Control System Engineering

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Control System Engineering

Topic:

Introduction To Control System And Classification

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MODULE-I : INTRODUCTION TO CONTROL SYSTEM

INTRODUCTION :-

→ Every activity in our day to day life is influenced by some sort of control system.

→ Concept of control system plays an important role in working of space vehicles, satelites, guided missiles, etc.

→ Control systems are found in no of practical applications like computerised control systems, transportation systems, power systems, temperature limiting systems, robotics etc.

BASIC CONCEPT OF CONTROL SYSTEMS :-

Definitions :-

SYSTEM :- It is a combination or an arrangement of different physical components which act together to achieve certain objective.

Ex :- → A classroom (comb^n of benches, blackboard, fans, etc)
→ A lamp (made up of glass, filament).

CONTROL :- It means to regulate, to direct or to command to get desired output.

CONTROL SYSTEM :- It is an interconnection of the physical components to provide a desired function, involving some kind of controlling action in it.

Ex :- ① If in a classroom, teacher is delivering his lecture, the combination becomes a control system.

② If lamp is switched ON or OFF using a switch, then the combination is a control system.

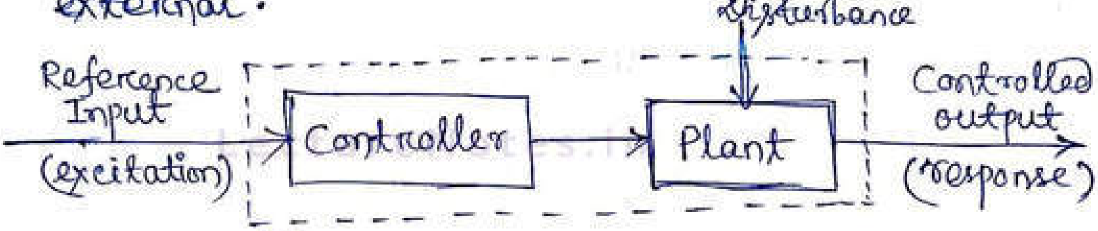
PLANT :- The portion of a system which is to be controlled or regulated is called the plant or the process.

CONTROLLER :- The element of the system itself or external to the system which controls the plant or process is called controller.

INPUT :- It is an applied signal or excitation to a control system in order to produce a specified output.

OUTPUT :- It is the actual response obtained from a control system when input is applied to it.

DISTURBANCES :- It is a signal which adversely affect the value of the output of a system. It may be internal or external.



[Fig. A Control System]

CLASSIFICATION OF CONTROL SYSTEM :-

(1) Natural control system : The biological systems inside human being are of natural type.

Ex : perspiration system (secreats sweat and regulate the temperature of human body).

(2) Manmade control systems : The systems which are being designed and manufactured by human beings are called manmade control systems.

Ex : An automobile system with gears, accelerator, braking syste

(3) Combinational control system : The control system having combination of natural and manmade together.

Ex : Driver driving a vehicle.

(4) Time Varying and Time-Invariant Systems :

→ Time varying control systems are those in which parameters of the systems are varying with time.

Ex :- Space vehicle system, in which mass of vehicle decreases with time, as it leaves earth.

→ Time Invariant control system are those in which the parameters of the system are not varying with time.

Ex :- Electrical network consisting of elements R, L & C is a time invariant system as the parameter values are not varying with time.

(5) Linear and Non-linear Systems:

→ A System is known as linear if and only if it posses both homogeneity and superposition properties.

- **Superposition property** :-

If an input $r_1(t)$ gives an output $c_1(t)$ i.e, $r_1(t) \to c_1(t)$ and another input $r_2(t) \to c_2(t)$ then if two inputs are applied together, then the o/p will be sum of two outputs i.e,

$$[r_1(t) + r_2(t)] \longrightarrow [c_1(t) + c_2(t)].$$

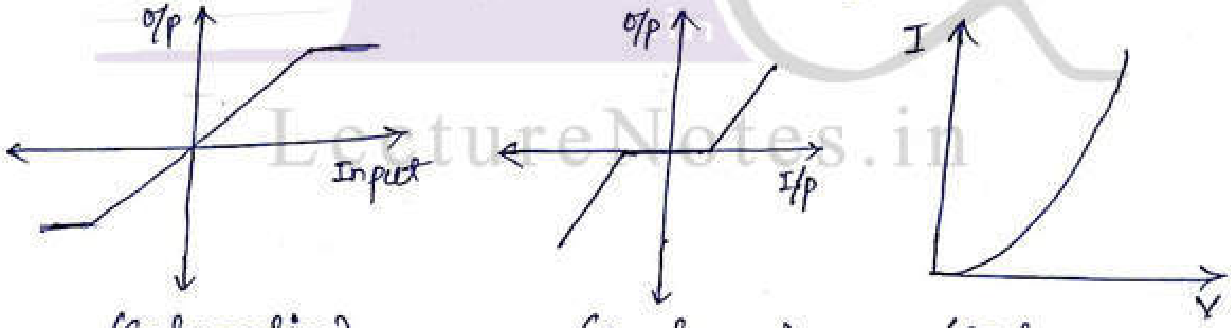
- **Homogeneity property** :-

If for any system the i/p magnitude is increased k time from $r_1(t)$ to $kr_1(t)$, the magnitude of o/p increased from $c_1(t)$ to $kc_1(t)$, then the property is called as homogeneity.

Ex:- A resistive network is a linear system.

→ A system is said to be non-linear if it doesn't satisfy the superposition & homogeneity.

Ex:- Diode circuit is a non-linear system, Saturation, Deadzone, etc.



(Saturation) (Dead zone) (Diode ckt response)

(6) Continuous Time & Discrete Time Control System :-

→ In continuous time control system, all system variables are the function of continuous time variable 't'.

Ex:- Speed control of a d.c. motor using a tachogenerator feedback.

→ In discrete time systems, one or more variables are known at certain discrete interval of time.

Ex:- Microprocessor or Computer based systems, use discrete time signals.

→ Continuous time systems uses continuous signals while discrete time systems uses discrete signal.

(7) Deterministic and Stochastic Control Systems :

→ If the response to input as well as behaviour of external disturbances is predictable & repeatable, then it is called as deterministic C/s.

→ If the response is unpredictable, then it is stochastic C/s.

(8) Lumped Parameter & Distributed Parameter C/s :-

→ If control system can be described by ordinary differential equations, then it is called lumped parameter control system.

Ex:- Electrical n/w with different parameters as resistance, inductance etc.

→ If Control System can be described by partial differential equations, then it is called distributed parameter C/s.

Ex:- Transmission line having its parameters resistance & inductance totally distributed along it.

Note:- The lumped parameters are physically separable, while the distributed parameters can't be physically separated.

(9) Single Input Single Output (SISO) & Multiple Input Multiple Output (MIMO) System :-

→ A system having only one input & one output is called as SISO system.

Ex:- Position Control System : one input (desired position) & one output (actual o/p position)

→ A system having multiple input & multiple output is called as MIMO system.

(10). Causal & Non-Causal System :-

→ The system where the o/p depends on past/current inputs but not on future inputs, is called causal system.

Ex:- $y(t) = x(t) + x(t-3)$.

→ If system o/p depends on some of future inputs, then it is called as Non-causal system.

Ex:- $y(t) = x(t+3) + x(t)$.

Note:- Anti Causal System :- If the o/p only depends on future i/p, then it is called as Anticausal System.

Ex:- $y(t) = x(t+10)$.

* Linear & Non-Linear System :-

→ A system is said to be linear, if satisfies superposition Property i.e., Additive & Homogenity.

→ Additive Property :-

If for input $x_1(t)$, o/p is $y_1(t)$ & for i/p $x_2(t)$, o/p is $y_2(t)$, then if two inputs are applied together, then the o/p will be sum of two outputs.

i.e, $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$.

→ Homogenity property :-

If input is scaled by a factor 'A', then the o/p must be scaled by the same factor.

i.e, $Ax(t) \longrightarrow Ay(t)$.

Examples :-

(i) $y(t) = x(t)$.

$x_1(t) \rightarrow y_1(t) = x_1(t)$

$x_2(t) \rightarrow y_2(t) = x_2(t)$

$y_1(t) + y_2(t) = x_1(t) + x_2(t)$ —— (1)

Replace, $x(t) = x_1(t) + x_2(t)$.

$\therefore y(t) = x_1(t) + x_2(t)$ —— (2)

As $y(t) = y_1(t) + y_2(t)$, so, Additive property satisfies.

Also, $Ax(t) \longrightarrow$ o/p $= Ax(t) = Ay(t)$.

∴ so, Linear system.

(ii) $y(t) = x^2(t)$.

$x_1(t) \longrightarrow y_1(t) = x_1^2(t)$

$x_2(t) \longrightarrow y_2(t) = x_2^2(t)$,

$y_1(t) + y_2(t) = x_1^2(t) + x_2^2(t)$ —— (1)

Replace, $x(t) = x_1(t) + x_2(t)$

$\therefore y(t) = [x_1(t) + x_2(t)]^2$ —— (2),

∴ As $y(t) \neq y_1(t) + y_2(t)$, so, Non-Linear system.

(3) $y(t) = \frac{d}{dt}x(t)$

$\therefore \frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t) = \frac{d}{dt}[x_1(t) + x_2(t)]$.

$\Rightarrow$ Linear. Also, $\frac{d}{dt}[Ax(t)] = A\frac{d}{dt}x(t) = Ay(t)$.

(4). $y(t) = \sin[x(t)]$. $\longrightarrow$ NL

(5) $y(t) = 2x(t) + 4$ $\longrightarrow$ NL.

(6) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t) = 2x(t)$ $\longrightarrow$ Linear.

(7) $\frac{dy(t)}{dt} + 3y(t) = 2x^2(t)$ $\longrightarrow$ NL.

* Time Variant & Time Invariant System :-

→ A system is said to be TI if its i/p & o/p ch. doesn't change with respect to the time. if

→ If the i/p is delayed by 't_0' seconds, then, the system o/p is also delayed by 't_0' seconds, then it is time invariant system.

Ex:- (1) $\boxed{y(t) = x(t)}$ $\longrightarrow$ LTI system.

i/p delay : $y(t) = x(t-t_0)$. —(1)

o/p delay : $y(t-t_0) = x(t-t_0)$ —(2)

$\therefore$ As eqn (1) = eqn (2) $\Rightarrow$ Time Invariant System.

(2) $y(t) = x^2(t)$

i/p delay : $y(t) = x^2(t-t_0)$. —(1)

o/p delay : $y(t-t_0) = x^2(t-t_0)$ —(2)

→ Time invariant system.

(3) $\boxed{y(t) = \frac{d}{dt}x(t)}$ $\longrightarrow$ LTI system

$\therefore \frac{d}{dt}x(t-t_0) = \frac{d}{d(t-t_0)}x(t-t_0) = \frac{d}{dt}x(t-t_0)$

→ TI system

(4) $y(t) = x(t-7)$.

$x(t-t_0-7) = x(t-t_0-7)$ $\longrightarrow$ LTI system.

(5) $y(t) = x(-t)$ $\longrightarrow$ TV.

(6) $y(t) = x(t)\cdot \cos\omega_0 t \longrightarrow$ TV.

(7) $\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 2y(t) = x(t) \longrightarrow$ LTI

(8) $\frac{d^2y(t)}{dt^2} + 2t\frac{dy(t)}{dt} + 2y(t) = x^2(t) \longrightarrow$ TV.

* Causal & Non-causal System :-

→ A system is said to be causal if the present o/p depends on the present i/p & on the past i/p & o/p, but not on the future values.

→ All causal systems are real-time systems & physically realizable.

→ Example : (1) $y(t) = x(t)$

$y(-1) = x(-1)$; $y(0) = x(0)$; $y(1) = x(1)$

→ causal system.

(2) $y(t) = x^2(t) \longrightarrow$ causal.

(3) $y(t) = x(2t)$

$y(-1) = x(-2)$; $y(0) = x(0)$; $y(1) = x(2)$.

→ Non-causal system.

(4) $y(t) = x(t-7) \longrightarrow$ causal.

(5) $y(t) = \frac{d}{dt}x(t) \longrightarrow$ causal.

** If any eqn contains addition of constant & if there is any mathematical operation on the input, then it is Non linear system.

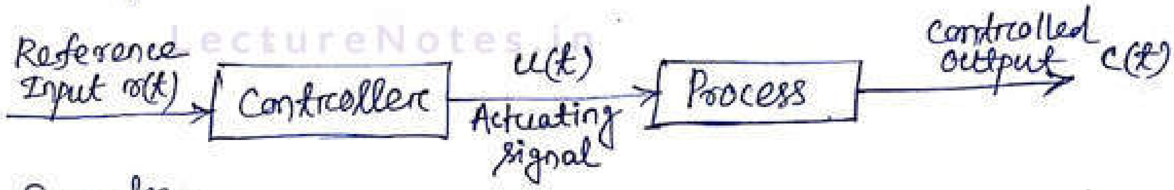
** If any eqn contains any matemadical operation on time or time is multiplied with any term, then it is TV, otherwise TIV.

(II) Open loop & Closed loop System :-

* OPEN LOOP SYSTEM :-

→ A system in which output is dependent on input but controlling or input is totally independent of the output or changes in output of the system, is called an Open loop system.

→ It is also called as non-feedback system.



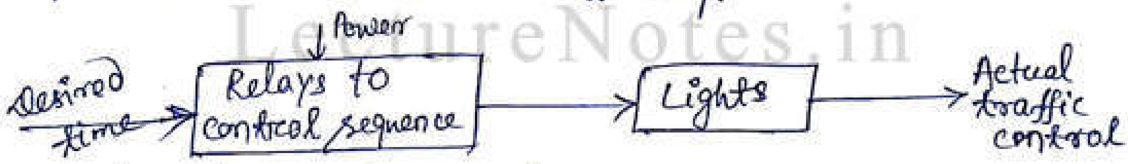
Examples :-

(i) Automatic Washing machine :- Here operating time is set manually. After the completion of time, the m/c will stop with the result we may or may not get the desired o/p (i.e, amount of cleanliness) of washed cloths because there is no feedback is provided.

(ii) A field control d.c motor

(iii) A bulb with electric switch. (Any change in light has no effect on ON-OFF pos^n of switch)

(iv) For automatic control of traffic lamp.



(v) Automatic Toaster system.

(vi) fan regulator.

Note :- The control system which operates on the time basis is open loop system. * OLCS are highly stable system, provided that external disturbances are zero.

Advantages :-

→ These Systems are simple in construction.

→ Less maintenance is required and easy also.

→ Not troubled with the problems of stability.

→ Systems are economical.

Disadvantages :-

→ Systems are inaccurate and unreliable.

→ Gives inaccurate results if there are varitions in the external environment.

→ Can't sense internal disturbances.

→ Optimization is not possible.

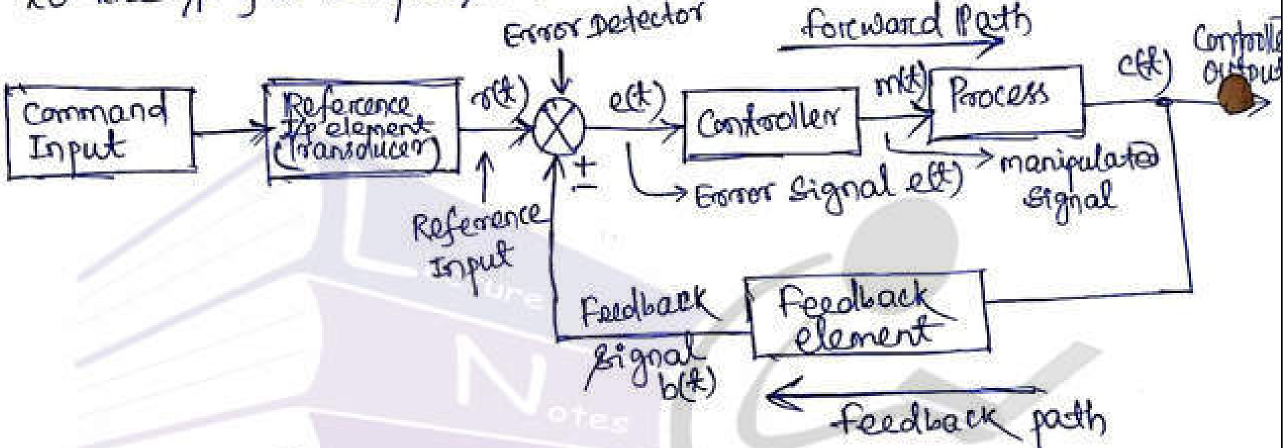
⑥

CLOSED LOOP CONTROL SYSTEM :-

→ It is also known as feedback control system.
→ A system in which control action dependent on the output or change in o/p is called closed loop c/s.

Feedback :- It is a propeirty of the system by which it permits the output to be compared with the reference i/p to generate the error signal based on which the appropriate controlling action can be decided.

→ So, in closed loop system, o/p or part of the output is feedback to the i/p for comparision with the reference input.

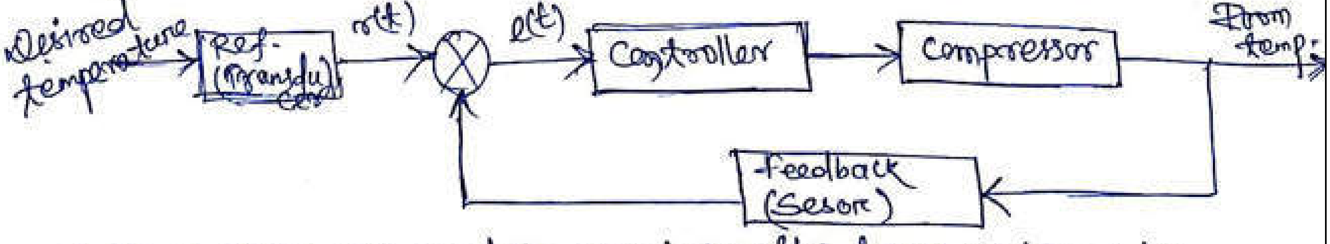


[Basic closed loop control System]

→ Here, error signal, $e(t) = r(t) \pm b(t)$. When feedback sign is positive, systems are called positive feedback system and if it is negative, systems are called negative feedback system.

Examples:-

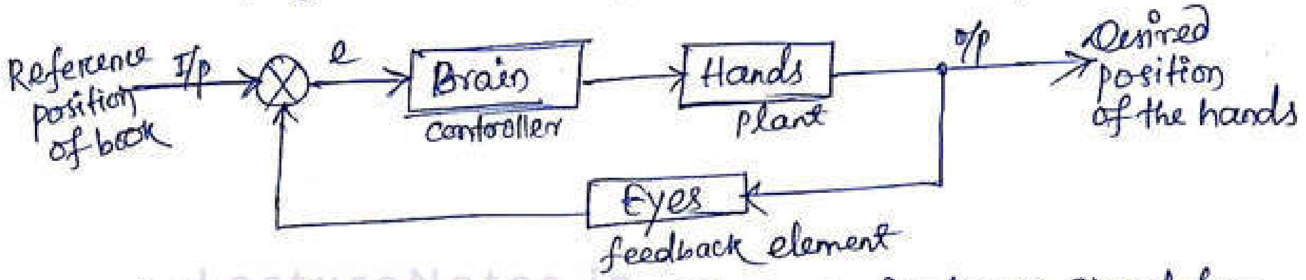
(i)



• In a room, we need to regulate the temp. and humidity. Acs are provided with thermostats. By measuring the actual room temp. & compared with the desired temp., an error signal is produced. The thermostat turns on/off the compresser.

(ii) Human being :-

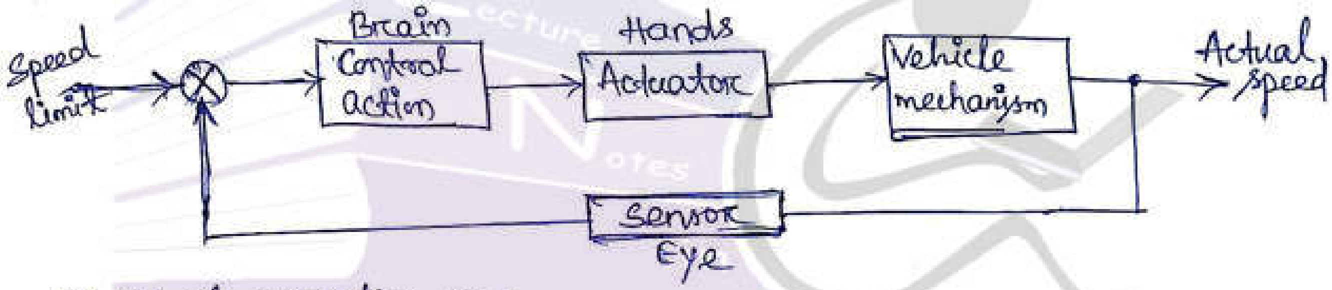
• If a person wants to reach for a book on the table, the closed loop system can be represented as following figure:-



→ Position of book is given as reference. Feedback signal from eyes compares the actual position of hands with reference position. Error signal is given to brain. Brain manipulates this error & gives to the hands. This process continues till the position of the hands get achieved appropriately.

(iii) Home heating system.

(iv) Manual speed control system for a locomotive:-



(v) Missile Launching System

(vi) Automatic Electric Iron.

(vii) Temperature control system.

Advantages :-

→ These systems are more reliable.

→ Accuracy of such systems is very high because controller modifies & manipulates the actuating signal such that error in the system will be zero.

→ These systems are faster.

→ It senses environmental changes or any disturbances & accordingly modifies the error.

→ optimization is possible.

Disadvantages :-

→ These systems are complicated & time consuming from design point of view and hence costlier.

→ Problem of stability is there due to feedback.

→ Maintenance is also difficult.

COMPARISON BETWEEN OPEN LOOP & CLOSED LOOP C/S :-

Open Loop C/S	Closed Loop C/S
1. Controlling action is independent of output.	1. Controlling action depends on the output.
2. Feedback element absent.	2. Feedback element is present.
3. Simple to construct & cheap.	3. Complicated to design & hence costly.
4. It is inaccurate and unreliable.	4. Highly accurate & reliable.
5. These are generally stable in nature.	5. Stability is the major consideration while designing.
6. Highly sensitive to disturbances.	6. Less sensitive to the disturbances.
7. Highly affected by non-linearities.	7. Reduced effect of nonlinearities.

SERVOMECHANISM :-

→ It is a feedback control system in which the output is a mechanical position or its time derivatives such as velocity or acceleration.

→ Ex :- position control system, power steering apparatus, Missile launchers, airplane autopilots.

→ This is also called as Tracking system.

REGULATING SYSTEMS (REGULATORS) :-

→ It is a feedback control system in which for a preset value of the reference input, the output is kept constant at its desired value.

→ In such systems, reference input remains constant for long periods. This fixed reference input is called set point.

→ The problems due to disturbances are mainly rectified by the regulator.

** → A regulator differs from a servomechanism in that the main funⁿ of regulator is to maintain a constant o/p for a fixed input while that of servomechanism is mostly to cause the o/p of the system to follow time varying input.

Control System Engineering

Topic:

Mathematical Model Of Physical Systems

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→ Examples :- Speed governer, temperature regulator, servostabilizer.

* MATHEMATICAL MODEL OF PHYSICAL SYSTEMS :-

→ The set of mathematical equations, describing the dynamic characteristics of a system is called mathematical model of the system.

→ Most of such mathematical equations are differential equations whether the system may be electrical, mechanical, thermal, hydraulic, etc.

MECHANICAL TRANSLATIONAL SYSTEM :-

→ In this, the motion is taking place along a straight line.

→ These systems are characterised by displacement, linear velocity & linear acceleration.

* → Newton's Law :

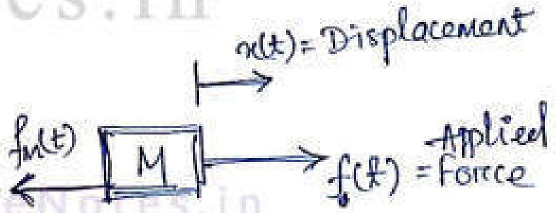
"Sum of forces applied on rigid body or system must be equal to sum of forces consumed to produce displacement, velocity & acceleration in various elements of the system."

→ There mainly three elements in translational motion systems.

(i) Mass (M) (ii) Spring (K) (iii) Friction (B)

1. MASS (M) :-

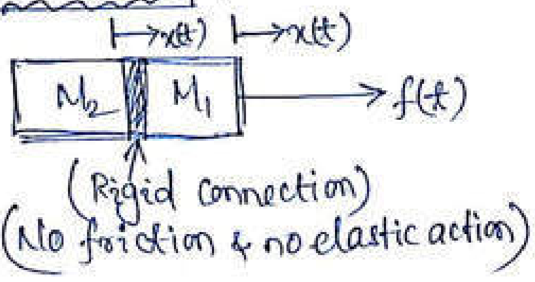
→ It stores kinetic energy of the translational motion. It can't store potential energy.



→ Let applied force f(t) produces displacement x(t) in the direction of force as shown in figure.

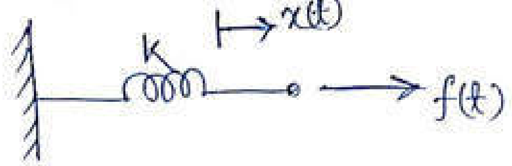
$$\therefore f(t) = Ma = M\frac{dv}{dt} \Rightarrow \boxed{f(t) = M\frac{d^2x(t)}{dt}}$$

→ The displacement of rigidly connected masses is always same.



2. SPRING (K) :-

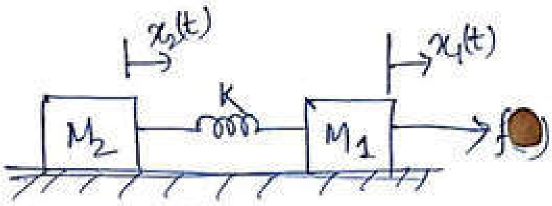
→ It may be an actual spring or spring action by elastic Cable or belt.



→ It stores potential energy

→ The force required to cause the displacement is proportional to the net displacement in the spring.

$\therefore \boxed{f(t) = K x(t)}$; K = spring constant.



→ Let's consider a spring is Connected in betn two masses.

→ M_1 will get displaced by $x_1(t)$ & M_2 will get displaced by $x_2(t)$, as Spring const. 'K' will store some potential energy.

→ opposing force by the spring is proportional to the net displacement i.e, $x_1(t) - x_2(t)$.

$\therefore \boxed{F_{spring} = K[x_1(t) - x_2(t)]}$.

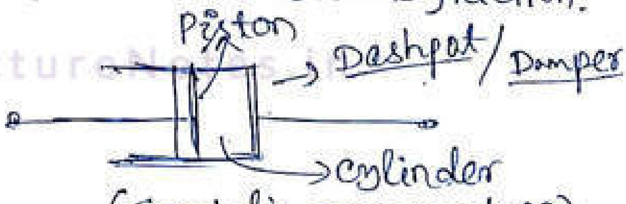
3. FRICTION (B) :-

→ Whenever there is a motion, there exists a friction.

→ It is non-linear in nature.

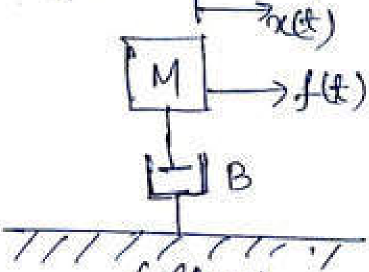
→ Divided into three types :-

(i) Viscous friction (ii) Static friction (iii) Coulomb friction.



(Symbolic representatn)

→ Consider a mass M having friction with a support with a const. 'B', represented by dash-point, as shown in fig(1).

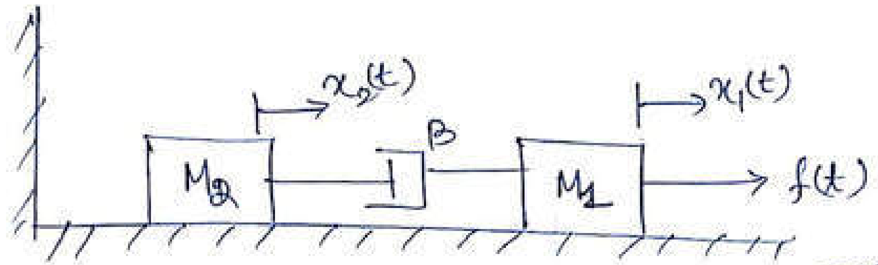


(fig 1)

→ Friction will oppose the motion of mass M & opposing force is proportional to velocity of 'M'.

$\therefore \boxed{F_{frictional} = B\frac{dx(t)}{dt}}$

where, B = Friction constant.

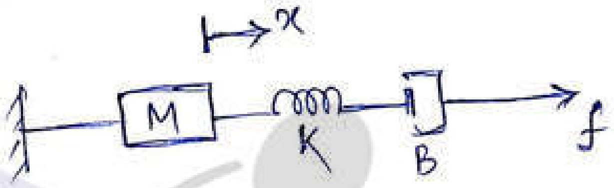


→ If friction is between two moving masses M_1 & M_2 as shown in above figure, then the opposition force is given by,

$$F_{frictional} = B\left[\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right].$$

Mass-Spring-Damper System :-

→ considering a mass spring damper system, as shown in figure, the mathematical model for this system is :-



$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t).$$

→ Taking Laplace transform :- (∵ Neglecting Initial condⁿˢ)

$$M s^2X(s) + BsX(s) + KX(s) = F(s)$$

$$\Rightarrow X(s)\left[Ms^2 + Bs + K\right] = F(s)$$

$$\Rightarrow \boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}}$$ → Transfer function of the translational mechanical system.

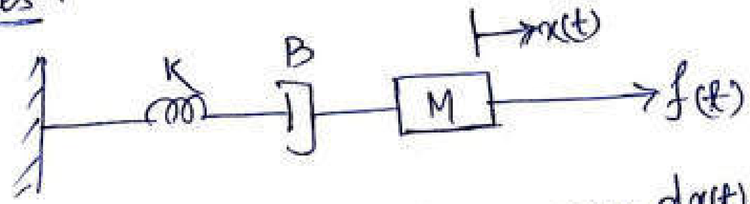
Equivalent Mechanical System (Node Basis) :-

Steps :-

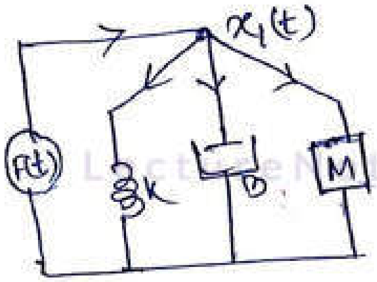
1. Identify the displacements in mechanical system due to applied force.
2. Identify the elements which are under the influence of different displacements.
3. Represent each displacement by separate node.
4. Show all the elements in parallel under the respective nodes which are under the influence of respective displacements.
5. Elements causing same change in displacement will get connected in parallel in betⁿ the respective nodes.
6.

Examples :-

①

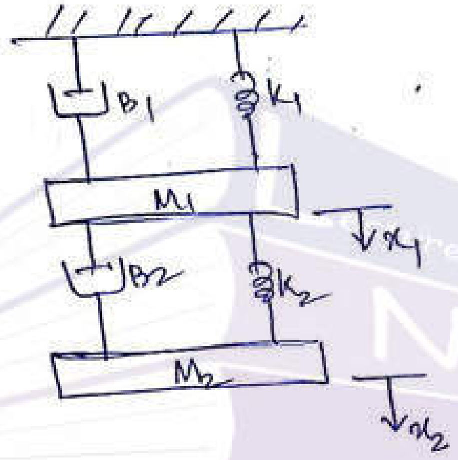


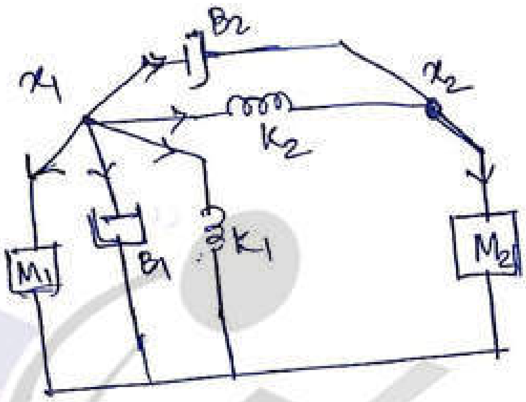
$$f(t) = K x(t) + B\frac{dx(t)}{dt} + M\frac{d^2x(t)}{dt}.$$



(Using nodal Analysis)

②



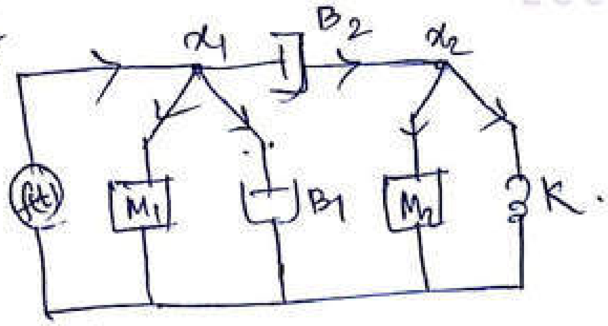


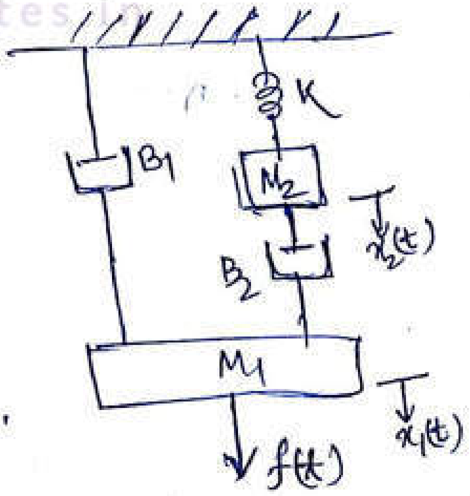
eqn ① $M_1\frac{d^2x_1}{dt} + B_1\frac{dx_1}{dt} + K_1 x_1 + B_2\frac{d}{dt}(x_1 - x_2) + K_2(x_1 - x_2) = 0.$

eqn ② $M_2\frac{d^2x_2}{dt^2} + B_2\frac{d}{dt}(x_2 - x_1) + K_2(x_2 - x_1) = 0$

③ Draw the equivalent mechanical system and write the mathematical model equations.

Soln:-





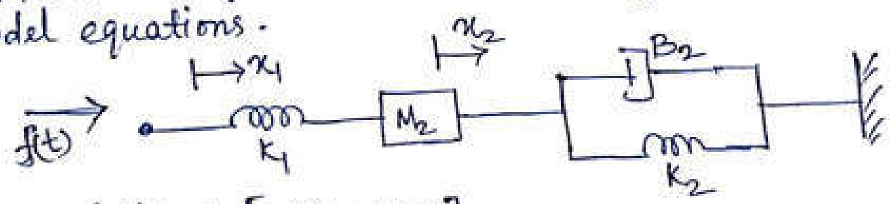
At node 1 :-

$$f(t) = M_1\frac{dx_1^2}{dt} + B_1\frac{dx_1}{dt} + B_2\frac{d}{dt}(x_1 - x_2).$$

At node 2 :-

$$0 = M_2\frac{dx_2^2}{dt} + K x_2 + B_2\frac{d}{dt}(x_2 - x_1).$$

④ Draw the equivalent mechanical system and write the mathematical model equations.

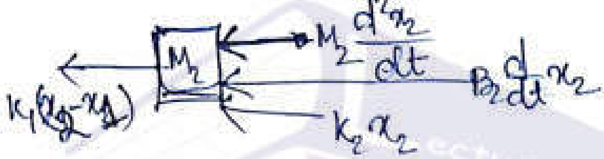


Ans :- $f_1(t) = K_1[x_1(t) - x_2(t)]$

$$0 = K_1[x_2(t) - x_1(t)] + M_2\frac{d^2x_2(t)}{dt^2} + K_2x_2(t) + B_2\frac{d}{dt}x_2(t).$$

By using freebody diagram method :-

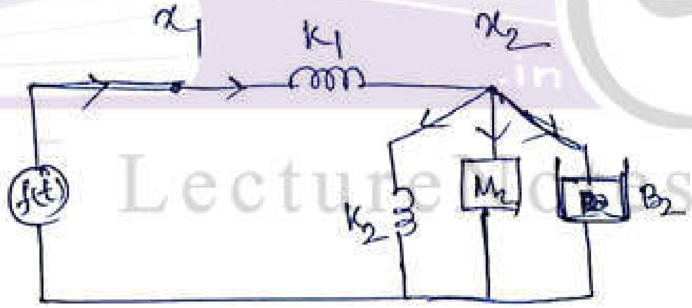
$\therefore f(t) = K_1(x_1 - x_2)$.



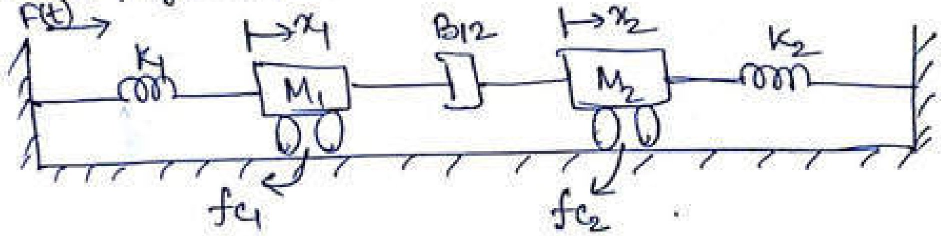
$\therefore K_1(x_2 - x_1) + M_2\frac{d^2x_2}{dt} + B_2\frac{dx_2}{dt} + K_2x_2 = 0$

$\Rightarrow 0 = K_1(x_2 - x_1) + M_2\frac{d^2x_2}{dt^2} + B_2\frac{d}{dt}x_2 + K_2x_2.$

By using Mechanical equivalent System method :-



⑤ Find the mathematical model equations for the following mechanical system in both the methods.



MECHANICAL ROTATIONAL SYSTEM :-

→ Rotational motion is the motion of a body about a fixed axis.

→ Here force is replaced by a moment about the fixed axis i.e, (force × distance from fixed axis) which is called as Torque.

→ Spring & friction behaves in same manner in rotational system. Only linear friction constant becomes torsional frictional constant (B) and linear spring constant becomes torsional spring constant (K).

→ The property of a system or body which stores kinetic energy in rotational system is called Inertia (J).

→ Opposing force due to inertia is proportional to the angular ~~displacement~~ acceleration (α).

$$\boxed{T_{\text{due to inertia}} = J\frac{d^2\theta(t)}{dt^2}}$$

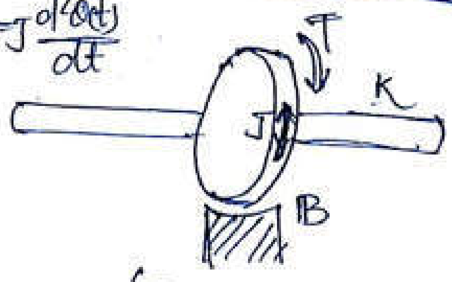
Analogous :-

Translational Motion	Rotational Motion
Mass (M)	Inertia (J)
Friction (B)	Friction (B)
Spring (K)	Spring (K)
Force (F)	Torque (T)
Displacement (x)	Angular displacement (θ)
Velocity, $v = \frac{dx}{dt}$	Angular velocity, $\omega = \frac{d\theta}{dt}$
Acceleration, $a = \frac{d^2x(t)}{dt}$	Angular acceleration, $\alpha = \frac{d^2\theta}{dt^2}$

→ Inertia Torque, $T_I(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt}$

Damping Torque, $T_B(t) = B\frac{d}{dt}\theta(t)$.

Spring Torque, $T_K(t) = K\theta(t)$.



(Basic Rotational System)

→ So, the mathematical model equation is :-

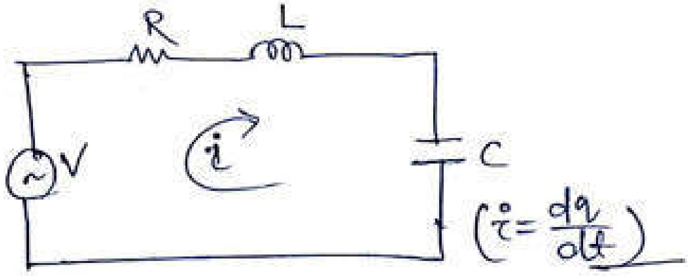
$$\boxed{T(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} + K\theta(t)}$$

(**) For any body, the algebraic sum of externally applied torques & the torques resisting rotation about any axis is zero.

ELECTRICAL SYSTEMS :-

→ Here, the dominant elements are :-
(i) Resistor (R) (ii) Inductor (L) (iii) capacitor (C).

→ Mathematical model equations are :-

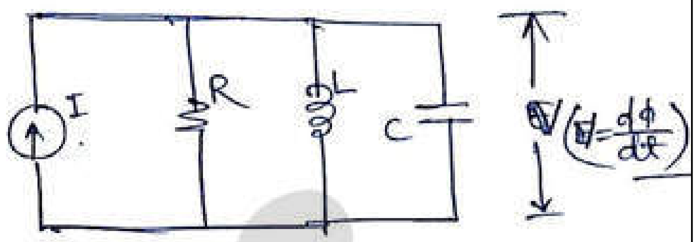


$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

$$\Rightarrow V = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}q.$$

$$I = \frac{V}{R} + \frac{1}{L}\int v\,dt + C\frac{dV}{dt}.$$

$$\Rightarrow I = \frac{1}{R}\left(\frac{d\phi}{dt}\right) + \frac{1}{L}\phi + C\frac{d^2\phi}{dt}$$



(Basic Electrical Systems)

ANALOGOUS SYSTEM :-

→ The systems whose differential equations are in the same form are called analogous system.

→ So, In between electrical & mechanical systems there exists analogy.

→ There are two methods of obtaining electrical analogous networks :
1) Force-Voltage Analogy (Direct Analogy)
2) Force-Current Analogy (Inverse Analogy)

1) Force-Voltage Analogy :-

$$F(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \longrightarrow \text{Mechanical Translational System}$$

$$V(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q \longrightarrow \text{Electrical System}.$$

$$T(t) = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta. \longrightarrow \text{Rotational System}$$

Table :-

Translational	Rotational	Electrical
Force (F)	Torque (T)	Voltage (V)
M	J	L
B	B	R
K	K	1/C
x	θ	q
$\dot{x} = dx/dt$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	$\dot{i} = \frac{dq}{dt}$

Force-Current Analogy :-

$$F(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \longrightarrow \text{Mechanical Translational system}$$

$$T(t) = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta \longrightarrow \text{Rotational system.}$$

$$I(t) = C\frac{d^2\phi}{dt} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi \rightarrow \text{Electrical System.}$$

Table:

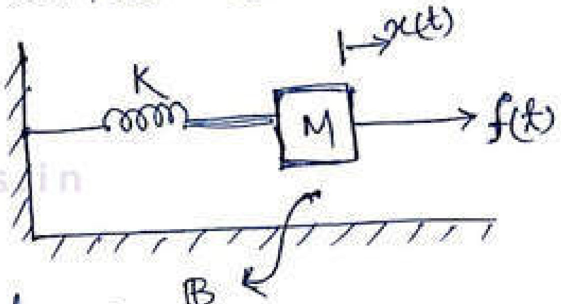
Translational	Rotational	Electrical
Force (F)	Torque (T)	Current (I)
M	J	C
B	B	$1/R$
K	K	$1/L$
x	θ	ϕ
$\dot{x}$ = Velocity = $\frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	voltage, $V = \frac{d\phi}{dt}$

SOLVING OF ANALOGOUS SYSTEM :-

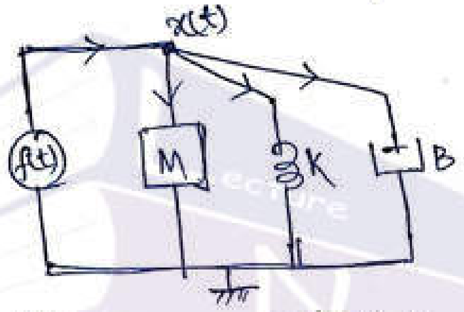
Steps :-

1. Identify all the displacements due to applied force.
2. Draw the equivalent mechanical system based on node basis.
3. Write the equilibrium equations.
4. In F-V analogy, use the following replacement :
   $F \to V$, $M \to L$, $B \to R$, $K \to 1/C$, $x \to q$, $\dot{x} = \frac{dx}{dt} = i$
5. Draw the electrical diagram using Loop method.
   (No of displacements = No of loop currents)
6. In F-I analogy, use the following replacement:
   $F \to I$, $M \to C$, $B \to 1/R$, $K \to 1/L$, $x \to \phi$, $\dot{x} = \frac{dx}{dt} = e$.
7. Draw the electrical n/w using node basis.
   (No of displacements = No of node voltages).

Q// For the physical system shown below draw its equivalent system & write equilibrium equations. Hence draw its electrical analogous circuits based on
(i) Force-voltage (ii) force-Current method.



Sol^n:- Equivalent mechanical system is :-

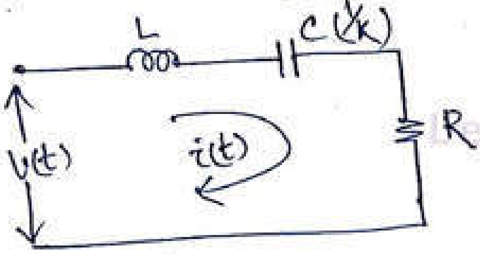


The equilibrium equations :-

$$f(t) = M\frac{d^2x(t)}{dt^2} + Kx(t) + B\frac{dx(t)}{dt}.$$

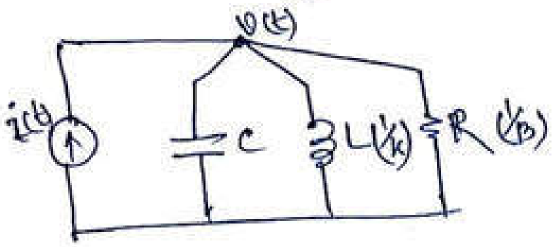
(i) F-V Analogy :-

$$v(t) = L\frac{di}{dt} + \frac{1}{C}\int i\,dt + Ri$$



(ii) F-I Analogy :-

$$i(t) = C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int v(t)dt.$$

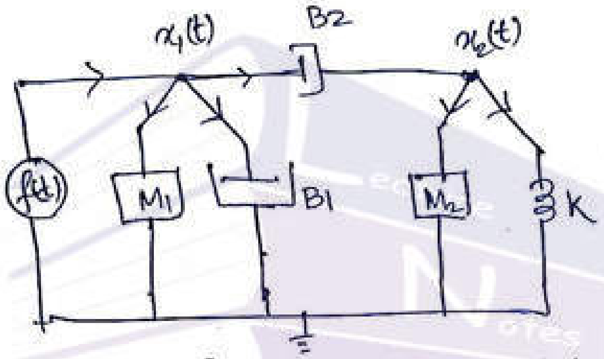


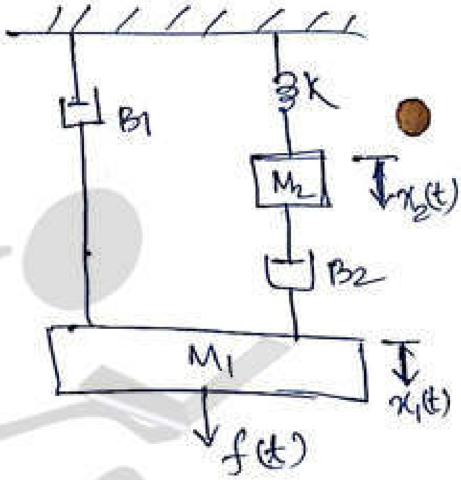
Note:-

* The elements in parallel in eq. mechanical system appears in series in F-V analogous system. Similarly elements in series in mechanical system appears in parallel in F-V analogy.
* The eq. mechanical system & the F-I analogous system are exactly identical.

Q// Draw the electrical analogous circuit for the following mechanical system using
(i) F-V analogy (ii) F-I analogy.

Soln :- Equivalent Mechanical System :-





$$f(t) = M_1\frac{d^2x_1}{dt^2} + B_1\frac{d}{dt}(x_1) + B_2\frac{d}{dt}(x_1 - x_2)$$

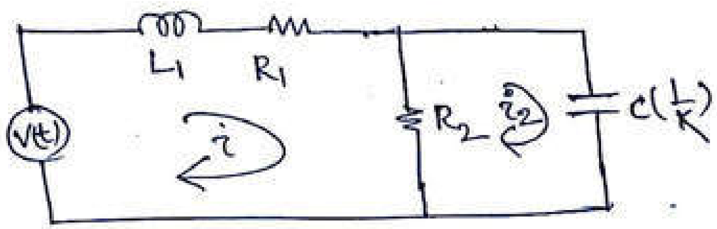
$$0 = M_2\frac{d^2x_2}{dt^2} + Kx_2 + B_2\frac{d}{dt}(x_2 - x_1).$$

(i) F-V analogy :-

$$V(t) = L_1\frac{di_1}{dt} + R_1 i_1 + R_2(i_1 - i_2)$$

$$0 = L_2\frac{di_2}{dt} + \left(\frac{1}{C}\right)\int i\,dt + R_2(i_2 - i_1)$$

Electrical circuit :- (Loop method) :-

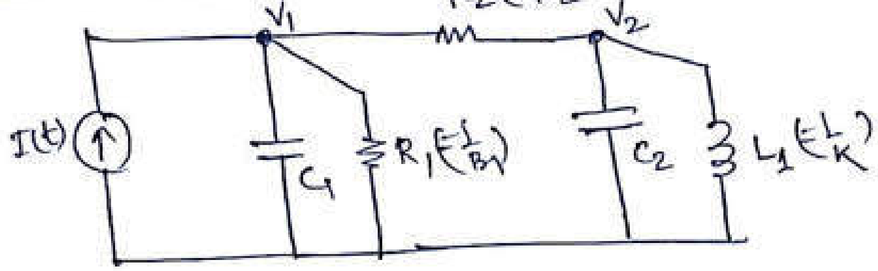


(ii) F-I Analogy :-

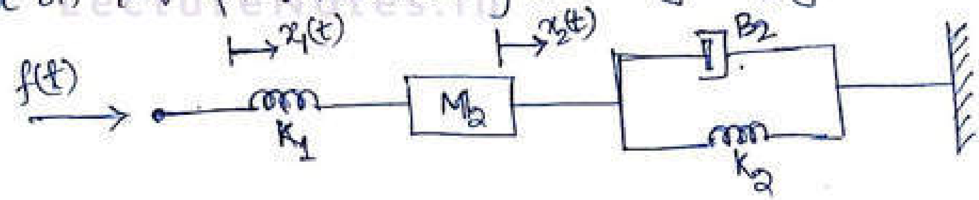
$$I(t) = C_1\frac{dV_1}{dt} + \frac{1}{R}(V_1) + \frac{1}{R_2}(V_1 - V_2).$$

$$0 = C_2\frac{dV_2}{dt} + \frac{1}{L}\int v\,dt + \frac{1}{R}(V_2 - V_1).$$

Electrical circuit :-



Q/ Draw the equivalent mechanical system and analogous system based on F-V & F-I methods for the given system.

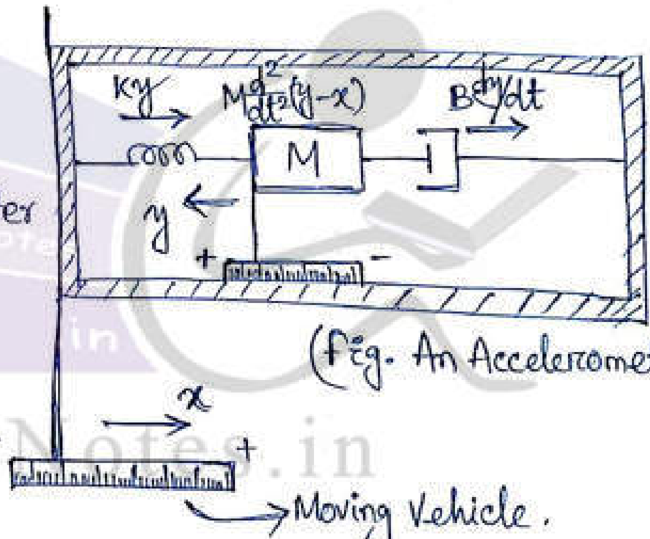


MECHANICAL ACCELEROMETER :-

→ An accelerometer consists of a spring-mass-dashpot system as shown in fig.

→ The frame of the accelerometer is attached to the moving vehicle.

→ When the moving vehicle accelerates, the frame of the accelerometer accelerates, so the mass attached to this is displaced in the opposite direction and the spring deflects.



(fig. An Accelerometer)

→ Let x = Displacement of the vehicle with respect to fixed reference frame.

y = Displacement of the mass with respect to accelerometer frame.

→ Force on the mass due to the spring is $-Ky$ & force due to viscous friction is $-B\,dy/dt$.

→ The motion of the mass with respect to fixed reference frame in the +ve direction $(y-x)$.

So, $M\frac{d^2(y-x)}{dt^2} + Ky + B\frac{dy}{dt} = 0$

$\Rightarrow M\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky = M\frac{d^2x}{dt^2} = Ma$

$\Rightarrow a = \frac{d^2y}{dt^2} + \frac{B}{M}\frac{dy}{dt} + \frac{K}{M}y$

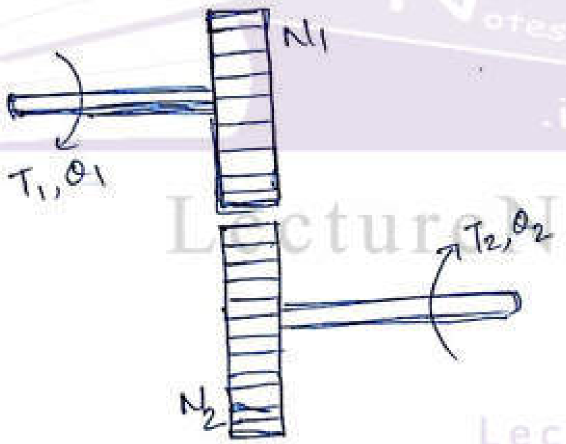
∴ If 'a' is constant, then 'y' will be also constant,

So, $a = 0 + 0 + \frac{K}{M}y \Rightarrow \boxed{a = (K/M)y}$

∴ Acceleration of the vehicle linearly varies with the displacement of the mass in accelerometer depending on the values of 'K' & 'M'.

GEAR TRAIN :-

→ It is a mechanical device which transfers energy from one part of the system to other part, in such a way that force, torque, speed & displacement may be altered.



[Gear System]

N = No of teeth on the circumference of gear wheel.

r = radius of gear wheel (m)

T = Torque (N-m)

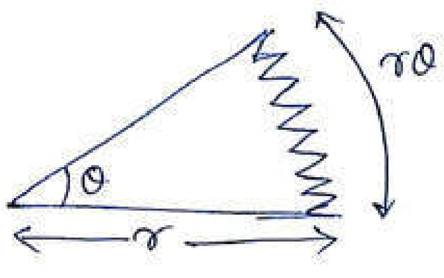
θ = Angular displacement (radians)

→ No of teeth are proportional to the radius of gear.

So, $r_1 N_2 = r_2 N_1 \Rightarrow \boxed{\frac{r_1}{r_2} = \frac{N_1}{N_2}}$

→ Distance travelled on each gear is same.

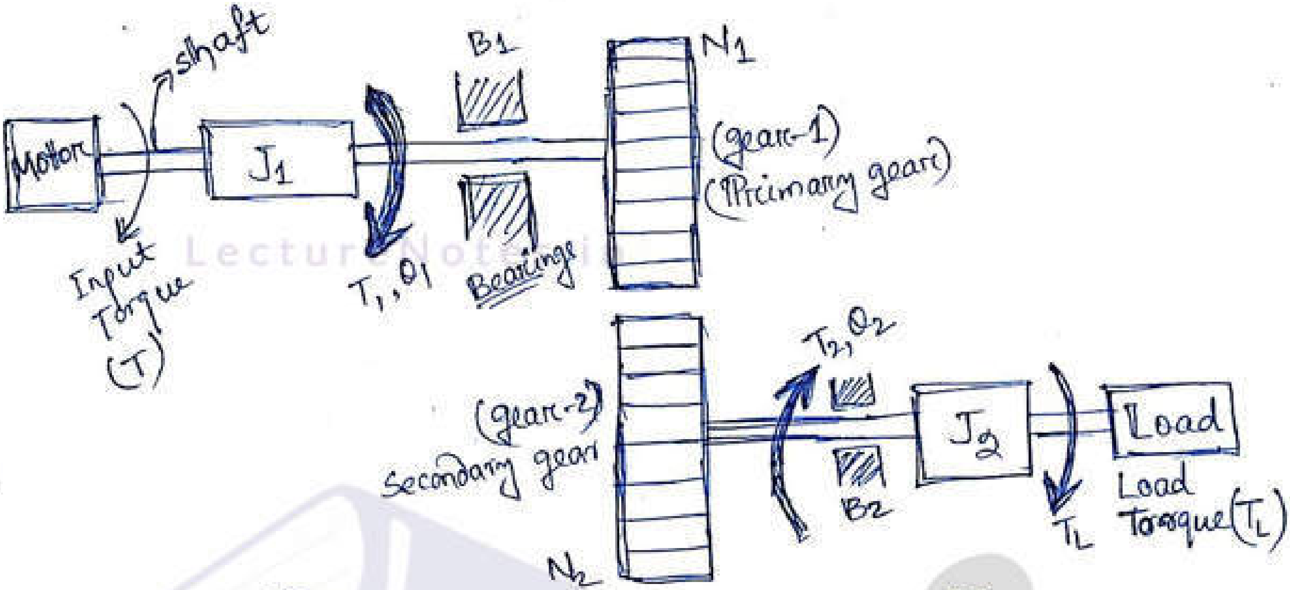
So, $\boxed{\theta_1 r_1 = \theta_2 r_2} \Rightarrow \boxed{\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2}}$



→ Work done by each gear is same.

∴ $T_1\theta_1 = T_2\theta_2 \Rightarrow \boxed{\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1}}$

→ Let's consider a practical gear arrangement connected to the load, as shown in figure.



[Practical Gear Train System]

→ Torque eqn of side 1 is :-

$$T = J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + T_1(t) \quad \text{——(1)}$$

Torque eqn of side 2 is :-

$$T_2 = J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + T_L(t) \quad \text{——(2)}$$

∴ We know that, $\frac{T_1}{T_2} = \frac{N_1}{N_2} = \frac{\theta_2}{\theta_1} \Rightarrow \boxed{T_2 = \frac{N_2}{N_1} T_1}$

Substituting 'T_2' in eqn (2) :-

$$\frac{N_2}{N_1} T_1 = J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + T_L$$

$$\Rightarrow T_1 = \frac{N_1}{N_2} J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L \quad \text{——(3)}$$

Substituting T_1 in eq(1) :-

$$T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L .$$

Substituting $\boxed{\theta_2 = \frac{N_1}{N_2} \theta_1}$.

$$\therefore T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \cdot \frac{N_1}{N_2} \frac{d^2\theta_1}{dt^2} + \frac{N_1}{N_2} B_2 \cdot \frac{N_1}{N_2} \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_L .$$

$$\Rightarrow T = \left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2\right]\frac{d^2\theta_1}{dt^2} + \left[B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2\right]\frac{d\theta_1}{dt} + \frac{N_1}{N_2}T_L.$$

$$\Rightarrow \boxed{T = J_{1e}\frac{d^2\theta_1}{dt^2} + B_{1e}\frac{d\theta_1}{dt} + \left(\frac{N_1}{N_2}\right)T_L}$$

Where, $J_{1e} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2$ = Equivalent Inertia referred to primary side.

$B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2$ = Eq. friction referred to primary side.

Similarly, by referring to load side, where applied torque gets transferred to load as $\left(\frac{N_2}{N_1}\right)T$, the eq^n will be :-

$$\boxed{\left(\frac{N_2}{N_1}\right)T = J_{2e}\frac{d^2\theta_2}{dt^2} + B_{2e}\frac{d\theta_2}{dt} + T_L}$$

Where, $J_{2e} = J_2 + \left(\frac{N_2}{N_1}\right)^2 J_1$ & $B_{2e} = B_2 + \left(\frac{N_2}{N_1}\right)^2 B_1$.

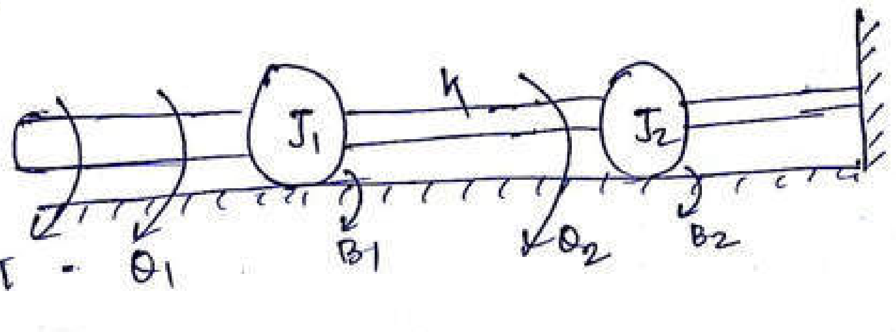
Use of Gear Train :-

→ In a control system, the load demands high torque at low speed while the motor supplies low torque at high speed.

→ In such cases, a gear train is used having a gear ratio $N_2/N_1 > 1$, N_1 = gear on the motor side
N_2 = gear on the load side.

→ Gears are used for stepping up or stepping-down either torque or speed. So, it is mechanically analogous to electrical transformer.

※ Q: Obtain the mathematical model equation for the following figure & draw the torque-voltage analogy.

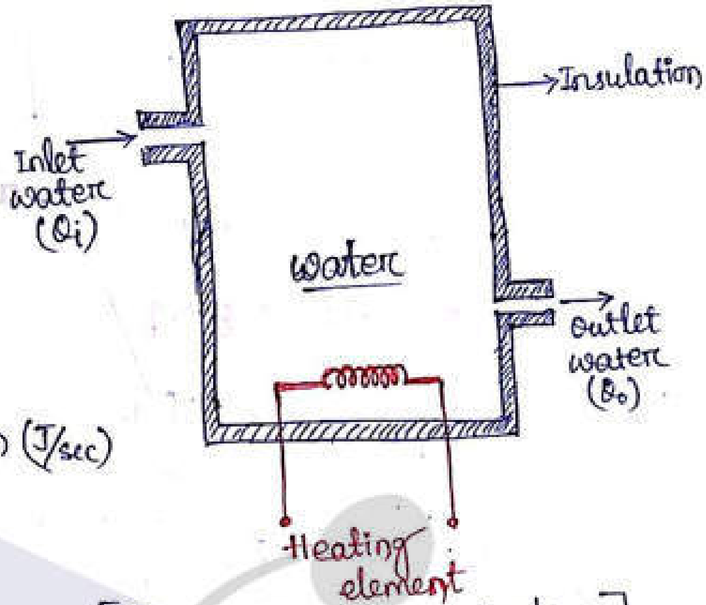


* DIFFERENT PHYSICAL COMPONENTS *

THERMAL SYSTEM :-

→ A thermal system is used for heating flow of water. There is transfer of heat from one substance to another substance.

→ Assume,
there is no heat store in the insulation & water having uniform temperature.



[Fig. Heat Transfer System]

Let, θ_i = temp of Inlet water (°C).

θ_o = temp. of outlet water (°C)

θ = temp. of surroundings (°C)

q = energy i/p to the system (J/sec)
= rate of heat flow from heating element.

C = thermal capacitance (J/°C)

R = thermal resistance (°C/J-s^{-1}).

∴ The rate of heat flow for the water in the tank is :→

$$q_i = C\frac{d\theta_o}{dt} \quad \text{--- (1)}$$

The rate of heat flow from the water to the surrounding is →

$$q_t = \frac{\theta_o - \theta}{R} \quad \text{--- (2)}$$

According to heat transfer principle :→

$$q = q_i + q_t = C\frac{d\theta_o}{dt} + \frac{\theta_o - \theta}{R}$$

$$\Rightarrow q = C\frac{d\theta_o}{dt} + \frac{\theta_o}{R} - \frac{\theta}{R}.$$

Neglecting 'θ/R' term as ambient temp. is very small than 'θ_o' :→

$$q = C\frac{d\theta_o}{dt} + \frac{\theta_o}{R} \quad \text{--- (3)}$$

Taking laplace transform of eqn (3) :→

$$Q(s) = sC\,\theta_o(s) + \frac{\theta_o(s)}{R} = \frac{\theta_o(s)}{R}(1+RsC)$$

So by relating the water temperature $[\theta_o(s)]$ as output and the input heat energy $[Q(s)]$, the transfer function of the system can be written as :→

$$\frac{Q(s)}{\theta_o(s)} = \frac{1+RCs}{R} \Rightarrow \boxed{\frac{\theta_o(s)}{Q(s)} = \frac{R}{(RCs+1)}}$$

↳ Transfer function of thermal system.

∴ Also, the time constant of thermal system is '<u>RC</u>'.

FLUID SYSTEM :-

→ In fluid system; if the fluid flow is linear or laminar the dynamics of the system is represented by ordinary linear differential equation & if the flow is turbulent, it is represented by non-linear differential eqⁿ.

→ In case of pipe flow the pressure drop in pipe section is :-

$$P = \frac{128 l\mu}{\pi D^4} Q = RQ \longrightarrow \text{for laminar flow.}$$

$$P = \frac{8K_t \rho l}{\pi^2 D^5} Q^2 = K_T Q^2 \longrightarrow \text{for turbulent flow.}$$

Where, L = Length of pipe section; D = diameter of pipe,
μ = viscocity ; Q = Volumetric flow rate.
ρ = density of the fluid.

→ The eqⁿ can be linearized as :-

$$P_o = K_T Q_o^2 \longrightarrow (1)$$

∴ Taylor series expansion of eqⁿ(1) is :-

$$P = P_o + \left.\frac{dP}{dQ}\right|_{(P_o, Q_o)} (Q - Q_o).$$

$$\Rightarrow P - P_o = 2K_T Q_o (Q - Q_o) \Rightarrow \boxed{\Delta P = R\Delta Q}$$

where, $R = 2K_T Q_o$ = turbulent flow resistance.

→ In fluid system, the another parameter is fluid capacitance.

→ The rate of fluid storage in the tank is $\boxed{A\frac{dH}{dt}}$.

$$\therefore \boxed{\frac{A}{\rho g}\frac{dP}{dt} = C\frac{dP}{dt}}$$

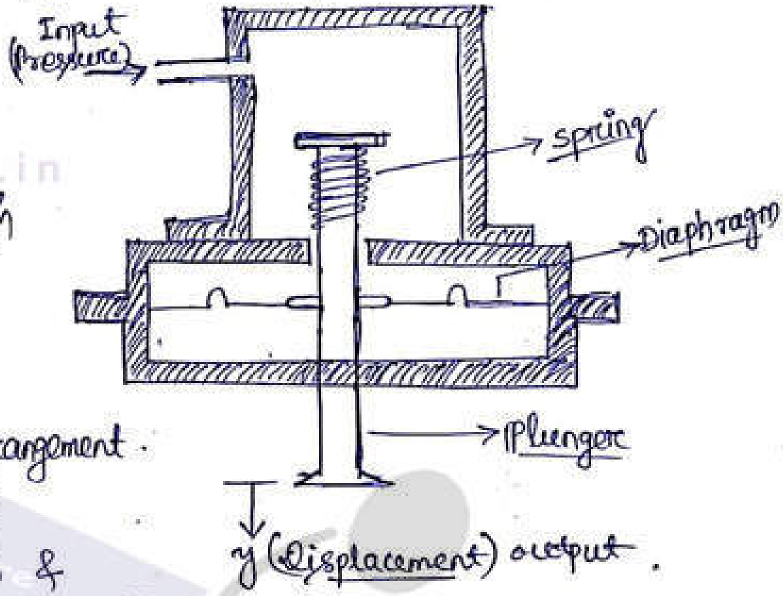
where, A = Cross-sectional area 'A'.
$C = A/\rho g$ = Capacitance of tank.

PNEUMATIC SYSTEM :-

→ Air medium is used in Pneumatic Control System.

→ Pneumatic actuating valve is used to obtain linear displacement of a plunger with pressurised air as input.

→ Air pressure pushes the diaphragm as well as plunger & thus the plunger is displaced through 'y' distance.



[fig. Pneumatic actuating valve]

Let, P_i = Air pressure

M = Mass of plunger arrangement.

f & K = Co-efficient of viscous friction & spring constant.

A = Area of diaphragm.

∴ So, force exerted on the system = AP_i.

→ This force is opposed by inertia force, viscous damping force & spring restoring force of the system.

So, $$M\frac{d^2y}{dt^2} + f\frac{dy}{dt} + Ky = AP_i \quad \text{——— (1)}$$

Taking Laplace transform of eqn (1) :-

$$Ms^2Y(s) + fsY(s) + KY(s) = AP_i(s)$$

∴ So, the transfer function relating the o/p displacement $Y(s)$ & input pressure $P_i(s)$ is given by :-

$$\boxed{\frac{Y(s)}{P_i(s)} = \frac{A}{Ms^2 + fs + K}}.$$

Control System Engineering

Topic:

Block Diagram And SFG

Contributed By:

Gyana Ranjan Biswal

BLOCK DIAGRAM

→ Block diagram is a pictorial representation of the functions performed by each component and of the flow of signals of a given system.

→ Advantages of block diagram representation of a system are:

(i) easy to form the overall block diagram & to find the transfer function.

(ii) possible to analyte the contribution of each component to the overall performance.

(iii) can be used as a tool for the analytic or computer solution of the system.

→ Any block diagram has the following four basic elements:

(i) Blocks (ii) Transfer functions of elements

(iii) Summing points (iv) Take off points.

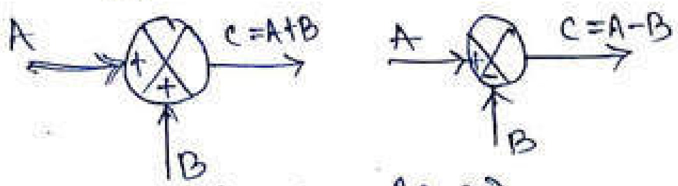
→ Blocks: The functional block or simple block is a symbol of an element of system for the mathematical operation on the input signal to the block that produces the o/p. The transfer functions of the components are usually inserted inside a block.

→ Transfer Function: It is the ratio of Laplace transform of the o/p variable to the laplace transform of the i/p variable by assuming all initial conditions are zero.

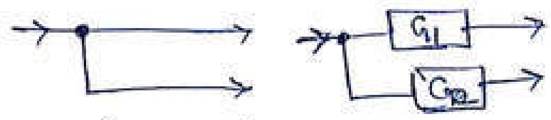
$$C(s) = R(s) \cdot G(s) \Rightarrow G(s) = \frac{C(s)}{R(s)} = \text{T.F. of the element.}$$

→ Summing Point:- The function of comparing the different signals is indicated by the summing point.

→ Take off Point:- It is a point from which the signal from a block goes unaltered to other blocks or summing points.



(Summing Point)



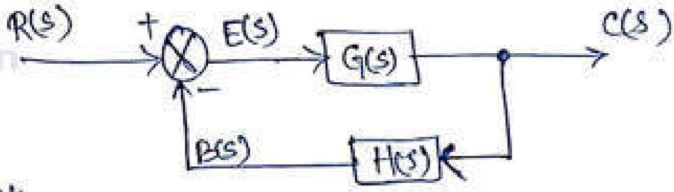
(Take off point)

②

Note:
1. Block diagram is an unilateral property of the system, as the signal can travel along the direction of arrow only.
2. The block diagram of a given system is not unique.

BLOCK DIAGRAM OF A CLOSED-LOOP CONTROL SYSTEM :-

→ A closed loop system is one in which o/p is fed back into an error detector and compared with the reference input. The feedback may be negative or positive.



[B.D. of closed Loop System]

⇒ Consider a negative feedback closed loop c/s as shown in figure.

$R(s)$ = Reference Input

$E(s)$ = Actuating signal or error signal

$G(s) = \frac{C(s)}{E(s)}$ = Forward path transfer function.

$C(s)$ = Controlled output signal.

$H(s)$ = Feedback transfer function.

$\therefore$ $C(s) = G(s) \cdot E(s)$ —(1)
$B(s) = H(s) \cdot C(s)$ —(2)
$E(s) = R(s) - B(s)$ —(3)
} from figure.

Put the value of $C(s)$ in eq^n(2) :-

$$B(s) = H(s) \cdot G(s) E(s) \Rightarrow \frac{B(s)}{E(s)} = G(s) \cdot H(s).$$

$$\therefore \boxed{\frac{B(s)}{E(s)} = G(s) H(s) = \text{open loop transfer function}}$$

Now, put $E(s)$ from eq^n(3) in eq^n(1) :-

$$C(s) = G(s)\left[R(s) - B(s)\right]$$

$$\Rightarrow C(s) = R(s) G(s) - G(s) B(s)$$

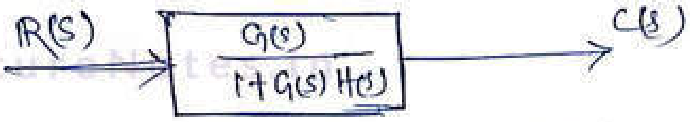
$$\Rightarrow C(s) = R(s) \cdot G(s) - G(s) \cdot H(s) \cdot C(s)$$

$$\Rightarrow C(s)\left[1 + G(s) H(s)\right] = R(s) G(s)$$

$$\Rightarrow \boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = T(s) = \text{closed loop Transfer function.}}$$

* For positive feedback system, $T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1-G(s)H(s)}$.

→ Now, the reduced block diagram is :-



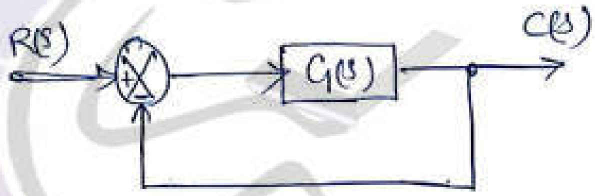
→ We have, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \Rightarrow \frac{G(s)E(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$.

$$\Rightarrow \boxed{\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \text{Error Ratio}}.$$

** For Unity feedback control system, $\boxed{H(s) = 1.}$

$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$, for negative feedback

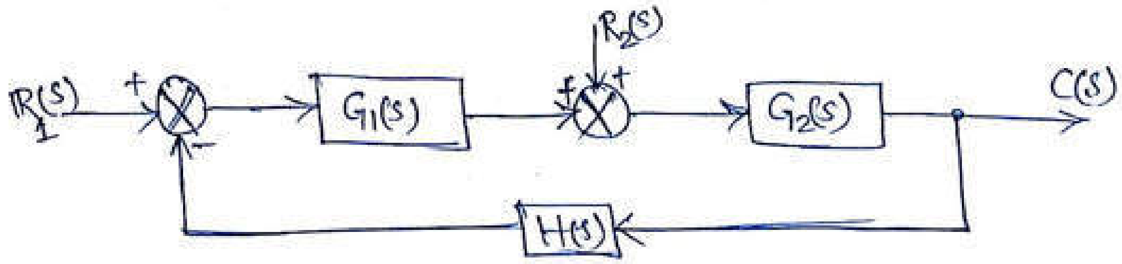
$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1-G(s)}$, for positive feedback.



[B.D. of Unity feedback c/s]

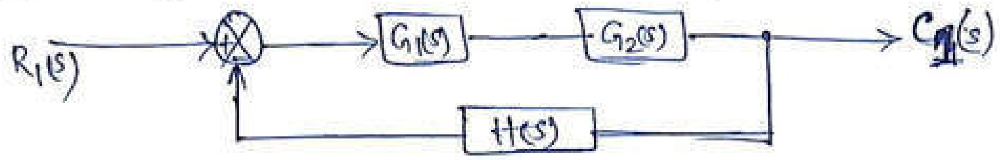
MULTI-INPUT-MULTI-OUTPUT SYSTEM (MIMO) :-

→ When multiple inputs are present in a linear system, each input can be treated independent of the others. Complete output of the system can be obtained by adding the effect of each input as per superposition.



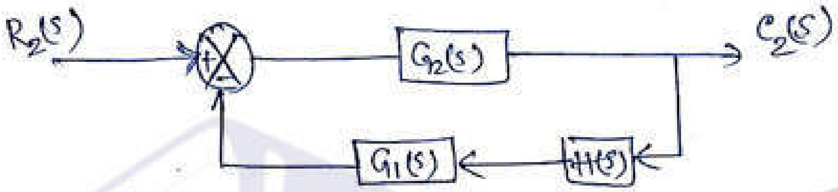
[Two Input Linear System]

→ By making $R_2(s) = 0$, the corresponding blockdiagram :-



$$\therefore \frac{C_1(s)}{R_1(s)} = \frac{G_1(s)}{R_1(s)} = \frac{G_1(s) \cdot G_2(s)}{1 + G_1(s) \cdot G_2(s) \cdot H(s)}$$

→ By making $R_1(s) = 0$, the corresponding block diagram :-



$$\therefore \frac{C_2(s)}{R_2(s)} = \frac{G_2(s)}{1 + G_1(s)\, G_2(s)\, H(s)}$$

→ The overall response due to both the inputs are :-

$$C(s) = C_1(s) + C_2(s)$$

$$\Rightarrow \boxed{C(s) = \frac{G_2(s)}{1 + G_1(s)\, G_2(s)\, H(s)} \left[R_1(s)\, G_1(s) + R_2(s) \right]}$$

BLOCK DIAGRAM REDUCTION :-

→ When a no of blocks are connected, the overall transfer fun^c can be obtained by block diagram reduction technique.

→ The rules for block diagram reduction are :-

Rule-1 : Blocks in Cascade

[Blocks in cascade & equivalence]

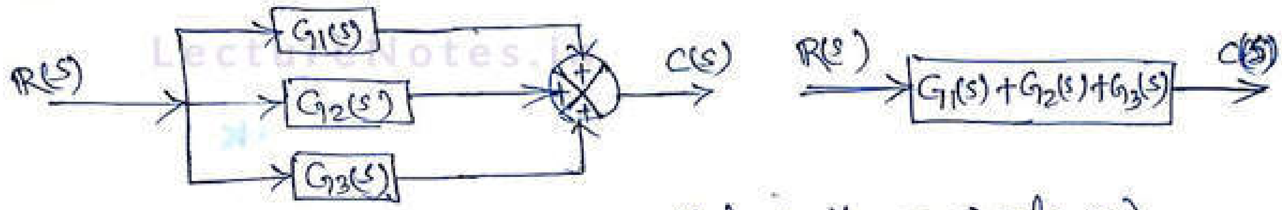
→ When two or more blocks are in cascade, the resultant block is a product of the individual block transfer function.

~~Rule-1~~ → From the above block diagram:-

$$\frac{C_1(s)}{R(s)} = G_1(s) \; ; \; \frac{C_2(s)}{C_1(s)} = G_2(s) \; ; \; \frac{C(s)}{C_2(s)} = G_3(s).$$

$$\therefore \frac{C(s)}{R(s)} = \frac{C_1(s)}{R(s)} \cdot \frac{C_2(s)}{C_1(s)} \cdot \frac{C(s)}{C_2(s)} = \boxed{G_1(s) \cdot G_2(s) \cdot G_3(s)}.$$

Rule-2 : Blocks in Parallel :-



(Blocks in parallel & its equivalence)

→ When two or more blocks are connected in parallel, then the resultant block is the sum of individual block transfer fun.

→ From figure:-

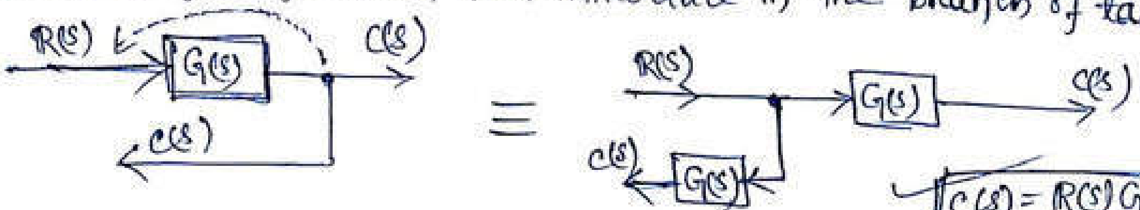
$$C(s) = R(s)G_1(s) + R(s)G_2(s) + R(s)G_3(s)$$
$$= R(s)\left[G_1(s) + G_2(s) + G_3(s)\right]$$

$$\therefore \boxed{\frac{C(s)}{R(s)} = G_1(s) + G_2(s) + G_3(s).}$$

Rule-3 : Interchanging two consecutive summing points.

Rule-4 : Moving a take off point ahead of a block.

→ If a take off point is moved ahead of a block, a block with same transfer function will introduce in the branch of takeoff point.

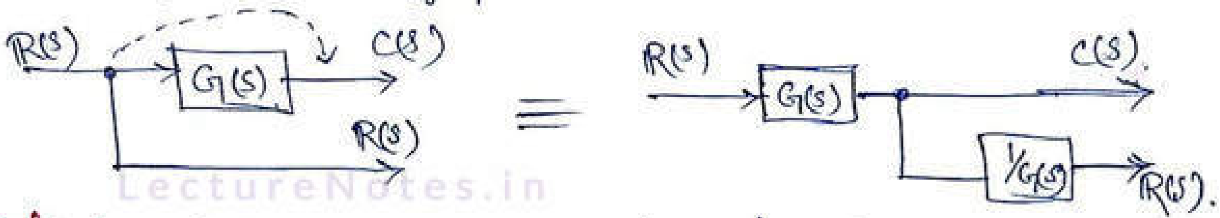


$$\boxed{C(s) = R(s)G(s)}$$

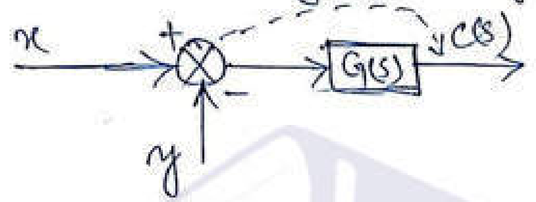
(6)

Rule-5: Moving a takeoff after the block.

→ If a takeoff point is moved after the block, a block with the reciprocal of the transfer function introduced in the branch of a take off point.



Rule-6: Moving a summing point beyond a block.



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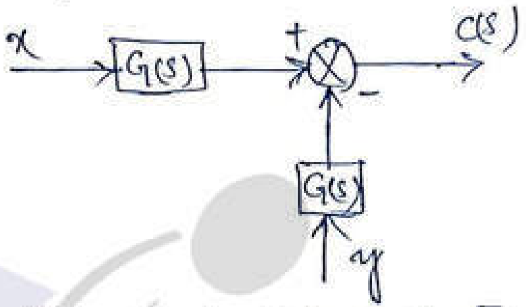
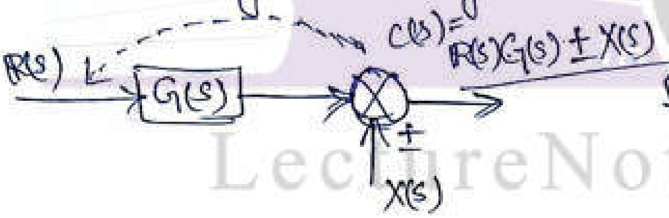
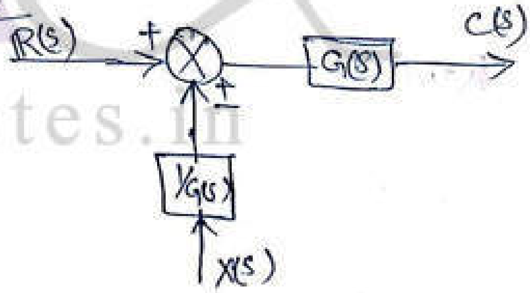


$C(s) = G(s)[x-y]$.

$C(s) = xG(s) - yG(s) = G(s)[x-y]$.

Rule-7: Moving a summing point before a block.

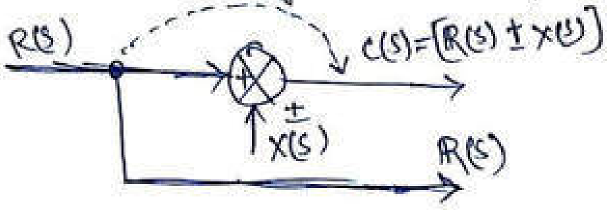


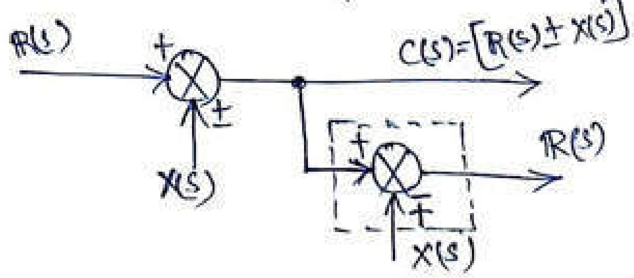


$C(s) = \left[R(s) \pm X(s) \cdot \frac{1}{G(s)}\right] G(s)$

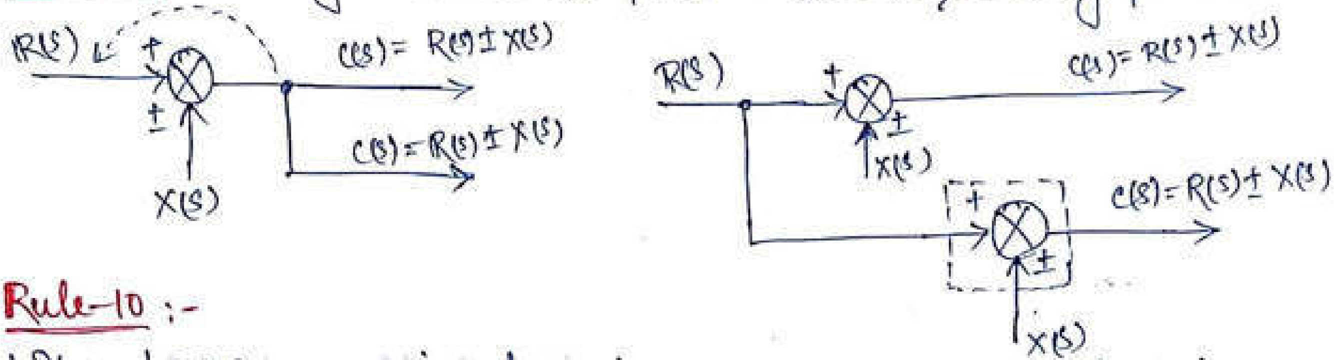
$\Rightarrow C(s) = R(s)\,G(s) \pm X(s)$.

Rule-8: Moving a take off point after the summing point.



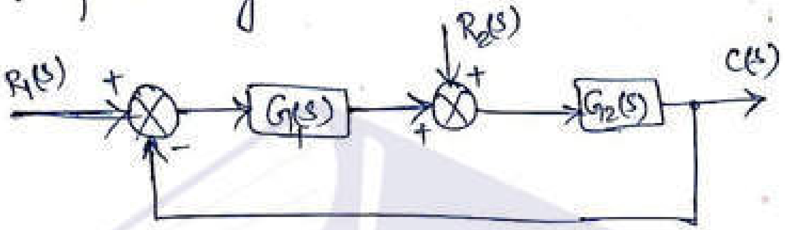


Rule-9: Moving a take off point before the summing point.



Rule-10 :-

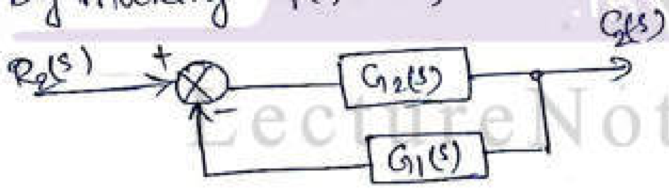
When two or more inputs act on a system, the total o/p is obtained by adding the effect of each individual input separately.



1. By making $R_2(s) = 0$;-

$$\therefore \frac{C_1(s)}{R_1(s)} = \frac{G_1(s)\, G_2(s)}{1 + G_1(s)\, G_2(s)} .$$

2. By making $R_1(s) = 0$;-



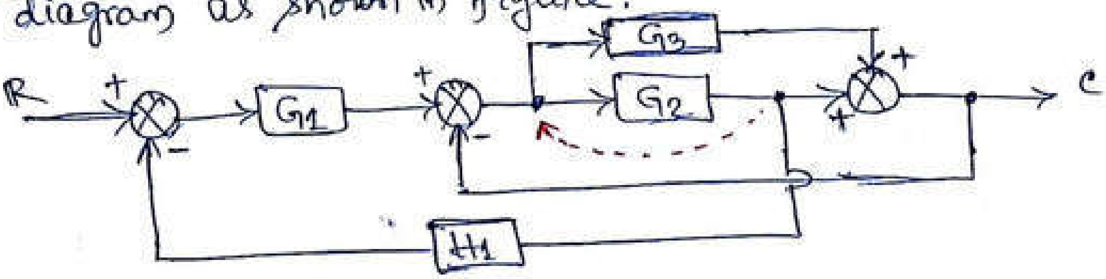
$$\therefore \frac{C_2(s)}{R_2(s)} = \frac{G_2(s)}{1 + G_1(s) G_2(s)} .$$

3. $C(s) = C_1(s) + C_2(s)$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{G_1(s) \cdot G_2(s)}{1 + G_1(s) \cdot G_2(s)} + \frac{G_2(s)}{1 + G_1(s) G_2(s)} = \frac{G_2(s)\left[1 + G_1(s)\right]}{1 + G_1(s) \cdot G_2(s)} .$$

Numericals :-

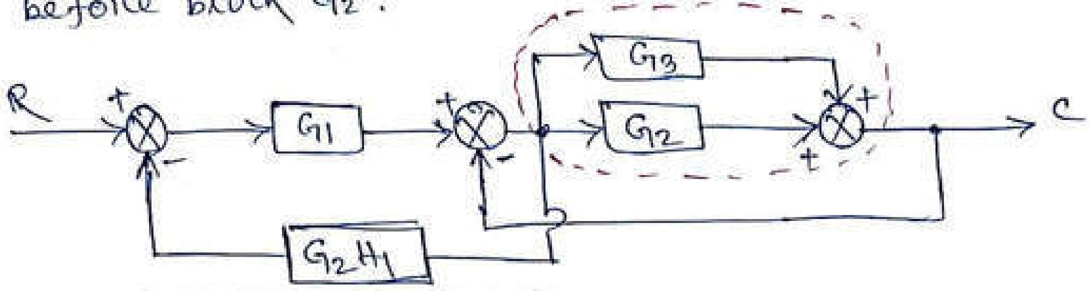
(1) Determine the transfer function C/R from the given block diagram as shown in figure.



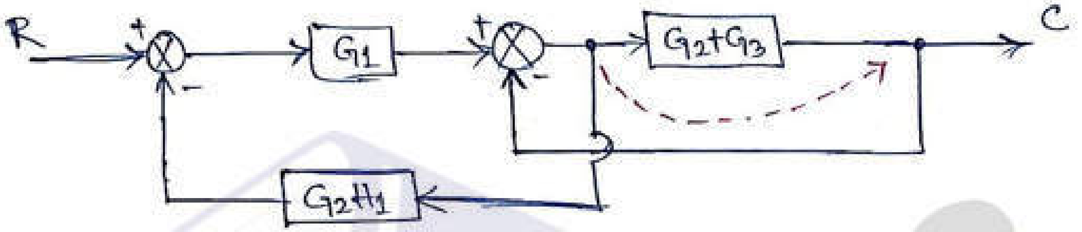
⑧

Solution :-

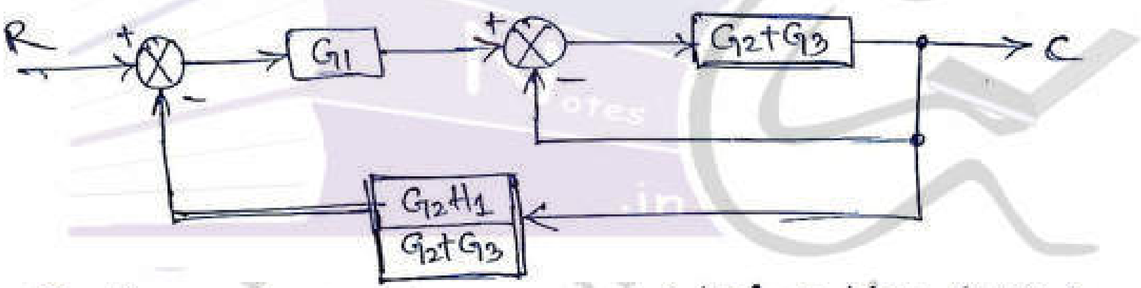
① shift the take off point after block G_2 to a position before block G_2.



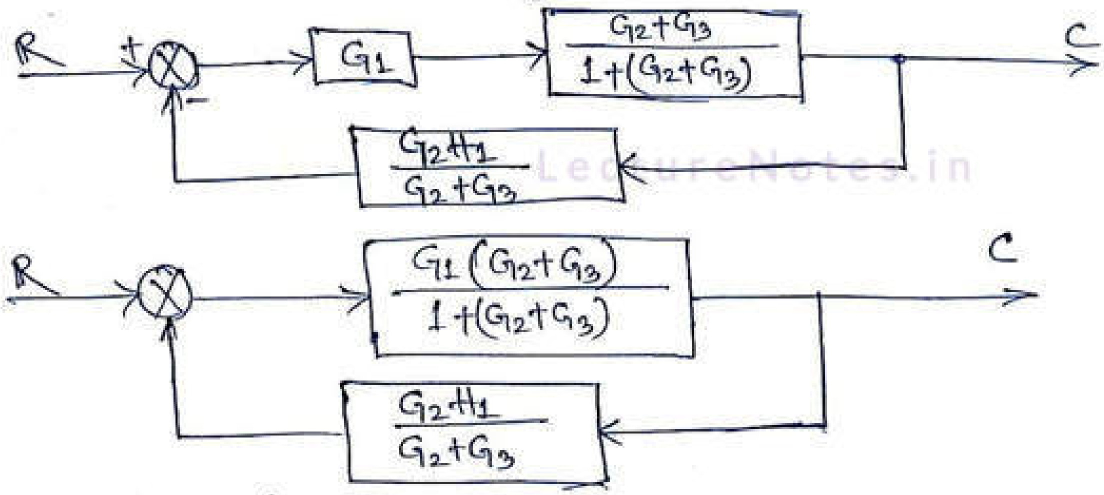
② Eliminate the summing point after block G_2 by Parallel Concept.



③ shift the take off point to a pos^n after block (G_2+G_3).

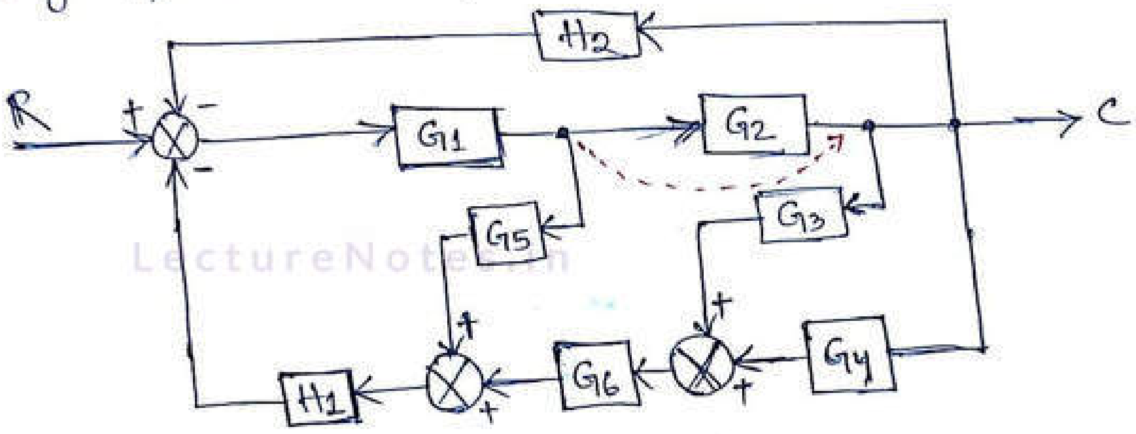


④ Eliminate the summing point before block (G_2+G_3).

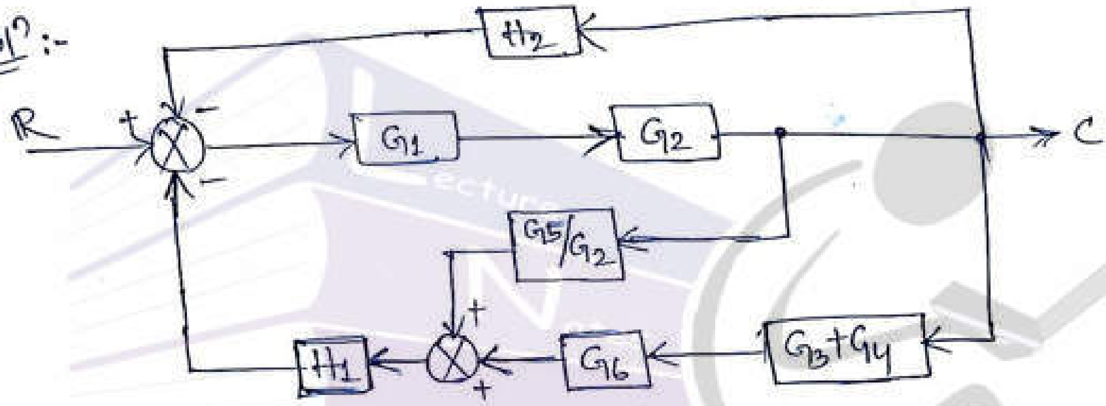


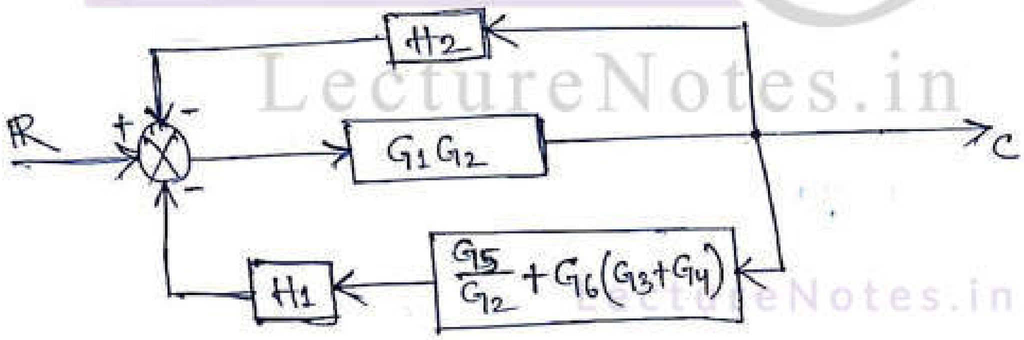
$$\therefore \frac{C}{R} = \frac{\dfrac{G_1(G_2+G_3)}{1+(G_2+G_3)}}{1+\left[\dfrac{G_1(G_2+G_3)}{1+(G_2+G_3)}\right]\cdot\left[\dfrac{G_2H_1}{G_2+G_3}\right]} = \boxed{\frac{G_1G_2+G_1G_3}{1+G_2+G_3+G_1G_2H_1}} \quad \text{(Ans)}$$

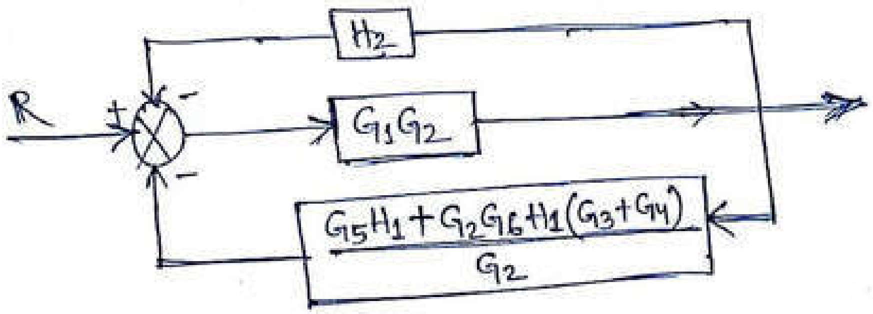
(Q) Determine the overall transfer function for the block diagram shown below,

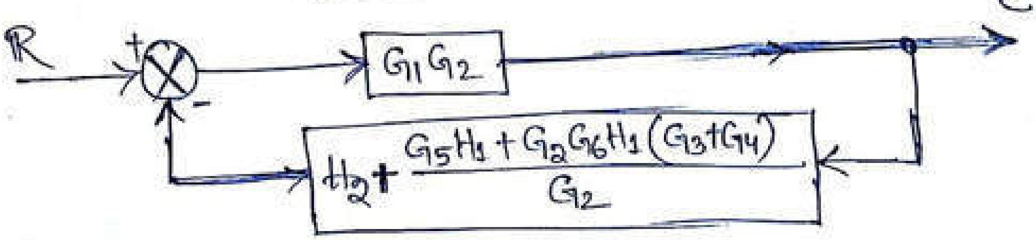


Sol^n :-





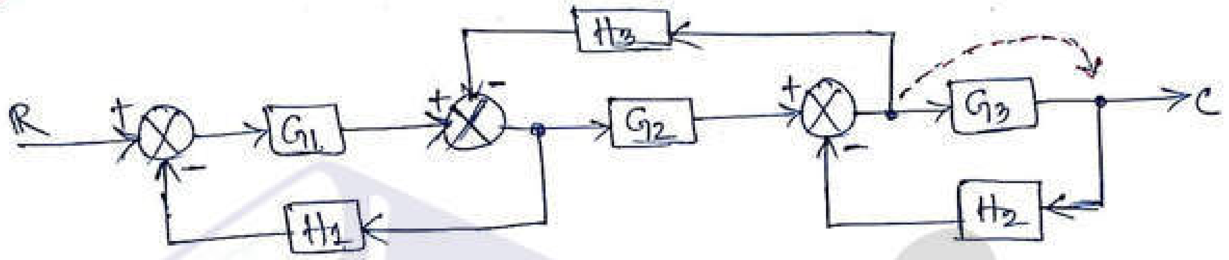




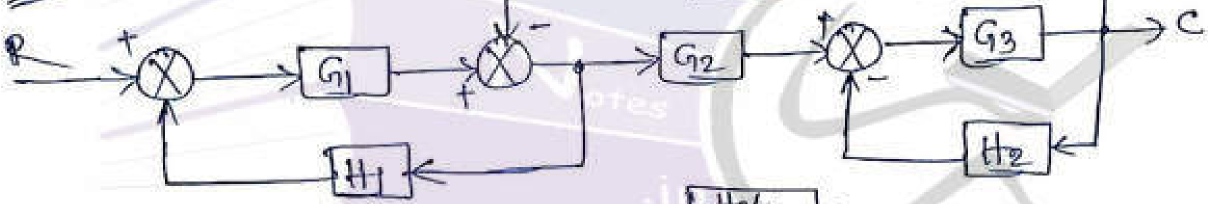
$$\therefore \frac{C}{R} = \frac{G_1G_2}{1+(G_1\cancel{G_2})\left[\frac{G_2H_2+G_5H_1+G_2G_6H_1(G_3+G_4)}{\cancel{G_2}}\right]}$$

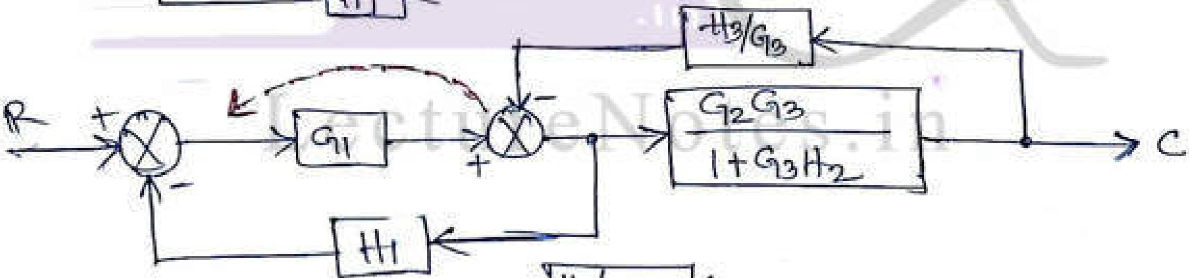
$$\Rightarrow \boxed{\frac{C}{R} = \frac{G_1G_2}{1+G_1G_2H_2+G_1G_5H_1+G_1G_2G_6H_1(G_3+G_4)}} \quad \text{. (Ans).}$$

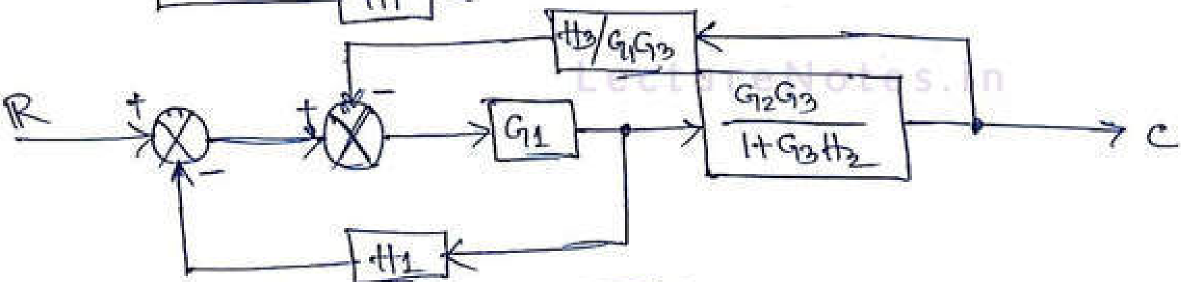
HW (3) Determine C/R for the block diagram given below:-

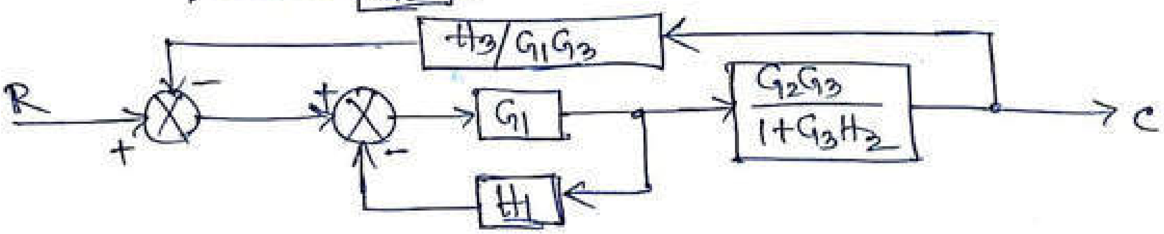


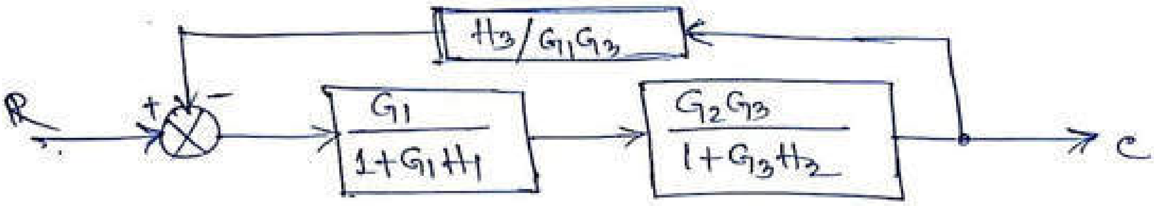
Sol^n:-







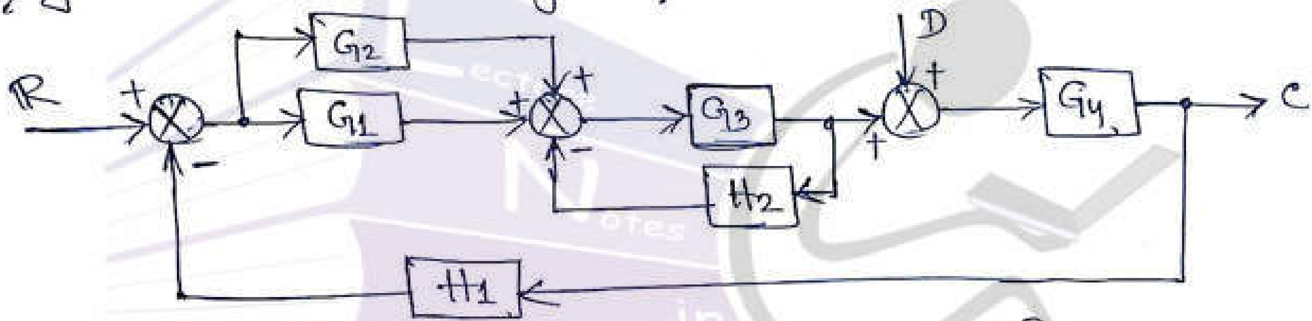




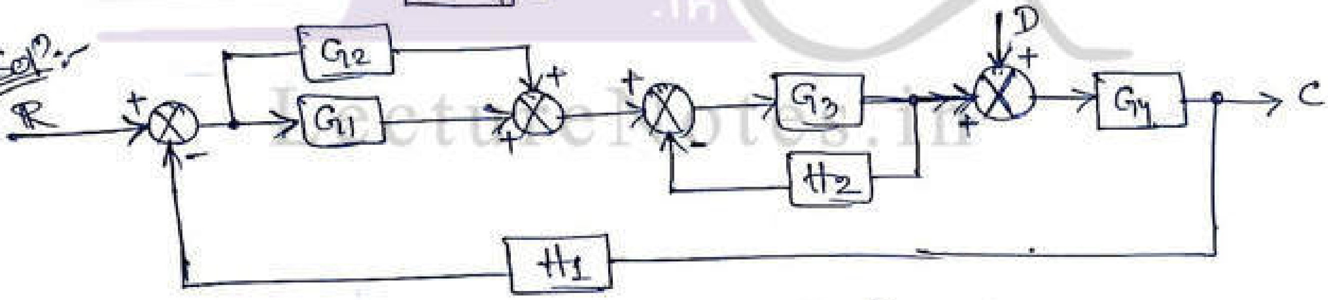
$$\therefore \frac{C}{R} = \frac{\frac{G_1G_2G_3}{(1+G_1H_1)(1+G_3H_2)}}{1+\left[\frac{G_1G_2G_3}{(1+G_1H_1)(1+G_3H_2)}\right]\left(\frac{H_1}{G_1G_3}\right)}$$

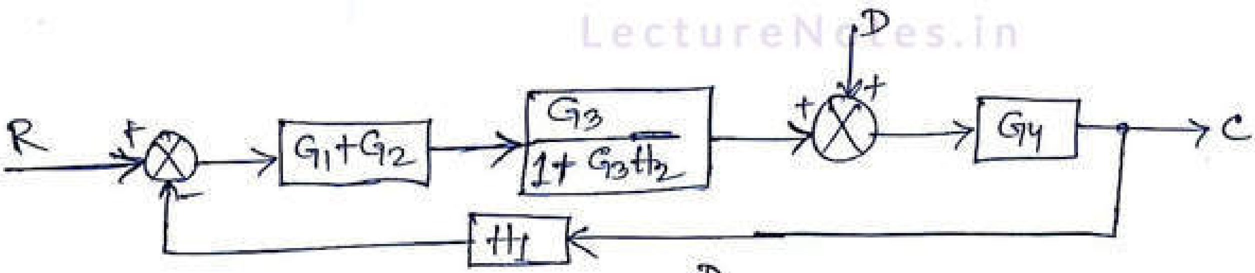
$$\Rightarrow \frac{C}{R} = \frac{G_1G_2G_3}{1+G_1H_1+G_2H_3+G_3H_2+G_1G_3H_1H_2} \quad \text{(Ans)}$$

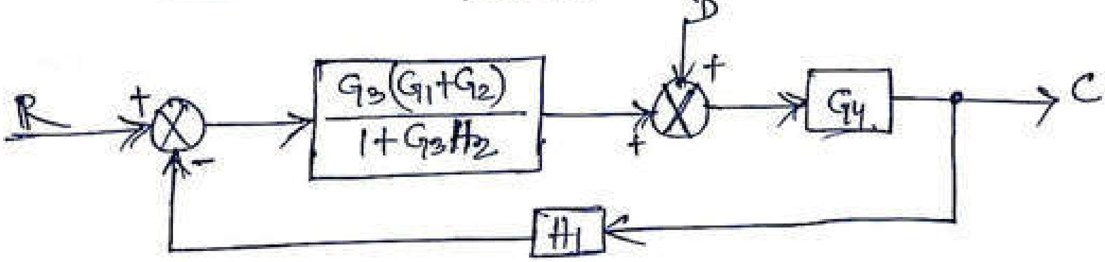
(4) Determine the ratio C/R, C/D & the total output for the system whose block diagram is as shown below :-



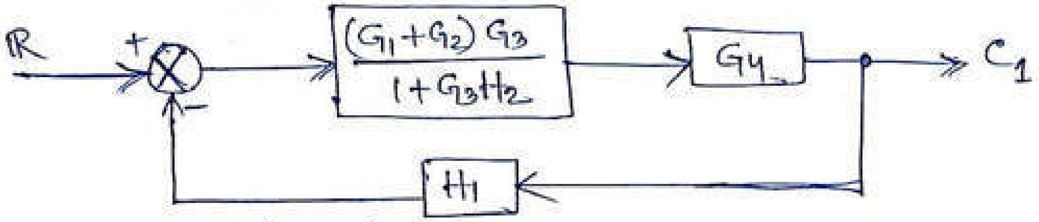
Sol^n:-







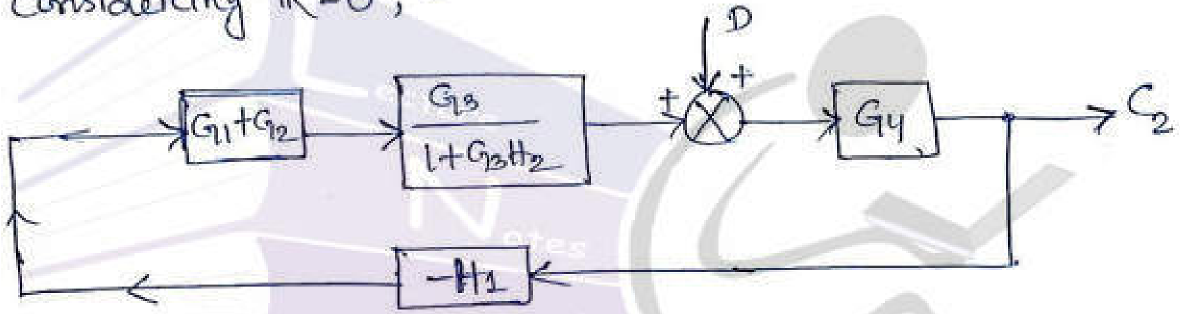
(i) Considering D=0 ;-

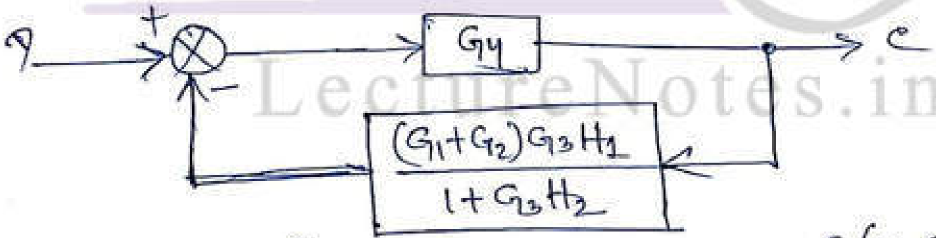


$$\therefore \frac{C_1}{R} = \frac{\frac{(G_1+G_2)G_3}{1+G_3H_2} \cdot G_4}{1+\left[\frac{(G_1G_2)G_3}{1+G_3H_2}\right](G_4).(H_1)}$$

$$\Rightarrow \frac{C_1}{R} = \frac{G_1G_3G_4 + G_2G_3G_4}{1+G_3H_2+G_1G_3G_4H_1+G_2G_3G_4H_1} \quad \text{(Ans)}$$

(ii) Considering R=0 ;-





$$\therefore \frac{C_2}{D} = \frac{G_4}{1+(G_4).\left[\frac{(G_1+G_2)G_3H_1}{1+G_3H_1}\right]} = \frac{G_4(1+G_3H_2)}{1+G_3H_2+G_1G_3G_4H_1+G_2G_3G_4H_1}$$

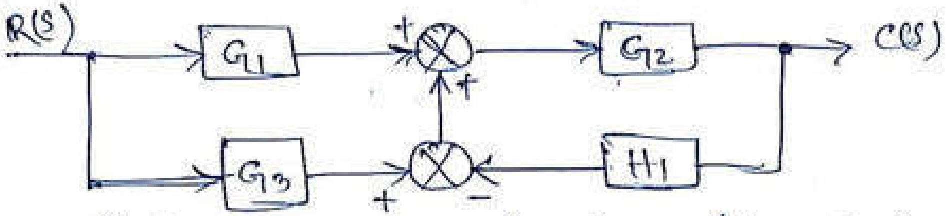
(iii) The total o/p is given by :-

$$\boxed{C = C_1R + C_2D}$$, where $C_1 = \frac{G_1G_3G_4+G_2G_3G_4}{1+G_3H_2+G_1G_3G_4H_1+G_2G_3G_4H_1}$

(Ans)

$C_2 = \frac{G_4(1+G_3H_2)}{1+G_3H_2+G_1G_3G_4H_1+G_2G_3G_4H_1}$

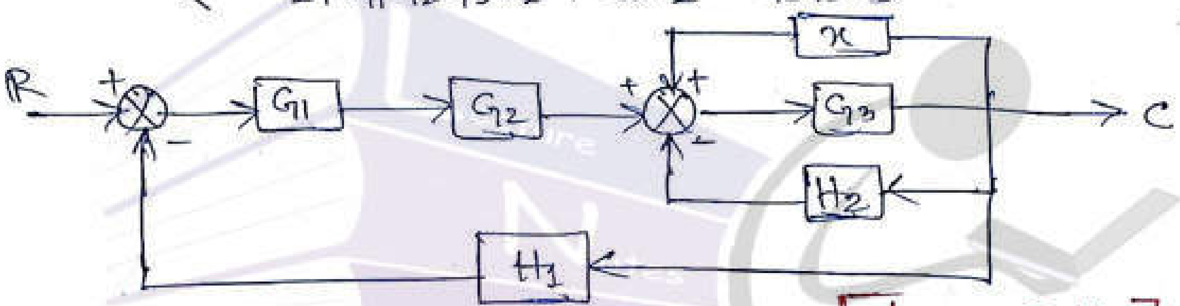
5. Consider the block diagram shown below :-



If the transfer function of the system is given by $T(s) = \frac{G_1G_2 + G_2G_3}{1+X}$, then 'x' is G_2H_1 .

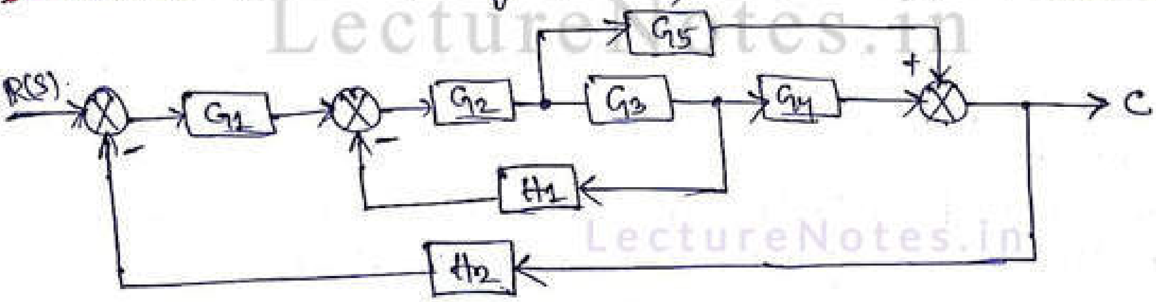
6. A system block diagram is shown in the given figure. The overall transfer function of the system is

$$\frac{C}{R} = \frac{G_1G_2G_3}{1+G_1G_2G_3H_1 + G_3H_2 - G_2G_3H_3}$$



determine the value of 'x' ? [Ans: G_2H_3]

7. Reduce the block diagram to its canonical form and obtain C/R.

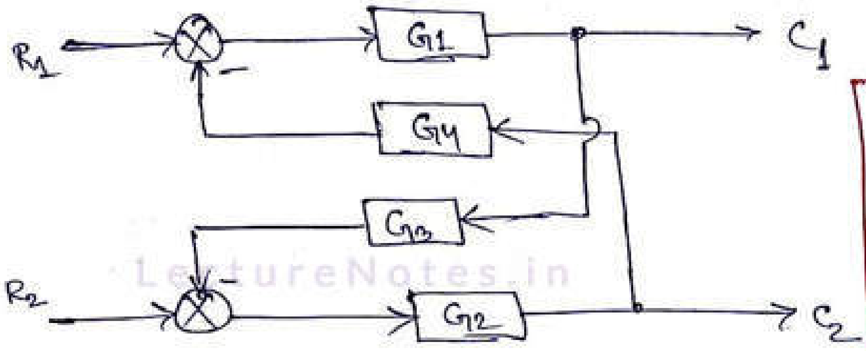


Ans: $\frac{C}{R} = \frac{G_1G_2(G_3G_4+G_5)}{1+G_2G_3H_1 + G_1G_2(G_3G_4+G_5)H_2}$.

Notes:-
- As far as possible try to shift take-off points towards right and summing points to the left.
- Unless & until it is the requirement of problem, do not use rule 8 and 9. These rules are called critical rules.
- Basic feedback loop is called as minor loop.

⑭

⑧ Obtain the expression for C_1 & C_2 for the given multiple input multiple output system.



Hint:

$\left.\frac{C_1}{R_1} \& \frac{C_2}{R_1}\right|_{R_2=0} = ?$

$\left.\frac{C_1}{R_2} \& \frac{C_2}{R_2}\right|_{R_1=0} = ?$

✻ SIGNAL FLOW GRAPH (SFG) REPRESENTATION :-

→ SFG is a graphical representation of variables of a set of linear algebraic equations representing the system.
→ Variables are represented by small circles called nodes.
→ The lines joining the nodes are called branches, which is associated with a transfer function & an arrow.

Ex:- Consider a eq: $V = IR$

R = Branch transfer fun^c or branch gain.

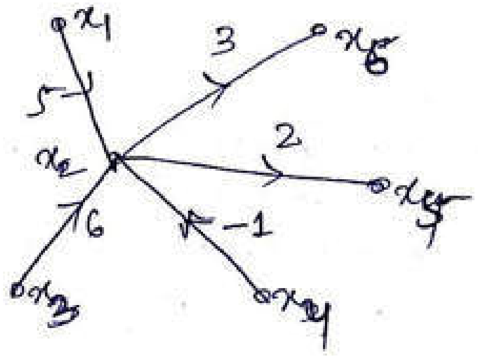
V & I = Variables represented by node.

Properties of SFG :-

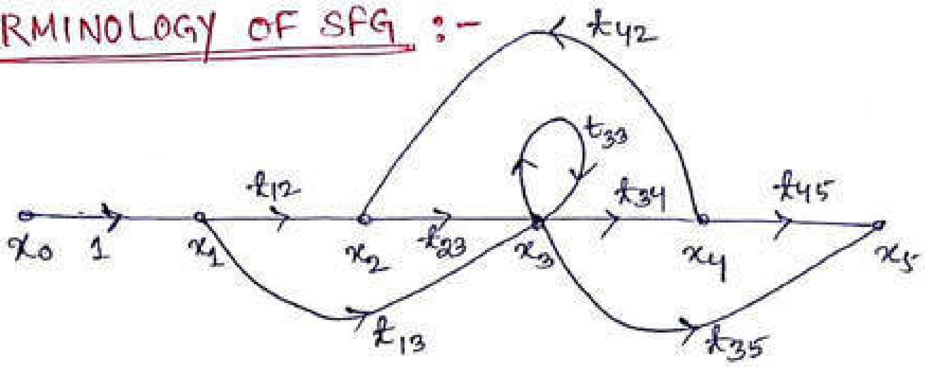
1. SFG is only applicable to Linear Time Invariant Systems.
2. The signal gets multiplied by branch gain when it travels along it.
3. The value of variable represented by any node is an algebraic sum of all the signals entering at the node.
4. The no of branches leaving a node does not affect the value of variable represented by that node.

Ex:- The value of variable x_2 can be written as :-

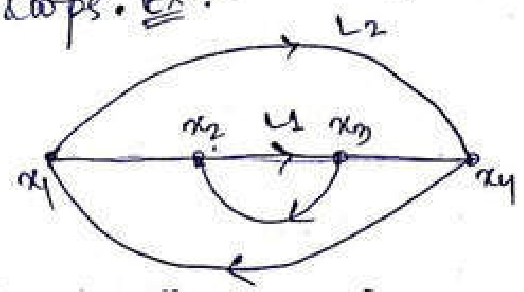
$$\boxed{x_2 = 5x_1 + 6x_3 - x_4}$$



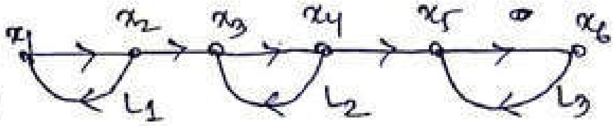
TERMINOLOGY OF SFG :-



1. **Node** :- It represents a system variable. Ex: x_0, x_1, x_2, x_3, x_4 & x_5.
2. **Source/Input node**: It is a node having only outgoing branches.
   Ex: x_0 is source node.
3. **Sink/Output node** : Node having only incoming branches.
   Ex: x_5 is sink node.
4. **Forward path** : A path from input to output node is called as forward path.
   Ex:- $x_0 - x_1 - x_2 - x_3 - x_4 - x_5$
   $x_0 - x_1 - x_3 - x_4 - x_5$
   $x_0 - x_1 - x_3 - x_5$
   $x_0 - x_1 - x_2 - x_3 - x_5$
   } forward paths.
5. **Feedback loop** :- It is a path which originates & terminates at the same node, without tracing any node twice.
   Ex:- $x_2 - x_3 - x_4 - x_2$.
6. **Self loop** : A feedback loop consisting of only one node is called self loop. Ex:- x_3 is a self loop.
7. **Path gain** :- The product of branch gains encountered in travelling a path.
   Ex:- $1 \times t_{12} \times t_{23} \times t_{34} \times t_{45}$ = Forward path gain.
8. **Non-touching Loops**:- If there is no node common in between the two or more loops, such loops are called as Non-touching loops. Ex:-



Loop 1 : $x_2 - x_3 - x_2$
Loop 2 : $x_1 - x_4 - x_1$
} Two Non-touching loops.



$L_1 : x_1 - x_2 - x_1$
$L_2 : x_3 - x_4 - x_3$
$L_3 : x_5 - x_6 - x_5$
} Three Non-touching loops.

9. Loop gain :- The product of all the gains of the branches forming a loop is called loop gain.

Ex:- For loop $x_2 - x_3 - x_4 - x_2$ from fig(1), the

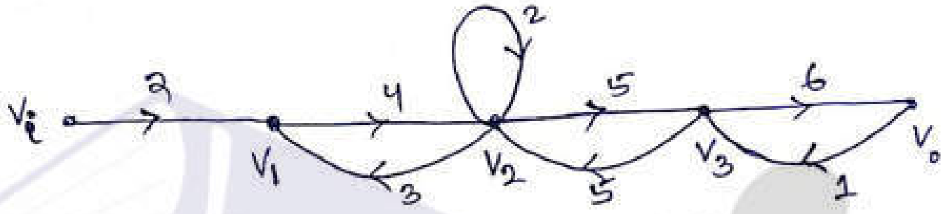
loop gain $= t_{23} \cdot t_{34} \cdot t_{42}$

METHODS TO OBTAIN SFG :-

(A) From equations :-

Given $V_1 = 2V_i + 3V_2$; $V_2 = 4V_1 + 5V_3 + 2V_2$

$V_3 = 5V_2 + V_0$; $V_0 = 6V_3$.



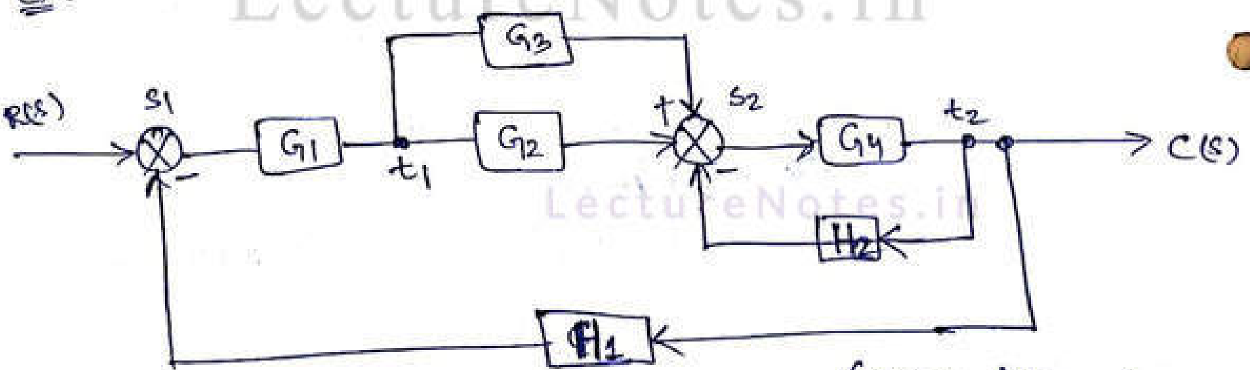
(B) From Block diagram :-

steps: (1) Represent each summing & take off point by a separate node in SFG.

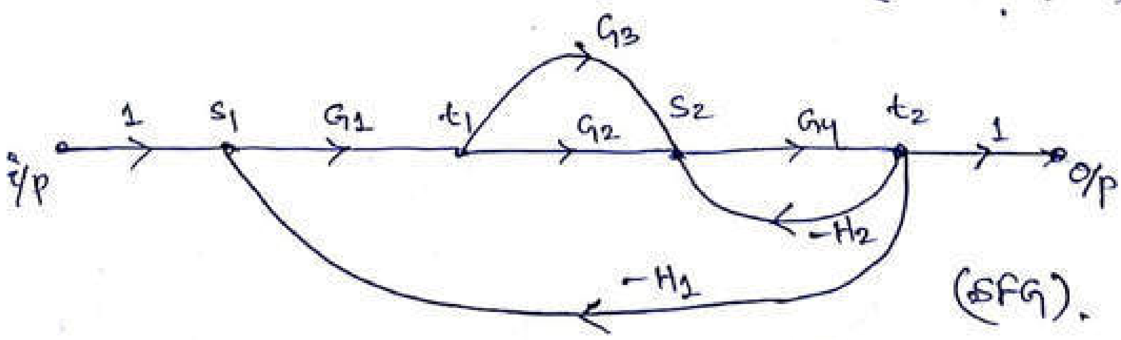
(2) Connect them by branches instead of blocks, indicating block transfer functions as the gains of the corresponding branches.

(3) Take off + SP $\rightarrow$ 2 node ; SP + take off $\rightarrow$ 2 node/1 node

Ex:-



(Block diagram)



(SFG).

MASON'S GAIN FORMULA :-

→ This formula is used to find the overall transfer function of a given system from its signal flow graph.

→ The formula can be stated as:-

$$\boxed{T = \frac{\sum_{k=1}^{n} P_k \Delta_k}{\Delta}}$$

n = No. of forward paths.

where, T = overall transfer function of the system.

P_k = Path gain of k^{th} forward path.

Δ = System determinant or characteristic function.

$= 1 - (\Sigma$ All loop gains of individual loops$)$

$+ (\Sigma$ gain products of all possible combn of two non-touching loops$)$

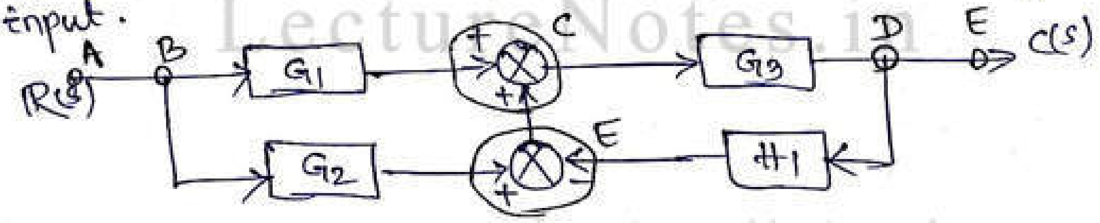
$- (\Sigma$ gain product of combn of three non-touching loops$) + \cdots$

Δ_k = Value of Δ for the part of the graph not touching with the k^{th} forward path.

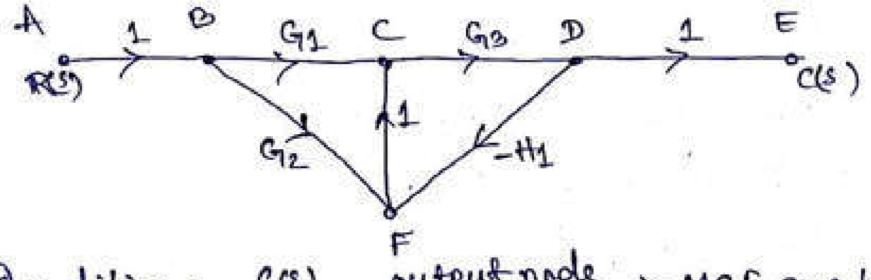
Δ_1 = It is that value of Δ which is obtained by removing all loops which are touching 1st forward path

Numericals :-

Q// From the block diagram shown in figure, draw the corresponding SFG & evaluate close-loop transfer function relating the o/p & input.



Soln:-



Condition :- $\frac{C(s)}{R(s)} = \frac{\text{output node}}{\text{Input node}}$ ⇒ MGF can be applied.

Forward path :-

	Gain
P_1 : ABCDE ⟶	G_1G_3
P_2 : ABFCDE ⟶	G_2G_3

Loop :-

	Gain
L_1 : CDFC ⟶	$-G_3H_1$

Non-touching loops $= 0$.

$\therefore \Delta = 1 - \{L_1\} = 1 + G_3H_1$.

$\Delta_1 = 1 - 0 = 1$; $\Delta_2 = 1 - 0 = 1$.

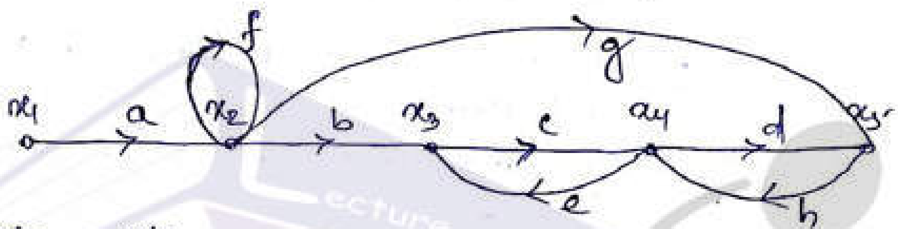
$$\therefore T/f = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_3 + G_2G_3}{1 + G_3H_1} \quad \text{(Ans)}.$$

Q// Represent the following set of equations by a SFG & determine the overall gain relating x_5 & x_1.

$x_2 = ax_1 + fx_2$; $x_3 = bx_2 + ex_4$

$x_4 = cx_3 + hx_5$; $x_5 = dx_4 + gx_2$.

Sol^n:-



check:- $\frac{x_5}{x_1} = \frac{O/P}{I/P} \Rightarrow$ MGF can be applied.

forward path :-

	gain
$P_1 : x_1x_2x_3x_4x_5 \longrightarrow$	$abcd$
$P_2 = x_1x_2x_5 \longrightarrow$	ag.

Loop :-

	gain
$L_1 : x_2x_2 \longrightarrow$	f
$L_2 : x_3x_4x_3 \longrightarrow$	ce
$L_3 : x_4x_5x_4 \longrightarrow$	dh

Non-touching loops :-

L_1L_2, L_1L_3, $\cancel{L_2L_3}$, $L_1\cancel{L_2}L_3$.

$\rightarrow$ Two non touching Loops are : L_1L_2, L_1L_3.

Gain : $L_1L_2 \longrightarrow fce$

$L_1L_3 \longrightarrow fdh$.

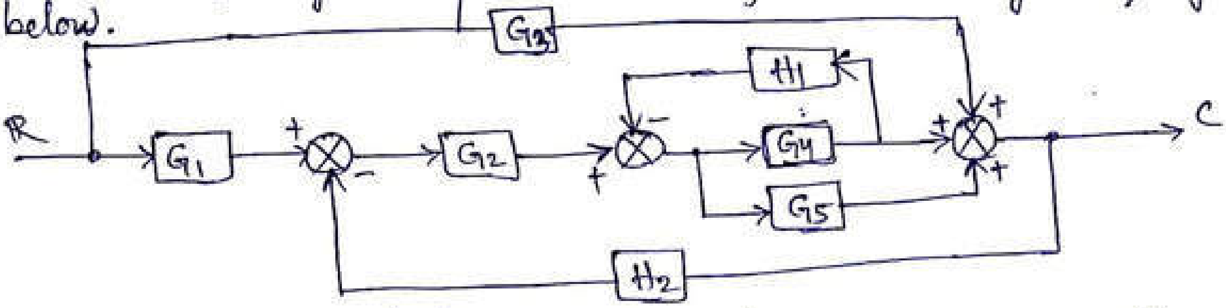
$\therefore \Delta = 1 - \{L_1 + L_2 + L_3\} + \{L_1L_2 + L_1L_3\}$

$\Rightarrow \Delta = 1 - (f + ce + dh) + (fce + fdh)$

$\therefore \Delta_1 = 1 - \{0\} = 1$; $\Delta_2 = 1 - L_2 = 1 - ce$.

$$\therefore T/f = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{(abcd)(1) + ag(1-ce)}{1 - f - ce - dh + fce + fdh} \quad \text{(Ans)}.$$

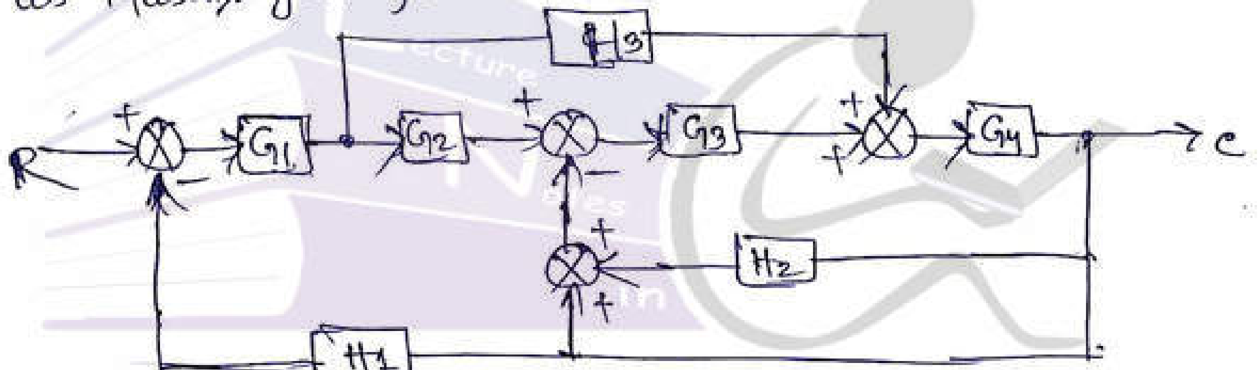
Q// The block diagram representation of a control system is given below.



Draw the signal flow graph to determine the overall transfer function.

$$\left[\text{Ans : } \frac{C}{R} = \frac{G_1G_2G_4 + G_1G_2G_5 + G_3(1+G_4H_1)}{1+G_4H_1+G_2G_4H_2+G_2G_5H_2}\right]$$

Q// Determine the overall transfere function C/R of the system shown below by block diagram reduction technique as well as Mason's gain formula.



$$\left[\text{Ans: } \frac{C}{R} = \frac{G_4G_1H_3 + G_4G_1G_2G_3}{1+G_3G_4+G_3G_4H_2+G_1G_4H_1H_3+G_1G_4G_2G_3H_1}\right]$$

* 2-forward paths
4-Loops.
0-NT loops

Q// Determine the t/f from the given SFG ?

$\frac{C}{R} = ?$.

$$\left[\text{Ans : } \frac{C}{R} = \frac{1}{1-H_2}\right].$$

* Try to avoid manipulations directly in SFG. unless until the loops are very far away from each other.

* $y_2 = a_1 y_1 \Rightarrow y_1 = \frac{1}{a_1} y_2$.

(X)

Control System Engineering

Topic:

Feedback Characteristic Of Control System

Contributed By:

Gyana Ranjan Biswal

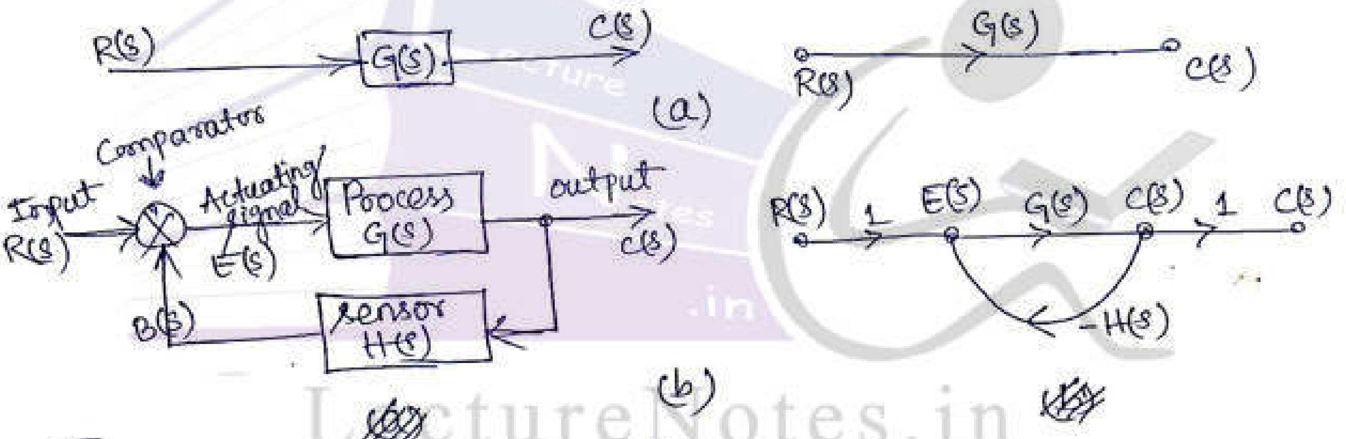
FEEDBACK CHARACTERISTIC OF CONTROL SYSTEM :-

→ Feedback means automatic regulation and control. This can be noticed in many physical, biological & soft systems.

→ for example : The body temperature of any living being is automatically regulated through a process which is essentially a feedback process.

→ In a non-feedback (open-loop system), there is no provision within the system for supervision of the output & no mechanism is provided to correct (or compensate) the system behaviour due to various disturbances.

→ In a feedback (closed loop system), there is a feedback signal derived from the output. This signal gives the capability to act as self correcting mechanism.



[Fig: Block diagram & SFG of
(a) A non-feedback (open-loop) System.
(b) A feedback (closed-loop) System.]

TYPES OF FEEDBACK

→ There are two types of feedback :

1. Positive feedback or Regenerative feedback
2. Negative feedback or degenerative feedback.

→ In positive feedback, the transfer function is given by :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

→ There is a possibility of zero value in the denominator when the loop gain $G(s)H(s) = 1$.

→ So for a given input, output goes to infinity, which makes the system unstable. Hence regenerative feedback systems are not used.

→ It is used in bistable multi vibrator circuit, oscillators for increasing the loop gain of the system, but not in the control system.

* EFFECTS OF DEGENERATIVE (-ve) FEEDBACK :-

→ The effects of negative feedback are :-

(i) Reduces the overall gain of the system.
(ii) Reduction of parameter variations.
(iii) Reduces sensitivity
(iv) Reduces disturbances
(v) Improves the time response of the system.
(vi) Improves the stability of the system.

(i) Effect on overall gain :-

For an open loop system, $\frac{C(s)}{R(s)} = G(s)$

For a closed loop negative feedback system,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}.$$

∴ So, by using feedback the overall gain of the system reduces by a factor of $[1+G(s)H(s)]$.

(ii) Effect on Parameter variations :-

→ In O.L. system, the $T/F = \frac{C(s)}{R(s)} = G(s)$

Let $\Delta G(s)$ = change in $G(s)$ due to parameter variations (temperature, pressure, environmental changes).

$\Delta C(s)$ = Corresponding change in output.

$$\therefore C(s) + \Delta C(s) = [G(s)+\Delta G(s)]\,R(s) = G(s)R(s) + \Delta G(s).R(s).$$

$$\Rightarrow \boxed{\Delta C(s) = \Delta G(s).R(s)}$$

→ In -ve feedback C.L. system, the transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}.$$

$$\Rightarrow C(s) = \frac{G(s)}{1+G(s)H(s)} R(s).$$

$$\therefore C(s) + \Delta C(s) = \left[\frac{G(s)+\Delta G(s)}{1+[G(s)+\Delta G(s)]H(s)}\right] R(s)$$

$$= \left[\frac{G(s)+\Delta G(s)}{1+G(s)H(s)+\Delta G(s)H(s)}\right] R(s).$$

Since, $\Delta G(s)H(s) << [1+G(s)H(s)]$, neglect $\Delta G(s)H(s)$.

$$\therefore C(s) + \Delta C(s) = \left[\frac{G(s)+\Delta G(s)}{1+G(s)H(s)}\right] R(s) = \frac{G(s)R(s)}{1+G(s)H(s)} + \frac{\Delta G(s)\cdot R(s)}{1+G(s)H(s)}.$$

$$\Rightarrow \boxed{\Delta C(s) = \frac{\Delta G(s)\cdot R(s)}{1+G(s)\cdot H(s)}}$$

→ It is clear that, the change in o/p is reduced due to parameter variations in $G(s)$ by $[1+G(s)H(s)]$. But in O.L. system there is no reduction because no feedback.

(iii) Effect of feedback on Sensitivity :-

→ Sensitivity is used to describe the relative variation in the overall transfer function $T(s)$ due to variation in $G(s)$ or $H(s)$ & is defined by :-

$$\text{Sensitivity} = \frac{\text{Percentage change in } T(s)}{\text{Percentage change in } G(s) \text{ or } H(s)}.$$

→ For small incremental variation in $G(s)$, the sensitivity will be written as :-

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \text{Sensitivity of 'T' w.r.t. 'G'}.$$

→ For an O.L. system, $S_G^T = \frac{\partial T}{\partial G} \times \frac{G}{T} = 1 \times \frac{G}{G} = 1$ $[\because T = G]$

→ For a C.L. system, $S_G^T = \frac{\partial T}{\partial G} \times \frac{G}{T} = \frac{\partial\left(\frac{G}{1+GH}\right)}{\partial G} \times \frac{G}{\left(\frac{G}{1+GH}\right)}$

$$\Rightarrow S_G^T = \frac{(1+GH)-GH}{(1+GH)^2} \times \frac{(1+GH)}{1} = \frac{1}{1+GH}.$$

→ It is clear that, the sensitivity of a C.L. system w.r.t. to variation in 'G' is reduced by a factor of $(1+GH)$.

$\therefore S_H^T \rightarrow$ sensitivity of 'T' w.r.t. 'H'.

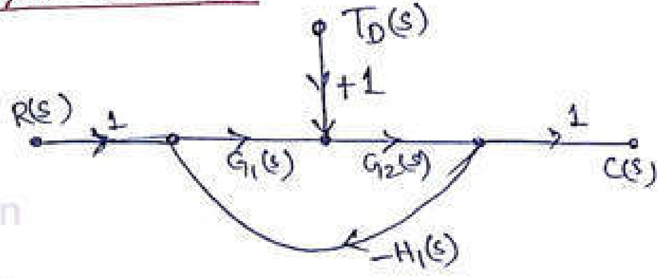
$$\therefore S_H^T = \frac{\partial T}{\partial H} \times \frac{H}{T} = \frac{-GH}{1+GH}.$$

→ The changes in 'H' directly affect the system output.

→ So, CLCS is more sensitive to variation in feedback path parameters than forward path parameters.

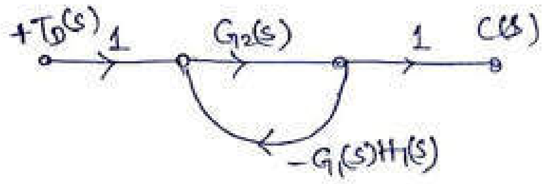
→ Therefore, it is important to use feedback elements which do not vary with environmental changes or maintained constant.

(iv) Effect of feedback on disturbances :-



→ Let's consider a disturbance $T_D(s)$ is added on a closed loop system.

→ The ratio of the o/p C(s) to the disturbance signal $T_D(s)$, when $R(s)=0$, is obtained from the SFG as shown in figure, & is given by :-



$$\frac{C_D(s)}{T_D(s)} = \frac{+G_2(s)}{1+H_1(s)G_1(s)G_2(s)}.$$

→ If $G_1(s)G_2(s)H_1(s) \gg 1$ over a working range of 's', then :

$$\frac{C_D(s)}{T_D(s)} = \frac{+G_2(s)}{G_1(s)G_2(s)H_1(s)} = \frac{+1}{H_1(s)G_1(s)}$$

Now, If $G_1(s)$ or $H_1(s)$ is made sufficiently large, then the effect of disturbance can be reduced.

(V) Effect of feedback on time response :-

Consider a 1st order system, where $G(s)=\frac{K}{1+ST}$, $K>1$.

→ open loop system, $\frac{C(s)}{R(s)} = G(s) = \frac{K}{1+ST}$.

If $R(s) = 1/s$ (unit step) is given, then :-

$$C(s) = \frac{1}{s}\cdot\frac{K}{1+ST} = \frac{A}{s} + \frac{B}{1+ST} = \frac{A}{s} + \frac{B}{T(s+1/T)}.$$

$$\therefore A = \frac{K}{1+ST}\Big|_{s=0} = K \;;\; B = \frac{K}{s}\Big|_{s=-1/T} = \frac{K}{-1/T} = -KT.$$

$$\therefore C(s) = \frac{K}{s} + \frac{-KT}{T(s+1/T)} = K\left[\frac{1}{s} - \frac{1}{s+1/T}\right]$$

$$\therefore \boxed{c(t) = K\left(1-e^{-t/T}\right)}.$$

→ For closed loop system having unity feedback :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K/1+ST}{1+K/1+ST} = \frac{K}{1+ST+K} = \frac{K}{ST+(1+K)}.$$

∴ For unit step input :-

$$C(s) = \frac{1}{S}\times\frac{K}{ST+(1+K)} = \frac{A}{S} + \frac{B}{ST+(1+K)}.$$

$$A = \frac{K}{1+K} \; ; \; B = \left.\frac{K}{S}\right|_{S=\frac{-(1+K)}{T}} = \frac{KT}{-(1+K)}.$$

$$\therefore C(s) = \frac{K/1+K}{S} + \frac{KT/-(1+K)}{ST+1+K} = \frac{K}{1+K}\left[\frac{1}{S} - \frac{1}{S+\left(\frac{1+K}{T}\right)}\right].$$

$$\therefore \boxed{C(t) = \frac{K}{1+K}\left[1-e^{-t/(T/1+K)}\right]}.$$

→ So, time constant for open loop system is 'T', whereas for C.L. system (unity feedback), it is $(T/1+K)$. So the the time constant for C.L. system is less, that means response is faster. Transient response decay very fast.

→ So, feedback improves the time response of the system. → Higher bandwidth.

(vi) Effect on stability of a system :-

Consider a 1st order system :-

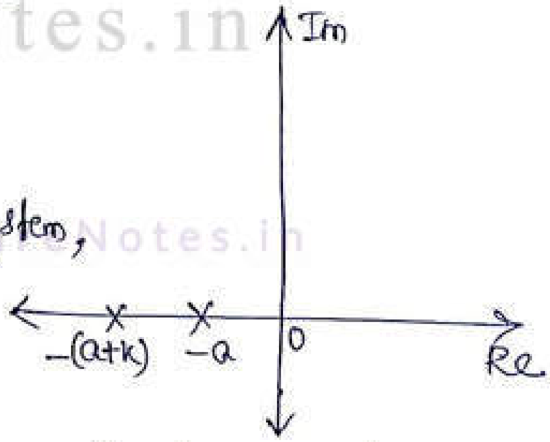
$$G(s) = \frac{K}{s+a}.$$

Open loop system/pole, $s=-a$.

→ Closed loop unity feedback system,

$$\frac{C(s)}{R(s)} = \frac{K/s+a}{1+K/s+a} = \frac{K}{s+(a+K)}.$$

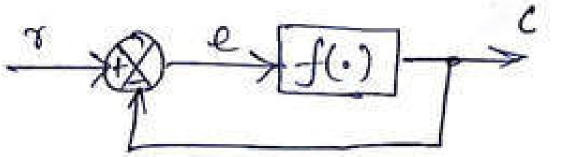
∴ closed loop poles, $s = -(a+K)$.



→ As stability depends on location of poles in s-plane, feedback may improves stability.

(vii) Linearlizing effect of feedback :-

Assume, the forward block function is nonlinear expression as, $c = f(e) = e^2$.



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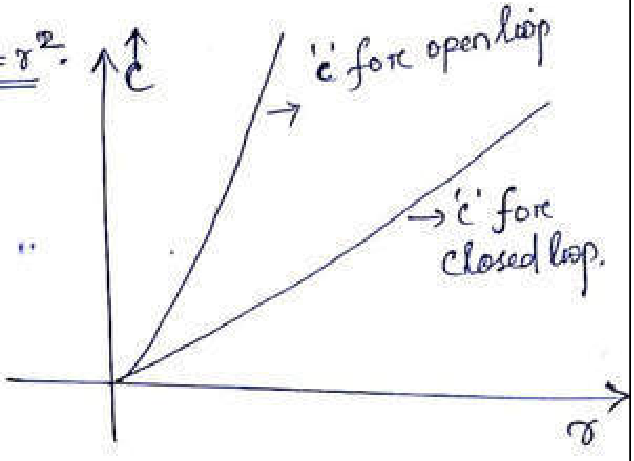
for open loop system, $e = r \Rightarrow \underline{\underline{c = r^2}}$.

For closed loop system, $e = r - c$

$\therefore$ So, $e = f(e) = (r-c)^2$.

→ From the comparison of the graphs, the i/p – o/p relation is approximately linear for the closed loop system compared to its open-loop behaviour.



LECTURENOTES.IN

Control System Engineering

Topic:

Time Response Analysis

Contributed By:

Gyana Ranjan Biswal

MODULE-2 :

TIME RESPONSE ANALYSIS

→ Time response of a control system means, how a system behaves with respect to time when a specified input test signal is applied.

→ Time response of a control system is divided into two parts namely :-

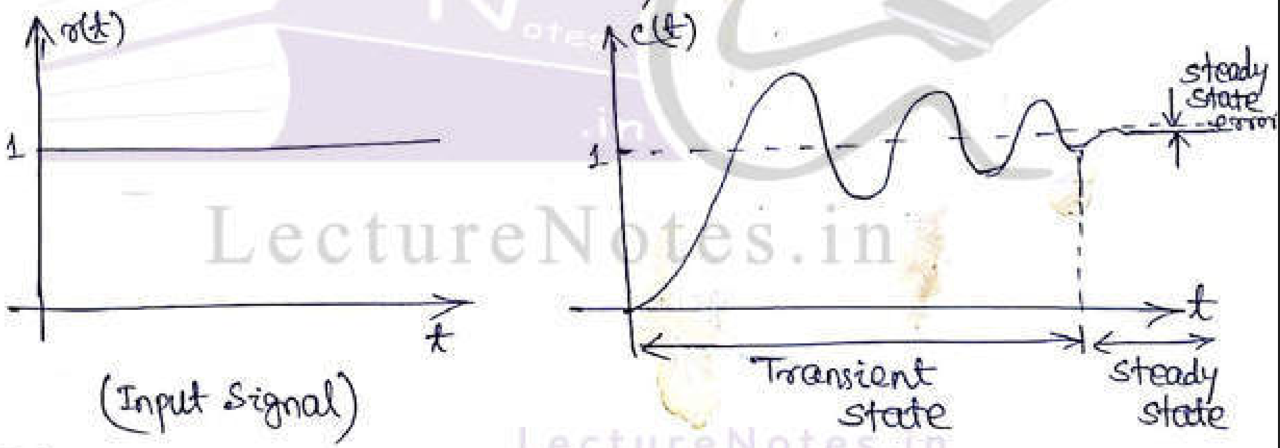
(i) Transient response

(ii) steady state response ($t \to \infty$).

→ In practice, the steady state period & transient period is identified in terms of time constants of a system.

→ Transient response is that part of total response, which becomes zero as time becomes large ($t \to \infty$).

→ steady state response is that part of total response which is left when transients are died out.



Note :

1. Transient part reveals the nature of response (i.e, underdamped or overdamped) & also indicate about its speed.

2. Steady state part reveals the accuracy of a C/s (i.e, whether the actual output exactly match with the i/p or not).

STANDARD OR TEST SIGNALS :

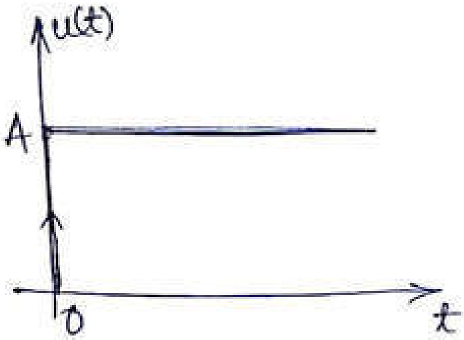
(1) Step Signal [u(t)] :-

→ It is also called as Heavside's function.

→ It is a sudden application of input signal.

→ It is defined as:-

$$u(t) = \begin{cases} A, & t > 0 \\ 0, & t < 0 \end{cases}$$



→ Also called displacement function.

→ It is not defined for $t=0$ (theoriticially).
Mathematically, $u(0) = \frac{1+0}{2} = \frac{1}{2}$ (Using Gibb's formula).

→ Laplace transform is $\mathcal{L}[u(t)] = A/s$.

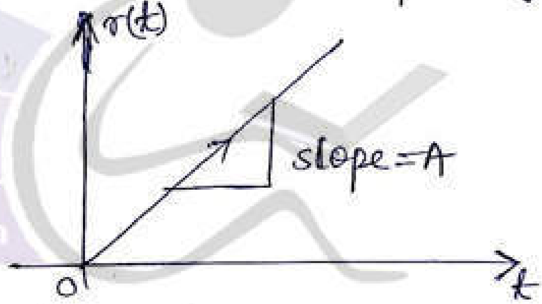
→ If $A=1$, then step function is called as Unit step function.

(2) Ramp function [$r(t)$]:-

→ It is described as gradual application of input signal.

→ It is defined as:-

$$r(t) = \begin{cases} At, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



→ It is also called as velocity function.

→ If A is 1, then it is called Unit ramp function.

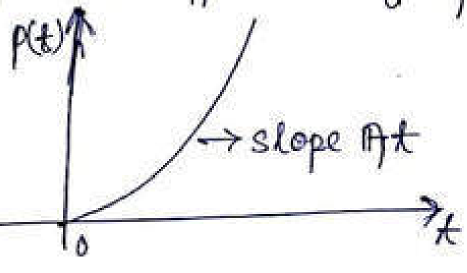
→ Laplace transform is $\mathcal{L}[r(t)] = A/s^2$.

(3) Parabolic function [$p(t)$]:-

→ It is described as more gradual application of input.

→ It is defined as:-

$$p(t) = \begin{cases} At^2/2, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



→ Also called Acceleration function.

→ If $A=1$, then it is called as unit parabolic function.

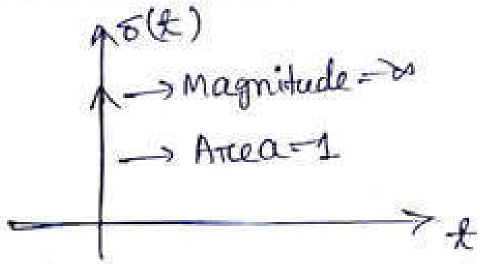
→ Laplace transform is $\mathcal{L}[p(t)] = A/s^3$.

(4) Impulse function $[\delta(t)]$:-

→ Also called as Dirac-Delta function.

→ It is defined as :-

$$\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t \neq 0 \end{cases}$$



→ It is suddenly applied as a shock for a very short duration of time.

→ Area under the function is '1'. So it is also called as Unit Impulse function.

→ Laplace transform is $\mathcal{L}[\delta(t)] = 1$.

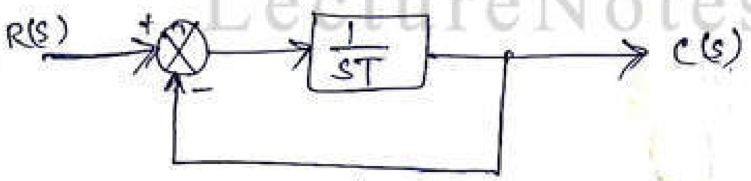
Note:

* $\dfrac{d^2 p(t)}{dt^2} = \dfrac{d}{dt} r(t) = u(t)$.

* $\dfrac{d\, u(t)}{dt} = \delta(t)$

TIME RESPONSE OF A FIRST ORDER SYSTEM :-

Order : It is the highest power of 's' in the denominator of a closed loop transfer function. (or) It is the highest power of 's' in the characteristic eq^n. (denominator = 0).



$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{1/ST}{1+1/ST} = \frac{1}{ST+1} \rightarrow 1^{st} \text{ order system.}$$

→ The output of the system is : $C(s) = R(s) \cdot \dfrac{1}{ST+1}$.

For Input Unit step

By giving input unit step, $u(t)$:-

$$C(s) = \frac{1}{s} \cdot \frac{1}{ST+1} = \frac{A}{s} + \frac{B}{ST+1}.$$

$$\therefore A = \frac{1}{ST+1}\Big|_{s=0} = 1 \;;\; B = \frac{1}{s}\Big|_{s=-1/T} = 1/(-1/T) = -T.$$

$$\therefore C(s) = \frac{1}{s} - \frac{T}{ST+1} = \frac{1}{s} - \frac{1}{s+1/T}$$

Taking Inverse Laplace transform :-

$$c(t) = (1 - e^{-t/T}) \cdot u(t).$$

→ The error is given by :-

$$e(t) = r(t) - c(t) = 1 - (1 - e^{-t/T}) = e^{-t/T}.$$

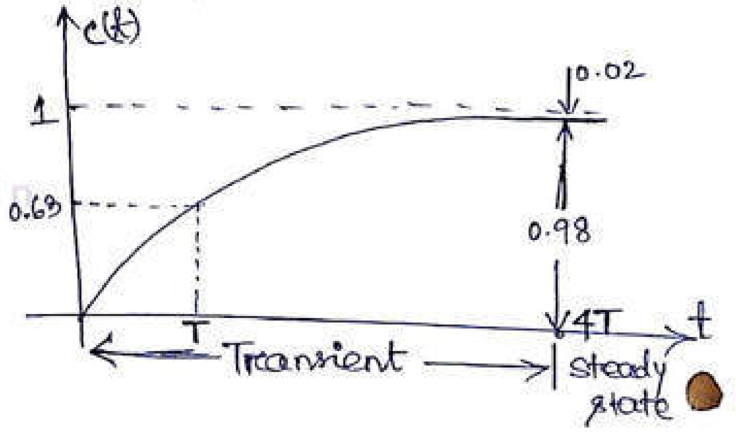
→ steady state error is, $e_{ss} = \lim_{t\to\infty} e^{-t/T} = 0$.

<u>observations :-</u>

→ At $t = T$, $c(t) = 0.63$.
So, 'T' is the time constant, which will make the exponential power '1'.

→ At $4T = t$, $c(t) = 0.98$.

→ steady state value of $c(t)\big|_{t\to\infty} = 1 - 0 = 1$.



→ So, time 4T is termed as settling time (t_s) of a control system. Because at this time, the actual o/p reaches within 2% of the desired value.

→ Lower the time constant, faster is the response.

Time Response of 1st order system for an unit ramp input:-

→ Input : $r(t) = t \Rightarrow R(s) = 1/s^2$

Output : $C(s) = R(s) \cdot \frac{1}{sT+1}$

$$\Rightarrow C(s) = \frac{1}{s^2} \cdot \frac{1}{sT+1} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{sT+1}$$

$$\therefore A = \frac{d}{ds}\left(\frac{1}{sT+1}\right)\Big|_{s=0} = \frac{-T}{(sT+1)^2}\Big|_{s=0} = -T.$$

$$B = \frac{1}{sT+1}\Big|_{s=0} = 1 \; ; \; C = \frac{1}{s^2}\Big|_{s=-1/T} = T^2.$$

$$\therefore C(s) = \frac{-T}{s} + \frac{1}{s^2} + \frac{T^2}{sT+1} = \frac{-T}{s} + \frac{1}{s^2} + \frac{T}{(s+1/T)}.$$

Taking Inverse Laplace transform :-

$$c(t) = \left(-T + t + T e^{-t/T}\right) \cdot u(t)$$

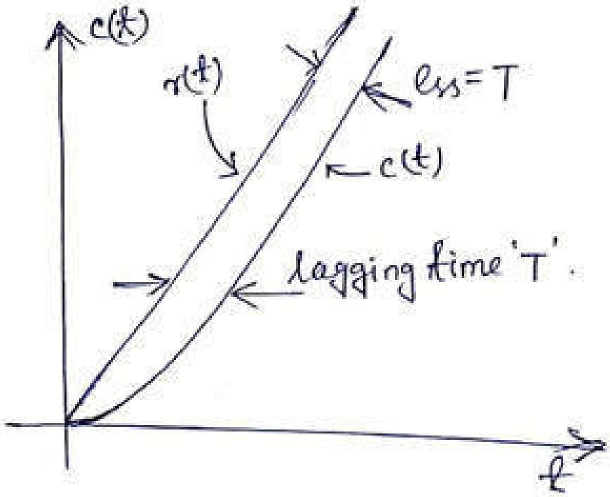
$$\Rightarrow c(t) = \left(t - T + T e^{-t/T}\right)$$

→ The error is given by :-

$$e(t) = r(t) - c(t)$$
$$= t - (t - T + Te^{-t/T})$$
$$= T - Te^{-t/T}$$

→ steady state error is :

$$e_{ss} = \lim_{t\to\infty} (T - Te^{-t/T}) = T.$$



Observation :-

→ During steady state the output velocity matches with the input velocity but lags behind the input by time 'T'.

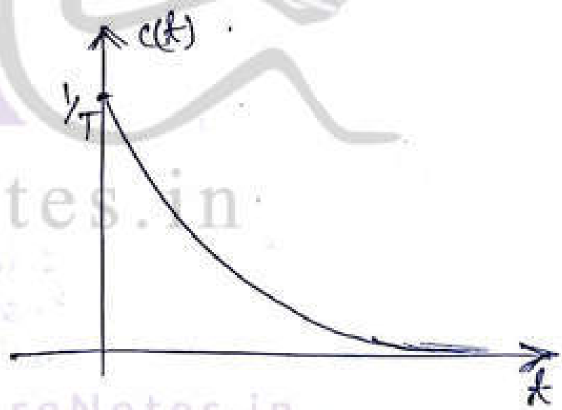
Time response of 1st order system for unit Impulse input :-

→ Input : $r(t) = \delta(t) \Rightarrow R(s) = 1.$

Output : $C(s) = R(s) \cdot \frac{1}{sT+1} = 1 \cdot \frac{1}{sT+1} = \frac{1}{sT+1} = \frac{1}{T(s+1/T)}.$

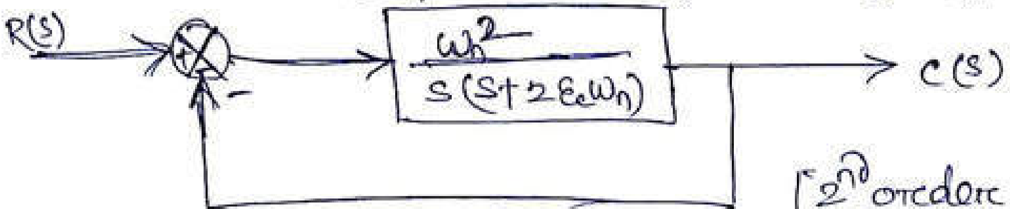
Taking Inverse Laplace transform :-

$$\therefore c(t) = \frac{1}{T} e^{-t/T}.$$



* TIME RESPONSE OF A 2ND ORDER CONTROL SYSTEM :-

→ A second order system means the highest power of 's' in the denominator of its transfer function is '2'.



[2nd order System with Unit feedback]

Open Loop t/f $= G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}.$

Closed loop t/f $= \frac{G(s)}{1+G(s)} = \boxed{\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$ → standard form of 2nd order System.

⑥

→ characteristic equation : $1+G(s)H(s)=0$

$\Rightarrow \boxed{s^2+2\xi\omega_n s+\omega_n^2=0}$.

↳ Roots of ch. eqⁿ are closed loop poles.

→ Practical example :-

1) All Indicating instruments (PMMC type).

2) RLC Network.

Time response of 2ⁿᵈ order System for Unit step Input :-

→ The output of the system is given by :-

$$C(s)=R(s)\cdot\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

for Unit step Input, $R(s)=1/s$.

$$\therefore C(s)=\frac{1}{s}\cdot\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}=\frac{A}{s}+\frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}.$$

$$\Rightarrow \frac{\omega_n^2}{\cancel{s(s^2+2\xi\omega_n s+\omega_n^2)}}=\frac{A(s^2+2\xi\omega_n s+\omega_n^2)+(Bs+C)s}{\cancel{s(s^2+2\xi\omega_n s+\omega_n^2)}}$$

$$\Rightarrow \omega_n^2=(A+B)s^2+(2A\xi\omega_n+C)s+A\omega_n^2$$

$\therefore (A+B)=0$; $2A\xi\omega_n+C=0$; $A\omega_n^2=\omega_n^2$

$\Rightarrow B=-1$ $\quad\Rightarrow C=-2\xi\omega_n$ $\quad\Rightarrow A=1$.

$$\therefore C(s)=\frac{1}{s}-\frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$=\frac{1}{s}-\frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$ (Making denominator Perfect square)

$$=\frac{1}{s}-\frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$ $\left[\because \omega_d=\omega_n\sqrt{1-\xi^2}\right]$

$$=\frac{1}{s}-\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}-\frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$\Rightarrow C(s)=\frac{1}{s}-\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}-\frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}.$$

Taking Laplace inverse transform :-

$$\therefore c(t)=1-e^{-\xi\omega_n t}\cos\omega_d t-\frac{\xi\omega_n}{\omega_d}\cdot e^{-\xi\omega_n t}\cdot\sin\omega_d t.$$

$$\Rightarrow c(t)=1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[(\sqrt{1-\xi^2})\cos\omega_d t+\xi\sin\omega_d t\right].$$

Put $\cos\phi = \xi \Rightarrow \sin\phi = \sqrt{1-\xi^2}$

$$\therefore c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}(\sin\phi \cdot \cos\omega_d t + \cos\phi \cdot \sin\omega_d t)$$

$$\Rightarrow \boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi).}$$

Where, $\omega_d = \omega_n\sqrt{1-\xi^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \cos^{-1}\xi$.

ω_d = damped frequency of oscillation

ω_n = Natural frequency of oscillation.

ξ = damping ratio

$\xi\omega_n$ = damping factor or actual damping or damping co-efficient.

→ As time response of 2nd order system is influenced by 'ξ', so, there are four cases possible for different values of 'ξ'.

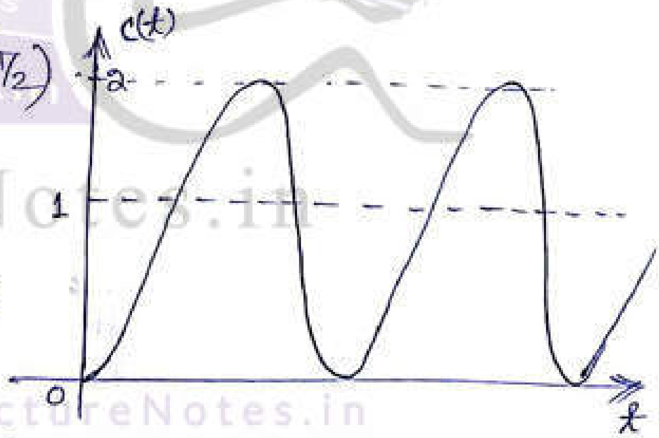
Case-I ($\xi = 0$) (Undamped Oscillation) :-

The time response for $\xi = 0$ will be :-

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2})t + \cos^{-1}\xi\right].$$

$$\Rightarrow c(t) = 1 - \frac{1}{1}\sin(\omega_n t + \pi/2)$$

$$\Rightarrow \boxed{c(t) = (1 - \cos\omega_n t)}$$



∴ So the response at $\xi = 0$, is called as undamped oscillation or sustained oscillation, with frequency 'ω_n'.

→ steady state value always lie between 0 to 2.

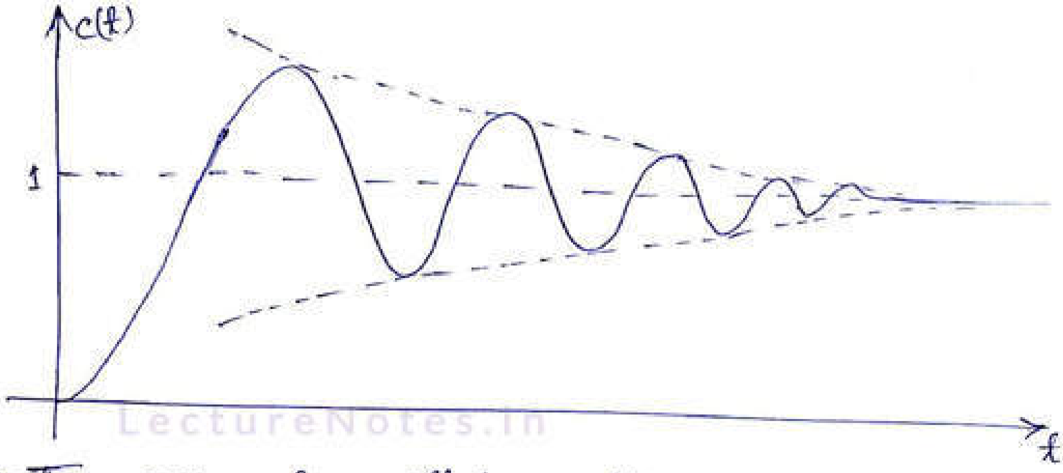
→ Error is given by $e(t) = r(t) - c(t) = \cos\omega_n t$

→ steady state error is between -1 to 1.

Case-II ($0 < \xi < 1$) (Underdamped Case) :-

The time response for $0 < \xi < 1$ will be :-

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$$



→ The response is oscillatory with oscillating frequency 'ω_d' but decreasing amplitude due to exponential term '$e^{-\xi\omega_n t}$'

→ This type of response is called underdamped response.

→ Steady state value = 1.

→ error is given by, $e(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$

→ steady state error is $e_{ss} = 0$.

→ Time constant is $\boxed{T = 1/\xi\omega_n}$.

Case-III ($\xi = 1$) (Critically damped case) :-

The time response at $\xi = 1$ will be :-

$$c(t) = \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2})t + \phi\right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\sin(\omega_n\sqrt{1-\xi^2})t \cdot \cos\phi + (\cos\omega_n\sqrt{1-\xi^2}\,t)\sin\phi \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[(\sin\omega_n\sqrt{1-\xi^2}\,t) \cdot \xi + (\cos\omega_n\sqrt{1-\xi^2}\,t)\sqrt{1-\xi^2} \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[(\omega_n\sqrt{1-\xi^2})t + 1 \cdot \sqrt{1-\xi^2} \right] \right\}$$

$$= \lim_{\xi \to 1} \left\{ 1 - \frac{e^{-\xi\omega_n t}}{\cancel{\sqrt{1-\xi^2}}} \left[\cancel{\sqrt{1-\xi^2}}\,(\omega_n t + 1) \right] \right\}$$

$$\Rightarrow \boxed{c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) = 1 - e^{-\omega_n t} + \omega_n t \cdot e^{-\omega_n t}}$$

→ So, output response consists of two exponentials but with same time constant $T = 1/\omega_n$.

→ steady state value is $c(t)|_{t\to\infty} = 1$.

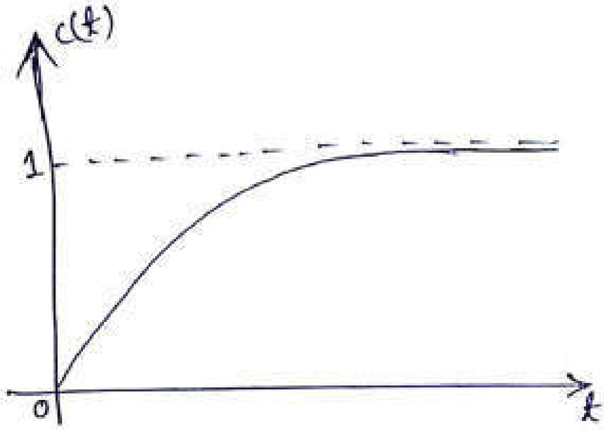
→ error is given by

$e(t) = r(t) - c(t)$

$= e^{-\omega_n t} + \omega_n t \cdot e^{-\omega_n t}$.

→ steady state error,

$e_{ss} = e(t)|_{t\to\infty} = 0$.



→ For $\xi = 1$, oscillations in output response are just dissappeared. This type of response is called as Critically damped response.

→ Characteristic eqⁿ: $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$\Rightarrow s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ or $-\xi\omega_n \pm j\omega_d$.

→ for $\xi = 1$; The roots are $-\omega_n, -\omega_n$.

→ System is Absolute stable.

Case-IV ($\xi > 1$) (Overdamped case) :-

The time response is given by

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{s} \frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_n^2(\xi^2-1)}$$

$$\Rightarrow C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_d^2} \qquad [\text{Put } \omega_d^2 = \omega_n^2(\xi^2-1)]$$

$$\Rightarrow C(s) = \frac{\omega_n^2}{s(s+\xi\omega_n+\omega_d)(s+\xi\omega_n-\omega_d)} = \frac{A}{s} + \frac{B}{s+\xi\omega_n+\omega_d} + \frac{C}{s+\xi\omega_n-\omega_d}.$$

$$\therefore A = \left.\frac{\omega_n^2}{(s+\xi\omega_n)^2 - \omega_d^2}\right|_{s=0} = \frac{\omega_n^2}{(\xi\omega_n)^2 - \omega_n^2(\xi^2-1)} = 1.$$

$$B = \left.\frac{\omega_n^2}{s(s+\xi\omega_n+\omega_d)}\right|_{s=-\xi\omega_n-\omega_d} = \frac{\omega_n^2}{(\xi\omega_n+\omega_d)(2\omega_d)} = \frac{\omega_n^2}{2\xi\omega_n\omega_d + 2\omega_d^2}$$

$$\Rightarrow B = \frac{\omega_n^2}{2\xi\omega_n \cdot \omega_n\sqrt{\xi^2-1} + 2\omega_n^2(\xi^2-1)} = \frac{1}{2\sqrt{\xi^2-1}\,(\xi + \sqrt{\xi^2-1})}.$$

Similarly, $C = \dfrac{-1}{2\sqrt{\xi^2-1}\,(\xi - \sqrt{\xi^2-1})}$.

Now, $$C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c+\sqrt{\varepsilon_c^2-1})(s+\varepsilon_c\omega_n+\omega_d)} - \frac{1}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c-\sqrt{\varepsilon_c^2-1})(s+\varepsilon_c\omega_n-\omega_d)}$$

$$\Rightarrow C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c+\sqrt{\varepsilon_c^2-1})\left[s+\varepsilon_c\omega_n+\omega_n\sqrt{\varepsilon_c^2-1}\right]} - \frac{1}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c-\sqrt{\varepsilon_c^2-1})\left[s+\varepsilon_c\omega_n-\omega_n\sqrt{\varepsilon_c^2-1}\right]}$$

$$\Rightarrow C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c+\sqrt{\varepsilon_c^2-1})\left[s+\omega_n(\varepsilon_c+\sqrt{\varepsilon_c^2-1})\right]} - \frac{1}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c-\sqrt{\varepsilon_c^2-1})\left[s+\omega_n(\varepsilon_c-\sqrt{\varepsilon_c^2-1})\right]}$$

Taking Inverse laplace transform :-

$$\boxed{c(t) = 1 + \frac{e^{-(\varepsilon_c+\sqrt{\varepsilon_c^2-1})\omega_n t}}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c+\sqrt{\varepsilon_c^2-1})} - \frac{e^{-(\varepsilon_c-\sqrt{\varepsilon_c^2-1})\omega_n t}}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c-\sqrt{\varepsilon_c^2-1})}}$$

→ Now above response has two time constants :-

$$T_1 = \frac{1}{(\varepsilon_c+\sqrt{\varepsilon_c^2-1})\,\omega_n} \quad \& \quad T_2 = \frac{1}{(\varepsilon_c-\sqrt{\varepsilon_c^2-1})\,\omega_n}$$

∴ T_1 is smaller than T_2, that means for 1st exponential term, the system is faster & for 2nd exponential term the system response is slower.

→ So the overall response is slow & the higher time constant i.e, T_2 is more dominant or significant.

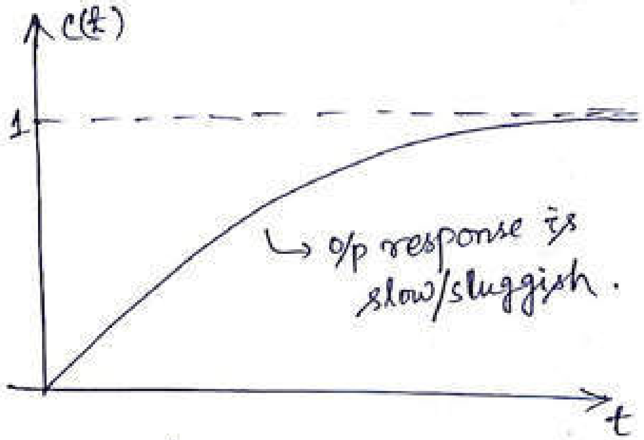
→ That means pole at $-(\varepsilon_c-\sqrt{\varepsilon_c^2-1})\omega_n$ is more dominant as compared to pole at $-(\varepsilon_c+\sqrt{\varepsilon_c^2-1})\omega_n$.

→ So, by neglecting the term having the pole at $-(\varepsilon_c+\sqrt{\varepsilon_c^2-1})\omega_n$,

$$\therefore c(t) = 1 - \frac{e^{-(\varepsilon_c-\sqrt{\varepsilon_c^2-1})\omega_n t}}{2\sqrt{\varepsilon_c^2-1}\,(\varepsilon_c-\sqrt{\varepsilon_c^2-1})}$$

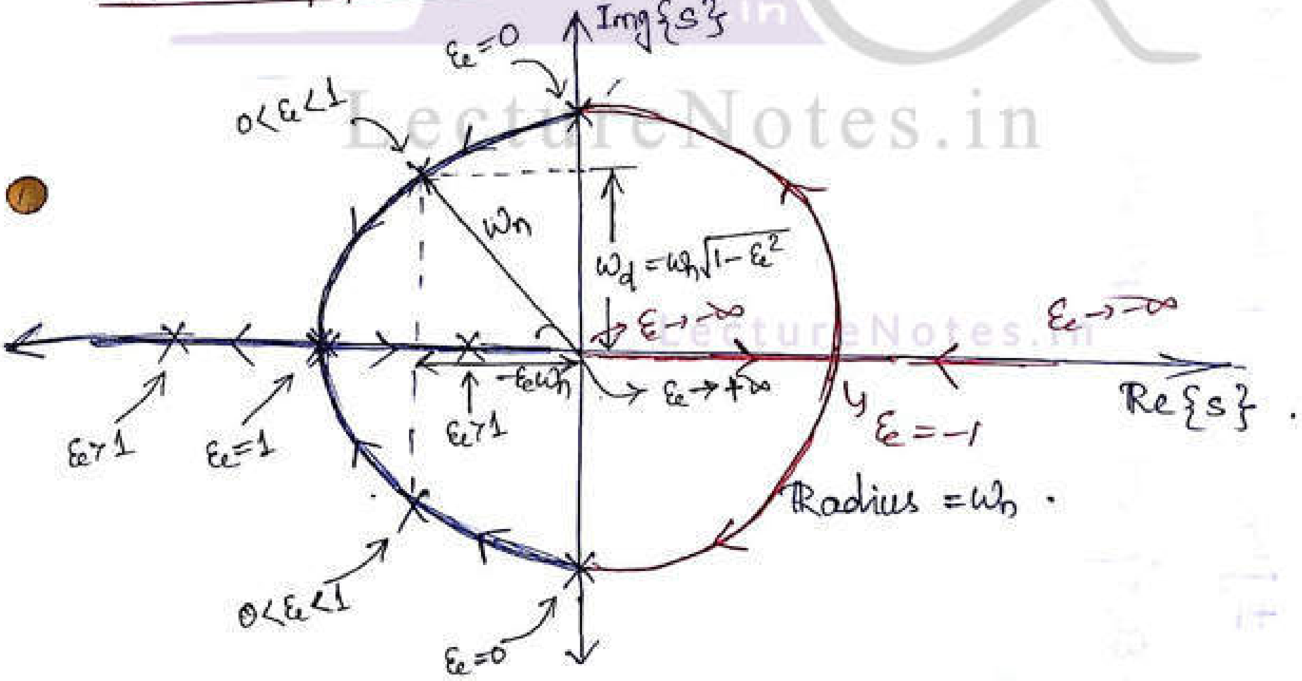
* The impulse response completely characterizes the LTI system. It gives the information about memory, causality, stability & invertibility of an LTI system.

Observations :-



1. steady state value
$$c(t)\big|_{t\to\infty} = 1 .$$
2. error is given by :-
$$e(t) = r(t) - c(t)$$
$$\Rightarrow e(t) = \frac{e^{-(\xi - \sqrt{\xi^2-1})\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi - \sqrt{\xi^2-1}\right)}$$
3. steady state error is $e_{ss} = e(t)\big|_{t\to\infty} = 0$.
4. The overall response is sluggish in nature for $\xi > 1$. This type of response is called as Overdamped response.
5. The closed loop poles are
$$s_1 \,\&\, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} = -\omega_n\left(\xi \pm \sqrt{\xi^2-1}\right).$$
6. As both the poles lies on the left half of s-plane, so this system is Absolute stable system.

Closed loop pole locations by varying ξ ($0 \le \xi < \infty$) :-



(+ve damping) (−ve damping)

ξ	CL poles	ω_d frequency	Oscillation	System type	Stability	Response	Curve
$\xi = 0$	$\pm j\omega_n$	$\omega_d = \omega_n$	Sustained oscill^n	Undamped	Marginal stable	$c(t) = 1 - \text{const}$	
$0 < \xi < 1$	$-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$	$\omega_d = \omega_n\sqrt{1-\xi^2}$	Oscill^n with decreasing amplitude	Underdamped	A. Stable	$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$	

Definitions :-

1. ω_n (Natural frequency/Undamped frequency) :-

→ It is the frequency of oscillation in output response when $\xi = 0$.

2. ω_d (damped frequency) :-

→ It is the frequency of oscillation in output response when $0 < \xi < 1$. $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$ & $\boxed{\omega_d < \omega_n}$

3. Damping factor ($\xi\omega_n$) :-

→ In an underdamped 2nd order system, damping is provided by the Real part of poles i.e, $-\xi\omega_n$. So this factor is called as damping factor or damping co-efficient or Actual damping.

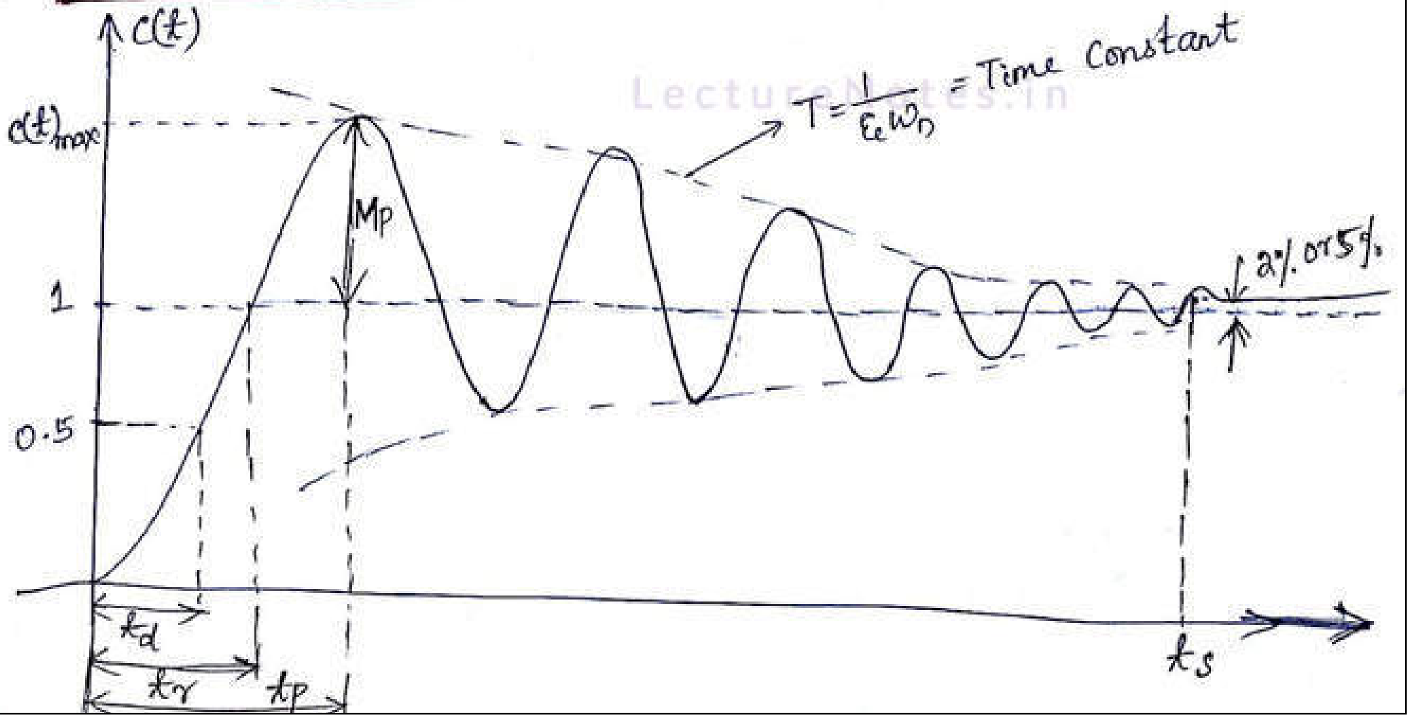
4. Damping ratio (ξ) :-

→ It is defined as the ratio of Actual damping to the Critical damping.

$$\text{Damping ratio} = \frac{\text{Actual damping}}{\text{Critical damping}} = \frac{\xi\omega_n}{1.\omega_n} = \xi.$$

→ It has no unit.

TRANSIENT RESPONSE SPECIFICATIONS OF 2nd ORDER SYSTEM:-



(14)

Delay time (t_d): It is the time required for the response to reach 50% of the final value in first time.

Rise time (t_r): It is the time required for the response to rise from 10% to 90% of its final value for overdamped systems and 0 to 100% for underdamped system.

Peak time (t_p) :- It is the time required for the response to reach the 1^{st} peak overshoot.

Peak Overshoot (M_p) :- It is the difference between the peak of the time response & steady output.

→ % peak overshoot is defined by :-

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 .$$

Settling time (t_s) :- It is the time required for the response to reach & stay within the specified range (2% or 5%) of its final value.

Expressions :-

1. t_d :-

Mathematically, $c(t_d) = \left[1 - \frac{e^{-\xi\omega_n t_d}}{\sqrt{1-\xi^2}} \sin(\omega_d t_d + \phi)\right] = 0.5$

∴ so solving for 't_d', mathematically is very difficult.

→ Graphically, $\boxed{t_d \simeq \frac{1+0.7\xi}{\omega_n}}$

2. t_r :-

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2})t + \phi\right]$$

Let response reaches 100% of desired value at $t = t_r$.

$$\therefore 1 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2})t + \phi\right]$$

$$\Rightarrow \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2})t + \phi\right] = 0$$

Since, $e^{-\xi\omega_n t} \neq 0$, $\therefore \sin\left[(\omega_n\sqrt{1-\xi^2})t + \phi\right] = 0$

$$\Rightarrow \sin\left[(\omega_n\sqrt{1-\varepsilon_e^2})t+\phi\right] = \sin n\pi$$

By putting $n=1$, $(\omega_n\sqrt{1-\varepsilon_e^2})t_r + \phi = \pi$

$$\Rightarrow \boxed{t_r = \frac{\pi-\phi}{\omega_n\sqrt{1-\varepsilon_e^2}} = \frac{3.14-\phi}{\omega_d}}$$

where, $\phi = \cos^{-1}\varepsilon_e = \tan^{-1}\left(\frac{\sqrt{1-\varepsilon_e^2}}{\varepsilon_e}\right)$

3. t_p :-

At $t=t_p$; $c(t)$ is having maximum value.

$\therefore$ For maximum, $\frac{d}{dt}[c(t)] = 0$

$$\therefore \frac{dc(t)}{dt} = \frac{-e^{-\varepsilon_e\omega_n t}}{\sqrt{1-\varepsilon_e^2}}\cos\left[(\omega_n\sqrt{1-\varepsilon_e^2})t+\phi\right]\cdot\omega_n\sqrt{1-\varepsilon_e^2}$$
$$+\sin\left[(\omega_n\sqrt{1-\varepsilon_e^2})t+\phi\right]\cdot\frac{\varepsilon_e\omega_n}{\sqrt{1-\varepsilon_e^2}}e^{-\varepsilon_e\omega_n t}$$

Since $e^{-\varepsilon_e\omega_n t} \neq 0$; so

$$\cos\left[(\omega_n\sqrt{1-\varepsilon_e^2})t+\phi\right]\sqrt{1-\varepsilon_e^2} = \sin\left[(\omega_n\sqrt{1-\varepsilon_e^2})t+\phi\right]\cdot\varepsilon_e$$

By putting $\sqrt{1-\varepsilon_e^2} = \sin\phi$ & $\cos\phi = \varepsilon_e$:-

$$\cos\left[(\omega_n\sqrt{1-\varepsilon_e^2})t+\phi\right]\sin\phi = \sin\left[(\omega_n\sqrt{1-\varepsilon_e^2})t+\phi\right]\cdot\cos\phi$$

$$\Rightarrow \sin(\omega_n\sqrt{1-\varepsilon_e^2})t = 0$$

$$\Rightarrow (\omega_n\sqrt{1-\varepsilon_e^2})t_p = n\pi \quad \text{where } n = 0,1,2,3,\ldots$$

$$\therefore \boxed{t_p = \frac{n\pi}{\omega_d}}$$

where, $n = 1,3,5 \rightarrow$ overshoot
$= 2,4,6 \rightarrow$ Undershoot.

4. M_p :-

$$M_p = c(t)|_{max} - 1 = c(t_p) - 1$$

$$\Rightarrow M_p = \left\{1 - \frac{e^{-\varepsilon_e\omega_n t_p}}{\sqrt{1-\varepsilon_e^2}}\sin(\omega_d t_p+\phi)\right\} - 1$$

$$\Rightarrow M_p = -\frac{e^{-\varepsilon_e\omega_n t_p}}{\sqrt{1-\varepsilon_e^2}}\sin(\omega_d t_p+\phi)$$

$\therefore$ But $t_p = \pi/\omega_d$; so, $M_p = \frac{-e^{-\varepsilon_e\omega_n t_p}}{\sqrt{1-\varepsilon_e^2}}\sin(\pi+\phi)$. $(n=1)$

$$\Rightarrow M_p = \frac{-e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}(-\sin\phi) \qquad [\because \sin(\pi+\phi) = -\sin\phi]$$

$$\Rightarrow M_p = \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\right) = e^{-\xi\omega_n t_p} \qquad [\because \sin\phi = \sqrt{1-\xi^2}]$$

$$\Rightarrow M_p = e^{-\xi\omega_n\left(\frac{\pi}{\omega_n\sqrt{1-\xi^2}}\right)}$$

$$\Rightarrow \boxed{M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}}$$

$\therefore$ In general, $\boxed{M_p = e^{-n\pi\xi/\sqrt{1-\xi^2}}}$

Where, $n = \begin{cases} 1, 3, 5 \rightarrow \text{overshoot} \\ 2, 4, 6 \rightarrow \text{undershoot.} \end{cases}$

$\Rightarrow$ % M_p = % peak overshoot $= 100 \times e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$.

5. t_s :-

At this time, the output is settled down with 2% or 5% of the final value.

$\rightarrow$ For 2% tolerance band,

$$c(t)\big|_{t=t_s} = 0.98.$$

$\rightarrow$ At this time, oscillatory term completely vanishes. So the only term which controls the amplitude is $e^{-\xi\omega_n t}$.

$$\therefore c(t)\big|_{t=t_s} = 1 - e^{-\xi\omega_n t_s} = 0.98$$

$$\Rightarrow e^{-\xi\omega_n t_s} = 1 - 0.98 = 0.02$$

$$\Rightarrow -\xi\omega_n t_s = \ln(0.02) = -3.912$$

$$\Rightarrow t_s = \frac{3.912}{\xi\omega_n} \Rightarrow \boxed{t_s \simeq \frac{4}{\xi\omega_n} \simeq 4T}$$

Where, $T = 1/\xi\omega_n$ = Time Constant of system.

$\rightarrow$ Similarly for 5% tolerance band,

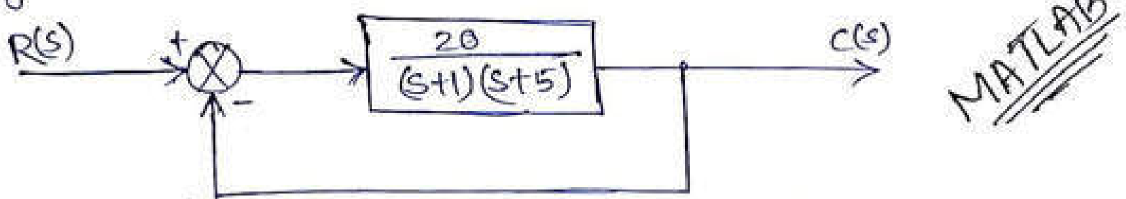
$$c(t)\big|_{t=t_s} = 0.95 = 1 - e^{-\xi\omega_n t_s}$$

$$\Rightarrow e^{-\xi\omega_n t_s} = 1 - 0.95 = 0.05$$

$$\Rightarrow t_s = \frac{2.995}{\xi\omega_n} \Rightarrow \boxed{t_s \simeq \frac{3}{\xi\omega_n} \simeq 3T}$$

Problems :-

① The block diagram of a unity feedback control system is shown in figure.



Determine the characteristic equation of the system, ω_n, ξ, ω_d, t_p, M_p, the time at which the 1st undershoot occurs, the time period of oscillations and the number of cycles completed before reaching the steady state.

Sol^n :- The overall transfer function is given by :-

$$\frac{C(s)}{R(s)} = \frac{\frac{20}{(s+1)(s+5)}}{1+\frac{20}{(s+1)(s+5)}\cdot 1} = \frac{20}{s^2+6s+25} = \frac{20}{25}\,\frac{25}{s^2+6s+25}$$

The characteristic eq^n is given by :-

$$\boxed{(s^2+6s+25)=0} \quad \text{(Ans)}$$

By comparing the ch. eq^n with the standard 2nd order system characteristic equation :-

$$\omega_n = \sqrt{25} = \boxed{5 \text{ rad/sec}} \quad \text{(Ans)}$$

$$\therefore\ 2\xi\omega_n = 6 \Rightarrow \xi = \frac{6}{10} \Rightarrow \boxed{\xi = 0.6} \quad \text{(Ans)}$$

$$\therefore\ \omega_d = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-(0.6)^2} = \boxed{4 \text{ rad/sec}} \quad \text{(Ans)}$$

Time at which maximum overshoot occurs is :-

$$t_p = \frac{\pi}{\omega_d} = \frac{3.141}{4} = \boxed{0.78 \text{ sec}} \quad \text{(Ans)}$$

The maximum overshoot is given by

$$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100 = e^{\frac{-0.6\pi}{\sqrt{1-(0.6)^2}}} \times 100 = \boxed{9.4\%} \quad \text{(Ans)}$$

The time at which the 1st undershoot occurs is :-

$$t = \frac{2\pi}{\omega_d} = \frac{2\times 3.141}{4} = \boxed{1.56 \text{ sec}} \quad \text{(Ans)}$$

$$\text{Time period of oscillation}(T) = \frac{2\pi}{\omega_d} = \frac{2\pi}{4} = \boxed{1.56 \text{ sec}} \quad \text{(Ans)}$$

This means, In 'T' seconds 1 cycle completes. $\left[\because \omega_d = 2\pi f = \frac{2\pi}{T}\right]$

⇒ In 1 second, '$1/T$' cycle completes.

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In 't_s' time, t_s/T cycles complete.

Now, before reaching steady state i.e, 2% tolerance band, the time will be settling time given by $t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.6\times5} = 1.33$ sec.

So, no. of cycles or no. of oscillations completed before reaching steady state is :-

$$\frac{\omega_d}{2\pi} \times \frac{4}{\xi\omega_n} = \frac{4}{2\pi} \times \frac{4}{0.6\times5} = \boxed{0.84 \text{ Cycle}} \quad \text{(Ans)}$$

② The maximum overshoot for a unity feedback control system having its forward path transfer function as $G(s) = \frac{K}{s(sT+1)}$ is to be reduced from 60% to 20%. The system i/p is a unit step function. Determine the factor by which 'K' should be reduced to achieve aforesaid reduction.

Solⁿ:- The overall transfer function is given by :-

$$\frac{C(s)}{R(s)} = \frac{K/s(sT+1)}{1+\frac{K}{s(sT+1)}\cdot 1} = \frac{K/T}{\left(s^2+\frac{1}{T}s+\frac{K}{T}\right)}.$$

The ch. eqⁿ is : $s^2 + \frac{1}{T}s + \frac{K}{T} = 0$.

$$\therefore \omega_n = \sqrt{K/T} \quad \& \quad 2\xi\omega_n = 1/T \Rightarrow \xi = \frac{1}{2\sqrt{KT}}.$$

Let 'K_1' be forward path gain when $M_{p_1} = 60\%$ & the corresponding damping ratio be 'ξ_1'.

$$\%M_{p_1} = e^{\frac{-\xi_1\pi}{\sqrt{1-\xi_1^2}}} \times 100\% \Rightarrow 60 = e^{\frac{-\xi_1\pi}{\sqrt{1-\xi_1^2}}} \times 100$$

$$\Rightarrow \log_e(0.6) = \frac{-\xi_1\pi}{\sqrt{1-\xi_1^2}} \Rightarrow -0.51 = \frac{-\xi_1\pi}{\sqrt{1-\xi_1^2}}$$

$$\Rightarrow \boxed{\xi_1 = 0.158.}$$

Let 'K_2' be the forward path gain when $M_{p_2} = 20\%$ & the corresponding damping ratio be 'ξ_2'.

$$\%M_{p_2} = e^{-\xi_2\pi/\sqrt{1-\xi_2^2}} \times 100 \Rightarrow 20 = e^{\frac{-\xi_2\pi}{\sqrt{1-\xi_2^2}}} \times 100$$

$$\Rightarrow \boxed{\xi_2 = 0.447}.$$

Assuming time constant 'T' to be constant :-

$$\frac{\xi_1}{\xi_2} = \frac{1}{2\sqrt{K_1T}} \times \frac{2\sqrt{K_2T}}{1} \Rightarrow \frac{K_2}{K_1} = \left(\frac{\xi_1}{\xi_2}\right)^2 = \left(\frac{0.158}{0.447}\right)^2 = \frac{1}{8}$$

$$\Rightarrow K_2 = 0.125\,K_1.$$

∴ So, 'K' should be reduced by a factor $\boxed{0.125}$. (Ans)

(3) The open loop transfer function of a unity feedback c/s is given by $G(s) = \frac{K}{s(sT+1)}$. By what factor the time const. 'T' should be multiplied so that the damping ratio is reduced from 0.6 to 0.3.

(4) The open loop transfer function for unity feedback is given by $G(s) = \frac{K(s+2)}{s^3+\alpha s^2+s+1}$. find the value of '$\alpha$' & 'K' such that $\xi = 0.2$ & $\omega_n = 3$ rad/sec.

Sol^n:- ch. equation : $1 + G(s)H(s) = 0$

$$\Rightarrow 1 + \frac{K(s+2)}{s^3+\alpha s^2+s+1} = 0$$

$$\Rightarrow \underbrace{s^3 + \alpha s^2 + (1+k)s + (1+2k) = 0.}_{\text{3rd order system.}}$$

Since, ξ & ω_n are given, that's why writing above ch. eq^n:-

$$s^3+\alpha s^2+(k+1)s+(2k+1) = (s+a)(s^2+2\xi\omega_n s+\omega_n^2)$$

$$\Rightarrow s^3+\alpha s^2+(k+1)s+(2k+1) = s^3+1.2s^2+9s+as^2+1.2as+9a$$

$$= s^3+(1.2+a)s^2+(9+1.2a)s+9a.$$

By comparing :-

$1.2+a=\alpha$; $9+1.2a = 1+k$; $9a = 1+2k$

$\Rightarrow \alpha = 1.2+a$

$= 1.2+2.57$

$\Rightarrow \boxed{\alpha = 3.77}$.

$\Rightarrow (1.2a-k = -8)\times 2$

$\quad \begin{array}{l} 9a - 2k = 1 \\ (-)\ \ (+)\ \ (-) \end{array}$

$(2.4-9)a = -16-1$

$\Rightarrow -6.6a = -17$

$\Rightarrow \boxed{a = 2.57}$

$\Rightarrow 9a-2k=1$

$\Rightarrow 2k = 9a-1$

$\Rightarrow k = \frac{(9\times 2.57)-1}{2}$

$\Rightarrow \boxed{k = 11.09}$. (Ans).

(5) A 2nd order c/s is represented by a transfer function given below: $\frac{Q_0(s)}{T(s)} = \frac{1}{Js^2+fs+K}$.

where Q_0 is the proportional output & T is the input torque. A step i/p of 10Nm is applied to the system & test results are given below:

(a) $M_p = 6\%$ (b) $t_p = 1$ sec & (c) the steady state value of the o/p is 0.5 radian. Determine the values 'J', 'f' and 'k'.

[Ans: $J = 1.13$ kg-m^2 ; $f = 6.26$ Nm/(rad/sec), $K = 20$N.]

Control System Engineering

Topic:

Steady State Error

Contributed By:

Gyana Ranjan Biswal

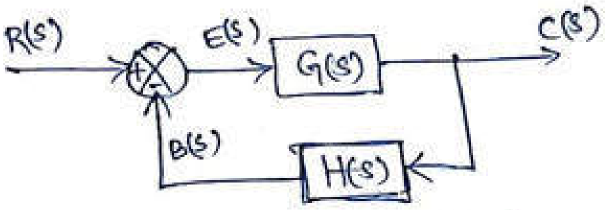
STEADYSTATE ERROR :-

→ If the actual output of a control system during steady state deviates from the reference input (i.e, desired output), the system is said to possess a steady state error.

→ As the steady state error is an index of accuracy of a control system, therefore it should be minimum as far as possible.

→ These errors arise from the nature of inputs, type of system & from non-linearities of system components.

→ Referring to the block diagram :-



[Basic block diagram of -ve feedback closed loop system]

$E(s) = R(s) - B(s)$

$= R(s) - C(s)H(s)$

$= R(s) - E(s)G(s)H(s)$

$\Rightarrow E(s) = \dfrac{R(s)}{1+G(s)H(s)}$.

→ steady state error, $e_{ss} = \lim_{t\to\infty} e(t)$

$\Rightarrow e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$ [∵ Using final value theorem]

$$\Rightarrow \boxed{e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}}$$

→ Now, it is clear that, the magnitude of steady state error in a closed loop c/s depends on its open loop transfer function [G(s) H(s)] & input signal R(s).

Type of a system :-

Generally the OLTF of a system is given by :-

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots}{s^N(1+sT_1)(1+sT_2)\ldots}$$

Where, K = forward path gain

$-\frac{1}{T_a}, -\frac{1}{T_b}$ are the zeros & $-\frac{1}{T_1}, -\frac{1}{T_2}, \ldots$ are the poles.

N = No. of poles at the origin.

→ The type of a system is defined as the number of poles at the origin possessed by an open-loop transfer function.

→ for $N=0$, Type-0 system

$N=1$, Type-1 system

$N=2$, Type-2 system & so on.

Ex:-

$$OLTF = \frac{2}{(S+1)} \longrightarrow \text{Type-0 system.}$$

$$G(s)\,H(s) = \frac{(s+5)}{s(s+6)(s+3)} \longrightarrow \text{Type-1 system.}$$

STATIC ERROR COEFFICIENTS :-

→ As the steady state error is considered during steady state period, the error is also called as static error.

→ There are three static error co-efficients associated with three different inputs such as unit step (displacement) or unit ramp (velocity) or unit parabolic (acceleration).

(1) Static positional error coefficient (K_p) :-

→ It is associated with unit step applied to a closed loop system & is determined as :-

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\times 1/s}{1+G(s)H(s)} \quad \left[\because R(s) = 1/s\right]$$

$$\Rightarrow e_{ss} = \frac{1}{1+\lim\limits_{s\to 0} G(s)H(s)} = \frac{1}{1+K_p} \Rightarrow \boxed{e_{ss} = \frac{1}{1+K_p}}$$

Where, K_p = static positional error coefficient

$$\Rightarrow \boxed{K_p = \lim_{s\to 0} G(s)H(s)}.$$

(2) Static velocity error coefficient (K_v) :-

→ It is associated with unit ramp input applied to a CL system & is determined as :-

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\times 1/s^2}{1+G(s)H(s)} \quad \left[\because R(s) = 1/s^2\right].$$

$$\Rightarrow e_{ss} = \lim_{s\to 0} \frac{1}{s+sG(s)H(s)} = \lim_{s\to 0} \frac{1}{sG(s)H(s)}$$

$$\Rightarrow e_{ss} = \frac{1}{\lim\limits_{s\to 0} sG(s)H(s)} = \frac{1}{K_v} \Rightarrow \boxed{e_{ss} = \frac{1}{K_v}}.$$

Where, K_v = static velocity error co-efficient

$$\Rightarrow \boxed{K_v = \lim_{s\to 0} sG(s)H(s)}$$

(iii) Static acceleration error coefficient :-

→ It is associated with unit parabolic input applied to a CL system & is determined as :-

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\times 1/s^3}{1+G(s)H(s)}$$

$$\Rightarrow e_{ss} = \lim_{s\to 0}\frac{1}{s^2+s^2G(s)H(s)} \Rightarrow e_{ss} = \frac{1}{\lim_{s\to 0} s^2G(s)H(s)}.$$

$$\Rightarrow \boxed{e_{ss} = \frac{1}{K_a}}.$$

Where, K_a = Static acceleration error co-efficient.

$$\Rightarrow \boxed{K_a = \lim_{s\to 0} s^2G(s)H(s).}$$

Effect of Open loop Transfer function on steady state error :-

(or) Generalized Static Error Coefficient Method

(1) For step Input :-

(i) Type-0 System :-

The O.L. Transfer function is given by :-

$$G(s)H(s) = \frac{K(sT_a+1)(sT_b+1)\ldots}{(1+sT_1)(1+sT_2)\ldots}$$

$$\therefore K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{K(sT_a+1)(sT_b+1)\ldots}{(1+sT_1)(1+sT_2)}$$

$$\Rightarrow K_p = K$$

$$\therefore \boxed{e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K}}.$$

(ii) Type-1 System :-

The O.L. Transfer function is given by :-

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots}{s(1+sT_1)(1+sT_2)\ldots}$$

$$\therefore K_p = \lim_{s\to 0} G(s)H(s) = \infty$$

$$\therefore \boxed{e_{ss} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0}.$$

(iii) Type-2 System :-

The O.L. Transfer function is given by :-

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\cdots}{s^2(1+sT_1)(1+sT_2)\cdots}$$

$$\therefore K_p = \lim_{s\to 0} G(s)H(s) = \infty .$$

$$\therefore \boxed{e_{ss} = \frac{1}{1+K_p} = 0}$$

(2) For unit ramp input :-

(i) Type-0 system :-

O.L. T/F is :- $G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$

$$\therefore K_v = \lim_{s\to 0} sG(s)H(s) = 0 .$$

$$\therefore \boxed{e_{ss} = 1/K_v = 1/0 = \infty}$$

(ii) Type-1 system :-

O.L. T/F is : $G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)}{s(1+sT_1)(1+sT_2)}$

$$\therefore K_v = \lim_{s\to 0} sG(s)H(s) = K$$

$$\therefore \boxed{e_{ss} = 1/K_v = 1/K}$$

(iii) Type-2 System :-

O.L. T/F is : $G(s)H(s) = \frac{K(sT_a+1)(1+sT_b)}{s^2(1+sT_1)(1+sT_2)}$

$$\therefore K_v = \lim_{s\to 0} sG(s)H(s) = \infty$$

$$\therefore \boxed{e_{ss} = 1/K_v = 1/\infty = 0}$$

(3) For unit Parabolic input :-

(i) Type-0 System :-

OLTF is : $G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)}{(1+sT_1)(1+sT_2)}$

$$\therefore K_a = \lim_{s\to 0} s^2 G(s)H(s) = 0 .$$

$$\therefore \boxed{e_{ss} = 1/K_a = 1/0 = \infty}$$

(ii) Type-1 System :-

OLTF is : $G(s)H(s) = \dfrac{K(1+sT_a)(1+sT_b)}{s(1+sT_1)(1+sT_2)}$.

$\therefore K_a = \lim\limits_{s \to 0} s^2 G(s)H(s) = 0$.

$$\boxed{\therefore e_{ss} = \frac{1}{K_a} = \infty}$$

(iii) Type-2 System :-

OLTF is : $G(s)H(s) = \dfrac{K(1+sT_a)(1+sT_b)}{s^2(1+sT_1)(1+sT_2)}$.

$\therefore K_a = \lim\limits_{s \to 0} s^2 G(s)H(s) = K$

$$\boxed{\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{K}}$$

Now summarizing above results, steady-state error (e_{ss}) in a table format will be :-

Type / Input	$A\,u(t)$	$A\,t\,u(t)$	$A\,\frac{t^2}{2}\,u(t)$
Type-0	$\frac{A}{1+K_p}$	∞	∞
Type-1	0	$\frac{A}{K}$	∞
Type-2	0	0	A/K

S.E.C K_p	K_v	
K	0	0
∞	K	0
∞	∞	K

Observations :-

① As type of the system increases, steady state error decreases for same input.

② As derivative of the input increases, steady state error (e_{ss}) increases for same type of system.

③ For e_{ss} non-zero finite number, steady state error decreases as K (process gain) increases.

Disadvantages Of Static Error Coefficient Method :-

→ This method can't give the error if inputs are other than the three standard test inputs.

→ It doesn't provide variation of error w.r.t. time.

→ This method is only applicable to stable systems.

Generalized Error Coefficient Method or (Dynamic Error Coefficients)

We know that, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Let us assume, $E(s) = f_1(s) \cdot f_2(s)$

Where, $f_1(s) = \frac{1}{1+G(s)H(s)}$ & $f_2(s) = R(s)$.

By using convolution Integral,

$$\mathcal{L}^{-1}[E(s)] = e(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau = \int_0^t f_1(\tau) R(t-\tau)\, d\tau$$

By expanding $R(t-\tau)$ using Taylor series form :-

$$R(t-\tau) = R(t) - \tau R'(t) + \frac{\tau^2}{2!} R''(t) - \frac{\tau^3}{3!} R'''(t) + \cdots$$

After substituting :-

$$e(t) = \int_0^t f_1(\tau)\left[R(t) - \tau R'(t) + \frac{\tau^2}{2!} R''(t) - \frac{\tau^3}{3!} R'''(t) + \cdots\right] d\tau$$

$$= \int_0^t R(t) \cdot f_1(\tau)\, d\tau - \int_0^t \tau R'(t) f_1(\tau)\, d\tau + \cdots$$

$$\therefore e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left[\int_0^t R(t) f_1(\tau)\, d\tau - \int_0^t \tau R'(t) f_1(\tau)\, d\tau + \cdots\right]$$

$$\Rightarrow e_{ss} = R(t)\int_0^\infty f_1(\tau)\, d\tau - R'(t)\int_0^\infty \tau f_1(\tau)\, d\tau + R''(t)\int_0^\infty \frac{\tau^2}{2!} f_1(\tau)\, d\tau \cdots$$

$$\Rightarrow \boxed{e_{ss} = K_0 R(t) + K_1 R'(t) + \frac{K_2}{2!} R''(t) + \cdots\cdots}$$

Where, K_0, K_1, K_2 are called dynamic error coefficients and these are defined by :-

$K_0 = \int_0^\infty f_1(\tau)\,d\tau$, $K_1 = -\int_0^\infty \tau f_1(\tau)\,d\tau$, $K_2 = \int_0^\infty \tau^2 f_1(\tau)\,d\tau \ldots$

To calculate :-

→ $K_0 = \int_0^\infty f_1(\tau)\,d\tau \Rightarrow K_0 e^{-s\tau} = \int_0^\infty f_1(\tau).d\tau\, e^{-s\tau} = F_1(s)$ — (1)

Take $\lim_{s\to 0}$ on both sides :-

$\lim_{s\to 0} K_0 e^{-s\tau} = \lim_{s\to 0} F_1(s) \Rightarrow \boxed{K_0 = \lim_{s\to 0} F_1(s)}$

where, $F_1(s) = \dfrac{1}{1+G(s)H(s)}$.

→ Take derivative w.r.t s of eqn(1) :-

$-\tau K_0 e^{-s\tau} = \dfrac{dF_1(s)}{ds} \Rightarrow -\tau \int_0^\infty f_1(\tau)d\tau.\, e^{-s\tau} = \dfrac{dF_1(s)}{ds}$

$\Rightarrow K_1 e^{-s\tau} = \dfrac{dF_1(s)}{ds}$.

$\Rightarrow \lim_{s\to 0} K_1 e^{-s\tau} = \lim_{s\to 0} \dfrac{dF_1(s)}{ds} \Rightarrow \boxed{K_1 = \lim_{s\to 0} \dfrac{dF_1(s)}{ds}}$.

→ **In general :-** $\boxed{K_n = \lim_{s\to 0} \dfrac{d^n F_1(s)}{ds}}$

Advantages :-

(1) It gives variation of error as a function of time

(2) For any input, other than standard i/p, error can be determined.

Problems :-

(1) A unity feedback system having open loop transfer function $G(s) = \dfrac{K(s+3)}{s(s^3+5s^2+6s)}$.

Find (i) Type of the system (ii) Error co-efficients

(iii) Find the value of K when the steady state error due to parabolic input is 0.3.

Solⁿ :- To determine type of system arrange G(s)H(s) in time-constant form.

$G(s)H(s) = \dfrac{K(1+s/3)3}{s^2(s+2)(s+3)} = \dfrac{K/2\,(1+s/3)}{s^2(1+s/2)(1+s/3)}$.

(i) System is Type-2 system.

(ii) Static error co-efficients :-

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K/2(1+s/3)}{s^2(1+s/2)(1+s/3)} = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} \frac{K/2(1+s/3)}{s(1+s/2)(1+s/3)} = \infty$$

$$K_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} \frac{K/2(1+s/3)}{(1+s/2)(1+s/3)} = K/2 .$$

(iii) Steady-state error due to parabolic input is :-

$$e_{ss} = \frac{1}{K_a} = 0.3 \text{ (given)}$$

$$\Rightarrow e_{ss} = \frac{1}{(K/2)} = \frac{2}{K} = 0.3 \Rightarrow K = \frac{2}{0.3} = \boxed{6.67} \text{ . (Ans)}$$

(2) Find K_p, K_v, K_a & steady-state error for a system with open loop transfer function as : $G(s)H(s) = \frac{10(s+2)(s+3)}{s(s+1)(s+5)(s+4)}$, where the input is $r(t) = 3+t+t^2$.

Solⁿ:- The open loop transfer function in time constant form :-

$$G(s)H(s) = \frac{10\times2\times3\,(1+s/2)(1+s/3)}{(5\times4)\,s\,(1+s)(1+s/5)(1+s/4)} = \frac{3(1+s/2)(1+s/3)}{s(1+s)(1+s/5)(1+s/4)} .$$

Now, static error co-efficients :-

$$K_p = \lim_{s\to 0} G(s)H(s) = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = 3 .$$

$$K_a = \lim_{s\to 0} sG(s)H(s) = 0 .$$

$\therefore$ Input $= 3+t+t^2 = 3+t+2\cdot\frac{t^2}{2}$.

So there are three standard inputs as :-

step having amplitude = 3
ramp " " = 1
Parabolic " " = 2.

for step i/p :-

$$e_{ss_1} = \frac{A}{1+K_p} = \frac{3}{1+\infty} = 0 .$$

For ramp i/p :-

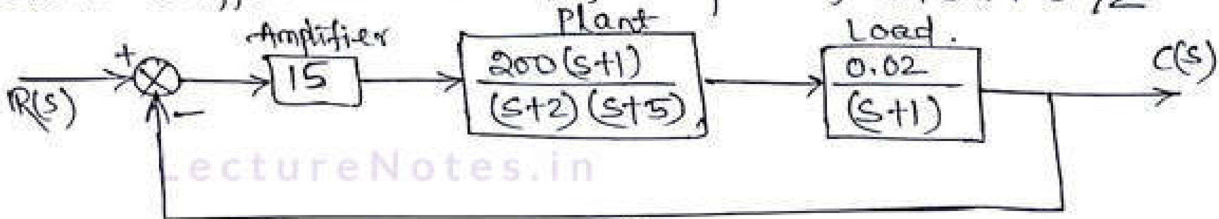
$$e_{ss_2} = \frac{A}{K_v} = \frac{1}{3} .$$

for Parabolic i/p :- $e_{ss_3} = \frac{A}{K_a} = \infty$.

$\therefore$ So, the steady-state error will be :-

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = 0 + \frac{1}{3} + \infty = \boxed{\infty} \text{ (Ans)}$$

③ For system shown in figure, find the error using dynamic error coefficient method for input of $6 + 5t + 6t^2/2$.



Sol^n:→ $G(s) = OLTF = \dfrac{15 \times 200 \cancel{(s+1)} \times 0.02}{\cancel{(s+1)}(s+2)(s+5)} = \dfrac{6}{(1+s/2)(1+s/5)}$

For dynamic error coefficient method :-

$$e_{ss}(t) = K_0 R(t) + K_1 R'(t) + \frac{K_2}{2!} R''(t) + \ldots\ldots$$

Where, $K_0 = \lim_{s \to 0} F_1(s)$, $K_1 = \lim_{s \to 0} \dfrac{dF_1(s)}{ds}$, $K_2 = \lim_{s \to 0} \dfrac{d^2 F_1(s)}{ds^2}$.

Where, $F_1(s) = \dfrac{1}{1 + G(s)H(s)} = \dfrac{1}{1 + \dfrac{6}{(1+s/2)(1+s/5)}} = \dfrac{s^2+7s+10}{s^2+7s+70}$

$\therefore K_0 = \lim_{s \to 0} F_1(s) = \frac{1}{7} = 0.1428$

$K_1 = \lim_{s \to 0} \dfrac{dF_1(s)}{ds} = \lim_{s \to 0} \dfrac{120s + 420}{(s^2+7s+70)^2} = \dfrac{420}{(70)^2} = 0.0857$.

$K_2 = \lim_{s \to 0} \dfrac{d^2 F_1(s)}{ds^2} = 7.346 \times 10^{-3}$.

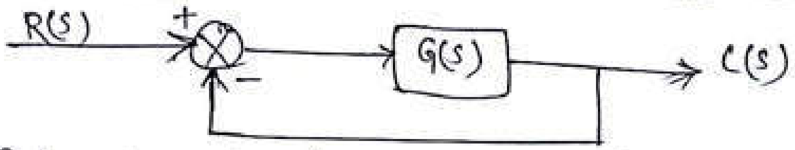
$\therefore e_{ss}(t) = K_0 R(t) + K_1 R'(t) + \dfrac{K_2}{2!} R''(t)$

$\therefore R(t) = 6 + 5t + 6t^2/2$; $R'(t) = 5 + 6t$; $R''(t) = 6$.

$\therefore e_{ss}(t) = 0.1428[6 + 5t + 3t^2] + 0.0857[5 + 6t] + \dfrac{7.346 \times 10^{-3}}{2!} \times 6$

$\Rightarrow \boxed{e_{ss}(t) = 0.4284t^2 + 1.2282t + 1.3073}$

④



Determine steady-state error for input $= u(t)$ & $G(s) = \dfrac{1}{s-2}$.

[Ans: $e(t) = 2u(t) - e^{t} u(t)$; $e_{ss} = -\infty$]

Control System Engineering

Topic:

Control Components

Contributed By:

Gyana Ranjan Biswal

(A)

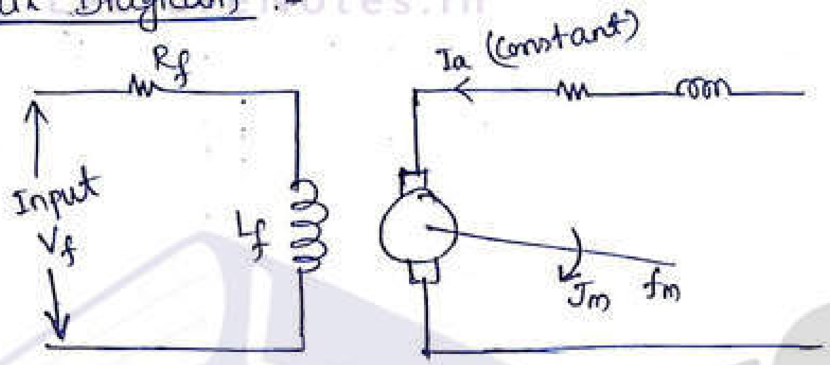
CONTROL COMPONENTS

(A) D.C. SERVO MOTOR :-

(1) Field Controlled D.C. Servo Motor :→

→ Transfer function of a field controlled d.c. motor relating angular shift in the shaft and the input field voltage.

Circuit Diagram :-



(Field controlled d.c. motor)

Let, V_f = Applied voltage to the field winding.

R_f, L_f = Resistance & inductance of field winding.

i_a = Armature Current

J_m, f_m = Moment of inertia & coefficient of friction at the motor shaft.

θ_m = Angular shift in the motor.

ω_m = Angular velocity.

T_M = developed motor torque.

i_f = field current.

→ Here, the armature current supplied is kept constant & thus the motor shaft is controlled by input voltage to the field i.e, V_f.

→ As i_a is kept constant, $T_M \propto i_f \Rightarrow \boxed{T_M = K_f i_f}$ ——— (1)

Where, K_f = Motor torque constant (Nm/A)

Applying KVL in field side :→

$$\boxed{V_f = R_f i_f + L_f \frac{di_f}{dt}} \text{ ——— (2).}$$

The torque developed in the motor shaft,

$$\boxed{T_M = J_m \frac{d^2\theta_m}{dt^2} + f_m \frac{d\theta_m}{dt}} \quad \text{——— (3)}$$

Taking laplace transform of the above equations:-

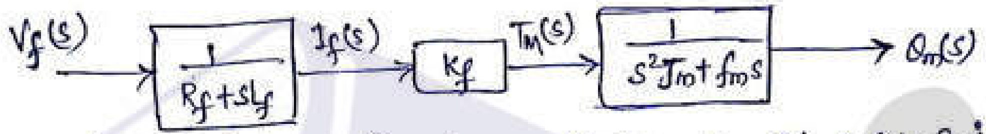
$$V_f(s) = R_f I_f(s) + s L_f I_f(s)$$

$$T_M(s) = K_f I_f(s)$$

$$T_M(s) = s^2 J_m \theta_m(s) + s f_m \theta_m(s)$$

$$\Rightarrow T_M(s) = s J_m \omega_m(s) + f_m \omega_m(s)$$

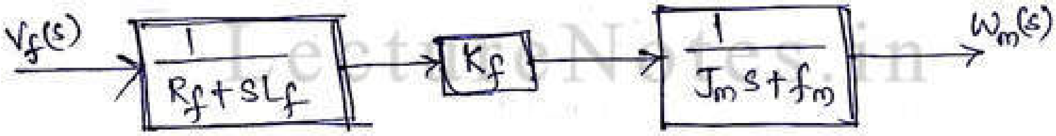
By representing the above equations in a block diagram:-



So, the transfer function relating the o/p $\theta_m(s)$ & i/p $V_f(s)$ is:-

$$\frac{\theta_m(s)}{V_f(s)} = \frac{K_f}{s(R_f+sL_f)(sJ_m+f_m)}$$

→ The block diagram representing the output $\omega_m(s)$ & the input $V_f(s)$ is given by:→



∴ So, the transfer function of field controlled dc servo motor:→

$$\boxed{\frac{\omega_m(s)}{V_f(s)} = \frac{K_f}{(R_f+sL_f)(sJ_m+f_m)}}$$

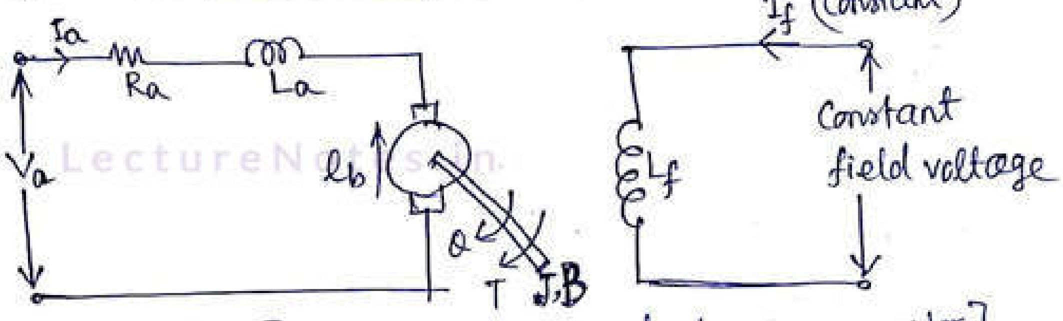
Notes:→

(1) It is an example of open loop control system.

(2) This is practically not preferable as its inductive circuit of field winding is having high time constant.

(2) Armature Controlled d.c. Servo motor :-

→ The transfer fun of an armature controlled d.c. motor relates the angular shift in the shaft & the input armature voltage. The circuit diagram is:-



[Armature Controlled d.c. servo motor]

→ Here, the field current 'if' supplied to field winding is kept constant by making the excitation voltage constant.

→ Assume, armature reaction is neglected & magnetic circuit is linear.

Let R_a, L_a = Resistance & self inductance of Armature.
i_a, i_f = Armature & field current.
e_b = back emf
V_a = applied voltage
T_M = Torque developed by the motor.
θ = Angular displacement
J, B = Moment of inertia & co-efficient of friction of the motor shaft.

→ Applying KVL in armature circuit :→

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + e_b \Rightarrow \boxed{V_a - e_b = R_a i_a + L_a \frac{di_a}{dt}} \quad \text{—(1)}$$

∴ Now the back emf induced in the motor is given by:-

$$e_b \propto \phi \omega \Rightarrow e = K_b \omega \quad (\because \phi = \text{Constant})$$

$$\Rightarrow \boxed{e_b = K_b \frac{d\theta}{dt}} \text{ —(2) where, } K_b = \text{back emf Constant}$$

∴ Torque developed by the motor is →

$$T_M \propto \phi i_a \Rightarrow \boxed{T_M = K_T i_a} \quad \text{— (3)} \quad (\because \phi = \text{Const.})$$

where, K_T = Motor torque Constant.

∴ The dynamic torque equation of the motor shaft is →

$$\boxed{T_M = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}} \quad \text{--- (4)}$$

Taking laplace transform of the above equations :-

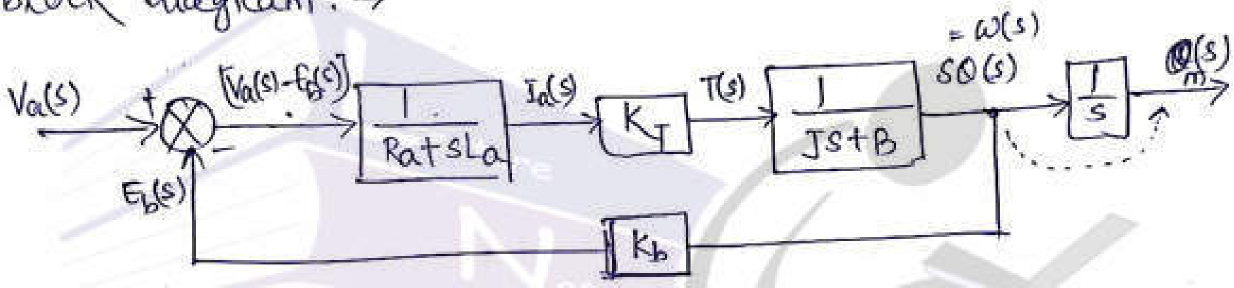
$$V_a(s) - E_b(s) = I_a(s)\left[R_a + sL_a\right]$$

$$E_b(s) = K_b\, s\theta(s)$$

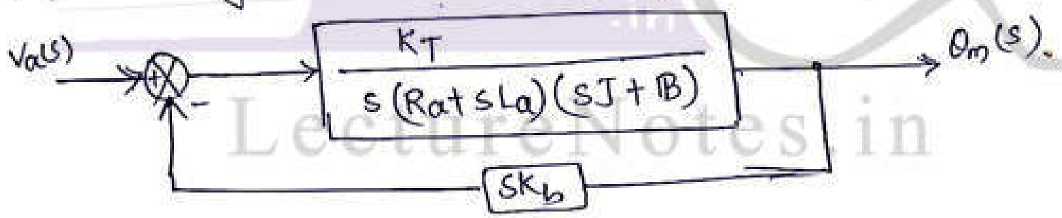
$$T_M(s) = K_T I_a(s)$$

$$T_M(s) = \left[s^2 J + sB\right]\theta(s). \Rightarrow T_M(s) = (sJ + B)s\theta(s).$$

By combining all the equations & representing them in the block diagram :→



By reducing the above block diagram :-



∴ The overall transfer function relating the output $\theta_m(s)$ & the input $V_a(s)$ is given by →

$$\frac{\theta_m(s)}{V_a(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K_T}{s(R_a+sL_a)(sJ+B)}}{1+\frac{K_T}{s(R_a+sL_a)(sJ+B)} \times sK_b}.$$

$$\Rightarrow \frac{\theta_m(s)}{V_a(s)} = \frac{K_T}{s(R_a+sL_a)(sJ+B) + sK_T K_b}$$

∴ By neglecting armature inductance, then the t/f is :→

$$\frac{\theta_m(s)}{V_a(s)} = \frac{K_T}{sR_a(sJ+B) + sK_T K_b} = \frac{K_T}{s(sR_aJ + R_aB + K_T K_b)}$$

⑧

$$\Rightarrow \frac{\theta_m(s)}{V_a(s)} = \frac{\left(\frac{K_T}{R_aB + K_TK_b}\right)}{s\left[\frac{sR_aJ}{R_aB+K_TK_b} + 1\right]} \Rightarrow \boxed{\frac{\theta_m(s)}{V_a(s)} = \frac{K_m}{s(1+sT_m)}}.$$

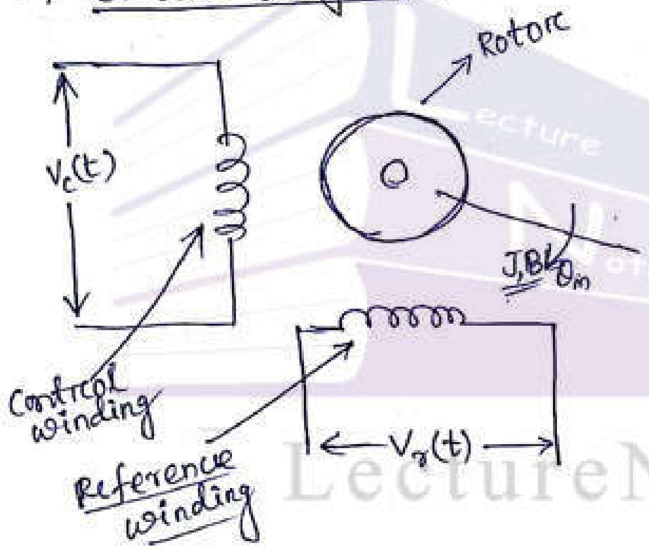
Where, $K_m = \dfrac{K_T}{R_aB + K_TK_b}$ = Motor gain constant

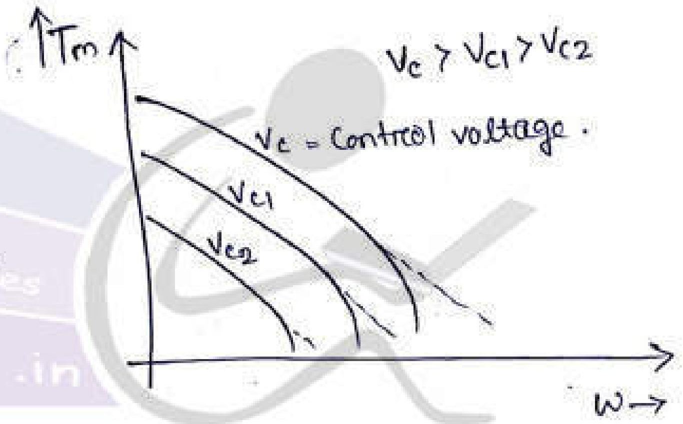
& $T_m = \dfrac{R_aJ}{R_aB + K_TK_b}$ = Motor time constant.

A.C. SERVOMOTOR :-

→ Transfer function of a two-phase a.c. servo motor relates the angular shift in the shaft & the input control voltage.

→ Circuit Diagram :-





[Torque - speed characteristic]

→ A two-phase a.c. servomotor is a two phase induction motor having drag cup type rotor construction.

→ The control voltage $V_c(t)$ is applied to the control winding & a fixed voltage is applied to the reference winding. These two windings are placed 90° spacially (in space) to each other.

→ The control voltage results in the developement of motor torque (T_M) and also the speed-torque ch. is shown above.

Let, J & B = Moment of Inertia & Co-efficient of friction of motor.

θ_m = Angular shift in motor shaft.

ω_m = Angular velocity in motor.

→ From the torque-speed characteristic, the relation betn motor torque & the speed is given by →

$$\boxed{T_M = m\omega_m + KV_c} \quad \text{——— (1).}$$

where, 'm' is the slope of the line & KV_c is the y-intercept or initial condition.

∴ so 'm' & 'K' can be found as follows :-

(i) If speed is zero, let the torque is T_0 (stalling torque) & this is directly proportional to control voltage (V_c).

$$\therefore T_0 = KV_c \Rightarrow \boxed{K = \frac{T_0}{V}} \quad (Nm/V).$$

(ii) slope of T-ω characteristic is →

$$\boxed{m = -\frac{T_0}{\omega_0}} \quad [Nm/(rad/sec)].$$

∴ so, now the torque eqn can be expressed as :→

$$\boxed{T_M = m\frac{d\theta_m}{dt} + KV_c} \quad \text{——— (2)}$$

∴ And the dynamic eqn of motor shaft is :-

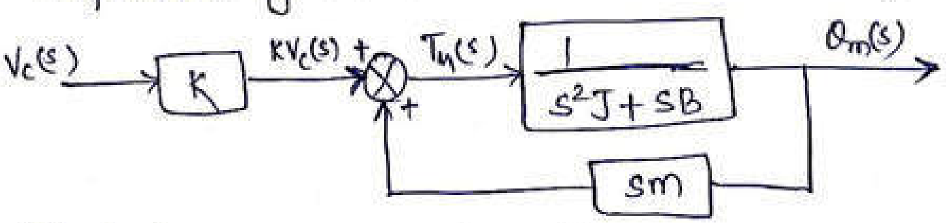
$$\boxed{T_M = J\frac{d^2\theta_m}{dt^2} + B\frac{d\theta_m}{dt}} \quad \text{——— (3)}$$

Taking Laplace transform of the above eqns :-

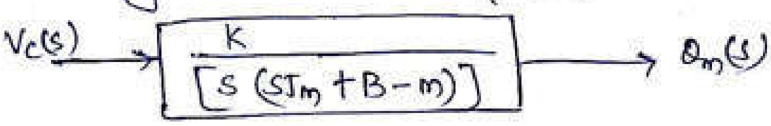
$$T_M(s) = sm\,\theta_m(s) + KV_c(s)$$

$$\& \quad T_M(s) = s^2J\,\theta_m(s) + sB\,\theta_m(s)$$

Representing the above two eqns in block diagram :-



Reducing the above block diagram →



So, the transfer function is given by →

$$\frac{\theta_m(s)}{V_c(s)} = \frac{K}{s(sJ+B-m)} = \frac{K/(B-m)}{s\left[1+\frac{sJ}{B-m}\right]}$$

$$\Rightarrow \boxed{\frac{\theta_m(s)}{V_c(s)} = \frac{K_m}{s(1+sT_m)}}$$

Where, $K_m = \frac{K}{B-m}$ = Motor gain Constant

$T_m = \frac{J}{B-m}$ = Motor time Constant.

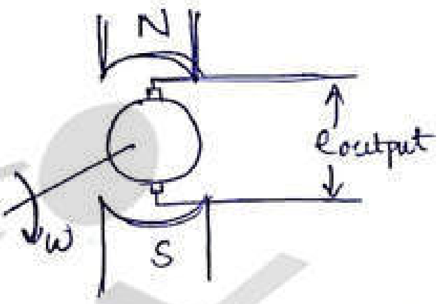
TACHOMETERS (or) TACHOGENERATORS :-

(i) D.C. Tachogenerator :-

→ It Converts a rotational speed into a proportional d.c. voltage, on the principle of d.c. generator.

→ It uses permanent magnet for producing magnetic field.

→ As magnetic flux is constant, the induced voltage at the terminals of armature is proportional to the speed.



[D.C. Tachogenerator]

i.e, $e \propto \omega \Rightarrow \boxed{e = K_{tg}\,\omega}$ — (1)

Where, K_{tg} = tachogenerator Const $[V/(rad/sec)]$.

Taking Laplace transform of eq^n(1) :→

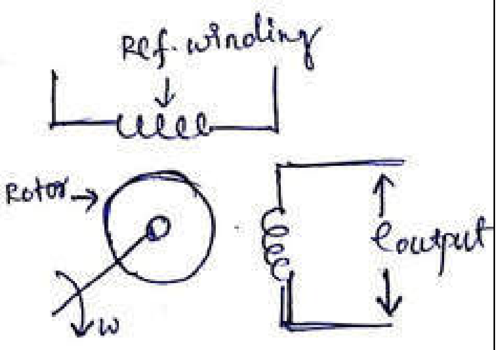
$E(s) = K_{tg} W(s) \Rightarrow \boxed{\frac{E(s)}{W(s)} = K_{tg}}$ → T/F of D.C. tachogenerator.

(ii) A.C. Tachogenerator :-

→ It Converts a rotational speed into a proportional a.c. voltage & works on the principle of induction generator.

→ The tachogenerator is coupled to a shaft to which speed is to be measured.

→ The ref winding is supplied by a ref. voltage & the o/p voltage 'e' is induced across the o/p winding.



→ So, the amplitude & the phase of the output voltage depends on the direction of rotation.

i.e, $e \propto \omega \Rightarrow \boxed{e = K_{tg}\omega}$. Where K_{tg} = tachogenerator constant V/(rad/sec)

Taking Laplace transform :→

$$E(s) = K_{tg}\, W(s) \Rightarrow \boxed{\frac{E(s)}{W(s)} = K_{tg}}$$

ERROR DETECTORS :-

→ These are used to measure the error signal in control system. It takes input as the difference in the actual output and the desired output and gives the corresponding output voltage (i.e, error signal).

→ In a way, error detectors also perform the function of a transducer.

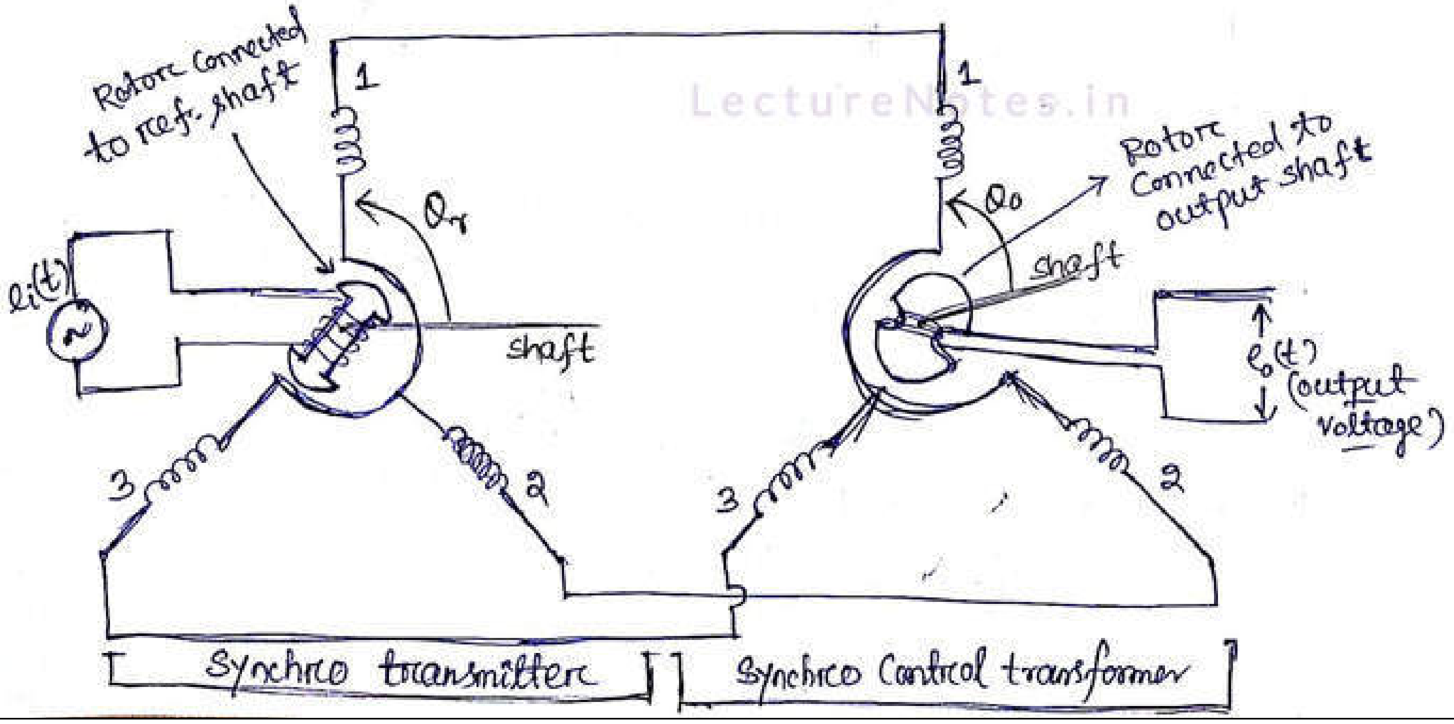
→ There are two types of error detectors :-

(i) potentiometer error detector

(ii) synchro (or selsyn) error detector.

SYNCHRO :-

→ It is the combination of synchro transmitter and synchro control transmitter arrangement for converting angular position difference into a proportional a.c. voltage.



→ The winding on the rotor of a synchro transmitter is connected to an a.c. supply voltage & the this rotor is held fixed at a desired angular position (θ_r).

→ The two stator windings of synchro transmitter & synchro control transformer are connected together.

→ The rotor of synchro transmitter is salient pole type & that of synchro control transformer is cylinderical type.

→ If the position of the o/p shaft (θ_0) is present, this results in an angular error in position given by $\theta_e = (\theta_r - \theta_0)$.

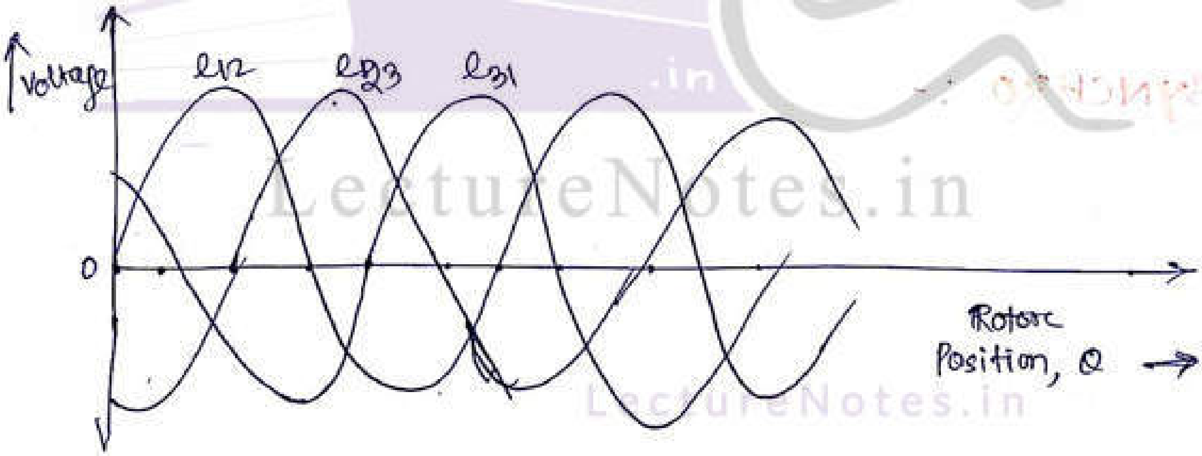
→ If the rotor of synchro transmitter shifts in clockwise direction, then the voltages in the stator coils are :-

$$e_{1n} = K E_m \sin \omega t \cdot \cos \theta .$$

$$e_{2n} = K E_m \sin \omega t \cdot \cos(120^\circ - \theta)$$

$$e_{3n} = K E_m \sin \omega t \cdot \cos(240^\circ - \theta)$$

→ from the above eqns, it is clear that, the magnitude of the voltages vary sinusoidally w.r.t. θ.



→ These three voltages are connected to three stator windings of the control transformer & produce a resultant flux in the air gap, which in turn induces a voltage across the rotor winding of the control transformer.

→ The magnitude of this induced voltage 'e' developed across the rotor of control transformer is given by :→

$$\boxed{e = K_s \sin(\theta_r - \theta_0)}$$

→ In practice, $(\theta_r - \theta_o)$ is very small, so we can write :→

$$e = K_s(\theta_r - \theta_o) \Rightarrow \boxed{e = K_s \theta_e} \quad \text{— (1)}$$

where, e = error output voltage.

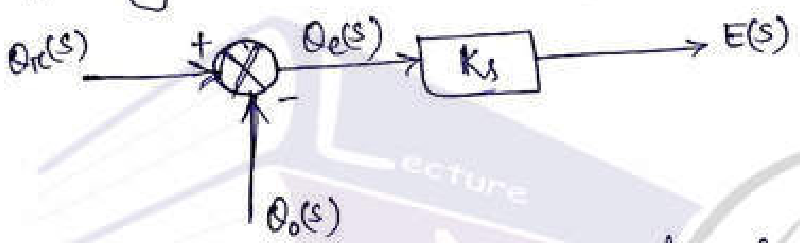
$\theta_e = \theta_r - \theta_o$ = Angular error.

K_s = Constant (V/rad).

→ Taking Laplace transform :→

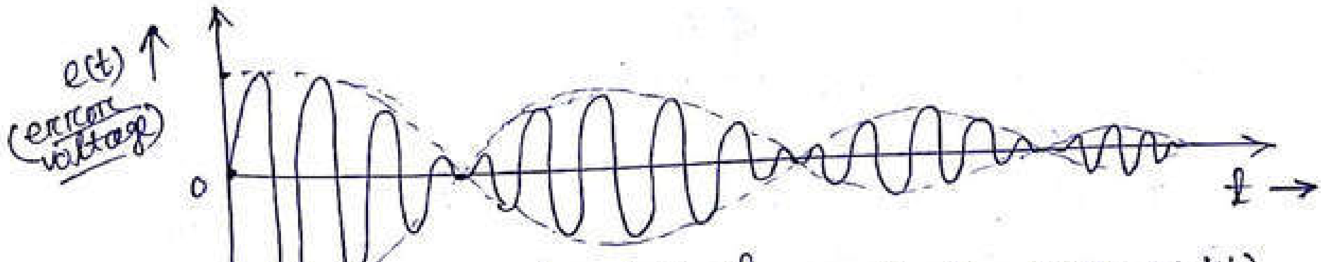
$$E(s) = K_s[\theta_r(s) - \theta_o(s)]. \Rightarrow \boxed{\frac{E(s)}{\theta_e(s)} = K_s}.$$

Showing the above relationship in block diagram :-



→ The variation of the magnitude of the o/p voltage 'e' of synchro error detector is a function of time is shown as below figure :-

(variation of Angular error (θ_e) w.r.t. 't')



(variation of error o/p voltage [e(t)] w.r.t. 't')

→ The above waveform shows that, the output error voltage varies linearly with respect to Angular error.

STEPPER MOTOR :-

→ It is an electromechanical device which converts electrical pulses into discrete mechanical movements. The shaft or spindle of a stepper motor rotates in discrete step increments when electrical command pulses are applied to it in the proper sequence.

→ Typical types of stepper motors can rotate 2°, 2.5°, 5°, 7.5° & 15° per input electrical pulse.

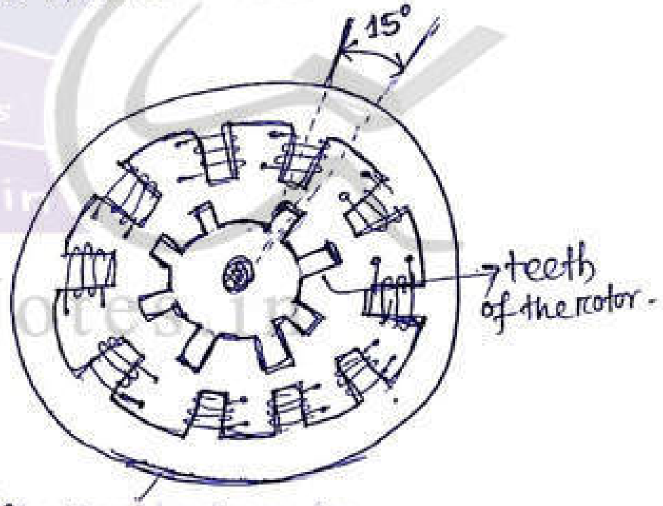
→ These are typically Brushless DC motors.

→ Stepper motors are usually operated in open loop mode, while most DC motors are operated as closed loop.

→ Mainly there are 3 types of stepper motors:-

(i) Variable-reluctance stepper motor
(ii) Permanent-magnet stepper motor
(iii) Hybrid stepper motor.

→ Permanent magnet motors use a permanent (PM) in the rotor and operate on the attraction or repulsion between the rotor PM and the stator electro-magnets.



[Variable Reluctance Motor]

→ Variable Reluctance (VR) motors have a plain iron rotor & operate based on the principle that minimum reluctance occurs with minimum gap, hence the rotor points are attracted toward the stator magnet poles.

→ Hybrid stepper motors are named because they use a combination of PM & VR techniques to achieve maximum power in a small package size.

→ There are two basic winding arrangements for the electromagnetic coils in a two phase stepper motor, those are:-
(i) bipolar & (ii) unipolar.

Advantages of stepper motor :-

1. Low cost for control achieved
2. High torque at startup & low speeds.
3. Simple in construction.
4. Can operate in open loop control system.
5. Low maintenance.
6. can be used in robotics in a wide scale.

Disadvantages :-

1. Require a dedicated control circuit
2. Use more Current than D.C. motors.
3. Torque reduces at higher speed.

Control System Engineering

Topic:

Stability Analysis And Routh Hurwitz Criteria

Contributed By:

Gyana Ranjan Biswal

* STABILITY ANALYSIS *

→ The analysis of, whether the given system can reach steady state; passing through the transients is called Stability Analysis of the system.

→ Total response of a system is divided into zero state Response (ZSR) & zero input Response (ZIR).

→ The response due to input only is called as ZSR & the response due to initial conditions only is called as ZIR.

→ There are two types of stability analysis are present :-
(1) Bounded Input Bounded Output (BIBO) Stability.
(2) Asymptotic stability.

(1) BIBO Stability :- A Linear Time Invariant (LTI) system is said to be BIBO stable if, for bounded input, the system produces a bounded output.

→ A signal is said to be bounded if the value is finite for large value of time. i.e, $|x(t)| < \infty$ for $t \to \infty$.

(2) Asymptotic stability :- A system is said to be asymptotic stable if output due to initial conditions becomes zero as time becomes large.

→ A stable system must be both BIBO stable & asymptotic stable.
→ In Control system / signal system subject, BIBO stability is considered. If nothing specified, by default take BIBO stability.

* For a system stability, four cases can be possible :-
(i) Absolute stable (ii) Marginally stable / critically stable
(iii) Unstable (iv) Conditionally stable.

→ Stability depends on closed loop poles i.e, roots of the characteristic equation.

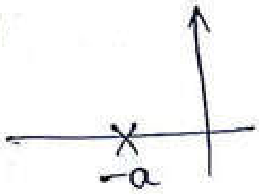
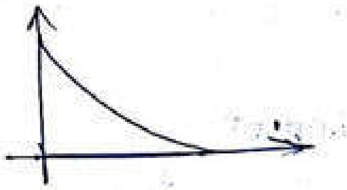
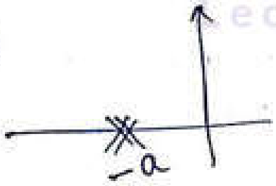
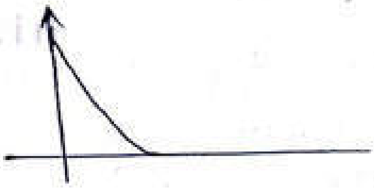
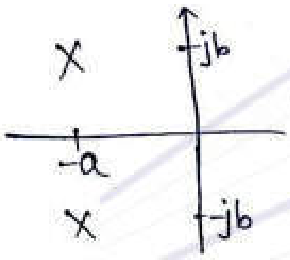
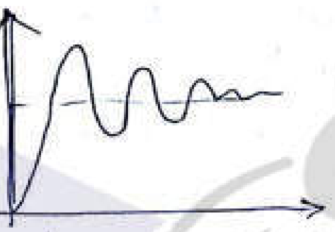
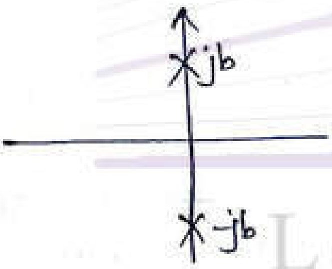
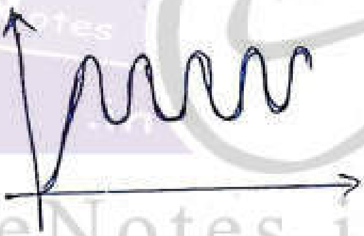
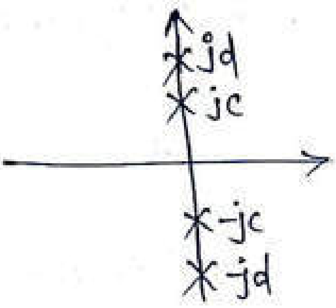
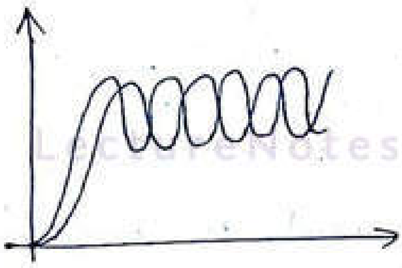
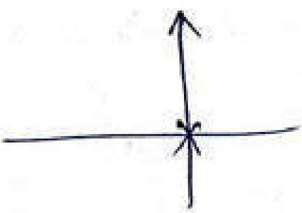
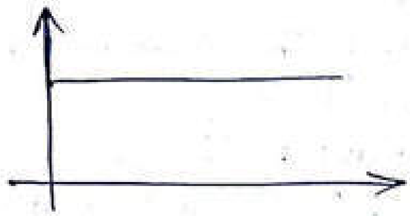
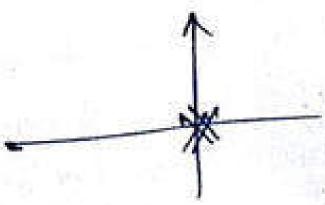
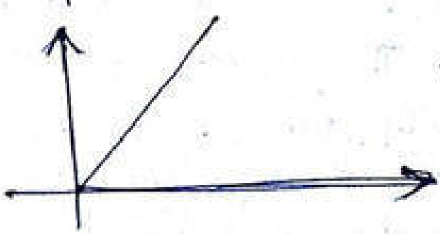
(i) Absolute stable :- The system o/p definitely have a finite value for a large time ($t \to \infty$).

(ii) Marginally stable :- The o/p have a finite value but having sustained oscillation having constant amplitude.

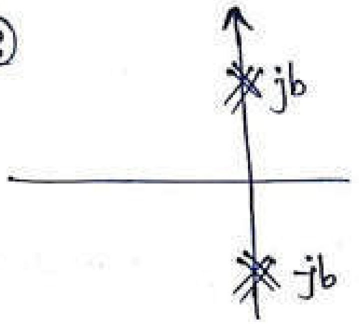
(iii) Unstable :- The system o/p will be definitely be infinite as $t \to \infty$

(iv) Conditionally stable :- If the stability of a system depends on certain condition of a parameter, then such type of s/s can be called as conditionally stable.

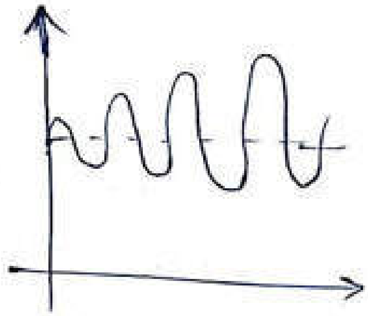
②

Pole location	Response expression	Response graph	stability condition
① 	Ae^{-at}		A. stable
② 	Ate^{-at}		A. stable
③ 	$Ae^{-at}\sin bt$		A. stable
④ 	$A\sin bt$		M. stable or BIBO Unstable
⑤ 	$A\sin ct + B\sin dt$		M. stable
⑥ 	A		M. stable
⑦ 	At		Unstable

8



$Atsinbt$



Unstable

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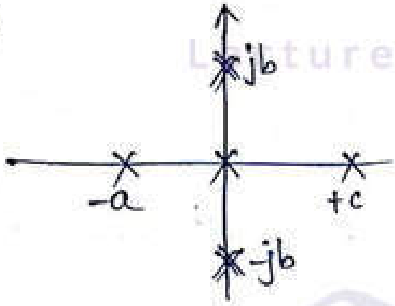
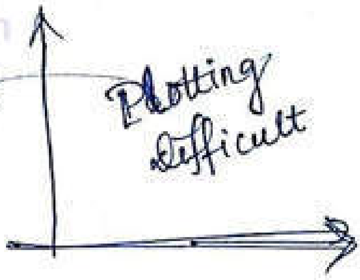
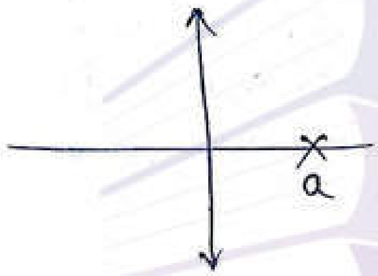


$Ae^{-at} + Bt\sin bt + C + De^{ct}$

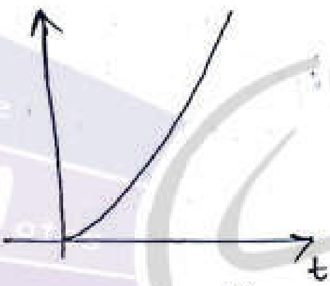


Unstable

10



Ae^{at}



Unstable

11

Ate^{at}

Unstable

* OBSERVATIONS :-

1. All poles are present in Left half of s-plane → Absolute stable
2. If some (or) all poles on jω-axis (not multiple) → Marginal stable (BIBO Unstable)
3. Rest all other combination → Unstable.

* Relative stability :-

→ The system is said to be relatively more stable or unstable on the basis of settling time.

→ Relative stability of the system improves, as the CL poles move away from the imaginary axis in the left half of s-plane.

ROUTH-HURWITZ CRITERIA :-

→ This criteria uses the closed Loop pole location, to determine the stability of the system.

→ The roots of the ch. eq^n are the closed loop poles.

→ Necessary & sufficient conditions for a system to be BIBO stable is :-

(i) the ch. eq^n must have all the co-efficients are non-zero and of same sign.

(ii) Normal Routh Hurwitz table can be formed without any special cases.

(iii) there will be no sign changes in the 1st column of the RH table.

→ RH table gives information about the number of poles lies on the right half of s-plane. i.e, No of sign changes is equal to the no of roots lying in the right half of the s-plane.

Method of Forming RH Table :-

→ For 'n' order system :-

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2	c_3	
$\vdots$				
s^0				

→ First two rows are filled with the help of characteristic eq^n.

→ The next rows can be obtained as follows :-

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \;;\; b_2 = \frac{a_1a_4 - a_0a_5}{a_1} \;;\; b_3 = \frac{a_1a_6 - a_0a_7}{a_1}$$

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \;;\; c_2 = \frac{b_1a_5 - a_1b_3}{b_1} \;, \dots$$

Problems :-

Q/① Determine the stability of a system whose characteristic equation is given by $P(s) = s^4 + 8s^3 + 18s^2 + 16s + 5$.

Sol^n :- Routh Table :-

s^4	1	18	5
s^3	$\cancel{8}$	$\cancel{16}$	0
s^3	1	2	0
s^2	16	5	0
s^1	$\frac{27}{16}$	0	
s^0	5		

$\}\div 8$ (for simplification, any row can be divided or multiplied by any +ve no.)

→ No. of sign changes in 1st column = 0.

→ No. of roots of ch. eq^n in RH = 0

∴ As normal RH & no sign change is 1st column, so the system is Absolute stable system. (Ans)

Special Cases :-

Case ① First element of any of the rows is zero & atleast one element is non-zero.

Q/ Comment upon stability for the ch. eq^n. Given ch. eq^n is :-

$P(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$.

Sol^n :- Routh Table :-

s^5	1	3	2
s^4	2	6	1
s^3	$\boxed{0}$	1.5	0

Remedy for case-1 :- Take 'a' be a very small +ve number.

s^5	1	3	2
s^4	2	6	1
s^3	a	1.5	0
s^2	$\frac{6a-3}{a}$	$\cancel{1}$	0
s^2	$6a-3$	a	0

$\} \times a$

⑥

$$\begin{array}{c|cc} s^1 & \dfrac{1.5(6a-3)-a^2}{(6a-3)} & 0 \\ s^0 & a & \end{array}$$

- As 'a' is very small +ve number, so
$(6a-3)$ is '-ve' & $\frac{1.5(6a-3)-a^2}{6a-3}$ is '+ve'. (By Inspection)

or $\lim_{a\to 0}(6a-3) = -3$ (-ve).

$$\lim_{a\to 0} \frac{1.5(6a-3)-a^2}{(6a-3)} = \lim_{a\to 0}(1.5) - \lim_{a\to 0}\frac{a^2}{(6a-3)} = 1.5 \ (+ve)$$

∴ By analyzing the 1st column of Routh Table :-
No of sign changes = 2 = No of CL poles on RH of s-plane.

∴ So the system is unstable. (Ans)

Case-2 :- All the elements of a row are zero.

Q// $P(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16$. Comment upon stability?

Soln :- RH Table is :-

$$\begin{array}{ccccc} s^6 & 1 & 8 & 20 & 16 \\ s^5 & 2 & 12 & 16 & 0 \\ s^4 & 2 & 12 & 16 & \\ s^3 & \boxed{0 \quad 0 \quad 0} & & & \rightarrow \text{case-2} \end{array}$$

Remedy for case-2 :-

Make Auxiliary eqn by taking the co-efficients of the Previous row of RH Table.

$\therefore A(s) = 2s^4 + 12s^2 + 16$

$\Rightarrow \frac{dA(s)}{ds} = 8s^3 + 24s$

By writing the RH table once again,

$$\begin{array}{ccccc} s^6 & 1 & 8 & 20 & 16 \\ s^5 & 2 & 12 & 16 & \\ s^4 & 2 & 12 & 16 & \\ s^3 & 8 & 24 & 0 & \end{array}$$

s^2	6	16	0
s^1	16	0	
s^0	1		

$\therefore$ No of sign changes in 1^{st} Column = No of roots of ch. eq^n in RH = 0.

* Roots of Auxiliary eq^n are also roots of ch. eq^n.

So, Roots of A.E. is given by :-

$$A(s) = 0 \Rightarrow 2s^4 + 12s^2 + 16 = 0$$

Let $s^2 = x$; $2x^2 + 12x + 16 = 0 \Rightarrow x = -2, -4$

$\therefore$ So roots of $A(s)$ are, $\pm 2j, \pm\sqrt{2}j$.

As two pair of poles are present on $j\omega$-axis & no pole on the RH of s-plane, so the system is Marginally stable with undamped frequency or frequency of sustained oscillation is $\boxed{2 \text{ \& } \sqrt{2} \text{ rad/sec}}$.

Notes:-

(1) Roots of Auxiliary equation are always symmetric with respect to origin.

(2) If Auxiliary eq^n or case-2 happens in RH table, then the system is marginally stable or unstable system & then check its roots.

Q// Comment upon stability for the given ch. eq^n:-

$$P(s) = s^4 + 2s^2 + 1.$$

Sol^n:→ RH Table

s^4	1	2	1
s^3	0	0	0

s^3 row $\boxed{0 \quad 0 \quad 0} \longrightarrow$ case-2

$\therefore A(s) = s^4 + 2s^2 + 1 \Rightarrow \frac{dA(s)}{ds} = 4s^3 + 4s$

$\therefore$ Again writing RH table,

s^4	1	2	1
s^3	4	4	
s^2	1	1	
s^1	0	0	

s^1 row $\boxed{0 \quad 0} \longrightarrow$ case-2

$\therefore A_1(s) = s^2+1 \Rightarrow \frac{dA_1(s)}{ds} = 2s$

$\therefore$ Final RH table will be :-

s^4	1	2	1
s^3	4	4	
s^2	1	1	
s^1	2	0	
s^0	1	0	

$\therefore$ No sign change in 1^{st} column = No of CL poles on RH side of s-plane.

* Auxiliary eqn with higher order is given priority. (wherever required).

So, roots of A(s) is :-

$$s^4+2s^2+1=0 \Rightarrow x^2+2x+1=0 \quad (\text{Let } s^2=x)$$

$$\Rightarrow x = -1, -1$$

So, the roots are :- $\pm j, \pm j$ $\rightarrow$ Multiple poles on $j\omega$-axis.

$\therefore$ The given system is Unstable system.

Q// Find the condition for which system is Absolute stable

$$P(s) = s^4+2s^3+3s^2+2s+K \longrightarrow \text{ch. eq}^n.$$

Soln :- Forming RH table :-

s^4	1	3	K
s^3	2	2	
s^2	2	K	
s^1	$\frac{4-2K}{2}$	0	
s^0	K		

Necessary condn for a A. stable system, $K>0$ (from ch. eqn)

Now, other conditions are (from table)

$$\frac{4-2K}{2} > 0 \Rightarrow 4-2K>0 \Rightarrow 2-K>0 \Rightarrow \boxed{K<2}.$$

So, for the system to be absolute stable system, the following condition is there, $\boxed{0 < K < 2}$. (Ans)

* The above said stability is called as Conditionally stable system as its stability depends upon conditions relating the gain (K).

Q// Find the value of 'a' & 'K' at which a unity feedback control system having OLTF $G(s) = \frac{K(s+1)}{s^3 + as^2 + 2s + 1}$ will have sustained oscillations at $\omega = 2$ rad/sec.

Soln :- Resonates at 2 rad/sec means undamped freq = 2 rad/sec.

ch. eqn : $1 + G(s)H(s) = 0$

$\Rightarrow s^3 + as^2 + 2s + 1 + Ks + K = 0$

$\Rightarrow s^3 + as^2 + (2+K)s + (1+K) = 0$

To obtain undamped frequency 2 rad/sec, system must be marginally stable system & minm order of Auxiliary equation must be '2'.

RH Table :-

s^3	1	$2+K$
s^2	a	$1+K$
s^1	$\frac{a(K+2)-(K+1)}{a}$	0
s^0	$K+1$	

Necessary conditions :-

$a > 0$; $K+2 > 0$; $K+1 > 0$ (from ch. eqn)

3rd row of table can be made case-2 if

$\boxed{a(K+2) - (K+1) = 0}$ —— (1)

$\therefore A(s) = as^2 + (K+1) = 0 \longrightarrow$ order '2'.

Roots of $A(s) \rightarrow s^2 = \frac{-(K+1)}{a} \Rightarrow s = \pm j\sqrt{\frac{K+1}{a}} = \pm j2$ (given)

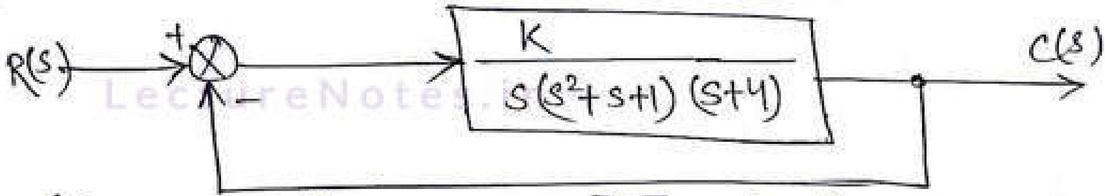
So, Now we got two eqns :-

$a(K+2) = (K+1)$ & $\sqrt{\frac{K+1}{a}} = 2 \Rightarrow \frac{K+1}{a} = 4 \Rightarrow K+1 = 4a$

$\Rightarrow a(4a+1) = 4a \Rightarrow \boxed{a = 3/4}$ $\Rightarrow \boxed{K = 2}$.

∴ So, the values of 'a' & 'K' will be 3/4 & 2 respectively, for which the system resonates at 2 rad/sec. (Ans).

Q// Determine the condition of sustained oscillation in the following control system and also find the freq. of sustained oscillation.



(Ans: $K = \frac{84}{25}$ & $\omega_n = \sqrt{\frac{4}{5}}$ rad/sec)

Q// Comment upon stability for $p(s) = s^4 + s^3 - 3s^2 - s + 2$.

(Ans: Unstable system)

LIMITATIONS of Routh-Hurwitz Criteria :-

1) valid only for Real co-efficients
2) valid only for polynomial of 's'.
   Ex:→ $e^s + \sin s + s^2 = 0$ (can't be determined by RH)
3) valid only for boundary $j\omega$-axis. Not valid for boundary unit circle. (for Discrete time signals).

Q// For the system given by the ch. eq^n $s^3 + 6s^2 + 10s + 12.4 = 0$. Determine the location of the roots by shifting the origin of the s-plane by one unit to left applying the Routh's criterion.

Control System Engineering

Topic:

Root Locus Technique

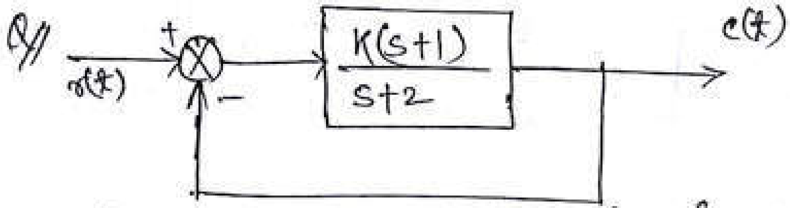
Contributed By:

Gyana Ranjan Biswal

ROOT LOCUS TECHNIQUE

Understanding of Root Locus :-

Q)



Determine & plot CL poles for $K = 0, 1, 2, 4, \infty$.

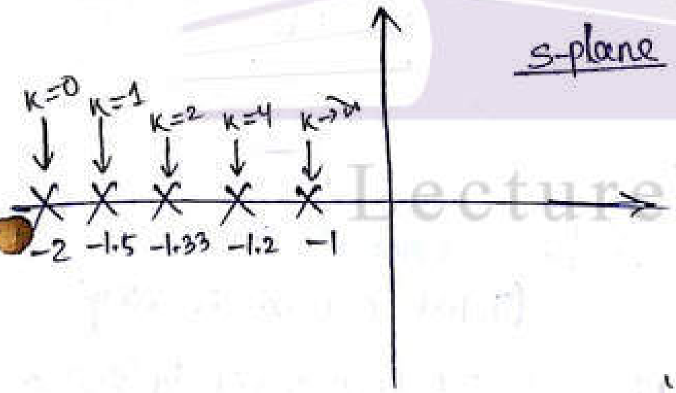
Solⁿ:- CL poles = Roots of the ch. eqⁿ

ch. eqⁿ : $1 + G(s)H(s) = 0 \Rightarrow s(K+1) + (K+2) = 0$.

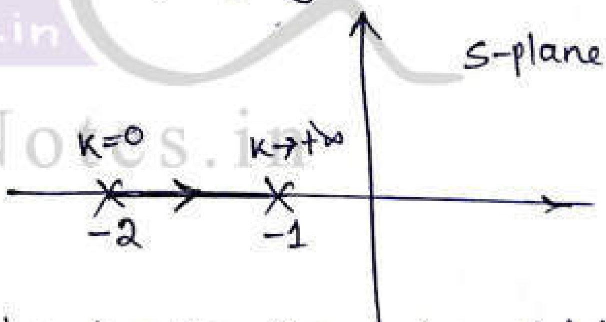
$\Rightarrow s = \frac{-(K+2)}{K+1}$ → Pole location depends on 'K'.

K	0	1	2	4	∞
s	−2	$-3/2$	$-4/3$	−1.2	−1

Representing all results in one plot :-



Simplifying the plot :-



'→' indicates ↑ing value of 'K'.

→ The above locus is the locus of closed loop poles by varying 'K' from 0 to +∞. This is known as Root Locus & this is ~~[illegible]~~ for −ve feedback & K>0.

→ Similarly, we can get the root locus by varying 'K' from −∞ to 0.

→ characteristic eqⁿ of a system is given by :-

$$1 \boxed{\pm \; K} G_1(s)H_1(s) = 0$$

± → Depends on feedback

K → k>0 or k<0

→ +ve ⇒ odd concept RL

→ −ve ⇒ even concept RL.

Odd Concept Root Locus :-

→ A point on real axis is a part of Root locus, if no of open loop poles & no of open loop zeros present in right hand side of that point is odd.

Even Concept Root Locus :-

→ A point on real axis is a part of root locus if no of open loop poles & no of OL zeros present in right hand side of that point is even.

OBSERVATION :-

Even RL ←(complementary w.r.t. each other)→ Odd RL

Even RL, Odd RL → Combined → Complete RL (Total real axis is occupied)

Defⁿ:-

→ RL is the variation of the closed loop poles in s-plane by varying K from −∞ to +∞.

→ This RL technique is given by W.R. Evan in 1948.

→ Used to know the stability of a system and also gives information about effect of parameter variation.

Properties of Root Locus :-

Open loop T/F = $G(s)H(s) = K\,G_1(s)H_1(s)$ → going to vary.

ch. eq^n: $1 \pm K G_1(s)H_1(s) = 0$ → Roots are closed loop poles.

Note:→
(1) Every system must have same no of open loop poles & open loop zeros. So for this, depending on OLTF, some OL poles/OL zeros may be present at infinite location → $|s| \to \infty$.

Let P = No of finite OL poles
Z = No of finite OL zeros.

① At $K=0$, closed loop poles = OL poles
$K \to \pm\infty$, CL poles = OL zeros

② Root Locus is symmetric with respect to real axis.

③ No of RL branches = order of ch. eq^n.
branches → paths available to move from 0 to +∞ or −∞ to 0.

④ Intersection of Root Locus with jω-axis.
Use Routh Hurwitz criteria.

⑤ Break Away point/Break-in point :-
Whenever for some value of 'K', if there exists multiple poles at same location, there exist Break-away point. ($K \neq 0$, $K \not\to \pm\infty$)

Steps to determine B.A. point :-

(i) characteristic eq^n: $1 \pm K G_1(s)H_1(s) = 0$
(ii) $K = f(s)$
(iii) $\frac{dK}{ds} = 0 \Rightarrow s = \begin{cases} s_1 \\ s_2 \\ s_3 \end{cases}$ → may be BA point
- either graphically
- or put in 'K' for which 'K' must come real number.

(6) Break Away angle (apart) $= \frac{180^\circ}{r}$, where r = No of poles at BA point.

(7) A point on real axis is part of RL if
even concept RL → No of O.L. poles & zeros right side of the point is even.
odd concept RL → No of OL poles & zeros right side of the point is odd.

(8) Behaviour of RL when $|s| \to \infty$:-

→ Behaviour of RL when $|s| \to \infty$ is approximated by st. lines Known as Asymptotes.

- to draw st. line, we need one point & angle(slope).
  ↳ centroid

Centroid :-
It is always present on real axis.

$$\text{Centroid} = \frac{\Sigma \text{ finite OL poles} - \Sigma \text{ finite OL zeros}}{P-Z} = 'a' (\text{let})$$

∴ co-ordinates of centroid → (a,0).

→ No of asymptotes $= |P-Z|$

→ Angle of asymptotes $= \frac{(2q+1)\times 180^\circ}{|P-Z|}$; q = 0,1,2 ... → odd concept RL.

$= \frac{(2q)\times 180^\circ}{|P-Z|}$; q = 0,1,2 .... → even concept RL.

(9) Arrow (→) indicates increasing value of 'k'.

(10) Angle of Arrival/Departure will be calculated only for multiple real OLP/OLZ, complex OLP/OLZ, multiple complex OLP/OLZ.

Steps :-

For even RL : $\Sigma \angle \text{zeros} - \Sigma \angle \text{poles} = 2q \times 180^\circ$
For odd RL : $\Sigma \angle \text{zeros} - \Sigma \angle \text{poles} = (2q+1)\times 180^\circ$
q = 0,1,2,...

→ Spirule Concept.

Note :-
* For odd concept RL, Root Locus starts from OL poles & terminates at OL zero.
* For even concept RL, Root Locus starts from OLZ & terminates at OL poles.

Q// $G(s)H(s) = \dfrac{K(s+2)(s+6)}{s(s+4)(s+8)}$. Draw Root Locus for $-\infty < K < \infty$.

Sol^n:→ Assuming -ve feedback :-

-ve f/b & $K>0$ → odd concept RL

-ve f/b & $K<0$ → Even concept RL

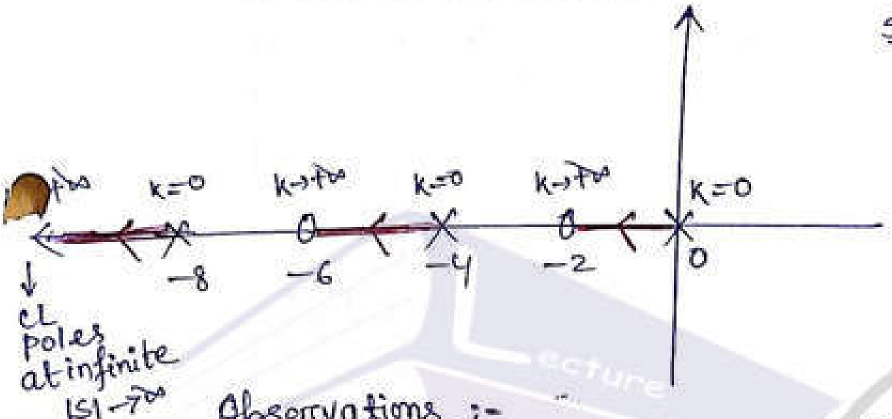
Steps

1- odd/even RL
2- OLP & OLZ; then plot
3- Mark 'K' value.
4- RL branches
5- check a pt present on RL.
6- Check BA point exist or not. if yes, calculate BA point & BA angle.
7- Asymptote calculation. If yes, no, centroid, angle.
8- check for intersection with $j\omega$-axis. If yes find the intersection point.
9- Check for Angle of Arrival or departure.

K>0

OL zeros $= -2, -6 \Rightarrow Z = 2$

OL poles $= 0, -4, -8 \Rightarrow P = 3$



Observations :-

① No valid BA point on real axis for $K>0$.

② Calculate Asymptotes :-

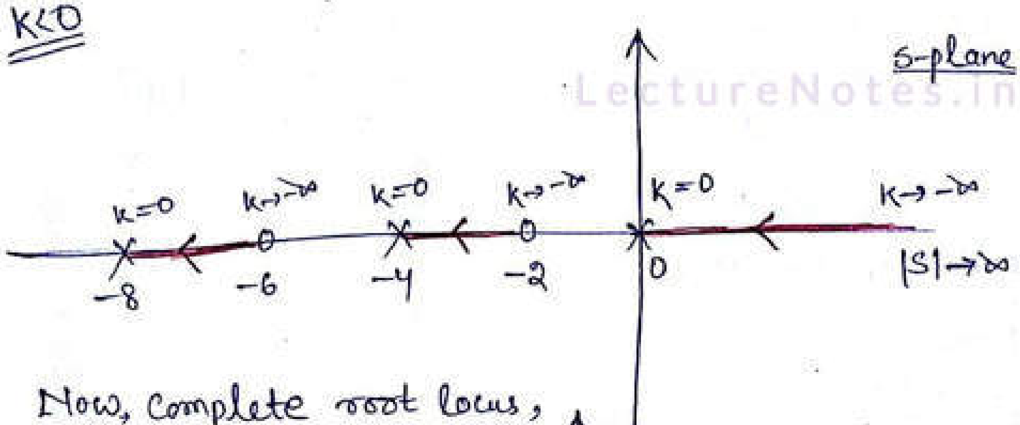
No of asymptotes $= |P-Z| = 1$.

$\therefore$ Centroid $= \dfrac{\{0+(-4)+(-8)\} - \{(-2)+(-6)\}}{3-2} = -4$.

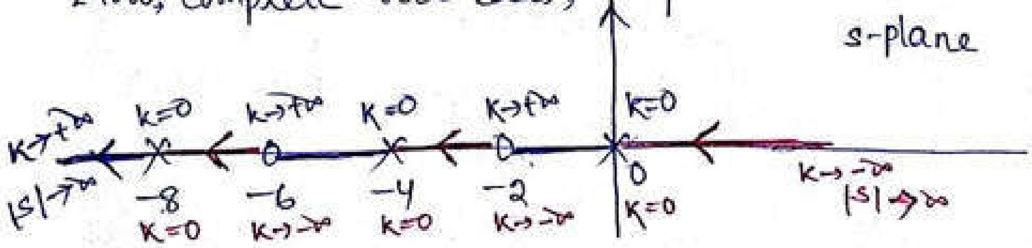
$\therefore$ co-ordinate of centroid $= (-4, 0)$.

Angle $= \dfrac{2q+1}{|P-Z|} \times 180° \Rightarrow \theta_1 = +180°$ for $q=0$

K<0



Now, complete root locus,



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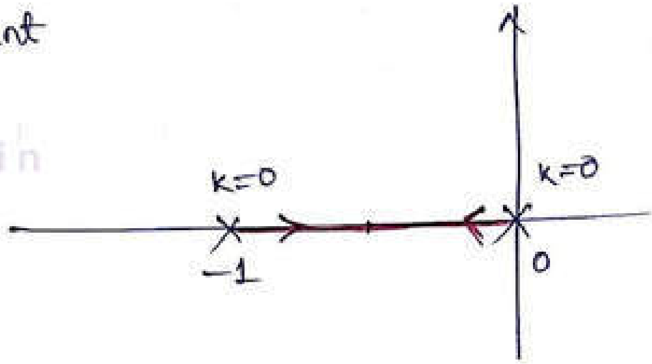
Q. $G(s)H(s) = \frac{k}{s(s+1)}$. Draw RL for $k>0$.

Solⁿ:- Assuming -ve feedback & $k>0$ (odd concept RL)

→ $P=2$, $Z=0$

→ No of RL branches = order of ch. eqⁿ = 2.

→ Definitely Break away point between -1 & 0.



BA point calculation

ch. eqⁿ: $1 + \frac{k}{s(s+1)} = 0$

$\Rightarrow s^2+s+k=0$

$\Rightarrow K = -(s^2+s)$.

$\therefore \frac{dk}{ds} = 0 \Rightarrow -(2s+1)=0 \Rightarrow s = -\frac{1}{2}$ valid BA point from graph.

Other method to check:-

Put $s=-\frac{1}{2}$ in 'K' value:-

$K = -\left[\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)\right] = -\left(\frac{1}{4}-\frac{1}{2}\right) = -\left(\frac{1-2}{4}\right) = \underline{1/4 > 0}$.

$\therefore$ Break away angle $= \frac{180^\circ}{r} = \frac{180^\circ}{2} = 90^\circ$ (apart)

Asymptote calculation

No of Asymptotes $= |P-Z| = 2$.

Centroid $= \frac{\{0+(-1)\}-0}{2} = -\frac{1}{2} = -0.5$

$\therefore$ co-ordinates of Centroid is $(-0.5, 0)$.

Angle :- $\theta_1 = \frac{(2q+1)\times 180^\circ}{|P-Z|}$, $q=0 \Rightarrow \theta_1 = \frac{180^\circ}{2} = +90^\circ$.

$\theta_2 = \frac{3}{2} * 180 = +270^\circ$. (for $q=1$).

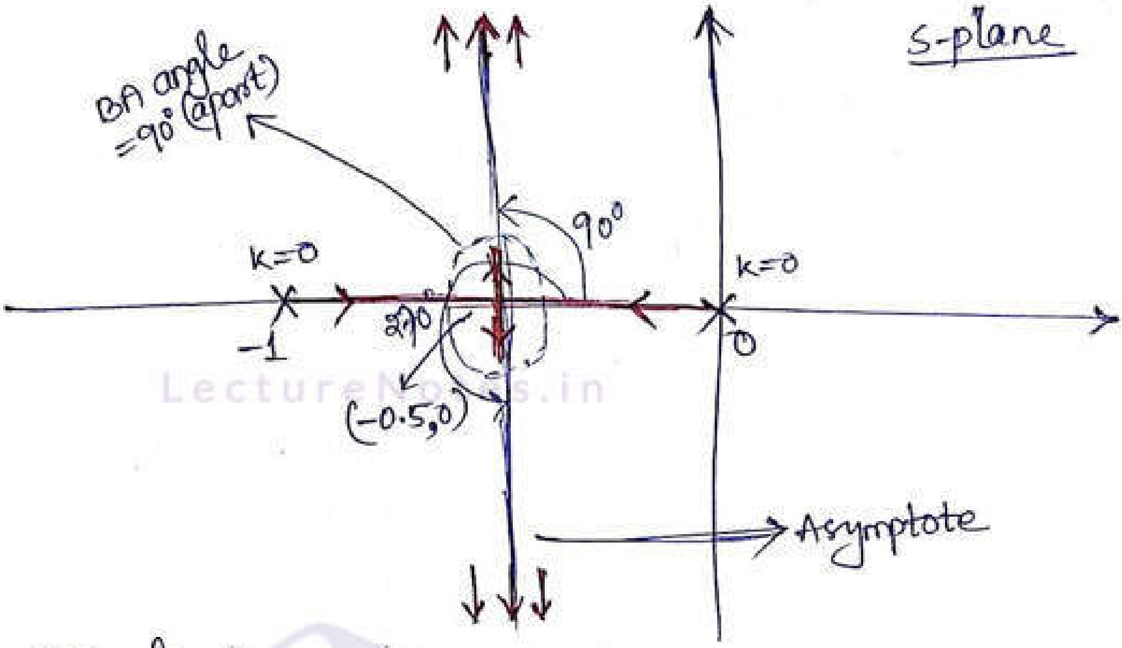
Intersection with jω-axis :-

Routh Array :-

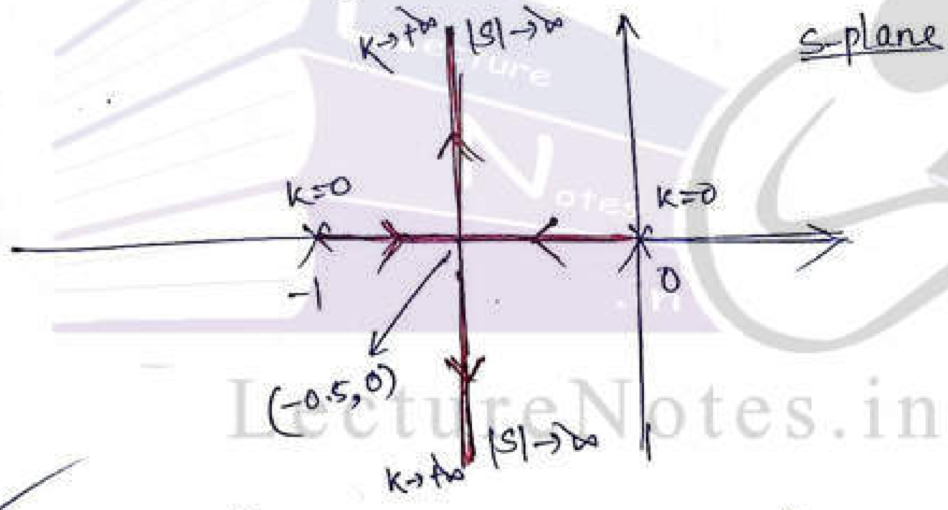
s^2	1	K
s^1	1	0
s^0	K	

⇒ No auxiliary eqⁿ for $K>0$

⇒ No intersection of RL ($K>0$) with jω-axis.



The final RL for K>0 is :→

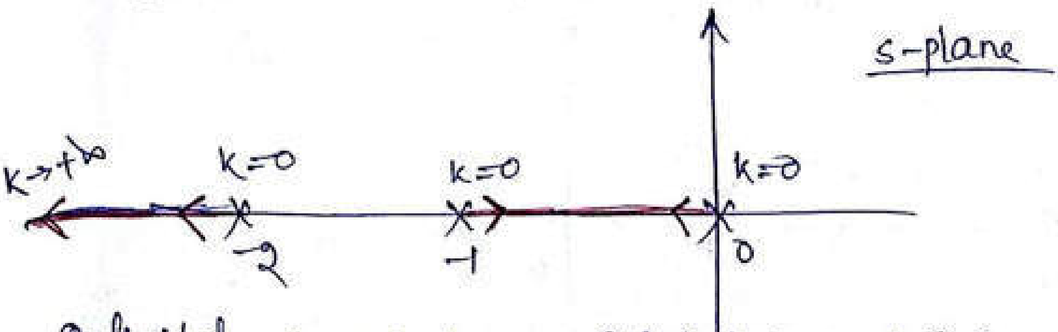


Q CW $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$. Draw RL for K>0 .

Sol^n → Assuming -ve f/b & K>0 ⇒ odd concept RL.

→ P=3 , Z=0

→ RL branches = order of ch. eq^n =3.



Definitely Break Away point between -1 & 0.

BA point Calculation →

→ ch. eq^n: $1+\frac{k}{s(s+1)(s+2)}=0$

$\Rightarrow (s^2+s)(s+2)+k=0 \Rightarrow k=-(s^3+3s^2+2s)$

$\therefore \frac{dk}{ds}=0 \Rightarrow -(3s^2+6s+2)=0$

$\Rightarrow 3s^2+6s+2=0 \Rightarrow s_1=-0.42$ → valid from graph.

$s_2=-1.57$

Other way:-

for $s_1=-0.42$; $k=0.38$ (valid as $k>0$).

$s_2=-1.57$; $k=-0.38$ × (Invalid)

$\therefore$ BA angle $=\frac{180°}{r}=\frac{180°}{2}=90°$ (apart).

Asymptote calculation :-

No. of asymptotes $=|P-Z|=3$.

Centroid $=\frac{\{0+(-1)+(-2)\}-0}{3}=(-1,0)$

Angle → $\theta_1=\frac{1}{3}\times 180°=60°$ for $q=0$

$\theta_2=\frac{1}{1}\times 180°=180°$ for $q=1$

$\theta_3=\frac{5}{3}\times 180°=300°$ for $q=2$.

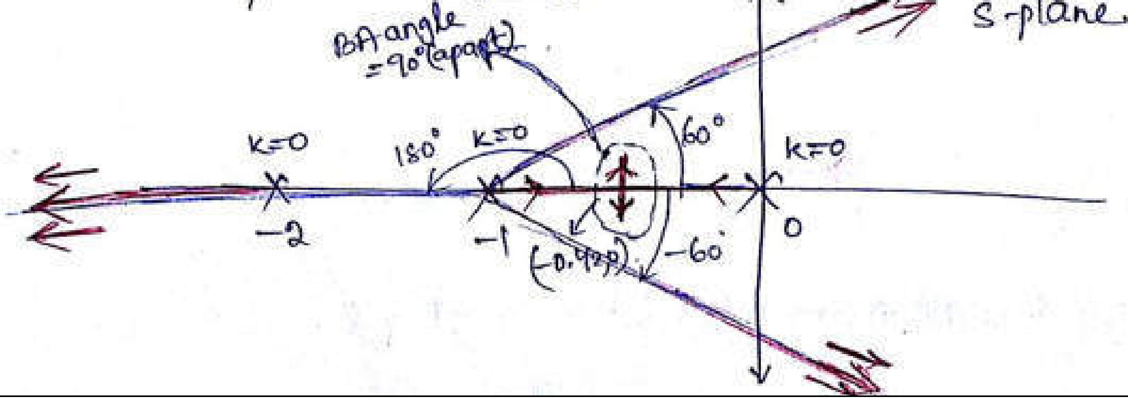
Intersection with jω-axis :-

s^3	1	2
s^2	3	K
s^1	6−K	0
s^2	K	

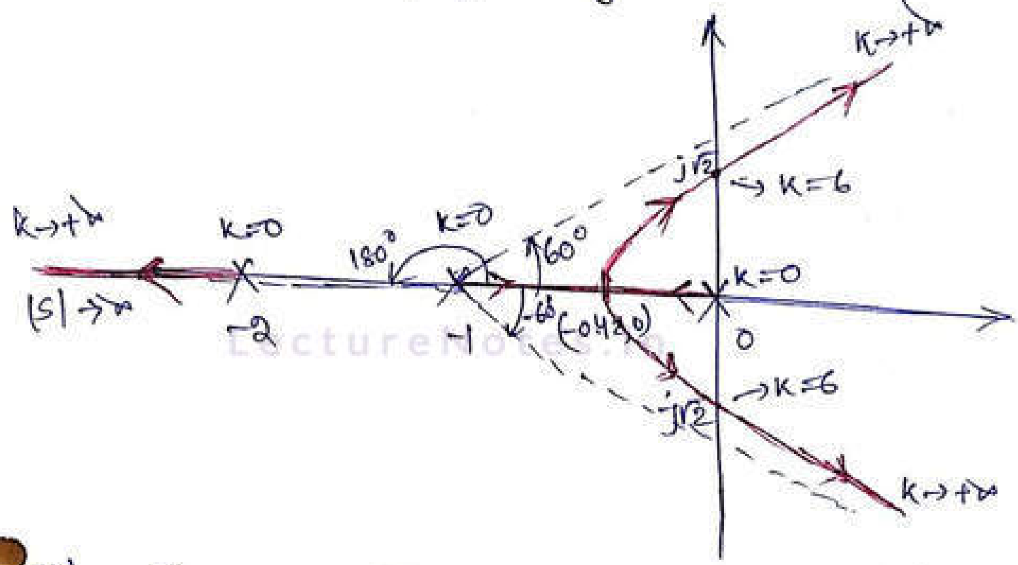
→ A(s) for $k=6$ i.e, $k>0$.

$\therefore A(s)=3s^2+k=3s^2+6=0$.

$\Rightarrow s^2=-2 \Rightarrow \boxed{s=\pm j\sqrt{2}}$ → at this point RL intersect.



Final Root Locus is given by :→



CW

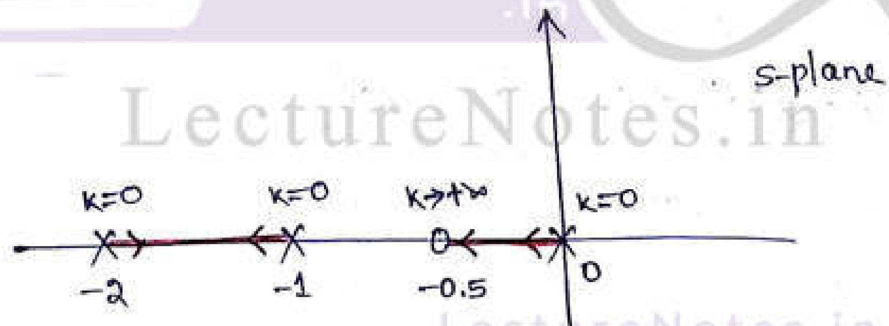
Q. $G(s)H(s) = \frac{K(s+0.5)}{s(s+1)(s+2)}$. Draw RL for $K>0$.

Solⁿ→ Assuming -ve feedback & $K>0$ ⇒ odd concept RL.

→ OL zeros = -0.5 ; Z = 1

OL poles = 0, -1, -2 ; P = 3.

→ RL branches = order of ch. eqⁿ = 3.



one breakaway point between (-1 & -2) on real axis.

BA point :-

ch. eqⁿ : $1 + \frac{K(s+0.5)}{s(s+1)(s+2)} = 0 \Rightarrow s^3 + 3s^2 + (2+K)s + 0.5K = 0$

$\Rightarrow K = \frac{-(s^3+3s^2+2s)}{(s+0.5)}$.

$\frac{dK}{ds} = 0 \Rightarrow \frac{d}{ds}\left(\frac{s^3+3s^2+2s}{s+0.5}\right) = 0 \Rightarrow 2s^3 + 4.5s^2 + 3s + 1 = 0$

$\Rightarrow s = -1.45, \ -0.39 \pm j0.432$

→ valid BA point (from graph)

Break away angle $= \frac{180}{2} = 90°$ (apart).

$\therefore$ No of asymptote $= |P-Z| = 2$.

Centroid $= \frac{\{0+(-1)+(-2)\}+(0.5)}{2} = \underline{(-1.25, 0)}$

Angle: $\theta_1 = \frac{1}{2} \times 180° = 90°$ (for $q=0$)

$\theta_2 = \frac{3}{2} \times 180° = 270° = -90°$ (for $q=1$)

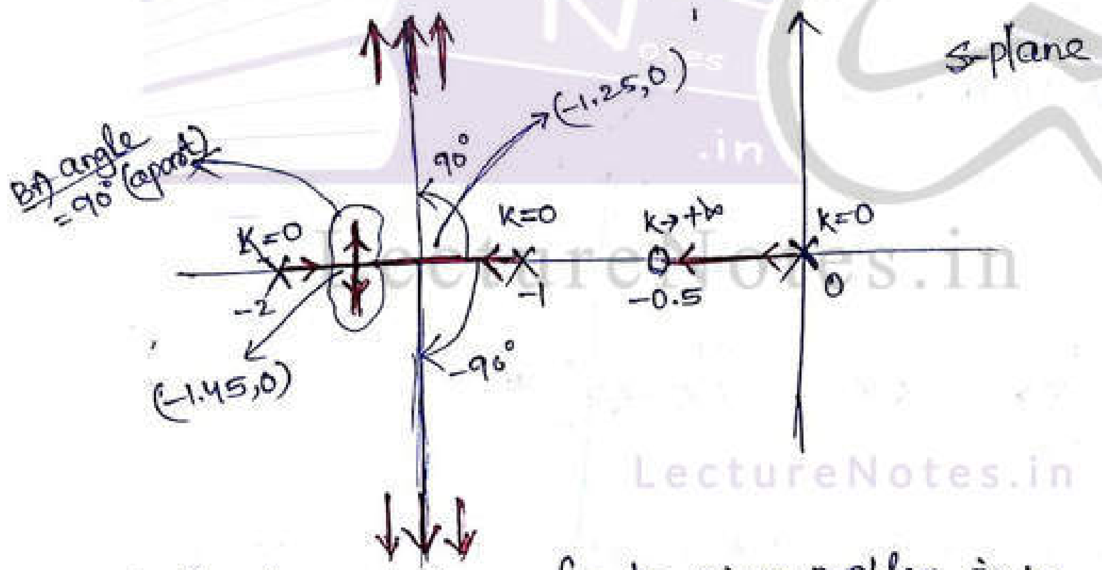
Intersection with $j\omega$-axis :-

<u>Routh Table</u>

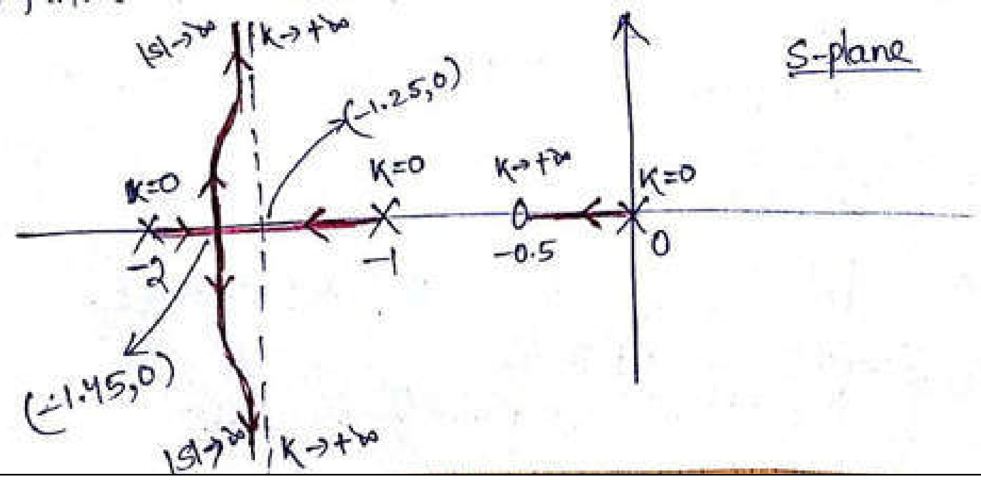
s^3	1	$2+K$
s^2	3	$0.5K$
s^1	$\frac{5K+2}{3}$	0
s^0	$0.5K$	0

No Auxiliary eq^n for $K>0$

$\Rightarrow$ No intersection for $\underline{K>0}$.



So, final Root locus for the given problem is :-



Observation from previous two problems :-

* Addition of zeros increases stability.

HW Q) $G(s)H(s) = \dfrac{2}{(s+7+K)s+10}$. Draw RL for $-\infty < K < +\infty$.

Sol^n → Ch. eq^n: $1+G(s)H(s)=0 \Rightarrow (s+7+K)s+10+2=0$

$\Rightarrow s^2+7s+12+Ks=0 \Rightarrow 1+\boxed{\dfrac{Ks}{s^2+7s+12}}$ → $G_1(s)H_1(s)$ (modified OLTF)

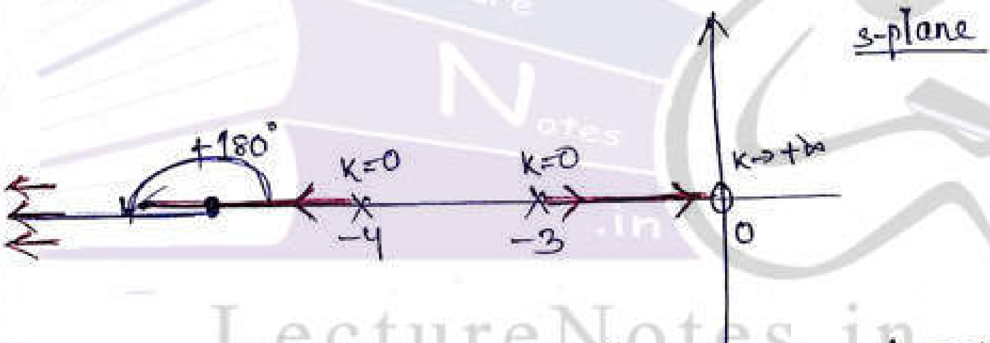
$\therefore G_1(s)H_1(s) = \dfrac{Ks}{(s+4)(s+3)}$

Assuming negative feedback & $K>0 \Rightarrow$ odd concept RL.

→ OLZ = 0 ; Z = 1

OLP = −4, −3 ; P = 2.

→ RL branches = order of ch. eq^n = 2.



→ No break-away point for $K>0$ on real axis.

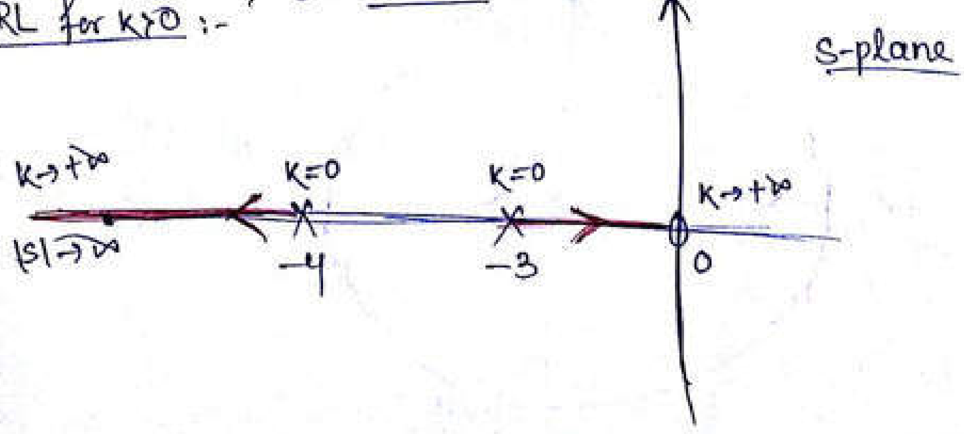
→ No of asymptotes $= |P-Z| = 1$.

Centroid $= \dfrac{\{(-4)+(-3)\}-0}{1} = (-7, 0)$.

Angle : $\theta = \dfrac{(2q+1)\times 180^\circ}{P-Z}$ $(q=0)$

$\Rightarrow \theta = \underline{180^\circ}$.

final RL for K>0 :-



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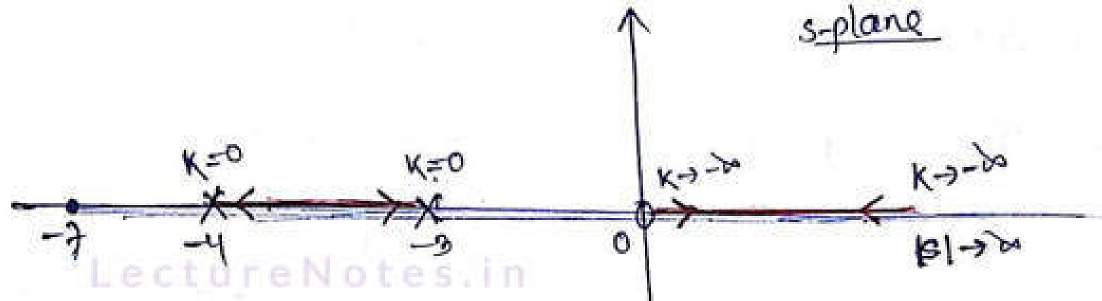
For K<0 & '-ve' feedback (even Concept RL) :-

→ P=2 & Z=1.



So, it is clear that, two break away points on real axis for K<0. (one lies in betⁿ -3 & -4 and other lies in betⁿ 0 to ∞).

BA point calculation :-

$$\frac{dK}{ds} = 0 \Rightarrow s^2-12=0 \Rightarrow s=\sqrt{12} = \pm 3.46 .$$

$\Rightarrow s = +3.46, -3.46$ → Both are valid points.

BA angle $= \frac{180°}{2} = 90°$ (apart)

∴ No of asymptotes $= |P-Z| = 1$.

centroid $= (-7,0)$.

Angle, $\theta = \frac{2q}{1} \times 180°$ $(q=0)$ $\Rightarrow \underline{\theta = 0°}$.

Intersection with Imaginary axis :-

Routh Table :→

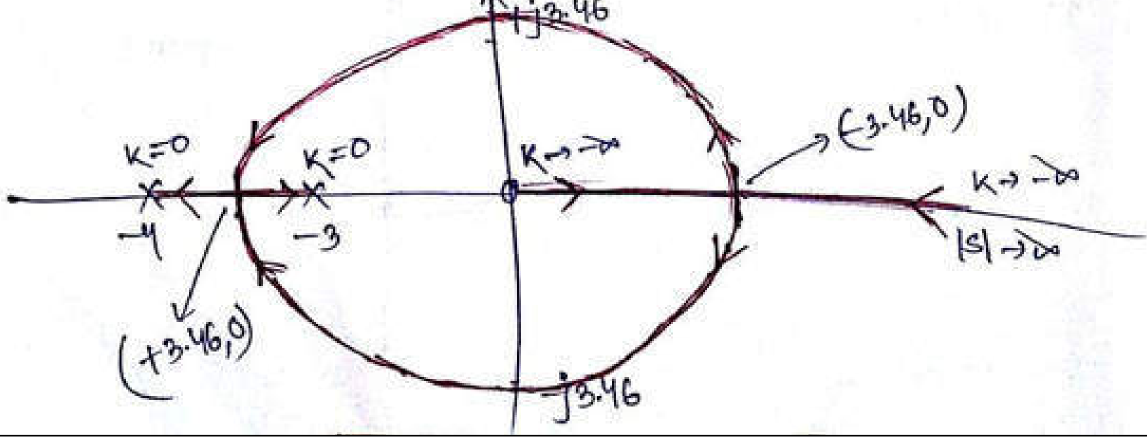
ch. eqⁿ: $1+\frac{KS}{(S+4)(S+3)} = 0$

$\Rightarrow s^2+(7+K)s+12=0.$

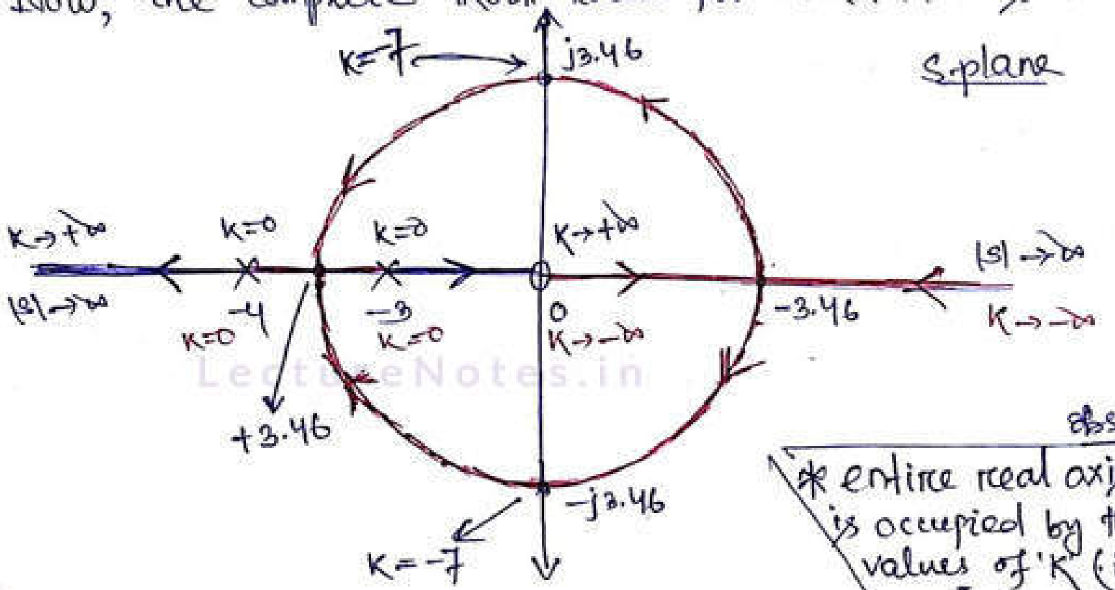
s^2	1	12
s^1	(7+K)	0
s^0	12	0

} → A(s) for K = -7.

∴ $A(s) = s^2+12 \Rightarrow s = \pm j2\sqrt{3} = \pm j3.46$.



Now, the Complete Root locus for $-\infty < K < +\infty$ is :-



Observation :-

* entire real axis of s-plane is occupied by the RL for all values of 'K' (i.e, complete RL drawn for $-\infty < K < \infty$)

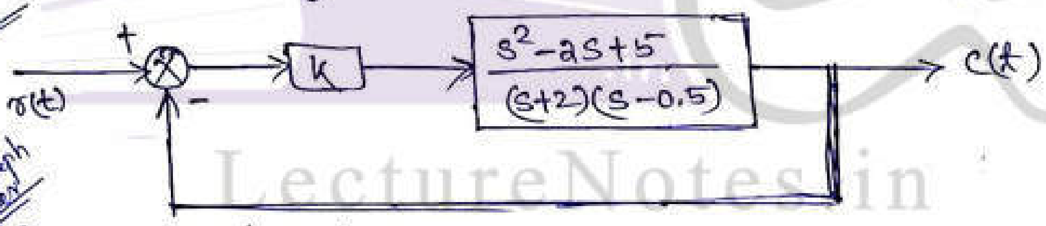
Observation :-

※ Whenever in OLTF, the followings are present

finite	
2P	2Z
2P	1Z
1P	2Z

} RL will be a complex path having a circle.

Q. H.W. Draw on Graph Paper



Draw RL for K > 0.

Soln → Assuming −ve f/b & K > 0 ⇒ odd concept Root Locus.

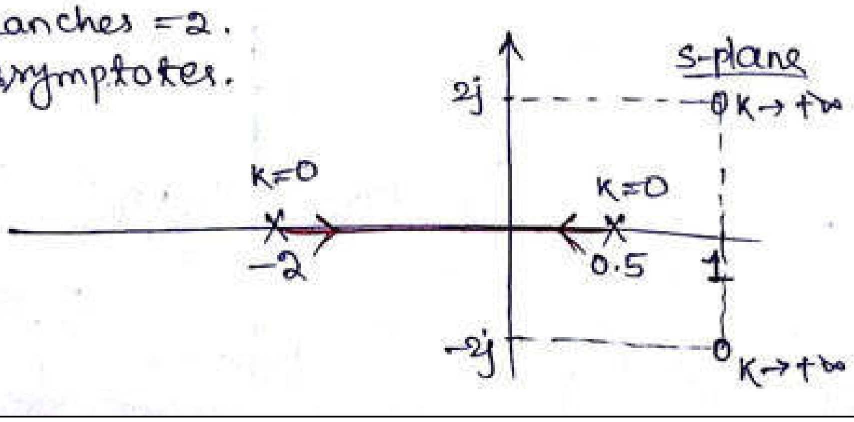
$$\therefore G(s) = \frac{K(s^2-2s+5)}{(s+2)(s-0.5)}.$$

→ OL zeros $= 1 \pm 2j$; $Z = 2$.

OL poles $= -2, 0.5$; $P = 2$.

→ RL branches $= 2$.

→ No asymptotes.



(29)

Definitely break away point is present in betⁿ (−2 to 0.5).

Ch. eqⁿ : $1+G(s)H(s)=0$

$$\Rightarrow K = \frac{-(s+2)(s-0.5)}{(s^2-2s+5)} = \frac{-(s^2+1.5s-1)}{(s^2-2s+5)}$$

$$\therefore \frac{dK}{ds}=0 \Rightarrow -3.5s^2+12s+5.5=0 \Rightarrow s=3.84$$

$= -0.41$ → valid BA point. (from graph)

$\therefore$ BA angle $= \frac{180}{2} = 90°$ (apart)

→ No assymptotes.

Intersection with jω-axis :-

Routh Table :-

Ch. eqⁿ : $1+G(s)H(s)=0$

$$\Rightarrow 1+\frac{K(s^2-2s+5)}{(s+2)(s-0.5)}=0 \Rightarrow (s+2)(s-0.5)+K(s^2-2s+5)=0$$

$$\Rightarrow s^2-0.5s+2s-1+Ks^2-2Ks+5K=0$$

$$\Rightarrow (1+K)s^2+s(1.5-2K)+(5K-1)=0.$$

RH Table

s^2	$(1+K)$	$(5K-1)$
s^1	$1.5-2K$	0
s^0	$5K-1$	

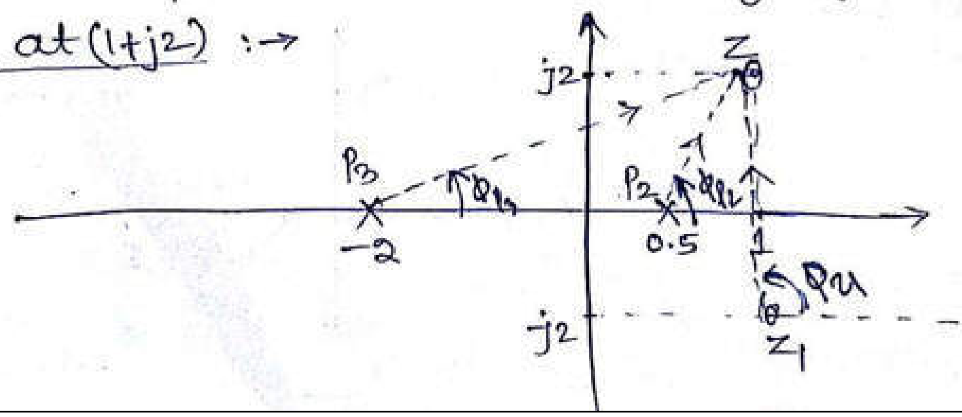
To have auxiliary eqⁿ ; $1.5-2K=0 \Rightarrow 2K=1.5 \Rightarrow \boxed{K=0.75}$.

$\therefore A(s) = (1+K)s^2+(5K-1)=0$

$\Rightarrow 1.75s^2+2.75=0 \Rightarrow \boxed{s=\pm j1.25.}$ → Intersection with jω-axis.

As odd concept RL, so, we have to find Angle of arrival (ϕ_A).

'ϕ_A' at $(1+j2)$:→



Formula to calculate 'ϕ_A' for odd concept RL :-

$$\Sigma \angle zeros - \Sigma \angle Poles = (2q+1)\times 180^\circ$$

$$\Rightarrow \{\phi_Z + \phi_{Z_1}\} - \{\phi_{P_2} + \phi_{P_3}\} = (2q+1)\times 180^\circ$$

$$\Rightarrow \phi_A + 90^\circ - 75.96^\circ - 33.69^\circ = 1\times 180^\circ \quad (q=0)$$

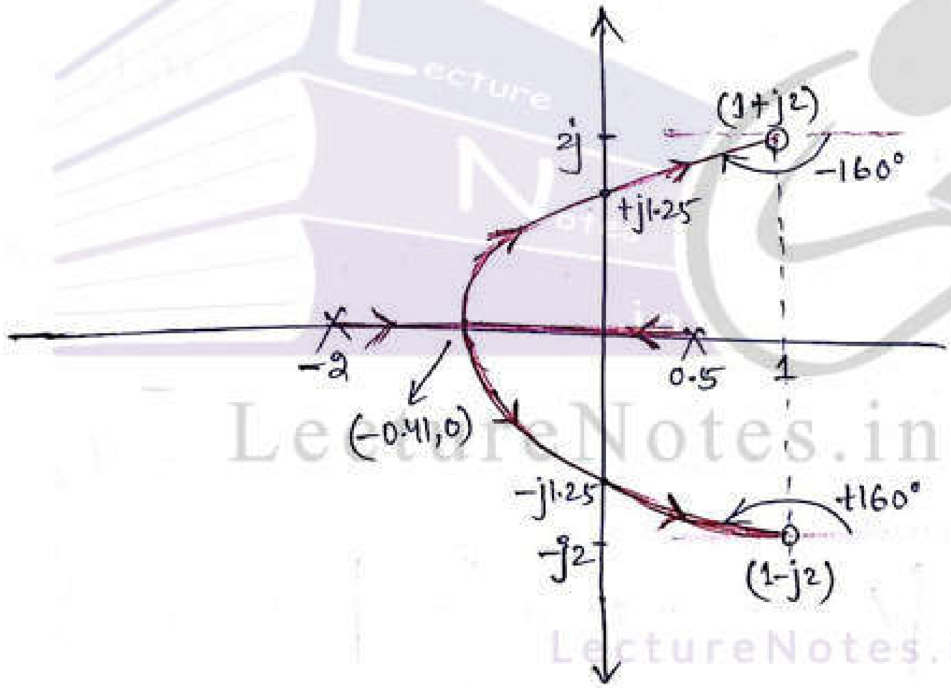
$\therefore \phi_{P_3} = \tan^{-1}(2/3) = 33.69^\circ$

$\phi_{P_2} = \tan^{-1}(2/0.5) = 75.96^\circ$.

$\phi_{Z_1} = 90^\circ$.

$\therefore \phi_A = 199.65^\circ \simeq -160^\circ$

So, ϕ_A at $1-j2$ is $\underline{+160^\circ}$.



Effect of Adding Poles and zeros to G(s) H(s) :-

* Adding of a pole to G(s)H(s) has the effect of pushing the root loci toward the right half. So, it decreases stability.
* Adding of zeros to G(s)H(s) generally has the effect of moving & bending the root loci toward the left half of s-plane. So it increases stability.

Some points on RL :-

① Centroid may or may not be part of Root locus.

② Break Away point can be real or complex but at break away point, 'K' value must always be real.

Angle & Magnitude Condition :-

ch. eqn: $1 + G(s)H(s) = 0 \Rightarrow \boxed{G(s)H(s) = -1 = -1 + j0}$.

→ For any value of 's' if it has to be on the root locus, it must satisfy the above equation.

→ So any point $s = s_1$ to be present on RL it should satisfy two conditions :-

(i) Magnitude Condition (ii) Angle condition.

(i) Magnitude Condn :-

For $s = s_1$ point to be present on RL, it must satisfy magnitude condition given by, $\boxed{|G(s_1)H(s_1)| = 1}$.

→ To determine the value of 'K' at any point of Root Locus, magnitude condition is mostly used.

(ii) Angle Condn :-

For $s = s_1$ point to be present on RL, it must satisfy angle condition which is given by :-

$$\boxed{\angle G(s_1)H(s_1) = \pm(2q+1) \times 180^\circ}, \quad q = 0, 1, 2, \ldots$$

Q// Consider the system with $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$. Find whether $s = -0.75$ & $-1+j4$ are on the root locus or not.

Soln → Using Angle Condition,

$$\angle G(s)H(s)\Big|_{at\ s=-0.75} = \frac{\angle K + j0}{(\angle -0.75 + j0)(\angle 1.25 + j0)(\angle 3.25 + j0)}$$

$$= \frac{0^\circ}{180^\circ \times 0^\circ \times 0^\circ} = \underline{-180^\circ}.$$

Now, $\angle G(s)H(s)\Big|_{s=-1+j4} = \dfrac{\angle k+j0}{(\angle -1+j4)(\angle 1+j4)(\angle 3+j4)}$

$$= \frac{0^\circ}{(104.03^\circ)(75.963^\circ)(53.13^\circ)} = -233.123^\circ.$$

∴ So from above result, it is clear that $s=-0.75$ is on root locus as it satisfies angle condition.

Q// In the above problem, find the value of 'K' at $s=-0.75$.

Sol^n → $|G(s)H(s)|_{s=-0.75} = 1$ (Magnitude Condition)

$$\Rightarrow \frac{|K|}{|-0.75|\,|1.25|\,|3.25|} = 1 \Rightarrow \boxed{K = 3.0468}.$$

Graphically determine 'K' :-

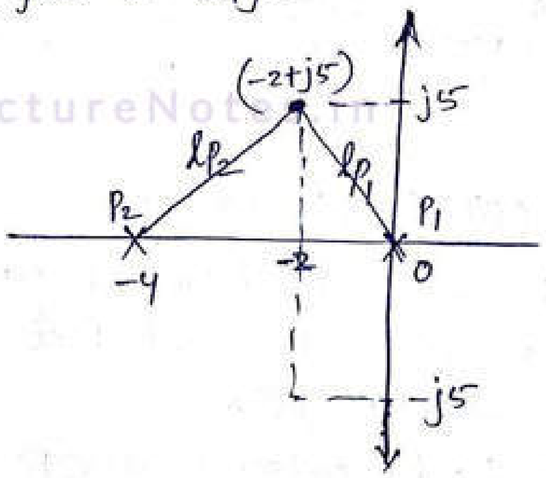
$$\text{'K' at any point} = \frac{l_{p_1} \times l_{p_2} \times \cdots}{l_{z_1} \times l_{z_2} \times \cdots}$$

Where, l = length from open loop pole/zero to that point.

Ex: $G(s)H(s) = \dfrac{K}{s(s+4)}$. Find K for $s=-2+j5$.

$l_{p_1} = \sqrt{2^2+5^2} = \sqrt{29}$.

$l_{p_2} = \sqrt{2^2+5^2} = \sqrt{29}$.



$$\therefore K \text{ at } s=-2+j5 = \frac{\sqrt{29}\times\sqrt{29}}{1}$$

$$\Rightarrow \boxed{K = 29}.$$

Determination of 'K' for different cases :-

Case-1 :- To find $K_{marginal}$?

It is that value of 'K' for which Auxiliary eq^n is possible.

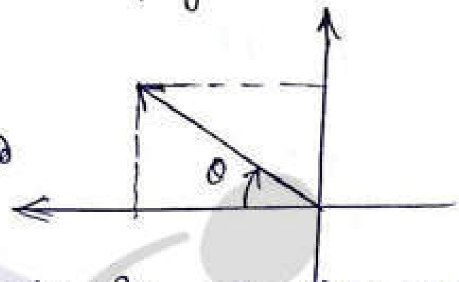
Case-2 :- To find $K_{critical}$? (or) 'K' at $\xi=1$?

Critical damping is present when the poles are real and multiple. It is only possible at breakaway point. Then find 'K' at breakaway point.

Case-3 :- To find 'K' for specified damping ratio 'ξ' :-

$\theta = \cos^{-1}(\xi)$

Draw a line at angle 'θ' from origin. 'θ' is measured from negative real axis in clockwise direction.



→ then find the 'K' at the point of intersection of line & RL using magnitude cond^n.

HW

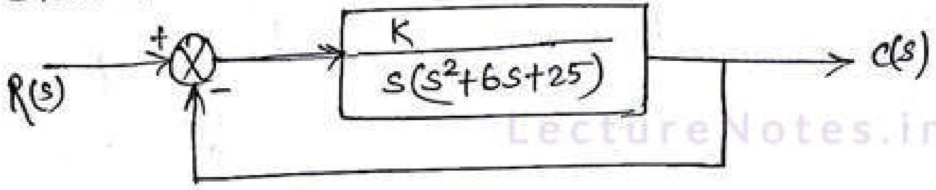
Q/ (A) Sketch the root locus of $G(s) = \dfrac{K}{s(s+3)(s+6)}$. Comment on its stability. Determine the value of 'K' for

(i) critical damping, (ii) marginal stability

(iii) for $\xi = 0.6$

Drawn on graph paper

Q/ (B) Sketch the root locus of the control system as shown in fig.



ROOT CONTOURS :-

→ When more than one parameter varies continuously from $-\infty$ to ∞, the root loci are referred to as the Root Contours (RC).

→ Root contours also possess the same properties as the single parameter root loci, so the methods of construction for RL also applies here.

(A) $G(s)H(s) = \dfrac{K}{s(s+3)(s+6)}$ · (for $K>0$, -ve f/b)

BA Point $= (-1.27, 0)$; BA angle $= 90°$ (apart)

Centroid $= (-3, 0)$; Angle of Assymptotes : $\theta_1 = 60°$; $\theta_2 = 180°$; $\theta_3 = 300°$.

Intersection with Imaginary axis : $s = \pm j4.22$ @ $K = 162$.

Results

1) 'K' (at critical damping) = 10.4

2) 'K' (at marginal stability) = 162

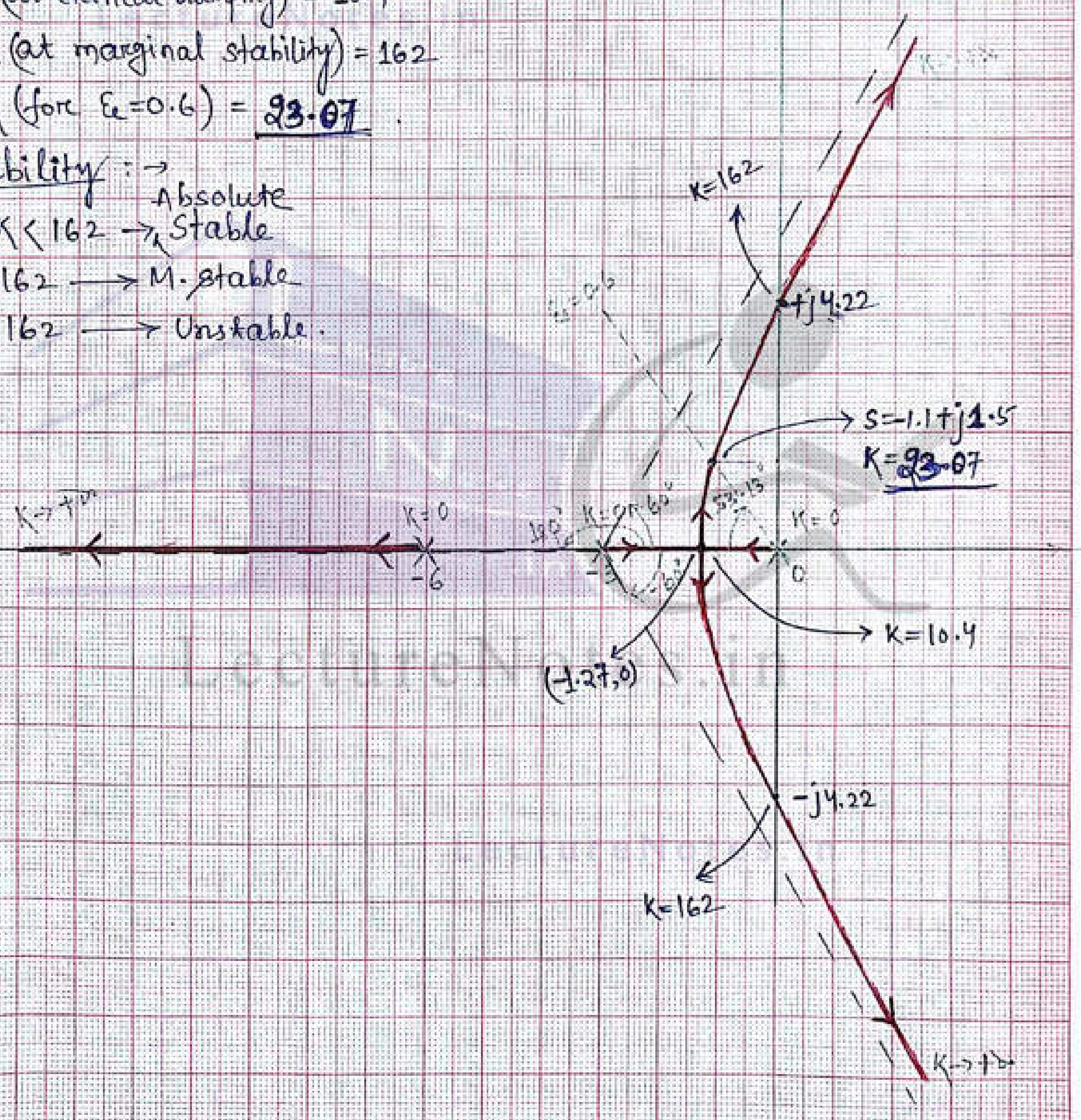
3) K (for $\xi = 0.6$) = **23.07**

Stability :→

$0 < K < 162$ → Absolute Stable

$K = 162$ → M. Stable

$K > 162$ → Unstable.



Manorama

(B) $G(s)H(s) = \frac{K}{s(s^2+6s+25)}$. (for $K>0$; -ve f/b)

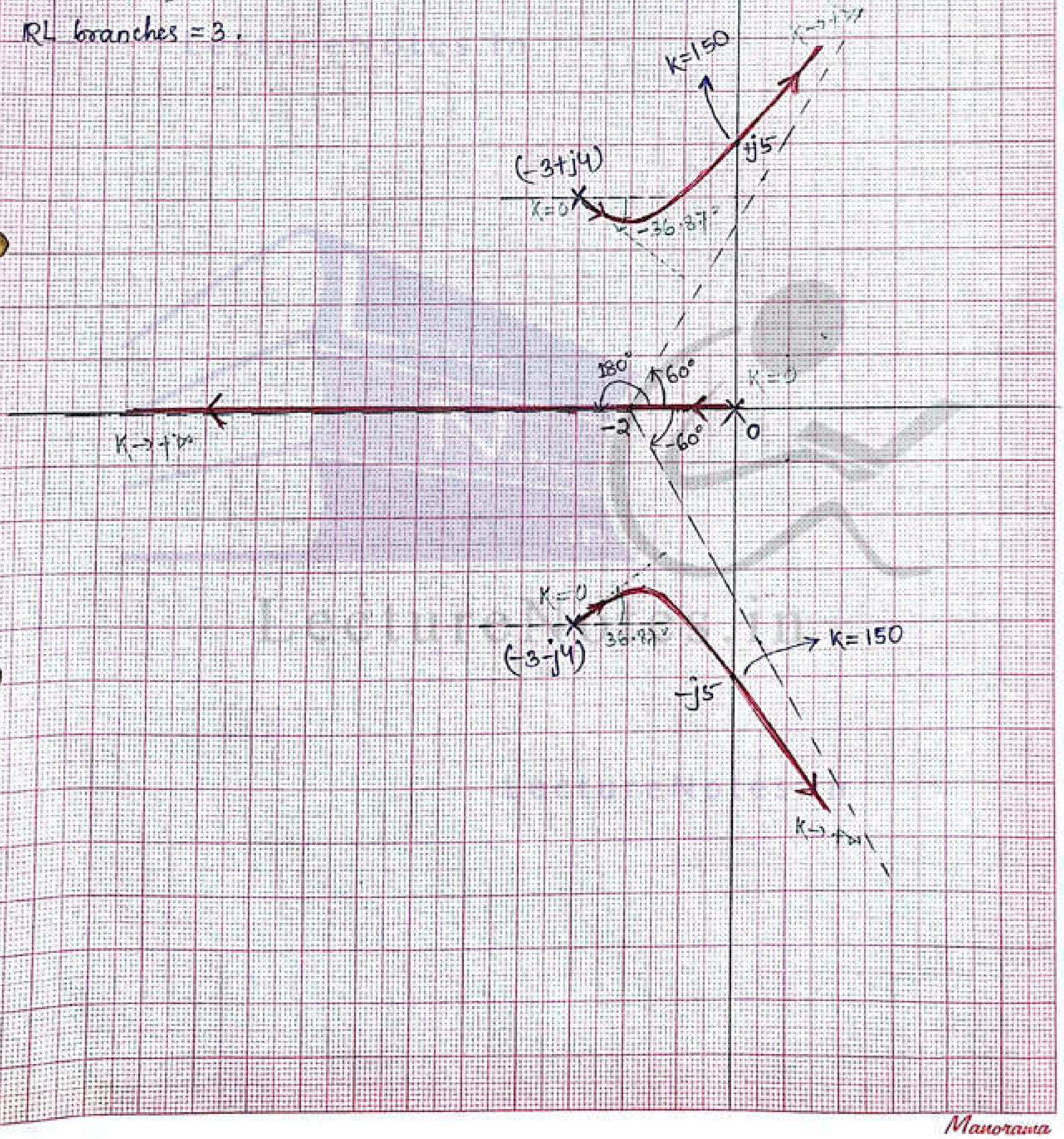
No BA Point. Angle of Assymptotes: $\theta_1 = 60°$; $\theta_2 = 180°$; $\theta_3 = 300°$.

Centroid $= (-2, 0)$. Intersection with Img-Axis $= \pm j5$.

ϕ_D (at $-3+j4$) $= -36.87°$

ϕ_D (at $-3-j4$) $= +36.87°$.

RL branches $= 3$.



Steps to draw RC :-

$$P(s) + k_1 Q_1(s) + k_2 Q_2(s) = 0.$$

Where, k_1 & k_2 are variable parameters

& $P(s)$, $Q_1(s)$ & $Q_2(s)$ are polynomials of 's'.

Step-I : Set one parameter to zero. (Let $k_2=0$)

$$P(s) + k_1 Q_1(s) = 0 \quad \text{--- (1)}$$

$$\Rightarrow 1 + \frac{k_1 Q_1(s)}{P(s)} = 0.$$

Step-II : Construct RL for $1 + k_1 G_1(s) H_1(s) = 0$.

Step-III : Restore the value of k_2, while consider k_1 fixed, and divide both sides by the terms that donot contain k_2.

$$1 + \frac{k_2 Q_2(s)}{P(s) + k_1 Q_1(s)} = 0.$$

Step-IV : Draw root contours by varying k_2 (while k_1 is fixed) for $1 + K_2 G_2(s) H_2(s) = 0$.

~~Step-V~~

Note :- The poles of $G_2(s) H_2(s)$ are identical to the roots of eq^n(1). So, the RC when k_2 varies must all start ($k_2=0$) at the points that lie on the Root loci of eq^n(1).

System with Transportation lag :-

→ In practice, after giving i/p to a system, there will be a time delay in the output (may be for fraction of second), which is known as transportation lag.

→ In time-domain, it is represented by $x(t-T)$ and in laplace domain $e^{-ST} X(s)$.

→ T.F. are common in industrial applications. They are often called dead time.

∴ So, transportain lag (e^{-sT}) can be expanded as :→

$$\left[e^{x} = 1 + x + \frac{x^2}{2!} + \dots\dots\right]$$

$$\therefore e^{-sT} = 1 - sT + \frac{s^2T^2}{2!} - \dots\dots$$

$\Rightarrow e^{-sT} \simeq 1 - sT$ [∵ Neglecting higher order terms]. as 'T' is very small

∴ Now using $e^{-sT} = 1 - sT$, we can able to draw the RL.

Q// $G(s)H(s) = \frac{Ke^{-s}}{s+2}$. Draw RL for $K>0$.

Sol^n → $G(s)H(s) = \frac{K(1-s)}{s+2} = \frac{-K(s-1)}{s+2}$.

for $K>0$, assuming negative f/b :- even concept RL.

$$1 + (-K)\frac{(s-1)}{s+2} = 0$$

(-ve)

Control System Engineering

Topic:

Frequency Response Analysis: Polar Plot

Contributed By:

Gyana Ranjan Biswal

FREQUENCY RESPONSE ANALYSIS

→ Whenever a signal is pass through linear system, then output frequency is same as input frequency, only magnitude and phase are affected.

$$\sin\omega t \longrightarrow \boxed{T(s)} \longrightarrow c(t)\Big|_{t\to\infty} = A\sin(\omega t+\phi).$$

stable LTI system

where, $A = |T(j\omega)|$
& $\phi = \angle T(j\omega)$.

→ For a sinusoidal i/p with variable frequency, the output at steady-state is given by :-

$$\sin\omega t \xrightarrow{} \boxed{T(s)} \longrightarrow c(t)\Big|_{t\to\infty} = A\sin(\omega t+\phi)$$

ω: 0^+ to ∞

stable LTI s/s
$T(j\omega)$

Where, $A = |T(j\omega)|$ Vs ω
$\phi = \angle T(j\omega)$ Vs ω.

→ Defn :-
Frequency response is the steady state response of a system to sinusoidal input by varying frequency 'ω' from 0^+ to ∞.

→ Magnitude & phase func are two plots in the frequency response Analysis.

$A = |T(j\omega)|$ Vs ω → Magnitude plot
$\phi = \angle T(j\omega)$ Vs ω → Phase plot.

Note: Why $s=j\omega$ for $\sin\omega t$?

Reason $\quad e^{st} = e^{\sigma t}\cdot e^{j\omega t}. \quad \sin\omega t = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$

So, practically $\sin\omega t$ is possible for which $e^{\sigma t}$ must be 1 & for which σ must be zero.

∴ So at steady state $\boxed{s = j\omega}$.

* Co-relation between Time Domain Analysis (TDA) & Frequency domain analysis (FDA) :-

→ The response in time domain for a 2^{nd} order prototype system is given by :→

$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2}.$$

Writing T(s) in time-constant form :→

$$\therefore T(s) = \frac{1}{1 + \frac{2\xi\omega_n}{\omega_n^2}s + \frac{1}{\omega_n^2}s^2}.$$

Putting $s = j\omega$:→

$$T(j\omega) = \frac{1}{1 + j2\xi\left(\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} = \frac{1}{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\} + j2\xi\left(\frac{\omega}{\omega_n}\right)}$$

Let $u = \frac{\omega}{\omega_n}$ = Normalised frequency

$$\therefore T(j\omega) = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}}$$

The magnitude & phase is given by:→

$$|T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}} \cdot \left[\phi = \tan^{-1}\frac{|2\xi u|}{|1-u^2|}\right.$$

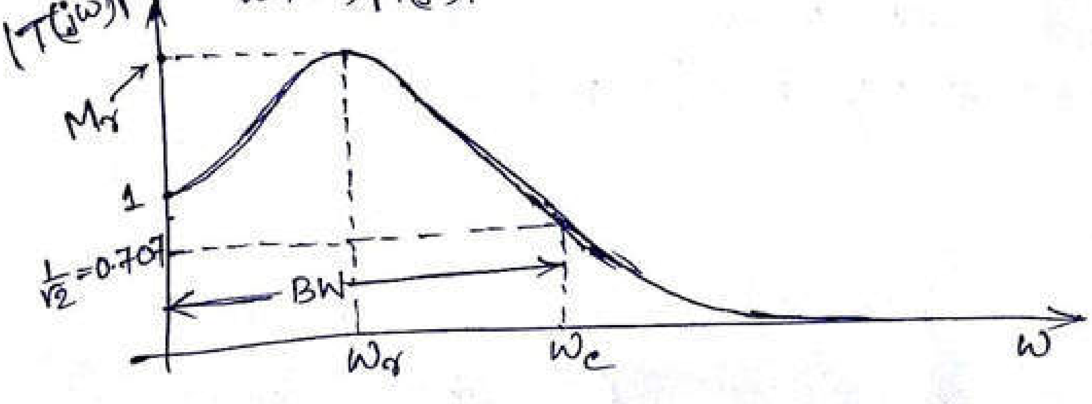
$$\angle T(j\omega) = \pi - \tan^{-1}\left[\frac{2\xi u}{(1-u^2)}\right], \quad \omega > \omega_n \ \& \ \xi > 0 \Rightarrow u > 1$$

$$= -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right), \quad \omega < \omega_n \ \& \ \xi > 0 \Rightarrow u < 1$$

Magnitude plot will be :→ $|T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}}$ Vs ω (0 to ∞)

At $\omega = 0$, $|T(j\omega)| = 1$.

$\omega \to \infty$, $|T(j\omega)| = 0$.



Frequency domain specifications are:-

1) Resonant frequency (ω_r) :-

$$\frac{dT(j\omega)}{d\omega} = 0 \Rightarrow \boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}}$$

~~2) Resonant Peak~~

$$\therefore \frac{dT(j\omega)}{du} = \frac{d}{du}\left\{\left[(1-u^2)^2 + 4\xi^2u^2\right]^{-1/2}\right\} = 0$$

$$\Rightarrow -\frac{1}{2}\left[(1-u^2)^2+4\xi^2u^2\right]^{-3/2} \cdot \frac{d}{du}\left[(1-u^2)^2+4\xi^2u^2\right] = 0$$

$$\Rightarrow -\frac{1}{2} \times \frac{1}{\left[(1-u^2)^2+4\xi^2u^2\right]^{3/2}}\left(-4u+4u^3+8\xi^2u\right) = 0$$

$$\Rightarrow 4u\left[u^2+2\xi^2-1\right] = 0$$

$$\therefore u = 0 \text{ or } u^2+2\xi^2-1 = 0 \Rightarrow u = \sqrt{1-2\xi^2}$$

$$\Rightarrow \boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}}$$

2) Resonant Peak (M_r) :-

$$M_r = |T(j\omega)|_{\omega=\omega_r} = \left.\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}\right|_{\omega=\omega_r=\omega_n\sqrt{1-2\xi^2}}$$

$$\Rightarrow M_r = \frac{1}{\sqrt{(1-1+2\xi^2)^2+4\xi^2(1-2\xi^2)}} = \frac{1}{\sqrt{+4\xi^4+4\xi^2-8\xi^4}}$$

$$\Rightarrow \boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}}$$

3) Cut off frequency (ω_c) :-

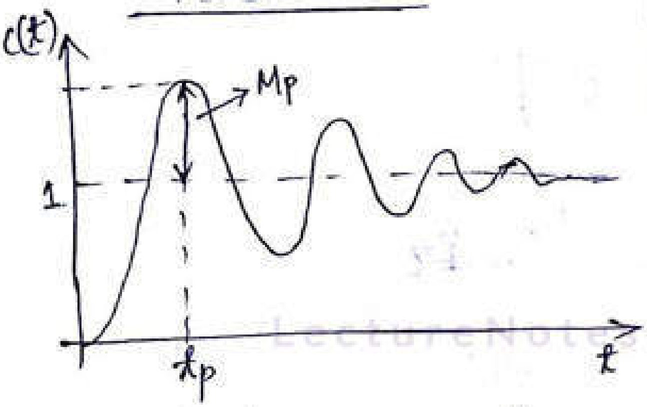
$$|T(j\omega)|_{\omega=\omega_c} = 1/\sqrt{2}$$

$$\Rightarrow \boxed{\omega_c = \omega_n\sqrt{-(2\xi^2-1)+\sqrt{(2\xi^2-1)^2+1}}} \text{ rad/sec}$$

4) Bandwidth (BW) :-

$$\boxed{BW = \omega_c} \text{ rad/sec.}$$

TDS



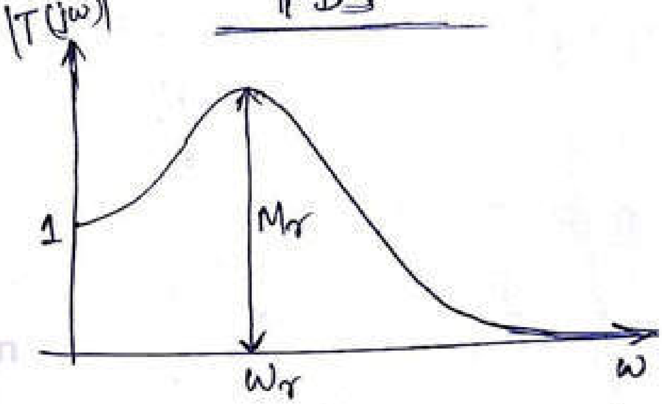
→ Unit step response for 2nd order prototype system.

→ $0 < \xi < 1$

→ $\omega_d = \omega_n\sqrt{1-\xi^2}$

→ $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$

FDS



→ Magnitude response for 2nd order Prototype system.

→ $0 < \xi < 1/\sqrt{2}$

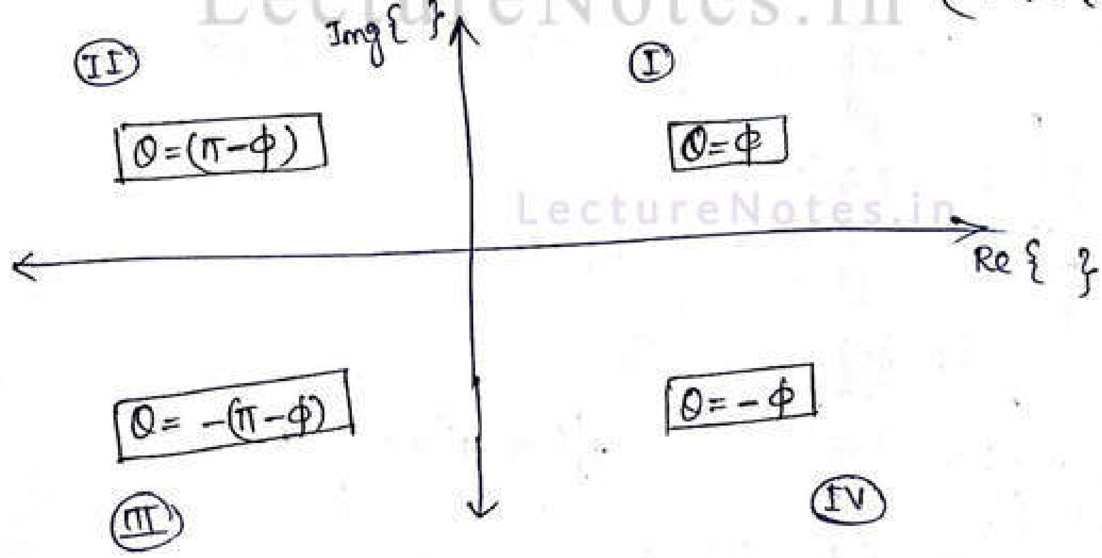
→ $\omega_r = \omega_n\sqrt{1-2\xi^2}$

→ $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

* Both analysis gives information about 'ξ'.

Complex No.

Complex No. → Cartesian $(x+jy)$
→ Polar $(re^{j\theta})$ → θ = Principal Argument $(-\pi < \theta \leq \pi)$



Where, $\phi = \tan^{-1}\dfrac{|y|}{|x|}$ = +ve (0 to 90°).

Ex :→ Given $Z = -2+j2$. write in polar form.

Q. Find Resonant peak (M_r), resonant frequency (ω_r) for a unity feedback system with $G(s) = \frac{100}{s(s+10)}$.

Sol^n :- $\frac{C(s)}{R(s)}$ = CL transfer function $= \frac{G(s)}{1+G(s)}$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{100/s(s+10)}{1+\frac{100}{s(s+10)}} = \frac{100}{s^2+10s+100}.$$

$\therefore \omega_n^2 = 100 \Rightarrow \omega_n = 10$. & $2\xi\omega_n = 10 \Rightarrow \xi = 0.5$.

$\therefore$ Resonant Peak $(M_r) = \frac{1}{2\xi\sqrt{1-\xi^2}} = \boxed{1.154}$

Resonant frequency $(\omega_r) = \omega_n\sqrt{1-2\xi^2} = \boxed{7.071 \text{ rad/sec}}$

Q. A 2nd order system has overshoot of 50% & period of oscillation 0.2 sec in step response. Determine resonant peak, resonant frequency & bandwidth.

Sol^n → $M_p = 50\%$ & $T = 0.2$ sec (given).

$\therefore M_p = 50\% \Rightarrow e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.5 \Rightarrow \xi = 0.2154$.

$\therefore T = 0.2 \Rightarrow f_d = 1/T = 5$ Hz.

$\therefore \omega_d = 2\pi f_d = 2\pi \times 5 = 31.41$ rad/sec.

So, $\omega_d = \omega_n\sqrt{1-\xi^2} = 31.41 \Rightarrow \omega_n = 32.165$ rad/sec.

$\therefore M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \boxed{2.377}$

$\omega_r = \omega_n\sqrt{1-2\xi^2} = 30.64$ rad/sec

$BW = \omega_n\left[1-2\xi^2+\sqrt{4\xi^4-4\xi^2+2}\right]^{1/2} = \boxed{48.3 \text{ rad/sec}}$

Advantages of Frequency Domain Analysis :-

→ FDA is very useful for higher order system.
→ More graphical methods are available to analyze the system response.
→ In frequency domain, it is easy to visualize the effects of noise disturbance & parameter variation.
→ Design & parameter adjustment can be carried out easily.
→ Sinusoidal test signals are easily available.

POLAR PLOT :-

→ It is the locus of tips of the phasors of various magnitudes plotted at the corresponding phase angles for different values of frequencies from 0^+ to ∞.

→ In polar plot, the magnitude & phase of $G(s)H(s)$ at $s=j\omega$ is plotted by varying ω.

1. Magnitude plot ($|G(j\omega)H(j\omega)|$ Vs ω)
2. phase plot ($\angle G(j\omega)H(j\omega)$ Vs ω)

Q// $G(s)H(s) = \frac{1}{s+1}$. Draw polar plot.

Soln → (1) write in time constant form.

(2) Take $s=j\omega$.

$$G(j\omega)H(j\omega) = \frac{1}{j\omega+1} = \frac{1}{1+j\omega}$$

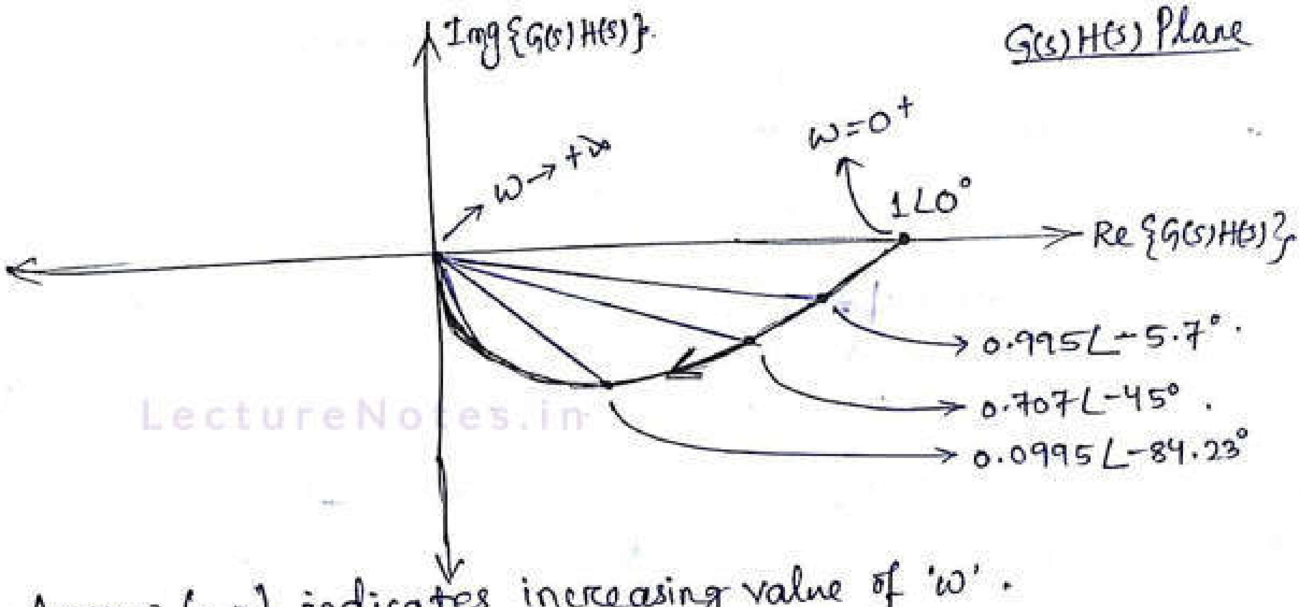
(3) Write magnitude & phase expression.

$$|G(j\omega)H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}} \quad \& \quad \angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega)$$

(4) Make a table of magnitude & phase by varying 'ω' from 0^+ to ∞.

	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$	
$\omega = 0^+$	1	0	→ (Marginal)
$\omega = 0.1$	0.995	$-5.71°$	→ IV
$\omega = 1$	0.707	$-45°$	→ IV
$\omega = 10$	0.0995	$-84.29°$	→ IV
$\omega = 100$	9.9995×10^{-3}	$-89.43°$	→ IV
$\omega \to \infty$	0	$-90°$	→ marginal.

(5) Select the quadrants where the plot will be present & then draw the plot in $G(s)H(s)$ plane.



* Arrow (→) indicates increasing value of 'ω'.

Q// $G(s)H(s) = \frac{K}{(1+s)(1+2s)}$. Draw polar plot.

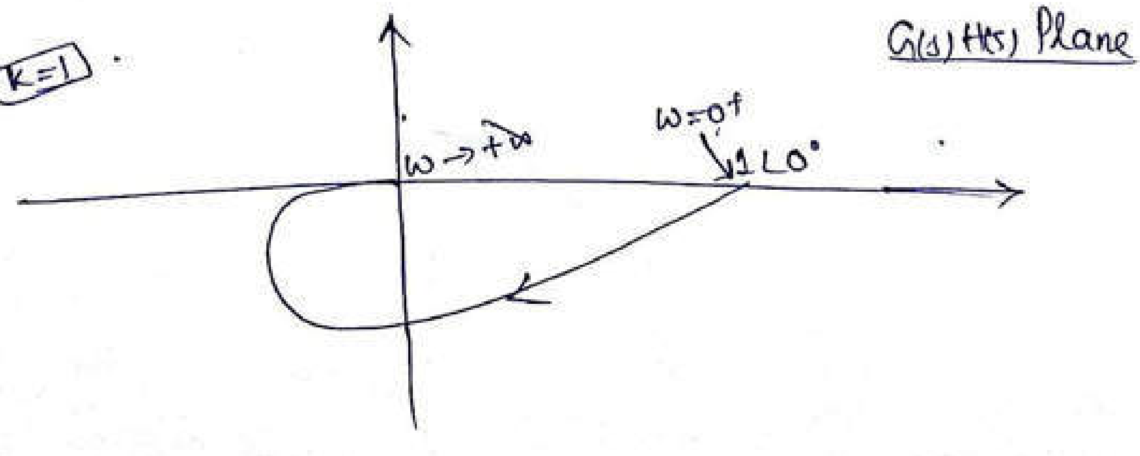
Sol^n:→ Type-0 & order-2.

Let k=1 : $G(j\omega)H(j\omega) = \frac{1}{(1+j\omega)(1+2j\omega)}$

$\therefore |G(j\omega)H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$ & $\angle G(j\omega)H(j\omega) = -\tan^{-1}\omega - \tan^{-1}(2\omega)$.

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$	
ω=0⁺	1	0	→ Marginal
ω=0.1	0.9757	-17.02	→ IV
ω=1			
ω=10	4.97×10^{-3}	-171.43°	→ III
ω=100	4.99×10^{-5}	-179.14	→ III
ω→∞	0	-180°	→ Marginal

K=1.



⑧

Q// $G(s)H(s) = \frac{K}{s(1+s)}$. Draw polar plot.

Sol^n → Type-1 & order-2 .

for K=1 ; $s=j\omega \rightarrow G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)}$.

$\therefore |G(j\omega)H(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}}$ & $\angle G(j\omega)H(j\omega) = -90° - \tan^{-1}\omega$.

ω	| |	$\angle$	
$\omega = 0^+$	$\boxed{\infty}$	$-90°$	→
$\omega = 0.1$	9.95	-95.71	→ III
$\omega = 10$	9.95×10^{-3}	-174.29	→ III
$\omega = 100$	9.99×10^{-5}	-179.43	→ III
$\omega \rightarrow \infty$	0	$-180°$	→ Marginal.

$$\therefore G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)} = \frac{-j}{\omega(1+j\omega)} = \frac{-j(1-j\omega)}{\omega(1+\omega^2)}$$

$$\Rightarrow G(j\omega)H(j\omega) = \frac{-j-\omega}{\omega(1+\omega^2)} = \frac{-\omega}{\omega(1+\omega^2)} - j\frac{1}{\omega(1+\omega^2)} .$$

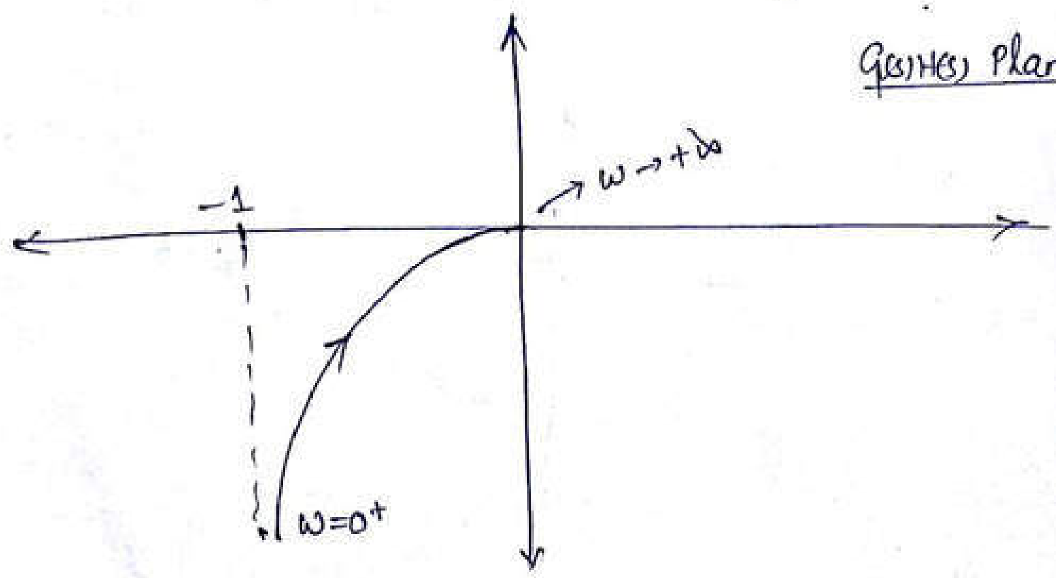
$$\therefore Re\{G(j\omega)H(j\omega)\} = \frac{-\omega}{\omega(1+\omega^2)} = \frac{-1}{1+\omega^2} .$$

$$Img\{G(j\omega)H(j\omega)\} = \frac{-1}{\omega(1+\omega^2)} .$$

At $\omega = 0^+$:→ $Re\{G(j\omega)H(j\omega)\} = -1$.

$Img\{G(j\omega)H(j\omega)\} = -ve$ very large quantity.

G(s)H(s) Plane



Q// $G(s)H(s) = \frac{K}{s^2(1+s)(1+2s)}$ • Draw polar plot.

Sol^n → Type-2 & order-4.

for K=1 : $|G(j\omega)H(j\omega)| = \frac{1}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$.

$\angle G(j\omega)H(j\omega) = -180° - \tan^{-1}\omega - \tan^{-1}2\omega$.

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$	
$\omega = 0^+$	$\boxed{\infty}$	$-180°$	→ X
$\omega = 0.1$	97.5714	-197.02	→ III
$\omega = 10$	4.969×10^{-5}	-351.43	→ IV
$\omega = 100$	4.99×10^{-9}	-359.14	→ IV
$\omega \to \infty$	0	$-360°$	→ Marginal

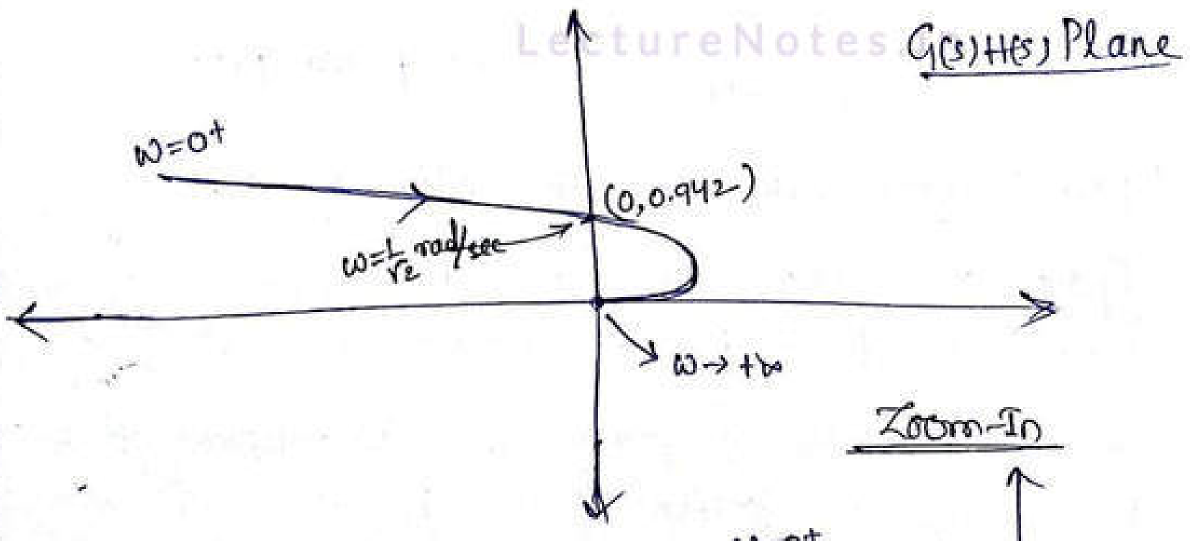
$\therefore \ Re\{G(j\omega)H(j\omega)\} = \frac{-(1-2\omega^2)}{\omega^2(1+\omega^2)(1+4\omega^2)}$

$Img\{G(j\omega)H(j\omega)\} = \frac{3}{\omega(1+\omega^2)(1+4\omega^2)}$

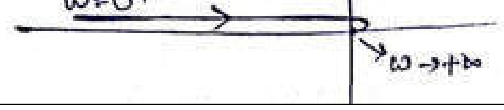
Putting $\omega = 0^+$:→

$Re\{\ \} = \lim_{\omega\to0^+} \left\{\frac{-1}{\omega^2}\right\}$ → -ve very very large

$Img\{\ \} = \lim_{\omega\to0^+} \left\{3/\omega\right\}$ → +ve very large



Zoom-In

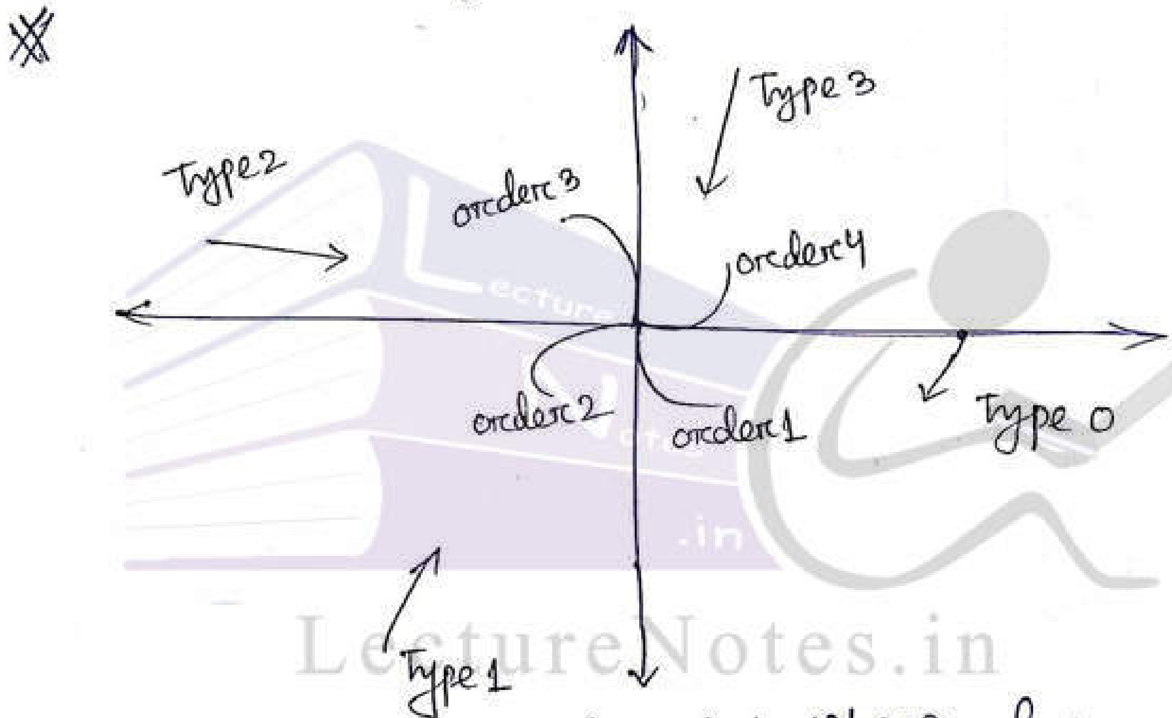


Observation :-

* Type decides starting point of polar plot & order decides termination of polar plot.

→ Addition of a pole at origin shifts the starting point of polar plot by 90° in clockwise direction.

→ Addition of a simple pole shifts the terminating point of polar plot by 90° in clockwise direction.



→ Above concept is only valid when ① no finite zeros & ② no poles present in right half.

Q// $G(s)H(s) = \frac{4s+1}{s^2(1+s)(1+2s)}$. Draw polar plot.

Gain Margin (GM), Phase Margin (PM) & Stability :-

Gain Cross over frequency (ω_{gc}) : → The frequency at which magnitude of $G(j\omega)H(j\omega)$ is unity i.e, 1 is called ω_{gc}.

Phase Cross over frequency (ω_{pc}):- The frequency at which phase angle of $G(j\omega)H(j\omega)$ is $-180°$ is called ω_{pc}.

Gain Margin (GM) :-

(i.e, multiplied)

→ It is the ~~margin~~ value of gain that can be introduced in the system till system reaches on the verge of instability.

→ Positive gain margin (>1) means system is in stable cond^n & the amount of margin can be introduced to make system unstable.

→ Negative gain margin (<1) means sys is in unstable & the amt of margin must be reduced to make system stable.

→ Math^l :→ $\boxed{GM = \dfrac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}}$

Phase Margin (PM) :-

→ ~~It is~~ ~~The~~ It is the additional phase lag (i.e, -ve angle) (i.e, added) which can be introduced in the system till system reaches on the verge of instability, is called PM.

→ positive PM means, some negative angle introduction is possible before system becomes unstable. Present cond^n stable.

→ Negative PM means, some +ve angle introduction is possible to make system stable. Present condition Unstable.

→ Math^l :→ $\boxed{PM = 180^\circ + \angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}}$

** PM & GM are directly proportional to stability.
i.e, GM ↑es ⇒ stability ↑es & PM ↑es ⇒ stability ↑es.

~~Stability~~

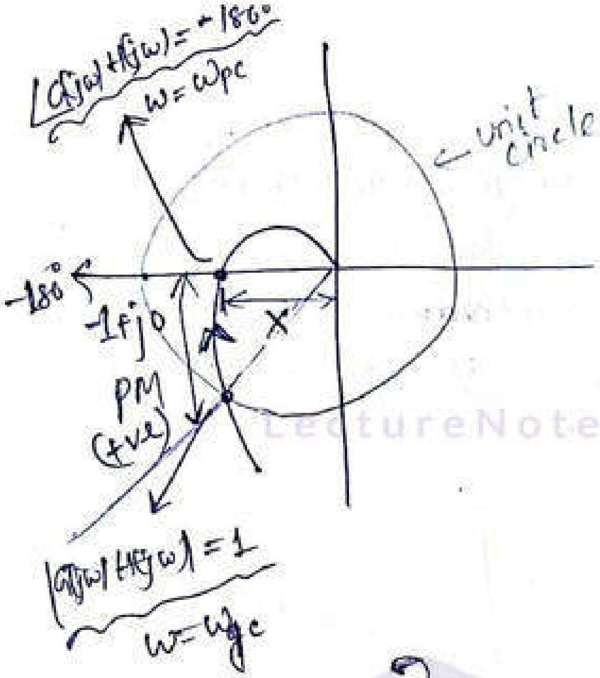
Stability :-

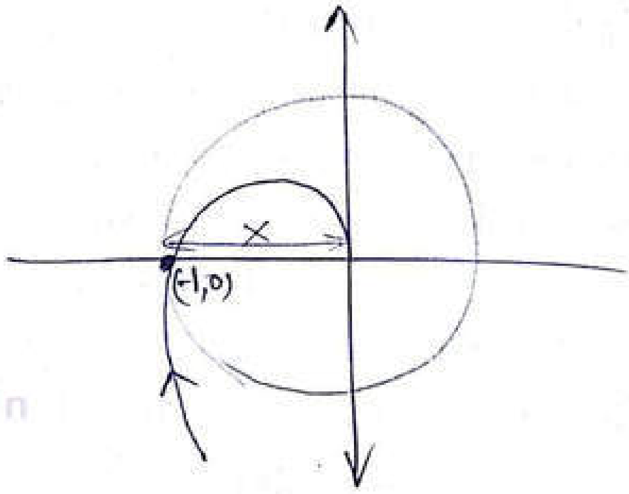
→ It is based on concept of enclosement
↳ A point is said to be enclosed if it is present in the right hand side of the direction contour (closed path)

Ex :→

// x not enclosed
y is enclosed.

// x is enclosed
y not enclosed.





$\boxed{PM = 0}$; $\boxed{GM = 1}$.

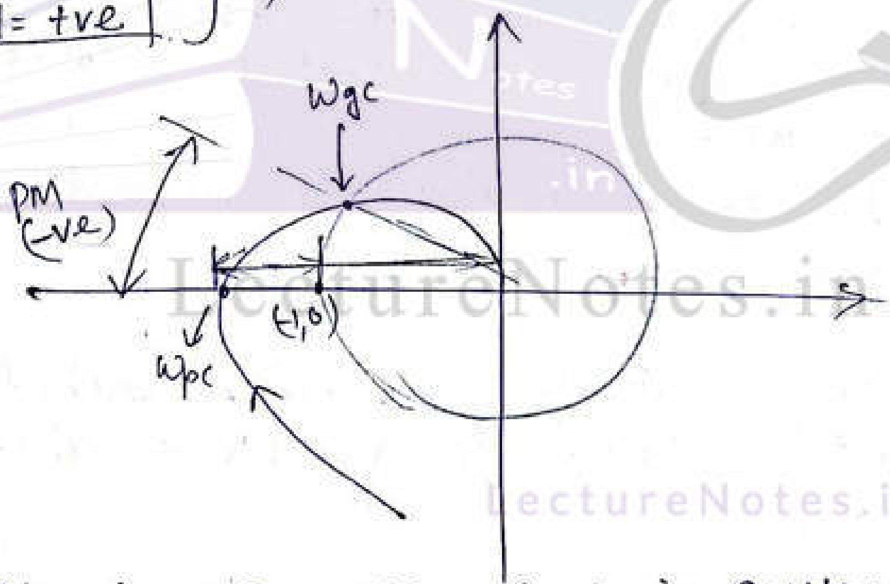
$\boxed{\omega_{pc} = \omega_{gc}}$

Marginal stable

$\boxed{\omega_{pc} > \omega_{gc}}$

PM = +ve
GM = +ve

} Absolute stable



$\omega_{pc} < \omega_{gc}$
PM = -ve
GM = -ve

Unstable

* Critical point $= -1+j0 = (-1,0)$ in $G(s)H(s)$ plane .
* A. stable → Critical point not enclosed.
  M. stable → on polar plot.
  Unstable → Critipoint ~~...~~ is enclosed .

※ Calculation of GM & PM :-

Gain Margin (GM)	Phase Margin (PM)
→ find ω_{pc} = ? at $\angle G(j\omega)H(j\omega) = -180°$.	→ find ω_{gc} = ? at $\lvert G(j\omega)H(j\omega)\rvert = 1$.
→ Put $\omega = \omega_{pc}$ in magnitude, $X = \lvert G(j\omega)H(j\omega)\rvert_{\omega=\omega_{pc}}$	→ Put $\omega = \omega_{gc}$ in phase, $\phi/\theta = \angle G(j\omega)H(j\omega)\rvert_{\omega=\omega_{gc}}$
→ $GM = \frac{1}{X}$ (Absolute) ∴ $GM = 20\log\left(\frac{1}{X}\right)$ (dB)	→ $PM = 180 - \lvert\phi\rvert$ or $\theta - 180°$ → degree.

Where, • ω_1 = Phase cross-over frequency (ω_{pc})
ω_2 = Gain Cross-over frequency (ω_{gc})

Q// $G(s)H(s) = \frac{K}{s(s+1)(s+2)}$. Sketch polar plot. Determine ω_{pc} & GM. Also find the condn on 'K' for which CL system is absolute stable.

Solⁿ→ Time Constant form :- $G(s)H(s) = \frac{K/2}{s(1+s)(1+s/2)}$

for K=1 :- $\Rightarrow G(j\omega)H(j\omega) = \frac{1/2}{j\omega(1+j\omega)(1+j\frac{\omega}{2})}$

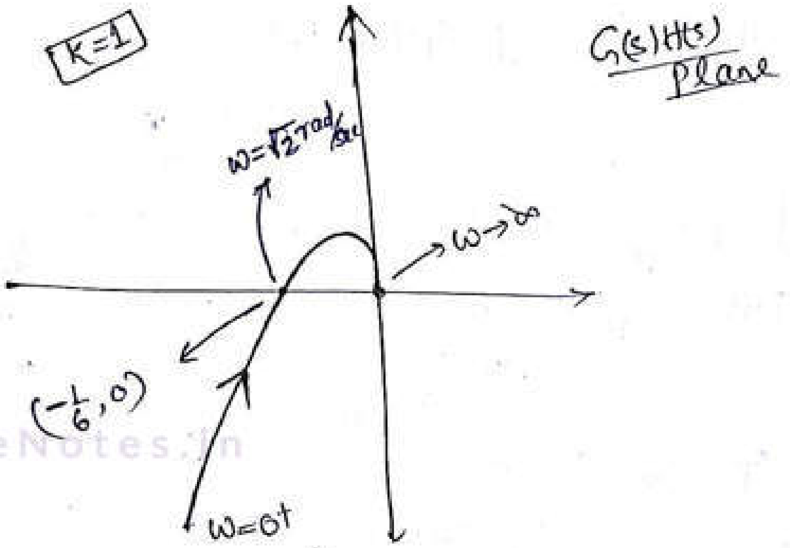
$$|G(j\omega)H(j\omega)| = \frac{1/2}{\omega\sqrt{1+\omega^2}\sqrt{1+\frac{\omega^2}{4}}}$$

$$\angle G(j\omega)H(j\omega) = -90° - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

Table :→

ω	$\lvert\ \rvert$	$\angle$	
$\omega = 0^+$	$\boxed{\infty}$	$-90°$	→ X
$\omega = 0.1$	4.969	-98.573	→ III
$\omega = 1$	0.316	-161.565	→ III
$\omega = 10$	9.76×10^{-4}	-252.98	→ II
$\omega = 100$	9.997×10^{-7}	-268.28	→ II
$\omega \to \infty$	0	$-270°$.	→ Marginal

(14)



At $\omega = 0.000001$ (i.e, $\omega = 0^+$) :→

$$Re\{G(j\omega)H(j\omega)\} = -0.75 - 5000000j$$

→ −ve Real (very small)

−ve Imaginary (very large)

ω_{pc} :-

$$\angle G(j\omega)H(j\omega) = -180^\circ$$

$$\Rightarrow -90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2} = -180^\circ \Rightarrow \tan^{-1}\omega + \tan^{-1}\frac{\omega}{2} = 90^\circ$$

$$\Rightarrow \tan\left[\tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{2}\right)\right] = \tan(90^\circ)$$

$$\Rightarrow \frac{\tan[\tan^{-1}(\omega)] + \tan\left[\tan^{-1}\left(\frac{\omega}{2}\right)\right]}{1 - \{\tan[\tan^{-1}(\omega)]\}\cdot\left\{\tan\left[\tan^{-1}\left(\frac{\omega}{2}\right)\right]\right\}} = \infty = \frac{x}{0}.$$

$$\Rightarrow \frac{\omega + \frac{\omega}{2}}{1 - \frac{\omega^2}{2}} = \frac{x}{0} \Rightarrow 1 - \frac{\omega^2}{2} = 0 \Rightarrow \omega^2 = 2 \Rightarrow \omega = \sqrt{2} = 1.414 \text{ rad/sec}.$$

$$\therefore \boxed{\omega_{pc} = 1.414 \text{ rad/sec}}.$$

$$\therefore |G(j\omega)H(j\omega)|_{\omega=\omega_{pc}} = \frac{1/2}{(\sqrt{2})\sqrt{1+2}\sqrt{1+\frac{1}{2}}} = \frac{1}{6} = \underline{0.1667} = X$$

$$\therefore \underline{\text{Gain Margin}} = \frac{1}{X} = \frac{1}{(1/6)} = 6.$$

$\underline{\omega_{gc}}$:-

$|G(j\omega)H(j\omega)| = 1 \Rightarrow \dfrac{1/4}{\omega^2(1+\omega^2)(1+\frac{\omega^2}{4})} = 1.$

$\Rightarrow \omega^2(1+\omega^2)(1+\frac{\omega^2}{4}) = 1/4 \Rightarrow (\omega^2+\omega^4)(1+\frac{\omega^2}{4}) = 1/4$

$\Rightarrow \omega^2 + \frac{\omega^4}{4} + \omega^4 + \frac{\omega^6}{4} = 1/6 \Rightarrow \omega^2 + 1.25\omega^4 + 0.25\omega^6 = 1/6$

~~Let ω²~~ $\Rightarrow \omega^2(1+1.25\omega^2+0.25\omega^3) = 1/6.$

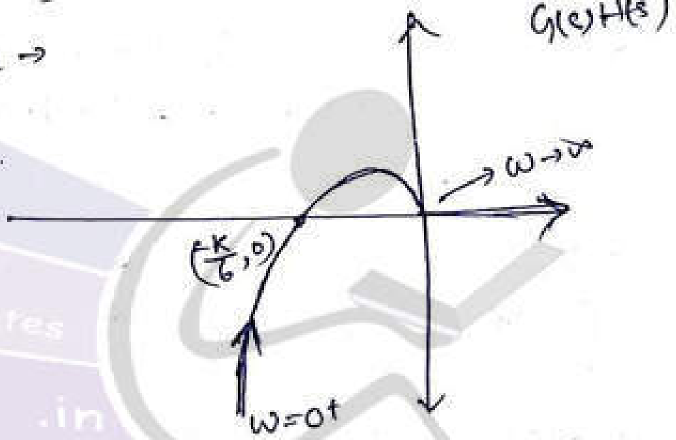
OR $G(j\omega)H(j\omega) = \dfrac{1/2}{(j\omega)(1+j\omega)(1+j\omega/2)}$

For $K>0$: Polar plot will be →

for absolute stable system,

$\frac{-K}{6} > -1 \Rightarrow \frac{K}{6} < 1$

$\Rightarrow \boxed{K<6} \rightarrow$ A. stable.



Q// $G(s)H(s) = \dfrac{2\sqrt{3}}{s(s+1)}$. Determine Phase Margin, ω_{gc} for CL System:

Sol^n → $|G(j\omega)H(j\omega)| = \dfrac{2\sqrt{3}}{\omega\sqrt{1+\omega^2}}$ & $\angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}(\omega)$.

To find ω_{gc} :-

$|G(j\omega)H(j\omega)| = 1 \Rightarrow \dfrac{2\sqrt{3}}{\omega\sqrt{(1+\omega^2)}} = 1 \Rightarrow \dfrac{4\times 3}{\omega^2(1+\omega^2)} = 1.$

$\Rightarrow \omega^2 + \omega^4 - 12 = 0 \Rightarrow \omega^4 + \omega^2 - 12 = 0.$

Let $x = \omega^2$; $x^2 + x - 12 = 0 \Rightarrow x = 3, -4.$

$\therefore \omega = \pm\sqrt{3}, \pm\sqrt{-4} = \pm j2$ X

$\therefore \boxed{\omega_{gc} = \sqrt{3}\ \text{rad/sec}}.$

$\therefore \angle|_{\omega=\omega_{gc}} = -150^\circ$ (clockwise).

$\therefore PM = 180^\circ - |\phi| = \boxed{+30^\circ}$. or $PM = 180^\circ + (-150^\circ) = +30^\circ$.

∴ CL system stable system.

(16)

Q// $G(s)H(s) = \frac{1}{s(s^2+s+1)}$. sketch Polar Plot & determine GM & PM for CL system. Also Comment upon stability.

Note:→

* for Absolute stable system:-

$\alpha < 1$; GM > 1 ; GM (+ve) & PM (+ve) ; $\omega_{pc} > \omega_{gc}$

& critical point not enclosed.

* for Marginal stable system:-

$\alpha = 1$; $\underbrace{GM}_{=0}$ (~~is +ve~~) & PM = 0 ; $\omega_{pc} = \omega_{gc}$ &

critical point is on the polar plot.

* for unstable system:-

$\alpha > 1$; GM (-ve) & PM (-ve) ; $\omega_{pc} < \omega_{gc}$ &

critical point is enclosed.

⟹ The above results are always true for minimum phase functions. It is may or may not be true for non-minimum phase function.

* Minimum Phase function:→

↳ No poles or zeros are present in right half of s-plane, otherwise non-minimum phase func.

Limitations:

(1) If system is unstable, then we can't determine no of CL poles in RH.

(2) Can't be always true for non-minimum phase plot.

→ Remedy to the above limitations is Concept of encirclement.

↳ Principle of mapping

↳ Principle of argument.

Control System Engineering

Topic:

Nyquist Plot

Contributed By:

Gyana Ranjan Biswal

NYQUIST PLOT

→ It is based on the polar plot which can be conveniently applied to the stability analysis of any kind of system.

→ It also gives the information about the no of CL poles present on right half of s-plane.

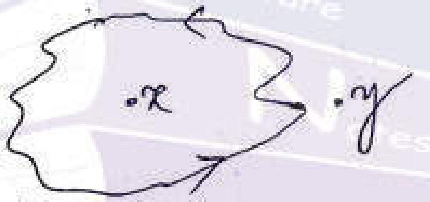
→ The stability of a system from Nyquist plot can be determined by using concept of encirclement.

→ This concept of encirclement is based on:
(i) principle of mapping
(ii) Principle of Argument.

Concept of encirclement :-

→ A point is said to be encircled by a closed path if it is found to lie inside that closed path.

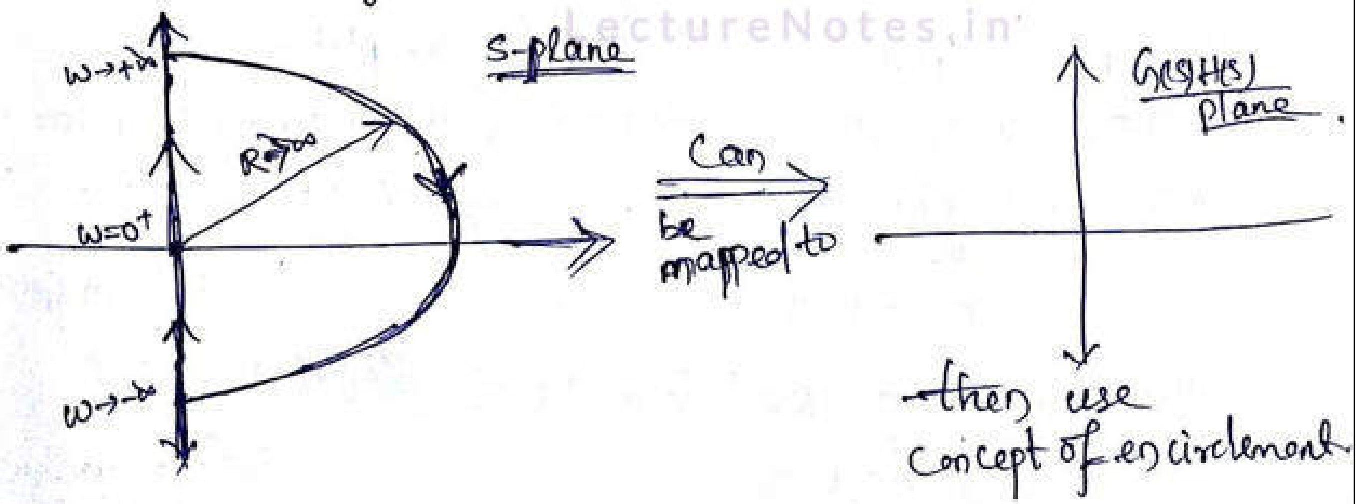
Ex :→



'x' is encircled in ACW dir^n,
& 'y' is not encircled.

Principle of Mapping

→ Principl of mapping states that the entire right half of s-plane is drawn in G(s)H(s) plane by mapping rules. Then use concept of encirclement to determine the stability.

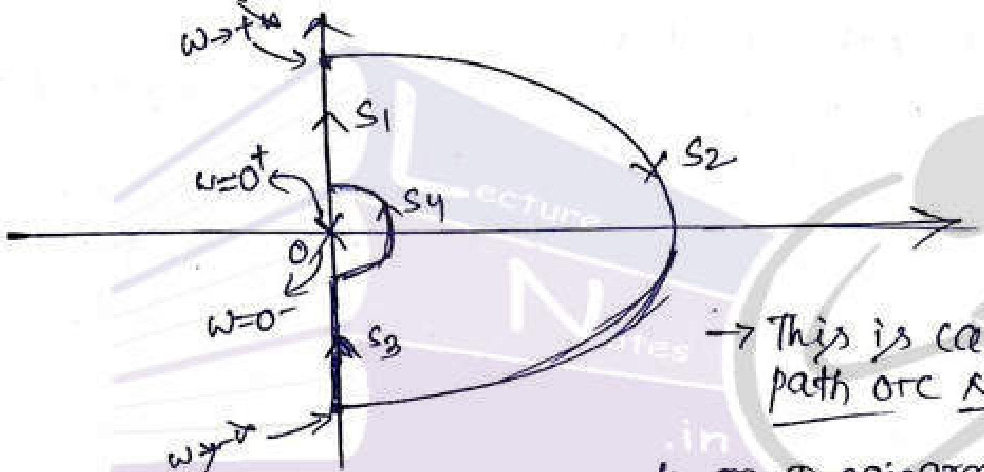


Principle of Argument :-

→ It states that, a system is said to be Absolute stable if $\boxed{N = P - Z}$. This cond^n is called as Nyquist stability criteria.

Where, N = No. of encirclement of critical point (from mapping)
P = No. of OL poles in RH of s-plane (from OLTF)
Z = No. of CL poles in RH of s-plane (to det. stability)

Mapping of Right Half of s-plane :-



→ This is called as Nyquist path or Nyquist contour. (in s-plane),

* If OLP or zeros present on Imaginary axis, then S_4 region will be present, otherwise only mapping of S_1, S_2 & S_3 is needed.

Mapping of different regions :-

(1) $S_1 \longrightarrow S_1 = j\omega \rightarrow 0^+$ to $+\infty$ (Polar plot)

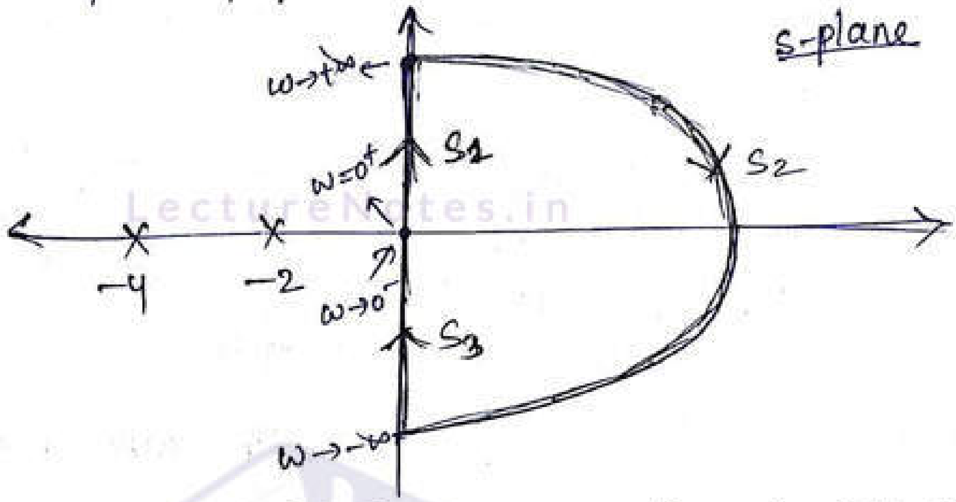
(2) $S_3 \longrightarrow S_3 = j\omega \rightarrow -\infty$ to 0^- (Mirror Image of polar plot)

(3) $S_2 \longrightarrow \lim_{\substack{R \to \infty \\ \theta \to \pi/2 \text{ to } -\pi/2}} \{Re^{j\theta}\} = S_2 \Rightarrow \underbrace{G(S_2)\,H(S_2) = |\;|\angle}_{\text{Plot in } G(s)H(s) \text{ plane}}$

(4) $S_4 \rightarrow \lim_{\substack{R \to 0 \\ \theta \to -\pi/2 \text{ to } \pi/2}} \{Re^{j\theta}\} = S_4 \Rightarrow \underbrace{G(S_4)\,H(S_4) = |\;|\angle}_{\text{plot in } G(s)H(s) \text{ plane.}}$

Q// OLTF is : $G(s)H(s) = \frac{10}{(s+2)(s+4)}$ · Comment on CL stability.

Sol^n → step-1 :→ Draw the Nyquist path (in s-plane) by observing the open loop poles & zeros.



step-2 :→ Plot all the regions in G(s)H(s) Plane. (use principle of mapping).

S_1 : Polar plot ($\omega = 0^+$ to $+\infty$).

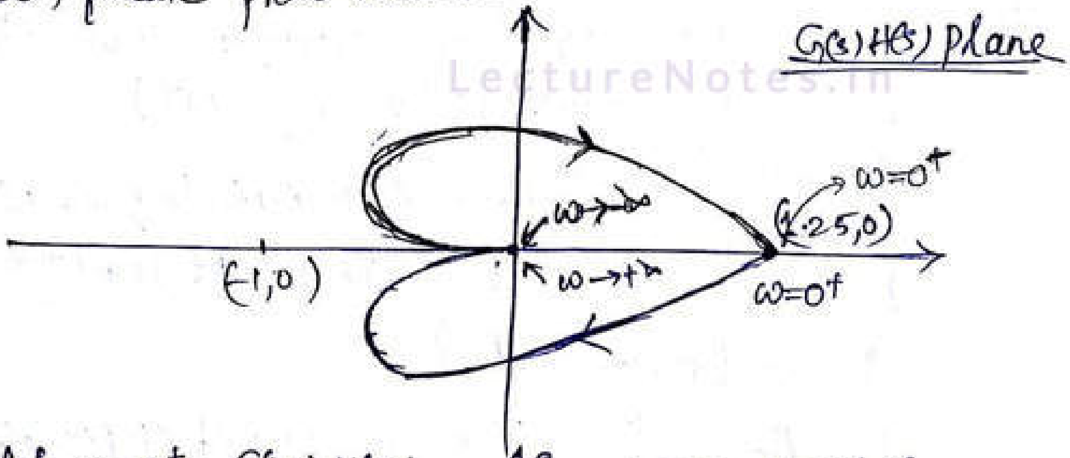
$$G(j\omega)H(j\omega) = \frac{10/8}{(1+\frac{s}{2})(1+\frac{s}{4})} = \frac{5/4}{(1+s/2)(1+s/4)} = \frac{5/4}{(1+j\frac{\omega}{2})(1+j\frac{\omega}{4})}$$

→ Time Constant form.

Method-1 : find $|G(j\omega)H(j\omega)|$ & $\angle G(j\omega)H(j\omega)$ & then plot.

M-2 : Type 0 & order-2.

So, polar plot will be :→



At $\omega = 0^+$, $G(j\omega)H(j\omega) = \frac{10}{8} = 1.25 = 1.25\angle 0°$.

S_3 : Mirror image of polar plot with respect to horizontal axis.

S_2 :

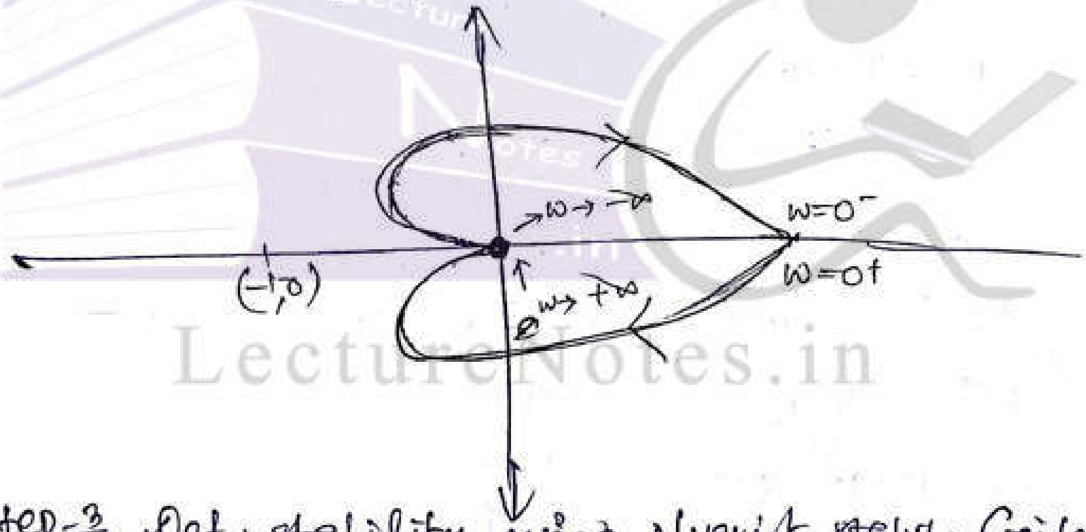
$$S_2 = \lim_{\substack{R\to\infty \\ \theta \to +\pi/2 \text{ to } -\pi/2}} \{Re^{j\theta}\} \Rightarrow |S_2| \text{ is very large.}$$

$$\therefore G(S_2)H(S_2) = \frac{10}{(S_2+2)(S_2+4)} \simeq \frac{10}{S_2^2} \quad (\because S_2 \text{ is very high})$$

$$\therefore G(S_2)H(S_2) = \lim_{\substack{R\to\infty \\ \theta \to +\pi/2 \text{ to } -\pi/2}} \frac{10}{(Re^{j\theta})^2} = \lim_{\substack{R\to\infty \\ \theta \to \pi/2 \text{ to } -\pi/2}} \left(\frac{10}{R^2}\right) \cdot e^{-j(2\theta)}.$$

$$\Rightarrow G(S_2)H(S_2) = 0\, e^{-j(-\pi \text{ to } \pi)} = \underline{\text{Origin in } G(s)H(s) \text{ plane}}$$

$\therefore$ so, final Nyquist plot is :→



step-3 Det. stability using Nyquist stability Criteria (or using principle of argument)

→ find No of encirclements in Anticlockwise = N.

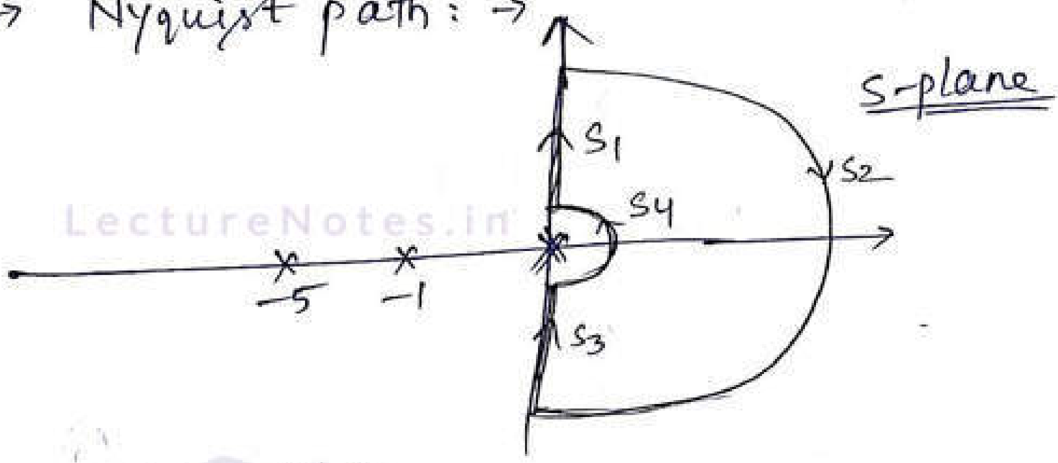
so, here, N = 0 i.e, critical pt is not encircled.

$P = 0$ (from OLTF)

$\therefore N = P - Z \Rightarrow \boxed{Z = 0}$ ∴ so no of CL poles in RH side of s-plane. ⇒ Absolute stable system.

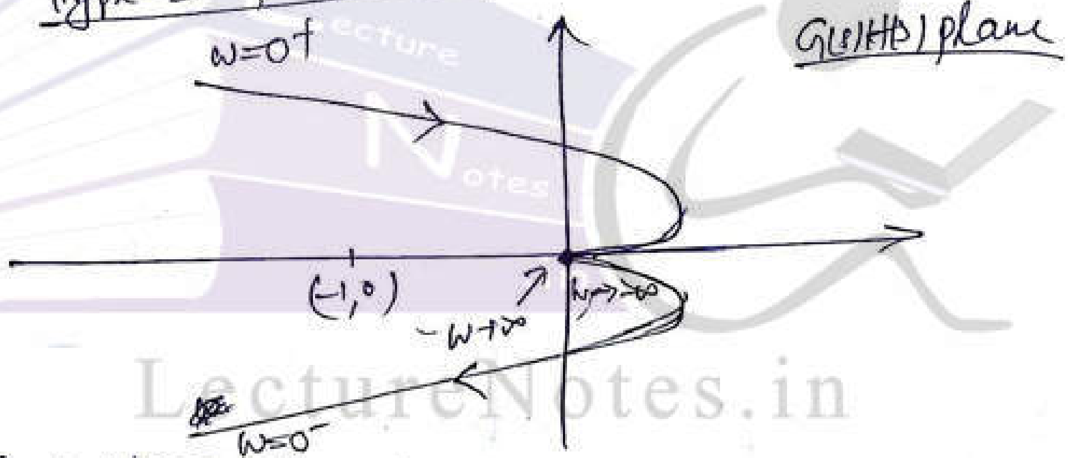
Q// Draw Nyquist plot for OLTF $G(s)H(s) = \frac{10}{s^2(s+1)(s+5)}$ & also determine the CL stability.

Sol^n → Nyquist path : →



S_1 : Polar Plot:

Type-2 & order-4



S_3 : Mirror image.

S_2 : As deg (Nr) < deg (Dr), the mapping of 's_2' is an origin in G(s)H(s) plane.

S_4 : $S_4 = \lim_{\substack{R \to 0 \\ \theta \to -\pi/2 \text{ to } \pi/2}} (Re^{j\theta})$ ⇒ $|S_4|$ is very small.

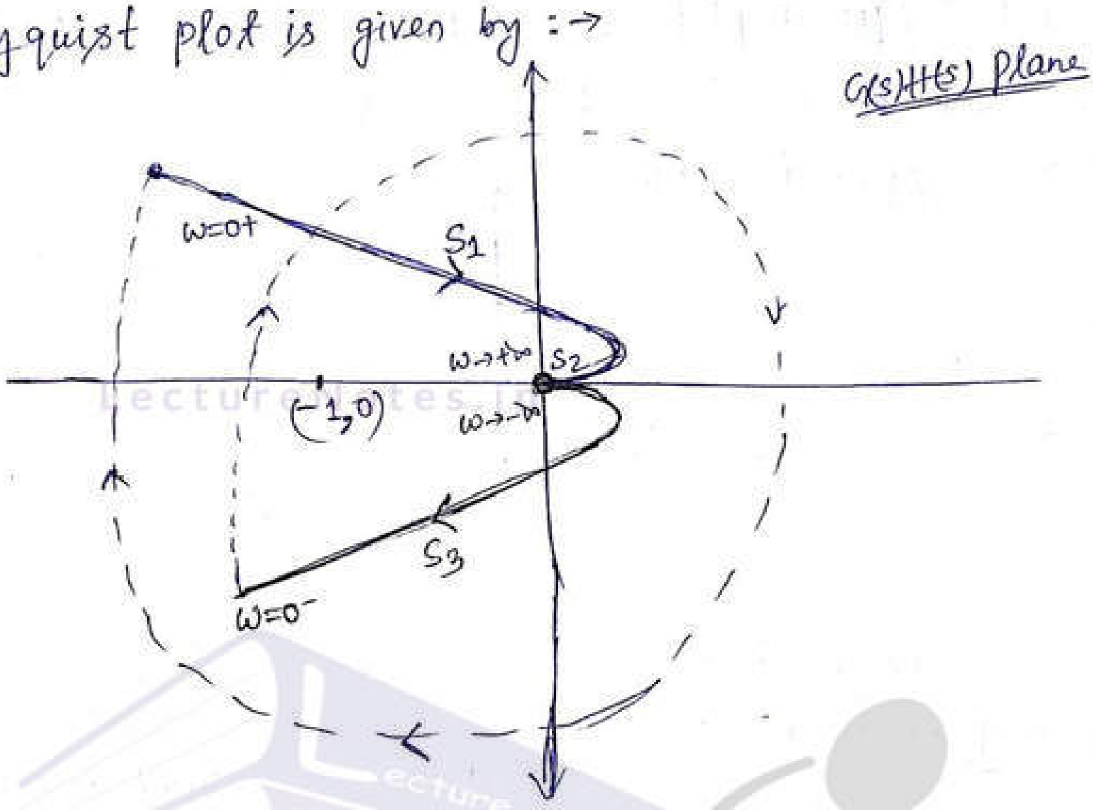
$\therefore G(S_4)H(S_4) = \frac{10}{S_4^2 (S_4+1)(S_4+5)} \simeq \frac{2}{S_4^2}$

$\therefore G(S_4)H(S_4) = \lim_{\substack{R \to 0 \\ \theta \to -\pi/2 \text{ to } +\pi/2}} \frac{2}{(Re^{j\theta})^2} = \frac{2}{R^2} e^{j(-2\theta)}$

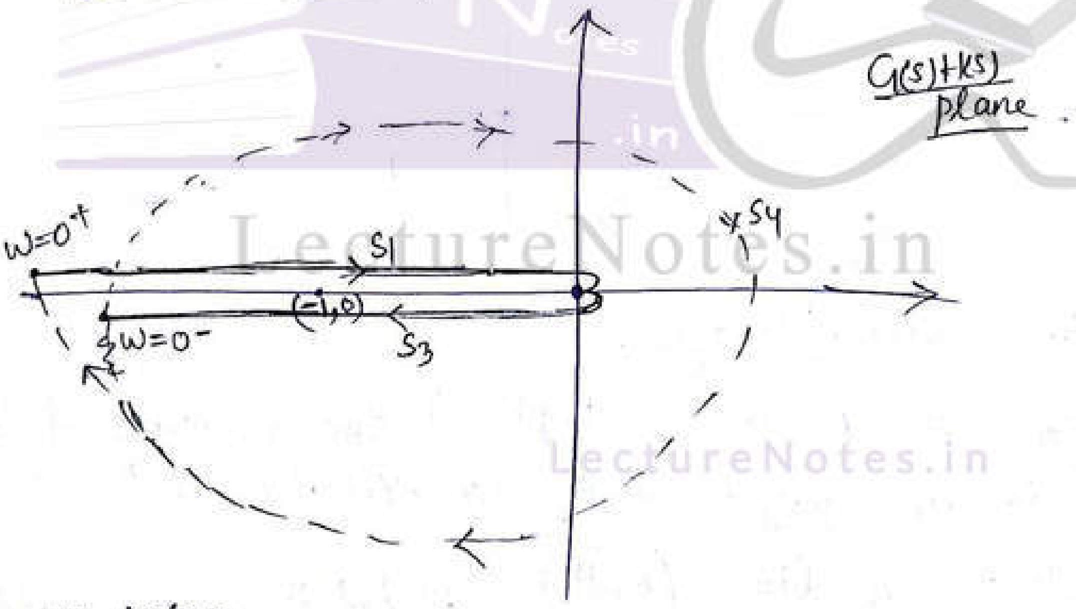
$= \infty \, e^{j(\pi \text{ to } -\pi)}$

(22)

Nyquist plot is given by :→



Zoom-In view →



Stability :-

$N = 2\ (CW) = -2\ (ACW)$

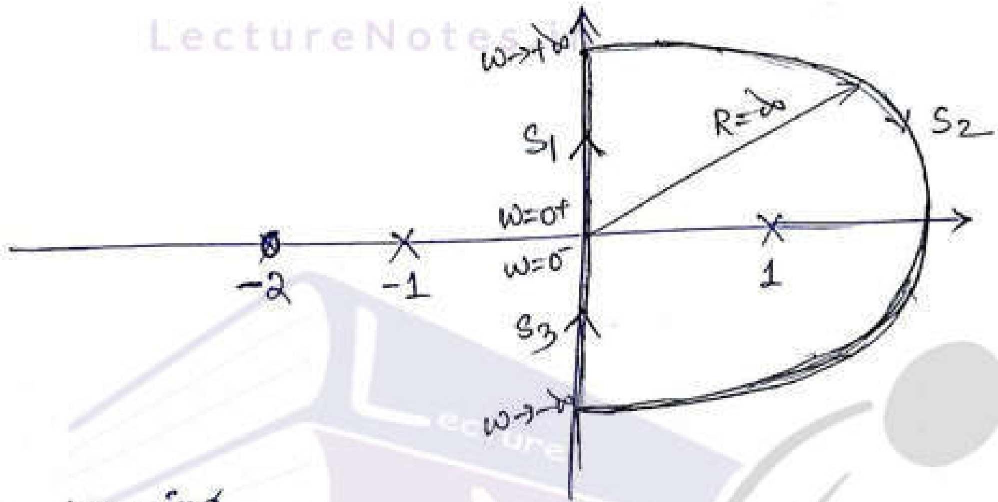
$P = 0$ (from OLTF)

$\therefore N = P - Z \Rightarrow \boxed{Z=2} \Rightarrow$ Unstable System . (Ans)

Q// By applying Nyquist stability criterion, find the value of 'K' for which the system is just stable.

$$G(s) = \frac{K(s+2)}{(s+1)(s-1)} \quad ; H(s) = 1.$$

Soln: → Nyquist path :-



Mapping

S1 Polar plot of $G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$ (Assuming K=1) by varying $\omega = 0^+$ to $+\infty$.

$$G(j\omega)H(j\omega) = \frac{2(1+s/2)}{(1+s)(s-1)}\Bigg|_{s=j\omega} = \frac{2\left(1+j\frac{\omega}{2}\right)}{(1+j\omega)(j\omega-1)}$$

$\phi = \tan^{-1}\left(\frac{\omega}{1}\right)$
$\Rightarrow \phi = \tan^{-1}(\omega)$

$$|G(j\omega)H(j\omega)| = \frac{2\sqrt{1+\omega^2/4}}{\sqrt{1+\omega^2}\sqrt{1+\omega^2}} = \frac{2\sqrt{1+\omega^2/4}}{(1+\omega^2)}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \cancel{\tan^{-1}(\omega)} - \left(\pi - \cancel{\tan^{-1}(\omega)}\right)$$

$$= -180^\circ + \tan^{-1}\left(\frac{\omega}{2}\right)$$

ω=0+	2	−180°	→ Marginal
ω=0.1	1.982	−177.14°	→ III
ω=10	0.101	−101.31	→ III
ω=100	0.01	−91.146	→ III
ω→∞	0	−90°	→ Marginal

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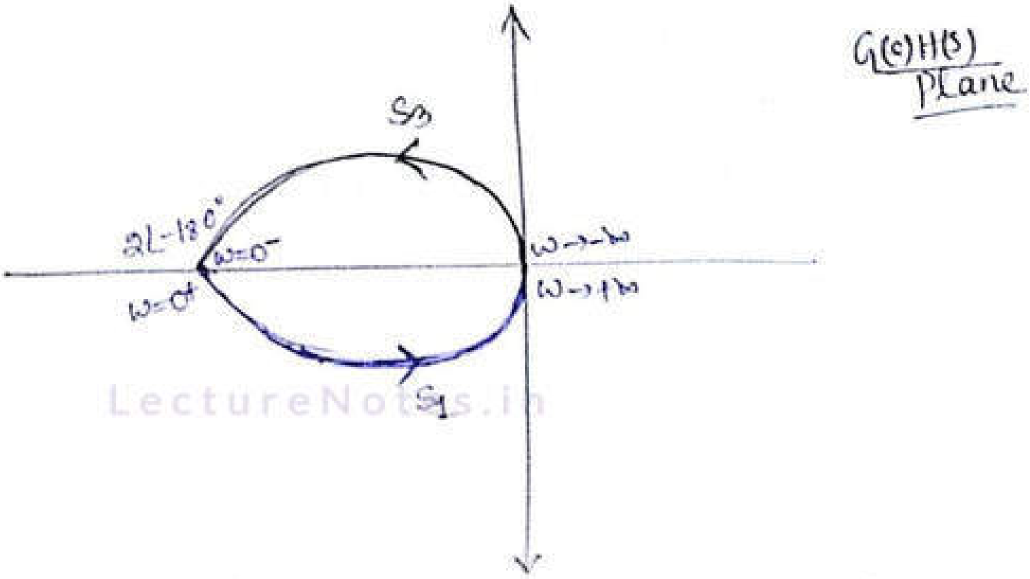


Mapping of s_3:-

Mirror image of 's_1'.

Mapping of 's_2':-

$|s|$ is very large.

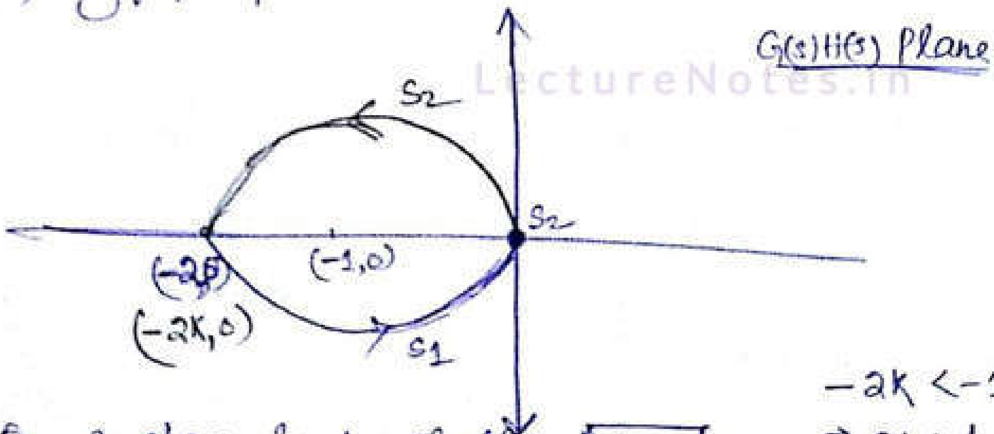
$$\therefore G(s_2)H(s_2) = \frac{2\left(1+\frac{s_2}{2}\right)}{(s_2+1)(s_2-1)} = \frac{2\left(\frac{s_2}{2}\right)}{(s_2)(s_2)} = \frac{s_2}{s_2^2} = \frac{1}{s_2}.$$

Put $s_2 = Re^{j\theta}$

$$\therefore G(s_2)H(s_2) = \lim_{\substack{R\to\infty \\ \theta \to \pi/2 \text{ to } -\pi/2}} \frac{1}{(Re^{j\theta})} = 0e^{j(\pi/2 \text{ to } -\pi/2)}.$$

origin in G(s)H(s) plane.

So, Nyquist plot is :→



for system to be stable, $\boxed{Z=0}$

$\therefore P = 1$ (from open loop T/F)

$\therefore N = P - Z \Rightarrow N = +1 \Rightarrow$ critical point must be encircled once in ACW.

$-2K < -1$

$\Rightarrow 2K > 1$

$\Rightarrow \boxed{K > 1/2}$ As stable system.

Q// sketch Nyquist plot & comment upon stability,

$$G(s)H(s) = \frac{1}{s(s+1)}.$$

Q// sketch the Nyquist plot for a system with,

$$G(s)H(s) = \frac{K}{(s+2)(s^2+4)}$$

Comment on the stability of the system.

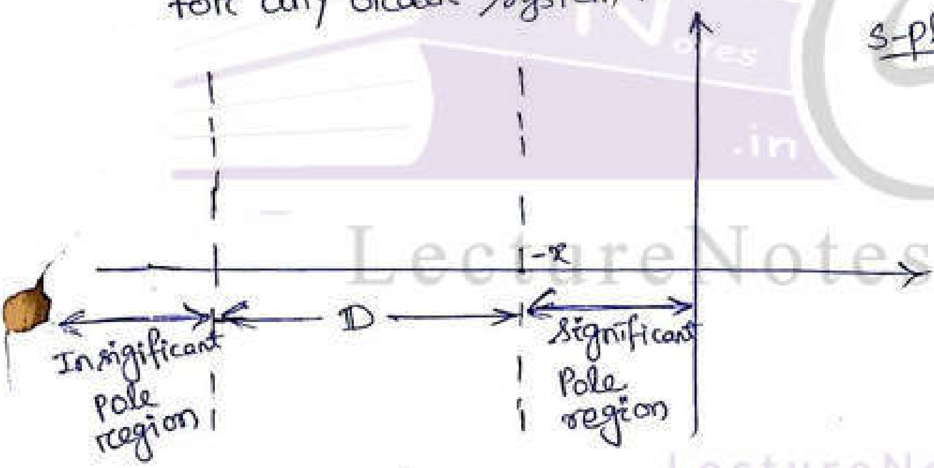
※ CONCEPT :-

(1) Dominant Pole :→

The poles near to jω-axis are more dominant as compared to poles away from jω-axis in case of overdamped response.

(2) Insignificant pole :→

for any order system :-

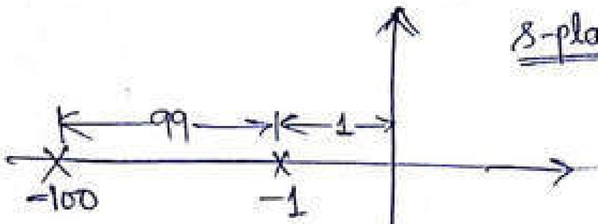


Condition is :-

$D > 5(-x)$

$\Rightarrow \boxed{D > 5|x|}$.

Ex :→ $T(s) = \frac{10}{(s+1)(s+100)}$



$5|x| = 5$

$\therefore D = 99$.

$D > 5|x| \Rightarrow 99 > 5$.

$\therefore$ so, $s = -100$ → Insignificant pole & can be discarded as compared to $s = -1$.

※ $T(s) = \underbrace{\dfrac{10}{(s+1)\,\cancel{(s+100)}}}_{\text{Pole-zero form}} = \underbrace{\dfrac{10/100}{(1+s)\,\cancel{(1+s/100)}}}_{\text{time constant form}}$

→ Always use time constant form to neglect insignificant poles.

Q// $T(s) = \dfrac{100}{(s+1)(s+100)}$. For unit step input, determine settling time for 2% criteria.

Control System Engineering

Topic:

Bode Plot

Contributed By:

Gyana Ranjan Biswal

BODE PLOT

→ It is also another graphical method for determining the stability of a control system based on sinusoidal frequency response.

→ By varying $\omega = 0$ to ∞, the magnitude and phase is plotted with respect to frequency.

→ Bode plot consists of two plots. Those are :-

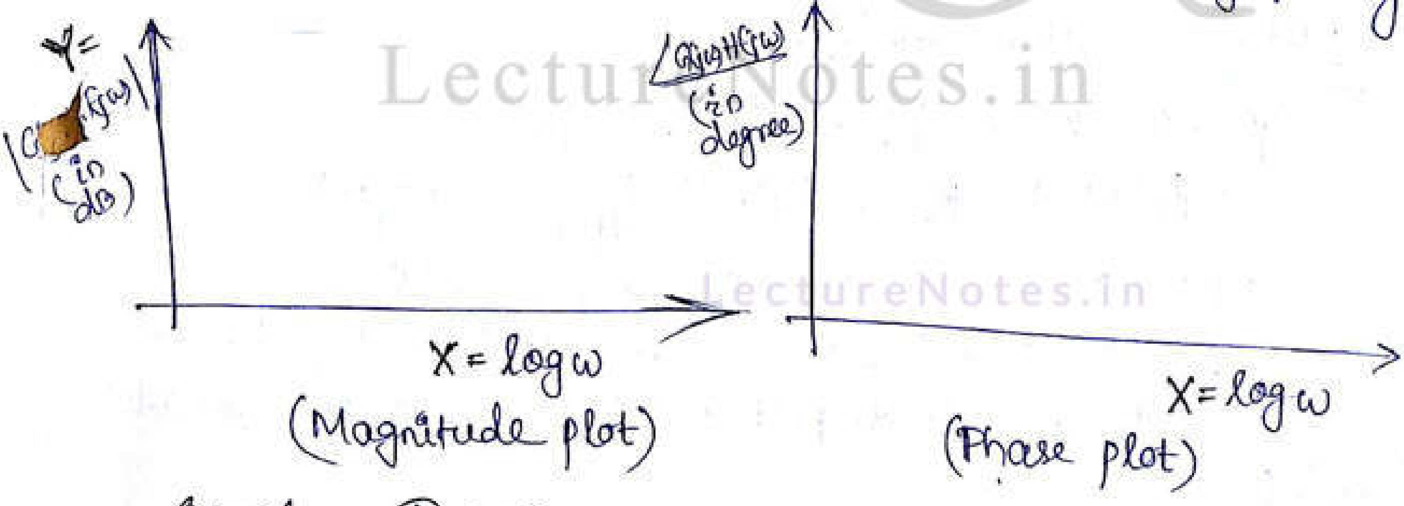
(1) Magnitude expressed in logarithmic values (i.e, in dB) with respect to logarithmic values of frequency called magnitude plot.

(2) phase angle in degrees wrt log of frequency called phase plot.

→ Why Bodeplot :-

(1) It shows wide range in variations in magnitude for wide range of frequencies on a single paper (ie from $\omega=0$ to $\omega=10^4$).

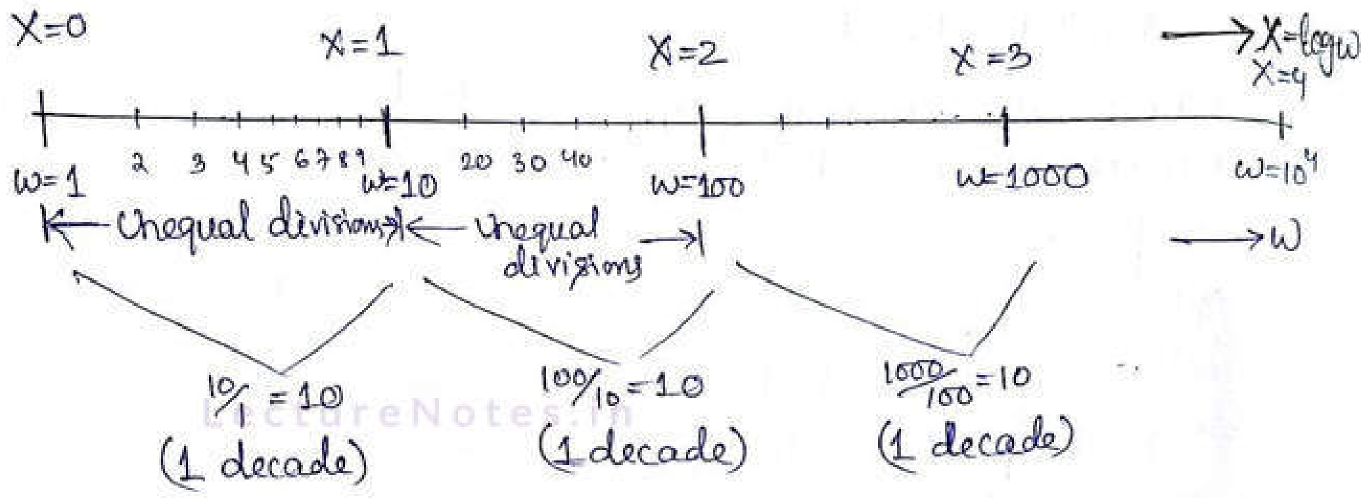
(2) Phase Margin & Gain Margin can be found graphically.



Semilog Paper :-

→ This is nothing but a X-Y graph paper, where X-axis is divided into a logarithmic scale which having unequal divisions within one decade, while Y-axis is divided into equal divisions as per linear scale. That's why it is called as Semilog paper.

2



※ <u>Straight line eqn</u> :→ $Y = mX + C$

→ To draw a straight line m (slope) & c (one point) is required.

Y (in dB), m (dB/dec), X (decade), c (dB).

<u>Magnitude Plot</u> :-

General form of an open loop transfer func :-

$$G(s)H(s) = \frac{K(1+sT_1)^n}{s^n(1+sT_2)^n\ (s^2+2\xi\omega_n s+\omega_n^2)^n}$$

<u>Analysis for different factors</u> :-

<u>factor 1</u> : <u>System Gain 'K'</u> :-

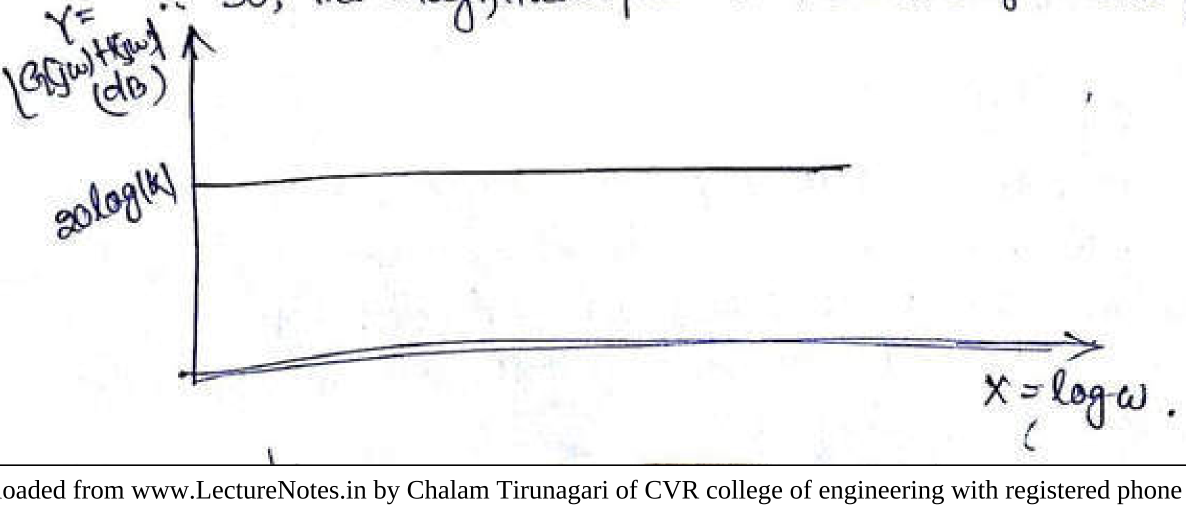
<u>Magnitude of plot</u> $G(s)H(s) = K \Rightarrow G(j\omega)H(j\omega) = K = K + j0$

$\therefore |G(j\omega)H(j\omega)| = K \Rightarrow |G(j\omega)H(j\omega)|_{(dB)} = 20\log|K|$

Comparing with $Y = MX + C \Rightarrow \boxed{Y = 0 \times X + C}$

$Y = |G(j\omega)H(j\omega)|$; $M = 0$ dB/dec, & $C = 20\log|K|$.

$\therefore$ So, the magnitude plot will be straight line parallel to 'X'.



So, for this factor, slope is 0dB/dec, No asymptotic approximation & no corner frequency.

Phase plot:-

$G(j\omega)H(j\omega) = K + j0 \Rightarrow \phi = \angle G(j\omega)H(j\omega) = 0^\circ$.

Factor 2:- Simple

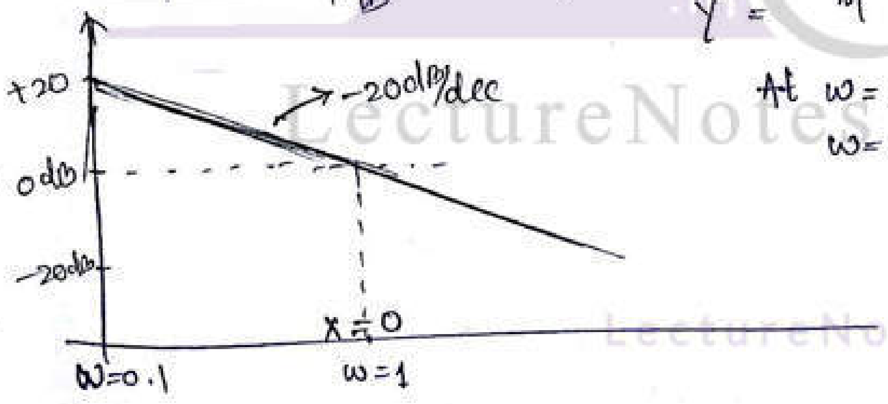
(i) 1/s (Pole at origin) :-

$G(s)H(s) = \frac{1}{s} \Rightarrow G(j\omega)H(j\omega) = \frac{1}{j\omega} = \frac{1}{0+j\omega}$

$\therefore |G(j\omega)H(j\omega)| = \frac{1}{\omega} \Rightarrow |G(j\omega)H(j\omega)|_{\text{in dB}} = 20\log\frac{1}{\omega}$

$\Rightarrow |G(j\omega)H(j\omega)|_{dB} = 20\log(\omega)^{-1}\ dB = -20\log\omega$

$y = \quad m \quad x \quad + \quad c$



At $\omega = 0.1$, $y = +20dB$

$\omega = 1$, $y = 0dB$.

(ii) $\frac{1}{s^2}$ (two poles at origin) :-

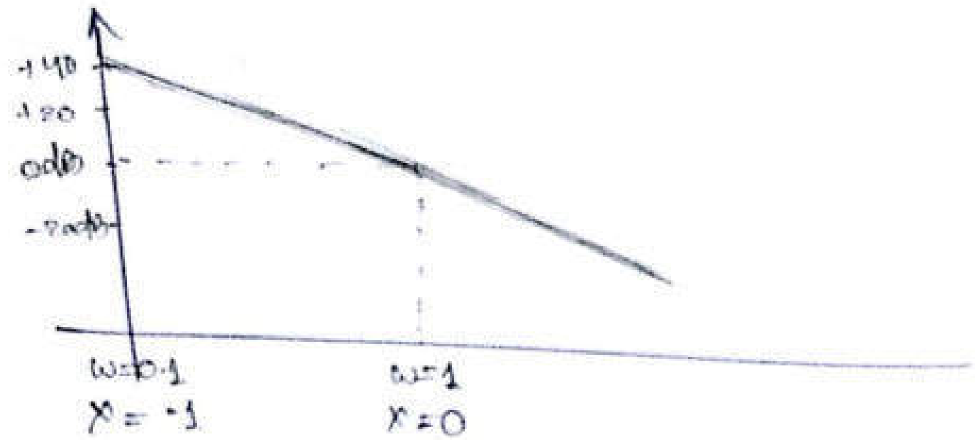
$G(s)H(s) = \frac{1}{s^2} \Rightarrow G(j\omega)H(j\omega) = \frac{1}{(j\omega)^2} \Rightarrow |G(j\omega)H(j\omega)| = \frac{1}{|-\omega^2|} = \frac{1}{\omega^2}$

$\therefore |G(j\omega)H(j\omega)|_{dB} = 20\log\left(\frac{1}{\omega^2}\right) = -40\log\omega.$ → Passes through origin.

At $\omega = 0.1$, $y = +40dB$.

$\omega = 1$, $y = 0dB$.

(4)



→ So, for $\frac{1}{s^n}$, slope is $-20n$ dB/dec & has no corner frequency and no assymptotic approximation.

(iii) S (Simple zero at origin) :-

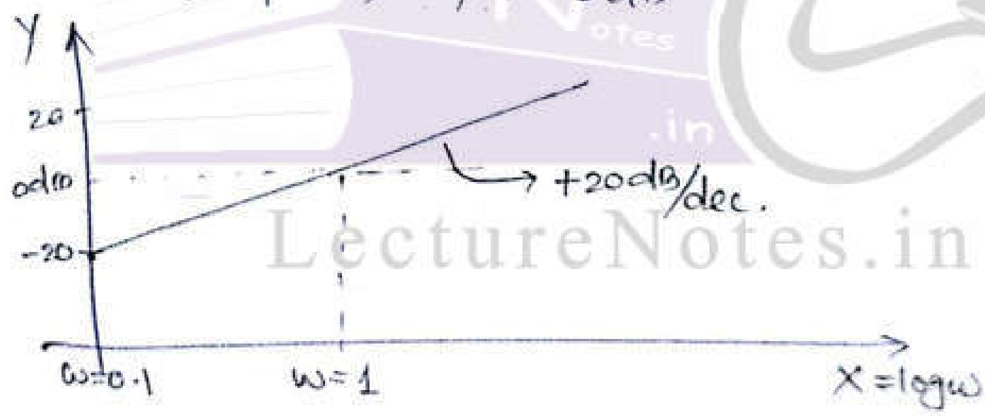
$G(s)H(s) = s$

$G(j\omega)H(j\omega) = j\omega \Rightarrow G(j\omega)H(j\omega) = \omega \Rightarrow |G(j\omega)H(j\omega)| = \omega$

$\therefore |G(j\omega)H(j\omega)|_{dB} = 20\log\omega$

At $\omega = 0.1 \Rightarrow |G(j\omega)H(j\omega)|_{dB} = -20dB$

$\omega = 1 \Rightarrow Y = 0dB$



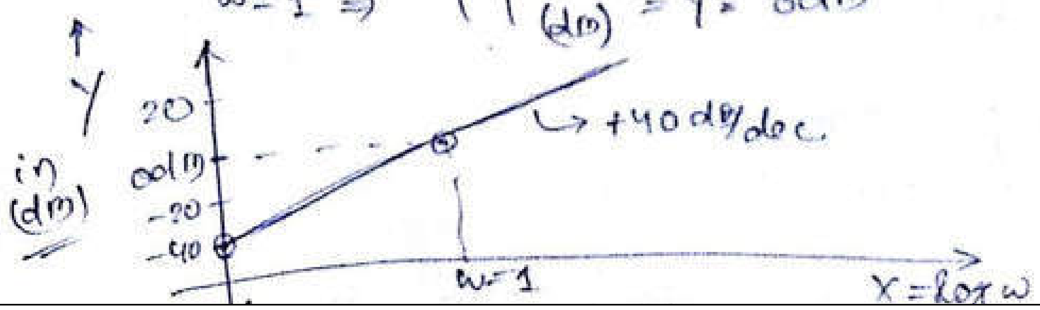
(iv) s^2 (multipole zeros at origin) :-

$G(s)H(s) = s^2 \Rightarrow G(j\omega)H(j\omega) = (j\omega)^2 = -\omega^2$.

$\therefore |G(j\omega)H(j\omega)|_{(dB)} = 20\log\omega^2 = 40\log\omega$.

At $\omega = 0.1 \Rightarrow |\ |_{(dB)} = -40dB$

$\omega = 1 \Rightarrow |\ |_{(dB)} = Y = 0dB$



Factor-3 : Simple pole or zero

(i) Simple pole : $\left(\frac{1}{1+sT_1}\right)$.

$$G(s)H(s) = \frac{1}{1+sT_1} \Rightarrow G(j\omega)H(j\omega) = \frac{1}{1+j\omega T_1}.$$

$$\therefore |G(j\omega)H(j\omega)| = \frac{1}{\sqrt{1+(\omega T_1)^2}}$$

$$\Rightarrow |G(j\omega)H(j\omega)|_{dB} = 20 \log \frac{1}{\sqrt{1+(\omega T_1)^2}}$$

$$= 20\left[\log(1) - \log\left(\sqrt{1+(\omega T_1)^2}\right)\right]$$

$$= -20 \log \sqrt{1+(\omega T_1)^2}$$

$$\Rightarrow |G(j\omega)H(j\omega)|_{dB} = -10 \log\left[1+(\omega T_1)^2\right].$$

For Low Frequency range ($\omega T_1 << 1$) :-

$$|G(j\omega)H(j\omega)|_{(dB)} = 0 \text{ dB}.$$

$$\Rightarrow \boxed{Y \text{ (in dB)} = 0 \text{dB}}$$

For High frequency range ($\omega T_1 >> 1$) :-

$$|G(j\omega)H(j\omega)|_{(dB)} = -20 \log \omega T_1$$

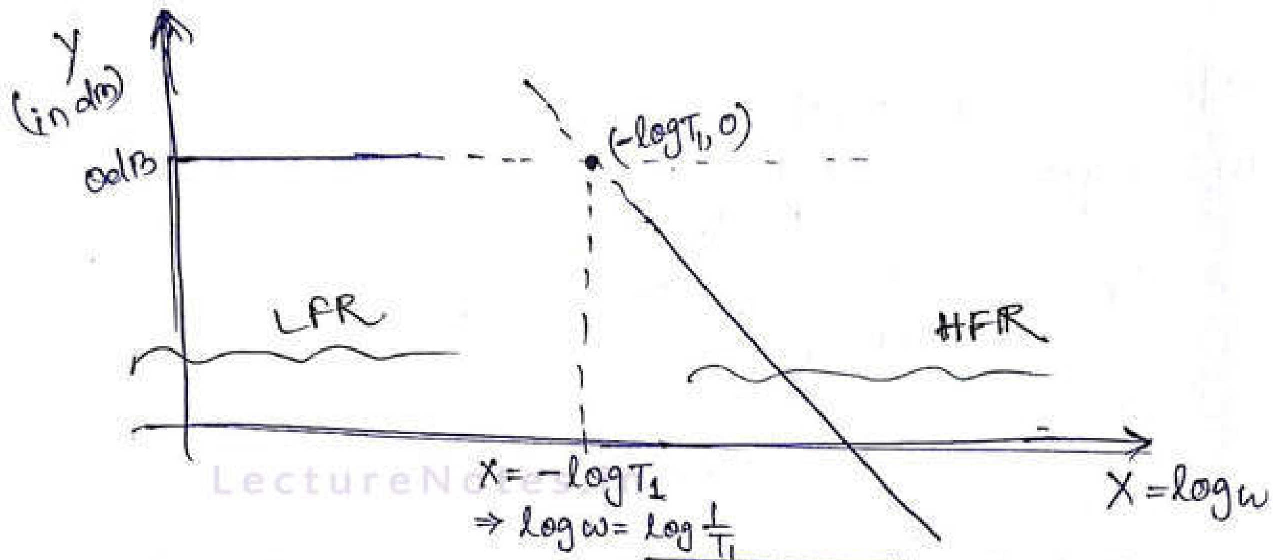
$$\Rightarrow |G(j\omega)H(j\omega)|_{(dB)} = \underset{\Downarrow}{-20}\ \underset{\Downarrow}{\log\omega}\ \underset{\Downarrow}{-20\log T_1}$$

$$\Rightarrow Y \text{ (in dB)} = M \quad X \quad + \quad C$$

So, two straight lines :→

① 0 dB line ② $Y = -20X + C = -20X - 20\log T_1$.

(6)



$x = -\log T_1$

$\Rightarrow \log \omega = \log \frac{1}{T_1}$

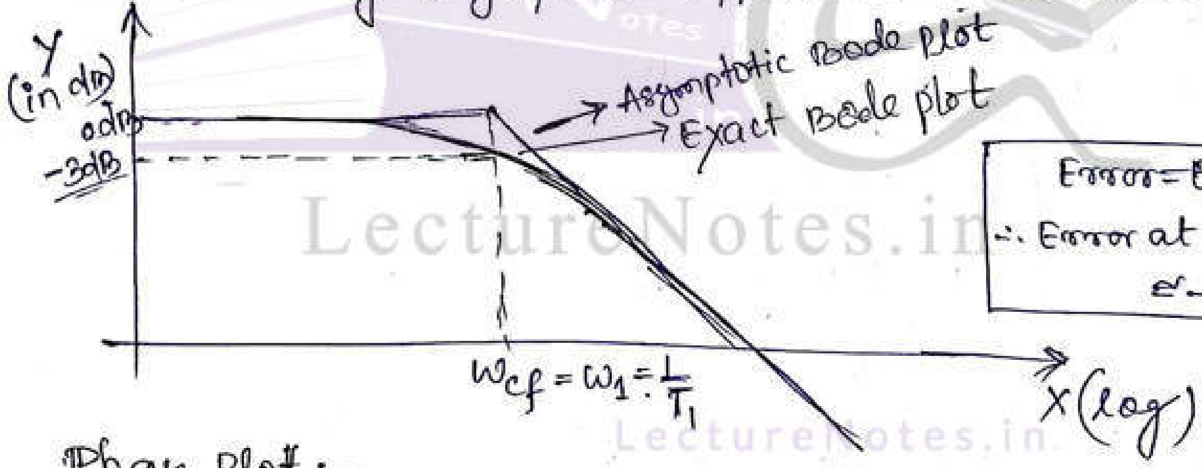
<u>Intersection Point</u> :- $\Rightarrow \boxed{\omega = 1/T_1 = \omega_{cf}}$ → corner frequency.

$\boxed{Y = 0}$; $Y = -20X - 20\log T_1$

$\Rightarrow 0 = -20X - 20\log T_1$

$\Rightarrow \boxed{X = -\log T_1}$

→ The above plot is called as Asymptotic bode plot, because we are using asymptotic approximation at corner frequency.



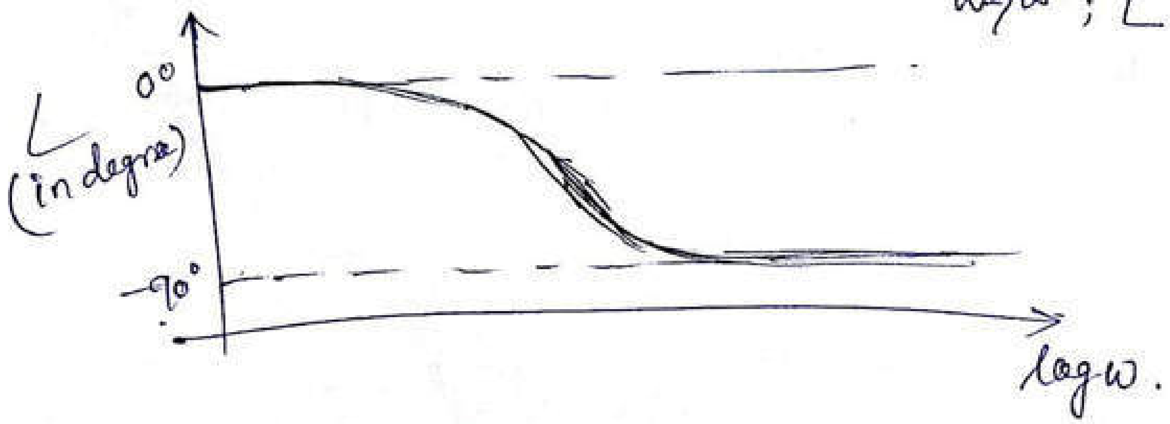
Error = Exact - Approx

∴ Error at ω_{cf} $\simeq -3dB$

<u>Phase Plot</u> :-

$\angle G(j\omega)H(j\omega) = -\tan^{-1}\omega T_1$; At $\omega = 0$; $\angle = 0°$

$\omega \to \infty$; $\angle = -90°$.



(ii) Multiple pole $\left[\frac{1}{(1+sT_1)^2}\right]$.

(iii) Simple zero $(1+sT_1)$:-

(iv) Multiple zero $(1+sT_1)^2$:-

$$G(s)H(s) = (1+sT_1)^2 \Rightarrow G(j\omega)H(j\omega) = (1+j\omega T_1)^2 = (1+j\omega T_1)(1+j\omega T_1)$$

$$\therefore |G(j\omega)H(j\omega)|_{(dB)} = 1+(\omega T_1)^2 = \sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_1)^2}$$

$$\Rightarrow |G(j\omega)H(j\omega)|_{(dB)} = 20\log\{1+(\omega T_1)^2\}$$

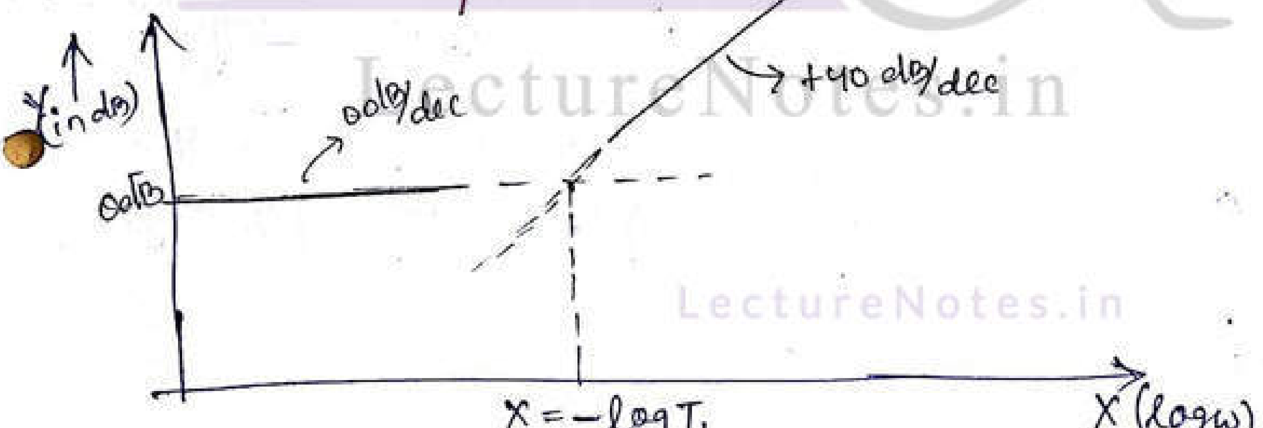
LFR $(\omega T_1 << 1)$

$$|\ \ |_{(dB)} = 20\log 1 = 0\,dB.$$

HFR $(\omega T_1 >> 1)$

$$|\ \ |_{(dB)} = 20\log(\omega T_1)^2 = 40\log\omega T_1$$

$$\Rightarrow \underbrace{|G(j\omega)H(j\omega)|(\text{in dB})}_{Y} = \underbrace{40}_{M}\,\underbrace{\log\omega}_{X} + \underbrace{40\log T_1}_{C}.$$



$x = -\log T_1$

$\Rightarrow \log\omega = \log\frac{1}{T_1}$

$\Rightarrow \omega = 1/T_1 \rightarrow$ corner frequency. (ω_{cf}).

(8)

Factor-4 :- Quadratic factors

$$G(s)H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$$\Rightarrow G(s)H(s) = \frac{1}{1 + \frac{2\xi}{\omega_n}s + \frac{s^2}{\omega_n^2}} \longrightarrow \text{Time Const. form.}$$

$$\therefore G(j\omega)H(j\omega) = \frac{1}{1 + \frac{2\xi}{\omega_n}(j\omega) + \frac{1}{\omega_n^2}(j\omega)^2}.$$

$$|G(j\omega)H(j\omega)|_{(dB)} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow |G(j\omega)H(j\omega)|_{(in\ dB)} = -20\log\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}\ \text{dB}.$$

For LFR $\left(\frac{\omega}{\omega_n} << 1\right)$

$$\boxed{|\ \ |_{(dB)} = 0\ \text{dB}.}$$

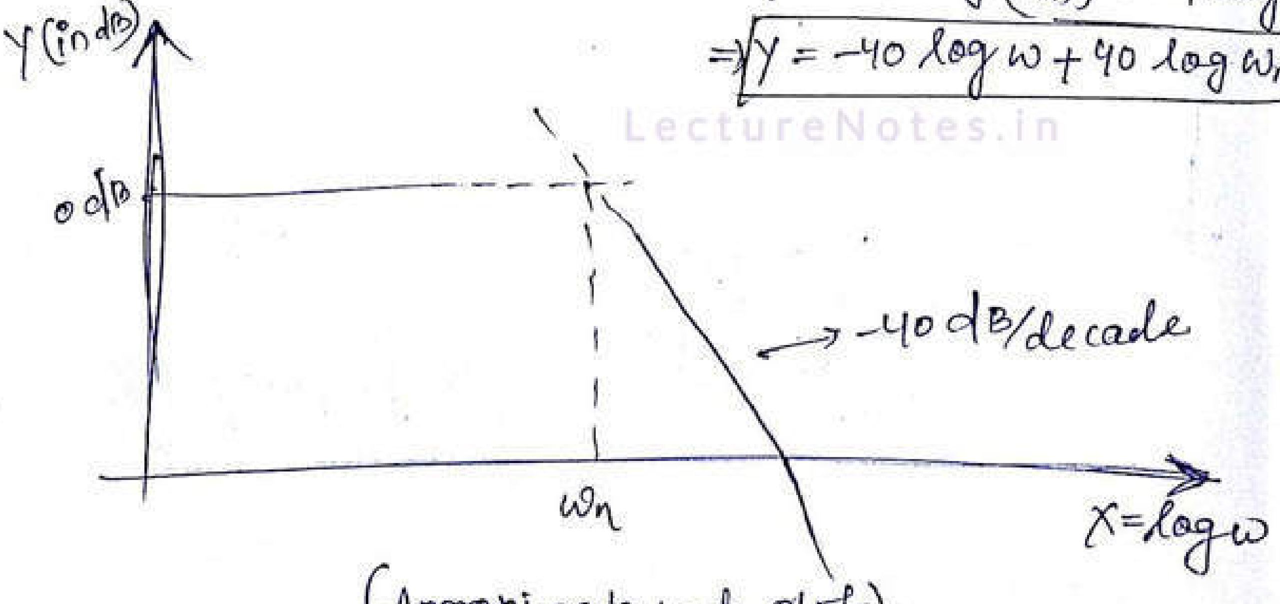
For HFR $\left(\frac{\omega}{\omega_n} >> 1\right)$

$$4\xi^2\left(\frac{\omega}{\omega_n}\right)^2 << \left(\frac{\omega}{\omega_n}\right)^4$$

$$\therefore |\ \ |_{(dB)} = -20\log\sqrt{\left[\left(\frac{\omega}{\omega_n}\right)^2\right]^2}$$

$$\Rightarrow Y = -20\log\left(\frac{\omega}{\omega_n}\right)^2 = -40\log\frac{\omega}{\omega_n}.$$

$$\Rightarrow \boxed{Y = -40\log\omega + 40\log\omega_n.}$$



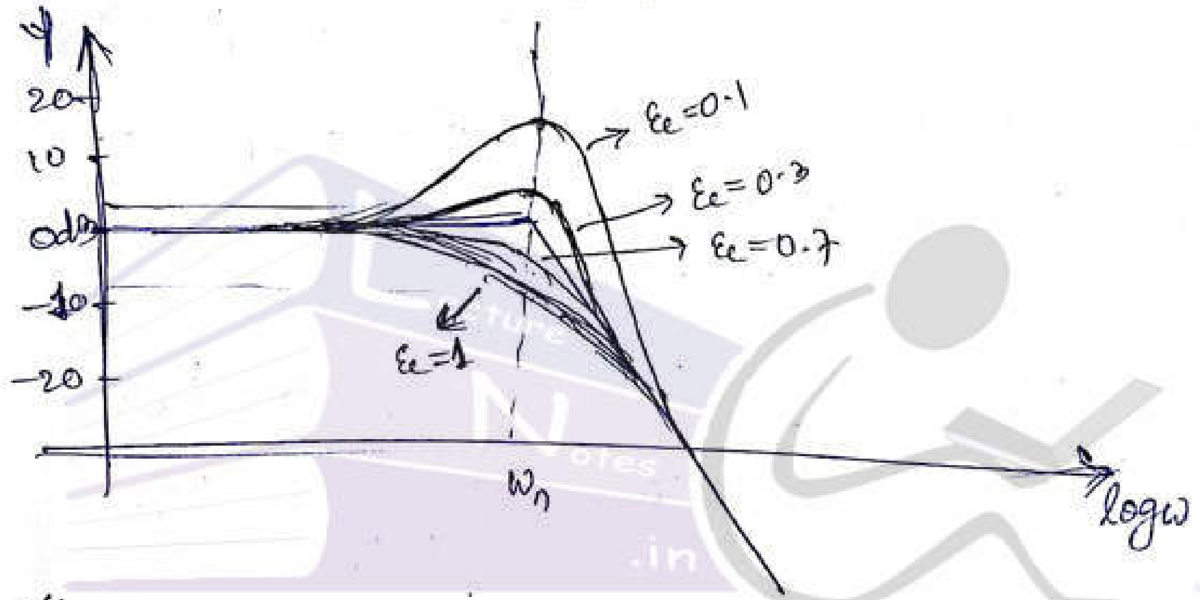
(Approximate bode plot)

Exact Bode Magnitude Plot

At $\frac{\omega}{\omega_n} = 1 \Rightarrow \omega = \omega_n$.

$\therefore$ Actual magnitude in dB $= -20 \log\sqrt{4\xi^2} = -20\log 2\xi$.

ξ	Actual magnitude
0.1	+13.97
0.3	+4.43
0.7	-2.92
1	-6.02



OBSERVATIONS :-

(1) one pole $\rightarrow$ -20 dB/dec

one zero $\rightarrow$ $+20$ dB/dec

(2)

Factors (n→integers)	ω_{cf} (rad/sec)	slope (dB/dec)
K	NA	0 dB/dec
$s^{\pm n}$	NA	$(20n)$ dB/dec
$(1+sT_1)^n$	$1/T_1$	$(20n)$ dB/dec
$\left(1+\frac{2\xi}{\omega_n}s+\frac{s^2}{\omega_n^2}\right)^n$	ω_n	$(40n)$ dB/dec

(3) At corner frequencies, there is change in slope.
→ Before corner frequency, magnitude in decibels (dB) for that factor is zero.

Example →

- $\dfrac{1}{(1+sT_1)^3}$ ⟶ Multiple first order pole
  ↳ Three factors (−60 dB/dec)

- $\dfrac{1}{(s^2+2\xi\omega_n s+\omega_n^2)^2}$ → Multiple quadratic pole
  ↳ Two factors (−40 dB/dec)

Q1. Draw Bode plot for OLTF $G(s)H(s) = \dfrac{100}{s(s+10)}$.

Soln: → Time constant form: →

$$G(s)H(s) = \frac{10}{s\left(1+\frac{s}{10}\right)}$$

factors	ω_{cf} (rad/sec)	slope (dB/dec)
$\dfrac{10}{s}$	NA	−20 dB/dec
$\dfrac{1}{1+\frac{s}{10}}$	10 rad/sec	−20 dB/dec

−20 dB/dec

−40 dB/dec

- Starting of magnitude bode plot :-

Let eqn of st. line $Y = MX + C$

$M = -20$ dB/dec.

C can be found by using, $\boxed{C = 20\log K}$

$\therefore C = 20\log 10 = 20$ dB.

$\therefore Y = -20X + 20$

At $\omega = 0.1 \Rightarrow X = \log 0.1 = -1$; $Y = -20(-1)+20 = 40\text{dB} \rightarrow (-1, 40)$

$\omega = 1 \Rightarrow X = \log 1 = 0$; $Y = -20(0)+20 = 20\text{dB} \rightarrow (0, 20)$.

Initial st. line is valid upto $\omega_{cf} = 10$ rad/sec.

for $\omega > 10$ rad/sec :-

Let eq^n of st. line, $Y = \underbrace{M}_{\to -40\ dB/dec}X + C$.

'c' can be found by taking a point present on st. line.

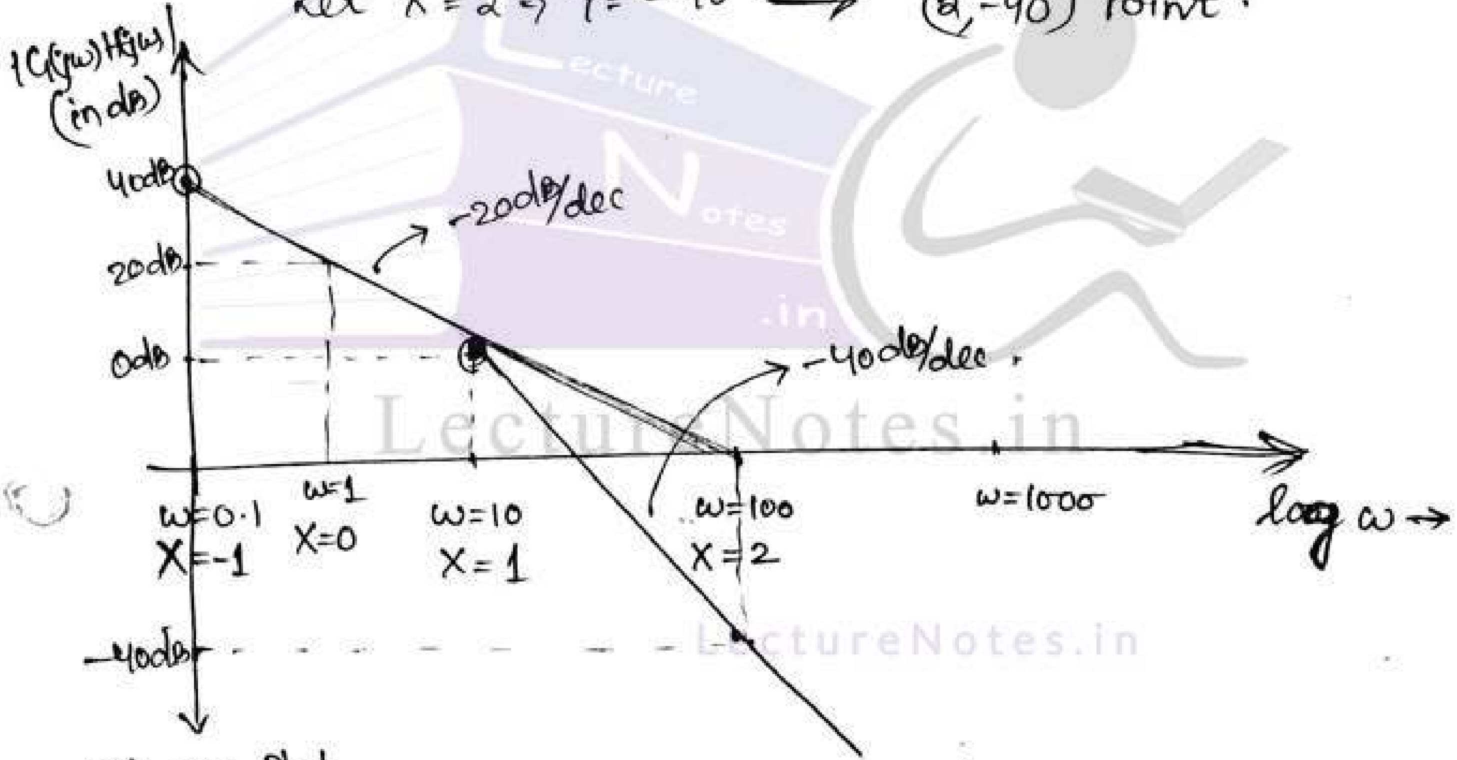
$\therefore C = Y - MX = 0 - (-40) = +40$ dB.

So, $Y = -40X + 40$.

Two points are required: one is point 'c'

for another point :-

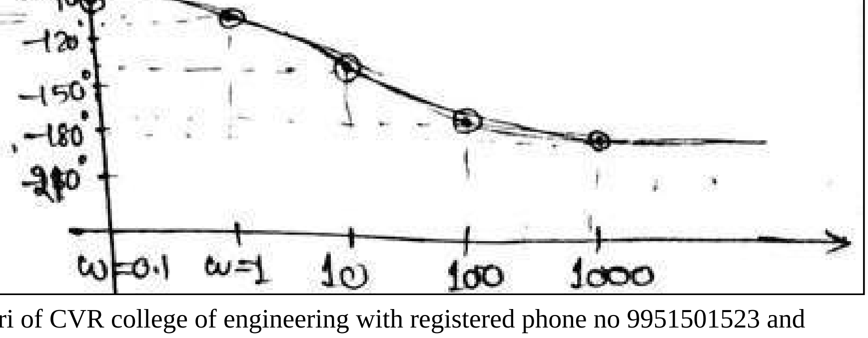
Let $X = 2 \Rightarrow Y = -40 \rightarrow (2, -40)$ Point.



Phase Plot :→

$G(j\omega)H(j\omega) = \dfrac{10}{(j\omega)(1 + j\frac{\omega}{10})}$. $\angle G(j\omega)H(j\omega) = -90° - \tan^{-1}\left(\frac{\omega}{10}\right)$.

$\omega =$	$\angle G(j\omega)H(j\omega)$
0.1	−90.57°
1	−95.71°
10	−135°
100	−174.29°
1000	−179.43°



(2) Q/ Draw Bode plot. $G(s)H(s) = \frac{3(s+1)(s+700)}{s^2(s^2+18s+400)}$

Sol^n → Time constant form:-

$$G(s)H(s) = \frac{3\times700(1+s)(1+\frac{s}{700})}{400s^2(1+\frac{18}{400}s+\frac{s^2}{400})} = \frac{5.25(1+s)(1+\frac{s}{700})}{s^2(1+\frac{18}{400}s+\frac{s^2}{400})}.$$

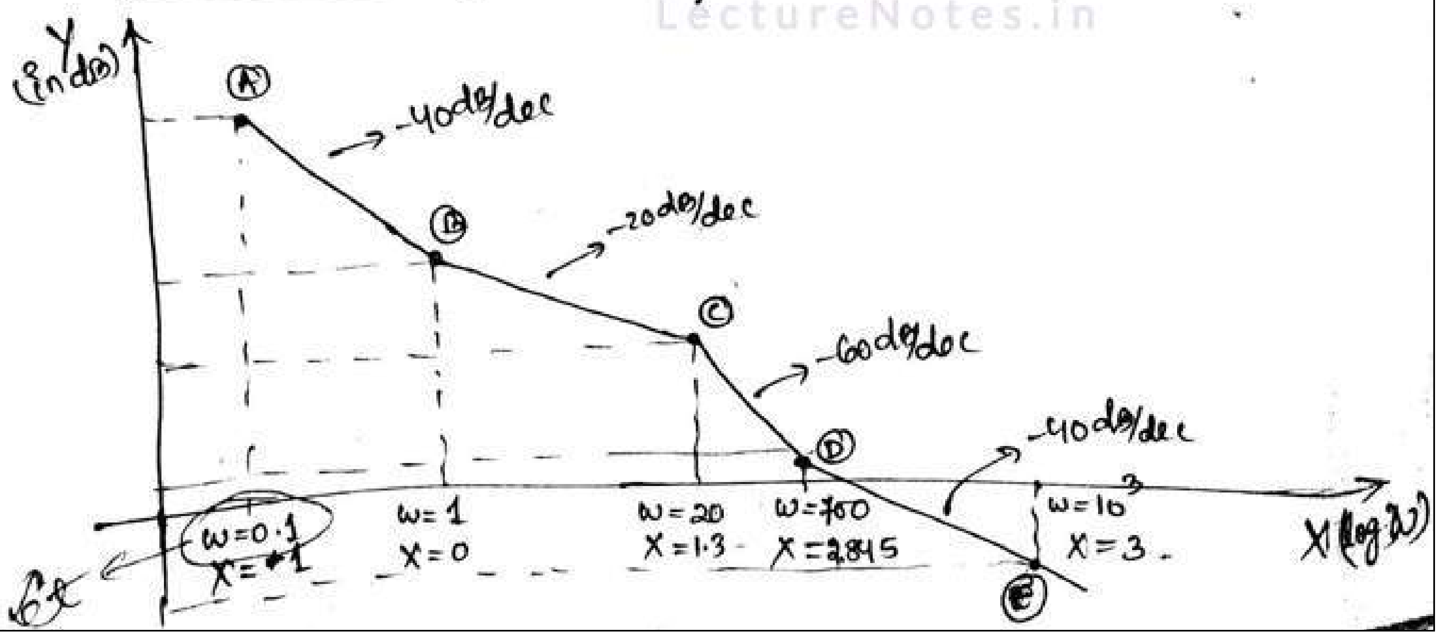
The corner frequencies are:→

1 rad/sec, 20 rad/sec, 700 rad/sec.

The corresponding table is:-

factors	ω_{cf} (rad/sec)	slope (dB/dec)	
1) $\frac{5.25}{s^2}$	NA	-40dB/dec	-40
2) $(1+s)$	1	-20dB/dec	+20
3) $\frac{1}{(1+\frac{18}{400}s+\frac{s^2}{400})}$	20	-60dB/dec	-40
4) $(1+\frac{s}{700})$	700	-40dB/dec	+20

Rough Bode magnitude plot:-



To determine the coordinates :-

① At $\omega < 1$ rad/sec

Let initial st. line eqⁿ → $Y = \underbrace{M}_{-40\,dB/dec}X\,\boxed{C} \rightarrow$ from $c = 20\log K$

$\therefore C = 20\log K = 20\log 5.25 = 14.4\,dB$.

Now, the eqⁿ st. line → $\boxed{Y = -40X + 14.4}$

To draw the line, minᵐ two points are required.

At $\omega = 0.1 \Rightarrow X = -1$; $Y = -40(-1) + 14.4 = 54.4\,dB$.

$\omega = 1 \Rightarrow X = 0$; $Y = -40(0) + 14.4 = 14.4\,dB$.

$\therefore$ So the two points are, $\omega = 0.1$, $Y = 54.4\,dB \rightarrow (-1, 54.4)$ Ⓐ.

& $\omega = 1$, $Y = 14.4\,dB \rightarrow (0, 14.4)$ Ⓑ

② $\omega = 1$ rad/sec to 20 rad/sec :→

$Y = \underbrace{M}_{-20\,dB/dec}X + \boxed{C} \rightarrow$ can be found from the point $(0, 14.4)$ Ⓑ.

$\therefore C = Y - MX = 14.4 - (-20)\times(0) = 14.4\,dB$.

Now, the eqⁿ of st. line is → $\boxed{Y = -20X + 14.4}$

One point is already known i.e, Ⓑ point, to find other point i.e, Ⓒ :→

At $\omega = 20 \Rightarrow X = 1.3$; $Y = (-20)(1.3) + 14.4 = -11.6\,dB$.

So, the point Ⓒ is $\omega = 20$, $Y = -11.6\,dB \Rightarrow (1.3, -11.6)$ Ⓒ.

③ $20 < \omega < 700$ rad/sec :→

$Y = \underbrace{M}_{-60\,dB/dec}X + \boxed{C} \rightarrow$ can be found from the point Ⓒ $(1.3, -11.6)$.

$\therefore C = Y - MX = -11.6 - (-60)(1.3) = 66.4\,dB$.

So, the eqⁿ of st. line → $\boxed{Y = -60X + 66.4}$

Now, other point is :→

At $\omega = 700 \Rightarrow X = 2.84$; $Y = (-60)(2.84) + 66.4 = -104\,dB$.

So, the point Ⓓ is $\omega = 700$, $Y = -104\,dB \Rightarrow (2.84, -104)$ Ⓓ.

(14)

(4) $\omega > 700$ rad/sec :-

$$Y = MX + \boxed{C} \longrightarrow \text{can be found from point (D) } (2.84, -104)$$

$M \to -40$ dB/dec

$\therefore C = Y - MX = -104 - (-40)(2.84) = 9.6$.

$\therefore$ The eqn of st. line becomes $\boxed{Y = (-40X) + 9.6}$.

Let $\omega = 10^3 \Rightarrow X = 3$; $Y = (-40)(3) + 9.6 = -110.4$ dB.

So, the point (E) is, $\omega = 10^3, Y = -110.4 \Rightarrow (3, -110.4)$ (E).

$\therefore$ Now put the points on semilog paper & draw the Bode magnitude plot.

Bode Phase plot :-

$$G(s)H(s) = \frac{5.25(1+s)\left(1+\frac{s}{700}\right)}{s^2\left(1+\frac{18}{400}s+\frac{s^2}{400}\right)}$$

At $s = j\omega$, $$G(j\omega)H(j\omega) = \frac{5.25(1+j\omega)\left(1+j\frac{\omega}{700}\right)}{(j\omega)^2\left[\left(1-\frac{\omega^2}{400}\right)+j\frac{18}{400}\omega\right]} \qquad \phi = \tan^{-1}\frac{\left|\frac{18}{400}\omega\right|}{\left|\left(1-\frac{\omega^2}{400}\right)\right|}$$

$$\therefore \angle G(j\omega)H(j\omega) = \begin{cases} \tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{700}\right) - 180^\circ - \phi, & \omega < 20 \left(\because \frac{\omega^2}{400} < 1\right) \\ \tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{700}\right) - 180^\circ - 90^\circ, & \omega = 20 \\ \tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{700}\right) - 180^\circ - (\pi - \phi), & \omega > 20. \end{cases}$$

Where, $\phi = \tan^{-1}\left(\frac{18\omega}{400-\omega^2}\right)$.

Table →

ω (in rad/sec)	∠ G(jω)H(jω)	
ω = 0⁺	−180°	
ω = 0.1	−174.54°	→ ω = 0.5 ; ∠ = −154.68°
ω = 1	−137.5°	→ ω = 10 ; ∠ = −125.85°
ω = 20	−181.22°	
ω = 100	−251.82°	→ ω = 500 ; ∠ = −232.5°
ω = 700	−223.61°	
ω = 1000	−214.02°	→ ω = 5000 ; ∠ = −187.77°
ω → ∞	−180.0°	

Phase Margin & Gain Margin calculation :-

Gain Margin (Ab) :→

It is that value of gain, which when multiplied to a existing system, makes system unstable from stable system.

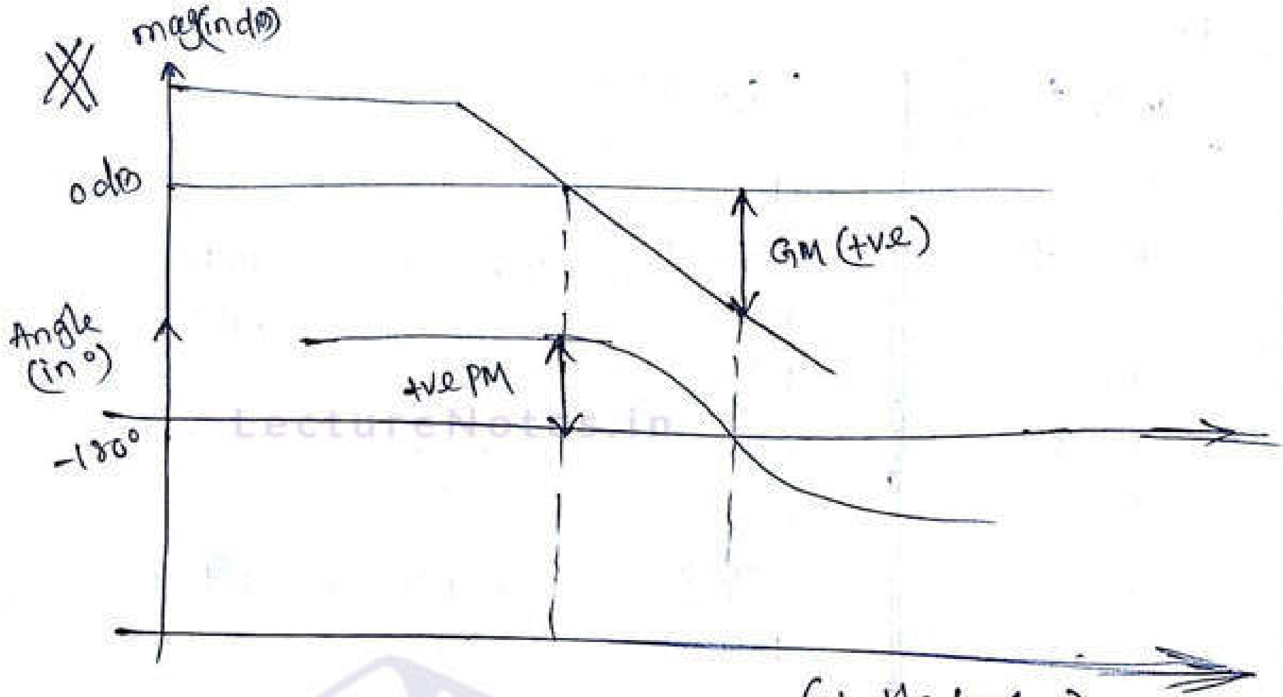
Gain Margin (dB) :→

It is that value of gain in dB, which when added to a existing system, makes system unstable from stable system.

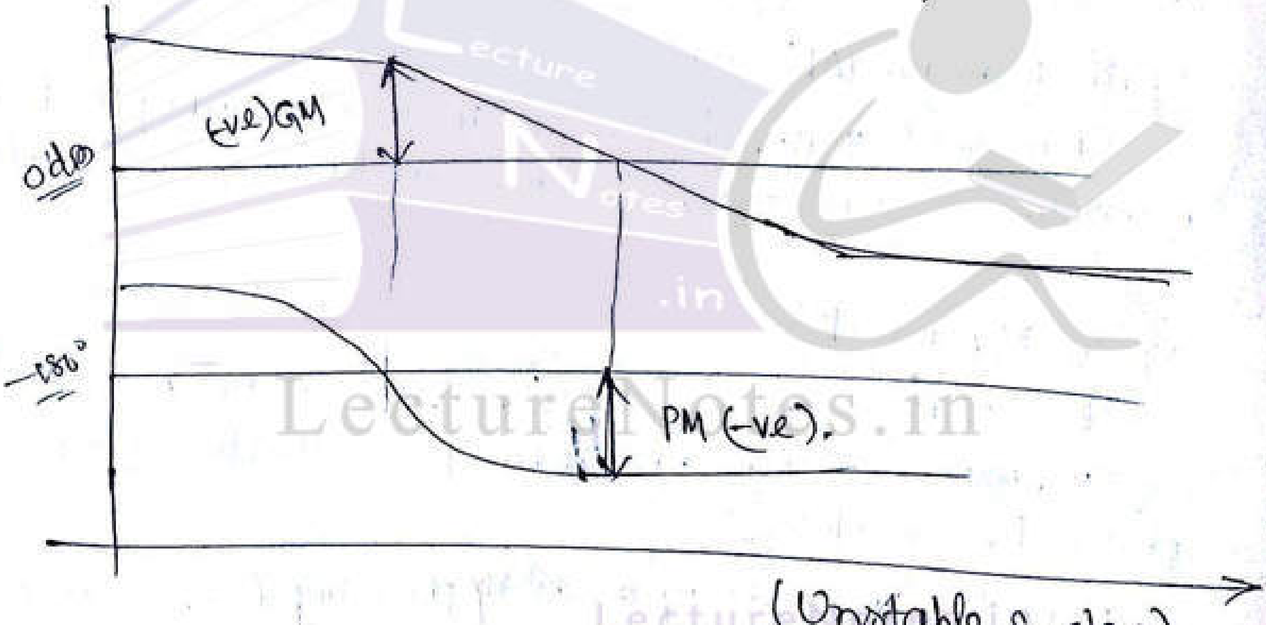
Phase Margin (dB) :-

It is the phase delay or ~~delay~~ phase lag (i.e, negative angle) which when added to a system, makes a system unstable from stable system.

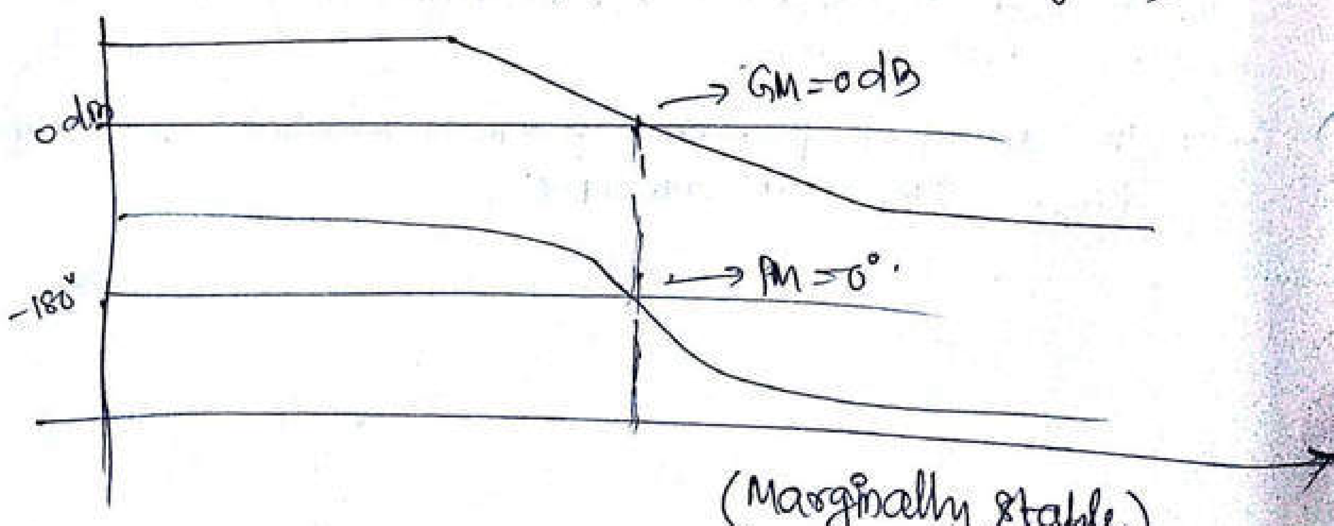
→ The ~~go~~ above problem; system is Absolute system as $\omega_{pc} > \omega_{gc}$, PM & GM are (+ve).



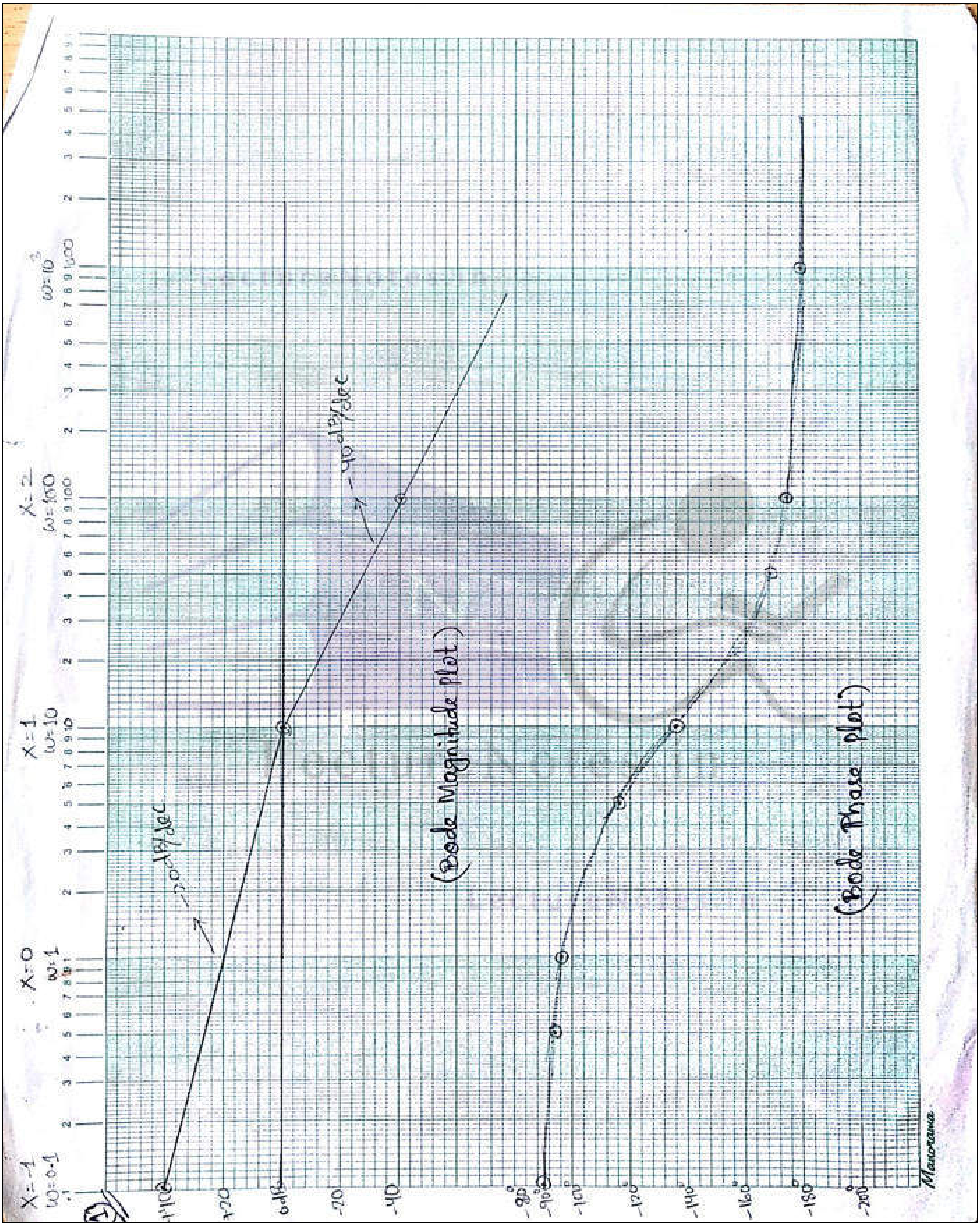
(stable system)

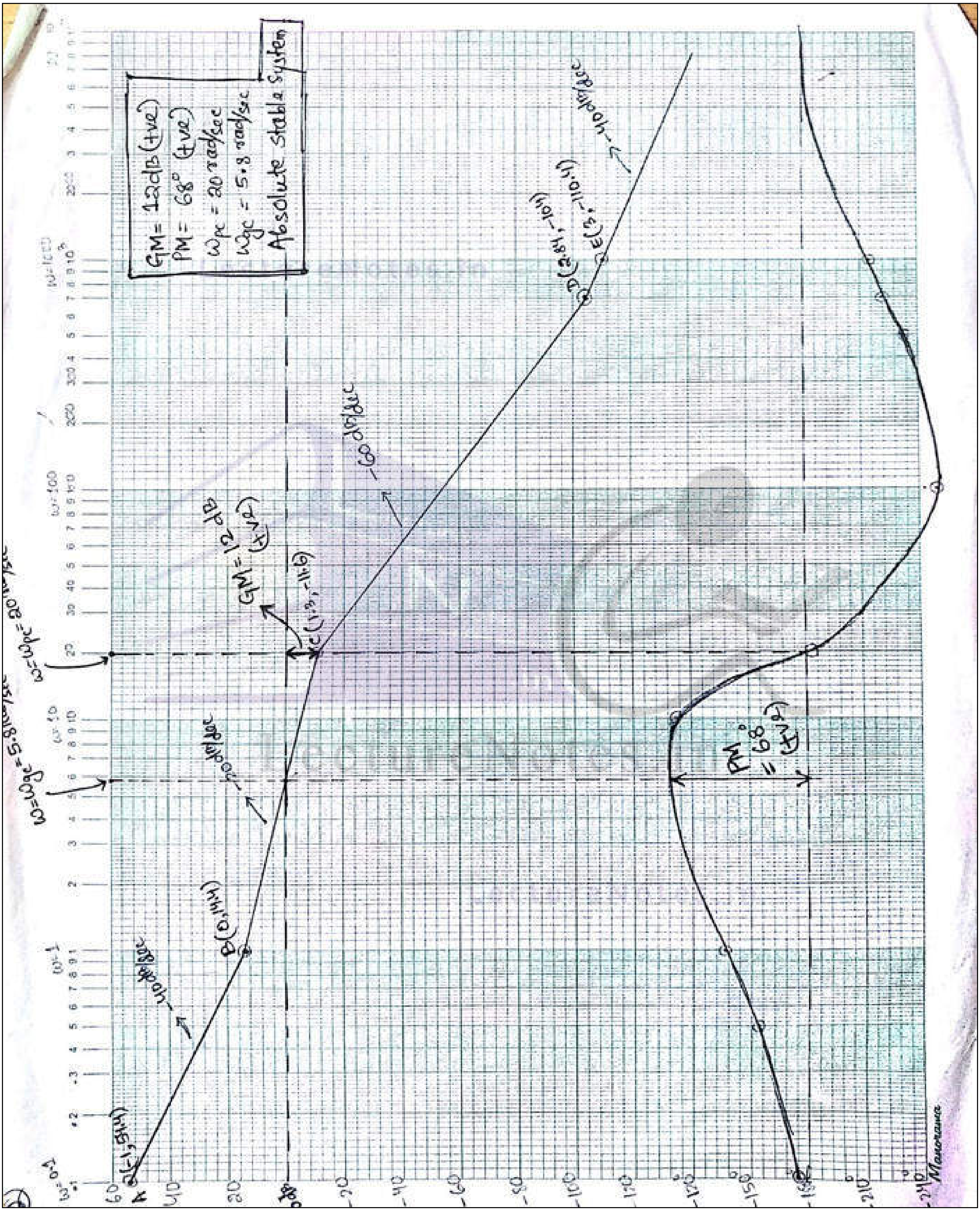


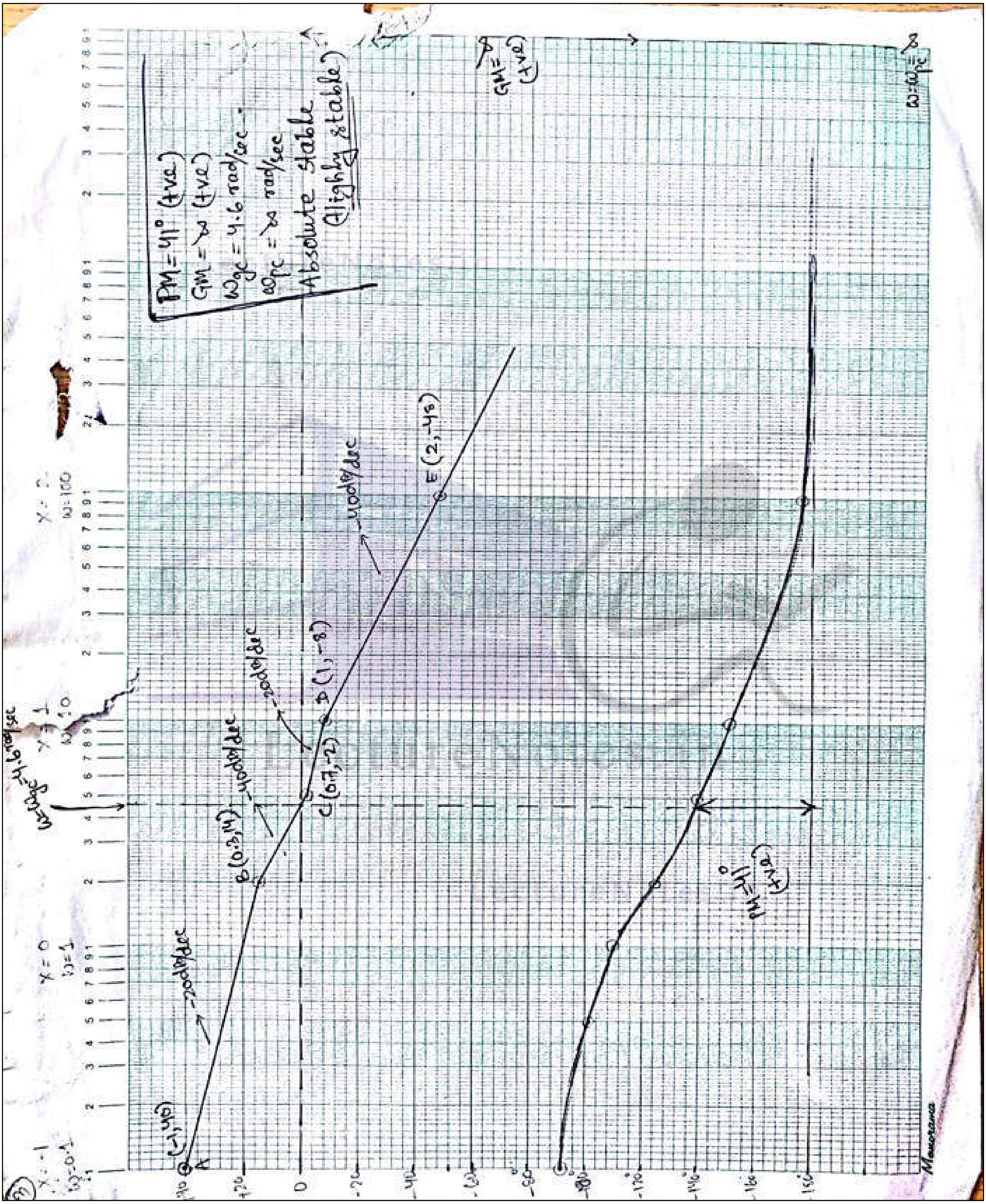
(Unstable system)



(Marginally stable)


(Bode Magnitude plot)
(Bode Phase plot)
-20dB/dec
-40dB/dec
X=-1
ω=0.1
X=0
ω=1
X=1
ω=10
X=2
ω=100
+40
+20
0dB
-20
-40
-90°


GM = 12dB (+ve)
PM = 68° (+ve)
ωpc = 20 rad/sec
ωgc = 5.8 rad/sec
Absolute Stable System
ω=ωgc=5.8 rad/sec
ω=ωpc=20 rad/sec
GM = 12 dB (+ve)
PM = 68° (+ve)
-20db/dec
-40db/dec
-60db/dec
-40db/dec
-20db/dec



PM = 41° (+ve)
GM = ∞ (+ve)
ωgc = 4.6 rad/sec
ωpc = ∞ rad/sec
Absolute stable
(highly stable)
A(-1, 40)
B(0.3, 14)
C(0.7, 2)
D(1, -8)
E(2, -48)
-20db/dec
-40db/dec
-20db/dec
-40db/dec
ωgc = 4.6 rad/sec
PM = 41° (+ve)
GM = ∞ (+ve)
ω = ωpc = ∞

For Absolute stable system :→

$$\angle G(j\omega)H(j\omega)\Big|_{\omega=\omega_{gc}} > -180^\circ$$

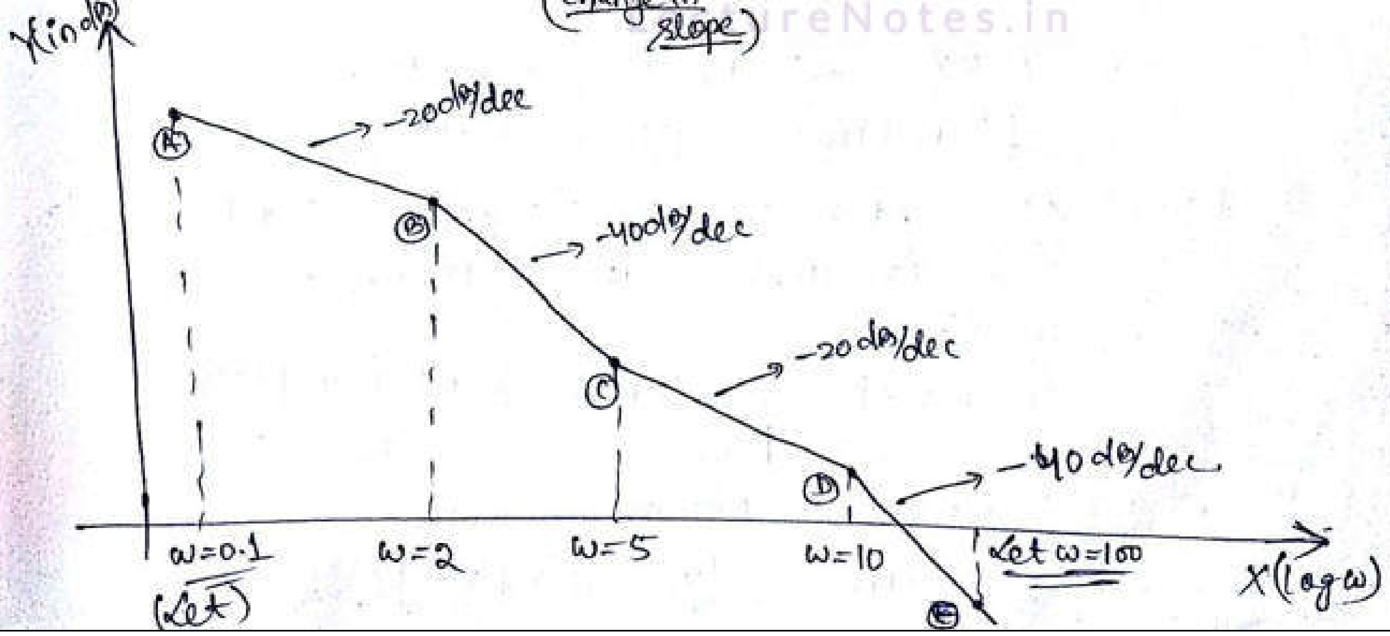
$$|G(j\omega)H(j\omega)| \text{ (in dB)}\Big|_{\omega=\omega_{pc}} < 0\text{ dB}.$$

Q (3) For a unity feedback system with OLTF : $G(s)=\dfrac{40(s+5)}{s(s+10)(s+2)}$. Draw Bode plot. Determine gain margin, phase margin, ω_{gc}, ω_{pc}. Comment on the stability of the system.

Solⁿ :→ Time Constant form :→

$$G(s)H(s)=\frac{40\times5\left(1+\frac{s}{5}\right)}{(10\times2)\,s\left(1+\frac{s}{10}\right)\left(1+\frac{s}{2}\right)}=\frac{10(1+0.2s)}{s(1+0.1s)(1+0.5s)}.$$

factors	ω_{cf} (rad/sec)	slope (dB/dec)
$\frac{10}{s}$	NA	-20 dB
	(−20)	
$\frac{1}{(1+\frac{s}{2})}$	2	-20
	(−40)	
$(1+\frac{s}{5})$	5	$+20$
	(−20)	
$\frac{1}{(1+\frac{s}{10})}$	10	-20
	(−40)	
	(change in slope)	



① $\underline{\text{At } \omega < 2 \text{ rad/sec}}$:→

$$Y = \underbrace{M}_{-20\text{ dB/dec}}X + C$$

$\therefore C = 20\log K = 20\log 10 = 20\text{dB}$.

$\therefore$ eqn of st. line → $\boxed{Y = -20X + 20}$

Point Ⓐ → (let) At $\omega = 0.1 \Rightarrow X = -1$; $Y = (-20)(-1) + 20 = 40\text{dB}$.

→ $(\omega = 0.1, Y = 40\text{dB})$ or $(-1, 40)$.

Point Ⓑ → At $\omega = 2 \Rightarrow X = 0.3$; $Y = (-20)(0.3) + 20 = 14\text{dB}$.

→ $(\omega = 2, Y = 14\text{dB})$ or $(0.3, 14)$.

② $\underline{2 < \omega < 5 \text{ rad/sec}}$:→

$$Y = \underbrace{M}_{-40\text{ dB/dec}}X + \boxed{C} \longrightarrow \text{can be found from pt Ⓑ}.$$

$\therefore C = Y - MX = 14 - (-40)(0.3) = 26\text{ dB}$.

$\therefore$ eqn of st. line → $\boxed{Y = -40X + 26}$.

Point Ⓒ → At $\omega = 5 \Rightarrow X = 0.7$; $Y = (-40)(0.7) + 26 = -2\text{dB}$.

→ $(\omega = 5, Y = -2\text{dB})$ or $(0.7, -2)$.

③ $\underline{5 < \omega < 10 \text{ rad/sec}}$:→

$$Y = \underbrace{M}_{-20\text{ dB/dec}}X + \boxed{C} \longrightarrow \text{can be found from pt Ⓒ}.$$

$\therefore C = Y - MX = -2 - (-20)(0.7) = 12\text{ dB}$.

$\therefore$ eqn of st. line → $\boxed{Y = -20X + 12}$.

Point Ⓓ → At $\omega = 10 \Rightarrow X = 1$; $Y = (-20)\times(1) + 12 = -8\text{dB}$.

→ $(\omega = 10, Y = -8\text{dB})$ or $(1, -8)$.

④ $\underline{\omega > 10 \text{ rad/sec}}$:→

$$Y = \underbrace{M}_{-40\text{ dB/dec}}X + \boxed{C} \longrightarrow \text{can be found from pt Ⓓ}.$$

$\therefore C = Y - MX = -8 - (-40)(1) = 32\text{ dB}$.

$\therefore$ eqn of st. line → $\boxed{Y = -40X + 32}$.

Point (E) → At $\omega = 100\, rad/sec \Rightarrow x = 2$; $y = (-40)(2) + 32 = -48\, dB$.
→ $(\omega = 100, y = -48\, dB)$ or $(2, -48)$.

→ Now by taking the above points the bode magnitude plot is plotted in semilog paper.

Phase plot :-

$$G(s)H(s) = \frac{10\left(1+\frac{s}{5}\right)}{s\left(1+\frac{s}{10}\right)\left(1+\frac{s}{2}\right)} \Rightarrow G(j\omega)H(j\omega) = \frac{10\left(1+j\frac{\omega}{5}\right)}{(j\omega)\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{2}\right)}.$$

$$\therefore \angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{\omega}{5}\right) - 90^\circ - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{2}\right).$$

Phase angle table :→

ω	$\angle G(j\omega)H(j\omega)$
$\omega = 0^+$	-90°
$\omega = 0.1$	-92.3°
$\omega = 0.5$	-101.2°
$\omega = 1$	-110.96°
$\omega = 2$	-124.51°
$\omega = 5$	-139.76°
$\omega = 10$	-150.25
$\omega = 100$	-176.1°
$\omega \to \infty$	-180°.

Tutorial-5

: 19 Determine the value of K in the transfer function given below such that (a) The gain margin is 20 dB
(b) the phase margin is 30°.

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j0.1\omega+1)(j0.05\omega+1)}$$

Sol^n :-

$$G(s)H(s) = \frac{K}{s(0.1s+1)(0.05s+1)} \quad \text{(in Time Constant form)}$$

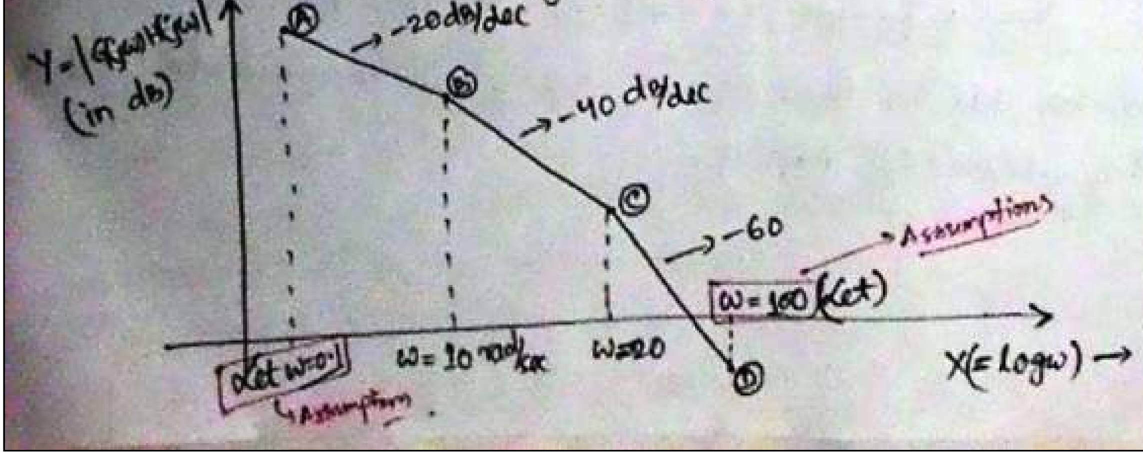
* As 'K' is unknown, so draw the Bode magnitude plot without considering K.
* The effect of K is to shift the magnitude plot upward or downward by $\underline{20\log K}$ dB.

Bode magnitude plot :-

→ The various factors are :→

Factors	Corner frequency (in rad/sec)	change in slope	slope (in dB/dec)
1. $\frac{1}{s}$	NA	-20 dB/dec	-20
2. $\frac{1}{0.1s+1}$	10	-40 dB/dec	-20
3. $\frac{1}{0.05s+1}$	20	-60 dB/dec	-20

→ The rough Bode magnitude plot is :-



Co-ordinates of the points :-

① for $\omega < 10$ rad/sec :-

Let eqⁿ of st. line : $y = \underbrace{m}_{-20\,dB/dec}x + c$ (Initial st. line).

$\therefore c = 20\log K = 20\log 1 = 0.$

$\therefore$ So, eqⁿ of st. line is $y = mx \Rightarrow y = -20x$.

Point Ⓐ $\Rightarrow$ at $\omega = 0.1 \Rightarrow x = \log 0.1 = -1$; $y = -20(-1) = 20$ dB.

$\longrightarrow$ ($\omega = 0.1$, $y = 20$ dB) or $(-1, 20)$.

Point Ⓑ $\Rightarrow$ at $\omega = 10 \Rightarrow x = \log 10 = 1$, $y = -20(1) = -20$ dB.

$\longrightarrow$ ($\omega = 10$, $y = -20$ dB) or $(1, -20)$.

② $10 < \omega < 20$ rad/sec :-

Let eqⁿ of st. line : $y = \underbrace{m}_{-40\,dB/dec}x + c \longrightarrow$ canbe found from pt Ⓑ.

$\therefore c = y - mx = -20 - (-40)1 = 20$ dB.

$\therefore$ So, eqⁿ of st. line is : $y = -40x + 20$.

Point Ⓒ $\Rightarrow$ at $\omega = 20 \Rightarrow x = 1.3$; $y = (-40)(1.3) + 20 = -32$ dB.

$\longrightarrow$ ($\omega = 20$, $y = -32$ dB) or $(1.3, -32)$.

③ $\omega > 20$ rad/sec :-

Let eqⁿ of st. line : $y = \underbrace{m}_{-60\,dB/dec}x + c \longrightarrow$ canbe found from pt Ⓒ.

$\therefore c = y - mx = -32 - (-60)(1.3) = 46$ dB.

$\therefore$ So, eqⁿ of st. line is : $y = -60x + 46$.

Point Ⓓ $\Rightarrow$ let at $\omega = 100 \Rightarrow x = 2$; $y = (-60)(2) + 46 = -74$ dB.

$\longrightarrow$ ($\omega = 100$, $y = -74$ dB) or $(2, -74)$.

$\therefore$ Now, by taking the points, the bode magnitude plot is drawn on the semilog paper.

Bode phase plot :-

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j0.1\omega+1)(j0.05\omega+1)}$$

$\therefore \angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}(0.1\omega) - \tan^{-1}(0.05\omega)$

So, the different angles for different 'ω' are :-

ω	$\angle G(j\omega)H(j\omega)$
$\omega = 0^+$	-90°
$\omega = 0.1$	-90.86°
$\omega = 0.4$	-93.44°
$\omega = 1$	-98.57°
$\omega = 3$	-115.23°
$\omega = 10$	-161.56°
$\omega = 20$	-198.43°
$\omega = 50$	-236.89°
$\omega = 100$	-252.98°
$\omega \to \infty$	-270.0°.

∴ So, taking the above points, the Bode phase plot is drawn in semilog paper.

* So, from the Bode plot (drawn in semilog paper) :-

→ For any value of 'K', ω_{pc} is unaffected, because there is no effect of 'K' value on the phase expression of $G(j\omega)H(j\omega)$.

∴ So, $\boxed{\omega_{pc} = 14.5 \text{ rad/sec}}$ (the freq. at which bode phase plot crosses -180° line)

Steps for finding K for given G.M. :-

(1.) Draw the horizontal line below 0 dB at a distance of given G.M. The intersection point of ω_{pc} line & this G.M. line is "A'".

(2.) Draw the vertical line from ω_{pc}, till it intersects magnitude plot without 'K'. This point is 'A'. And this will give you the actual GM without 'K'.

(3.) To match the given GM, AA' shift is required i.e, the Contribution of K which is $20\log K$ dB.

(4.) Upward shift (taken positive) & downward shift (negative).

$\therefore 20\log K = \text{shift }(AA')$.

So, for the given problem,

$$20 \log K = +7 \text{ dB (upward)}$$

$$\Rightarrow \log K = \frac{7}{20} = 0.35$$

$$\Rightarrow \boxed{K = 2.2387} \text{ . (Ans)}$$

Steps to find K for given P.M. :-

(1.) Draw a horizontal line above $-180°$ line at a distance of given P.M., till it ~~cross~~ intersects phase plot. This point "B".

(2.) Draw a vertical line, which will intersect the magnitude plot at "C" and intersect the 0dB line at "C'". This line gives you the ω_{gc} (Because PM is calculated at $\omega = \omega_{gc}$)

(3.) But at ω_{gc}, the ~~gain~~ magnitude must be 0 dB. So to achieve this, CC' shift is required in upward direction, which is provided by 'K' i.e,

$$20 \log K = \text{shift}(CC').$$

∴ So, for this problem :-

$$20 \log K = +18 \text{ (upward)}$$

$$\Rightarrow \log K = \frac{18}{20} = 0.9$$

$$\Rightarrow \boxed{K = 7.943} \text{ (Ans)}$$

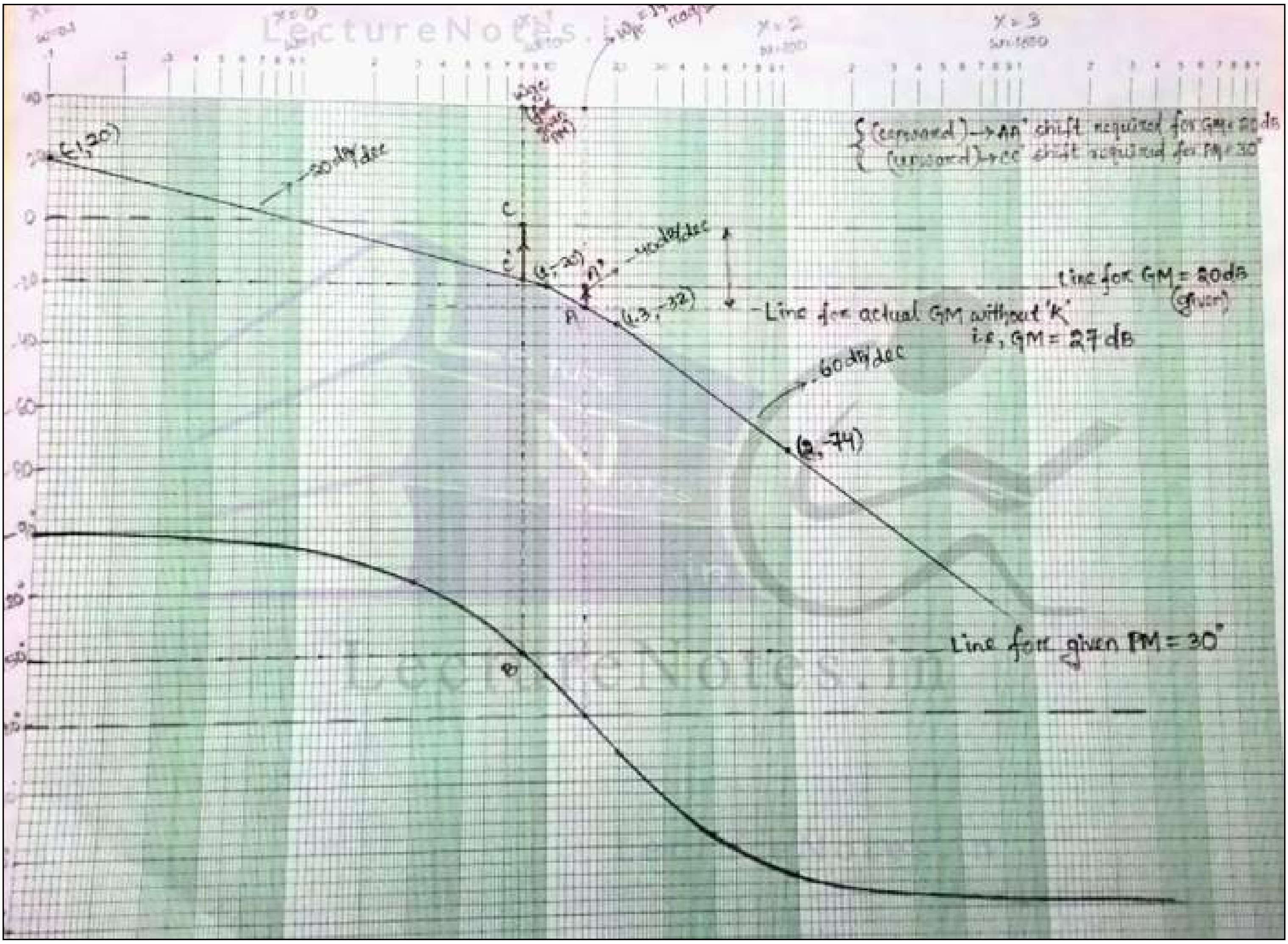
(a) So, the value of 'K' for gain margin 20 dB, is $\boxed{2.2387}$ and

(b) the value of 'K' for phase margin $30°$, is $\boxed{7.943}$. (Ans)

Note :-

* This type of questions are very rarely coming in BPUT exam. Still practice it only by understanding the concepts of PM & GM.

* Don't write the steps in exam copy.


(1,20)
-20 dB/dec
-40 dB/dec
-60 dB/dec
(1,-20)
(1.3,-32)
(2,-74)
Line for GM = 20dB (given)
Line for actual GM without 'K' i.e, GM = 27 dB
Line for given PM = 30°

LECTURENOTES.IN

Control System Engineering

Topic:

Closed Loop Frequency Response

Contributed By:

Gyana Ranjan Biswal

CLOSED LOOP FREQUENCY RESPONSE :-

→ The closed loop frequency response specifications are Resonanant frequency (ω_r), Resonant peak (M_r) & Bandwidth (BW).

→ To determine the above specifications directly from the graph, following graphical methods (in frequency domain) are used.

1. M circles 2. N circles 3. Nichols chart.

→ Let G(s) is the forward path transfer function of a unity feedback system.

The closed-loop transfer function is :→

$$T(s) = \frac{G(s)}{1+G(s)}$$

By replacing, $s=j\omega$, G(s) becomes :-

$$G(j\omega) = \text{Complex number} = Re[G(j\omega)] + j\,Img[G(j\omega)].$$
$$= x + jy.$$

for simplicity, $x = Re[G(j\omega)]$ & $y = Img[G(j\omega)]$.

∴ The magnitude of closed-loop transfer function is written as :→

$$\boxed{|T(j\omega)| = \left|\frac{G(j\omega)}{1+G(j\omega)}\right| = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}} \quad \text{——— (1)}$$

And phase is given by :→

$$\angle T(j\omega) = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right) = \phi \quad \text{——— (2)}$$

$$\Rightarrow \tan\phi = \tan\left[\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right)\right]$$

$$= \frac{y/x - y/(1+x)}{1+\left(\frac{y}{x}\right)\left(\frac{y}{1+x}\right)} = \frac{(1+x)y - xy}{x(1+x)+y^2}$$

$$\Rightarrow \boxed{\tan\phi = \frac{y}{x^2+x+y^2}} \quad \text{——— (3)}.$$

→ If magnitude [T(jω)] is plotted w.r.t G(jω) [i.e, Re{G(jω)} + j Im{G(jω)}] then the plot is called as M-circles.

→ If phase $[\angle T(j\omega)]$ is plotted w.r.t. $G(j\omega)$, then the plot is called as N-circles.

→ The plot between $|T(j\omega)|$ Vs $|G(j\omega)|\angle G(j\omega)$ and $\angle T(j\omega)$ Vs $|G(j\omega)|\angle G(j\omega)$ in one graph is called as Nichol's chart.

※ Note :→

$\boxed{x^2+y^2+2gx+2fy+c=0}$ → eqⁿ of circle in x-y plane.

Where, center is $(-g,-f)$ & Radius is $\sqrt{g^2+f^2-c}$.

M-CIRCLES :- (or) (Constant Magnitude Loci) :-

→ It is the graph between magnitude of closed-loop t/f (i.e, $|T(j\omega)|$) with respect to the rectangular form of $G(j\omega)$ (i.e, $Re\{G(j\omega)\}+j\,Img\{G(j\omega)\}$).

$$\therefore\ |T(j\omega)| = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} = M \quad (\text{Let})$$

$$\Rightarrow M^2[(1+x)^2+y^2] = x^2+y^2 \Rightarrow M^2(1+x^2+2x+y^2) = x^2+y^2$$

$$\Rightarrow M^2+M^2x^2+2xM^2+y^2M^2 = x^2+y^2$$

$$\Rightarrow (1-M^2)x^2+(1-M^2)y^2-2M^2x-M^2 = 0 \quad \text{——— (1)}$$

Where, 'M' is always +ve.

Case 1 :- $\boxed{M=1}$;

The eqⁿ(1) becomes; $\boxed{-2x-1=0}$ → eqⁿ of st. line in x-y plane.

Case 2 :- $\boxed{M\neq 1}$;

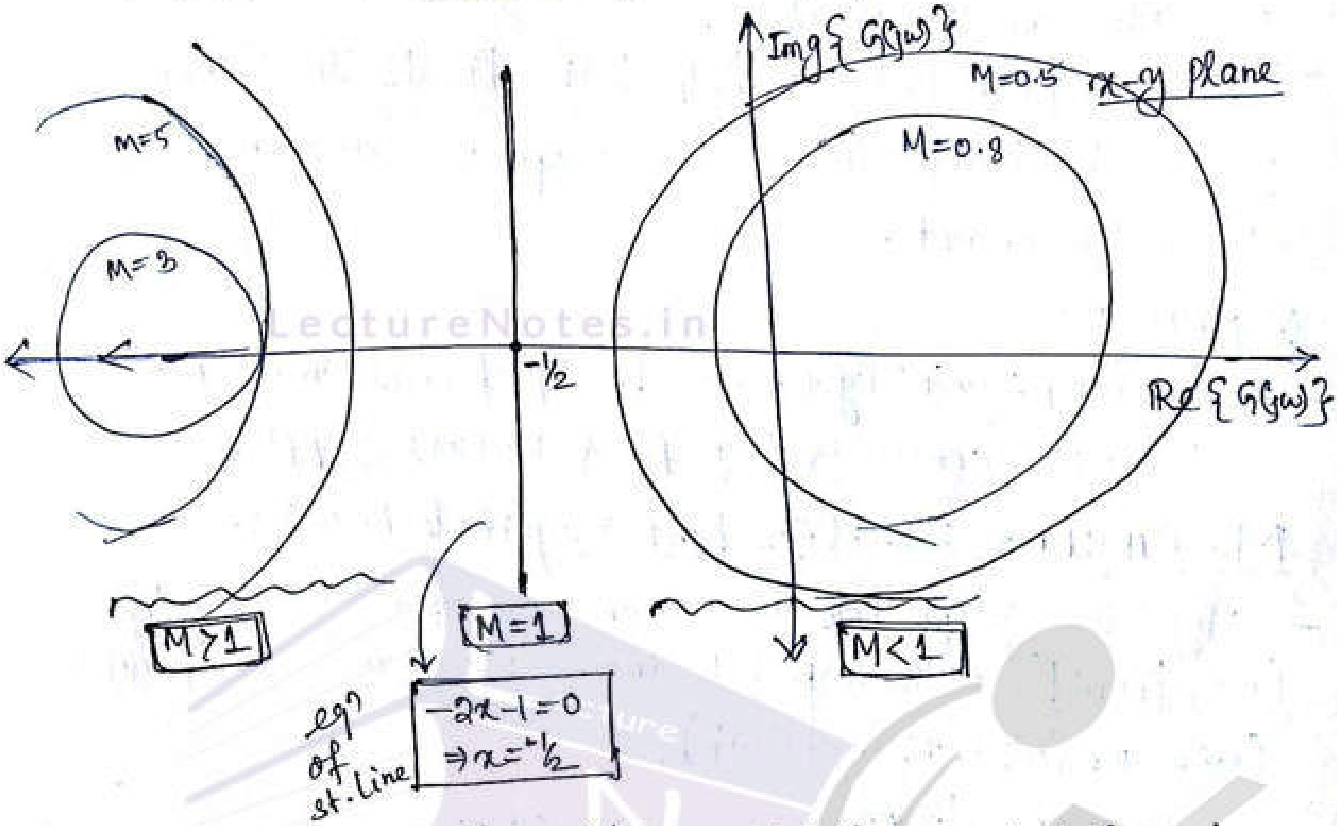
The eqⁿ(1) becomes :-

$$\boxed{x^2+y^2-\frac{2M^2}{(1-M^2)}x-\frac{M^2}{1-M^2}=0} \rightarrow \text{eq}^n \text{ of circle in x-y plane.}$$

The circle is having →

Center $\left(\frac{-M^2}{M^2-1}, 0\right)$ & Radius $= \left|\frac{M}{1-M^2}\right|$.

→ Center of '<u>M-circle</u>' is always present on x-axis.



→ The intersection of G(jω) plot (Nyquist plot) and Constant M-circle gives the value of Magnitude 'M'.

→ The M-circle, which is tangent to the G(jω) plot will give the value of resonant peak (M_r) & resonance frequency (ω_r).

N-CIRCLES :-

→ It is the plot between phase of the closed loop transfer function [i.e, $\angle T(j\omega)$] with respect to the rectangular form of G(jω) [i.e, $Re\{G(j\omega)\} + j\, Img\{G(j\omega)\}$].

$$\therefore \tan\phi = \frac{y}{x^2+y^2+x} = N \quad (\text{Let})$$

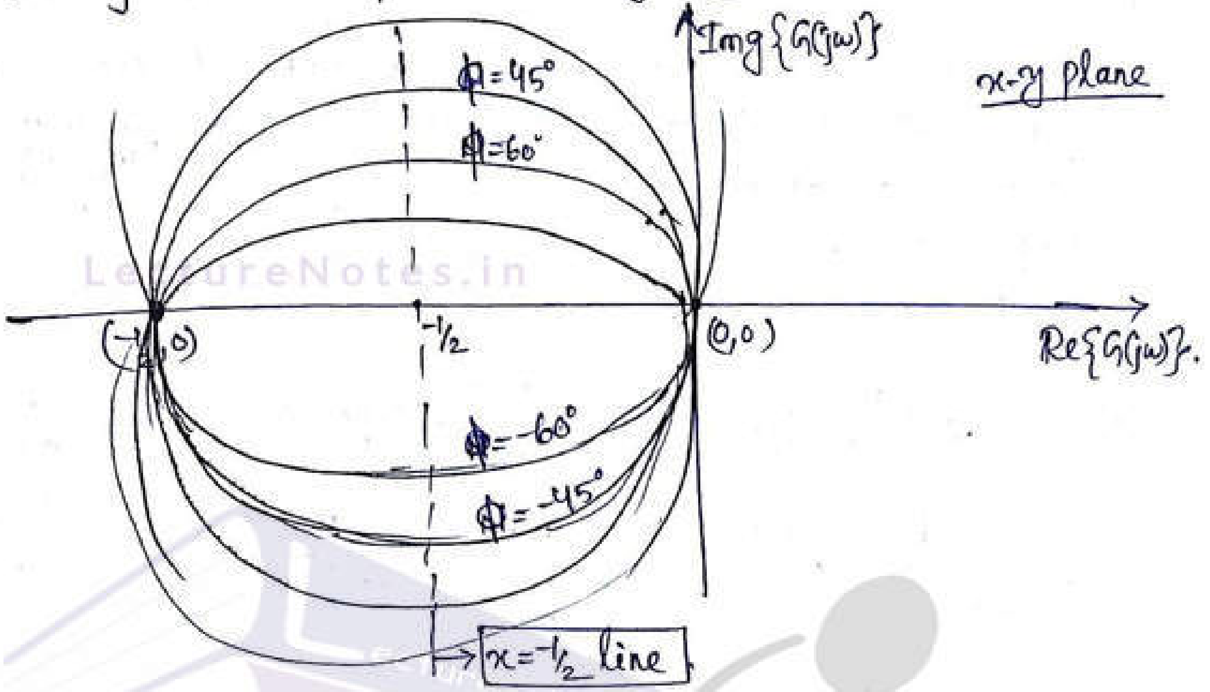
$$\Rightarrow \boxed{x^2+y^2+x-\frac{1}{N}y=0} \rightarrow \text{eq}^n \text{ of circle in } \underline{x\text{-}y \text{ plane}}.$$

The circle is having →

Center $\left(-\frac{1}{2}, +\frac{1}{2N}\right)$ & Radius $=\sqrt{\frac{1}{4}+\frac{1}{4N^2}} = \frac{\sqrt{1+N^2}}{2N}$.

→ so, the center of the circle will lies on $x = -\frac{1}{2}$ line.

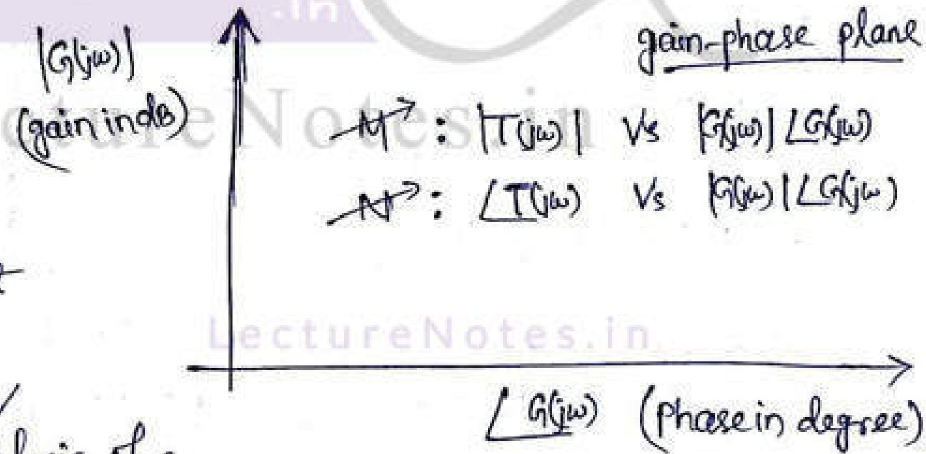
→ every N-circle passes through (0,0) and (−1,0).



NICHOLS CHART :-

→ The plot of M and N circles in gain-phase plane is known as Nichols chart.

→ The phase angle of Nicholas chart varies from 0° to −360°, but the region of $\angle G(j\omega)$ generally used for analysis of a system in between −90° to −270°.



Control System Engineering

Topic:

Controllers

Contributed By:

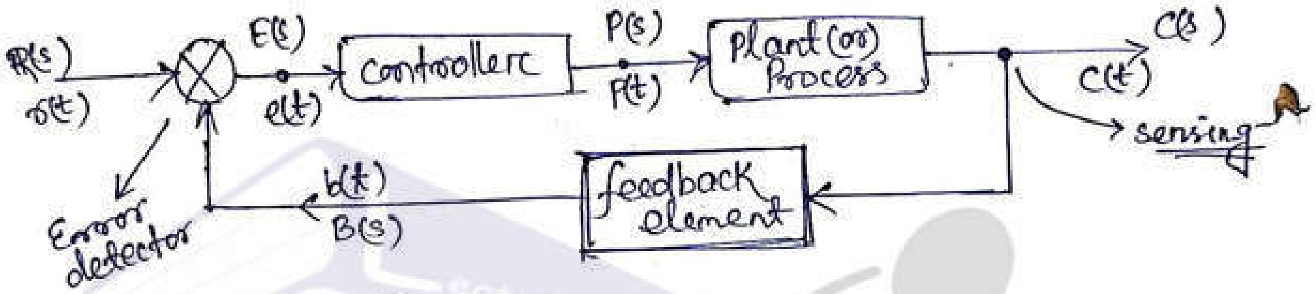
Gyana Ranjan Biswal

-: CONTROLLERS :-

INTRODUCTION :-

→ It is used to reduce the deviation to zero, which is present due to the difference betn actual value and desired value. Hence the system performance/characteristic can be improved by using Controllers.

→ The general block diagram is as follows :-



CLASSIFICATION :-

Controllers

- Dis Continuous mode
  - ON/OFF Controller (two position mode)
  - Multiposition Controllers
- Continuous mode
  - Proportional (P) Controller
  - Integral (I) controller
  - Derivative (D) Controller

* Continuous mode controllers can be combined to form Composite controllers. Ex :→ PI, PD and PID Controllers.

ON-OFF Controller :-

→ It is most common & simplest mode of controllers.

→ It has to control two positions of control element, either ON or OFF.

→ It has two possible output states namely 0% or 100%.

Mathematically,

$$\boxed{\begin{aligned} P &= 0\% \quad \text{for } e_p < 0 \\ &= 100\% \quad \text{for } e_p > 0 \end{aligned}}$$

P = Controller o/p.
e_p = error

→ Example :- A Room heater.

→ If temp. drops below a set point, the heater is turned on & if the temp. increases above a set point, the heater is turned off.

PROPORTIONAL CONTROLLER (P-Controller) :-

→ Here, the o/p of the Controller is simple proportional to the error e(t). Mathl :→ $p(t) \propto e(t)$

$$\Rightarrow \boxed{p(t) = K_p\, e(t)}.$$

where, K_p = Proportional gain constant.

→ The range of error which covers 0% to 100% controller output is called proportional band.

Mathl :→ $\boxed{PB = \frac{100}{K_p}}$.

→ When load changes are small, then P-controller is best suitable.

→ In Laplace domain, $P(s) = K_p E(s) \Rightarrow \boxed{\frac{P(s)}{E(s)} = K_p}$ → T/f of P-controller

→ In block diagram, $\xrightarrow{P(s)} \boxed{K_p} \longrightarrow E(s)$

INTEGRAL CONTROLLER (I-Controller) : (or) Reset Action Controller :-

→ Here, the o/p of the Controller p(t) is changed at a rate which is proportional to the error signal e(t).

Mathl → $\frac{dp(t)}{dt} \propto e(t) \Rightarrow p(t) \propto \int_0^t e(t)\,dt$

$$\Rightarrow \boxed{p(t) = K_I \int_0^t e(t)\,dt}.$$

Where, K_I = Integral Constant

→ The o/p depends on the history of the error, not on the instantaneous value of the error.

→ It is a relatively slow error. So, sometimes, T_i is specified. $\boxed{T_i = 1/K_i = \text{Integral time}}$.

In P-Controller, error reduces but can't go to zero. It finally produces an offset error. Whereas I-Controller acts till the error reduces to zero.

※ P-Controller acts immediately, but I-Controller acts slowly. So P-Controller improves transient response, whereas I-Controller improves steady-state response.

→ In practical, I-controller never used alone.

→ In Laplace domain, $P(s) = \frac{K_I}{s} E(s) \Rightarrow \boxed{\frac{P(s)}{E(s)} = \frac{K_I}{s}}$ → T/F of I-controller

⇒ The block diagram representation, $\xrightarrow{P(s)} \boxed{K_I/s} \rightarrow E(s)$

DERIVATIVE CONTROLLER (D-Controller) :-

⇒ Here, o/p of the controller depends on the time rate of change of the actual errors.

→ It is also called as Rate action mode or anticipatory action mode

Math^{l} → $P(t) \propto \frac{de(t)}{dt} \Rightarrow \boxed{P(t) = K_d \frac{de(t)}{dt}}$

where, K_d = Derivative gain constant.

→ D-Controller anticipates the actuating error, initiates the early corrective action & tends to increase stability of the system improving the transient response.

⇒ When the error is constant, the controller o/p is zero & when the error is zero, again the controller o/p is zero. So it never used alone.

→ Its gain should be small because faster rate of change of error can cause very large sudden change of controller output.

⇒ In Laplace domain, $P(s) = sK_D E(s) \Rightarrow \boxed{\frac{P(s)}{E(s)} = sK_D}$ → T/F of D-controller

→ The block diagram :→ $\xrightarrow{P(s)} \boxed{sK_D} \rightarrow E(s)$

Proportional + Integral (PI) Controller :-

→ The mathematical expression is →

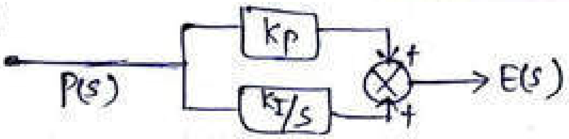
$$\boxed{P(t) = K_p e(t) + K_i \int_0^t e(t)\, dt}$$

→ The main advantage of this control, offset gets eliminated by P-Controller & steady state error also reduced to zero by the I-Controller.

* Characteristics of PI Controller :-

⋇ In Laplace domain, $P(s) = \left(K_p + \frac{K_I}{s}\right) E(s) \Rightarrow \boxed{\frac{P(s)}{E(s)} = K_p + \frac{K_I}{s}} \rightarrow$ T/F of PI Controller

→ The block diagram :→



* characteristic :-

1. As addition of pole is there, so stability decreases.
2. Improves steady state accuracy. (i.e, e_{ss} ↓es)
3. Increases rise time & settling time, so system response becomes slow.
4. Decreases bandwidth of the system.
5. Makes response more oscillatory.
6. Filters out high frequency noise.

PROPERTIONAL + DERIVATIVE MODE (PD Controller) :-

→ Mathematical expression is →

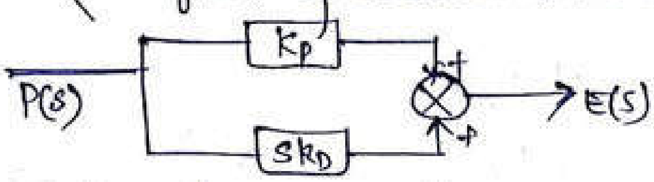
$$\boxed{P(t) = K_p\, e(t) + K_d \frac{de(t)}{dt}}$$

→ Here, derivative mode causes sudden increase or decrease in the output corresponding to decrease or increase in the error. But this mode can not eliminate the offset error.

*→ In Laplace domain,

$$P(s) = [K_p + sK_D]\, E(s) \Rightarrow \boxed{\frac{P(s)}{E(s)} = [K_p + sK_D]} \rightarrow \text{T/F of PD-Controller.}$$

→ The block diagram representation :-



* Characteristics of PD Controller :-

1. As there is addition of zeros, so the stability increases.
2. Steady-state accuracy decreases (i.e, e_{ss} ↑es)
3. transient response will improve, that means overshoot reduces, rise time decrease & t_s also decreases.
4. So the system response is fast.

5. → Improves bandwidth of the system.
6. → Improves damping (ξ ↑es).
7. It may make noise dominant at high frequencies.

PID Controller :-

→ The composite controller including the combination of the proportional, integral & derivative control mode is called PID controller. It is also called as three mode controller.

→ It is very complex to design but very powerful in action.

→ Mathematical expression is :→

$$\boxed{p(t) = K_p e(t) + K_i \int_0^t e(t)\,dt + K_d \frac{de(t)}{dt}}$$

→ Main advantage is, the integral mode eliminates the offset error & the response is also very fast due to derivative mode.

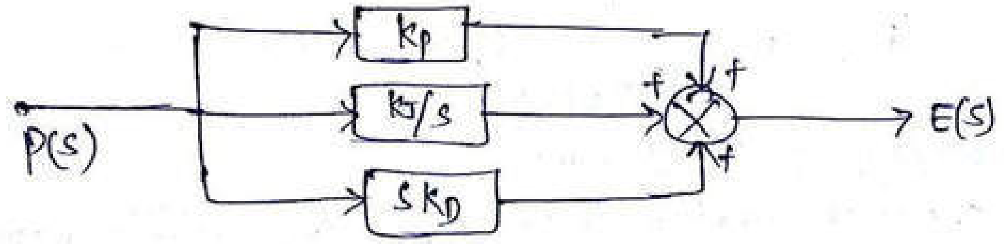
→ With PID control action, there is no offset, no oscillations with least settling time. So there is improvement in both transient as well as steady state response. Thus PID is the ultimate process composite controller.

→ In Laplace domain →

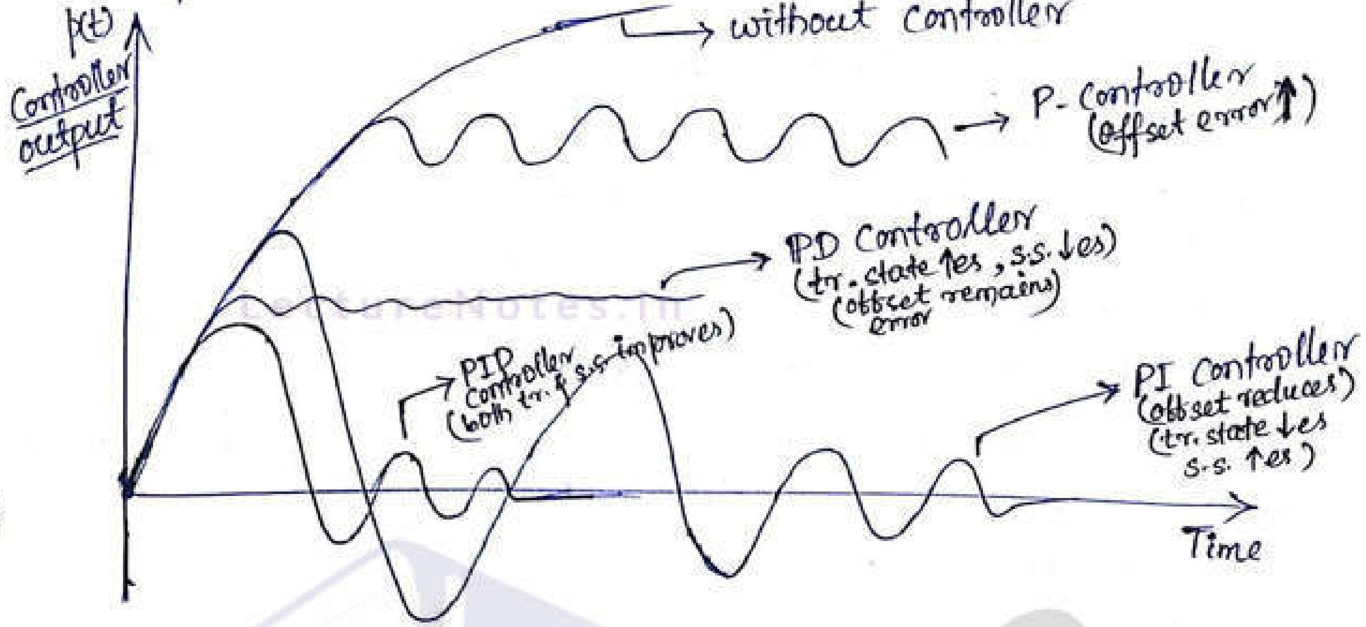
$$P(s) = \left(K_p + \frac{K_I}{s} + sK_D\right)E(s) \Rightarrow \boxed{\frac{P(s)}{E(s)} = K_p + \frac{K_I}{s} + sK_D}$$

↳ T/f of PID Controller

→ The block diagram representation :→



* Response of various control modes to unit step load change :-



* ZIEGLER-NICHOLS METHOD FOR CONTROLLER TUNING :-

→ Effect of various controllers depends upon proper tuning of parameters. Tuning means, at which value of the parameters (i.e, K_p, T_i & T_d), the system will give better response.

→ In this method, there are two approaches for the tuning of controllers:-

① First Approach :-

→ Here, plant dynamics is known to us, we just calculate the overall transfer function & the response.

→ If the controller is P-controller, then system is asymptotically stable in the range $\boxed{0 \leq K_p < K_c}$, where K_p = proportional gain, K_c = critical value of the gain.

→ If $K_p > K_c$, then it is unstable.

and if $K_p = K_c$, then system is marginally stable.

Types of Controller	K_p	T_i	T_d
P	$0.5K_c$	∞	0
PI	$0.45K_c$	$0.83T$	0
PID	$1.6K_c$	$0.5T$	$0.125T$

→ Use the appropriate value of K_p, T_i & T_d for different type of controllers from the above table.

<u>Calculation of K_c & T</u> :-

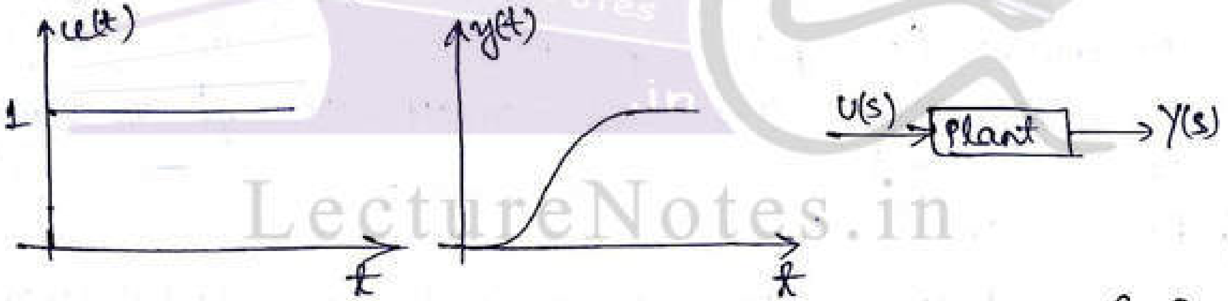
- Under pure P-Control, find the system close loop transfer function & characteristic equation.
- Findout auxilliary polynomial from Routh array. then find out the oscillating frequency and the corresponding time period; $\omega = \frac{2\pi}{T} \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$.
- The value of 'K' at which the system has auxiliary equation, that value will be $\boxed{K_c}$.

(2) SECOND Approach :-

→ This method is applicable, when plant dynamic is not known to us.

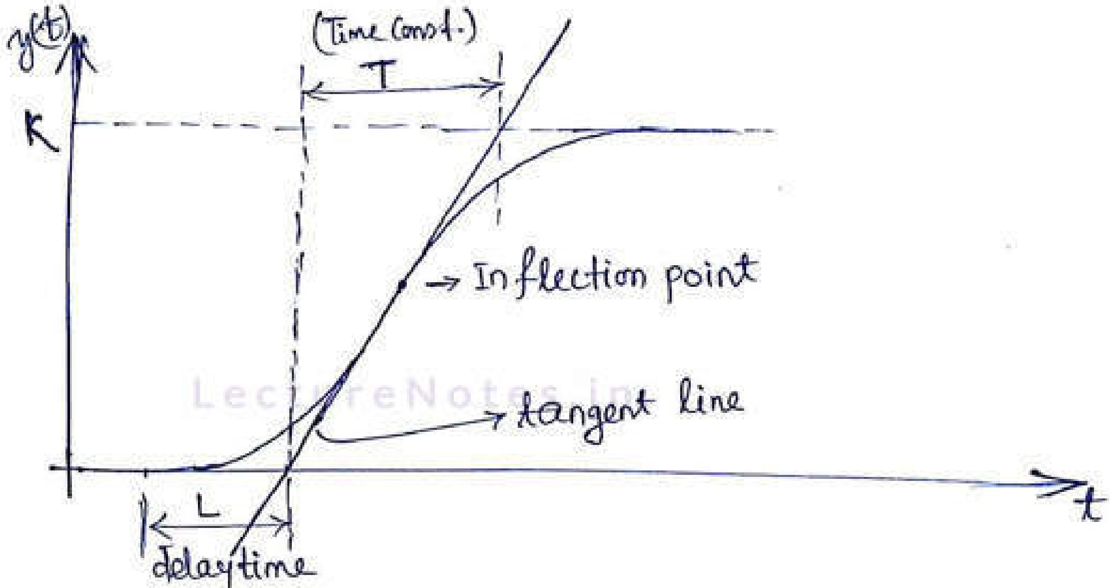
<u>Procedure</u> :-

1. Obtain the unit step transient response for open loop plant as shown in figure.



(Fig- Open loop plant with i/p & o/p waveform)

2. If the plant involves neither the integrator nor the dominant complex conjugate poles, then a unit step response must look like <u>s-shaped</u> curve as shown in above figure. Otherwise this method is not applicable.
3. Then the s-shaped curve is characterized by delay time (L) and time constant (T) by drawing a tangent at the inflection point. Inflection point is the point at which the tangent gives maximum slope.



4. Then determine the intersection point of the tangent line with the time axis and $y(t)=K$ line, as shown in the above figure. Find the L & T from the graph.

5. Now, Transfer function $Y(s)/U(s)$ may be approximated by a 1st order system with transportation lag.

$$\frac{Y(s)}{U(s)} = \frac{K e^{-Ls}}{1+sT}$$

6. If $T \gg L$, the process is easily controllable with a simple controller, otherwise difficulty may arise.

7.

Types of Controller	K_p	T_i	T_d
P	T/L	∞	0
PI	$0.9(T/L)$	$L/0.3$	0
PID	$1.2(T/L)$	$2L$	$0.5L$

Choose the appropriate value of K_p, T_i & T_d for different controllers from the above table. These values will give you the improved response.

Control System Engineering

Topic:

State Variable Analysis

Contributed By:

Gyana Ranjan Biswal

STATE VARIABLE ANALYSIS

INTRODUCTION :-

→ The classical control theory based on a simple input-output description of the plant, usually expressed as a transfer function.

→ Modern control theory is based on the description of system equations in terms of 'n' first-order differential equations.

→ So the concept, which uses the total internal state of the system considering all initial conditions is called as state variable Analysis or state space analysis.

Advantages of state Variable Analysis :-

→ It takes into account the effect of all initial conditions.

→ It can be applied to non-linear system.

→ It can be applied to time-variant system.

→ It can be applied to Multiple Input Multiple Output (MIMO) system.

→ Any type of input can be considered for designing the system.

→ The vector matrix notation greatly simplifies the mathematical representation of the system.

Definitions :-

State :- The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time $t = t_0$ (initial time), together with the knowledge of input for time $t > t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

→ In state-determined system models, the no of state variables 'n' is equal to the no of independent energy storage elements.

→ State variables represented as $x_1(t)$, $x_2(t)$, ... $x_n(t)$.

State Vector :- The 'n' state variables necessary to describe the complete behavior of the system can be considered as 'n' components of a vector, called as state vector.

→ It is represented as a column vector & denoted by $X(t)$.

State Space :- The n-dimensional space whose co-ordinate axes consist of 'x_1' axis, 'x_2' axis, ..., 'x_n' axis is called a state space.

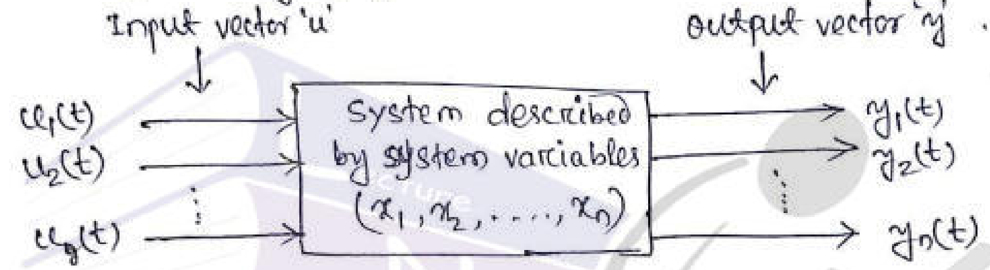
→ For any dynamical system, the state space remains unique, but the state variables are not unique.

State trajectory :- It is the locus of the tips of the state vectors, with time as implicit variable.

* STATE MODEL :-

State Equation :-

→ Consider multiple input multiple output, n^{th} order system as shown in following figure.



→ In the standard form of the mathematical description of the system is expressed as a set of 'n' coupled first-order ordinary differential equations, known as the state equations.

→ In state equations, the time derivative of each state variable is expressed in terms of the state variables $x_1(t)$, $x_2(t), ..., x_n(t)$ & the system inputs $u_1(t), ..., u_r(t)$

→ Mathematically, $\dot{x}_1(t) = f_1(x, u, t)$

$\dot{x}_2(t) = f_2(x, u, t)$ & so on.

⇒ For an LTI system of order 'n' & with 'r' inputs, a set of 'n' coupled first-order differential eq^n with constant coefficients are :-

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1r}u_r.$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2r}u_r.$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nr}u_r.$$

where, a_{ij} & b_{ij} = constant coefficients that describe the system

→ So, the above equations can be written in vector matrix form:

$$\boxed{\dot{X}(t) = A\,X(t) + B\,U(t)} \longrightarrow \text{state equation.}$$

Where, $X(t)$ = state vector (i.e, column vector of order $n \times 1$)

$U(t)$ = Input vector (i.e, column vector of order $r \times 1$)

A = System matrix (or) evolution matrix (i.e, square matrix of order $n \times n$).

B = Input matrix (or) control matrix (i.e, order $n \times r$).

$\dot{X}(t) = \frac{d}{dt}[X(t)]$ = derivative of state vector (order $n \times 1$).

Output Equation :-

→ Output variable in a system can be expressed as the linear combination of the input variables & state variables as:-

$$y_1(t) = c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n + d_{11}u_1 + \dots + d_{1r}u_r$$

$$\vdots$$

$$y_m(t) = c_{m1}x_1 + c_{m2}x_2 + \dots + c_{mn}x_n + d_{m1}u_1 + \dots + d_{mr}u_r.$$

→ Representing the above equation is vector matrix form :-

$$\boxed{Y(t) = C\,X(t) + D\,U(t)} \longrightarrow \text{output equation.}$$

Where, $Y(t)$ = Output vector (i.e, column vector of order $m \times 1$)

C = Output matrix or observation matrix (i.e, of order $m \times n$).

D = Direct transmission matrix (i.e, of order $m \times r$).

** The two vector equations together is called as the state model of the LTI system.

$$\boxed{\begin{aligned} [\dot{X}(t)]_{n\times1} &= [A]_{n\times n}[X(t)]_{n\times1} + [B]_{n\times r}[U(t)]_{r\times1} \\ [Y(t)]_{m\times1} &= [C]_{m\times n}[X(t)]_{n\times1} + [D]_{m\times r}[U(t)]_{r\times1} \end{aligned}} \begin{aligned} &\longrightarrow \text{state eq}^n \\ &\longrightarrow \text{output eq}^n \end{aligned} \Bigg\} \text{ state model.}$$

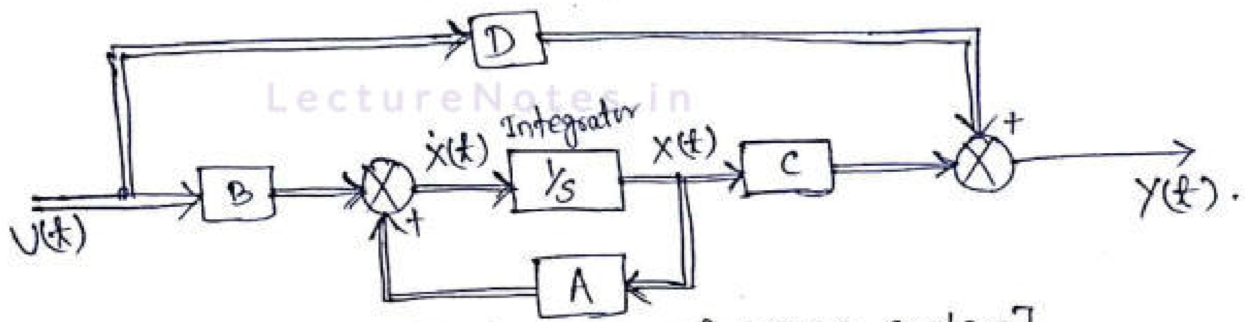
Note: for many physical systems, the matrix 'D' is a null matrix.

State Diagram of Standard State Model :-

→ The standard state Model equations are :-

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

→ The Block diagram representation of above two eq^{ns} are :-



[State diagram of MIMO system].

→ There must be 'n' parallel integrators for 'n' state variables.

→ The thick arrows indicate, there are multiple number of input, output & state variables.

Note :- To obtain the state model from state diagram, always choose output of each integrator as a state variable. No of integrators always equals the order of the system.

TRANSFER FUNCTION FROM STATE MODEL :-

The standard state Model equations are :-

$$\dot{X}(t) = AX(t) + BU(t)$$
$$\& \quad Y(t) = CX(t) + DU(t).$$

Taking Laplace transform of both sides :-

$$[SX(s) - X(0)] = AX(s) + BU(s)$$
$$\& \quad Y(s) = CX(s) + DU(s).$$

for transfer function, initial condition is zero i.e, $X(0)=0$.

$\therefore \quad SX(s) = AX(s) + BU(s) \Rightarrow SX(s) - AX(s) = BU(s).$

$\Rightarrow SIX(s) - AX(s) = BU(s) \qquad (\because I = \text{Identity matrix of order } n\times n)$

$\Rightarrow [SI-A]X(s) = BU(s).$

$\Rightarrow [SI-A]^{-1}[SI-A]X(s) = [SI-A]^{-1}BU(s)$

$\Rightarrow X(s) = [SI-A]^{-1}BU(s)$ ———— (1)

Substituting the eqⁿ(1) in output equation :→

$$Y(s) = C[SI-A]^{-1} B\,U(s) + D\,U(s)$$

$$\Rightarrow Y(s) = \{C[SI-A]^{-1}B + D\}\,U(s)$$

$$\Rightarrow \boxed{\text{Transfer function} = \frac{Y(s)}{U(s)} = C[SI-A]^{-1}B + D}$$

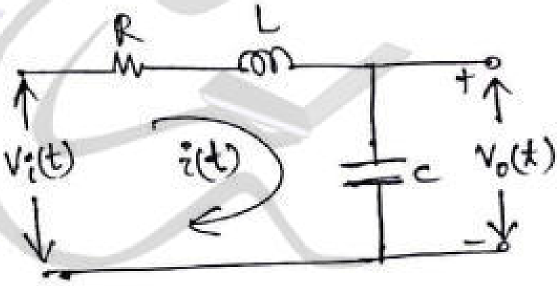
where, $[SI-A]^{-1} = \dfrac{Adj\,[SI-A]}{|SI-A|}$.

→ The characteristic eqⁿ of the system is $\boxed{|SI-A| = 0}$, as it is the denominator of the transfer function. The roots of this equation are the closed loop poles of the system.

Problems

Q1. Obtain the state Model of the given electrical system. Also find Transfer function from state Model.

solⁿ: There are two energy storage elements 'L' & 'C'. So two state variables are current through inductor $i(t)$ & voltage across capacitor i.e, $v_0(t)$.



So, $x_1(t) = i(t)$ & $x_2(t) = v_0(t)$.

$u_1(t) = V_i(t)$ = Input variable.

Putting KVL, $v_i(t) = R\,i(t) + L\dfrac{di(t)}{dt} - v_0(t)$

$$\Rightarrow \frac{di(t)}{dt} = \frac{1}{L}v_i(t) - \frac{R}{L}i(t) - \frac{1}{L}v_0(t)$$

$$\Rightarrow \dot{x}_1(t) = \frac{-R}{L}x_1(t) - \frac{1}{L}x_2(t) + \frac{1}{L}u_1(t) \quad \text{(Using state variables)}$$

———— (1)

voltage across capacitor is :→

$$v_0(t) = \frac{1}{C}\int i(t)\,dt \Rightarrow \frac{dv_0(t)}{dt} = \frac{1}{C}i(t)$$

$$\Rightarrow \dot{x}_2(t) = \frac{1}{C}x_1(t) \quad \text{———— (2)}.$$

By writing eqⁿ(1) & (2) in matrix form :→

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} [u_1(t)]$$

i.e, $\dot{X}(t) = AX(t) + BU(t) \longrightarrow$ State equation . (Ans)

Output variable can be written as, $y(t) = V_o(t) = x_2(t)$.
By writing in matrix form:-

$$y(t) = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0][U(t)]$$

i.e, $y(t) = CX(t) + DU(t) \longrightarrow$ output eq^n.

By comparing with the standard state Model equations:-

$$[A] = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}_{2\times2} ; [B] = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}_{2\times1} ; [C] = [0 \quad 1]_{1\times2} ; [D] = [0]_{1\times1}.$$

The transfer function can be written as:-

$$\frac{V_o(s)}{V_i(s)} = C[SI-A]^{-1}B + D.$$

$$\therefore [SI-A] = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} = \begin{bmatrix} (S+R/L) & 1/L \\ -1/C & S \end{bmatrix}.$$

$$\therefore [SI-A]^{-1} = \frac{1}{\left(S^2+\frac{R}{L}S+\frac{L}{LC}\right)}\begin{bmatrix} S & -1/L \\ +1/C & (S+R/L) \end{bmatrix}$$

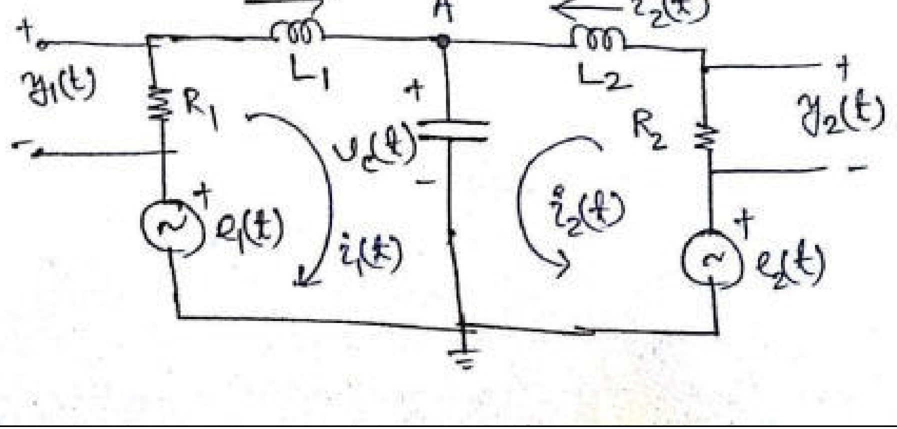
$$\therefore C[SI-A]^{-1}B = \begin{bmatrix} \frac{1}{C\left(S^2+\frac{R}{L}S+\frac{L}{LC}\right)} & \frac{1}{\left(S+\frac{R}{L}\right)\left(S^2+\frac{R}{L}S+\frac{L}{LC}\right)} \end{bmatrix}_{1\times2} \begin{bmatrix} 1/L \\ 0 \end{bmatrix}_{2\times1}$$

$$\Rightarrow C[SI-A]^{-1}B = \frac{1}{S^2LC + SRC + 1}.$$

$$\therefore \boxed{\text{Transfer function} = \frac{V_o(s)}{V_i(s)} = \frac{1}{S^2LC + SRC + 1}}. \text{ (Ans)}$$

Q2. Find the state Model of given electrical system as shown in figure.

Sol^n → There are:→
2 inputs, 2 outputs and 3 state variables as 3 no of energy storage elements present.



Inputs are : $e_1(t)$ & $e_2(t)$.

Outputs are : $y_1(t)$ & $y_2(t)$.

State variables are :- $i_1(t)$, $i_2(t)$ & $v_c(t)$.

$$\therefore X(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Putting KVL in Mesh-1 :→

$$e_1(t) - R_1 i_1(t) - L_1 \frac{di_1(t)}{dt} - v_c(t) = 0$$

$$\Rightarrow \frac{di_1(t)}{dt} = \frac{-R_1}{L_1} i_1(t) - \frac{1}{L_1} v_c(t) + \frac{1}{L_1} e_1(t) \quad \text{——— (1)}$$

$$\Rightarrow \dot{x}_1(t) = \frac{-R_1}{L_1} x_1(t) + 0 \cdot x_2(t) - \frac{1}{L_1} x_3(t) + \frac{1}{L_1} e_1(t) + 0 \cdot e_2(t)$$

(Using state variable notation).

Putting KVL in Mesh-2 :→

$$e_2(t) - R_2 i_2(t) - L_2 \frac{di_2(t)}{dt} - v_c(t) = 0$$

$$\Rightarrow \frac{di_2(t)}{dt} = \frac{-R_2}{L_2} i_2(t) - \frac{1}{L_2} v_c(t) + \frac{1}{L_2} e_2(t) \quad \text{——— (2)}$$

$$\Rightarrow \dot{x}_2(t) = 0 \cdot x_1(t) + \left(\frac{-R_2}{L_2}\right) x_2(t) - \frac{1}{L_2} x_3(t) + 0 \cdot e_1(t) + \frac{1}{L_2} e_2(t)$$

(Using state variable notation).

Putting KCL at node 'A' →

$$i_1(t) + i_2(t) = C \frac{dv_c(t)}{dt}$$

$$\Rightarrow \frac{dv_c(t)}{dt} = \frac{1}{C} i_1(t) + \frac{1}{C} i_2(t) \quad \text{——— (3)}$$

$$\Rightarrow \dot{x}_3(t) = \frac{1}{C} x_1(t) + \frac{1}{C} x_2(t) + 0 \cdot x_3(t) + 0 \cdot e_1(t) + 0 \cdot e_2(t)$$

(Using state variable notation).

Representing the above equations in vector matrix form :-

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}_{3\times1} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & -1/L_2 \\ \frac{1}{C} & 1/C & 0 \end{bmatrix}_{3\times3} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}_{3\times1} + \begin{bmatrix} 1/L_1 & 0 \\ 0 & 1/L_2 \\ 0 & 0 \end{bmatrix}_{3\times2} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}_{2\times1}$$

$$\Rightarrow \boxed{\dot{X} = AX + BU} \rightarrow \text{state eq}^n.$$

The output equation is :→

$$y_1(t) = -R_1 i_1(t) = -R_1 i_1(t) + 0 \cdot i_2(t) + 0 \cdot v_c(t) + 0 \cdot e_1(t) + 0 \cdot e_2(t) \quad \text{—— (1)}$$

$$y_2(t) = -R_2 i_2(t) = 0 \cdot i_1(t) - R_2 i_2(t) + 0 \cdot v_c(t) + 0 \cdot e_1(t) + 0 \cdot e_2(t) \quad \text{—— (2)}$$

(06)

By writing the above equations in vector matrix form :-

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}_{2\times1} = \begin{bmatrix} -R_1 & 0 & 0 \\ 0 & -R_2 & 0 \end{bmatrix}_{2\times3} \begin{bmatrix} i_1(t) \\ i_2(t) \\ V_c(t) \end{bmatrix}_{3\times1} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}_{2\times1}$$

$\Rightarrow \boxed{Y = CX + DU} \longrightarrow$ output eqn.

So, from the above two equations of state Model :-

$$[A] = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & -1/L_2 \\ 1/C & 1/C & 0 \end{bmatrix}_{3\times3}, \quad [B] = \begin{bmatrix} 1/L_1 & 0 \\ 0 & 1/L_2 \\ 0 & 0 \end{bmatrix}_{3\times2} \quad \text{(Ans)}$$

$$[C] = \begin{bmatrix} -R_1 & 0 & 0 \\ 0 & -R_2 & 0 \end{bmatrix}_{2\times3}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$$

* STATE TRANSITION MATRIX :-

→ If 'A' is a constant matrix & input forces are zero, then $\boxed{\dot{X}(t) = AX(t)}$ is called as <u>homogeneous equation</u>.

→ If 'A' is a constant matrix & $U(t)$ is non-zero vector, then $\boxed{\dot{X}(t) = AX(t) + BU(t)}$ is called <u>non-homogeneous equation</u>.

→ The solution of the state equation (or) total response of the system is given by the combination of Zero Input Response & zero state Response.

→ zero Input response is called as Natural response, where as zero state response is called as forced response.

Solution of Non-homogeneous equation :-

Non-homogeneous eqn is given by $\dot{X}(t) = AX(t) + BU(t)$

$$\Rightarrow \dot{X}(t) - AX(t) = BU(t).$$

$$\Rightarrow e^{-At}\left[\dot{X}(t) - AX(t)\right] = e^{-At} \cdot BU(t) \quad \text{——— (1)}$$

But, $\frac{d}{dt}\left[e^{-At}X(t)\right] = e^{-At}\dot{X}(t) - e^{-At} \cdot AX(t)$.

So, equation-(1) becomes :-

$$\frac{d}{dt}\left[e^{-At}X(t)\right] = e^{-At}\cdot BU(t)$$

Assuming initial time as $t=0$ & integrating both sides from $t=0$ to t

$$e^{-At}X(t)\Big|_0^t = \int_0^t e^{-A\tau} BU(\tau)\,d\tau$$

$$\Rightarrow e^{-At}X(t) - X(0) = \int_0^t e^{-A\tau}\cdot BU(\tau)\,d\tau$$

$$\Rightarrow e^{At}\cdot e^{-At}X(t) - e^{At}\cdot X(0) = e^{At}\int_0^t e^{-A\tau} BU(\tau)\,d\tau$$

$$\Rightarrow \boxed{X(t) = e^{At}\cdot X(0) + \int_0^t e^{A(t-\tau)}\cdot BU(\tau)\,d\tau}$$

$\therefore$ The above solution contains zero input Response (ZIR) i.e, $e^{At}\cdot X(0)$ and zero state Response (ZSR) i.e, $\int_0^t e^{A(t-\tau)}\cdot BU(\tau)d\tau$.

→ $\boxed{e^{At} = \phi(t)}$ is called as <u>state transition matrix (STM)</u>.

→ Properties of STM are :-

(i) $\phi(0) = I$ = Identity matrix.

(ii) $\phi^{-1}(t) = \phi(-t)$

(iii) $\phi(t_1+t_2) = \phi(t_1)\cdot\phi(t_2)$

(iv) $[\phi(t)]^n = \phi(nt)$.

(v) $\phi(t_2-t_1)\cdot\phi(t_1-t_0) = \phi(t_2-t_0)$

(vi) $\phi(t)$ is a <u>non-singular matrix</u> for all finite values of 't'.

<u>Solution of state equation by Laplace Transform</u> :-

Non-homogeneous eq^n is $\dot{X}(t) = AX(t) + BU(t)$.

Taking Laplace transform ; $SX(s) - x(0) = AX(s) + BU(s)$

$$\Rightarrow SX(s) - AX(s) = x(0) + BU(s)$$

$$\Rightarrow [SI-A]X(s) = x(0) + BU(s)$$

$$\Rightarrow X(s) = \underbrace{[SI-A]^{-1}x(0)}_{ZIR} + \underbrace{[SI-A]^{-1}[B][U(s)]}_{ZSR}$$

Taking Laplace Inverse transform :-

$$x(t) = \mathcal{L}^{-1}\{[SI-A]^{-1}x(0)\} + \mathcal{L}^{-1}\{[SI-A]^{-1}[B][U(s)]\}.$$

$$\Rightarrow \boxed{x(t) = [\phi(t)][x(0)] + \mathcal{L}^{-1}\{[\phi(s)]\cdot[B]\cdot[U(s)]\}}.$$

↳ state transition equation.

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Putting $x(t)$ in output eqⁿ of state Model ⇒

$$y(t) = [C][\phi(t)][x(0)] + [C]\mathcal{L}^{-1}\{[\phi(s)][B][U(s)]\} + [D][U(t)].$$

$$\Rightarrow \boxed{\underbrace{y(t)}_{\text{Total Response}} = \underbrace{[C][\phi(t)][x(0)]}_{\text{ZIR (or) Natural Response}} + \mathcal{L}^{-1}\underbrace{\left[\{[C][\phi(s)][B]+[D]\}[U(s)]\right]}_{\text{ZSR (or) Forced Response}}}.$$

→ $\boxed{\phi(s) = [SI-A]^{-1}}$ is called <u>Resolvant Matrix</u>.

→ $\boxed{\mathcal{L}^{-1}[\phi(s)] = \phi(t) = e^{At}}$ = State Transition Matrix.

<u>Q.</u> Find state transition matrix for state model given by

$$A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$$

<u>Solⁿ</u>:-

$$[SI-A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}.$$

$$\therefore [SI-A]^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\Rightarrow \phi(s) = [SI-A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\therefore \phi(t) = e^{At} = \mathcal{L}^{-1}[\phi(s)] = \mathcal{L}^{-1}\begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{-1}{s+1} + \frac{1}{s+2} \\ \frac{2}{s+1} - \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\Rightarrow \phi(t) = \text{State Transition matrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}. \text{(Ans)}$$

* CONTROLLABILITY & OBSERVABILITY :-

→ A system is said to be completely state controllable if it is possible to transfer the internal state of the system from any initial state to any other final state in a finite time interval, by an external input or control vector.

→ In some cases, completely output controllable is taken for the design aspect.

Kalman's Test for Controllability :-

→ Consider n^{th} order multi input LTI system represented by its state eq^n $\dot{X}(t) = AX(t) + BU(t)$, is completely state controllable iff the rank of the composite matrix 'Q_c' is 'n', where 'Q_c' is given by

$$\boxed{Q_c = [B : AB : A^2B : \ldots\ldots : A^{n-1}B]}$$

Where, n = No of state variables.

$B, AB, A^2B, \ldots$ = various columns.

Note :-

* The matrix has a rank 'r' means the determinant of order $r \times r$ of the matrix has non-zero value & any determinant having order $(r+1)$ or more than that has zero value.

Kalman's Test for observability :-

→ A system is said to be completely observable, if every state $X(t_0)$ can be completely identified by measurements of the outputs $Y(t)$ over a finite interval of time $(t_0 \le t \le t_1)$.

→ Consider n^{th} order multi input LTI system with m dimensional output vector

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t) .$$

The system is completely observable iff the rank of the composite matrix 'Q_o' is 'n', where 'Q_o' is given by

$$\boxed{Q_o = [C^T : A^TC^T : \ldots\ldots : (A^T)^{n-1}C^T]}$$

Where, n = No of state variables.

$C^T, A^TC^T, \ldots$ = are various rows.

or $Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Q : The state Model of a system is given by,

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

check controllability & observability.

Sol^n **Controllability** :-

No of state variables, $n = 2$. ($\because$ as order of 'A' is 2×2).

$\therefore Q_c = [B \vdots AB]$.

$$\therefore AB = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\text{So, } Q_c = \begin{bmatrix} 0 & \vdots & 1 \\ 1 & \vdots & -3 \end{bmatrix} \Rightarrow Q_c = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$\therefore |Q_c| = -1 \neq 0 \Rightarrow$ Rank of 'Q_c' is '2', which is also equals to 'n' state variables.

$\therefore$ So the system is state controllable. (Ans)

Observability :-

$$Q_o = [C^T \vdots A^T C^T] \text{ or } Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$\therefore CA = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \end{bmatrix}.$$

$$\therefore Q_o = \begin{bmatrix} 1 & 1 \\ \cdots & \cdots \\ 2 & -2 \end{bmatrix} \Rightarrow Q_o = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$\therefore |Q_o| = -4 \neq 0 \Rightarrow$ Rank of 'Q_o' is '2', which is same as no of state variables.

$\therefore$ So the system is state observable. (Ans)

Q: Evaluate the Controllability of the system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$