Computational Methods for Reachability Analysis of Stochastic Hybrid Systems

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Abstract. Stochastic hybrid system models can be used to analyze and design complex embedded systems that operate in the presence of uncertainty and variability. Verification of reachability properties for such systems is a critical problem. Developing algorithms for reachability analysis is challenging because of the interaction between the discrete and continuous stochastic dynamics. In this paper, we propose a probabilistic method for reachability analysis based on discrete approximations. The contribution of the paper is twofold. First, we show that reachability can be characterized as a viscosity solution of a system of coupled Hamilton-Jacobi-Bellman equations. Second, we present a numerical method for computing the solution based on discrete approximations and we show that this solution converges to the one for the original system as the discretization becomes finer. Finally, we illustrate the approach with a navigation benchmark that has been proposed for hybrid system verification.

1 Introduction

Stochastic hybrid system models can be used to analyze and design complex embedded systems that operate in the presence of uncertainty and variability because they incorporate complex dynamics, uncertainty, multiple modes of operations, and they can support high-level control specifications that are required for design of autonomous or semi-autonomous applications. Reachability analysis for such systems is a critical problem because of the interaction between the discrete and continuous stochastic dynamics. Reachability properties are usually expressed as formulas in appropriate logics. Given a specification formula encoding a property, the task is to determine whether the formal model of the system satisfies the property or generate a counterexample that violates the formula. In this paper, we proposed a probabilistic method for reachability analysis. Instead of encoding the reachability property with a logical formula that can be evaluated to be true or false, we consider a representation using measurable functions taking values in $[0, 1]$ that characterize the probability that the system will satisfy the property.

In this paper, we show that reachability for stochastic hybrid systems can be represented by a measurable function that is the viscosity solution of a system of coupled Hamilton-Jacobi-Bellman (HJB) equations. This function is similar to

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the value function for the exit problem of a standard stochastic diffusion but the running and terminal costs depend on the function itself. Using a non-degeneracy assumption for the diffusion term of the stochastic dynamics, we show that the viscosity solution is continuous and bounded which allows us to extend standard results for Markov diffusions to stochastic hybrid systems.

One of the advantages of characterizing reachability properties using viscosity solutions is that for computational purposes we can employ numerical methods based on discrete approximations. We use an approximation method based on finite differences and we present an iterative algorithm based on dynamic programming for computing the solution. We show that the algorithm converges for appropriate initial conditions. Further, we show that the solution based on the discrete approximations converges to the one for the original stochastic hybrid system as the discretization becomes finer. Finally, we illustrate the approach with a navigation benchmark which has been proposed for hybrid system verification.

In this paper, we adopt the model presented in [3] which can be viewed as an extension of the stochastic hybrid systems described in [12]. An important characteristic of this model used in our analysis is that it satisfies the strong Markov property [3]. Related models have been presented in [11] with the emphasis on modeling and analysis of communication networks and in [1] for simulation of concurrent systems. Stochastic hybrid systems can be viewed as an extension of piecewise-deterministic processes [6] that incorporate stochastic continuous dynamics. Reachability of such systems has been studied in [4]. Communicating piecewise Markov processes have been presented in [20] with an emphasis on concurrency. Viscosity solution techniques in optimal control of piecewise deterministic processes have been studied in [7]. Our approach extends the results of [7] for reachability analysis of stochastic hybrid systems.

Reachability properties for continuous and hybrid systems have been characterized as viscosity solutions of variants of HJB equations in [16, 17]. Extensions of this approach to stochastic hybrid systems and a toolbox based on level set methods have been presented in [18]. Level set methods are also based on a discretization of the state space but they may offer computational advantages since the computation is limited to a boundary of the reachable set. The dynamic programming approach described in this paper is simpler to implement and capture the dependency of the value function between discrete modes. The approach allows us to show the convergence of solution obtained using the numerical solutions to the solution of the stochastic hybrid system.

Discrete approximation methods based on finite differences have been studied extensively in [15] and the references therein. Based on discrete approximations, the reachability problem can be solved using algorithms for discrete processes [19, 5, 8]. The approach has been applied for optimal control of stochastic hybrid systems given a discounted cost criterion in [14]. For reachability analysis, the discount term cannot be used and convergence of the value function can be ensured only for appropriate initial conditions. A grid-based method for safety analysis of stochastic systems with applications to air traffic management has been presented in [13]. Our approach is similar but using viscosity solutions we show the convergence of the discrete approximation methods.

The paper is organized as follows. Section 2 describes the stochastic hybrid system model. Section 3 formulates the reachability problem and characterizes its solution. Section 4 presents and analyzes the numerical methods based on discrete approximations. Section 5 illustrates the approach using a navigation benchmark and Section 6 concludes the paper.

2 Stochastic Hybrid Systems

We adopt the General Stochastic Hybrid System (GSHS) model presented in [3]. We briefly describe the model to establish the notation.

Let Q be a set of discrete states. For each $q \in Q$, we consider the Euclidean space $\mathbb{R}^{d(q)}$ with dimension $d(q)$ and we define an invariant as an open set $X^q \subseteq \mathbb{R}^{d(q)}$. The hybrid state space is denoted as $S = \bigcup_{q\in Q}\{q\} \times X^q$. Let $\bar{S} = S \cup \partial S$ and $\partial S = \bigcup_{q\in Q}\{q\} \times \partial X^q$ denote the completion and the boundary of S respectively. The Borel σ-field in S is denoted as $\mathcal{B}(S)$.

Definition 1. *A GSHS is defined as* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- Q *is a set of discrete states,*
- $d : Q \to \mathbb{N}$ *is a map that defines the continuous state space dimension for each* $q \in Q$*,*
- $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ *is a map that describes the invariant for each* $q \in Q$ *as an open set* $X^q \subseteq \mathbb{R}^{d(q)}$*,*
- $b : Q \times X^q \to \mathbb{R}^{d(q)}$ *and* $\sigma : Q \times X^q \to \mathbb{R}^{d(q)\times p}$ *are drift vectors and dispersion matrices respectively,*
- $Init : \mathcal{B}(S) \to [0, 1]$ *is an initial probability measure on* S*,*
- $\lambda : \bar{S} \to \mathbb{R}_+$ *is a nonnegative transition rate function, and*
- $R : \bar{S} \times \mathcal{B}(\bar{S}) \to [0, 1]$ *is a transition measure.*

To define the execution of the system, denote $(\Omega, \mathcal{F}, P)$ the underlying probability space and consider an $\mathbb{R}^p$-valued Wiener process $w(t)$ and a sequence of *stopping times* $\{t_0 = 0, t_1, t_2, \ldots\}$ that represent the times when the continuous and discrete dynamics interact. Let the state at time t_i be $s(t_i) = (q(t_i), x(t_i))$[1] with $x(t_i) \in X^{q(t_i)}$. While the continuous state stays in $X^{q(t_i)}$, $x(t)$ evolves according to the stochastic differential equation (SDE)

$$dx = b(q, x)dt + \sigma(q, x)dw \tag{1}$$

where the discrete state $q(t) = q(t_i)$ remains constant and the solution of (1) is understood using the Itô stochastic integral.

[1] When there is no confusion, we will use interchangeably the notation (q, x) and s for the hybrid state to simplify complex formulas and often we will use the notation $s_{t_i} = (q_{t_i}, x_{t_i})$ for brevity.

Let $t_{i+1}^* = \inf\{t \geq t_i, x(t) \in \partial X^{q(t_i)}\}$. The next stopping time t_{i+1} is defined as the minimum between t_{i+1}^* and a stopping time τ_{i+1} with survivor function $\exp\left(-\int_{t_i}^t \lambda(q(t_i), x_z(\omega)dz,\right)$, $\omega \in \Omega$. Thus, the survivor function of t_{i+1} can be written as

$$F(t,\omega) = I_{(t<t_{i+1}^*)} \exp\left(-\int_{t_i}^t \lambda(q(t_i), x_z(\omega))dz\right)$$

where I denotes the indicator function. If $t_{i+1} = \infty$, the system continues to evolve according to (1) with $q(t) = q(t_i)$. If $t_{i+1} < \infty$, the system jumps at t_{i+1} to a new state $s(t_{i+1}) = (q(t_{i+1}), x(t_{i+1}))$ according to the transition measure $R(s(t_{i+1}^-), A)$ with $A \in \mathcal{B}(S)$. The evolution of the system is then governed by (1) with $q(t) = q(t_{i+1})$ until the next stopping time.

The following assumptions are imposed on the model. The functions $b(q, x)$ and $\sigma(q, x)$ are bounded and Lipschitz continuous in x for every q, and thus the SDE (1) has a unique solution. The transition rate function λ is a bounded and measurable function which is assumed to be integrable for every $x_t(\omega)$. For the transition measure, it is assumed that $R(\cdot, A)$ is measurable for all $A \in \mathcal{B}(S)$, $R(s, \cdot)$ is a probability measure for all $s \in \bar{S}$, and $R((q, x), dz)$ is a stochastic continuous kernel.

Let $N_t = \sum_i I_{t \geq t_i}$ denote the number of jumps in the interval $[0, t]$. It is assumed that $E_s[N_t] < \infty$ for every initial state $s \in S$. Sufficient conditions for ensuring finitely many jumps can be formulated by imposing restrictions on the transition measure $R(s, A)$ [1].

Additionally, in this paper we consider the two following assumptions:

Assumption 1: Non-degeneracy. The boundaries ∂X^q are assumed to be sufficiently smooth and the trajectories of the system satisfy a non-tangency condition with respect to the boundaries. A sufficient condition for the non-tangency assumption is that the diffusion term is non-degenerate, i.e. $a(q, x) = \sigma(q, x)\sigma^T(q, x)$ is positive definite. This assumption is used to show the continuity of the viscosity solution close to the boundaries [10]. It should be noted that it is possible to show the continuity of the viscosity solution close to the boundaries even with degenerate variance by imposing appropriate conditions [10, 15].

Assumption 2: Boundness. It is assumed that the set Q is finite and that X^q is bounded for every q. This is a reasonable assumption for many systems that have finitely many modes and saturation constraints on the continuous state. Even if the state space is unbounded, often it is desirable to approximate it for applying numerical methods. By defining appropriately the boundary conditions, it can be shown that the effect of the numerical cutoff is small [10]. This assumption is used for approximating the hybrid system by a finite Markov chain and employing numerical methods based on dynamic programming.

We refer to the class of GSHS that satisfies the assumptions above simply as stochastic hybrid systems (SHS).

3 Probabilistic Reachability

In this section, we show that the probability that a state will reach a set of target states while avoiding an unsafe set can be characterized as the viscosity solution of a system of coupled HJB equations.

Let $T = \cup_{q \in Q_T} \{q\} \times T^q$ and $U = \cup_{q \in Q_U} \{q\} \times U^q$ be subsets of S representing the set of target and unsafe states respectively. We assume that T^q and U^q are proper subsets of X^q for each q, i.e. $\partial T^q \cap \partial X^q = \partial U^q \cap \partial X^q = \emptyset$ and the boundaries ∂T^q and ∂U^q are sufficiently smooth. We define $\Gamma^q = X^q \setminus (T^q \cup U^q)$ and $\Gamma = \cup_{q \in Q} \{q\} \times \Gamma^q$. The initial state (which, in general, is a probability distribution) must lie outside the sets T and U. The transition measure $R(s, A)$ is assumed to be defined so that the system cannot jump directly to U or T.

Consider the stopping time $\tau = \inf\{t \geq 0 : s(\tau^-) \in \partial T \cup \partial U\}$. Let s be an initial state in Γ, then we define the function $V : \bar{\Gamma} \to \mathbb{R}_+$ by

$$V(s) = \begin{cases} E_s[I_{(s(\tau^-) \in \partial T}], & s \in \Gamma \\ 1, & s \in \partial T \\ 0, & s \in \partial U \end{cases}.$$

The function $V(s)$ can be interpreted as the probability that a trajectory starting at s will reach the set T while avoiding the set U.

Inspired by [6], we add a new state Δ and we denote $\Gamma' = \Gamma \cup \Delta$. The system transitions to Δ according to the measure

$$R(s, \Delta) = \begin{cases} 1, & \text{if } s \in \partial T \cup \partial U \\ 0, & \text{otherwise} \end{cases}.$$

The new process is indistinguishable from the original process $s(t)$ for $t < \tau$ and at time τ it jumps to Δ and stays there forever. The system dies immediately after transitioning to Δ, i.e. $b(\Delta) = \sigma(\Delta) = \lambda(\Delta) = 0$. Finally, we extend V to Γ' by defining $V(\Delta) = 0$ which agrees with the probabilistic interpretation of V. By abuse of notation, we will denote the new process also by $s(t)$.

Given the assumptions on the sets T and U and their boundaries, we can construct a bounded function $c : \bar{S} \to \mathbb{R}_+$ continuous in x such that

$$c(q, x) = \begin{cases} 1, & \text{if } s = (q, x) \in \partial T^q \\ 0, & \text{if } s = (q, x) \in \partial U^q \cup \partial X^q \end{cases}.$$

Then, the value function V can be written as

$$V(s) = E_s \left[\int_0^\infty c(q_{t^-}, x_{t^-}) dp^*(t) \right] \tag{2}$$

where $p^*(t) = \sum_{i=1}^\infty I_{(t \geq t_i)} I_{((q_{t_i^-}, x_{t_i^-}) \in \partial S)}$ is a counting process counting the number of times the trajectory hits the boundary and jumps.

Consider the set of nonnegative Borel measurable functions $\mathcal{B}(S)_+$ and define the operator $\mathcal{G} : \mathcal{B}(S)_+ \to \mathcal{B}(S)_+$ by

$$Gg(q, x) = E_s[c(q_{t_1^-}, x_{t_1^-}) I_{(t_1 = t_1^*)} + g(q_{t_1}, x_{t_1})]. \tag{3}$$

where t_1 is the stopping time of the first jump. We will show that V is a fixed point of $\mathcal{G}$.

Lemma 1. $G^n g(q,x) = E_s\left[\int_0^{t_n} c(q_{t^-}, x_{t^-}) dp^*(t) + g(q_{t_n}, x_{t_n})\right].$

Proof. By the strong Markov property [3] and the construction of the SHS process we have[2]

$$\begin{aligned} E_s[c(q_{t_2^-}, x_{t_2^-}) I_{(t_2=t_2^*)} + g(q_{t_2}, x_{t_2}) | \mathcal{F}_{t_1}] &= E_s[c(q_{t_1}, x_{t_2^-}) I_{(t_2=t_2^*)} + g(q_{t_2}, x_{t_2}) | \mathcal{F}_{t_1}] \\ &= E_s[g(q_{t_1}, x_{t_1})]. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{G}^2 g(q,x) = \mathcal{G}(\mathcal{G}g(q,x)) &= E_s[c(q_{t_1^-}, x_{t_1^-}) I_{(t_1=t_1^*)} + \mathcal{G}g(q_{t_1}, x_{t_1})] \\ &= E_s[c(q_{t_1^-}, x_{t_1^-}) I_{(t_1=t_1^*)} + E_s[c(q_{t_2^-}, x_{t_2^-}) I_{(t_2=t_3^*)} + g(q_{t_2}, x_{t_2}) | \mathcal{F}_{t_1}]] \\ &= E_s[c(q_{t_1^-}, x_{t_1^-}) I_{(t_1=t_1^*)} + c(q_{t_2^-}, x_{t_2^-}) I_{(t_2=t_2^*)} + g(q_{t_2}, x_{t_2})]. \end{aligned}$$

By induction, we get

$$\begin{aligned} G^n g(q,x) &= E_s[\sum_{i=1}^{n} c(q_{t_i^-}, x_{t_i^-}) I_{(t_i=t_i*)} + g(q_{t_n}, x_{t_n})] \\ &= E_s[\int_0^{t_n} c(q_{t^-}, x_{t^-}) dp^*(t) + g(q_{t_n}, x_{t_n})]. \end{aligned}$$

Theorem 1. *The value function V is a fixed point of the operator $\mathcal{G}$.*

Proof. By definition of $\mathcal{G}$, for any $\psi_1 \leq \psi_2$ we have $\mathcal{G}\psi_1 \leq \mathcal{G}\psi_2$. Let $v^0(q,x) = 0$ for every q and every x and set $v^{n+1}(q,x) = \mathcal{G}v^n(q,x)$. Then $\{v^n\}$ increases monotonically and v^n takes values in $[0,1]$ for every n. Therefore, $\lim_{n\to\infty} v^n(q,x) = v(q,x)$ exists. Note that convergence is not guaranteed for other choices of v^0.

Since $v \geq v^n$, we have $\mathcal{G}v \geq \mathcal{G}v^n$ and thus $\mathcal{G}v \geq v^{n+1}$ for all n, therefore $\mathcal{G}v \geq v$. In addition, $\mathcal{G}v^n = v^{n+1} \leq v \leq \mathcal{G}v$ and $\lim_{n\to\infty} v^n = v$, therefore $\mathcal{G}v \leq v \leq Gv$ and $v = \lim_{n\to\infty} v^n$ is a fixed point of $\mathcal{G}$.

Finally by Lemma 1, $v = \lim_{n\to\infty} G^n v^0 = E_s[\int_0^\infty c(q_{t^-}, x_{t^-}) dp^*(t)]$ therefore V is a fixed point of $\mathcal{G}$, i.e. $V(s) = \mathcal{G}V(s)$.

Next, we show that the value function V can be represented as a discounted cost criterion with a target set where the running and the terminal cost depend on V itself.

Theorem 2. *Consider the value function $V(s)$ defined by (2) and define $L^V(q,x) = \lambda(q,x)\int_\Gamma V(y)R((q,x),dy)$ and $\psi^V(q,x) = c(q,x) + \int_\Gamma V(y)R((q,x),dy)$. Then, for $s \in \Gamma$*

$$V(s) = E_s\left[\int_0^{t_1^*} \Lambda(t) L^V(q_{t^-}, x_{t^-}) dt + \Lambda(t_1^*) \psi^V(q_{t_1^*}, x_{t_1^*})\right]. \tag{4}$$

[2] $\mathcal{F}_t$ denotes the filtration of the SHS process.

Proof. The SHS satisfies the strong Markov property [3], and therefore, the Markov property can be applied not only for constant times but also for random stopping times. Let t_1 be the time of the first jump and $t_1^* = \inf\{t \geq 0 : x(t) \in \partial X^{q(t_0)}\}$, then, using a standard dynamic programming argument, we can write

$$V(s) = E_s \left[I_{(t_1<t_1^*)} \int_\Gamma V(y) R((q_{t_1^-}, x_{t_1^-}), dy) dt \right.$$
$$\left. + \; I_{(t_1=t_1^*)} \left(c(q_{t_1^*}, x_{t_1^*}) + \int_\Gamma V(y) R((q_{t_1^*}, x_{t_1^*}), dy) \right) \right]. \tag{5}$$

By construction of the transition rate λ, t_1 and x_t are not independent (unless λ is constant). Denote $\mathcal{F}_\infty$ the σ-field $\sigma(x_t, t \geq 0)$ generated by x_t. The conditional distribution of t_1 given $\mathcal{F}_\infty$ is $P[t_1 > t|\mathcal{F}_\infty] = I_{t<t_1^*} \Lambda(t)$, where $\Lambda(t) = \exp\left\{ - \int_0^t \lambda(q_0, x_z) dz \right\}$, and the conditional density is

$$\frac{dP[t_1 \leq t|\mathcal{F}_\infty]}{dt} = \lambda(q_0, x_t) \Lambda(t) I_{(t<t_1^*)} + \Lambda(t_1^*) \delta(t - t_1^*).$$

Thus, equation (5) can be written as

$$V(s) = E_s \left[E_s \left[I_{(t_1<t_1^*)} \int_\Gamma V(y) R((q_{t_1^-}, x_{t_1^-}), dy) dt \right.\right.$$
$$\left.\left. + \; I_{(t_1=t_1^*)} \left(c(q_{t_1^*}, x_{t_1^*}) + \int_\Gamma V(y) R((q_{t_1^*}, x_{t_1^*}), dy) \right) |\mathcal{F}_\infty \right]\right]$$
$$= E_s \left[\int_0^{t_1^*} \lambda(q_t, x_t) \Lambda(t) \int_\Gamma V(y) R((q_{t^-}, x_{t^-}), dy) dt \right.$$
$$\left. + \; \Lambda(t_1^*) c(q_{t_1^*}, x_{t_1^*}) + \Lambda(t_1^*) \int_\Gamma V(y) R((q_{t_1*}, x_{t_1*}), dy) \right].$$

Using the definitions of $L^V(q, x)$ and $\psi^V(q, x)$ we have

$$V(s) = E_s \left[\int_0^{t_1^*} \Lambda(t) L^V(q_{t^-}, x_{t^-}) dt + \Lambda(t_1^*) \psi^V(q_{t_1^*}, x_{t_1^*}) \right].$$

Assuming that the transition measure $R(s, A)$ is a continuous stochastic kernel, the map $(q, x) \to \int_\Gamma f(y) R((q, x), dy)$ is bounded uniformly continuous for every bounded and continuous function f [2]. Then, if V is continuous in $\bar{X}^{q(t_0)}$, equation (4) is very similar to the discounted cost criterion with a target set [15]. The main difference is that the running cost $L^V(q, x)$ and the terminal cost $\psi^V(q, x)$ depend on the value function. Since the SHS satisfies the strong Markov property, the same procedure can be repeated every time a jump occurs. Next, we show that under the non-degeneracy assumption V is continuous.

Theorem 3. *V is bounded and continuous in x on $\bar{\Gamma}$.*

Proof. The $\mathcal{G}$ operator defined by (3) can be written as

$$\mathcal{G}g(q,x) = E_s\left[\int_0^{t_1} c(q_{t^-}, x_{t^-})dp^*(t) + g(q_{t_1}, x_{t_1})\right].$$

Since the SHS satisfies the strong Markov property, we can apply the same transformation as in Theorem 2 to get

$$\mathcal{G}q(q,x) = E_s\left[\int_0^{t_1^*} \Lambda(t)L^g(q_{t^-}, x_{t^-})dt + \Lambda(t_1^*)\psi^g(q_{t_1^*}, x_{t_1^*})\right]. \tag{6}$$

Therefore

$$\begin{aligned} v^{n+1}(q,x) &= \mathcal{G}v^n(q,x) \\ &= E_s\left[\int_0^{t_n^*} \Lambda(t)L^{v^n}(q_{t^-}, x_{t^-})dt + \Lambda(t_1^*)\psi^{v^n}(q_{t_n^*}, x_{t_n^*})\right]. \end{aligned}$$

Because of the non-degeneracy assumption, the exit times t_i^* are continuous at the sample paths of the process [15]. Therefore, all the functions in the sequence v^n are continuous and further, we have $v^n \geq v^0$ for every n. By applying the results of [2] (Chapter 7) we can conclude that $V = \lim_{n\to\infty} v^n$ is lower semi-continuous and bounded below.

Next, define a new function $\tilde{V} : \bar{\Gamma} \to \mathbb{R}_+$ by

$$V(s) = \begin{cases} E_s[I_{(s(\tau^-)\in\partial U)}, & s \in \Gamma \\ 1, & s \in \partial U \\ 0, & s \in \partial T \end{cases}.$$

The function $\tilde{V}$ can be interpreted as the probability that a trajectory starting at s will reach U before T and it can be written as

$$\tilde{V}(s) = E_s\left[\int_0^{\infty} \tilde{c}(q_{t^-}, x_{t^-})dp^*(t)\right]$$

where

$$\tilde{c}(q,x) = \begin{cases} 0, & \text{if } s = (q,x) \in \partial T^q \cup \partial X^q \\ 1, & \text{if } s = (q,x) \in \partial U^q \end{cases}.$$

From the non-degeneracy assumption, we have that $\tilde{V} = 1 - V(s)$. By applying the argument given in the first part of the proof to $\tilde{V}$, it follows that $\tilde{V}$ is lower semi-continuous and bounded below and therefore, $V = 1 + (-\tilde{V})$ is upper semi-continuous and bounded above. Thus, V is continuous and bounded in $\bar{\Gamma}$.

Next, we prove the main result of this section that characterizes V as the viscosity solution of a system of HJB equations. We use the results of [15] to derive the HJB equations (similar results can be found also in [10]).

Theorem 4. *Assume that f and σ are continuously differentiable w.r.t. x in Γ^q for each q and for suitable C_1 and C_2 satisfy $|f_x| \leq C_1$, $|\sigma_x| \leq C_1$, and $|f(q,0)| + |\sigma(q,x)| \leq C_2$. Then V is the unique viscosity solution of the system of equations*

$$\mathcal{H}_V\left((q,x), V, D_x V, D_x^2 V\right) = 0 \text{ in } \Gamma^q,\ q \in Q \tag{7}$$

with boundary conditions

$$V(q,x) = \psi^V(q,x) \text{ on } \partial\Gamma^q,\ q \in Q \tag{8}$$

where

$$\mathcal{H}_V\left((q,x), V, D_x V, D_x^2 V\right) = f(q,x)D_x V + \frac{1}{2} tr(a(q,x)D_x^2 V) + \lambda(q,x)V + L^V(q,x).$$

Proof. Consider the function

$$v(q,x) = \begin{cases} \mathcal{G}g(q,x) & \text{in } \Gamma^q \\ \psi^g(q,x) & \text{on } \partial\Gamma^q \end{cases}$$

where $g \in \mathcal{B}(S)_+$ is a continuous and bounded function. From (6), it follows that $v(q,x)$ is the value function of an exit-time problem in Γ^q for the diffusion (1) where $L^g : \Gamma \to \mathbb{R}_+$ and $\psi^g : \partial\Gamma \to \mathbb{R}_+$ are bounded continuous functions. Under the assumptions of f and σ, we can apply the results for standard Markov diffusions [10] (Thm V.2.1 and Cor. V.3.1) and therefore, $v(q,x)$ a viscosity solution of

$$\mathcal{H}_g\left((q,x), V, D_x V, D_x^2 V\right) = 0 \text{ in } \Gamma^q \tag{9}$$

$$V(q,x) = \psi^g(q,x) \text{ on } \partial\Gamma^q. \tag{10}$$

By Theorem 3, V is bounded and continuous. Therefore,

$$\bar{V}(q,x) = \begin{cases} \mathcal{G}V(q,x) & \text{in } \Gamma^q \\ \psi^V(q,x) & \text{on } \partial\Gamma^q \end{cases}$$

is a viscosity solution of

$$\mathcal{H}_V\left((q,x), \bar{V}, D_x \bar{V}, D_x^2 \bar{V}\right) = 0 \text{ in } \Gamma^q$$

$$\bar{V}(q,x) = \psi^V(q,x) \text{ on } \partial\Gamma^q.$$

where V is considered known and $\bar{V}$ unknown. But V is a fixed point of $\mathcal{G}$, and thus $V = \mathcal{G}V = \bar{V}$ in Γ^q and $\psi^V = \psi^{\bar{V}}$ on $\partial\Gamma^q$, which means $V = \bar{V}$ is a viscosity solution of (7 - 8). Further, V is continuous, and therefore, is the unique viscosity solution which is continuous in $\bar{\Gamma}$.

4 Numerical Methods for Reachability Analysis

4.1 Locally Consistent Markov Chains

In this section, we employ the finite difference method presented in [15] to compute locally consistent Markov chains (MCs) that approximate the SHS while preserving local mean and variance. We consider a discretization of the state space denoted by $\bar{S}^h = \cup_{q \in Q}\{q\} \times \bar{S}^h_q$ where $\bar{S}^h_q$ is a set of discrete points approximating X^q and $h > 0$ is an approximation parameter characterizing the distance between neighboring points. By abuse of notation, we denote the sets of boundary and interior points of $\bar{S}^h_q$ by ∂S^h_q and S^h_q respectively. The state of the approximating MC is denoted by $s^h_n = (q^h_n, \xi^h_n),\ n = 0, 1, 2, \ldots$.

Consider the continuous evolution of the SHS between jumps and assume that the state is (q, x). The local mean and variance given the SDE (1) on the interval $[0, \delta]$ are

$$E[x(\delta) - x] = b(q(t), x(t))\delta + o(\delta)$$
$$E[(x(\delta) - x)(x(\delta) - x)^T] = a(q(t), x(t))\delta + o(\delta).$$

Let $\{q^h_n = q, \xi^h_n\}$ describe the MC on $S^h_q \subset X^q$ with transition probabilities denoted by $p^h_D((q, x), (q, y))$. A locally consistent MC must satisfy

$$E[\Delta\xi^h_n] = b(q, x)\Delta t^h(q, x) + o(\Delta t^h(q, x))$$

$$E[(\Delta\xi^h_n - E[\Delta\xi^h_n])(\Delta\xi^h_n - E[\Delta\xi^h_n])^T] = \\ a(q(t), x(t))\Delta t^h(q, x) + o(\Delta t^h(q, x))$$

where $\Delta\xi^h_n = \xi^h_{n+1} - \xi^h_n, \xi^h_n = x$ and $\Delta t^h(q, x)$ are appropriate interpolation intervals (or the "holding times") for the MC.

The diffusion transition probabilities $p^h_D((q, x), (q', x'))$ and the interpolation intervals can be computed systematically from the parameters of the SDE (details can be found in [15]). If the diffusion matrix $a(q, x)$ is diagonal and we consider a uniform grid with e_i denoting the unit vector in the i^{th} direction, the transition probabilities are

$$p^h_D((q, x), (q, x \pm he_i)) = \frac{a_{ii}(q, x)/2 + hb^{\pm}_i(q, x)}{Q(q, x)}$$

and the interpolation intervals are $\Delta t(q, x) = h^2/Q(q, x)$ where $Q(q, x) = \sum_i [a_{ii}(q, x) + h|b_i(q, x)|]$ and $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$ denote the positive and negative parts of a real number.

Next, consider the jumps with transition rate $\lambda(q, x)$ and transition measure $R((q, x), A)$. Suppose that at time t the state has just changed to $\{q^h_n = q, \xi^h_n = x\}$. The probability that a jump will occur on $[t, t + \delta)$ conditioned on the past data can be approximated by

$$P[(q, x) \text{ jumps on } [t, t + \delta)|q(s), x(s), w(s), s \leq t] = \lambda(q, x)\delta + o(\delta).$$

The i^{th} jump of the approximating process is denoted by $\zeta((q, x), \rho_i)$ where ρ_i are independent random variables with distribution $\bar{R} = \{\rho : \zeta((q, x), \rho_i) \in A\} = R((q, x), A)$ with compact support Θ. Let ζ_h be a bounded measurable function such that $|\zeta_h((q, x), \rho) - \zeta(q, x), \rho)| \to 0$ uniformly in x for each ρ and which satisfies $\zeta_h((q,x),\rho) \in \bar{S}^h$. If $x \in S_q^h$, then with probability $1-\lambda(q,x)\Delta t^h(q,x)-o(\Delta t^h(q,x))$ the next state is determined by the diffusion probabilities p_D^h and with probability $\lambda(q,x)\Delta t^h(q,x)+o(\Delta t^h(q,x))$ there is a jump and the next state is $(q_{n+1}^h, \xi_{n+1}^h) = \zeta((q,x),\rho_i)$. For the points in ∂S_q^h, the next state is determined by $\zeta((q,x),\rho_i)$ with probability 1. Therefore, the transition probabilities are defined by

$$p^h((q,x),(q',x')) = \begin{cases} (1-\lambda(q,x)\Delta t^h(q,x) - o(\Delta t^h(q,x)))p_D^h((q,x),(q',x')) & \\ +(\lambda(q,x)\Delta t^h(q,x) + o(\Delta t^h(q,x)))\bar{R}\{\rho : \zeta_h((q,x),\rho) = (q', x'-x)\} & \text{if } x \in S_q^h \\ \bar{R}\{\rho : \zeta_h((q,x),\rho) = (q', x'-x)\} & \text{if } x \in \partial S_q^h \end{cases} \tag{11}$$

4.2 Iterative Methods for Reachability Analysis

This section describes the approximation of the value function, formulates the problem for the discrete approximations, and presents the convergence results for the numerical methods.

Consider the approximating MC $\{s_n^h\} = \{\xi_n^h, q_n^h\}$ with transition probabilities $p^h((q,x),(q',x'))$ defined in (11). Let $\bar{T}^h = \bar{S}^h \cap \bar{T}$ and $\bar{U}^h = \bar{S}^h \cap \bar{U}$ and denote by n_i the jump times and ν_h the stopping time representing that $(q_n^h, \xi_n^h\} \in \bar{T}^h \cup \bar{U}^h$, then the value function V can be approximated by

$$V^h(s) = E_s\left[\sum_{n=0}^{\nu_h} c(q_n^h, \xi_n^h) I_{(n=n_i)}\right].$$

The function V^h can be computed using a value iteration algorithm. To show the convergence of the algorithm, we consider a terminal state Δ similar to Section 3. The state space of the MC becomes $\tilde{S}^h = \bar{S}^h \cup \{\Delta\}$ and the transition probabilities are defined so that $\tilde{p}^h((q,x),\Delta) = 1$ if $x \in \bar{T}^h \cup \bar{U}^h$, $\tilde{p}^h(\Delta,\Delta) = 1$, and $\tilde{p}^h((q,x),(q',x')) = p^h((q,x),(q',x'))$ otherwise. This means that when the state reaches T or U, it transitions to Δ and stays there for ever. Consider the function $\tilde{c} : \tilde{S}^h \to \mathbb{R}_+$ with $\tilde{c}(\Delta) = 1$ and $\tilde{c}(q,x) = 0$ for every (q,x) and the value function

$$\tilde{V}^h(s) = E_s[\sum_{n=0}^{\infty} \tilde{c}(s_n)]. \tag{12}$$

Clearly, this sum is well-defined, bounded, and we have $\tilde{V}^h = V^h$.

Proposition 1. *Let $\tilde{V}_0^h(q,x) = 0$ for every (q,x), then the iteration*

$$\tilde{V}_{n+1}^h(q,x) = \left[\sum_{q',x'} \tilde{p}^h((q,x),(q',x'))\tilde{V}_n^h(q',x')\right] \tag{13}$$

converges pointwise and monotonically to $\tilde{V}^h = V^h$.

Proof. Consider the value function defined by (12) for $\{s_n\}$. We have that $\tilde{V}^h(q,x) \in [0,1] < \infty$ and $\tilde{c}(s) \geq 0$ for all $s \in \tilde{S}^h$. Therefore, computing $\tilde{V}$ is a special case of the total expected reward criterion for positive models [19]. If v is a fixed point of the iteration (13), then $v + k[1,\ldots,1]^T,\ k > 0$ is also a fixed point. Thus, the iteration may have multiple fixed points but if we pick $\tilde{V}_0^h = 0$ it converges to the least fixed point $\tilde{V}$ [19] (Thm 7.2.12).

4.3 Convergence Results

Finally, we show that the value function V^h obtained using the approximating MC converges to the value function V of the SHS as $h \to 0$. Let $g \in \mathcal{B}(S)_+$ be a continuous and bounded function and suppose that V is the unique viscosity solution of (9-10) that is bounded and continuous in $\bar{\Gamma}^q$. Consider $\bar{\Sigma}_q^h$ to be a discretization of $\bar{\Gamma}^q$ and denote Σ_q^h and $\partial\Sigma_q^h$ the set of interior and boundary points respectively. Using the approximation described in Subsection 4.1, the dynamic programming equation for $\bar{\Sigma}_q^h$ can be written as

$$V^h(q,x) = \begin{cases} F_g^h[V^h(\cdot)](q,x) & \text{if } x \in \Sigma_q^h \\ \psi_g^h(q,x) & \text{if } x \in \partial\Sigma_q^h \end{cases}$$

where

$$\begin{aligned} &F_g^h[V^h(\cdot)](q,x) = \\ &\quad (1-\lambda(q,x)\Delta t^h(q,x) - o(\Delta t^h(q,x)) \sum_{q',x'} p_D^h((q,x),(q',x'))V^h(q',x') \\ &\quad +(\lambda(q,x)\Delta t^h(q,x) + o(\Delta t^h(q,x)) \int_\Theta g(\zeta_h((q,x),\rho))\bar{R}(d\rho) \end{aligned}$$

and

$$\psi_g^h(q,x) = c(q,x) + \int_\Theta g(\zeta_h((q,x),\rho))\bar{R}(d\rho).$$

Lemma 2. $\lim_{y\to x, h\to 0} V^h(q,y) = V(q,x)$ *uniformly in* $\bar{\Gamma}^q$.

Proof. V is continuous and bounded viscosity solution of (9-10) and $\psi^g(q,x)$ is continuous. Therefore, for each q we have a standard exit problem from Γ^q for the SDE (1) and by applying the results of [10] (Sec. IX.5) we have that V^h converges uniformly to V.

To show convergence of V^h for the SHS, we replace g by V and we follow an argument similar to the proof of Theorem 1.

Theorem 5. *Let*

$$V^h(q,x) = \begin{cases} F_V^h[V^h(\cdot)](q,x) & \text{if } x \in \Sigma_q^h \\ \psi_V^h(q,x) & \text{if } x \in \partial\Sigma_q^h \end{cases}$$

then $\lim_{y\to x, h\to 0} V^h(q,y) = V(q,x)$.

Proof. Assume that V is given and define

$$\bar{V}^h(q,x) = \begin{cases} F_V^h[\bar{V}^h(\cdot)](q,x) & \text{if } x \in \Sigma_q^h \\ \psi_V^h(q,x) & \text{if } x \in \partial\Sigma_q^h \end{cases}.$$

By Lemma 2, since V is bounded and continuous we have $\lim_{y\to x,h\to 0} \bar{V}^h(q,y) = \bar{V}(q,x)$. Assume that for each h, $\bar{V}^h$ is computed by a value iteration algorithm with $v^0 = 0$. Then, V^h is a fixed point of F_V^h and therefore, $\bar{V}^h = V^h$ for every h and $\bar{V} = V$.

5 Navigation Benchmark

This section illustrates the approach using a stochastic version of the navigation benchmark presented in [9]. The benchmark describes an object moving within a bounded 2-dimensional region partitioned into cells X^q, $q \in \{0,1,\dots,N\}$ as shown in Figure 1. Let $[x_1,x_2]^T$ and $v = [v_1,v_2]^T$ denote the position and the velocity of the object respectively. The behavior is defined by the ODE $\dot{v} = A(v - v_d^q)$ where $A \in \mathbb{R}^{2\times 2}$ and $v_d^q = [sin(q\pi/4), cos(q\pi/4)]^T$. Selecting the matrix A and adding a diffusion term, the dynamics of the object are described by the SDE

$$dx = (\tilde{A}x + \tilde{B}u_d^q)dt + \Sigma dw$$

where $x = [x_1,x_2,v_1,v_2]^T$, $u_d^q = [0,0,v_d^q]^T$, $w(t)$ is an $\mathbb{R}^4$-valued Wiener process,

$$\tilde{A} = \begin{bmatrix} 0 & I_2 \\ 0 & A \end{bmatrix},\ A = \begin{bmatrix} -1.2 & 0.1 \\ 0.1 & -1.2 \end{bmatrix},\ \text{and } \Sigma = 0.1 I_4.$$

Consider the target set T and the unsafe set U shown in Figure 1. Given initial state $s_0 = (q_0, x_0)$, we want to compute the probability that the state will reach T while avoiding U. Sample trajectories are shown in Figure 1. In order to apply the approach described in this paper, we under-approximate each cell X^q by $\tilde{X}^q$ by considering a smooth boundary $\partial\tilde{X}^q$. We also define a transition

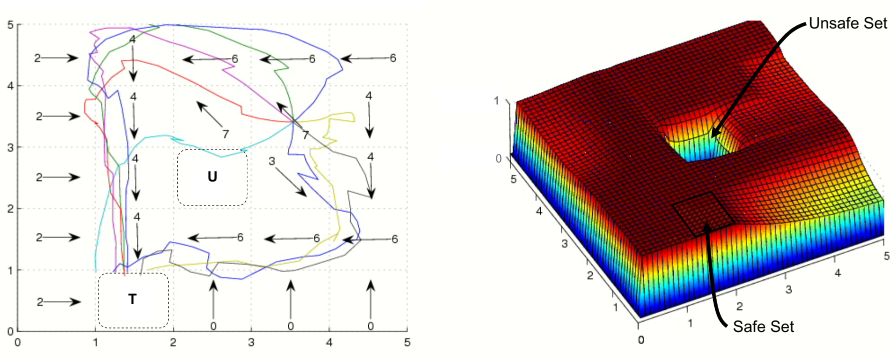

Fig. 1. The navigation benchmark, sample trajectories, and the value function

Table 1. Performance Data

h	Time (minutes)	Number of States
.5	.5	2500
.25	7	32400
.1	200	1147041
.05	5110	17147881

measure $R((q,x),A)$ so that the state jumps into an adjacent cell if it hits an "inner" boundary and jumps into the same cell if it hits on "outer" boundary. The transition rate is assumed to be zero. We discretize the state space using a uniform grid with approximation parameter $h > 0$ and apply the method described in Section 4 to compute $V^h(q,x)$. As $h \to 0$, $V^h(q,x)$ converges to the solution $V(q,x)$ of the stochastic approximation of the benchmark problem.

Since the continuous state space of the example is 4-dimensional, we select to plot a projection of V^h for initial velocity $v_0 = [0,0]^T$. Figure 1 shows this projection for $h = 0.1$ that describes the probability that a trajectory starting from $(q, [x_1, x_2, 0, 0]^T)$ will reach T while avoiding U. The computational performance of the algorithm is illustrated in Table 1. All data was collected using a 3.0 GHz desktop computer with 1 Gb RAM. A more exact characterization is more involved since the operator F_V^h of the value iteration algorithm is not a contraction mapping and convergence is guaranteed only for $V_0^h = 0$.

6 Conclusions and Future Work

The paper characterizes reachability of stochastic hybrid systems as a viscosity solution of a system of coupled Hamilton-Jacobi-Bellman equations and employs a numerical method based on discrete approximations for reachability analysis. The main advantage of the approach is that it guarantees the convergence of the solution based on the discrete approximation to the solution of the original problem. The approach can be extended to controlled stochastic hybrid systems by imposing appropriate conditions for admissible controls. Convergence of the discrete approximation methods can be investigated using relaxed controls. Characterization of error bounds and convergence rates is an important and challenging problem especially since convergence is not based on contraction mappings but it is guaranteed only for appropriate initial conditions. Another fundamental challenge is to develop scalable numerical methods that can be applied to large systems. Towards this goal, currently we are investigating methods based on variable resolution grids and parallel algorithms as well as methods based on value function approximation.

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