possibility to draw parallels between the computations based on the extended generator of a PDMP and the Hamilton-Jacobi equations. A start point is to consider, for any Borel set, its indicator function $I_E : \mathbb{S} \to \{0, 1\}$ and to set

$$\begin{aligned} P[Reach_T(E)] &= \mathbb{E}[\max\nolimits_{t \in [0,T]} I_E(\alpha_t)] \\ P[Reach_\infty(E)] &= \mathbb{E}[\max\nolimits_{t \geq 0} I_E(\alpha_t)]. \end{aligned}$$

Because $Reach_T(E)$ and $Reach_\infty(E)$ are measurable, it follows that the expectations $E[\max_{t \in [0,T]} I_E(\alpha_t)]$ and $E[\max_{t \geq 0} I_E(\alpha_t)]$ are well-defined. Since I_E is binary the maximum in the last equations exists. Using similar techniques as those developed in [11], we believe that it is possible to characterise the previous probabilities as viscosity solutions to a partial differential equation ([3]).

A further step of our work will be to improve our stochastic hybrid system models, namely the PDMPs, requiring some randomness between jumps. This aim could be accomplished asking the continuous motion between jumps to be led by some "nice" stochastic differential equations.

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