Symmetry Reduction of a Class of Hybrid Systems*

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Abstract. The optimal control problem for a class of hybrid systems (switched Lagrangian systems) is studied. Some necessary conditions of the optimal solutions of such a system are derived based on the assumption that there is a group of symmetries acting uniformly on the domains of different discrete modes, such that the Lagrangian functions, the guards, and the reset maps are all invariant under the action. Lagrangian reduction approach is adopted to establish the conservation law of certain quantities for the optimal solutions. Some examples are presented. In particular, the problems of optimal collision avoidance (OCA) and optimal formation switching (OFS) of multiple agents moving on a Riemannian manifold are studied in some details.

1 Motivation

In this paper we study the optimal control problem of a class of hybrid systems which we called *switched Lagrangian systems*. Roughly speaking, a switched Lagrangian system is a hybrid system with a set of discrete modes, and associated with each discrete mode, a domain which is a manifold (possibly with boundary) together with a Lagrangian function defined on it. The continuous state of the system evolves within one of the domains, and upon hitting certain subsets (*guards*), can trigger a jump in the discrete mode, in which case the continuous state is reset inside the domain of the new discrete mode according to some prescribed rules (*reset maps*). Thus a typical execution of the system can be partitioned into a number of curves in distinctive domains. The cost of the execution is then the sum of the costs of these curves, with the cost of each curve being the integral along it of the corresponding Lagrangian function. Given two points which might lie in different domains, we try to find the executions that steer the system from one point to the other with minimal cost.

If there is only one discrete mode, then the problem is a classical variational problem whose solutions are characterized by the Euler-Lagrangian equations [1] in any coordinate system of the domain. In particular, if the Lagrangian function

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is quadratic and positive definite on each fiber, it can be used to define a riemannian metric on the domain, and the optimal solutions are geodesics under this metric [5]. For a general system with multiple discrete modes, these conclusions still hold for the segments of an optimal solution restricted on each individual domain. Overall speaking, however, it is usually a very tricky issue to determine how these individual segments can be pieced together to form an optimal solution, especially when the guards and the reset maps are complicated. In [11], it is shown that if all the Lagrangian functions are quadratic and positive definite, then, under some additional mild assumptions, successive segments must satisfy at the switching points a necessary condition that is analogous to the *Snell Law* in optics, provided that segments of optimal solutions in different domains are thought of as light rays traveling in heterogeneous media.

In this paper, we try to derive necessary conditions on an optimal solution that apply *both* on its segments inside each domain *and* at its switching points where discrete jumps occur. We do so under the additional assumption that the switched Lagrangian system admits a group of symmetries, i.e., there exists a Lie group G acting uniformly on all of the domains, with respect to which the Lagrangian functions, the guards, and the reset maps are invariant. By using perturbations generated by the group action, we can establish through variational analysis the conservation law of certain quantities (*momentum maps*) taking values in the dual of the Lie algebra of G throughout the duration of the optimal solution. It should be pointed out that this approach has extensive applications in geometry and mechanics when the underlying state space is smooth, and is often presented from the more elegant symplectic point of view in the literature [1, 12]. What is new in this paper is its reformulation and application in the context of switched Lagrangian systems, which are nonsmooth in nature. This nonsmoothness also justifies the Lagrangian point of view adopted here.

The results in this paper will be illustrated through examples. Two important examples are the *Optimal Collision Avoidance* (OCA) and the *Optimal Formation Switching* (OFS) of multiple agents moving on a Riemannian manifold [10]. In either case, one has to steer a group of agents from a starting configuration to a destination configuration on the manifold with minimal cost, such that their joint trajectory satisfies certain separation constraints throughout the process. We will show how these two problems can be formulated as the optimal control problems for suitably chosen switched Lagrangian systems, and how the conserved quantities can be derived for various choices of the Riemannian manifold.

A price we pay for the general applicability of the necessary conditions is that they in general only partially characterize the optimal solutions, since the number of symmetries presented in a system is usually much smaller than the dimensions of the domains, and that the conserved quantities are usually not integrable. Nonetheless, in certain simple cases, the derived necessary conditions can indeed help to characterize the optimal solutions completely [10].

In this paper, we only consider holonomic motions of the agents. For nonholonomic motion planning, see [2, 6]. Some relevant results can also be found in [4, 13], to name a few.

This paper is organized as following. In Section 2, we define the notions of switched Lagrangian systems, G-symmetry, and their optimal control problem. In Section 3, Lagrangian variational approach is adopted to derive a necessary condition for the optimal solutions. Two important examples of switched Lagrangian systems, the OCA and the OFS problems on Riemannian manifolds, are introduced in Section 4. In particular, we study the cases when the underlying manifold is $\mathbf{SO}_n$ and the Grassmann manifold. Finally, some concluding remarks are presented in Section 5.

2 Switched Lagrangian Systems

First we define the notions of switched Lagrangian systems and their executions. For the definition of general hybrid systems, see [8].

Definition 1 (Switched Lagrangian Systems). *A switched Lagrangian system $\mathcal{H}$ is specified by the following:*

1. *A set Γ of discrete modes;*
2. *For each $l \in \Gamma$, a domain M_l which is a manifold, possibly with boundary ∂M_l, and a Lagrangian function $L_l : TM_l \to \mathbb{R}$ defined on the tangent bundle of M_l. We assume that the domains M_l, $l \in \Gamma$, are disjoint, and write $M = \cup_{l \in \Gamma} M_l$;*
3. *A set of discrete transitions $E_d \subset \Gamma \times \Gamma$;*
4. *For each $(l_1, l_2) \in E_d$, a subset $D_{(l_1,l_2)} \subset M_{l_1}$, called the* guard *associated with the discrete transition (l_1, l_2), and a continuous transition relation $E_c(l_1, l_2) \subset D_{(l_1,l_2)} \times M_{l_2}$ such that for each $q_1 \in D_{(l_1,l_2)}$, there exists at least one $q_2 \in M_{l_2}$ with $(q_1, q_2) \in E_c(l_1, l_2)$. In other words, $E_c(l_1, l_2)$ specifies a one-to-many map from $D_{(l_1,l_2)}$ to M_{l_2}.*

Definition 2 (Hybrid Executions). *A* hybrid execution *(or simply a* path*) of $\mathcal{H}$ defined on some time interval $[t_0, t_1]$ can be described as the following: there is a finite partition of $[t_0, t_1]$, $t_0 = \tau_0 \leq \ldots \leq \tau_{m+1} = t_1$, $m \geq 0$, and a succession of discrete modes $l_0, \ldots, l_m \in \Gamma$ and arcs $\gamma_0, \ldots, \gamma_m$, such that*

- *$(l_j, l_{j+1}) \in E_d$ for $j = 0, \ldots, m-1$;*
- *$\gamma_j : [\tau_j, \tau_{j+1}] \to M_{l_j}$ is a continuous and piecewise C^∞ curve[1] in M_{l_j} for $j = 0, \ldots, m$;*
- *For each $j = 0, \ldots, m-1$, $\gamma_j(\tau_{j+1}) \in D_{(l_j,l_{j+1})}$, and $(\gamma_j(\tau_{j+1}), \gamma_{j+1}(\tau_{j+1})) \in E_c(l_j, l_{j+1})$.*

We will denote such a path by γ, and call $\gamma_0, \ldots, \gamma_m$ the *segments* of γ. The *cost* of γ is defined by

$$J(\gamma) = \sum_{j=0}^{m} \int_{\tau_j}^{\tau_{j+1}} L_{l_j}(\dot{\gamma}_j)\, dt.$$

[1] All curves in this paper are assumed to be continuous and piecewise C^∞.

Intuitively speaking, a path is the trajectory of a point moving in M such that whenever the point is in M_{l_1} and it reaches a point in a guard, say, $q_1 \in D_{(l_1,l_2)}$, it has the option of jumping to a point q_2 in M_{l_2} according to the continuous transition relation $(q_1, q_2) \in E_c(l_1, l_2)$, and continuing its motion in M_{l_2}, and so on. We call such a jump a *transition*, which consists of a discrete transition $l_1 \to l_2$ and a continuous transition $q_1 \to q_2$. Note that during a transition, there is possibly more than one position the point can jump to due to two reasons: guards for different discrete transitions may intersect; and for a single discrete transition, the continuous transition relation is a one-to-many map.

It is allowed in the definition of the path γ that some of $\tau_0, \ldots, \tau_{m+1}$ are identical, implying that more than one transition may occur at the same epoch, though in a sequential way. This can cause trouble since it is possible that all the legitimate transitions from a certain point lead to infinite number of jumps occurring at the same epoch, thus blocking the system from further evolving. This kind of hybrid systems is usually called *blocking*. To ensure that the switched Lagrangian systems studied in this paper are nonblocking, we make the following (usually stronger) assumption.

Assumption 1 (Connectness) *Assume that the switched Lagrangian system $\mathcal{H}$ is connected in the sense that for any two points $a, b \in M$, there exists at least one path γ connecting them.*

The problem of interest to us can be formulated as

Problem 1 (Optimal Control on $\mathcal{H}$) *Given $a, b \in M$ and a time interval $[t_0, t_1]$, find the path (or paths) from a to b defined on $[t_0, t_1]$ with minimal cost.*

Assumption 1 alone can not guarantee the existence of solutions to Problem 1, which requires some completeness conditions on the space consisting of all the paths from a to b defined on $[t_0, t_1]$. In addition, in the context of hybrid systems, one needs to ensure that the solutions are not *zeno* (i.e., exhibiting infinite number of transitions within a finite time interval). These difficulties are sidestepped in this paper by the following assumption.

Assumption 2 (Existence of Solutions) *Assume that $\mathcal{H}$ is chosen such that solutions to Problem 1 exist.*

We focus on a special class of switched Lagrangian systems. Let G be a Lie group.

Definition 3 (G-symmetry). *A switched Lagrangian system $\mathcal{H}$ is G-symmetric if it has the following properties:*

- *G acts on the domain M_l from the left for each $l \in \Gamma$. We shall denote this smooth action uniformly by $\Phi : G \times M_l \to M_l$ for all l;*
- *For each discrete transition $(l_1, l_2) \in E_d$, the guard $D_{(l_1,l_2)}$ and the continuous transition relation $E_c(l_1, l_2)$ are both invariant under the action of G, i.e., for any $g \in G$, $\Phi(g, D_{(l_1,l_2)}) \subset D_{(l_1,l_2)}$, and $(q_1, q_2) \in E_c(l_1, l_2)$ if and only if $(\Phi(g, q_1), \Phi(g, q_2)) \in E_c(l_1, l_2)$;*

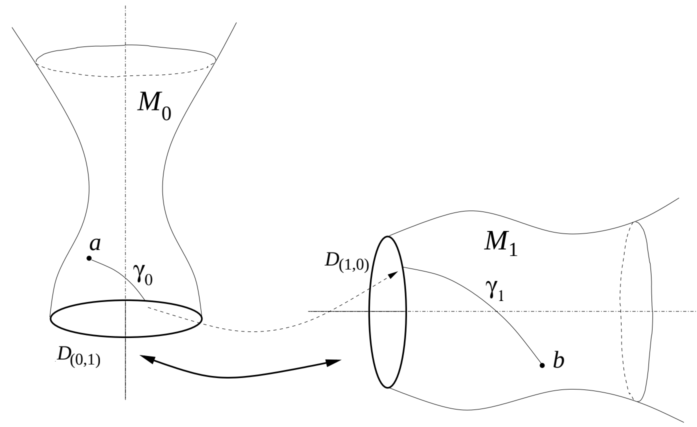

Fig. 1. An example of an $\mathbf{S}^1$-symmetric switched Lagrangian system.

– *For each $l \in \Gamma$, the Lagrangian function L_l is invariant under G, i.e., $L_l \circ d\Phi_g = L_l$, $\forall g \in G$. Here $\Phi_g : M_l \to M_l$ is the map defined by $\Phi_g : q \mapsto \phi(g, q)$, $\forall q \in M_l$, and $d\Phi_g : TM_l \to TM_l$ is its tangential map.*

Assumption 3 (G-symmetry of $\mathcal{H}$) *There is a Lie group G such that the switched Lagrangian system $\mathcal{H}$ is G-symmetric.*

We illustrate the above concepts by two simple examples.

Example 1. Shown in Figure 1 is an example of a switched Lagrangian system that is $\mathbf{S}^1$-symmetric. Here $\Gamma = \{0, 1\}$, and the domains of the two discrete modes are two disjoint surfaces of revolution M_0 and M_1 in $\mathbb{R}^3$, each with a boundary obtained as the cross section of the surface with a plane perpendicular to its rotational axis. Let $E_d = \{(0,1), (1,0)\}$, $D_{(0,1)} = \partial M_0$, $D_{(1,0)} = \partial M_1$. Define the continuous transition relation $E_c(0,1)$ as the graph of a map φ from ∂M_0 to ∂M_1 that is invariant under the rotations, i.e., φ rotates the circle ∂M_0 by a certain angle and "fits" it into ∂M_1. Define $E_c(1,0)$ to be the graph of φ^{-1}. Let M_0 be equipped with the riemannian metric inherited from $\mathbb{R}^3$, and define a Lagrangian function L_0 on it by $L_0(v) = \frac{1}{2}\|v\|_q^2$, $\forall v \in T_q M_0$, where $\|\cdot\|_q$ is the norm on the tangent space $T_q M_0$ determined by the riemannian metric. In the following we shall simply write $L_0 = \frac{1}{2}\|\cdot\|^2$. Similarly $L_1 = \frac{1}{2}\|\cdot\|^2$, where the norm now corresponds to the riemannian metric on M_1 inherited from $\mathbb{R}^3$. The action of $\mathbf{S}^1 = \{z \in \mathbb{Z} : \|z\| = 1\}$ on M_0 can be defined as: each $e^{j\theta} \in \mathbf{S}^1$ corresponds to a rotation of M_0 along its axis by an angle θ. Similarly for the action of $\mathbf{S}^1$ on M_1. By properly choosing the directions of rotation in the above definitions, one can easily check that the resulting system $\mathcal{H}$ is $\mathbf{S}^1$-symmetric.

Example 2. Consider the following system $\mathcal{H}$. Let $\Gamma = \{0, 1\}$. For each $l \in \Gamma$, $M_l = \{l\} \times (\mathbb{R}^2 \setminus \{0\}) \subset \mathbb{R}^3$ is a plane with the origin removed, and the

Lagrangian function L_l is defined in the polar coordinate of $M_l \simeq \mathbb{R}^2 \setminus \{0\}$ as: $L_l(v) = \frac{1}{2} v^t A_l(r) v$, for $v = (\dot{r}, \dot{\theta})^t \in T_q M_l$, $q \in M_l$. Here $A_l(r)$ is a 2-by-2 positive definite matrix whose entries are smooth functions of r, and $A_l(r) \to \infty$ as $r \to 0$. Let $E_d = \{(0,1),(1,0)\}$, $D_{(0,1)} = M_0$, $D_{(1,0)} = M_1$. Choose $E_c(0,1) = \{((0,x),(1,x)) : x \in \mathbb{R}^2, x \neq 0\}$, $E_c(1,0) = \{((1,x),(0,x)) : x \in \mathbb{R}^2, x \neq 0\}$. Therefore, a point moving in $\mathcal{H}$ can freely switch between two copies of $\mathbb{R}^2 \setminus \{0\}$ with different Lagrangian functions. $\mathbf{S}^1$ acts on M_l, $l \in \Gamma$, in the following way. Each $e^{j\theta} \in \mathbf{S}^1$ corresponds to a rotation counterclockwise of M_l by an angle of θ. $\mathcal{H}$ thus defined can be verified to be $\mathbf{S}^1$-symmetric.

3 Conservation Laws

Suppose we are given a switched Lagrangian system $\mathcal{H}$ that satisfies the assumptions in the previous section. We shall now derive necessary conditions that optimal solutions to Problem 1 must satisfy.

Since $\mathcal{H}$ is G-symmetric, an important observation is

Proposition 1. *If γ is a path of $\mathcal{H}$ defined on $[t_0, t_1]$, then for each C^∞ curve $g : [t_0, t_1] \to G$, $g\gamma$ is also a path of $\mathcal{H}$ defined on $[t_0, t_1]$.*

To be precise, $g\gamma$ is defined in the following way. Let $\tau_0, \ldots, \tau_{m+1}$ and $l_0, \ldots, l_m$ be as in the definition of γ, and let $\gamma_0, \ldots, \gamma_m$ be the corresponding segments of γ. Then $g\gamma$ is a sequence of arcs, $g\gamma_j \triangleq \Phi_g(\gamma_j) = \Phi(g(\cdot), \gamma_j(\cdot))$, $j = 0, \ldots, m$, whose intervals of definition and corresponding discrete modes coincide with those of γ_j, $j = 0, \ldots, m$, respectively. The proof of Proposition 1 is straightforward, hence omitted.

Assume that for given $a, b \in M$, γ is an optimal solution to Problem 1 defined on $[t_0, t_1]$ connecting a and b. Denote with $c_e : [t_0, t_1] \to G$ the constant map mapping every $t \in [t_0, t_1]$ to the identity $e \in G$. Let g be a *proper variation* of c_e, i.e., $g : (-\epsilon, \epsilon) \times [t_0, t_1] \to G$ is a C^∞ map satisfying $g(\cdot, t_0) = g(\cdot, t_1) = g(0, \cdot) \equiv e$ for some small positive number ϵ. Then for each $s \in (-\epsilon, \epsilon)$, $g_s(\cdot) \triangleq g(s, \cdot)$ is a C^∞ curve in G both starting and ending at e, hence by Proposition 1 can be used to define a path $\gamma_s = g_s \gamma$ in $\mathcal{H}$ that starts from a and ends in b. Note that $\gamma_0 = \gamma$ since $g_0 = c_e$. Define $J(s) = J(\gamma_s)$, $s \in (-\epsilon, \epsilon)$. Then a necessary condition for γ to be optimal is that J achieves its minimum at $s = 0$, which in turn implies $\frac{dJ}{ds}(0) = 0$.

For each $(s, t) \in (-\epsilon, \epsilon) \times [t_0, t_1]$, introduce the notations

$$\dot{g}_s(t) = \dot{g}(s,t) = \frac{\partial g}{\partial t}(s,t), \quad g'_s(t) = g'(s,t) = \frac{\partial g}{\partial s}(s,t),$$

where we follow the convention in [5] of using dot and prime to indicate differentiations with respect to time t and variation parameter s respectively. Both $\dot{g}(s,t)$ and $g'(s,t)$ are tangent vectors of G at $g(s,t)$. We pull them back via left multiplication to the tangent space of G at the identity e. Thus we define

$$\xi_s(t) = \xi(s,t) = g(s,t)^{-1} \dot{g}(s,t),$$
$$\eta_s(t) = \eta(s,t) = g(s,t)^{-1} g'(s,t).$$

Here to simplify notation we use $g(s,t)^{-1}\dot{g}(s,t)$ to denote $dm_{g(s,t)^{-1}}[\dot{g}(s,t)]$ (for any $g \in G$, $m_g : G \to G$ stands for the left multiplication by g, while dm_g is its tangent map). Similarly for $g(s,t)^{-1}g'(s,t)$. This kind of notational simplifications will be carried out in the following without further explanation. Both $\xi(s,t)$ and $\eta(s,t)$ belong to $\mathfrak{g} = T_eG$, the Lie algebra of G. The fact that g is a proper variation implies that $g'(\cdot,t_0) = g'(\cdot,t_1) = 0$, hence $\eta(\cdot,t_0) = \eta(\cdot,t_1) = 0$. Moreover, $g(0,\cdot) = e$ implies that $\dot{g}(0,\cdot) = 0$, hence $\xi(0,\cdot) = 0$.

Lemma 1. *Let $\xi'(s,t) = \frac{\partial \xi}{\partial s}(s,t)$ and $\dot{\eta}(s,t) = \frac{\partial \eta}{\partial t}(s,t)$. Then*

$$\xi' = \dot{\eta} + [\xi,\eta] \tag{1}$$

for all $(s,t) \in (-\epsilon,\epsilon) \times [t_0,t_1]$, where $[\xi,\eta]$ is the Lie bracket of ξ and η.

Proof. A general proof can be found in, for example, [3]. In the case when G is a matrix Lie group, the proof is particularly simple ([12]): differentiating $\xi = g^{-1}\dot{g}$ with respect to s and $\eta = g^{-1}g'$ with respect to t, we get

$$\begin{aligned}
\xi'(s,t) &= -g^{-1}g'g^{-1}\dot{g} + g^{-1}\frac{\partial^2 g}{\partial s \partial t} = -\eta\xi + g^{-1}\frac{\partial^2 g}{\partial s \partial t}, \\
\dot{\eta}(s,t) &= -g^{-1}\dot{g}g^{-1}g' + g^{-1}\frac{\partial^2 g}{\partial s \partial t} = -\xi\eta + g^{-1}\frac{\partial^2 g}{\partial s \partial t}.
\end{aligned}$$

Their difference gives $\xi' - \dot{\eta} = \xi\eta - \eta\xi = [\xi,\eta]$.

Define

$$\omega(t) = \xi_0'(t) = \xi'(0,t), \quad \forall t \in [t_0,t_1]. \tag{2}$$

By letting $s = 0$ in (1), we have $\omega = \dot{\eta}_0 + [\xi_0,\eta_0] = \dot{\eta}_0$ since $\xi_0 = 0$. So $\int_{t_0}^{t_1} \omega(t)\,dt = \eta_0(t_1) - \eta_0(t_0) = 0$. Conversely, for each C^∞ map $\omega : [t_0,t_1] \to \mathfrak{g}$ with $\int_{t_0}^{t_1} \omega(t)\,dt = 0$, we can define $\alpha(t) = \int_{t_0}^{t} \omega(t)\,dt$, which satisfies $\alpha(t_0) = \alpha(t_1) = 0$. By choosing $g(s,t) = \exp[s\alpha(t)]$, where exp is the exponential map of G, one can verify that g is indeed a proper variation of c_e such that $\omega = \xi_0'$, where $\xi = g^{-1}\dot{g}$. Therefore,

Lemma 2. *The necessary and sufficient condition for a C^∞ map $\omega : [t_0,t_1] \to \mathfrak{g}$ to be realized as $\omega = \xi_0'$ where $\xi = g^{-1}\dot{g}$ for some C^∞ proper variation g of c_e is*

$$\int_{t_0}^{t_1} \omega(t)\,dt = 0. \tag{3}$$

Suppose one such g is chosen. For each $(s,t) \in (-\epsilon,\epsilon) \times [t_0,t_1]$, and each segment γ_j of γ, $j = 0,\dots,m$, we have[2]

$$L_{l_j}[\frac{d}{dt}(g_s\gamma_j)] = L_{l_j}[\dot{g}_s\gamma_j + g_s\dot{\gamma}_j] = L_{l_j}[g_s(\xi_s\gamma_j + \dot{\gamma}_j)] = L_{l_j}[\xi_s\gamma_j + \dot{\gamma}_j], \tag{4}$$

[2] Since γ_j is only piecewise C^∞, this and all equations that follow should be understood to hold only at those t where $\dot{\gamma}_j$'s are well defined.

where the last equality follows by the G-invariance of L_{l_j}. Here $\dot{g}_s\gamma_j$ denotes $d\Phi^{\gamma_j}(\dot{g}_s)$, where $d\Phi^{\gamma_j}$ is the differential of the map $\Phi^{\gamma_j}: G \to M_{l_j}$ that maps each $g \in G$ to $\Phi(g, \gamma_j)$, and $g_s\dot{\gamma}_j$ denotes $d\Phi_{g_s}(\dot{\gamma}_j)$. Both $\dot{g}_s\gamma_j$ and $g_s\dot{\gamma}_j$ are tangent vectors in $T_{g_s\gamma_j}M_{l_j}$. The cost of $\gamma_s = g_s\gamma$ is then

$$J(s) = \sum_{j=0}^{m} \int_{\tau_j}^{\tau_{j+1}} L_{l_j}[\xi_s\gamma_j + \dot{\gamma}_j]\,dt. \tag{5}$$

For a vector space V, denote with $(\cdot,\cdot): V^* \times V \to \mathbb{R}$ the natural pairing between V and its dual V^*, i.e., $(f, v) = f(v), \forall f \in V^*, v \in V$. Differentiating (5) with respect to s at $s = 0$, and using the fact that $\xi_0 = 0$ and $\xi_0' = \omega$, we have

$$J'(0) = \sum_{j=0}^{m} \int_{\tau_j}^{\tau_{j+1}} ((dL_{l_j})_{\dot{\gamma}_j}, d\Phi^{\gamma_j}(\omega))\,dt = \sum_{j=0}^{m} \int_{\tau_j}^{\tau_{j+1}} ((d\Phi^{\gamma_j})^*(dL_{l_j})_{\dot{\gamma}_j}, \omega)\,dt. \tag{6}$$

Here $(dL_{l_j})_{\dot{\gamma}_j}$ is in fact the differential at $\dot{\gamma}_j$ of the restriction of L_{l_j} on the fiber $T_{\gamma_j}M_{l_j}$. We identify the tangent space at $\dot{\gamma}_j$ of $T_{\gamma_j}M_{l_j}$ with $T_{\gamma_j}M_{l_j}$ itself, so $d\Phi^{\gamma_j}(\omega) \in T_{\gamma_j}M_{l_j}$ and $(dL_{l_j})_{\dot{\gamma}_j} \in T^*_{\gamma_j}M_{l_j}$. In addition, $(d\Phi^{\gamma_j})^*: T^*_{\gamma_j}M_{l_j} \to \mathfrak{g}^*$ is the dual of $d\Phi^{\gamma_j}: \mathfrak{g} \to T_{\gamma_j}M_{l_j}$ defined by

$$((d\Phi^{\gamma_j})^* f, v) = (f, d\Phi^{\gamma_j}(v)),\ \forall f \in T^*_{\gamma_j}M_{l_j},\ v \in \mathfrak{g}. \tag{7}$$

From (6) and Lemma 2, the condition that $J'(0) = 0$ for all g is equivalent to that

$$\sum_{j=0}^{m} \int_{\tau_j}^{\tau_{j+1}} ((d\Phi^{\gamma_j})^*(dL_{l_j})_{\dot{\gamma}_j}, \omega)\,dt = 0, \tag{8}$$

for all C^∞ map $\omega: [t_0, t_1] \to \mathfrak{g}$ such that $\int_{t_0}^{t_1} \omega\,dt = 0$. Since $(d\Phi^{\gamma_j})^*(dL_{l_j})_{\dot{\gamma}_j}$ is piecewise continuous (though not necessarily continuous) in $\mathfrak{g}^*$, (8) implies that $(d\Phi^{\gamma_j})^*(dL_{l_j})_{\dot{\gamma}_j}$ is constant for all t and all j whenever $\dot{\gamma}_j$'s are well defined, for otherwise one can always choose an ω with $\int_{t_0}^{t_1} \omega\,dt = 0$ such that (8) fails to hold. Therefore,

Theorem 1 (Noether). *Suppose γ is an optimal solution to Problem 1, and let $\gamma_0, \dots, \gamma_m$ be its segments. Then there exists a constant $\nu_0 \in \mathfrak{g}^*$ such that*

$$(d\Phi^{\gamma_j})^*(dL_{l_j})_{\dot{\gamma}_j} = \nu_0, \quad \forall t \in [\tau_j, \tau_{j+1}],\ j = 0, \dots, m. \tag{9}$$

A simple way of writing equation (9) is $(d\Phi^\gamma)^* dL_{\dot{\gamma}} \equiv \nu_0$.

If for each $l \in \Gamma$, there is a riemannian metric $\langle\cdot,\cdot\rangle_l$ on M_l such that $L_l = \frac{1}{2}\|\cdot\|_l^2$, then under the canonical identification of $T_{\gamma_j}M_{l_j}$ with $T^*_{\gamma_j}M_{l_j}$ via the metric $\langle\cdot,\cdot\rangle_l$, $(dL_{l_j})_{\dot{\gamma}_j}$ is identified with $\dot{\gamma}_j$, and equation (9) becomes

$$(d\Phi^{\gamma_j})^*\dot{\gamma}_j = \nu_0, \quad \forall t \in [\tau_j, \tau_{j+1}],\ j = 0, \dots, m, \tag{10}$$

where $(d\Phi^{\gamma_j})^*: T_{\gamma_j}M_{l_j} \to \mathfrak{g}^*$ now is defined by

$$((d\Phi^{\gamma_j})^* u, v) = \langle u, d\Phi^{\gamma_j}(v)\rangle_{l_j},\ \forall u \in T_{\gamma_j}M_{l_j},\ v \in \mathfrak{g}. \tag{11}$$

Example 3. Consider Example 1 with a, b as shown in Figure 1. Then an optimal solution γ from a to b consists of two segments $\gamma_0 \subset M_0$ and $\gamma_1 \subset M_1$. It can be shown that for γ_0, the conserved quantity ν_0 is the component of the angular momentum $(\gamma_0 - c_0) \times \dot{\gamma}_0 \in \mathbb{R}^3$ along the rotational axis of M_0. Here c_0 is an arbitrary point on the rotational axis. Similarly we can obtain the conserved quantity for γ_1. Theorem 1 states that these two quantities are identical.

4 OCA and OFS Problems

In this section, we will describe very briefly two related classes of switched Lagrangian systems. For more details, see [9, 10].

Let M be a C^∞ Riemannian manifold[3]. Denote with $\langle \cdot, \cdot \rangle$ the riemannian metric and with $\| \cdot \|$ the associated norm on TM. The arc length of a curve $\alpha : [t_0, t_1] \to M$ is defined as $\int_{t_0}^{t_1} \|\dot{\alpha}(t)\| \, dt$. The distance between two arbitrary points q_1 and q_2 in M, which we denote as $d_M(q_1, q_2)$, is by definition the infimum of the arc length of all the curves connecting q_1 and q_2. A geodesic in M is a locally distance-minimizing curve. In this paper, we always assume that M is connected and complete, and that all the geodesics are parameterized proportionally to arc length.

Let $L : TM \to \mathbb{R}$ be a smooth nonnegative function that is convex on each fiber. As an example one can take $L = \frac{1}{2}\| \cdot \|^2$. For each curve $\alpha : [t_0, t_1] \to M$, its *cost* is defined as $J(\alpha) = \int_{t_0}^{t_1} L[\dot{\alpha}(t)] \, dt$.

Consider an (ordered) k-tuple of points of M, $\langle q_i \rangle_{i=1}^k = (q_1, \dots, q_k)$. We say that $\langle q_i \rangle_{i=1}^k$ satisfies the *r-separation condition* for some positive r if and only if $d_M(q_i, q_j) \geq r$ for all $i \neq j$. Let $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$ be two k-tuples of points of M, each of which satisfies the r-separation condition. $\langle a_i \rangle_{i=1}^k$ is called the *starting position* and $\langle b_i \rangle_{i=1}^k$ the *destination position*. Let $h = (h_1, \dots, h_k)$ be a k-tuple of curves in M defined on $[t_0, t_1]$ such that $h_i(t_0) = a_i, h_i(t_1) = b_i$, for $i = 1, \dots, k$. h is said to be *collision-free* if the k-tuple $\langle h_i(t) \rangle_{i=1}^k$ satisfies the r-separation condition for each $t \in [t_0, t_1]$.

Problem 2 (OCA) *Among all collision-free $h = \langle h_i \rangle_{i=1}^k$ that start from $\langle a_i \rangle_{i=1}^k$ at time t_0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_1, find the one (or ones) minimizing*

$$J(h) = \sum_{i=1}^{k} \lambda_i J(h_i). \tag{12}$$

Here $\langle \lambda_i \rangle_{i=1}^k$ is a k-tuple of positive numbers.

To introduce the second problem we need some notions. Let $\langle q_i \rangle_{i=1}^k$ be a k-tuple of points of M satisfying the r-separation condition. Then a graph (V, E) can be constructed as following: the set of vertices is $V = \{1, \dots, k\}$; the set

[3] M here should not be confused with M in Section 2, where it is used to denote the union of the domains of $\mathcal{H}$.

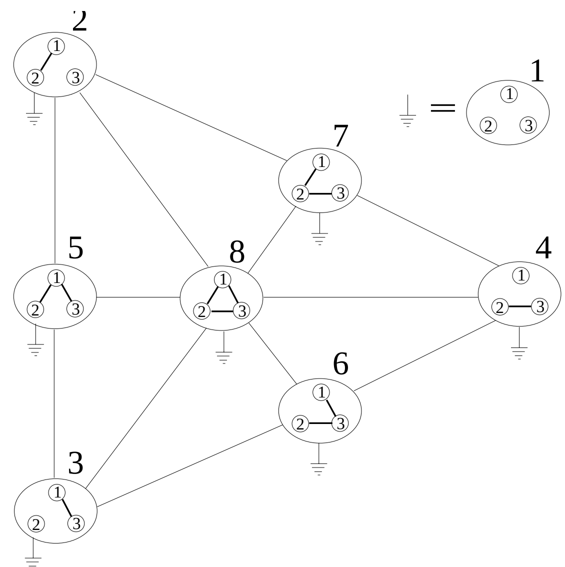

Fig. 2. Formation adjacency graph ($k = 3$, $M = \mathbb{R}^2$).

E of edges is such that an edge e_{ij} between vertex i and vertex j exists if and only if $d_M(q_i, q_j) = r$. (V, E) is called the *formation pattern* of $\langle q_i \rangle_{i=1}^k$. Let $h = \langle h_i \rangle_{i=1}^k$ be a collision-free k-tuple of curves in M defined on $[t_0, t_1]$. Then for each $t \in [t_0, t_1]$, the formation pattern of h at time t is defined to be the formation pattern of $\langle h_i(t) \rangle_{i=1}^k$.

Depending on M, r, and k, not all graphs with k vertices can be realized as the formation pattern of some $\langle q_i \rangle_{i=1}^k$. Two possible formation patterns are called *adjacent* if one is a strict subgraph of the other. This adjacency relation can be used to define a graph G_{adj} called the *formation adjacency graph*, whose set of vertices is the set of all possible formation patterns, and whose set of edges is such that an edge exists between two formation patterns if and only if they are adjacent. Figure 2 shows G_{adj} in the case $M = \mathbb{R}^2$ and $k = 3$, where the attachments of the "ground" symbol to vertices $2, \dots, 8$ signal their adjacency with vertex 1.

Now we are ready to define the OFS problem.

Problem 3 (OFS) *Let G'_{adj} be a connected subgraph of G_{adj} such that the formation patterns of $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$ are both vertices of G'_{adj}. Among all collision-free h that start from $\langle a_i \rangle_{i=1}^k$, end in $\langle b_i \rangle_{i=1}^k$, and satisfy the additional constraint that the formation pattern of h at any time $t \in [t_0, t_1]$ belongs to the vertices of G'_{adj}, find the one (or ones) minimizing the cost (12).*

For some choices of G'_{adj}, the OFS problem may not have a solution for all $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$. This difficulty is removed if we assume that G'_{adj} is *closed*, i.e., for each formation pattern (V, E) belonging to the vertices of G'_{adj}, any formation pattern (V', E') containing (V, E) as a subgraph is also a vertex of G'_{adj}. In the example shown in Figure 2, one can choose G'_{adj} to be the subgraph obtained by removing vertices 1, 2, 3, 4 and all the edges connected to them, thus imposing the constraint that all three agents, each of which is of radius $\frac{r}{2}$, have to "contact" one another either directly or indirectly via the third agent at any time in the joint trajectory. As another example, G'_{adj} can be taken to the subgraph of G_{adj} consisting of vertices 2, 5, 7, 8, and all the edges among them. So agent 1 and agent 2 are required to be bound together at all time, and the OFS problem becomes the optimal collision avoidance between agent 3 and this two-agent subsystem.

The OFS (hence OCA) problem can be naturally described as a switched Lagrangian system $\mathcal{H}$. Its discrete modes correspond to the vertices of G'_{adj}, i.e., the allowed formation patterns. For each such formation pattern (V, E), the corresponding domain is the subset of $M^{(k)} = M \times \ldots \times M$ consisting of points $(q_1, \ldots, q_k)$ such that $d_M(q_i, q_j) = r$ if $e_{ij} \in E$ and $d_M(q_i, q_j) \geq r$ otherwise, and the Lagrangian function is given by $\sum_{i=1}^{k} \lambda_i L \circ dP_i$, where dP_i is the differential of the projection P_i of $M^{(k)}$ onto its i-th component. The set E_d of discrete transitions is exactly the set of edges of G'_{adj}. For each discrete transition, its guard is the intersection of the domain of the source discrete mode with that of the target discrete mode, its continuous transition relation is given by the graph of the identity map. One anomaly of this definition is that the domains of different discrete modes may intersect each other. But this can be removed by introducing an additional index dimension, as is the case in Example 2.

We make the following two assumptions:

1. $\Phi : G \times M \to M$ is a C^∞ left action of a Lie group G on M by isometries.
2. The function L is G-invariant.

As before, for each $g \in G$, define $\Phi_g : M \to M$ to be the map $q \mapsto gq$, $\forall q \in M$. Similarly, for each $q \in M$ define $\Phi^q : G \to M$ to be the map $g \mapsto gq$, $\forall g \in G$. Therefore, for each $g \in G$, the first assumption implies that Φ_g is an isometry of M, while the second assumption implies that $L \circ d\Phi_g = L$. A very important observation is that, under these two assumptions, the switched Lagrangian system $\mathcal{H}$ is G-symmetric. Therefore, by Theorem 1 we have

Theorem 2. *Suppose $h = \langle h_i \rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem. Then there exists a constant $\nu_0 \in \mathfrak{g}^*$ such that*

$$\nu \triangleq \sum_{i=1}^{k} \lambda_i (d\Phi^{h_i})^* dL_{\dot{h}_i} \equiv \nu_0 \tag{13}$$

for all $t \in [t_0, t_1]$ where $\dot{h}_i$'s are well defined.

Example 4 ($G = \mathbf{SO}_n$, $M = \mathbf{S}^{n-1}$). Let $M = \mathbf{S}^{n-1} = \{(x_1, \ldots, x_n)^t \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1\}$ be the unit $(n-1)$-sphere, and let $G = \mathbf{SO}_n = \{Q \in \mathbb{R}^{n \times n} :$

$Q^tQ = I, \det Q = 1\}$ be the group of orientation-preserving n-by-n orthogonal matrices. G acts on M by left matrix multiplication. For each $q \in \mathbf{S}^{n-1}$, the tangent space $T_q\mathbf{S}^{n-1} = \{v \in \mathbb{R}^n : v^t q = 0\}$ is equipped with the standard metric $\langle u, v\rangle = u^t v$, $\forall u, v \in T_q\mathbf{S}^{n-1}$, which is invariant under the action of $\mathbf{SO}_n$. The Lie algebra of $\mathbf{SO}_n$, $\mathfrak{so}_n$, is the set of all n-by-n skew-symmetric matrices, i.e., $\mathfrak{so}_n = \{X \in \mathbb{R}^{n\times n} : X + X^t = 0\}$, where the Lie bracket is given by $[X, Y] = XY - YX$, $\forall X, Y \in \mathfrak{so}_n$. Choose $L = \frac{1}{2}\|\cdot\|^2$. Suppose that $h = \langle h_i\rangle_{i=1}^k$ is a k-tuple of curves on $\mathbf{S}^{n-1}$ that solves the OCA (or OFS) problem. At any time $t \in [t_0, t_1]$, let $u \in T_{h_i}\mathbf{S}^{n-1}$ and $v \in \mathfrak{so}_n$ be arbitrary. Then

$$\langle u, d\Phi^{h_i} v\rangle = \langle u, vh_i\rangle = u^t v h_i = \mathrm{tr}(u^t v h_i) = \mathrm{tr}(h_i u^t v) = \langle u h_i^t, v\rangle_F,$$

where $\langle\cdot,\cdot\rangle_F$ is the Frobenius inner product on $\mathbb{R}^{n\times n}$ defined by $\langle X, Y\rangle_F = \mathrm{tr}(X^tY)$ for any two n-by-n matrices X and Y. Since v is skew-symmetric, it is easily checked that $\langle u h_i^t, v\rangle_F = \frac{1}{2}\langle u h_i^t - h_i u^t, v\rangle_F$. Therefore

$$((d\Phi^{h_i})^* u, v) = \langle u, d\Phi^{h_i} v\rangle = \frac{1}{2}\langle u h_i^t - h_i u^t, v\rangle_F, \quad \forall v \in \mathfrak{so}_n. \tag{14}$$

Note that $u h_i^t - h_i u^t$ is skew-symmetric, hence belongs to $\mathfrak{so}_n$. We use the restriction of $\frac{1}{2}\langle\cdot,\cdot\rangle_F$ to establish a metric on $\mathfrak{so}_n$, hence identifying $\mathfrak{so}_n^*$ with $\mathfrak{so}_n$. Then equation (14) can be written as

$$(d\Phi^{h_i})^* u = u h_i^t - h_i u^t.$$

Hence the conservation law (13) becomes

$$\sum_{i=1}^k \lambda_i(\dot{h}_i h_i^t - h_i \dot{h}_i^t) \equiv \nu_0 \in \mathfrak{so}_n, \tag{15}$$

If we write each h_i in coordinates as $h_i = (h_{i,1}, \ldots, h_{i,n})^t \in \mathbf{S}^{n-1} \subset \mathbb{R}^n$, then (15) is equivalent to $\sum_{i=1}^k \lambda_i(\dot{h}_{ij_1} h_{ij_2} - h_{ij_1}\dot{h}_{ij_2}) \equiv C_{j_1j_2}$, $\forall t \in [t_0, t_1]$, $1 \le j_1 < j_2 \le n$, where $C_{j_1j_2}$'s are constants in $\mathbb{R}$. In particular, if $n = 2$ ($M = \mathbf{S}^2$, $G = \mathbf{SO}_3$), then equation (15) can be written compactly as $\sum_{i=1}^k \lambda_i(h_i \times \dot{h}_i) \equiv \Omega_0$ for some $\Omega_0 \in \mathbb{R}^3$, where $\times$ is the vector product. This is exactly the conservation of total angular momentum.

Example 5 (Grassmann Manifold). Let $\mathbf{O}_n$ be the set of orthogonal n-by-n matrices equipped with the standard metric inherited from $\mathbb{R}^{n\times n}$, which can be shown to be bi-invariant. Let p be an integer such that $1 \le p \le n$. Define H_p to be the subgroup of $\mathbf{O}_n$ consisting of all those matrices of the form $\begin{bmatrix} Q_p & 0 \\ 0 & Q_{n-p}\end{bmatrix}$, where Q_p and Q_{n-p} are p-by-p and $(n-p)$-by-$(n-p)$ orthogonal matrices respectively. Let $G_{n,p} = \{QH_p : Q \in \mathbf{O}_n\}$ be the set of all left cosets of H_p in $\mathbf{O}_n$. Alternatively, $G_{n,p}$ is the set of all equivalence classes of the equivalence relation $\sim$ defined on $\mathbf{O}_n$ by: $\forall P, Q \in \mathbf{O}_n$, $P \sim Q$ if and only if $P = QA$ for some $A \in H_p$.

Elements in $G_{n,p}$ are denoted by $[[Q]] = QH_p = \{QA : A \in H_p\}$, $\forall Q \in \mathbf{O}_n$, and correspond in a one-to-one way to the set of all p-dimensional subspaces of $\mathbb{R}^n$. As a quotient space of $\mathbf{O}_n$, $G_{n,p}$ admits a natural differential structure, and is called the Grassmann manifold. At each $Q \in \mathbf{O}_n$, the tangent space of $\mathbf{O}_n$ can be decomposed as the direct sum of two parts: the vertical space $\mathrm{vert}_Q\mathbf{O}_n$ and the horizontal space $\mathrm{hor}_Q\mathbf{O}_n$. $\mathrm{vert}_Q\mathbf{O}_n$ is the tangent space of QH_p at Q, which consists of all those matrices of the form $Q\begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix}$ for some p-by-p skew symmetric matrix Y and some $(n-p)$-by-$(n-p)$ skew symmetric matrix Z; $\mathrm{hor}_Q\mathbf{O}_n$ is the orthogonal complement in $T_Q\mathbf{O}_n$ of $\mathrm{vert}_Q\mathbf{O}_n$, and consists of all those matrices of the form $Q\begin{bmatrix} 0 & -X^t \\ X & 0 \end{bmatrix}$ for some $(n-p)$-by-p matrix X. Define a metric on $\mathrm{hor}_Q\mathbf{O}_n$ by

$$\langle Q\begin{bmatrix} 0 & -X_1^t \\ X_1 & 0 \end{bmatrix}, Q\begin{bmatrix} 0 & -X_2^t \\ X_2 & 0 \end{bmatrix}\rangle = \frac{1}{2}\langle Q\begin{bmatrix} 0 & -X_1^t \\ X_1 & 0 \end{bmatrix}, Q\begin{bmatrix} 0 & -X_2^t \\ X_2 & 0 \end{bmatrix}\rangle_F = \mathrm{tr}(X_1^t X_2), \tag{16}$$

for all $X_1, X_2 \in \mathbb{R}^{(n-p)\times p}$. An important observation is that $\mathrm{hor}_Q\mathbf{O}_n$ provides a representation of the tangent space of $G_{n,p}$ at $[[Q]]$, and the metric defined in (16) is independent of the choice of Q in $[[Q]]$, as long as one equates $Q\begin{bmatrix} 0 & -X^t \\ X & 0 \end{bmatrix}$ in $\mathrm{hor}_Q\mathbf{O}_n$ with $Q\begin{bmatrix} 0 & -X^t \\ X & 0 \end{bmatrix}A$ in $\mathrm{hor}_{QA}\mathbf{O}_n$ for arbitrary $A \in H_p$. Note that here we use the fact that the metric on $\mathbf{O}_n$ is bi-invariant. Therefore, (16) induces a metric on $G_{n,p}$, which is easily verified to be invariant with respect to the action of $\mathbf{O}_n$. Under this metric, the distance between $[[Q_1]]$ and $[[Q_2]]$, $Q_1, Q_2 \in \mathbf{O}_n$, can be calculated as $\sqrt{\sum_{i=1}^p \theta_i^2}$, where $\cos\theta_i$, $i = 1, \ldots, p$, are the singular values of the p-by-p matrix $\begin{bmatrix} I_p & 0 \end{bmatrix} Q_1^t Q_2 \begin{bmatrix} I_p \\ 0 \end{bmatrix}$. Here I_p is the p-by-p identity matrix ([7]).

Suppose $L = \frac{1}{2}\|\cdot\|^2$. Let $h = \langle h_i\rangle_{i=1}^k$ be a k-tuple of curves in $G_{n,p}$ which is a solution to the OCA (or OFS) problem. For each $i = 1, \ldots, k$, let q_i be a *lifting* of h_i in $\mathbf{O}_n$, i.e. q_i is a curve in $\mathbf{O}_n$ such that $[[q_i(t)]] = h_i(t)$, $\forall t \in [t_0, t_1]$. In other words, $q_i(t)$ is an orthogonal matrix in $\mathbf{O}_n$ whose first p columns span the subspace $h_i(t) \in G_{n,p}$. Also implicit in this definition is that q_i is continuous and piecewise C^∞. At any time $t \in [t_0, t_1]$, from the previous paragraph we can identify $T_{h_i}G_{n,p}$ with $\mathrm{hor}_{q_i}\mathbf{O}_n$. So for any $u \in T_{h_i}G_{n,p} \subset T_{q_i}\mathbf{O}_n$ and $v \in \mathfrak{o}_n$,

$$\langle u, d\Phi^{h_i} v\rangle_{T_{h_i}G_{n,p}} = \langle u, P_{q_i}(vq_i)\rangle_{\mathrm{hor}_{q_i}\mathbf{O}_n} = \langle u, vq_i\rangle_{T_{q_i}\mathbf{O}_n} = \langle uq_i^{-1}, v\rangle_{\mathfrak{o}_n}.$$

Here for clarity we indicate in the subscript the associated tangent space of each inner product. P_{q_i} is defined as the orthogonal projection of $T_{q_i}\mathbf{O}_n$ onto the subspace $\mathrm{hor}_{q_i}\mathbf{O}_n$. More specifically, each $w \in T_{q_i}\mathbf{O}_n$ can be written as $q_i\begin{bmatrix} Y & -X^t \\ X & Z \end{bmatrix}$ for some p-by-p skew symmetric matrix Y, $(n-p)$-by-$(n-p)$ skew symmetric matrix Z, and $(n-p)$-by-p matrix X, then $P_{q_i}(w) = q_i\begin{bmatrix} 0 & -X^t \\ X & 0 \end{bmatrix}$.

As a result, we see that $(d\Phi^{h_i})^* u = u q_i^{-1} = u q_i^t$, $\forall u \in \mathbf{T}_{h_i} G_{n,p}$. Finally, notice that $\dot{h}_i = P_{q_i}(\dot{q}_i)$. Therefore, the conserved quantity is

$$\nu_0 = \sum_{i=1}^{k} \lambda_i P_{q_i}(\dot{q}_i) q_i^t \in \mathfrak{o}_n. \tag{17}$$

5 Conclusions

We study the optimal control problem of switched Lagrangian systems with a group of symmetries. Necessary conditions are given for the optimal solutions. In particular, we show that the OCA and the OFS problems for multiple agents moving on a Riemannian manifold are special occasions of such a problem. Several examples are presented to illustrate the results.

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