Digital Communication Techniques

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Topic:

Sampling

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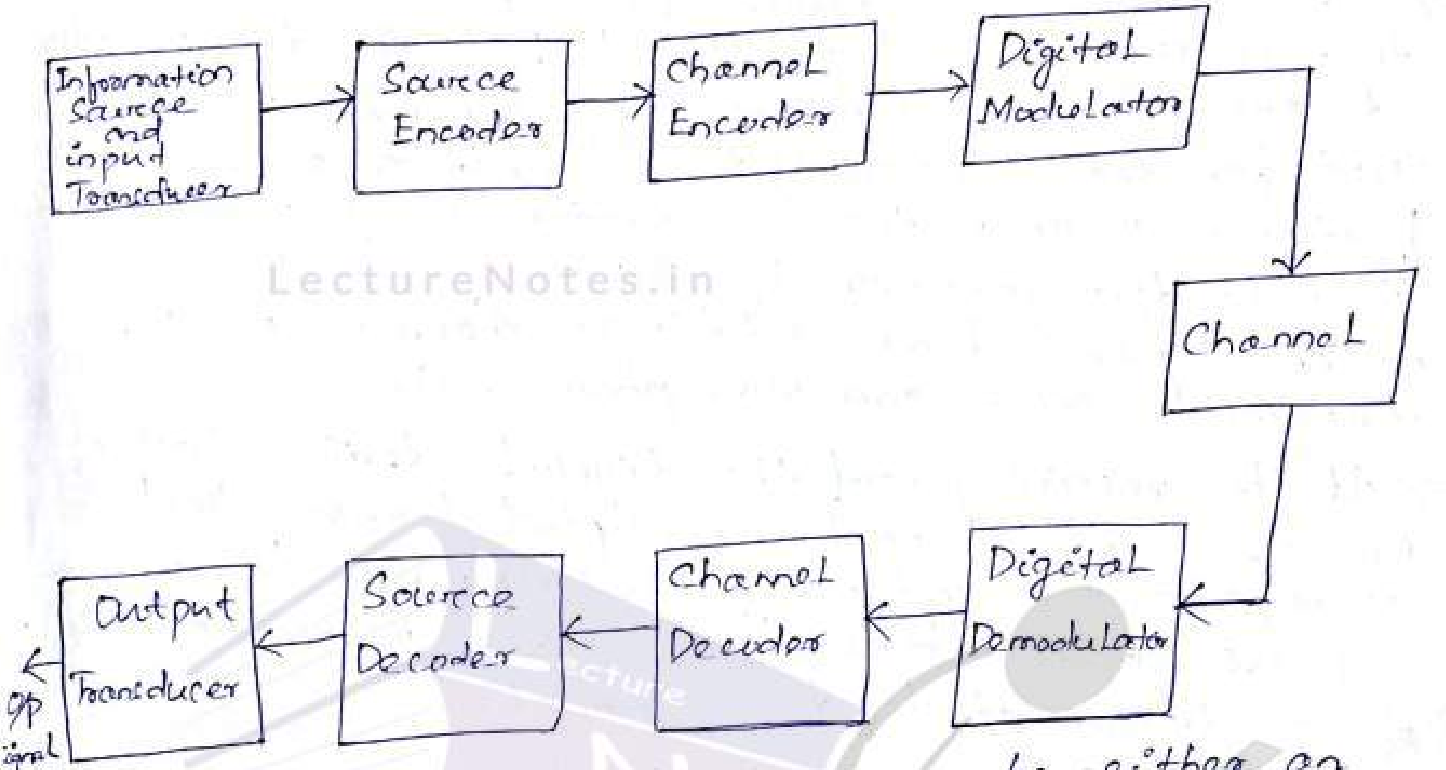
INTRODUCTION

ELEMENTS OF AN ELECTRICAL COMMUNICATION SYSTEM



→ The information source output may be either an analog signal, such as an audio or video signal, or a digital signal, such as the output of a computer.

→ The source encoder encodes each of the messages produced by the information source into a sequence of binary digits.

→ The sequence of binary digits from the source encoder, which we call the information sequence, is passed to the channel encoder.

→ The purpose of the channel encoder is to introduce, in a controlled manner, some redundancy in the binary information sequence that can be used at the receiver to overcome the effects of noise and interference encountered in the channel.

→ The binary sequence at the output of the channel encoder is passed to the digital modulator. The digital modulator may simply map the binary digit '0' into a waveform $S_0(t)$ and binary digit '1' into a waveform $S_1(t)$. In this manner each bit from the channel encoder is transmitted separately. We will call this as binary Modulation.

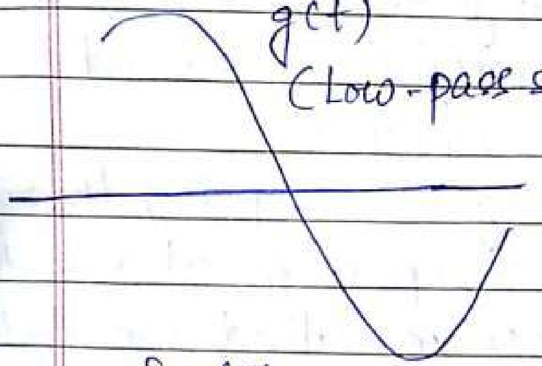
→ The communication channel is the physical medium that is used to send the signal from the transmitter to the receiver. Whatever the physical medium used for transmission of the information, the essential feature is that the transmitted signal is corrupted in a random manner by a variety of possible mechanisms, such as additive thermal noise, man made noise and atmospheric noise.

→ At the receiving end the digital demodulator processes the channel-corrupted transmitted waveform and reduces the waveforms to a sequence of numbers that represent estimates of the transmitted data symbols.

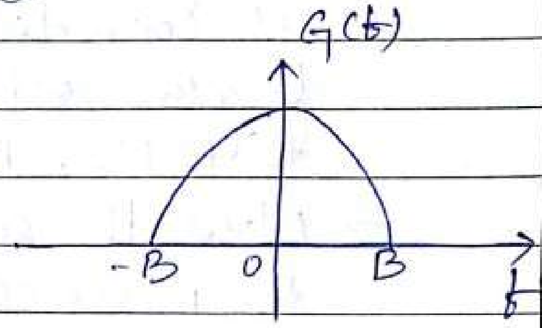
→ The average probability of a bit-error at the output of the decoder is a measure of performance of the demodulator-decoder combination.

→ As a final step, when an analog output is desired, the source decoder accepts the output sequence from the channel decoder, from the knowledge of the source encoding method used attempts to reconstruct the original signal from the source.

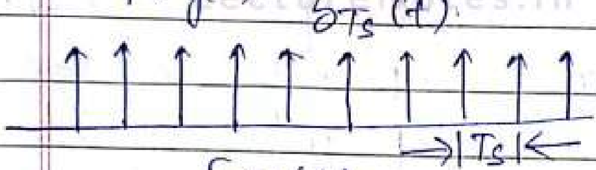
SAMPLING THEOREM :-



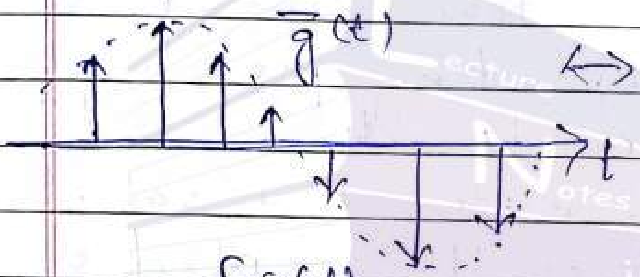
Fig(a).



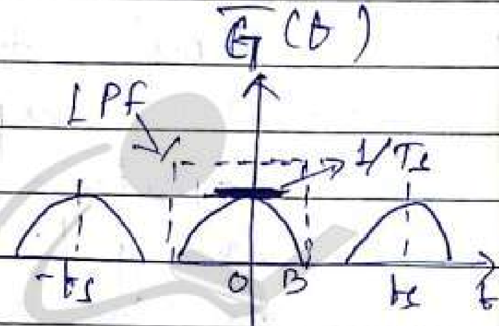
Fig(b).



Fig(c).



Fig(d)



Fig(e).

→ Let us consider a Low-pass signal whose spectrum is band-limited to "B" Hz i.e. $G(f) = 0$, for $|f| > B$.

⇒ This signal can be reconstructed exactly (without any error) at the receiver side from its discrete time samples taken uniformly at a rate of "R" samples per sec, and the condition is that $R > 2B$.

⇒ In other words, the minimum sampling frequency for perfect signal recovery is "$f_s = 2B$" Hz.

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PROOF :-

→ To prove the sampling theorem, let us consider a signal $g(t)$ shown in fig (a), whose spectrum is band-limited to "B" Hz, shown in fig.(b).

→ Now the signal $g(t)$ is going through the process of sampling at a rate of "f_s" Hz which means that we take f_s uniform samples per second.

→ This uniform sampling can be obtained by multiplying $g(t)$ by an impulse train $\delta_{T_s}(t)$, shown in fig (c).

→ This impulse train consists of unit impulses repeating periodically every T_s seconds where $T_s = 1/f_s$, known as sampling interval.

→ This results in the sampled signal $\bar{g}(t)$ shown in figure (d).

→ The sampled signal $\bar{g}(t)$ can be expressed as:-

$$\bar{g}(t) = g(t)\delta_{T_s}(t) = \sum_n g(nT_s)\delta(t-nT_s) \quad \text{——— (1).}$$

→ Since the impulse train $\delta_{T_s}(t)$ is a periodic signal of period T_s, it can be expressed as an exponential fourier series as:-

$$\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} e^{jn\omega_s t} \quad \text{——— (2)}$$

$$\text{where } \omega_s = \frac{2\pi}{T_s} = 2\pi f_s.$$

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→ Therefore, $\bar{g}(t)$ can be expressed as

$$\bar{g}(t) = g(t) \cdot \delta_{T_s}(t)$$
$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} g(t) e^{jn2\pi f_s t} \quad \text{——— (3).}$$

→ In order to find out $\bar{G}(f)$, we take the Fouriere Transform of $\bar{g}(t)$ as in equation (3).

→ Based on the frequency-shifting property, the Fourier-Transform of the nth term is shifted by nf_s.

→ Therefore,

$$\bar{G}(f) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} G(f - nf_s) \quad \text{——— (4).}$$

→ This means that the spectrum $\bar{G}(f)$ consists of $G(f)$, scaled by a constant $1/T_s$, repeating periodically with period $f_s = 1/T_s$ Hz, shown in fig(e).

→ After uniform sampling, a vital question becomes: Can $g(t)$ be reconstructed from $\bar{g}(t)$ without any Loss or distortion?

→ If we are able to reconstruct $g(t)$ from $\bar{g}(t)$, equivalently in the frequency domain, we should be able to recover $G(f)$ from $\bar{G}(f)$.

→ The perfect recovery is possible if there is no overlap among the replicas of spectrum in $\bar{G}(f)$.

→ Fig(e) clearly shows that this requir

$$f_s \geq 2B \text{ ——— } (5).$$

→ Therefore $T_s \leq 1/2B$ — (6).

→ Hence, as long as the sampling frequency f_s is greater than or equal to the twice of signal bandwidth B Hz, $\bar{G}(f)$ will consist of non overlapping repetitions of $G(f)$.

→ When this is true, fig(e) shows that $g(t)$ can be recovered from its samples $\bar{g}(t)$ by passing the sampled signal through an ideal low-pass filter of bandwidth "B" Hz.

→ The minimum sampling rate $f_s = 2B$ required to recover $g(t)$ from its samples $\bar{g}(t)$ is called as the NYQUIST RATE for $g(t)$, and the corresponding sampling interval $T_s = 1/2B$ is called the NYQUIST INTERVAL for the low-pass signal $g(t)$.

SIGNAL RECONSTRUCTION FROM UNIFORM SAMPLES

→ The process of reconstruction of continuous time signal $g(t)$ from its uniform samples is known as INTERPOLATION.

→ The sampling theorem is already

proved from where we are concluded that if a signal $g(t)$ band-limited to B Hz can be reconstructed or interpolated exactly from its uniform samples.

→ This means not only that uniform sampling at above the Nyquist rate preserves all the signal information but also that simply passing the sampled signal through an ideal Low-pass filter of bandwidth 'B' Hz, will reconstruct the original message.

→ In order to recover $g(t)$, the sampled signal $\bar{g}(t) = \sum_n g(nT_s)\,\delta(t-nT_s)$ must be passed through an ideal Low-pass filter of bandwidth 'B' Hz and gain T_s.

→ Such an ideal filter has the transfer function given as

$$H(f) = T_s\,\Pi\left(\frac{\omega}{4\pi B}\right) = T_s\,\Pi\left(\frac{f}{2B}\right) \quad \text{—— (7)}.$$

IDEAL RECONSTRUCTION:-

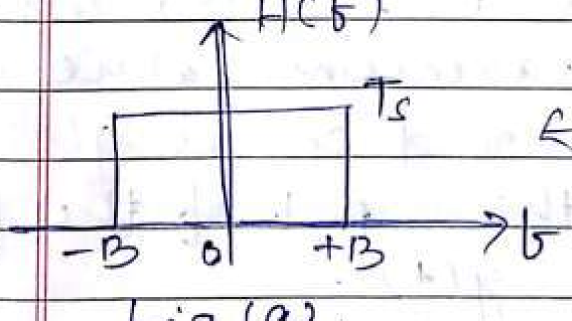

fig (a).

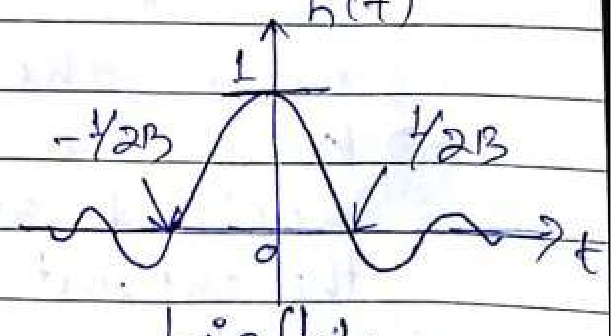

fig (b).

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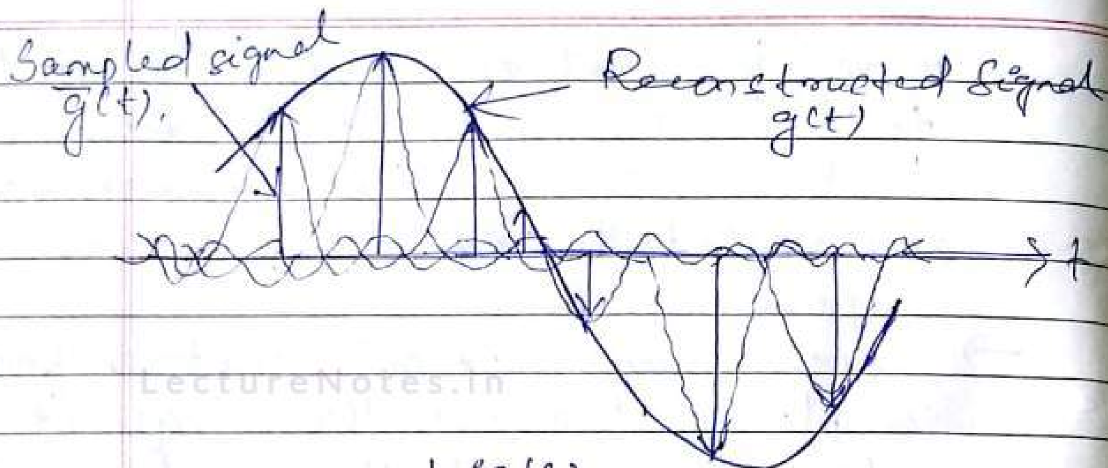


fig(c).

→ To recover the analog signal from its uniform samples, the ideal reconstruction or interpolation filter transfer function found in eq^n (7) is shown in fig(a) above.

→ The impulse response of this filter i.e. the inverse Fourier Transform of H(f), is

$h(t) = 2BT_s \operatorname{sinc}(2\pi Bt)$ —— (8).

→ Assuming the use of Nyquist sampling rate, i.e., $2BT_s = 1$, then

$h(t) = \operatorname{sinc}(2\pi Bt)$ — (9).

→ This h(t) is shown in fig(b).

→ Observe the very interesting fact that $h(t) = 0$ at all Nyquist sampling instants i.e. at $t = \pm n/2B$, and it attains the maximum value at $t = 0$.

→ When the sampled signal $\bar{g}(t)$ is applied at the input of this filter, the output is $g(t)$.

→ Each sample in $\bar{g}(t)$, being an

impulse, generates a sinc pulse of height equal to the strength of the impulse, as shown in fig(c) above.

→ Addition of the sinc pulses generated by all the samples results in $g(t)$.

→ Let us consider a kth sample of the input $\bar{g}(t)$ is the impulse $g(kT_s)\,\delta(t-kT_s)$.

→ Hence, the Low-pass filter output to $\bar{g}(t)$, which is $g(t)$, can now be expressed as a sum

$$g(t) = \sum_k g(kT_s)\, h(t-kT_s)$$

$$\Rightarrow g(t) = \sum_k g(kT_s)\,\mathrm{sinc}\left[2\pi B(t-kT_s)\right] \quad \text{—— (10.a)}$$

$$\Rightarrow g(t) = \sum_k g(kT_s)\,\mathrm{sinc}(2\pi Bt - k\pi) \quad \text{—— (10.b)}$$

→ Eq (10) is the INTERPOLATION FORMULA.

PROBLEMS BASED ON SAMPLING:-

Q(1). An analog signal is expressed by the equation $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$.

Calculate the Nyquist rate for this signal?

Ans:- The given signal is expressed as

$$x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t.$$

Here $\omega_1 = 50\pi \Rightarrow f_1 = 25\,Hz$.

$\Rightarrow \omega_2 = 300\pi \Rightarrow f_2 = 150\,Hz$.

$\Rightarrow \omega_3 = 100\pi \Rightarrow f_3 = 50\,Hz$.

∴ Therefore, the maximum frequency present in $x(t)$ is

$f_2 = 150\,Hz$.

∴ Nyquist rate, $f_s = 2f_2 = 300\,Hz$.

Q.(2) Find the Nyquist rate and the Nyquist interval for the signal given $x(t) = \frac{1}{2\pi}\cos(4000\pi t).\cos(1000\pi t)$?

Ans:- $x(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t)$

$= \frac{1}{4\pi}\left[2\cos(4000\pi t).\cos(1000\pi t)\right]$

$= \frac{1}{4\pi}\left[\cos 5000\pi t + \cos 3000\pi t\right]$

Here $\omega_1 = 5000\pi \Rightarrow f_1 = 2500\,Hz$.

and $\omega_2 = 3000\pi \Rightarrow f_2 = 1500\,Hz$.

∴ Therefore, the maximum frequency present in $x(t)$ is $f_1 = 2500\,Hz$.

∴ Nyquist rate, $f_s = 2f_1 = 5000\,Hz$

$= 5\,KHz$.

and the Nyquist Interval is

$T_s = \frac{1}{2f_1} = \frac{1}{2\times 2500} = \frac{1}{5000}$

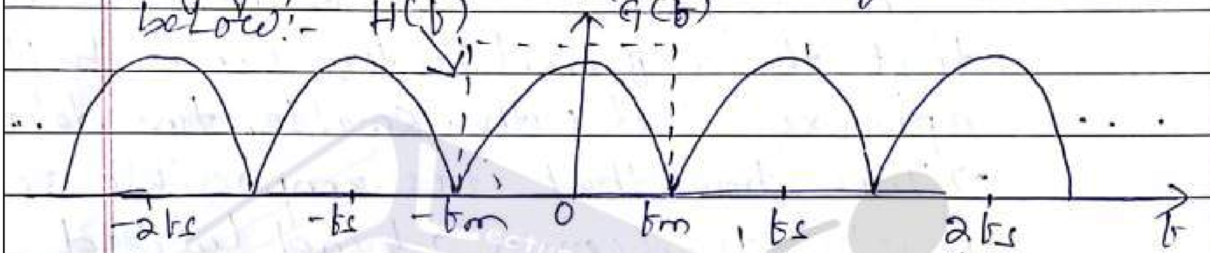
$\Rightarrow T_s = 0.2\times 10^{-3}\,secs = 0.2\,m\,sec$.

SAMPLING AND ANALOG-TO-DIGITAL CONVERSION.

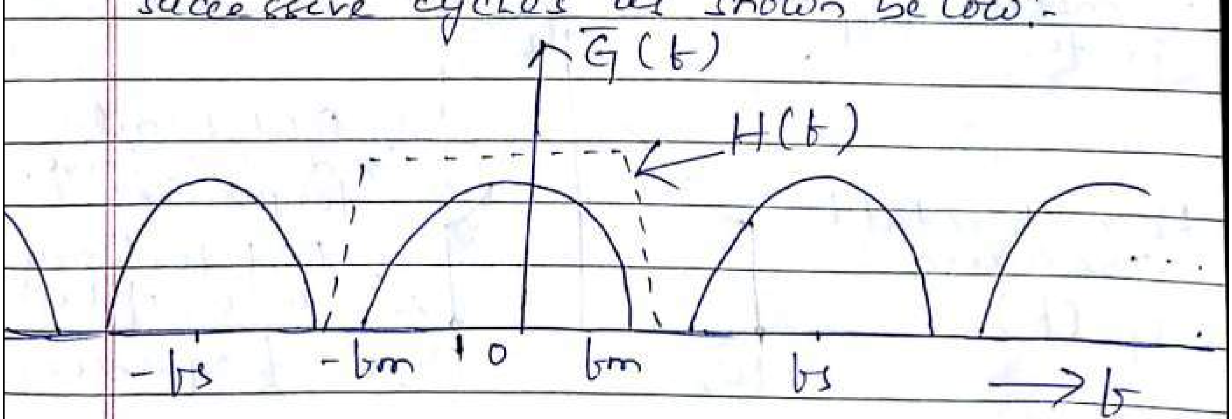
PRACTICAL ISSUES IN SIGNAL SAMPLING AND RECONSTRUCTION:-

(a). Realizability of Reconstruction Filters

→ If a signal is sampled at the Nyquist rate $f_s = 2f_m$ Hz, then the spectrum $\bar{G}(f)$ consists of repetitions of $G(f)$ without any gap between successive cycles as shown below:-



In order to recover $g(t)$ from $\bar{g}(t)$, we need to pass the sampled signal $\bar{g}(t)$ through an ideal low-pass filter, and such a filter is unrealizable in practice. A practical solution to this problem is to sample the signal at a rate higher than the Nyquist rate i.e. $f_s > 2f_m$ consisting of repetitions of $G(f)$ with a finite band gap between successive cycles as shown below:-



→ We can now recover $g(t)$ from $\bar{g}(t)$ by using a Low-pass filter with a gradual cutoff characteristics as shown by dotted area.

→ According to the Paley-Wiener criterion, it is impossible to realize even this filter also.

→ The only advantage in this case is that the required filter can be better approximated with a smaller time delay.

→ This shows that it is impossible in practice to recover a band-limited signal $g(t)$ exactly from its samples, even if the sampling rate is higher than the Nyquist rate.

→ However, as the sampling rate increases the recovered signal approaches the desired signal more closely.

(b) ALIASING PROBLEM:-

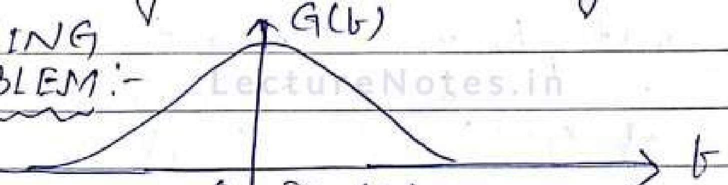


Fig.(a).

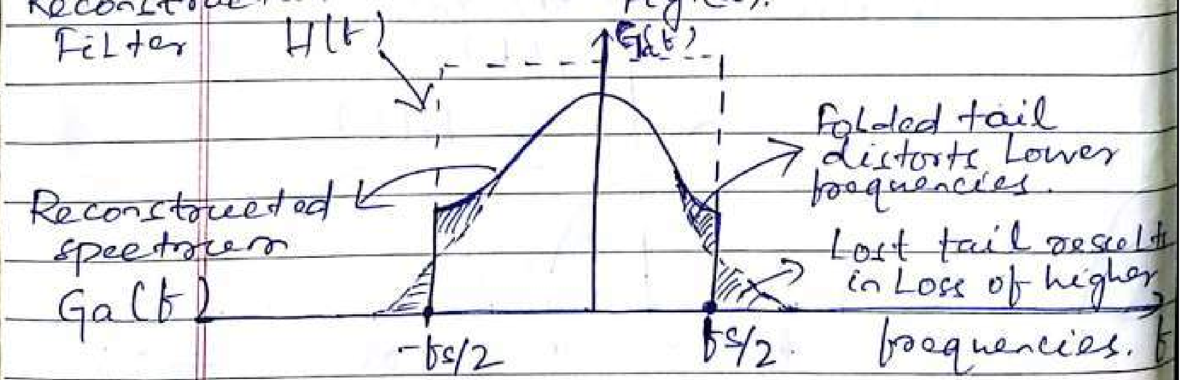


Fig.(b)

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→ The sampling theorem was proved on the assumption that the signal $g(t)$ is band-limited.

→ All practical signals are time-limited, that is they are of finite duration or width.

→ A time-limited signal cannot be band-limited, and vice versa.

→ Clearly, all practical signals, which are necessarily time-limited, are non-band-limited, as shown in fig (a) i.e. they have infinite bandwidth, and the spectrum $\bar{G}(f)$ consists of overlapping cycles of $G(f)$ repeating every f_s Hz.

→ Because of the infinite bandwidth in this case, the spectral overlap is unavoidable, regardless of the sampling rate.

→ If the sampled signal is passed through an ideal low-pass filter of cut-off frequency $f_s/2$ Hz, then the output is not $G(f)$ but rather it is $G_a(f)$ as shown in fig (b), which is a version of $G(f)$ distorted as a result of two separate causes:

(a). The loss of the tail of $G(f)$ beyond $|f| > f_s/2$ Hz.

(b). The reappearance of this tail folded back onto the spectrum.

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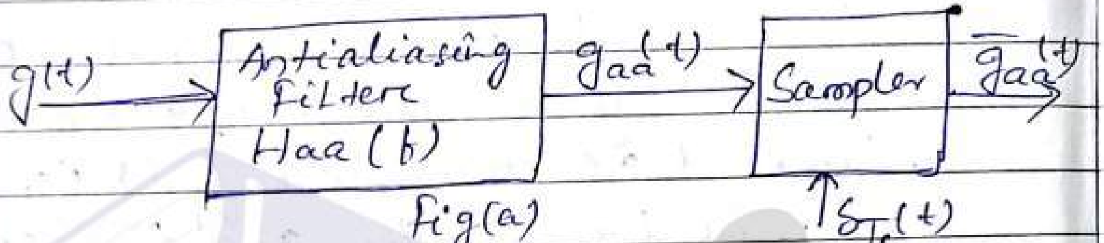
→ The spectra cross at frequency $f_s/2$ Hz, is called the "Folding frequency".
→ The tail inversion or folded back is known as spectral folding or aliasing.

(C) THE ANTIALIASING FILTER :-



Fig(a)

→ In order to overcome the aliasing problem, we should eliminate or suppress the potential defectors, which are all the frequency components beyond the folding frequency $f_s/2$ Hz.

→ Such suppression of higher frequencies can be accomplished by an ideal Low-pass filter of cutoff $f_s/2$ Hz, which is known as Antialiasing filter.
→ Antialiasing filtering is performed before sampling.

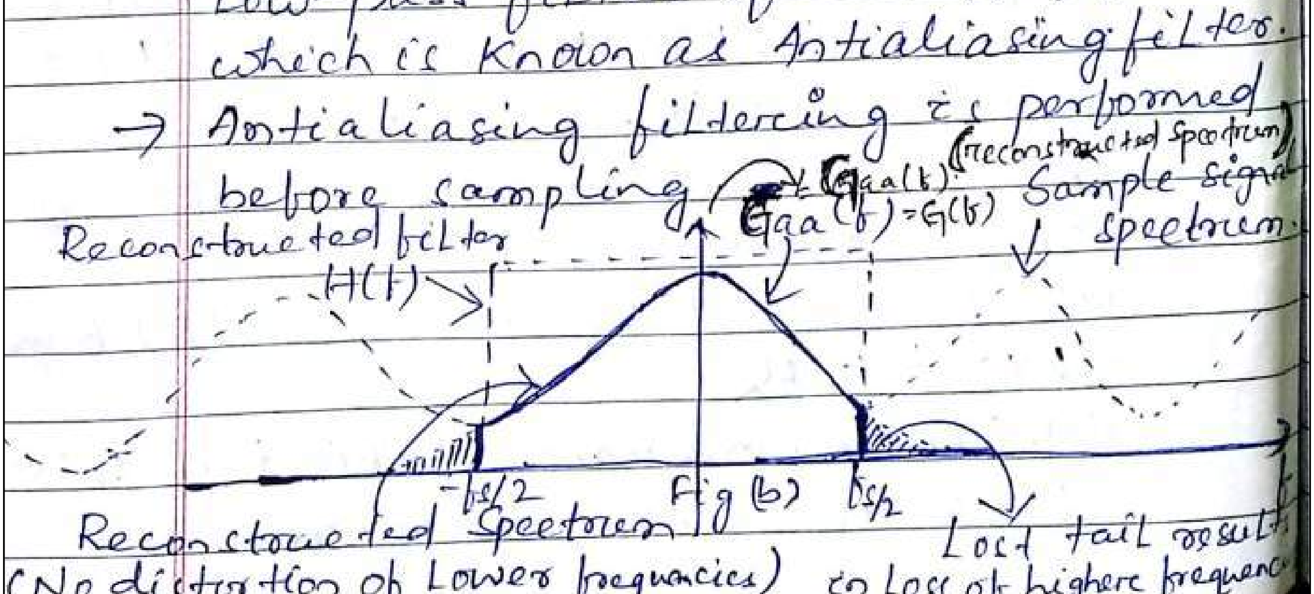


Fig (b)

→ Fig(b) shows the sampled signal spectrum and the reconstructed signal $G_{aa}(f)$ when the antialiasing scheme is used.

→ An antialiasing filter essentially band-limits the signal $g(t)$ to $f_s/2$ Hz.

→ This way, we lose only the components beyond the folding frequency $f_s/2$ Hz.

→ These suppressed components now cannot reappear, corrupting the components of frequencies below the folding frequency.

→ Clearly, use of an antialiasing filter results in the reconstructed signal spectrum $\tilde{G}_{aa}(f) = G(f)$ for $|f| < f_s/2$ Hz.

→ Thus, although we lost the spectrum beyond $f_s/2$ Hz, the spectrum for all the frequencies below $f_s/2$ remains intact or undistorted.

Maximum Information Rate: Two Pieces of Information per Second per Hertz

→ A knowledge of the maximum rate at which information can be transmitted over a channel of bandwidth 'B' Hz is of fundamental importance in digital communication.

→ A maximum of 2B independent pieces of information per second can be

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transmitted, error free, over a noiseless channel of bandwidth B Hz.

→ We need to show that any sequence of independent data at the rate 2B Hz can come from uniform samples of a Low-pass signal with bandwidth B Hz.

→ Suppose a sequence of independent data samples is denoted as $\{g_n\}$. Its rate is 2B samples per sec.

→ Then there always exists a signal $g(t)$ such that $g_n = g(nT_s)$, $T_s = 1/2B$. —— (1)

→ Because of aliasing the ideal sampled signal is expressed as

$$\bar{g}(t) = \sum_n g(nT_s)\,\delta(t - nT_s)$$

$$= \sum_n g_a(nT_s)\,\delta(t - nT_s) \quad \text{—— (2)}$$

where $g_a(t)$ is the aliased Low-pass signal whose samples $g_a(nT_s)$ equal to the samples of $g(nT_s)$.

→ Hence, $g_n = g(nT_s) = g_a(nT_s)$ —— (3)

→ Also, from the sampling theorem, a Low-pass signal $g_a(t)$ with bandwidth 'B' can be reconstructed from its uniform samples as

$$g_a(t) = \sum_n g_n \operatorname{sinc}(2\pi B t - n\pi) \quad \text{—— (4)}$$

→ Assuming no noise, this signal can be transmitted over a distortionless channel of bandwidth 'B' error free.

→ At the receiver, the data sequence $\{g_n\}$ can be recovered from the Nyquist samples of the distortionless channel output $g_a(t)$ as the desired information data.

Nonideal Practical Sampling Analysis

→ Thus far, we have mainly focused on ideal uniform sampling that can use an ideal impulse sampling pulse train.

→ In practice, no physical device can carry out such a task.

→ Practical samplers take each signal sample over a short time interval T_p around $t = KT_s$.

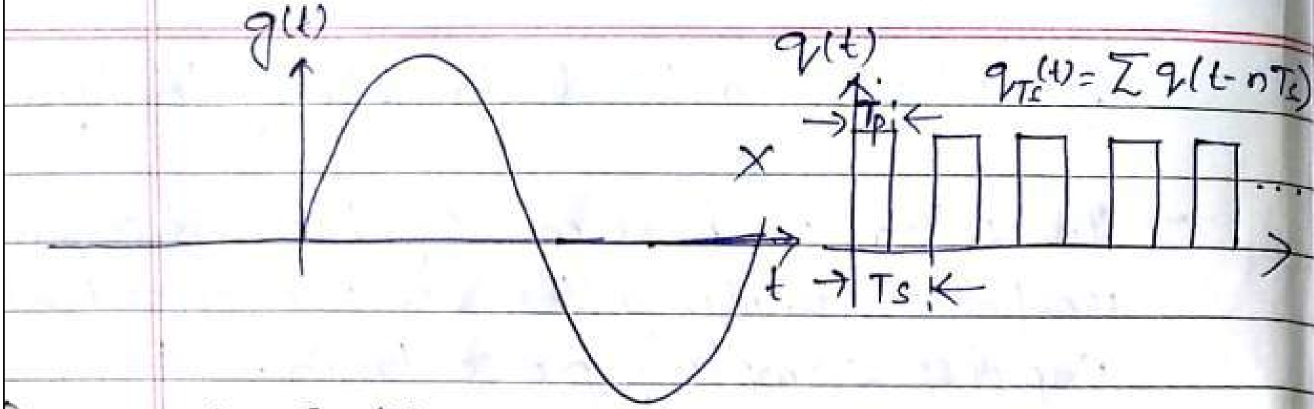
→ In other words, every T_s seconds the sampling device takes a short snapshot of duration T_p from the signal $g(t)$ being sampled.

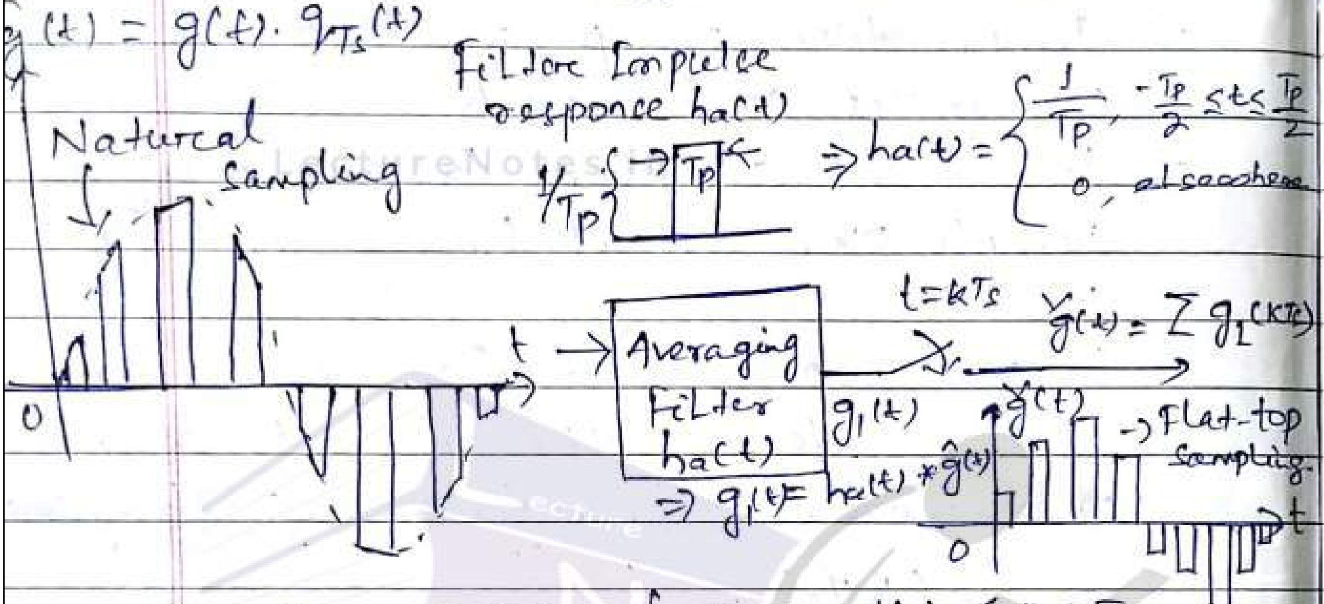
→ $$g_1(KT_s) = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} g(KT_s + t)\,dt \quad \text{— (1)}$$

The above equation shows the practical sampler would generate a sample value at $t = KT_s$ by averaging the values of signal $g(t)$ over the window T_p.

→ Depending on the actual device, this averaging may be weighted by a device-dependent averaging function $q(t)$ such that

$$g_1(KT_s) = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} q(t)\, g(KT_s + t)\,dt \quad \text{— (2)}$$





where $q_1(t) = \begin{cases} 1, & |t| \leq 0.5T_p \\ 0, & |t| > 0.5T_p \end{cases}$ —— (3)

→ Thus practical samplers in fact generate sampled signal of the form

$$\check{g}(t) = \sum g_1(kT_s)\,\delta(t - kT_s) \quad \text{—— (4)}$$

Some Applications of Sampling Theorem

→ The sampling theorem is very important in signal analysis, processing, and transmission because it allows us to replace a continuous time signal by a discrete sequence of numbers.

→ This leads directly into the area of digital filtering.

→ This opens doors to many new techniques of communicating continuous time signals by pulse trains.

→ We may vary the amplitudes, widths, or positions of these pulses, which leads to pulse amplitude modulation (PAM), pulse width modulation (PWM), and pulse position modulation (PPM) techniques.

→ The most important form of pulse modulation today is pulse code modulation (PCM).

→ One advantage of using pulse modulation is that it permits the simultaneous transmission of several signals on a time-sharing basis - Time Division Multiplexing (TDM).

→ A pulse modulated signal occupies only a part of the channel time, we can transmit several pulse modulated signals on the same channel by interweaving them.

→ Another method of transmitting several baseband signals simultaneously is Frequency Division Multiplexing (FDM), in which various signals are multiplexed by sharing the channel bandwidth.

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Digital Communication Techniques

Topic:

Digital Representation Of Analog Signal And Non-uniform Quantization

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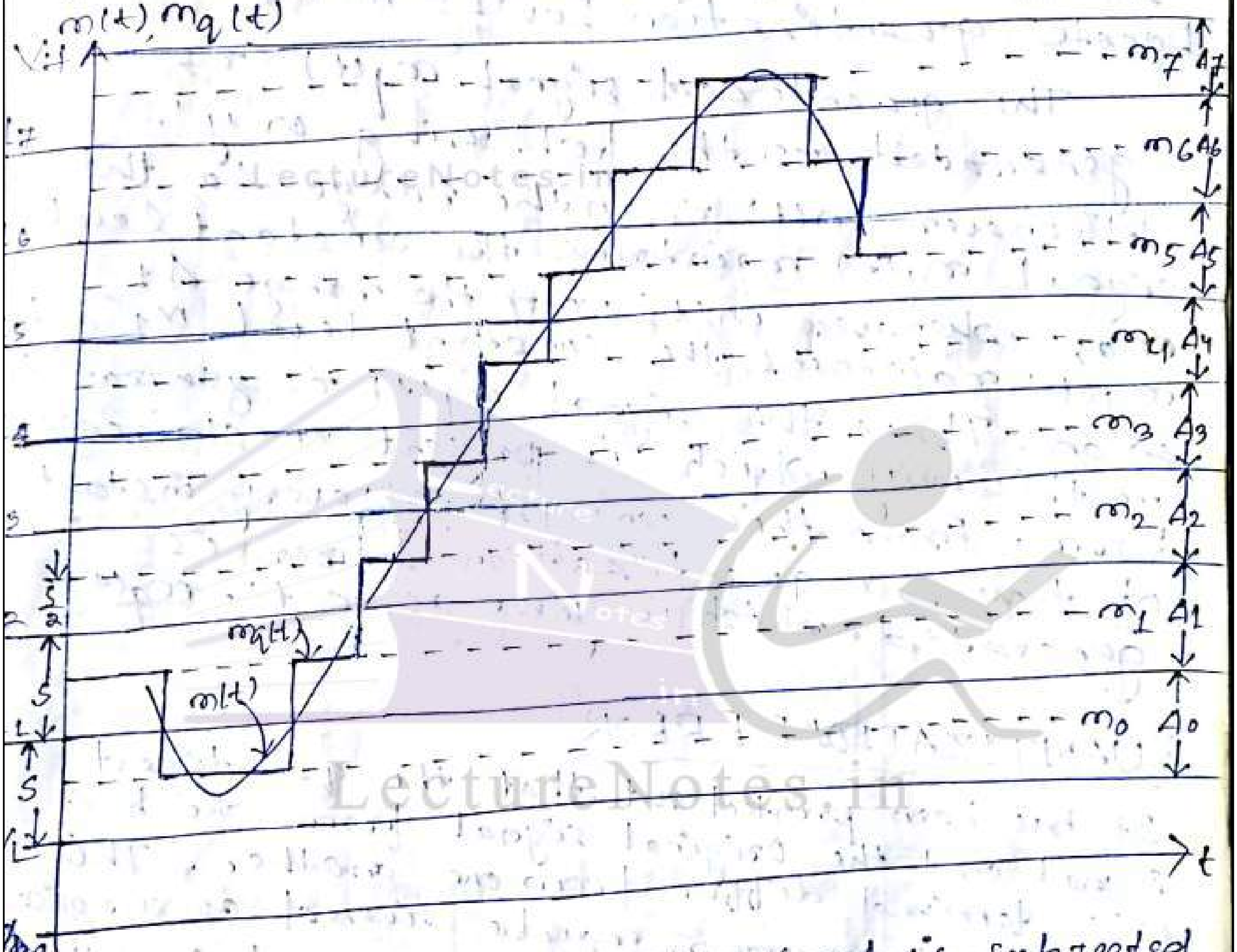
DIGITAL REPRESENTATION OF ANALOG SIGNALS

In order to make the digital representation of analog signals the first step is sampling the time axis i.e. discrete-time representation of analog signals, then the next step is to discretize the amplitude axis and then represent that signal in terms of binary digits to complete the process.

QUANTIZATION OF SIGNALS



We now describe how the signal is subjected to the operation of quantization. When quantizing a signal m(t) we create a new signal mq(t), which is approximation to the m(t). However, the quantized signal mq(t) has the greater merit that it is separable from the additive noise.

The operation of the quantization is described above. Here, we are taking a signal whose peak to peak is confined in the range from V_L to V_H. We have divided this total range in equal intervals, each of size 'S'. These Levels are known as transition Levels. Let there are 'M' no. of transition levels, then

$$\text{step size } 'S' = \frac{V_H - V_L}{M} \quad \text{——— (1)}.$$

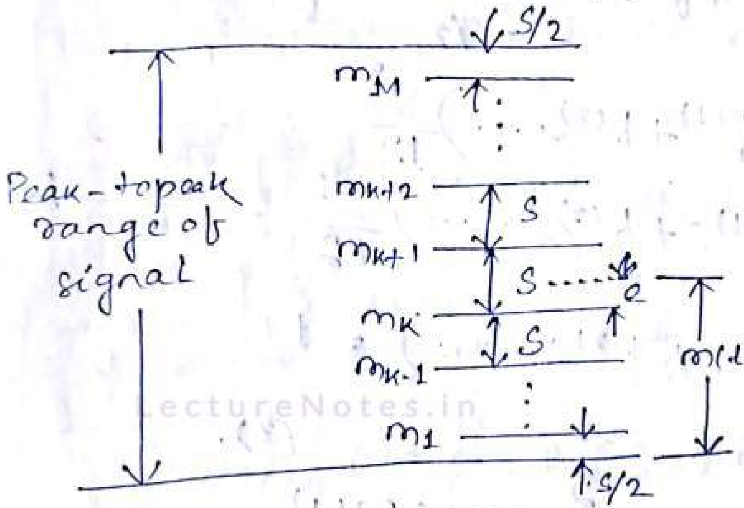
In the center of each of these steps, we Locate quantization Levels $m_0, m_1, \ldots m_7$.

The quantized signal $m_q(t)$ is generated in the following way:- Whenever $m(t)$ is in the range A_0, the signal $m_q(t)$ maintains the constant level m_0, whenever $m(t)$ is in the range A_1, $m_q(t)$ maintains the constant level m_1 and so on. Thus the signal $m_q(t)$ is generated in this way which is present in staircase form. from this concept at every instant of time, a quantization error is generated. which is given as $m(t) - m_q(t)$.

QUANTIZATION ERROR

It has been pointed out that the quantized signal and the original signal from which it was derived differ from one another. This difference or error may be viewed as a noise due to quantization process and is called quantization error. We now calculate the mean-square quantization error $\overline{e^2}$, where 'e' is the difference between the original and quantized signal voltages.

Let us divide total peak-to-peak range of the message signal $m(t)$ into M equal voltage intervals, $m_1, m_2, \ldots m_M$. It is shown in the figure. According to the figure, $m(t)$ happens to be closest to the level m_K, the quantizer output will be m_K. The error is $e = m(t) - m_K$.



Let $f(m)\,dm$ be the probability that $m(t)$ lies in the voltage range $m - S/2$ to $m + S/2$. Then the mean-square quantization error is

$$\overline{e^2} = \int_{m_1 - S/2}^{m_1 + S/2} f(m)(m - m_1)^2\,dm + \int_{m_2 - S/2}^{m_2 + S/2} f(m)(m - m_2)^2\,dm + \dots \quad (1).$$

If we take more and more quantization levels, then the step-size 'S' is small in comparison with the peak-to-peak range of the message signal. In this case it is certainly reasonable to make the approximation that $f(m)$ is constant within each quantization range. Then in eq^n (1), in 1st term we set $f(m) = f^{(1)}$, a constant. In the 2nd term, $f(m) = f^{(2)}$, etc. We may now remove $f^{(1)}, f^{(2)}$, etc, from inside the integral sign. If make the substitution

$x \equiv m - m_K$, then

eq^n (1) becomes

$$\overline{e^2} = \left(f^{(1)} + f^{(2)} + \dots\right)\int_{-S/2}^{S/2} x^2\,dx$$

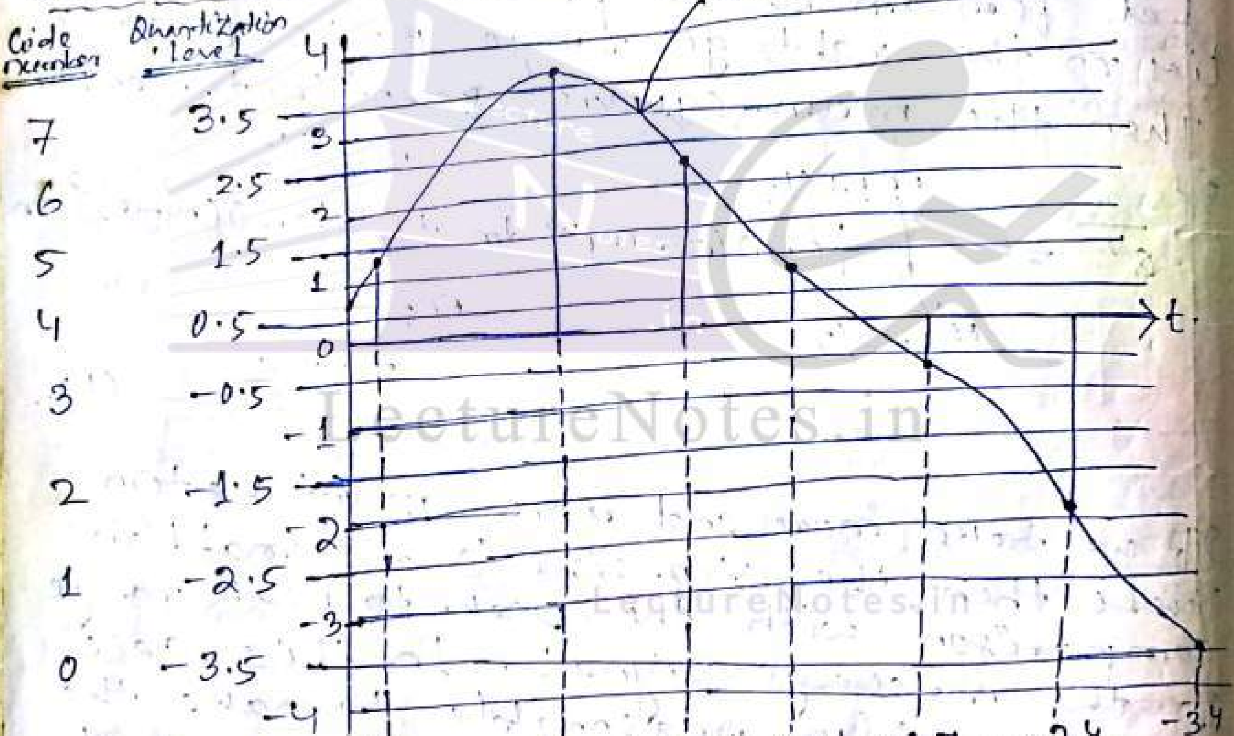
$$= \left(f^{(1)} + f^{(2)} + \dots\right)\frac{S^3}{12}$$

$$= \left(f^{(1)}S + f^{(2)}S + \dots\right)\frac{S^2}{12} \qquad (2)$$

But $\left(f^{(1)}S + f^{(2)}S + \dots\right) = 1$

$\therefore$ so $\overline{e^2} = \dfrac{S^2}{12}$ ——— (3).

PULSE CODE MODULATION (PCM)



Sample value →	1.3	3.6	2.3	0.7	-0.7	-2.4	-3.4
Nearest quantization level →	1.5	3.5	2.5	0.5	-0.5	-2.5	-3.5
code number →	5	7	6	4	3	1	0
Binary Representation →	101	111	110	100	011	001	000

DIGITAL REPRESENTATION OF ANALOG SIGNALS :-

QUANTIZATION OF SIGNALS :-

→ The process of discretization of the given analog signal in the amplitude axis, is known as Quantization.

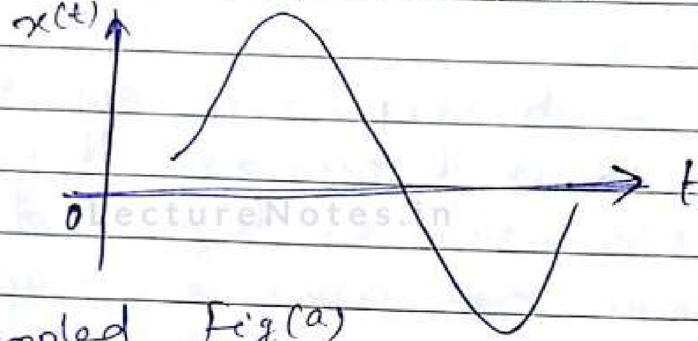


Fig (a)

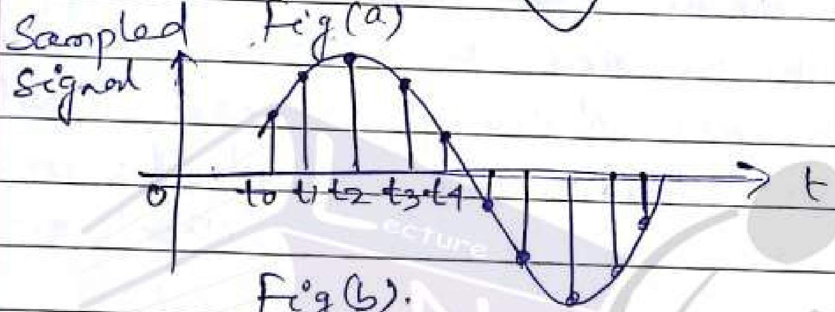


Fig (b).

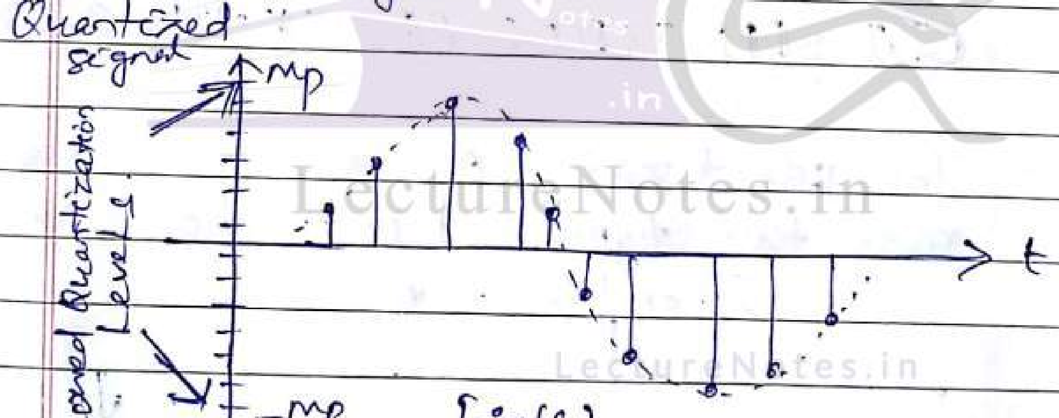


fig (c).

→ Let us consider an analog signal as shown in fig (a).

→ Fig (b) shows a representation of the analog signal in terms of its samples.

→ As shown in fig (C), amplitude of the signal $x(t)$ lie in the range "$(-M_p, M_p)$" which is partitioned into "L" intervals, each of magnitude $\Delta v = S = 2m_p/L$, known as stepsize.

→ Now, each sample is approximated or rounded off to the nearest quantization level.

→ Since each sample is now approximated to one of the 'L' numbers, therefore the information is digitized.

→ The quantized signal is an approximation of the original one. We can improve the accuracy of the quantized signal to any desired degree simply by increasing the number of quantization levels 'L'.

CLASSIFICATION OF QUANTIZATION

→ The quantization process is classified into two types and it is shown below:-

- Quantization
  - Uniform Quantization
    - Mid tread type
    - Mid-rise type
  - Nonuniform Quantization

→ This classification is based on the step-size as defined earlier.

(i) **Uniform Quantizer**:- A uniform Quantizer is that type of quantizer

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in which 'stepsize' remains same throughout the input range.

(ii) Non-uniform Quantizer:- A non-uniform quantizer is that type of quantizer in which the 'step size' varies according to the input signal values.

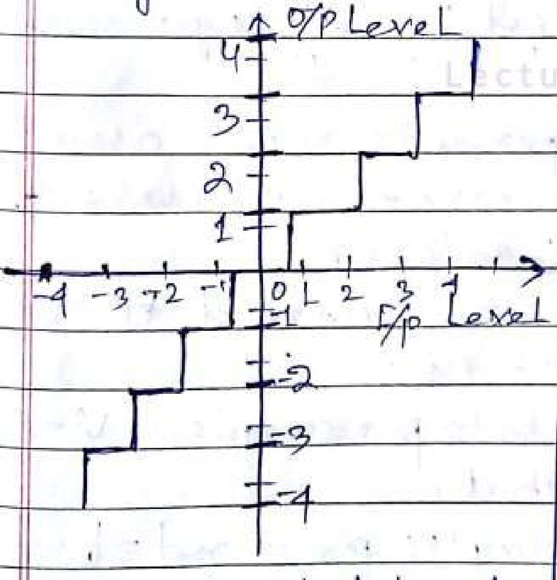
UNIFORM QUANTIZER:-

As discussed earlier, a quantizer is called as an uniform quantizer if the step-size remains constant throughout the input range.

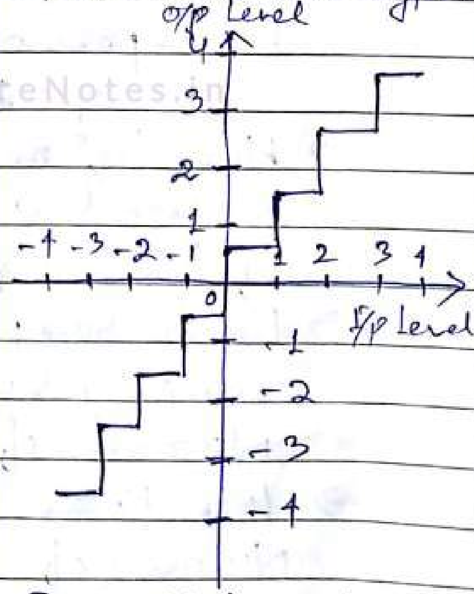
TYPES OF UNIFORM QUANTIZER:-

→ There are two types of uniform quantizer as under:-

(i) Symmetric quantizer of the midtread type.
(ii) Symmetric quantizer of the mid-rise type.



Fig(a). Mid-tread type



Fig(b) Mid-rise type.

→ The above fig(a) shows the input-output characteristics of a uniform quantizer of the mid-tread type, which is also called because the origin lies in the middle of the tread of the staircase like graph.

→ The above fig(b) shows the corresponding input-output characteristics of a uniform quantizer of the mid-rise type, in which the origin lies in the middle of a rising part of the staircase like graph.

TRANSMISSION BANDWIDTH IN A PCM SYSTEM:-

→ Let us assume that the quantizer uses "v" number of binary digits to represent each level.

→ Then, the number of levels that may be represented by "v" digits will be,

$$q = 2^{v} \quad \text{--- (1)}$$

→ Here "q" represents total number of digital quantization levels of a q-level quantizer.

→ Each sample is converted to "v" binary bits i.e.

Number of bits per sample $= v$ --- (2)

→ We know that,

Number of samples per second $= f_s$ --- (3)

→ Therefore, Number of bits per

second is expressed as
(Number of bits per sec) = (Number of bits per samples) × (Number of samples/sec)

$$\Rightarrow \quad r = v f_s \text{ ——— } (3).$$

where 'r' represents the number bits/sec and is known as signaling rate or bit rate of PCM.

→ Since bandwidth needed for PCM transmission is given by half of the signaling rate, therefore, we have Transmission Bandwidth in PCM system

$$BW \geq \frac{1}{2} r \text{ ——— } (4).$$

→ But we know that

$$r = v f_s$$

$$\therefore BW \geq \frac{1}{2} v f_s$$

again, since $f_s \geq 2 f_m$

$$\therefore BW \geq v f_m \text{ ——— } (5).$$

→ Eq^n(5) is the required expression for bandwidth of a PCM system.

QUANTIZATION ERROR IN PCM:-

→ Because of quantization process, there is a deviation of approximation value of the signal from the actual value of the signal.

→ This deviation results in an error, which is known as Quantization error.

→ As defined earlier, the quantization error is given as

$$\varepsilon = x_q(nT_s) - x(nT_s) \quad \text{——— (1)}.$$

→ Let us assume that the input $x(nT_s)$ to a Linear or uniform quantizer has continuous amplitude in the range $-x_{max}$ to $+x_{max}$.

→ Now, if the total amplitude range is divided into "q" levels of quantizer, then the step-size "Δ" will be

$$\Delta = \frac{x_{max} - (-x_{max})}{q} = \frac{2x_{max}}{q} \quad \text{——— (2)}$$

→ Again, now if the signal $x(t)$ is normalized to minimum and maximum values equal to 1, then we have, $x_{max} = 1$, $-x_{max} = -1$, therefore, step-size would be

$$\Delta = \frac{2}{q} \text{ (for normalized signal)} \quad \text{——— (3)}.$$

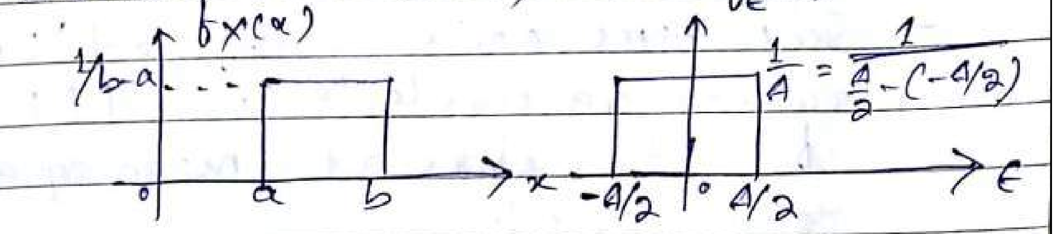
→ Now, if the step-size "Δ" is considered as sufficiently small, then it may be assumed that the quantization error "ε" will be an uniformly distributed random variable.

→ We also know that the maximum quantization error is given as

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right|$$

$$\text{i.e. } -\frac{\Delta}{2} \leq \varepsilon_{max} \leq \frac{\Delta}{2} \quad \text{——— (4)}.$$

→ Hence, over the interval $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$, quantization error may be assumed as an uniformly distributed random variable, whose "PDF" is given as $f_e(\epsilon)$



→ The probability density function (PDF) for quantization error 'ϵ' may be defined as:-

$$f_e(\epsilon) = \begin{cases} 0, & \text{for } \epsilon < -\Delta/2 \\ 1/\Delta, & \text{for } -\frac{\Delta}{2} \le \epsilon \le \Delta/2 \\ 0, & \text{for } \epsilon > \Delta/2 \end{cases} \quad \text{—— (5)}$$

→ From the above figure, it may be observed that quantization error "ϵ" has zero average value.

→ In other words, the mean "m_e" of the quantization error is zero.

→ Further, we know that the signal to quantization noise ratio of the quantizer is defined as

$$\frac{S}{N} = \frac{\text{Signal Power (normalized)}}{\text{Noise Power (normalized)}} \quad \text{—— (6)}$$

→ If we assume that the input signal $x(t)$ is a voltage signal, then it is possible to calculate signal power.

→ The noise power is expressed as

$$\text{Noise power} = \frac{V^2_{noise}}{R} \quad \text{—— (7)}.$$

→ Here, V^2_{noise} is taken as the mean square value of noise voltage.

→ Since, here noise is defined by random variable "ε" and PDF by $f_\varepsilon(\varepsilon)$; therefore its mean square value is given as :-

mean square value $= E[\varepsilon^2] = \overline{\varepsilon^2} = V^2_{noise}$ —— (8).

→ As we know that $E[\varepsilon^2] = \int_{-\infty}^{+\infty} \varepsilon^2 f_\varepsilon(\varepsilon) d\varepsilon$ —— (9).

→ Using eqn (5), the above eqn (9) may be written as

$$E[\varepsilon^2] = \int_{-\Delta/2}^{+\Delta/2} \varepsilon^2 \times \frac{1}{\Delta} d\varepsilon$$

$$= \frac{1}{\Delta}\left[\frac{\varepsilon^3}{3}\right]_{-\Delta/2}^{+\Delta/2}$$

$$= \frac{1}{\Delta}\left[\frac{(\Delta/2)^3}{3} + \frac{(\Delta/2)^3}{3}\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right]$$

$$\therefore E[\varepsilon^2] = \Delta^2/12 \text{ —— (10).}$$

→ Now, using eqn (7), the mean square value of noise voltage would be

$$V^2_{noise} = E[\varepsilon^2] = \Delta^2/12$$

→ Also, if load resistance, $R = 1\,\Omega$, then the noise power is normalized i.e.

Noise power (normalized) $= \frac{V^2_{noise}}{1} = \Delta^2/12$

$\therefore$ Quantizer error = Noise power (normalized)

$= \Delta^2/12$. (Ans)

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Date
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NUMERICALS:-

Q. (1). A PCM system uses a uniform quantizer followed by a 'v' bit encoder. Show that rms signal to quantization noise ratio is approximately given as $(1.8 + 6v)$ dB.

Solution:- Let us assume that the modulating signal is a sinusoidal voltage, having a peak amplitude equal to A_m.

→ Then, the power of this signal will be,

$$P = V^2/R$$

where V = rms value = $A_m/\sqrt{2}$

→ $\therefore P = \frac{A_m^2}{2} \times \frac{1}{R}$

→ In case when $R = 1$ ohm, the power 'P' is normalized i.e. $P = \frac{A_m^2}{2}$.

→ We know that, signal to quantization noise ratio is given as

$$\frac{S}{N} = \frac{\text{Normalized signal Power}}{\text{Normalized Noise Power}}$$

$$\Rightarrow \frac{S}{N} = \frac{\text{Normalized signal Power}}{\Delta^2/12}$$

where $\Delta = \frac{2x_{max}}{q} = \frac{2x_{max}}{2^v}$

$$\rightarrow \frac{S}{N} = \frac{\text{Normalized signal Power}}{\left(\frac{2x_{max}}{2^v}\right)^2 \times \frac{1}{12}}$$

$$\Rightarrow \frac{S}{N} = \frac{P}{\frac{4x_{max}^2}{2^{2v}} \times \frac{1}{12}} = \frac{3P}{x_{max}^2} \times 2^{2v}$$

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→ Here, $P = \frac{Am^2}{2}$ and $x_{max} = Am$.

→ Substituting these values in S/N equation, we get,

$$\frac{S}{N} = \frac{3 \times \frac{Am^2}{2} \times 2^{2v}}{Am^2} = \frac{3}{2} \times 2^{2v}$$

$$\therefore \frac{S}{N} = 1.5 \times 2^{2v}$$

→ Expressing (S/N) in dB, we get

$$\left(\frac{S}{N}\right)_{dB} = 10\log_{10}\left(\frac{S}{N}\right)$$

$$\Rightarrow \left(\frac{S}{N}\right)_{dB} = 10\log_{10}(1.5 \times 2^{2v})$$

$$\Rightarrow \left(\frac{S}{N}\right)_{dB} = 10\log_{10}(1.5) + 10\log_{10}(2^{2v})$$

$$\Rightarrow \left(\frac{S}{N}\right)_{dB} = 1.76 + 2v \times 10\log_{10}(2)$$

$$\therefore \left(\frac{S}{N}\right)_{dB} = 1.76 + 2v \times 10 \times 0.3$$

$$\therefore (S/N)_{dB} \simeq 1.8 + 6v \text{ (Proved)}.$$

Q(2) A Television signal having a bandwidth of 4.2 MHz is transmitted using binary PCM system. Given that the number of quantization levels is 512. Determine
(i) Code word length.
(ii) Transmission bandwidth.
(iii) Final bit rate.
(iv) Output signal to quantization noise ratio.

Solution:- Given that $f_m = 4.2$ MHz.
$q = 512$ = no. of quantization levels.

(i) We know that the no. of bits and

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quantization levels are related in binary PCM as under:

$$q = 2^v$$

$$\Rightarrow 2^v = 512 \Rightarrow \log_{10} 2^v = \log_{10} 512$$

$$\Rightarrow v \log_{10} 2 = \log_{10} 512$$

$$\Rightarrow v = \log_{10} 512 / \log_{10} 2 = 9 \text{ bits.}$$

Hence code word length is 9 bits.

(ii) We know that the transmission channel bandwidth is given as

$$BW \geq v f_m \geq 9 \times 4.2 \times 10^6 \text{ Hz} \geq 37.8 \text{ MHz.}$$

(iii) The final bit rate is equal to signaling rate. We know that the signaling rate is given as

$$r = v f_s \quad \text{and} \quad f_s \geq 2 f_m$$

$$\therefore r = 9 \times 2 \times 4.2 \text{ MHz} = 75.6 \times 10^6 \text{ bits/sec.}$$

(iv) $(S/N)_{dB} = (1.8 + 6v) \, dB$

$$= 6 \times 9 + 1.8 = 55.8 \text{ dB (Ans).}$$

Q.(3). The bandwidth of an input signal to the PCM is restricted to 4kHz. The input signal varies in amplitude from −3.8 V to +3.8 V and has the average power of 30mW. The required signal to noise ratio is given as 20dB. The PCM modulator produces binary output. Assuming uniform quantization,

(i) Find the number of bits required per sample.

(ii) Outputs of 30 such PCM coders are time multiplexed. What would be the minimum required transmission bandwidth for this multiplexed signal?

Solution:- Given that $(S/N)_{dB} = 20$

$$\Rightarrow 10\log_{10}(S/N) = 20$$

$$\Rightarrow \frac{S}{N} = 10^2 = 100.$$

(i) We know that

$$\frac{S}{N} = \frac{3P \cdot 2^{2\nu}}{x^2_{max}}$$

Here $x_{max} = 3.8V$ $P = 30mW$.

and $S/N = 100$

$$\therefore 100 = \frac{3 \times 30 \times 10^{-3} \times 2^{2\nu}}{(3.8)^2}$$

$$\Rightarrow \nu \simeq 6.98 \text{ bits} \simeq 7 \text{ bits (Ans)}.$$

(ii) The maximum frequency is given as $f_m = 4kHz$.

$\therefore$ Transmission bandwidth

$$BW \geq \nu f_m$$

Since there are 30 PCM coders which are time multiplexed, the transmission bandwidth must be

$$BW \geq 30 \times \nu f_m \geq 30 \times 7 \times 4kHz$$

$$BW \geq 840kHz \text{ (Ans)}.$$

PCM SYSTEM :- Fig(a) shows a PCM Transmitter. The baseband signal is sampled at Nyquist rate by the sampler. The sampled pulses are then quantized in the quantizer. The encoder (an A/D convertor) encodes these quantized pulses into bits which are then transmitted over the channel.

Fig(b) shows a PCM Receiver. The first block is again a quantizer. But this quantizer is different from the transmitter quantizer because it has to take a decision about the presence or absence of a pulse. The output of the quantizer goes to the decoder which performs the inverse operation of the encoder. The decoder output is a sequence of quantized pulses. The original baseband signal is reconstructed in the holding ckt. and LPF.

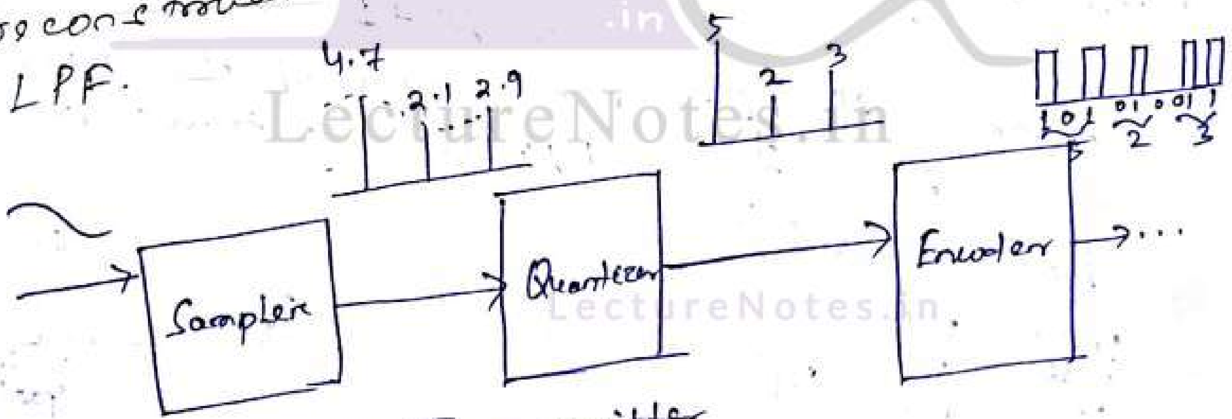
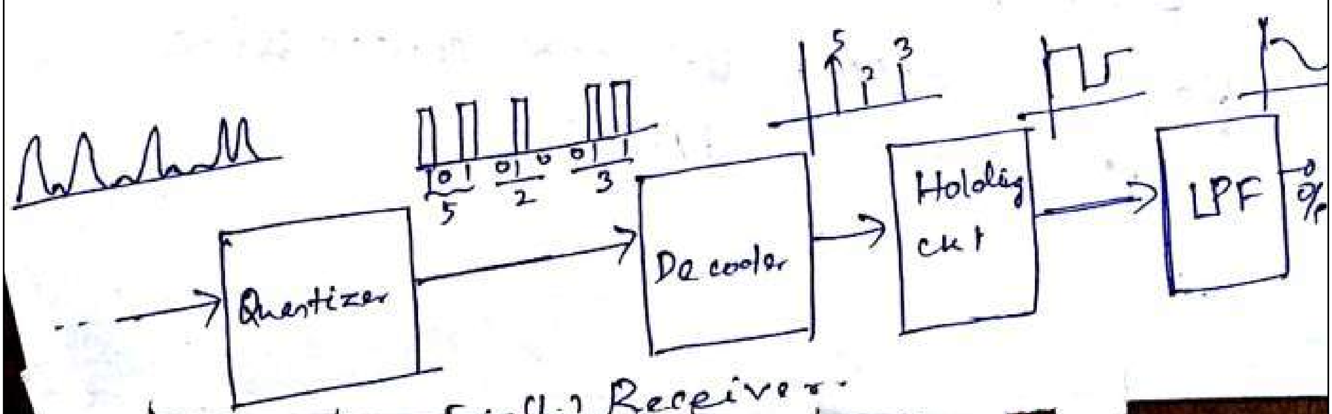


Fig (a) Transmitter



Fig(b) Receiver.

Problem(2) : Twenty-four voice signals are sampled uniformly and then time-division multiplexed. The sampling operation uses flat-top samples with 1 μs duration. The multiplexing operation includes provision for synchronization by adding an extra pulse of appropriate amplitude and 1 μs duration. The highest frequency component of each voice signal is 3.4 kHz.

(i) Assuming a sampling rate of 8kHz calculate the spacing between successive pulses of the multiplexed signal.

(ii) Repeat

Ans :- Sampling rate = 8kHz = 8000 samples/sec.

There are 24 voice signals + 1 synchronizing pulse.

Pulse width of each voice channel and synchronizing pulse is 1μsec.

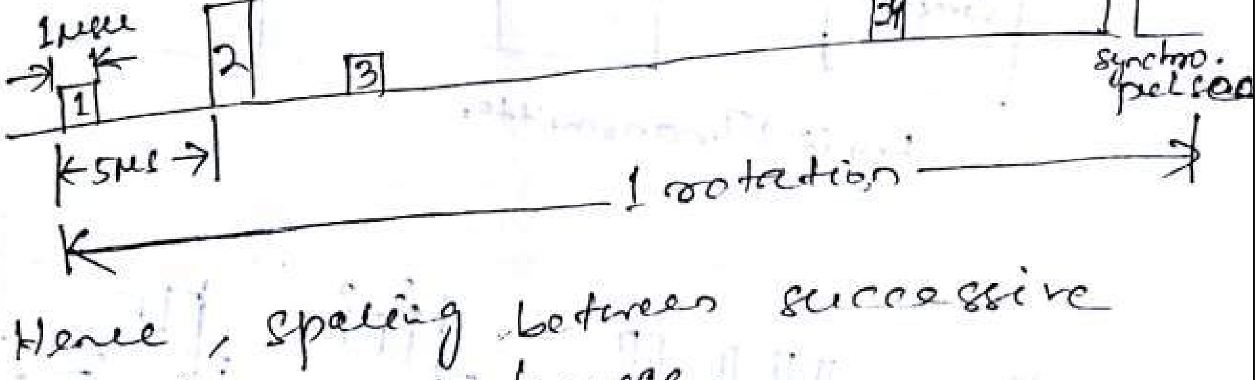
Now, time taken by the commutator for 1 rotation

$$= \frac{1}{8000} = 125\ \mu sec.$$

Number of pulses produced in 1 rotation

$$= 24+1 = 25.$$

Therefore, the leading edges of the pulses are at $= \frac{125}{25} = 5\mu$ secs. distance as shown below.



Hence, spacing between successive pulses = 5 - 1 = 4 μsec.

Problem(1): A Television signal having a bandwidth of 4.2MHz is transmitted using binary PCM system. Given that the number of quantization levels are 512. Determine:

(i) Code word length.

(ii) Transmission bandwidth.

(iii) final bit rate.

(iv) Output signal to quantization noise ratio.

Solution:- Given that the bandwidth is 4.2MHz. This means that highest frequency component will have frequency of 4.2 MHz i.e. $f_m = 4.2 MHz$.

Also, given that Quantization Levels $q = 512$.

(i) We know that the no. of bits and quantization levels are related in binary PCM as under:

$$q = 2^{\upsilon}$$

where υ = code word length in bits.

$$\Rightarrow 512 = 2^{\upsilon}$$

$$\Rightarrow \log_{10} 512 = \upsilon \log_{10} 2$$

$$\Rightarrow \upsilon = \frac{\log_{10} 512}{\log_{10} 2} = 9 \text{ bits}$$

(ii) Transmission channel bandwidth is given as

$$B.W. = \upsilon f_m = 9 \times 4.2 \times 10^6 Hz = 37.8 MHz.$$

(iii) final bit rate is the signaling rate.

we also know that signaling rate $= \upsilon f_s$

$$f_s \geq 2f_m \geq 2 \times 4.2 MHz = 8.4 MHz.$$

$$\therefore \text{ Signaling rate} = \upsilon f_s = 9 \times 8.4 \times 10^6$$
$$= 75.6 \times 10^6 \text{ bits/sec.}$$

(iv) The output signal to noise ratio is expressed

Ans:- $\left(\frac{S}{N}\right) dB = (6\nu + 4.8)\, dB$, $(6\nu + 1.8)\, dB$: for sinusoidal signals.

$= (6 \times 9 + 4.8)\, dB$

$= 58.8\, dB$.

(2) Twenty four voice signals are sampled uniformly and then have to be time division multiplexed. The highest frequency component for each voice signal is equal to 3.4 kHz. Now

(i) If the signals are pulse amplitude modulated using Nyquist rate, what would be the minimum channel bandwidth required.

(ii) If the signals are pulse code modulated with an 8 bit encoder, what would be the sampling rate? The bit rate of system is given as 1.5×10^6 bits/sec.

Ans:- (i) Minimum channel B.W $= N f_m$

$= 24 \times 3.4\, kHz = 81.6\, kHz$.

(ii) The signaling rate of the system is given as

$r = 1.5 \times 10^6$ bits/sec.

Since there are 24 no. of channels so the signaling rate or bit rate of an individual channel is

$$r(\text{one channel}) = \frac{1.5 \times 10^6}{24} = 62{,}500 \text{ bits/sec.}$$

Further, since each sample is encoded using 8 bits, the samples per second will be

$$\text{samples/sec} = \frac{r(\text{one channel})\ \text{bits/sec}}{\text{bits/sample}}$$

$$\Rightarrow \text{Sampling frequency}, f_s = \frac{62{,}500 \text{ bits/sec}}{8 \text{ bits/sample}}$$

$= 7812.5\, Hz$.

(3) A PCM system uses a uniformly quantizer followed by a 7-bit binary-encoder. The bit rate of the system is 50×10^6 bits/sec.

(i) What is the maximum message signal B.W. for which the system operates satisfactorily?

(ii) Calculate the output signal to quantization noise ratio when a full load sinusoidal modulating wave of frequency 1MHz is applied to the i/p.

Ans:- (i) $f_s \geq 2f_m$

The no. of bits given as $v = 7$ bits.

We know that, the signaling rate is

$$r = v f_s$$

$$\Rightarrow r = 7\times 2f_m$$

$$\Rightarrow 50\times10^6 = 7\times2\times f_m = 14 f_m$$

$$\Rightarrow f_m = \frac{50\times10^6}{14} = 3.57 \text{ MHz}.$$

(ii) The modulating wave is sinusoidal. For such signal, the signal to quantization noise ratio is expressed as

$$\left(\frac{S}{N}\right)dB = 6v + 1.8$$

$$= 6\times7 + 1.8 = 43.8 \text{ dB}.$$

(4) The discrete samples of an analog signal is to be uniformly quantized for PCM system. If the maximum value of the analog sample is to be represented within 0.1% of accuracy, find the minimum no. of binary digits required?

Given that $V_{max} = 1$ volt.

Ans Uniformly quantized PCM system.

Sampling accuracy $= 0.1\% = 0.001$

So $\frac{s}{2} = 0.001$

$\Rightarrow s = 0.002$

$\Rightarrow \frac{2V_{max}}{q} = 0.002$ [$\because$ where s = step size, q = quantized level]

$\Rightarrow \frac{V_{max}}{q} = 0.001$

$\Rightarrow \frac{V_{max}}{2^v} = 0.001$

$\Rightarrow \frac{1}{2^v} = \frac{1}{1000}$

$\Rightarrow v \simeq 10$ bits.

Sol^n (2):- Let 'A' be the maximum value of the discrete samples.

Error tolerated is 0.1% i.e. 0.001A.

If 's' is the stepsize, then possible maximum error = $s/2$

$\Rightarrow s/2 = 0.001A$

$\Rightarrow A/s = \frac{1000}{2} = 500$

$\Rightarrow$ Thus no. of levels req^d = 500

$\Rightarrow q = 500$

$\Rightarrow 2^v = 500$

$\Rightarrow v \simeq 9$ bits.

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$$\left[\frac{S_0}{N_q}\right]_{dB} = 10\log\left[\frac{S_0}{N_q}\right] = 10\log_{10} 2^{2N} \approx 6N$$

$$\Rightarrow \boxed{\left[\frac{S_0}{N_q}\right]_{dB} \simeq 6N}$$

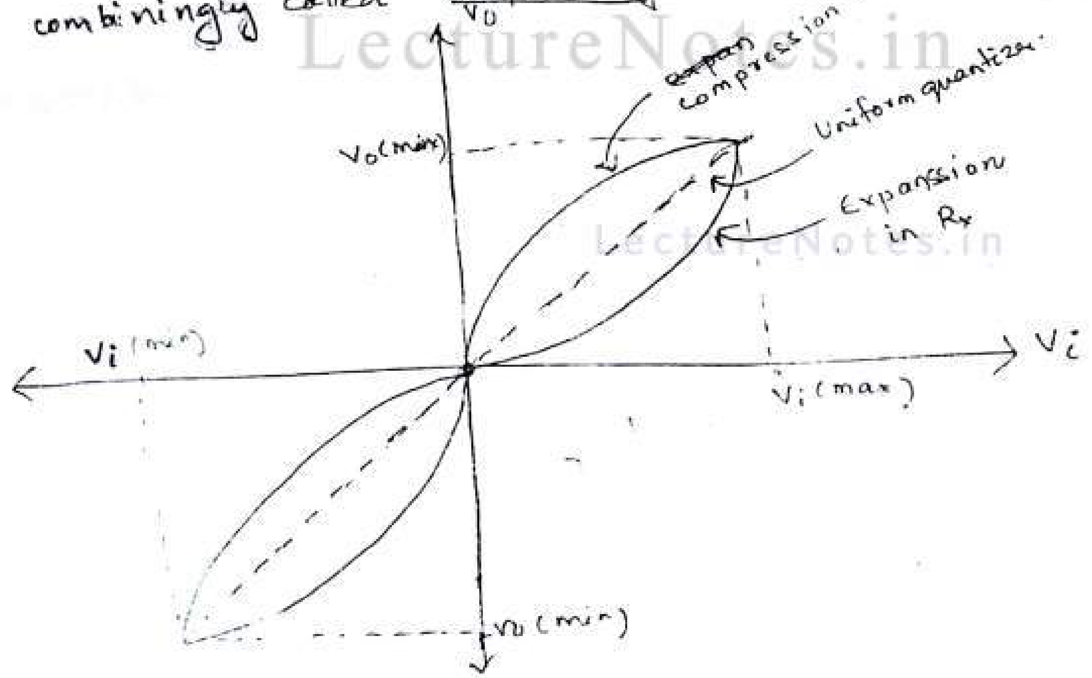
Concept of non-uniform quantization (Companding in PCM):

→ The quantization error depends upon the step size.

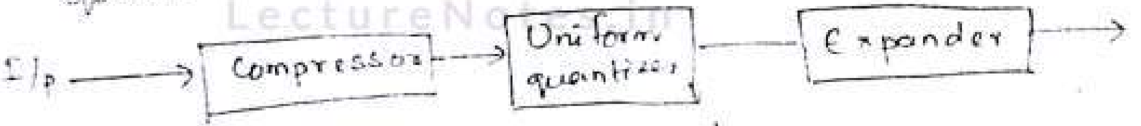
→ Hence during small amplitude of sample SN_qR is low and at higher amplitude SN_qR is high since in both cases the denominator is constant which is quantization noise.

→ So for a fixed quantization level inorder to have a uniform SN_qR, the step size has to be adjusted in such a manner that the ratio remain constant. i.e. the step size must be small for small amplitude signals and large for large amplitude signals. The reverse has to be done at the receiver.

→ Hence we undergoes the process of compression of signal at transmitter and expanssion at receiver which combiningly called companding.



The process of non-uniform quantization is done by compressing the signal first the passing the compressed signal through ~~an~~ uniform quantizer.

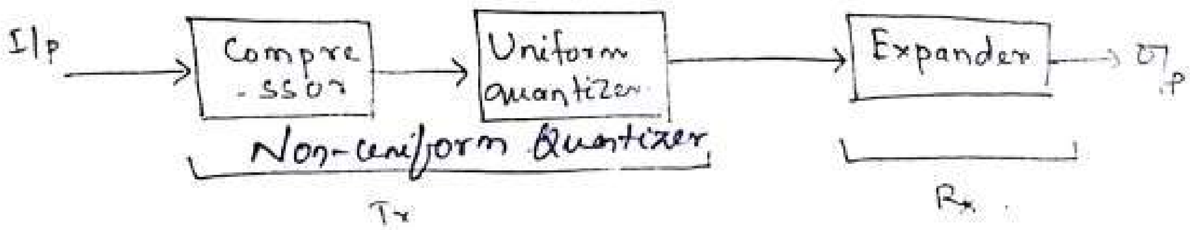


And expanding at receiving side.

→ ~~Compander~~ Compression at transmitter is done by amplifying the signal at low signal level and attenuating at high signal level. After this the signal is feed to the uniform quantizer. Due to this process for low signal level there are more quantization levels hence low step size will occure and for high signal lever there are less quantization levels hence high step size will occure. It ensures uniform S/N(?) level for all signal.

→ Reverse is done at receiver which is refered as expansion.

→ The combination of a compressor and an expander is called compander. In actual PCM system, the combination of compressor and uniform quantizer is located ~~at~~ in transmitter and expander is in receiver. The model of a non-uniform quantizer is shown below



$$\left[\frac{S_o}{N_q}\right]_{dB} = 10\log\left[\frac{S_o}{N_q}\right] = 10\log_{10} 2^{2N} \simeq 6N$$

$$\Rightarrow \boxed{\left[\frac{S_o}{N_q}\right]_{dB} \simeq 6N}$$

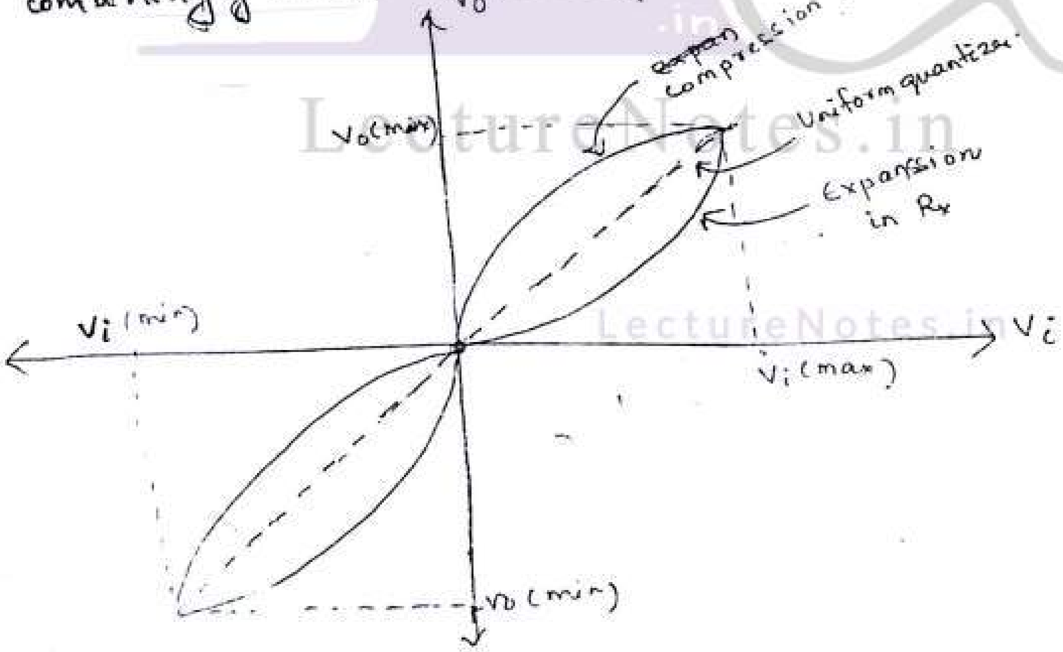
Concept of non-uniform quantization (Companding in PCM):-

→ The quantization error depends upon the step size.

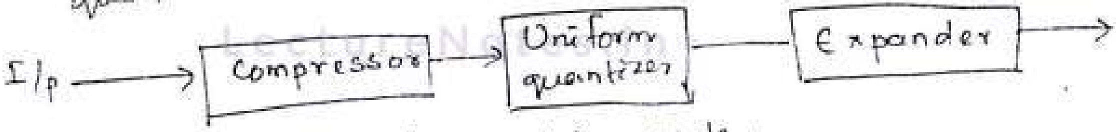
→ Hence during small amplitude of sample SN_qR is Low and at higher amplitude SN_qR is high since in both cases the denominator is constant which is quantization noise.

→ So for a fixed quantization level inorder to have a uniform SN_qR, the step size has to be adjusted in such a manner that the ratio remain constant. i.e. the step size must be small for small amplitude signals and large for large amplitude signals. The reverse has to be done at the receiver.

→ Hence we undergoes the process of compression of signal at transmitter and expanssion at receiver which combiningly called companding.



The process of non-uniform quantization is done by compressing the signal first the passing the compressed signal through uniform quantizer.

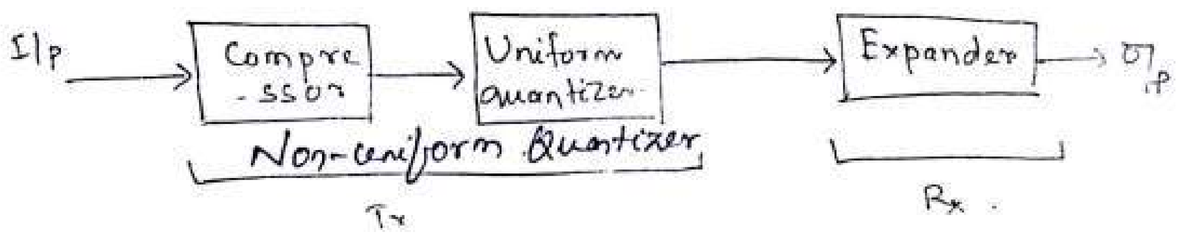


And expanding at receiving side.

→ ~~Compandor~~ Compression at transmitter is done by amplifying the signal at low signal level and attenuating at high signal level. After this the signal is feed to the uniform quantizer. Due to this process for low signal level there are more quantization levels hence low step size will occure and for high signal lever there are less quantization levels hence high step size will occure. It ensures uniform SNqR level for all signal.

→ Reverse is done at receiver which is refered as expansion.

→ The combination of a compressor and an expander is called compander. In actual PCM system, the combination of compressor and uniform quantizer is located in transmitter and expander is in receiver. The model of a non-uniform quantizer is shown below



COMPANDING IN PCM SYSTEM:-

→ Among several choices, two compression laws have been accepted as desirable standards by the ITU: (1) The μ-Law used in North America and Japan, and (2) the A-Law used in Europe and the rest of the world. Both the μ-Law and the A-Law curves have odd symmetry about the vertical axis.

→ The μ-Law (for positive amplitudes) is given as

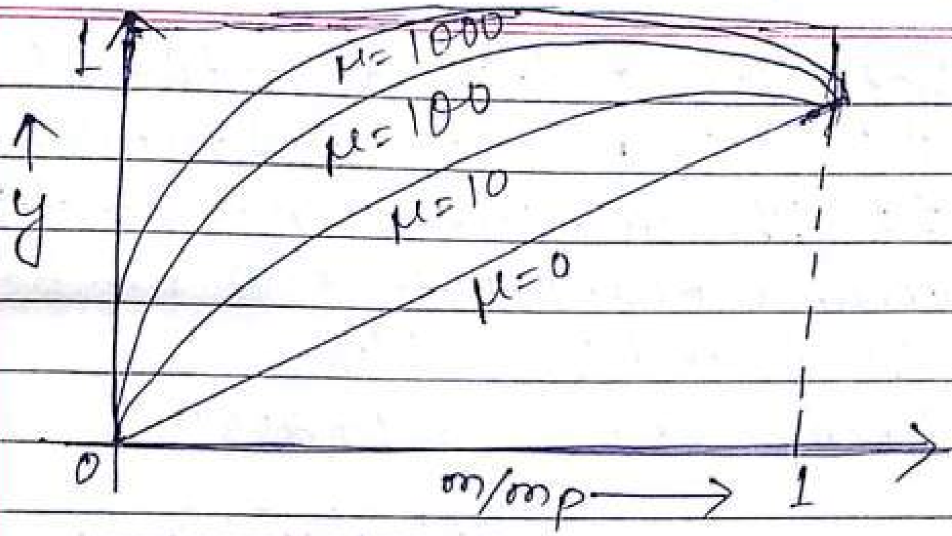
$$y = \frac{1}{\ln(1+\mu)} \ln\left(1 + \frac{\mu m}{m_p}\right), \quad 0 \le \frac{m}{m_p} \le 1 \qquad (1)$$

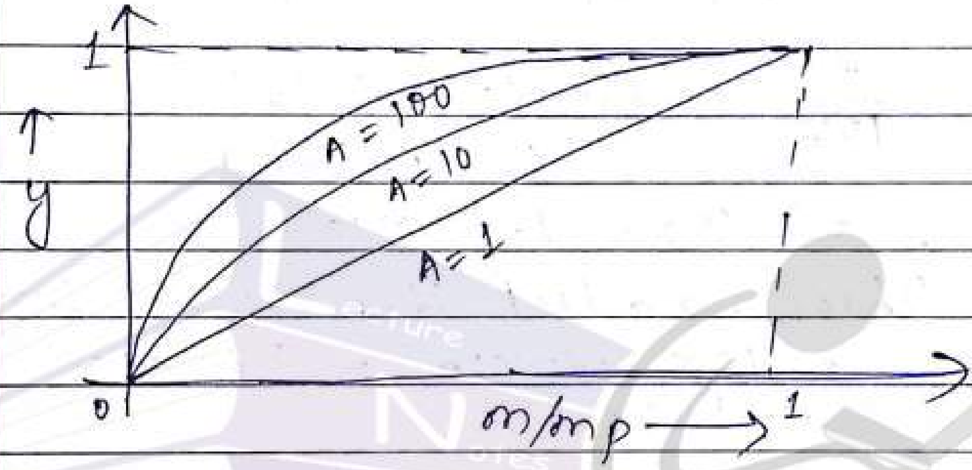
where y = Normalized o/p
m/m_p = Normalized i/p.
and μ = a positive constant.

→ The A-Law (for positive amplitudes) is given as

$$y = \begin{cases} \dfrac{A}{1+\ln A}\left(\dfrac{m}{m_p}\right), & 0 \le \dfrac{m}{m_p} \le \dfrac{1}{A} \\[2ex] \dfrac{1}{1+\ln A}\left(1 + \ln \dfrac{Am}{m_p}\right), & \dfrac{1}{A} \le \dfrac{m}{m_p} \le 1 \end{cases} \qquad (2)$$

where A is a +ve constant.





Q:- A μ-Law compander is defined as

$$y = \pm \frac{\ln(1+\mu|x|)}{\ln(1+\mu)}, \ |x| \le 1, \text{ where}$$

x is input and y is output. The + sign is used when x is +ve and - sign is used when x is negative. If the peak of input is 10V and no. of bits available for quantization are 8 then find the smallest and largest separation between levels. Consider $\mu = 255$.

Ans:- The y-axis is uniformly quantized with step size $= \frac{1}{\left(\frac{2^8}{2}\right) - 1} = \frac{1}{127}$ in both +ve and -ve direction between ±1 when peak of input varies between ±1. The smallest step in x-direction occurs nearest to $x=0$ i.e. between $y_1 = 0$ and $y_2 = 1/127$

Then, $0 = \frac{\ln(1+255x_1)}{\ln(1+255)}$

$\Rightarrow x_1 = 0$ and

$\frac{1}{127} = \frac{\ln(1+255x_2)}{\ln(1+255)}$

$\Rightarrow x_2 = 1.75\times10^{-4}$

Thus, smallest step-size is $10\times(x_2 - x_1)$

$= 10\times 1.75\times10^{-4} = 1.75\times10^{-3}$

$= 1.75mV.$

The largest step-size occurs when x is at its extreme between $y_1 = 1 - \frac{1}{127} = \frac{126}{127}$ and $y_2 = 1$.

Again substituting

$(126/127) = \ln(1+255x_1)/\ln(1+255)$

$\Rightarrow x_1 = 0.957$

and $1 = \ln(1+255x_2)/\ln(1+255)$

$\Rightarrow x_2 = 1$

Thus, the largest step size is

$10\times(x_2 - x_1)$

$= 10\times(1-0.957)$

$= 0.43V$ (Ans).

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Digital Communication Techniques

Topic:

Certain Issues In Digital Transmission

Contributed By:

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LINE CODES :-

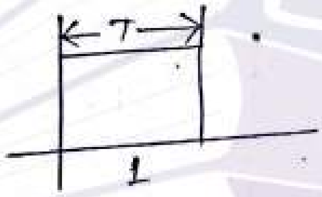
The digital data (0's and 1's) are transmitted over the line by means of 'Line Codes' (also known as 'Data Transmission Codes' or 'Modulation Codes'). Thus, they give electrical representation of symbols 0 and 1.

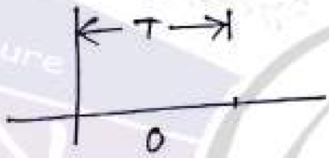
Types of Line codes :-

(1) UNRZ (Unipolar Non-Return to Zero) code :-

In this code, a '1' is represented by a positive pulse and a '0' is represented by no pulse. This is also known as 'ON-OFF code'.

Exm :-





(2) BNRZ (Bipolar Non-Return to Zero) Code :-

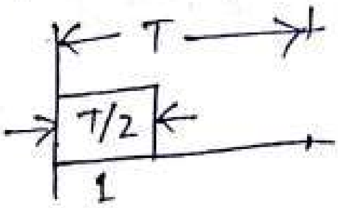
In this code, a '1' is represented by a positive pulse and a '0' is represented by a negative pulse.

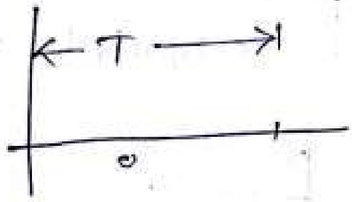
Exm :-

(3) URZ (Unipolar Return to Zero) Code :-

In this code, a '1' is represented by a positive pulse of half symbol-width and a '0' is represented by no pulse.

Exm :-





(4) BRZ (Bipolar Return to Zero) Code :-

In this code, a '1' is represented by a positive pulse of half symbol-width and a '0' is represented by a negative pulse of half symbol width.

Exm :-

(5) Split-Phase Code (Manchester Code) :-

In this code, a '1' is represented by a positive half-symbol width pulse followed by a negative half-symbol width pulse and a '0' is represented by a negative half symbol width pulse followed by positive half symbol width pulse.

Exm :-

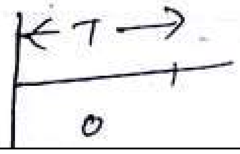
(6) Differential code or BRZ-AMI (Binary Return to Zero - Alternate Mark Inversion) Code :-

In this code, a '1' is represented alternatively by a positive pulse of half-width and a negative pulse of half width, where as a '0' is represented by no pulse.

Exm :-

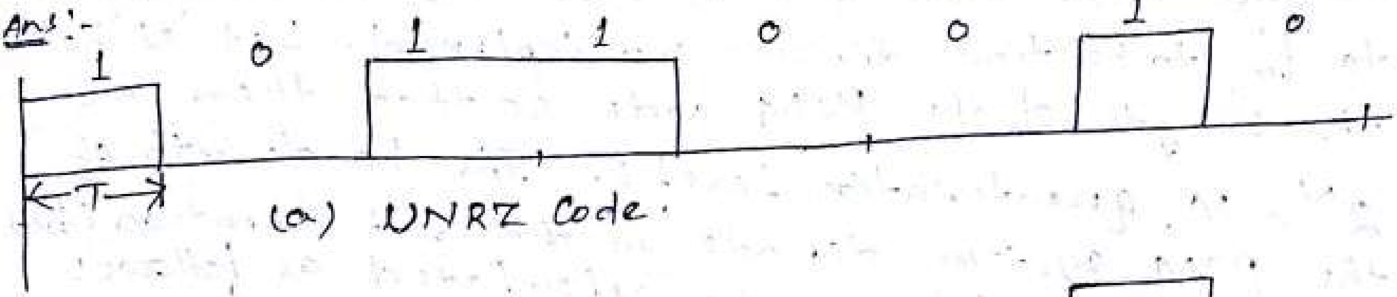
(symbols alternately occure)

1

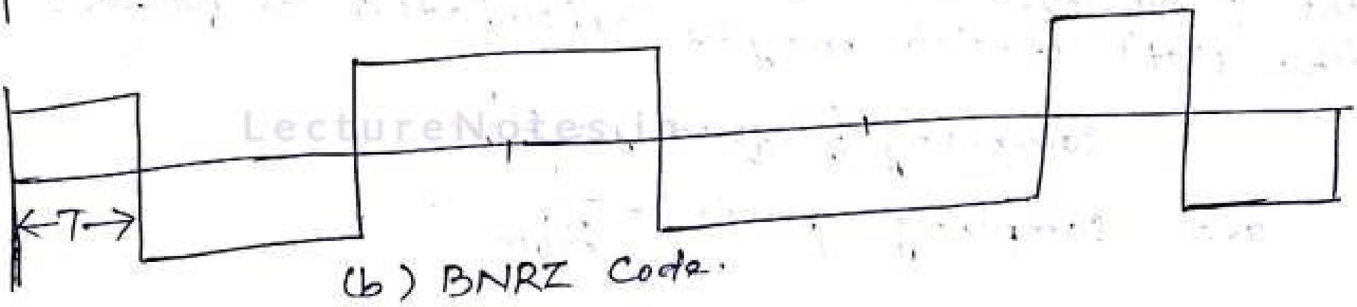


Problem:- Let us consider a binary sequence given as 1 0 1 1 0 0 1 0. Represent all the Line codes?

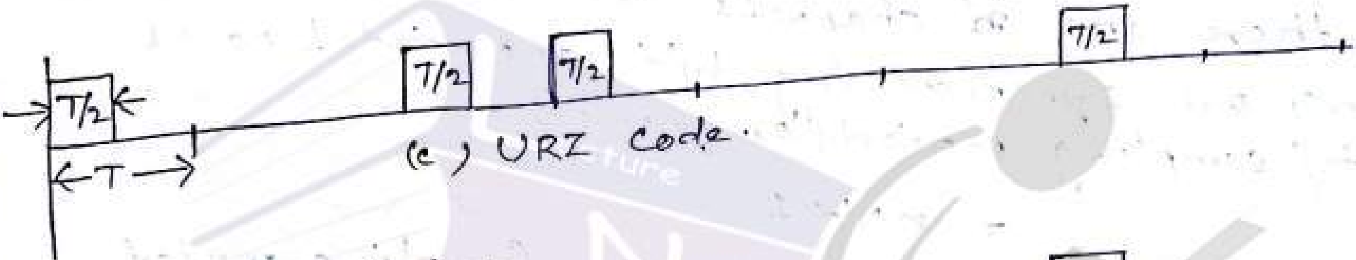
Ans:-



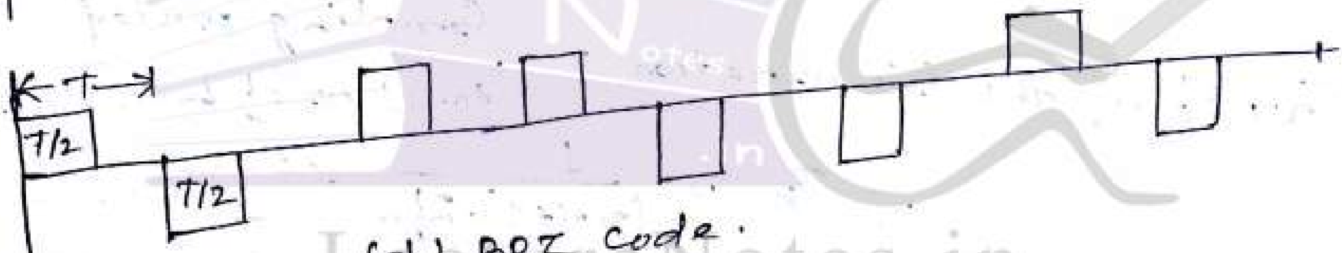
(a) UNRZ Code.



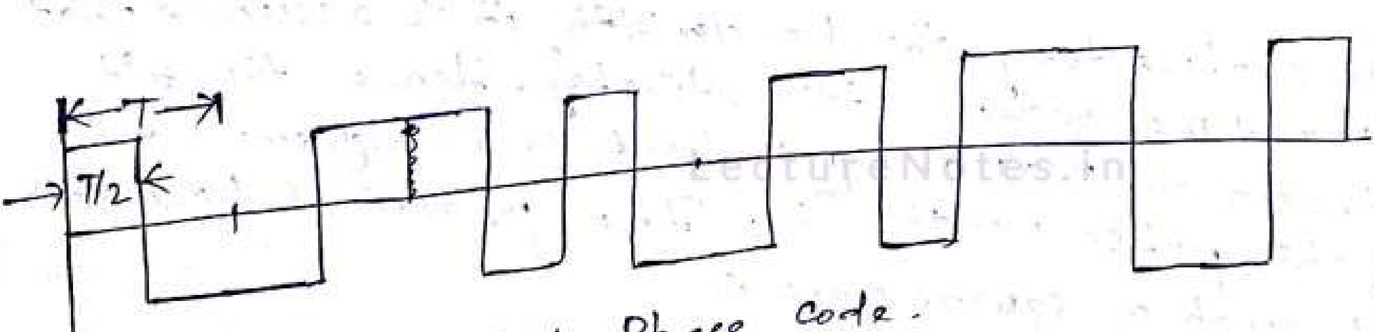
(b) BNRZ Code.



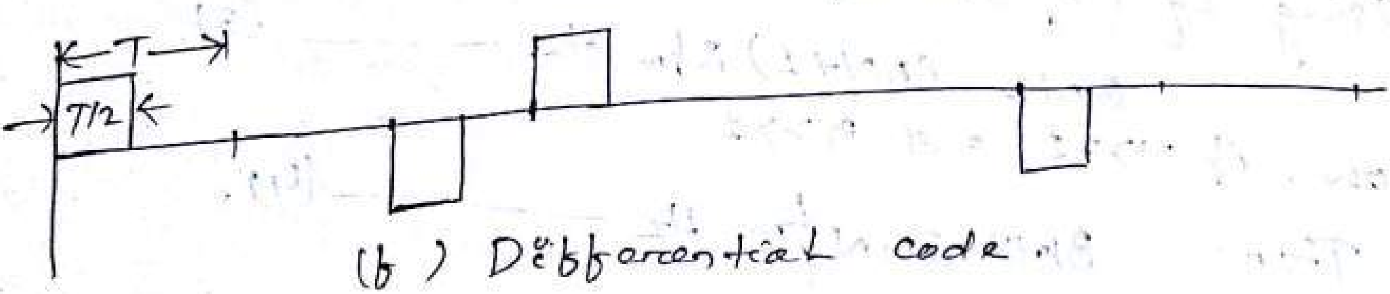
(c) URZ Code.



(d) BRZ Code.



(e) Split-Phase Code.



(f) Differential code

SCRAMBLING :-

→ Scrambling is a process by which data is randomized.

→ The application of scrambling is in digital transmission of television signals.

→ Scrambler also removes a pattern while randomizing the data.

→ Thus, it can remove long string of 1's or 0's which can help clock recovery and synchronization.

An Example :-

Let, the digital data input to the scrambler be represented by d(k) and its output is b(k).

→ This $b(k) = d(k) \oplus b(k-2) \oplus b(k-4)$ — ①

where, $b(k-j)$ represents j-th last value of b(k).

→ Let us consider that input data are all 1's and past values of b(k) to begin with are all zero, i.e. delay elements are all initialized with 0's.

K	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d(k)	1	1	1	1	1	1	1	1	1	1	1	1	1	1
b(k-4)	0	0	0	0	1	1	0	0	0	0	1	1	0	0
b(k-3)	0	0	0	1	1	0	0	0	0	1	1	0	0	0
b(k-2)	0	0	1	1	0	0	0	0	1	1	0	0	0	0
b(k-1)	0	1	1	0	0	0	0	1	1	0	0	0	0	1
b(k)	1	1	0	0	0	0	1	1	0	0	0	0	1	1

→ We see that the long string of 1's is broken by scrambler by periodic insertion of 1 after certain number of 0's.

→ The unscrambling process is the reverse process of scrambling that will be reverse of eq(1) in order to recover d(k).

→ The corresponding equation for this example is given as

$$\hat{d}(k) = b(k) \oplus b(k-2) \oplus b(k-4) \quad \text{--- (2)}$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
b(k)	1	1	0	0	0	0	1	1	0	0	0	0	1	1
b(k-1)	0	1	1	0	0	0	0	1	1	0	0	0	0	1
b(k-2)	0	0	1	1	0	0	0	0	1	1	0	0	0	0
b(k-3)	0	0	0	1	1	0	0	0	0	1	1	0	0	0
b(k-4)	0	0	0	0	1	1	0	0	0	0	1	1	0	0
$\hat{d}(k)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1

→ The following table i.e the above table can illustrate the recovery of original data and we see that $\hat{d}(k)$ is same as transmitted message d(k).

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Digital Communication Techniques

Topic:

DPCM: Differential Pulse Code Modulation

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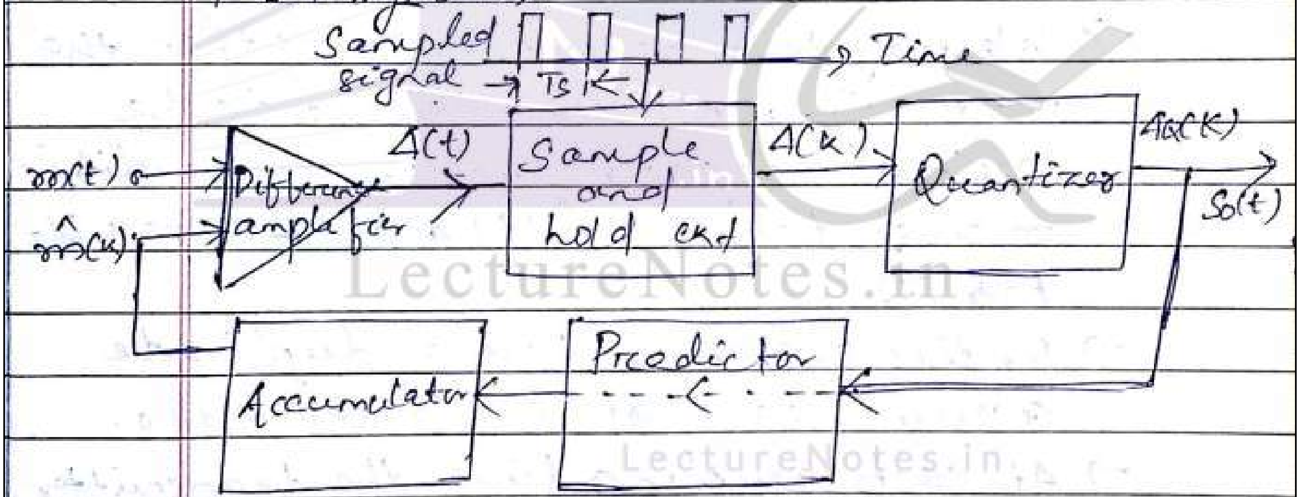
DPCM (DIFFERENTIAL PULSE CODE MODULATION)

→ This scheme doesnot transmit complete sample values at each sampling time rather it transmit the difference between present sample and the previous sample, i.e. at k^{th} instant it transmits the difference i.e

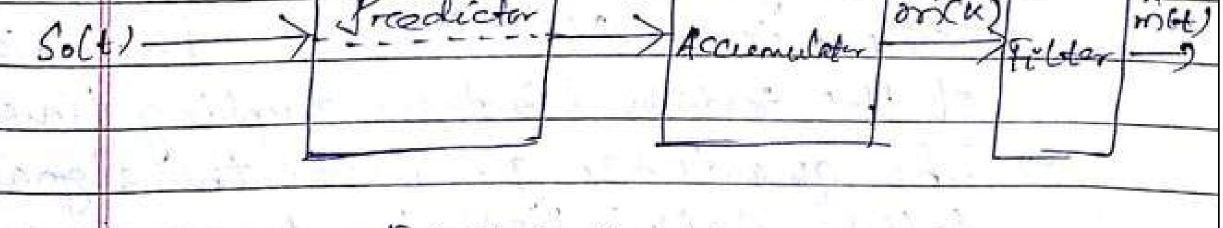
$$\Delta(k) = m(k) - m(k-1).$$

→ The receiver can generate an identical waveform $m(t)$ by only accumulating or adding up the change.

→ The main advantage of this type of scheme is that for most of the case the difference $\Delta(k) = m(k) - m(k-1) << m(k)$. and hence lesser bits are required to encode the difference and lesser bits per sample to transmit than conventional PCM system.



Fig(a). Transmitter.



Fig(b) Receiver.

→ The quantized differential transmission scheme is shown above.

→ The receiver consists of an accumulator which adds up the received quantized differences $\Delta_Q(k)$ and a filter which smooths out the quantization noise.

→ The output of the accumulator is the signal approximation $\hat{m}(k)$ which becomes $\hat{m}(t)$ at the filter output.

→ At the transmitter, we need to know whether $\hat{m}(t)$ is larger or smaller than $m(t)$, and by how much.

→ We may then determine whether the next difference $\Delta_Q(k)$ needs to be positive or negative and of what amplitude in order to bring $\hat{m}(t)$ as close as possible to $m(t)$.

→ For this reason we have a duplicate accumulator at the transmitter.

→ At each sampling time, the transmitter difference amplifier compares $m(t)$ and $\hat{m}(t)$ and the sample and hold circuitry holds the result of that comparison $\Delta(t)$ for the duration of the interval between sampling times.

→ The quantizer generates the signal $S_o(t) = \Delta_Q(k)$ both for transmission to the receiver as well as the input to the transmitter accumulator.

→ In a practical system, the quantized differences would first be encoded into a binary bit stream before transmission and decoded at the receiver. For simplicity the encoder and decoder are not included in the block diagram.

NEED FOR A PREDICTOR :-

→ When the sampling rate is set at the Nyquist rate, it generates unacceptably excessive quantization noise in comparison to PCM.

→ The quantization noise can be reduced by significantly increasing the sampling rate.

→ With increased rate, the differences from sample to sample and the rate of producing large quantization errors is reduced.

→ One of the disadvantage of the DPCM system is that the bit rate of DPCM exceeds that required for PCM system.

→ The situation in DPCM can be improved by recognizing that there is a correlation between successive samples of the signal $m(t)$ and of $\Delta(t)$, if the signal is sampled at a rate exceeding the Nyquist rate.

→ Hence a knowledge of past sample values or differences allows us to predict, with some probability of being correct.

→ To take the advantage of this correlation, a predictor is included in the DPCM system as shown in block diagram.

→ The predictor will generally be a moderately sophisticated system; it will need to incorporate the facility for storing past differences and for carrying out some algorithm to predict the next required increment.

LINEAR PREDICTOR DESIGN :-

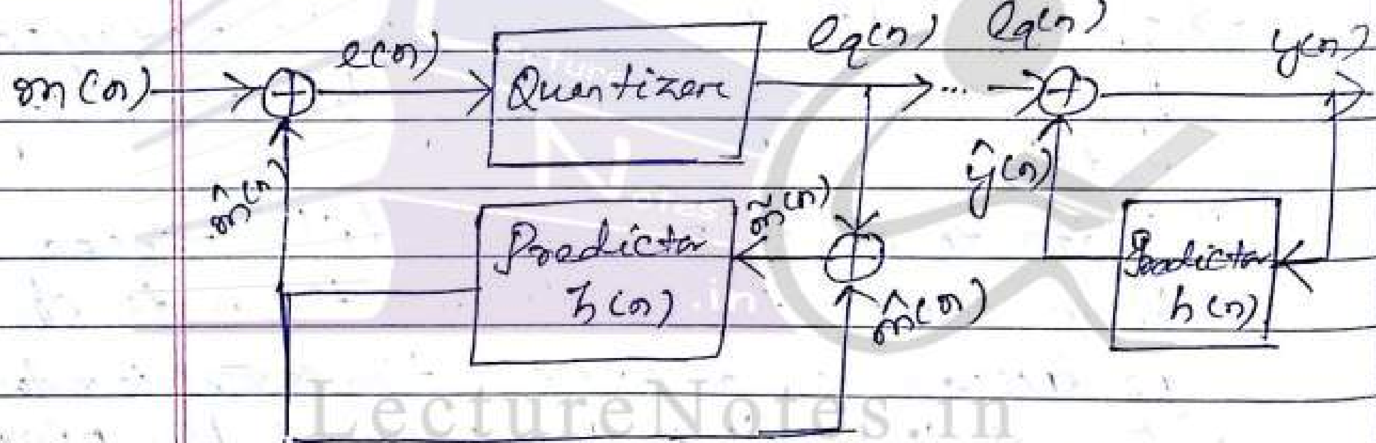


Figure: A discrete time representation of DPCM system.

→ A scheme to design a linear predictor is optimal in Mean Square Error (MSE) sense.

→ The signal processing in discrete domain uses transfer functions like $H(z)$ that is defined in $Z(= e^{j\omega})$ domain as

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \quad \text{--- (1)}$$

→ The quantity $h(n)$ represents impulse response of the system and in popular Finite Impulse Response (FIR) system, extend from $h(1), \ldots, h(N)$.

→ In linear predictor design, we shall consider it as a FIR system and optimality in MSE sense tries to find $h(n)$, $n = 1 \cdots N$ that minimizes some error criterion.

→ The convolution relation of two signals $x(n)$ and $h(n)$ in discrete time domain that generates $y(n)$ is given below:-

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k) \quad \text{——— (2).}$$

→ If $X(z)$ and $Y(z)$ are Z-transforms of $x(n)$, and $y(n)$ respectively, then we can write,

$$Y(z) = X(z)\, H(z) \quad \text{——— (3).}$$

→ In designing predictor for DPCM the main objective is to minimize the error $e(n)$ that is quantized and transmitted.

→ We write the equations that describes the above system, the quantities used are defined in the above block diagram by the sum and difference operators.

→ Quantization error, $q(n) = e_q(n) - e(n)$ ——— (4)

Prediction error, $e(n) = m(n) - \hat{m}(n)$ ——— (5)

→ Input to predictor, $\tilde{m}(n) = e_q(n) + \hat{m}(n)$ ——— (6)

→ Output equation

$$y(n) = e_q(n) + \hat{y}(n) \quad \text{--- (7)}$$

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Page ______

Linear Predictor output

$$\hat{m}(n) = \tilde{m}(n) \otimes h(n) \quad \text{--- (8)}$$

Converting above equation's in z-domain and using convolution and linearity property, we can write some useful eq^n's as :-

$$E_q(z) = Q(z) + E(z), \text{ from eq}^n (4)$$

$$\Rightarrow E_q(z) = Q(z) + M(z) - \hat{M}(z), \text{ from eq}^n (5)$$

$$\Rightarrow E_q(z) = Q(z) + M(z) - \tilde{M}(z).H(z) \quad \text{--- (9)}$$

---- from eq^n (8).

$$\Rightarrow \tilde{M}(z) = \hat{M}(z) + E_q(z), \text{ from eq}^n (6).$$

$$\Rightarrow \tilde{M}(z) = \cancel{\tilde{M}(z)H(z)} + [Q(z) + M(z) - \cancel{\tilde{M}(z)H(z)}] \quad \text{--- (10)}$$

$$\Rightarrow \tilde{M}(z) = M(z) + Q(z) \quad \text{--- (11)}.$$

$$Y(z) = \frac{E_q(z)}{1 - H(z)} \quad \left[\because Y(z) = E_q(z) + \hat{Y}(z) \Rightarrow Y(z) = E_q(z) + Y(z)H(z) \Rightarrow Y(z)[1 - H(z)] = E_q(z)\right]$$

$$= \frac{Q(z) + M(z) - H(z)\{M(z) + Q(z)\}}{1 - H(z)}$$

$$\Rightarrow Y(z) = \frac{M(z) + Q(z) - M(z)H(z) - Q(z).H(z)}{1 - H(z)}$$

$$\Rightarrow Y(z) = \frac{M(z)[1 - H(z)] + Q(z)[1 - H(z)]}{1 - H(z)}$$

$$\Rightarrow Y(z) = \frac{[M(z) + Q(z)]\cancel{[1 - H(z)]}}{\cancel{[1 - H(z)]}}$$

$$\Rightarrow Y(z) = M(z) + Q(z) \quad \text{--- (12)}$$

$$= \tilde{M}(z)$$

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DIFFERENTIAL PCM (DPCM):-

→ It may be observed that the samples of a signal are highly correlated with eachother.

→ This is due to the fact that any signal doesnot change fast or immediately.

→ This means that its value from present sample to next sample doesnot differ by Large amount.

→ The adjacent samples of the signal carry the same information with a little bit difference.

→ When these samples are encoded by a standard PCM system, the resulting encoded signal contains some redundant information.

→ figure given below illustrates this redundant inform.

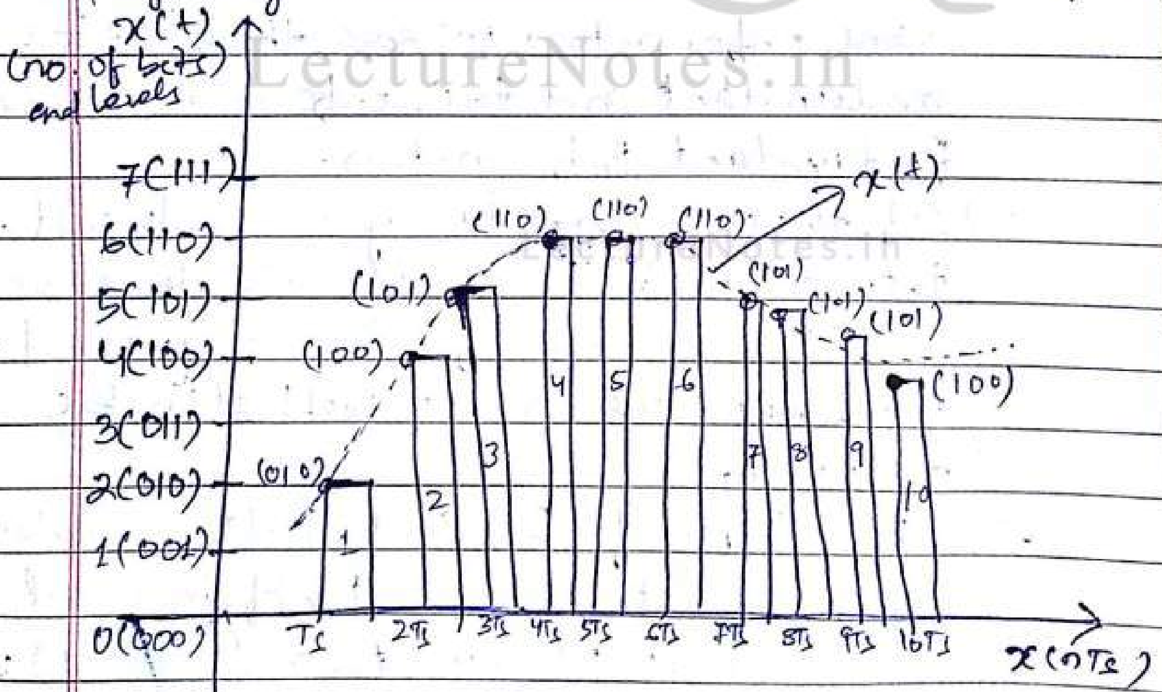

fig: Illustration of redundant information in PCM.

→ The above figure shows a continuous time signal $x(t)$ by dotted line.

→ This signal is sampled by flat-top sampling at intervals T_s, $2T_s$, $3T_s$, .. nT_s.

→ The sampling frequency is selected to be higher than Nyquist rate.

→ The samples are encoded by using 3 bits (8 levels) PCM system.

→ The sample is quantized to the nearest quantization level as shown by small circles in the above figure.

→ from the above figure it is observed that the samples taken at $4T_s$, $5T_s$ and $6T_s$ are encode to same value (110).

→ This information can be carried only by one sample.

→ But three samples are carrying the same information means that it is redundant and this information is called Redundant Information.

→ If this redundancy is reduced, then overall bit rate will decrease and the number of bits required to transmit one sample will also be reduced.

→ This type of Digital Pulse modulation scheme is known as Differential Pulse Code Modulation (DPCM).

→ In fact the differential PCM

works on the principle of prediction.

→ The value of the current sample or present sample is predicted from the past samples.

→ The prediction maynot be exact but it is very much close to the actual sample value.

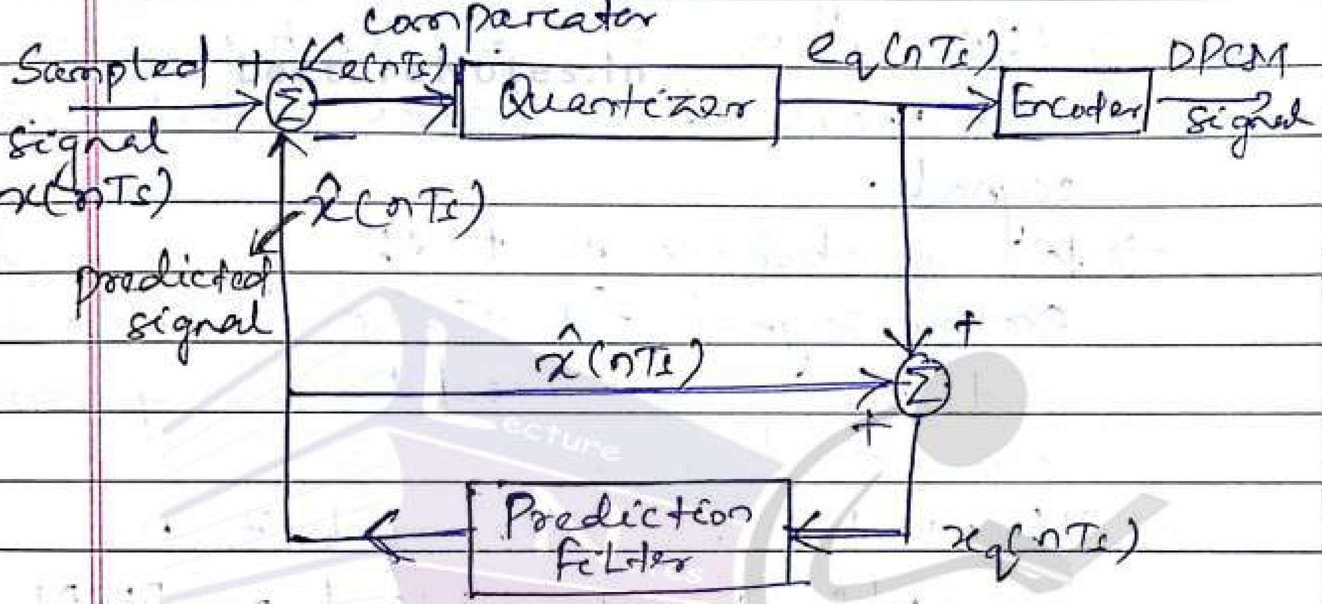

fig: A DPCM Transmitter.

→ The above figure shows the transmitter of DPCM system.

→ The sampled signal is denoted by $x(nT_s)$ and the predicted signal is denoted by $\hat{x}(nT_s)$.

→ The comparator finds out the difference between the actual sample value $x(nT_s)$ and predicated sample value $\hat{x}(nT_s)$.

→ This is known as Prediction error and it is denoted by $e(nT_s)$.

→ It can be defined as

$$e(nT_s) = x(nT_s) - \hat{x}(nT_s) \quad \text{——} \quad (1)$$

→ The predicted value is produced by using a prediction filter.

→ The quantizer output is $e_q(nT_s)$.

→ The quantizer output and previous prediction is added and given as input to the prediction filter.

→ This signal is called $x_q(nT_s)$.

→ This makes the prediction more and more close to the actual sampled signal.

→ We can observe that the quantized error signal $e_q(nT_s)$ is very small and can be encoded by using small number of bits.

→ Thus the number of bits per sample are reduced in DPCM.

→ The quantizer output can be written as

$$e_q(nT_s) = e(nT_s) + q(nT_s) \quad \text{—— (2).}$$

where $q(nT_s)$ is the quantization error.

→ The prediction filter input $x_q(nT_s)$ is obtained by sum of $\hat{x}(nT_s)$ and quantizer output.

$$\therefore x_q(nT_s) = \hat{x}(nT_s) + e_q(nT_s) \quad \text{—— (3).}$$

→ Substitute eq^n (2) in eq^n (3), we get

$$x_q(nT_s) = \hat{x}(nT_s) + e(nT_s) + q(nT_s) \quad \text{—— (4)}$$

→ From eq^n (1), we know that

$$e(nT_s) = x(nT_s) - \hat{x}(nT_s)$$
$$\Rightarrow \hat{x}(nT_s) + e(nT_s) = x(nT_s) \quad \text{--- (5)}$$

→ Substitute eqⁿ(5) in eqⁿ (4), we will get

$$\Rightarrow x_q(nT_s) = \hat{x}(nT_s) + e(nT_s) + q(nT_s)$$
$$\Rightarrow x_q(nT_s) = x(nT_s) + q(nT_s) \quad \text{--- (6)}$$

→ Hence, the quantized version of the signal $x_q(nT_s)$ is the sum of original sample value and quantization error.

RECONSTRUCTION OF DPCM SIGNAL:-

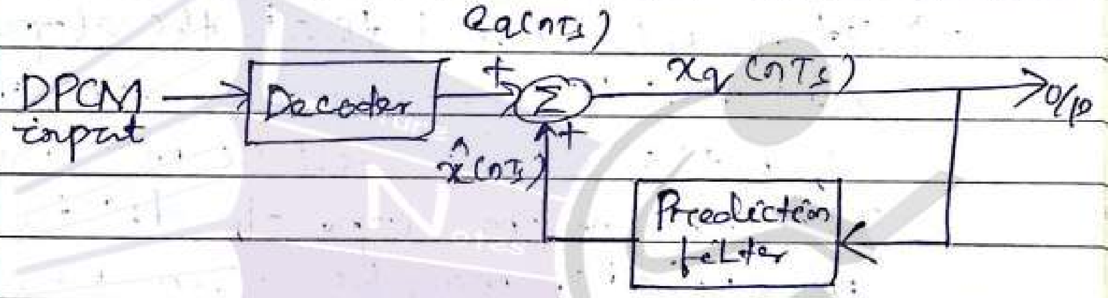

fig: DPCM Receiver.

→ The above figure shows the block-diagram of DPCM receiver.
→ The decoder first reconstructs the quantized error signal from the incoming binary signal.
→ The prediction filter output and quantized error signal are summed up to give the quantized version of the original signal.
→ Thus the signal at the receiver differs from the actual signal by quantization error $q(nT_s)$, which is introduced permanently in the reconstructed signal.

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Digital Communication Techniques

Topic:

Digital Multiplexing

Contributed By:

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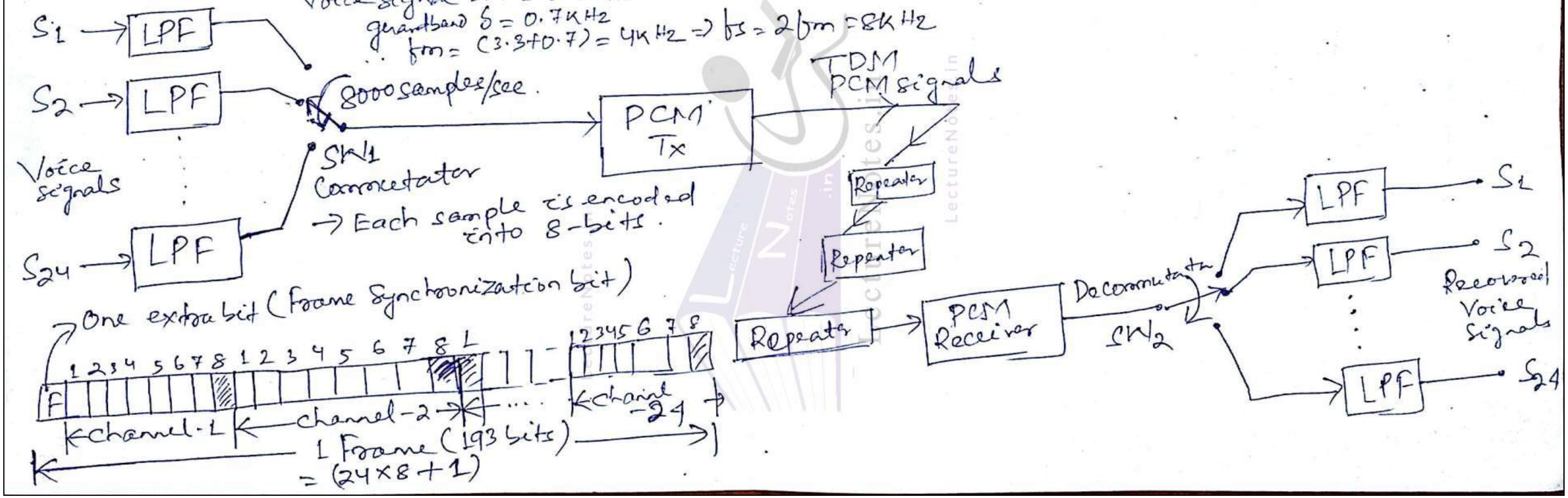
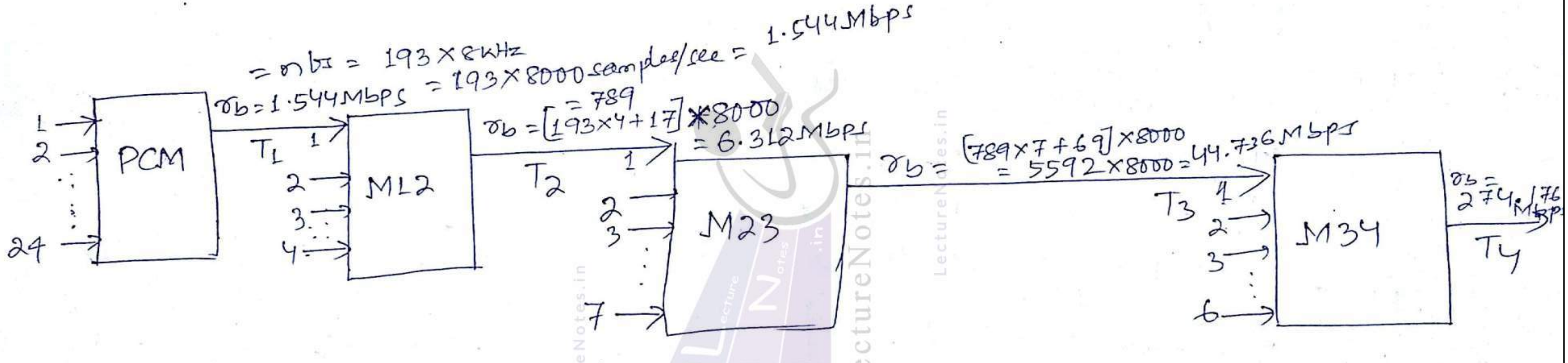
T_1 Digital System or T_1 Carrier System :-

Voice signal BW = 3.3KHz

guardband $\delta = 0.7KHz$

$\therefore f_m = (3.3+0.7) = 4KHz \Rightarrow f_s = 2f_m = 8KHz$





For T_4

$$r_b = (5592 \times 6 + 720) \times 8000$$
$$= 34272 \times 8000$$
$$= 274.176 \text{ Mbps}$$

Digital Multiplexing:-

T1 digital system:-

It is a time-division multiplexing scheme for convey multiple signal over telephone line using wide band co-axial cable. T1 carrier system accomodates 24 analog signals.

→ Each channel is band limited to 3.3 KHz.

$$f_m = 3.3\,KHz$$

Gaurd band $\delta = 0.7\,KHz$

$$f_s = 2(f_m+\delta) = 2(3.3+0.7) = 8\,KHz.$$

BITS/FRAME

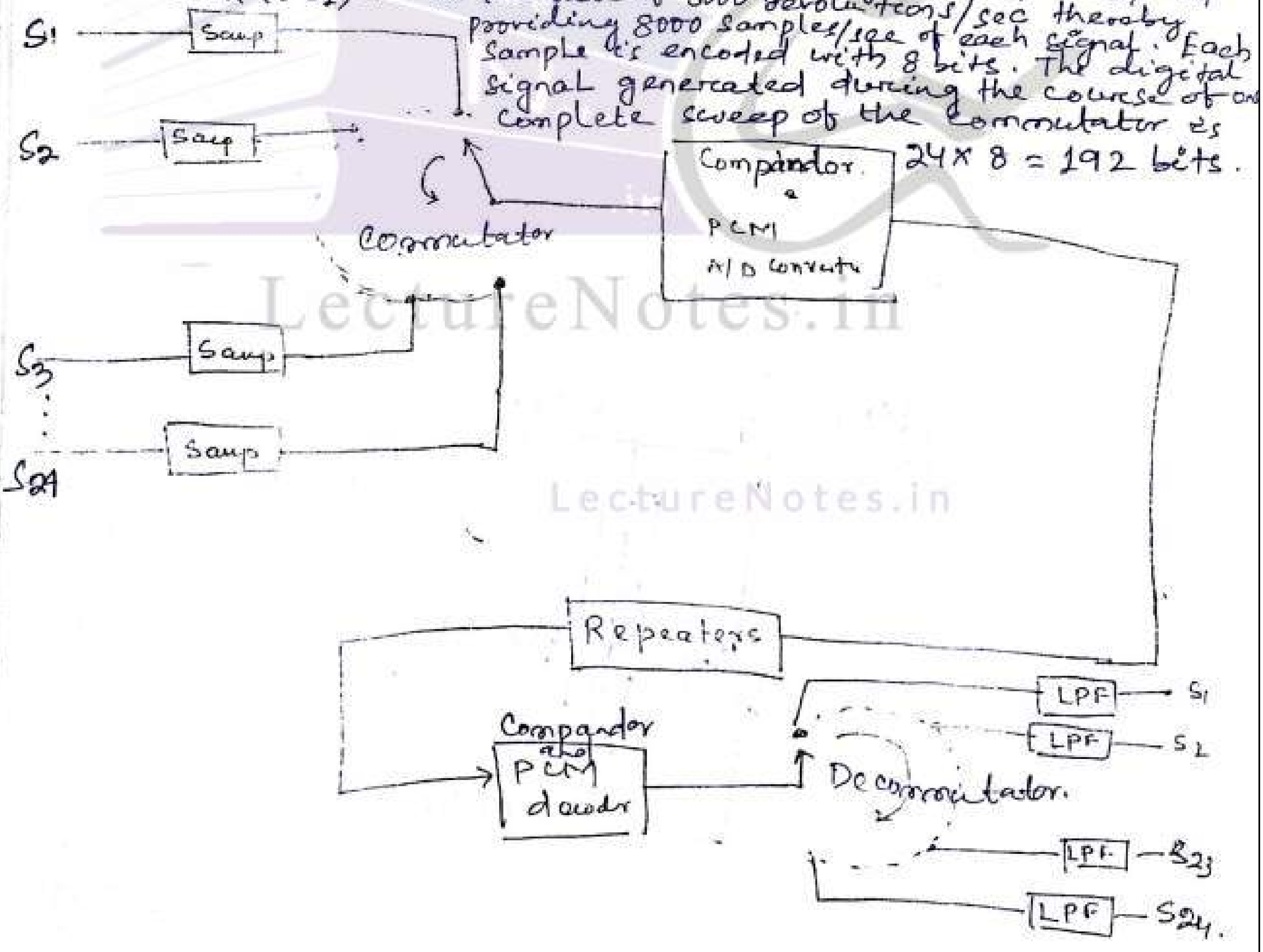
→ × Each sample is encoded with 8 bits.

× One sample of each signal is called frame.

× So no. of bits in each frame = 8×24 = 192 bits.

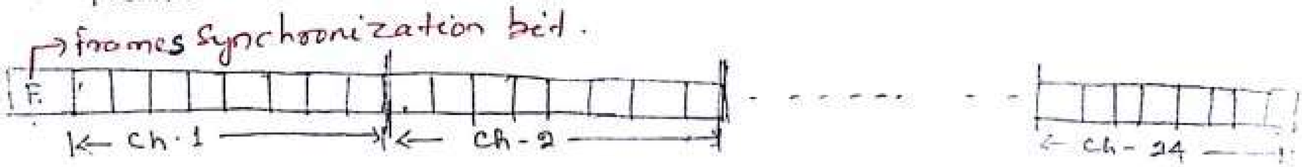
i.e. one complete sweep of commulator generates 192 bits

→ The commutator sweeps continuously from S_1 to S_{24} and back to S_1, etc. at the rate of 8000 revolutions/sec thereby providing 8000 samples/sec of each signal. Each sample is encoded with 8 bits. The digital signal generated during the course of one complete sweep of the commutator is 24×8 = 192 bits.



FRAME SYNCHRONIZATION :-

→ For synchronization, one bit is added at the start of the frame.



Twelve successive 'F' slots are used to transmit a 12-bit code given by - '110111001000'. This code is transmitted repetatively once every 12 frames and used at the receiver for synchronization.

So no. of bits per frame = 192 + 1 = 193 bits, is called frame

→ So no of bits transmitted per second

$r_b(T1) = 193 \times 8000 = 1.544$ Mbps.

Time for each frame = $\frac{1}{8 \times 10^3}$ = 125 μs.

frame synchronization code repeates at 125μs × 12 = 1.5 ms

and it repeats nearly 667 times per second.

Multiplexing T1 Lines – T2, T3, T4 Lines :-

It is the digital hireracy of multiplexing

(i) 4 T1 system is multiplexed to produce T2 system.

(ii) 7 T2 system is ←do→ T3 ,, .

(iii) 6 T3 ←do→ T4 ,, .

So T2 system carries 24×4 = 96 channels.

T3 –do———— 7×4 = 28 T1 system

or 96×7 = 28×24 = 672 channels.

28×6 = 168 T1 system

T4 ←do———→

⇒ 672×6 = 168×24 = 4032 channels.

∴ M12 Mux adds 17 extra bits for frame synchronization and pulse stuffing.

So no. of bits per frame (T2) = 193×4 + 17 = 789 bits/frame.

No. of bits transmitted per second

$r_b(T2) = 789 \times 8000 = 6.312$ Mbps.

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Digital Communication Techniques

Topic:

Delta Modulation, And Adaptive Delta Modulation.

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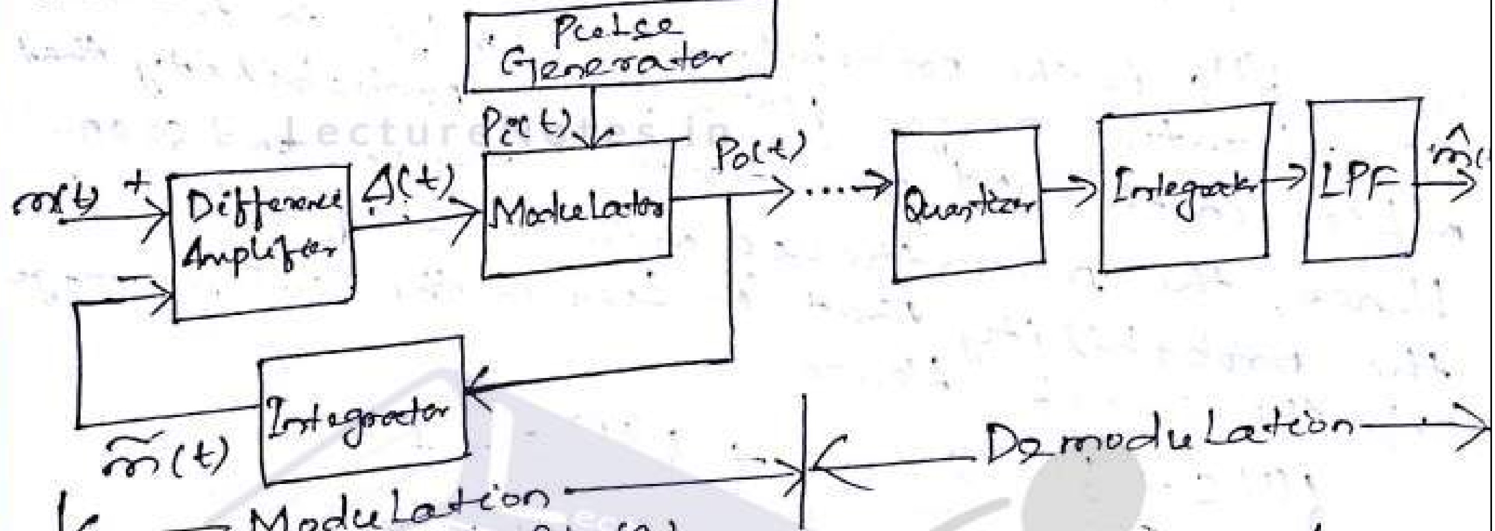
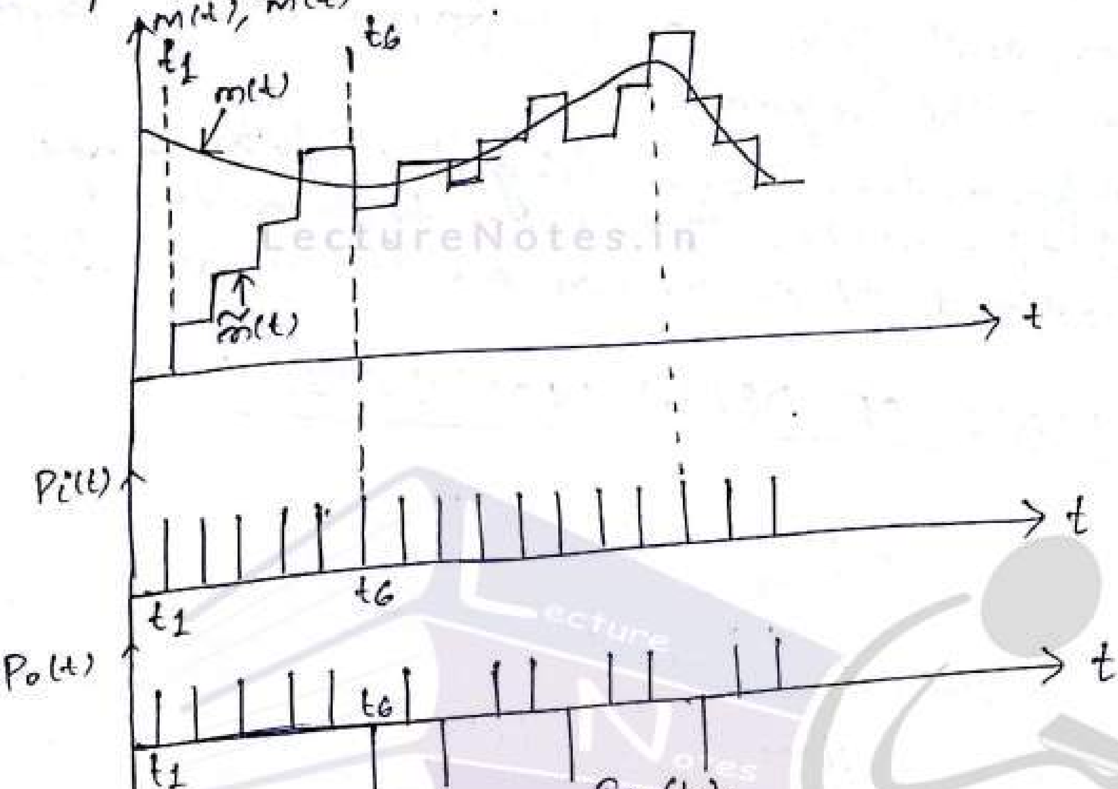
DELTA MODULATION :-



fig(a)

By the delta modulation technique, an analog signal can be encoded into bits. Hence, in one sense a delta modulation is also PCM.

The block diagram of a DM system is shown above. The pulse generator produces a pulse train $P_i(t)$ of positive pulses. The modulator receives $P_i(t)$ and $\Delta(t)$, the output of the difference amplifier. The modulator output $P_o(t)$ is the input pulse train $P_i(t)$ multiplied by +1 or -1 depending upon the polarity of the $\Delta(t)$. $P_o(t)$ is +ve when $\Delta(t)$ is +ve, and $P_o(t)$ is negative when $\Delta(t)$ is negative. The output of the modulator $P_o(t)$ is applied to an integrator whose output is $\tilde{m}(t)$. The input signal $m(t)$ and the integrator output $\tilde{m}(t)$ are compared in a difference amplifier whose output is $\Delta(t) = m(t) - \tilde{m}(t)$.



Fig(b).

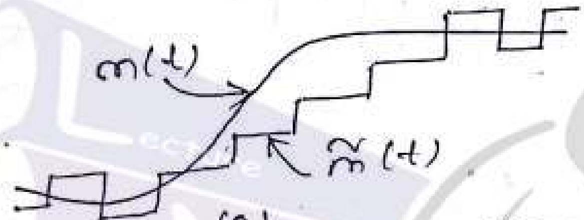
The above figure explains the operation of a delta modulator. The initial values of $m(t)$ and $\tilde{m}(t)$ have been assumed arbitrarily. At time t_1 of the first pulse in $P_i(t)$, the situation is such that $\Delta(t)$ is positive. Hence, the first pulse in $P_o(t)$ is positive. In the same way, the pulses in $P_o(t)$ are either positive or negative depending upon whether $\Delta(t)$ is positive or negative. The waveform $\tilde{m}(t)$ approaches $m(t)$ in the form of a staircase, and then closely follows it. Thus $\tilde{m}(t)$ is an approximation to the input signal $m(t)$.

The waveform $P_o(t)$ is transmitted. At the receiver side, the quantizer takes a decision whethere the received pulse is positive or negative.

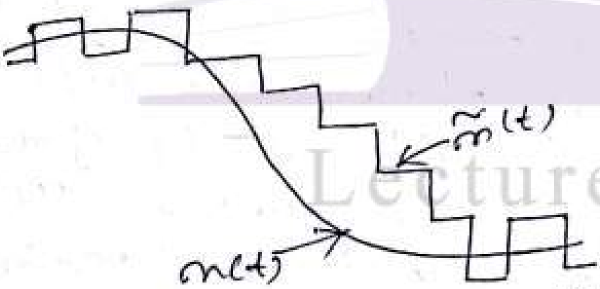
Hence, assuming no error, the output of the quantizer is the same as the waveform $P_0(t)$ and is fed to an integrator, whose output takes the form of the waveform $\tilde{m}(t)$. The Low pass filter (LPF), then smoothens the output of the integrator and gives a waveform $\hat{m}(t)$, which is similar to the signal $m(t)$.

As the information regarding the difference signal $\Delta(t) = m(t) - \tilde{m}(t)$ is transmitted in this method, it is known as Delta modulation.

LIMITATIONS OF DELTA MODULATION



(a)
slope overload (positive)



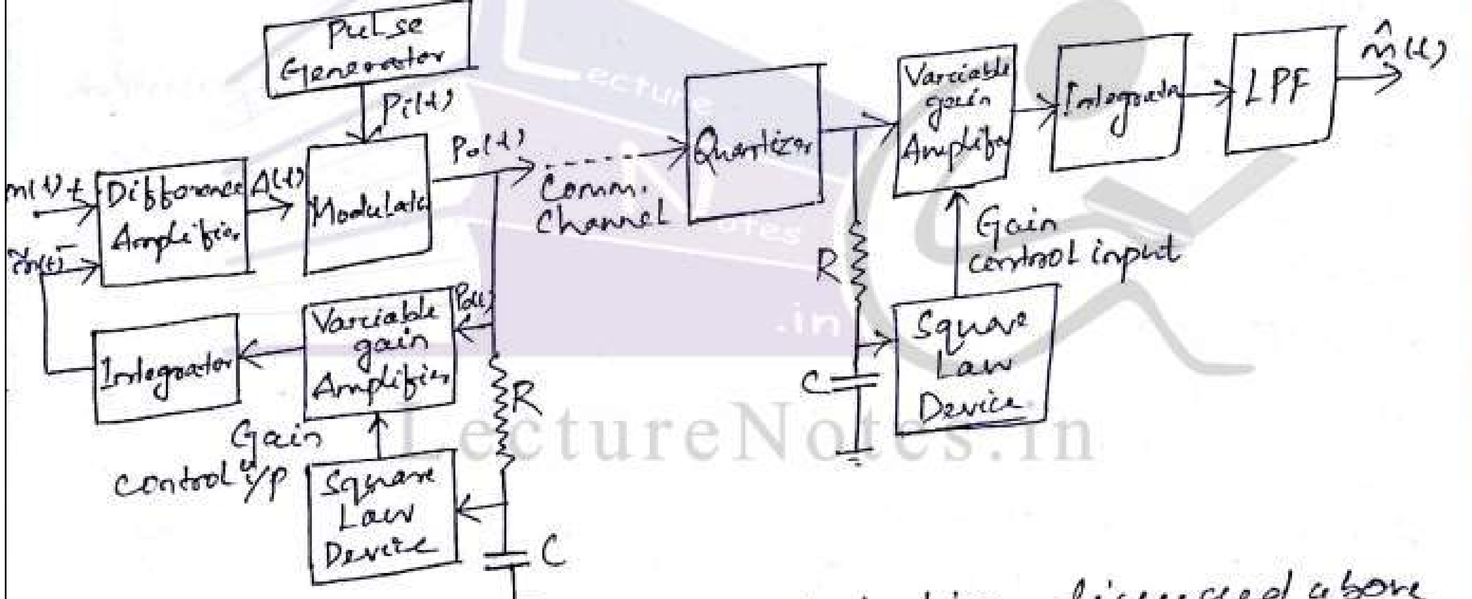
(b) slope Overload (negative)

(c)
Slow-varying signal

The waveform $\tilde{m}(t)$ needs to closely follow the waveform $m(t)$, only then the recovered waveform $\tilde{m}(t)$ resembles $m(t)$. Fig (a) shows a situation where the waveform $\tilde{m}(t)$ is unable to follow $m(t)$ because the slope of $m(t)$ is greater than the slope of $\tilde{m}(t)$. Same is true with fig (b), except that in this case the slopes are negative, and the slope of $m(t)$ is more negative than the slope of $\tilde{m}(t)$. In both the cases the recovered waveform will be distorted. The DM system, is said to be having with <u>slope overload</u>.

In fig(c), the variations in $m(t)$ are such that they are within the step size. Hence, the waveform $\tilde{m}(t)$ is like a square wave. This will be recovered as d.c., whereas the original signal $m(t)$ is not d.c. Thus in this case also, distortion is resulted and the noise is known as <u>granular noise</u>.

ADAPTIVE DELTA MODULATION



The Limitations of Delta modulation discussed above can be overcome by suitably changing the stepsize. Slope-overload can be overcome if the step size is increased in such a way that the magnitude of the slope of $\tilde{m}(t)$ becomes greater than the magnitude of the slope of $m(t)$, and when the signal variations are less than the step size is known as the solution to be made as the step size may be reduced to take care of the situation.

A DM system which adjusts its step size is known as the Adaptive Delta Modulation (ADM) system. An ADM system is shown above.

On the transmitter side, a variable gain amplifier is used before the integrator block with $P_o(t)$ as its input. The gain of this amplifier depends on the gain control input, which is obtained by first integrating $P_o(t)$ in an R-C network and then passing the integrator output through a square-law device. Under the slope-overload condition, $P_o(t)$ is a long sequence of either positive or negative pulses. The RC integrator integrates these pulses. Thus, the output of this integrator is either of a large positive or large negative value. The square law device output is of a large positive value, irrespective of whether the input is positive or negative. Thus, the gain control input of the variable gain amplifier is large and its gain increases. Hence the step size increases, which can then take care of the slope-overload difficulty.

When the signal variations are within the step size, $P_o(t)$ is a sequence of alternate positive and negative pulses. The RC integrator output in this case is zero and hence the gain control input of the variable gain amplifier is also zero. The gain of the variable gain amplifier decreases, resulting in a reduced step size, which takes care of the situation.

On the receiver side, the output of the quantizer is fed to a variable gain amplifier whose gain control input is derived from an RC integrator and a square law device. Thus an adaptive adjustment of the step size is obtained at the receiver, resulting in an undistorted reception of the transmitted signal.

NOISE IN DELTA MODULATION

Quantization Noise :-

The quantization error in DM is given by

$$\Delta(t) = m(t) - \tilde{m}(t)$$

The maximum quantization error in DM (as long as slope overload is avoided) is 'S', where 'S' is the step size. If it is assumed that the error $\Delta(t)$ takes on all values between $-S$ and $+S$ with equal likelihood, then the probability density function of $\Delta(t)$ is

$$f(\Delta) = \frac{1}{2S}, \quad -S \leq \Delta \leq +S.$$

The normalized power of $\Delta(t)$ is then

$$\overline{[\Delta(t)]^2} = \int_{-S}^{+S} \Delta^2 f(\Delta)\, d\Delta$$

$$= \int_{-S}^{+S} \frac{\Delta^2}{2S}\, d\Delta = \frac{S^2}{3}$$

In can be reasonably assumed that the frequency spectrum of $\Delta(t)$ is white over the range '0' to 'f_b', where $f_b = \frac{1}{\tau}$, 'τ' being the step duration.

The quantization noise power in the frequency range '0 to f_b' is $\frac{s^2}{3}$. Hence the output noise power in the baseband frequency range '0' to 'f_M' is

$$N_q = \frac{s^2}{3}\frac{f_M}{f_b} = \frac{s^2 f_M}{3 f_b} \quad \text{--- (1)}$$

Output Signal Power :-

In PCM, the signal excursion limits are $-MS/2$ to $+MS/2$, where 's' is the step size and 'M' is the number of quantization levels.

Let $m(t) = A \sin \omega_M t$

where $A \to$ amplitude.

and $\omega_M \to 2\pi f_M$, f_M being the maximum frequency component present in the baseband signal.

Then, the output signal power is

$$S_o = \overline{m^2(t)} = \frac{A^2}{2} \quad \text{--- (2)}$$

The maximum slope of $m(t)$ is $\omega_M A$. The maximum average slope of DM approximation $\tilde{m}(t)$ is

$\frac{S}{\tau} = s f_b$ $\left(f_b = \frac{1}{\tau}\right)$.

The limiting value of 'A' just before the start of the slope overload is then by the condition

$$\omega_M A = s f_b \quad \text{--- (3).}$$

$$\Rightarrow A = \frac{s f_b}{\omega_M}$$

$$\therefore \quad S_o = \frac{s^2 f_b^2}{2\omega_M^2} \quad \text{--- (4).}$$

OUTPUT SIGNAL TO QUANTIZATION NOISE RATIO (S/N_q)

$$\therefore \frac{S_o}{N_q} = \frac{s^2 f_b^2}{2(2\pi f_M)^2} \times \frac{3 f_b}{s^2 f_M}$$

$$= \frac{3}{8\pi^2}\left(\frac{f_b}{f_M}\right)^3 \cong \frac{3}{80}\left(\frac{f_b}{f_M}\right)^3 \quad \text{------} (5).$$

COMPARISON BETWEEN PCM AND DM :-

- DM needs a simple circuit as compared to PCM.
- Signal to Quantization noise ratio is less for DM as compared to PCM, because, in PCM, the maximum possible error due to quantization is $\frac{s}{2}$ whereas, it is 's' in DM.
- Moreover, it has been found experimentally that, for voice transmission, the bit rate needed by PCM is 56 kbps, whereas the bit rate needed by DM for same quality of voice transmission is much higher than 56 kbps.
- The bandwidth needed by DM is less than that needed by PCM.

PROBLEMS ON DELTA MODULATION:-

(1) Given a sine wave of frequency f_m and amplitude A_m applied to a delta modulator having step size Δ. Show that the slope overload distortion will occur if

$$A_m > \frac{\Delta}{2\pi f_m T_s}$$, here T_s is the sampling interval.

Solution:- Let us consider that the sine wave is represented as

$$x(t) = A_m \sin(2\pi f_m t)$$

→ It may be noted that the slope of $x(t)$ will be maximum when derivative of $x(t)$ w.r.t. "t" will be maximum.

→ The maximum slope of delta modulator may be given as,

$$\text{Maximum Slope} = \frac{\text{Step size}}{\text{Sampling Period}} = \frac{\Delta}{T_s} \quad \text{(1)}$$

→ We know that slope overload distortion will take place if slope of sine wave is greater than slope of delta modulator

i.e. $$\max\left|\frac{d\,x(t)}{dt}\right| > \frac{\Delta}{T_s}$$

$$\Rightarrow \max\left|\frac{d}{dt}(A_m \sin 2\pi f_m t)\right| > \frac{\Delta}{T_s}$$

$$\Rightarrow \max\left|A_m \cdot 2\pi f_m \cos(2\pi f_m t)\right| > \Delta/T_s$$

$$\Rightarrow A_m 2\pi f_m > \Delta/T_s$$

$$\Rightarrow \boxed{A_m > \Delta/2\pi f_m T_s} \quad \text{(Proved).}$$

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(2) A delta modulator system is designed to operate at five times the Nyquist rate for a signal having a bandwidth equal to 3KHz bandwidth. Calculate the maximum amplitude of a 2KHz input sinusoid for which the delta modulator doesnot have slope over load. Given that the quantizing stepsize is 250mV.

Solution :- As we know that slope overload distortion will occure only when

$$A_m > \frac{\Delta}{2\pi f_m T_s} \quad \text{--- (1)}$$

Thus, slope overload will not occur if

$$A_m \le \frac{\Delta}{2\pi f_m T_s} \quad \text{--- (2)}.$$

Given that $f_m = 3KHz$.

Hence, Nyquist rate $= 2f_m = 6KHz$.

$\therefore$ Sampling frequency, $f_s = 5 \times$ Nyquist rate

$\Rightarrow f_s = 5 \times 6 = 30KHz$.

$\therefore$ Sampling interval, $T_s = 1/f_s = \frac{1}{30 \times 10^3}$ secs.

$\therefore$ Given that step size, $\Delta = 250mV = 0.25V$

Again given that, $f_m = 2KHz = 2 \times 10^3 Hz$.

$$\therefore A_m \le \frac{\Delta}{2\pi f_m T_s}$$

$$\therefore A_m \le \frac{0.25}{2\pi \times 2 \times 10^3 \times \frac{1}{30 \times 10^3}}$$

$\therefore A_m \le 0.6$ volts (Ans).

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Digital Communication Techniques

Topic:

Digital Modulation Techniques

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DIGITAL MODULATION TECHNIQUES.

→ INTRODUCTION:- In digital communication the modulation process involves SWITCHING or KEYING the amplitude, frequency or phase of the carrier in accordance with the input data.

→ Thus, there are three basic modulation techniques for the Tx of digital data.

→ They are known as Amplitude-Shift Keying (ASK), Frequency Shift keying (FSK) and Phase shift keying (PSK) which can be viewed as special cases of amplitude modulation, frequency modulation and phase modulations.

→ Now, when it is required to transmit digital signals on a bandpass channel, the amplitude, frequency or phase of the sinusoidal carrier is varied in accordance with the incoming digital data.

→ Since the digital data is in discrete steps, so the modulation of the carrier signal is also done in discrete steps.

→ Due to this reason, this type of Digital modulation is also known as SWITCHING or SIGNALING.

→ Now, if amplitude of the carrier is switched depending on the input digital signal, then it is called

Amplitude Shift Keying (ASK).

- If the frequency of the sinusoidal carrier is switched depending upon the input signal then it is known as frequency shift keying (FSK).
- If the phase of the carrier is switched depending upon the input digital signal, then it is known as Phase shift Keying (PSK).
- Because of the constant amplitude envelope of FSK and PSK, the effect of non-linearities noise interference is minimum on signal detection. However, these effects are more pronounced on ASK.
- Therefore, FSK and PSK are preferred over ASK.

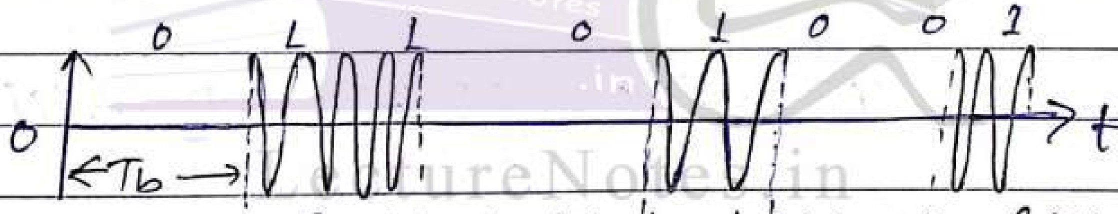


fig (a) Amplitude-shift keying (ASK).

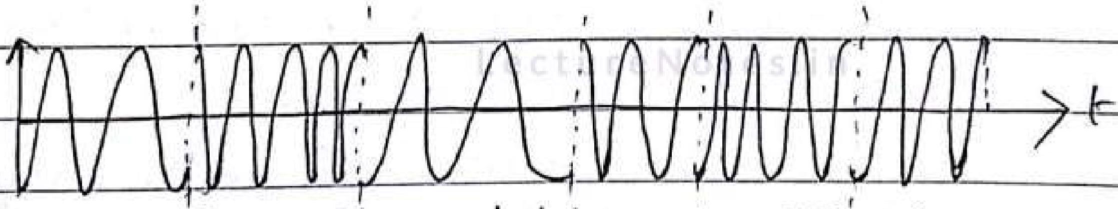


fig (b) Phase-shift keying (PSK).

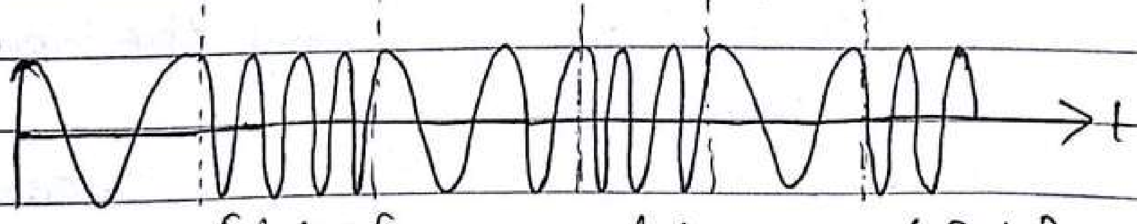


fig (c) Frequency-shift keying (FSK).

In digital modulations, instead of

transmitting one bit at a time, we transmit two or more bits simultaneously.

→ This is known as M-ary Transmission.

→ However, sometimes we use two quadrature carriers for modulation. This process is known as QUADRATURE MODULATION.

→ The digital modulation scheme should possess the following design characteristics:

(a) Maximum Data Rate.

(b) Minimum Error probability.

(c) Minimum transmitted power.

(d) Maximum channel bandwidth.

(e) Maximum resistance to interfering signals.

(f) Minimum circuit complexity.

BINARY AMPLITUDE SHIFT KEYING OR ON-OFF KEYING (ASK):-

→ Amplitude shift Keying (ASK) or ON-OFF keying is the simplest digital modulation technique.

→ In this method, there is only one unit energy carrier and it is switched ON or OFF depending the input binary sequence.

→ Let us assume that the carrier is given as

$$s(t) = A \cos(2\pi f_c t) \quad \text{— (1)}$$

Here 'A' represents peak value of

sinusoidal carrier. For the standard 1Ω Load resistor, the power (normalized) dissipated would be

$$P_s = \frac{1}{2}A^2 \Rightarrow A = \sqrt{2P_s} \quad \text{—— (2)}.$$

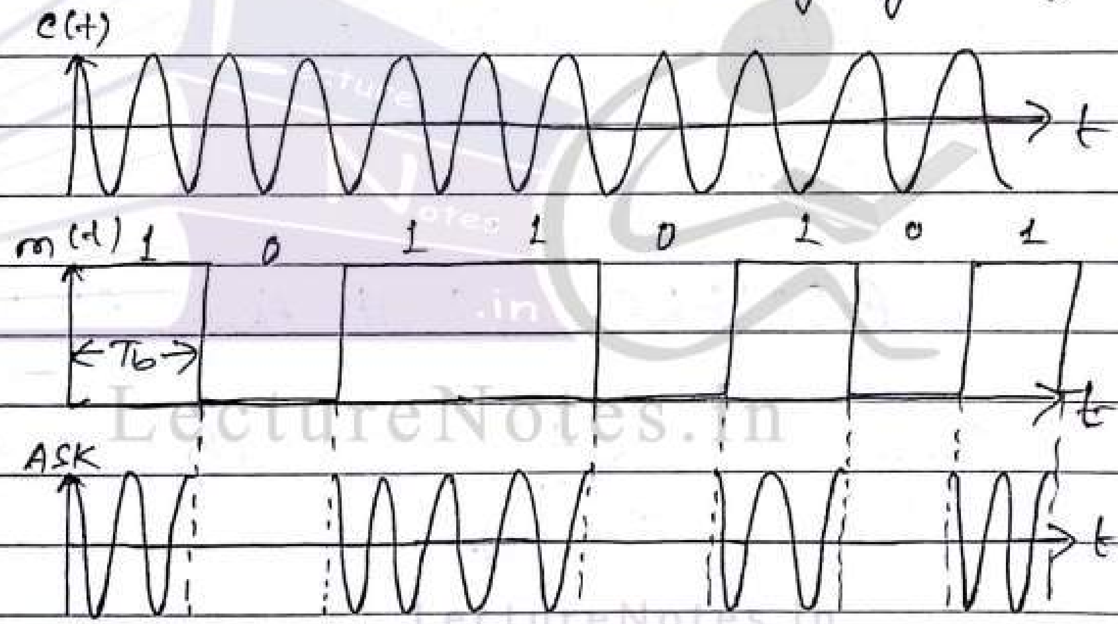
→ The ASK waveform may be represented as

$$s(t) = \sqrt{2P_s}\,\cos(2\pi f_c t) \quad \text{—— (3)}$$

(To represent "1").

→ To transmit symbol '0' the signal $s(t) = 0$, i.e. no signal is transmitted.

→ Hence, the ASK waveform Looks like an ON-OFF of the signal. Therefore, it is also known as ON-OFF Keying (OOK).



→ The above figure shows the ASK waveform.

SIGNAL SPACE DIAGRAM OF ASK:-

→ The ASK waveform of eq^n(3) for symbol "1" can be represented as

$$S(t) = \sqrt{P_s T_b} \times \sqrt{2/T_b}\,\cos(2\pi f_c t)$$

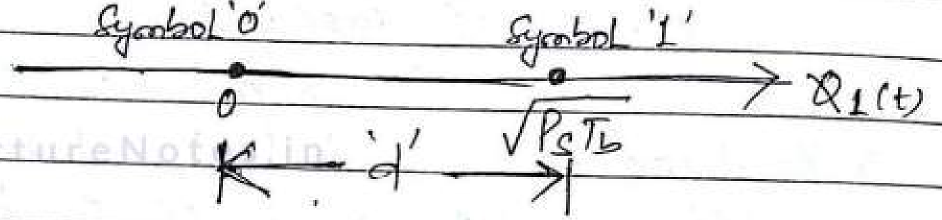
$$\Rightarrow S(t) = \sqrt{P_s T_b}\,\phi_1(t) \quad \text{—— (4)}.$$

→ This means that there is only one

carrier function $\varphi_1(t)$.

→ The signal space diagram will have two points on $\varphi_1(t)$.

→ One will be at zero and other will be at $\sqrt{P_s T_b}$.

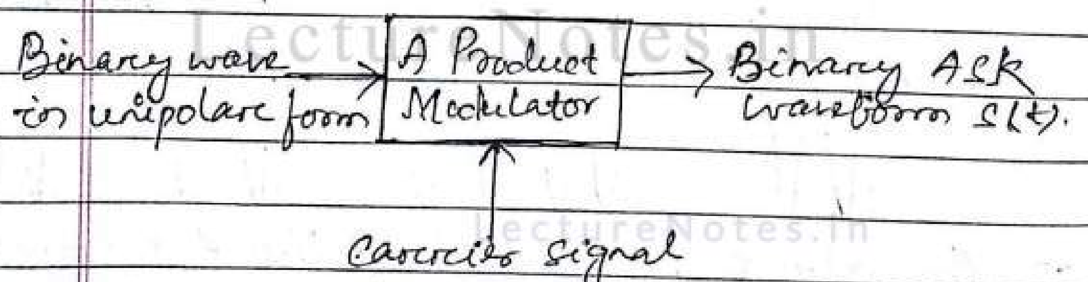


→ The above figure shows the signal space diagram or constellation diagram of ASK.

→ Thus, the distance between two signal points is known as Euclidean distance and it is given as

$$d = \sqrt{P_s T_b} = \sqrt{E_b} \quad \text{——— (5).}$$

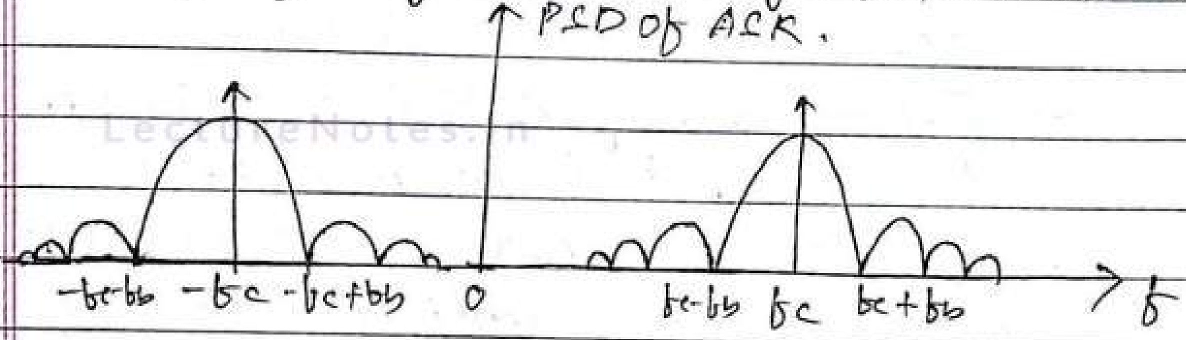
GENERATION OF ASK SIGNAL :-



→ ASK signal may be generated by simply applying the incoming binary data (represented in unipolar form) and the sinusoidal carrier to the two inputs of a product modulator.

→ The resulting output will be the ASK waveform, which is shown in above fig.

→ The ASK signal which is basically the product of the binary sequence and the carrier signal, has a power spectral density (PSD) same as that of input baseband on-off signal but shifted in the frequency domain by $\pm f_c$, shown below.

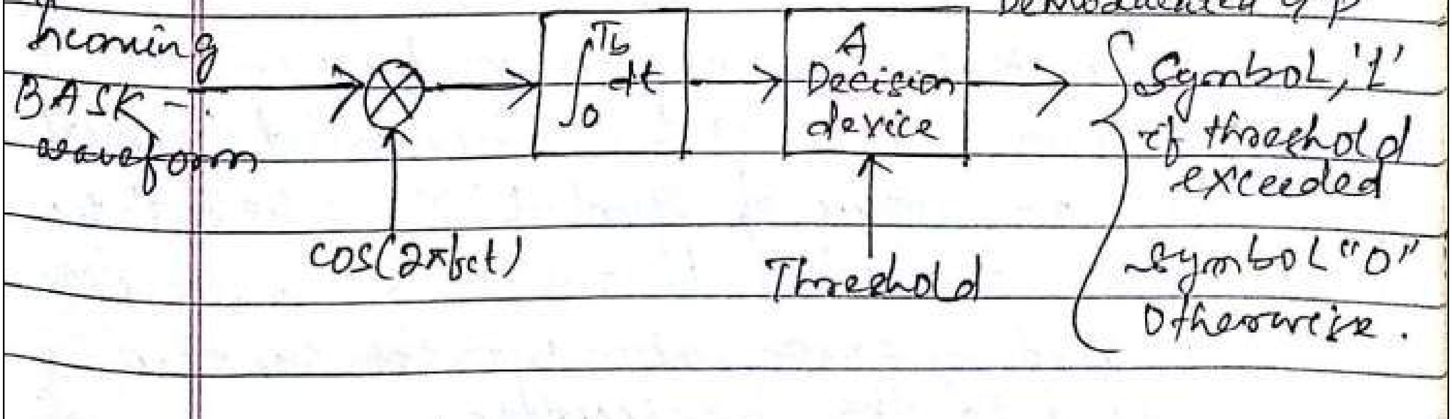


→ The spectrum of the ASK signal shows that it has infinite bandwidth.

→ It may be observed that for practical purposes, the bandwidth of ASK signal is approximately $3/T_b$ Hz.

→ The bandwidth of the ASK signal can however, be reduced by using smoothed versions of the pulse waveform instead of rectangular pulse waveform.

DEMODULATION OF BASK :-



→ The demodulation of binary ASK signal can be achieved with the help of coherent detector as shown above.
→ It consists of a product modulator followed by an integrator combinely known as CORRELATOR and a decision-making device.
→ The incoming ASK signal is applied to one input of the product modulator.
→ The other input of the product modulator is supplied with a carrier which is generated with the help of a Local oscillator.
→ The output of the product modulator goes to input of the integrator.
→ The integrator operates on the o/p of the multiplier for successive bit intervals and essentially performs a Low-pass filtering action.
→ The output of the integrator goes to the input of decision-making device.
→ Now, the decision-making device compares the output of the integrator with a preset threshold.
→ It makes a decision in favour of symbol '1' when the threshold exceeded and in favour of symbol '0', otherwise.
→ It is assumed that, the local carrier is in perfect synchronisation with the carriers used in the Transmitter.

DIGITAL MODULATION SCHEMES

The process of mapping a digital sequence to signals for transmission over a communication channel is called DIGITAL MODULATION or DIGITAL SIGNALING.

The mapping between the digital sequence (which we assume to be a binary sequence) and the signal sequence to be transmitted over the channel can be either MEMORYLESS or WITH MEMORY, resulting in memoryless modulation schemes and modulation scheme with memory.

In a memoryless modulation scheme, the binary sequence is divided into subsequences and each subsequence is mapped into one of the $s_m(t)$ where $1 \leq m \leq 2^k$. This modulation scheme is equivalent to a mapping from $M = 2^k$ messages to M possible signals.

In a modulation scheme with memory, the mapping is from the set of the current bits and the past bits to the set of possible $M = 2^k$ messages.

BINARY PHASE-SHIFT KEYING (BPSK):-

In BPSK, the transmitted signal is a sinusoid of fixed amplitude. It has one fixed phase when the data is at one level and when the data is at the other level the phase is different by 180°. If the sinusoid is of amplitude 'A' it has a power $P_s = \frac{1}{2}A^2$ so that $A = \sqrt{2P_s}$
$= \sqrt{\frac{2E_b}{T_b}}$

In BPSK system, the pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols '1' and '0', respectively, is defined by

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \quad \text{——— (1)}$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi)$$

$$= -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \quad \text{——— (2).}$$

where $0 \le t \le T_b$, and E_b is the transmitted signal energy per bit.

A pair of sinusoidal waves that differ only in a relative phase-shift of 180° as indicated by eq's (1) and (2) are referred to as antipodal signals.

In BPSK there is only one basis function of unit energy i.e.

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \quad \text{——— (3).}$$

then we express the transmitted signals $s_1(t)$ and $s_2(t)$ in terms of $\phi_1(t)$ as follows

$$\therefore s_1(t) = \sqrt{E_b}\, \phi_1(t) \quad \text{——— (4).}$$

$$s_2(t) = -\sqrt{E_b}\, \phi_1(t) \quad \text{——— (5).}$$

A BPSK system is characterized by having a signal space that is one-dimensional (i.e. $N=1$), with a signal constellation consisting of two message points (i.e. $M=2$).

The co-ordinates of message points are

$$s_{11} = \int_0^{T_b} s_1(t)\phi_1(t)\,dt = +\sqrt{E_b} \quad \text{——— (6).}$$

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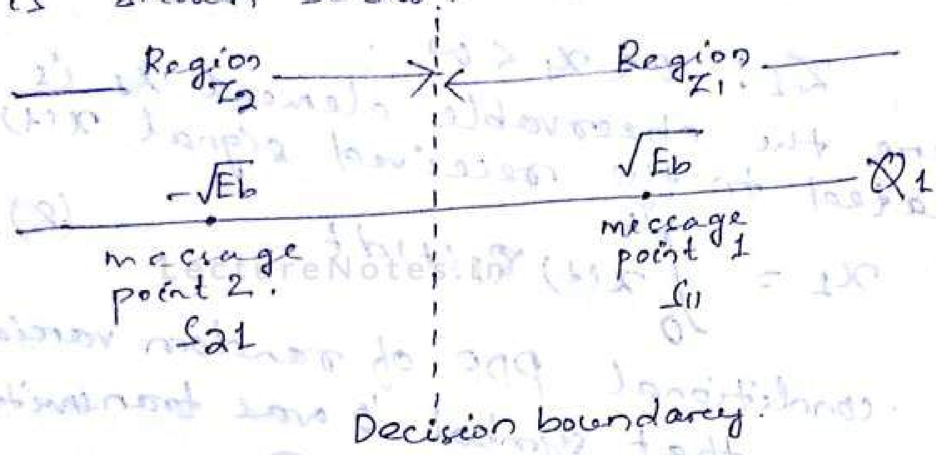
and

$$s_{21} = \int_0^{T_b} s_2(t)\,\phi_1(t)\,dt \quad \text{———} \quad (7).$$
$$= -\sqrt{E_b}$$

The corresponding signal-space diagram is shown below.



Error Probability of BPSK

The decision rule is simply to decide that signal $s_1(t)$ (i.e. binary symbol '1') was transmitted if the received signal point falls in the region Z_1 and decide that the signal $s_2(t)$ (i.e. binary symbol '0') was transmitted if the received signal point falls in region Z_2.

But during this course of tx and Rx, there are two kinds of error generated. signal $s_2(t)$ is transmitted, but the noise is such that the received signal point falls inside the region Z_1 and so the receiver decides in favour of signal $s_1(t)$.

Alternatively, signal $s_1(t)$ is transmitted but the noise is such that the received signal point falls inside region Z_2 and so the receiver decides in favour of signal $s_2(t)$.

To calculate the probability of making an error of the first kind, the decision region associated with symbol '1' or signal $s_1(t)$ is described by

$$Z_1 : 0 < x_1 < \infty$$

where the observable element x_1 is related to the received signal $x(t)$ by

$$x_1 = \int_0^{T_b} x(t)\, \phi_1(t)\, dt \quad \text{———— (8).}$$

The conditional PDF of random variable X_1, given that symbol '0' was transmitted is defined by

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{21})^2\right]$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] \quad \text{———— (9)}$$

The conditional probability of the receiver deciding in favour of symbol '1', given that symbol '0' was transmitted is

$$P_{10} = \int_0^{\infty} f_{X_1}(x_1|0)\, dx_1$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] dx_1 \quad \text{———— (10)}$$

Putting $Z = \frac{1}{\sqrt{N_0}}(x_1 + \sqrt{E_b})$ ———— (11). $\Rightarrow \frac{dZ}{dx_1} = \frac{1}{\sqrt{N_0}} \Rightarrow dx_1 = \sqrt{N_0}\, dZ$

$$P_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-Z^2)\, dZ$$

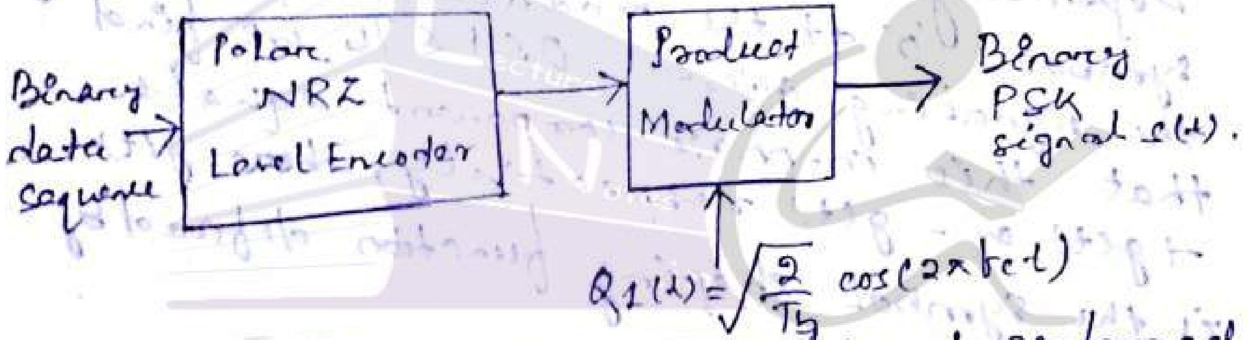
$$= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \quad \text{—— (12)}$$

Similarly we can also find out the conditional probability P_{01} which has the same value given by eqⁿ (12).

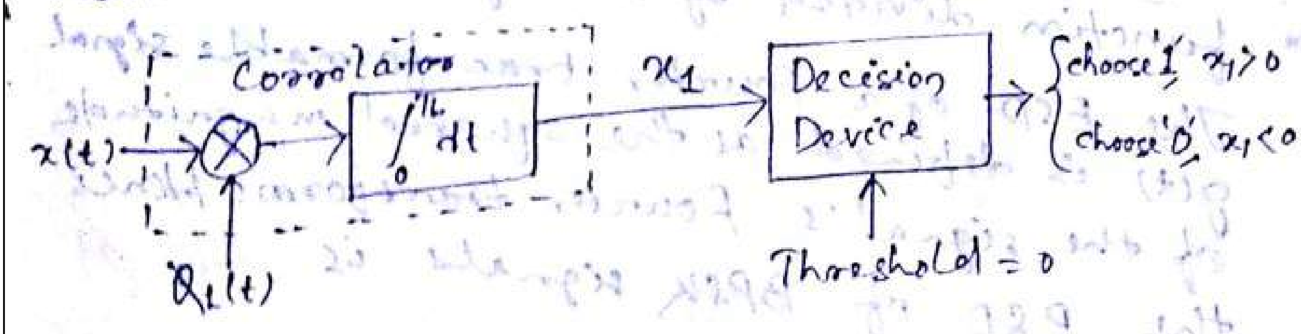
Thus, averaging the conditional error probability P_{10} and P_{01} we find that the average probability of symbol error or equivalently the bit error rate (BER) of BPSK is given as

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)} \quad \text{——— (13)}.$$

GENERATION AND DETECTION OF BPSK.



The signal transmission encoding is performed by a polar NRZ Level encoder. The resulting binary wave and a sinusoidal carrier $Q_1(t)$ are applied to a product modulator. The carrier and the timing pulses used to generate the BPSK signal obtained at the modulator output.



To detect the original binary sequence

of 1s and 0s, we apply the noisy PSK signal $x(t)$ to a correlator which is also supplied with the carrier $\phi_1(t)$. The correlator output x_1, is compared with a threshold of zero volts. If $x_1 > 0$, the receiver decides in favour of symbol '1'. On the other hand, if $x_1 < 0$, it decides in favour of symbol '0'. If x_1 is exactly zero the receiver makes a random guess in favour of '0' or '1'.

POWER SPECTRA OF BPSK SIGNALS

Depending on whether we have symbol '1' or symbol '0' at the modulator input during the signaling interval $0 \le t \le T_b$, we find that the in-phase component equals $+g(t)$ or $-g(t)$ respectively, where $g(t)$ is the symbol shaping function defined by

$$g(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}}, & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \qquad (14)$$

The P.S.D. of a random binary wave is equal to the ESD of the symbol shaping function divided by the symbol duration.

The ESD of a Fourier transformable signal $g(t)$ is defined as the squared magnitude of the signal's Fourier transform. Hence the PSD of BPSK signals is

$$S_B(f) = \frac{2E_b \sin^2(\pi T_b f)}{(\pi T_b f)^2} = 2E_b \, \text{sinc}^2(T_b f) \qquad (15)$$

QUADRIPHASE - SHIFT KEYING (QPSK) :-

In order to achieve highly reliable performance in the communication system, the most important things needed are low probability error, efficient utilization of bandwidth of the channel. This is obtained by a very bandwidth - conserving modulation scheme known as QPSK which is an example of quadrature - carrier multiplexing.

In QPSK, information carried by the transmitted signal is contained in the phase. In particular, the phase of the carrier takes on one of four equally spaced values, such as $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$. The Tx signal is defined as

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right], & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \quad \text{--- (1)}$$

where $i = 1, 2, 3, 4$. E is the transmitted signal energy per symbol and T is the symbol duration. Each possible value of the phase corresponds to a unique dibit. For example, we may choose the foregoing set of phase values to represent the Gray-encoded set of dibits: 10, 00, 01, and 11.

SIGNAL- SPACE DIAGRAM OF QPSK

The transmitted signal given by eq^n (1) can be represented as

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[(2i-1)\pi/4\right]\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin\left[(2i-1)\pi/4\right]\sin(2\pi f_c t) \quad \text{--- (2)}$$

Based on this representation we can make the following observations:

- There are two Orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ which are defined by a pair of quadrature carriers:

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad \text{——— (3)}$$

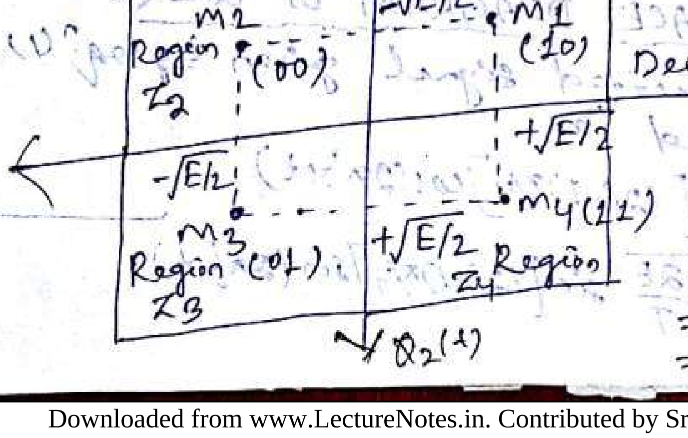
$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \quad \text{——— (4)}$$

- There are four message points and the associated signal vectors are defined by

$$s_i = \begin{bmatrix} \sqrt{E}\cos\left((2i-1)\pi/4\right) \rightarrow s_{i1} \\ -\sqrt{E}\sin\left((2i-1)\pi/4\right) \rightarrow s_{i2} \end{bmatrix} \quad i = 1,2,3,4 \quad \text{——— (5)}$$

Signal-space characterization of QPSK:-

Gray encoded i/p Dibit	Phase of QPSK signal (radian)	Co-ordinates of message points s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$ (m_1)
00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$ (m_2)
01	$5\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$ (m_3)
11	$7\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$ (m_4)



→ Region Z_1

Decision boundary $\rightarrow \phi_2(t)$

Euclidean distance

$$d = 2\sqrt{E_b}$$
$$= \sqrt{E/2} + \sqrt{E/2} = 2\sqrt{E/2}$$
$$= \sqrt{4E/2} = \sqrt{2E} = \sqrt{2 \cdot 2E_b} = 2\sqrt{E_b}$$

ERROR PROBABILITY of Q-PSK

In Q-PSK system, the received signal x(t) is

$$x(t) = S_i(t) + w(t) \quad \text{——— (6)}$$

where $i = 1, 2, 3, 4$ and $0 \leq t \leq T$.

$w(t)$ is the sample function of a white Gaussian noise process of zero mean and psd $N_0/2$.

The observation vector x has two elements i.e. x_1 and x_2, defined by

$$x_1 = \int_0^T x(t)\, \phi_1(t)\, dt.$$

$$= \sqrt{E} \cos\left[(2i-1)\pi/4\right] + w_1$$

$$= \pm\sqrt{\frac{E}{2}} + w_1 \quad \text{——— (7)}.$$

and $$x_2 = \int_0^T x(t)\, \phi_2(t)\, dt$$

$$= -\sqrt{E} \sin\left[(2i-1)\pi/4\right] + w_2$$

$$= \mp\sqrt{\frac{E}{2}} + w_2 \quad \text{——— (8)}.$$

x_1, x_2 are the sampled value of the independent random variable with mean values $\pm\sqrt{E/2}$ and $\mp\sqrt{E/2}$ and with a common variance equal to $N_0/2$.

Here the signal energy per bit i.e. $E_b = E/2$.

A QPSK system is in fact equivalent to two BPSK systems working in parallel and using two carriers that are inphase and quadrature phase.

In each channel (inphase or quadrature phase) average probability of error

$$p' = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{(E/2)}{N_0}} \quad (9)$$

$$\Rightarrow p' = \frac{1}{2} \operatorname{erfc} \sqrt{E/2N_0}$$

Average probability of correct decision is

$$P_c = 1 - p' = 1 - \frac{1}{2} \operatorname{erfc} \sqrt{E/2N_0} \quad (10)$$

(per channel)

The average probability of a correct decision resulting from the combined action of the two channels working together is

$$P_c = (1-p')^2$$

$$= \left(1 - \frac{1}{2} \operatorname{erfc} \sqrt{E/2N_0}\right)^2$$

$$= 1 + \frac{1}{4} \operatorname{erfc}^2 \sqrt{E/2N_0} - \operatorname{erfc} \sqrt{E/2N_0}$$

Then the average probability of symbol error

$$P_e = 1 - P_c$$

$$\Rightarrow P_e = \operatorname{erfc}\sqrt{E/2N_0} - \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{E/2N_0}\right)$$

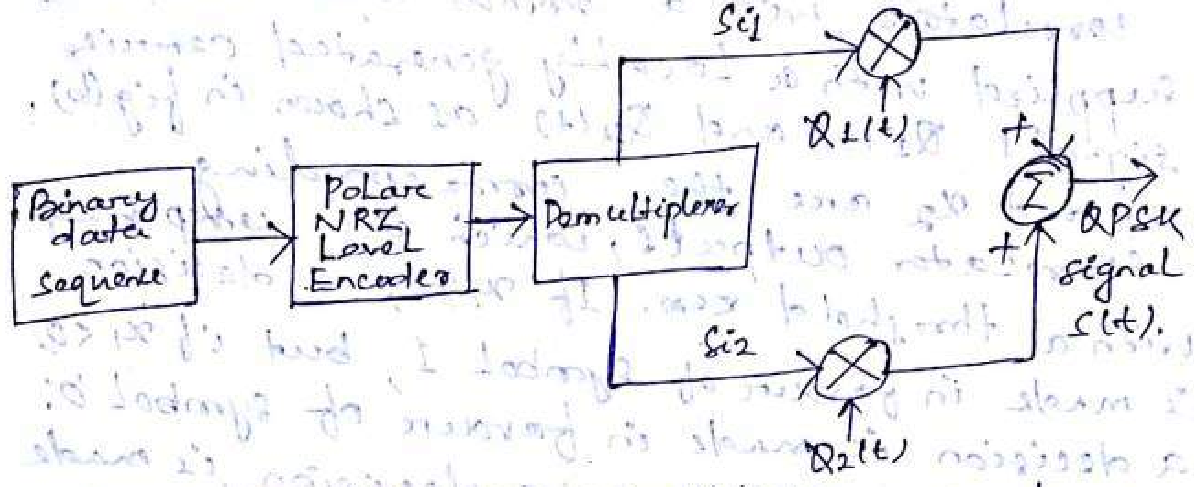
If $\sqrt{E/2N_0} >> 1$, quadratic term can be ignored.

$$\therefore P_e = \operatorname{erfc}\sqrt{(E/2N_0)} = \operatorname{erfc}\sqrt{\frac{E_b}{N_0}}$$

For each channel

$$B.E.R, \; P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{N_0}} \quad (11)$$

GENERATION AND DETECTION OF QPSK SIGNALS



Fig(a) QPSK Transmitter.

The incoming binary data sequence is first transformed into polar form by a NRZ level encoder. Thus, symbols '1' and '0' are represented by $+\sqrt{E_b}$ and $-\sqrt{E_b}$ respectively. This binary wave is next divided by means of a Demux into two separate binary waves consisting of the odd- and even-numbered input bits. These two binary waves are denoted by S_{i1} and S_{i2}. The two binary waves S_{i1} and S_{i2} are used to modulate with a pair of quadrature carriers i.e. $Q_1(t)$ and $Q_2(t)$. This results in producing two BPSK signals and then this two BPSK signals are added to produce the desired QPSK signal.

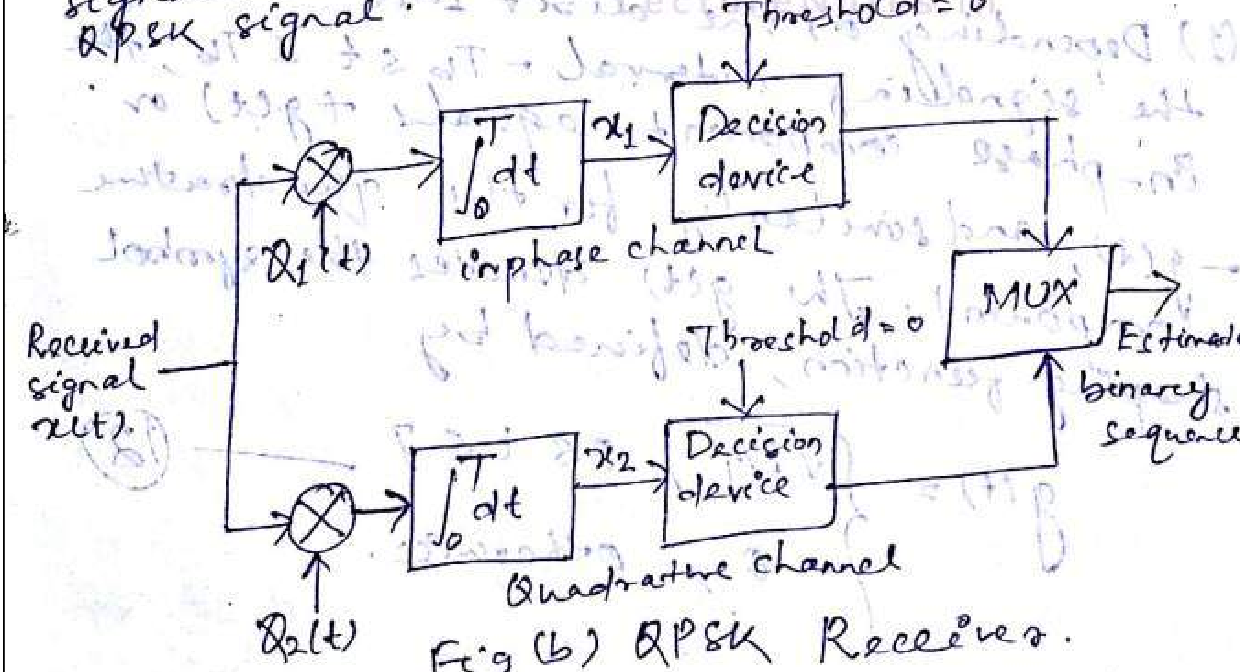


Fig (b) QPSK Receiver.

The QPSK receiver consists of a pair of correlators with a common input and also supplied with a locally generated carrier signals $Q_1(t)$ and $Q_2(t)$ as shown in fig(b). x_1 and x_2 are the corresponding correlator outputs, which are compared with a threshold zero. If $x_1 > 0$, a decision is made in favour of symbol 1, but if $x_1 < 0$, a decision is made in favour of symbol '0'. Similarly, if $x_2 > 0$, a decision is made in favour of symbol '1' for the quadrature channel output, but if $x_2 < 0$, a decision is made in favour of symbol 0. Finally, in-phase and quadrature channel outputs are combined in a multiplexer to reproduce the original binary sequence.

POWER SPECTRA OF QPSK SIGNALS

We make the following observations pertaining to the in-phase and quadrature components of a QPSK signal;

(1) Depending on the dibit sent during the signalling interval $-T_b \le t \le T_b$, the in-phase component equals $+g(t)$ or $-g(t)$, and similarly for the quadrature component. The $g(t)$ denotes the symbol shaping function, defined by

$$g(t) = \begin{cases} \sqrt{E/T}, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases} \qquad (12)$$

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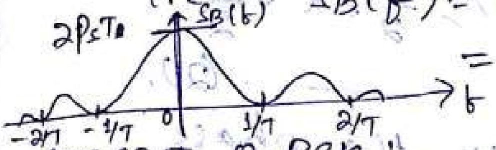
(2) The in-phase and quadrature components are statistically independent. The psd of the QPSK signal equals the sum of the individual psd of the in-phase and quadrature components.

i.e. $S_B(f) = 2E\,\text{sinc}^2(Tf)$ ——(13).

$= 4E_b\,\text{sinc}^2(2T_b f)$



$B/W_{(QPSK)} = \frac{1}{T} - (-1/T) = 2/T = \frac{2}{2T_b} = \frac{1}{T_b} = f_b$

OFFSET Q-PSK :-

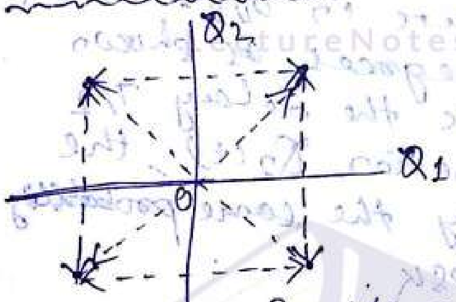


Fig (a) Possible path switching betn the message points in QPSK

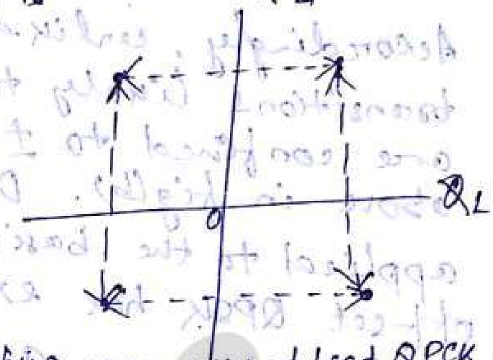


Fig. (b) offset QPSK.

(1) The carrier phase changes by ±180 degrees whenever both the in-phase and quadrature phase components of the QPSK signal changes sign.

(2) The carrier phase changes by ±90 degrees whenever the in-phase or quadrature component changes sign.

(3) The carrier phase is unchanged when neither in-phase component nor the quadrature component changes sign.

The extent of amplitude fluctuations exhibited by QPSK signals may be reduced by using offset QPSK. In offset QPSK, the bit stream responsible for generating the quadrature component is delayed (i.e. offset) by half a symbol interval wrt inphase component.

The two basis functions of offset QPSK are defined by

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \le t \le T \quad \text{---} (1)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad \frac{T}{2} \le t \le \frac{3T}{2} \quad \text{---} (2).$$

Accordingly, unlike QPSK, the phase transitions likely to occure in offset QPSK are confined to ±90 degrees, as shown above in fig(b). Despite the delay T/2 applied to the basis function $\phi_2(t)$, the offset QPSK has exactly the same probability of symbol error as QPSK.

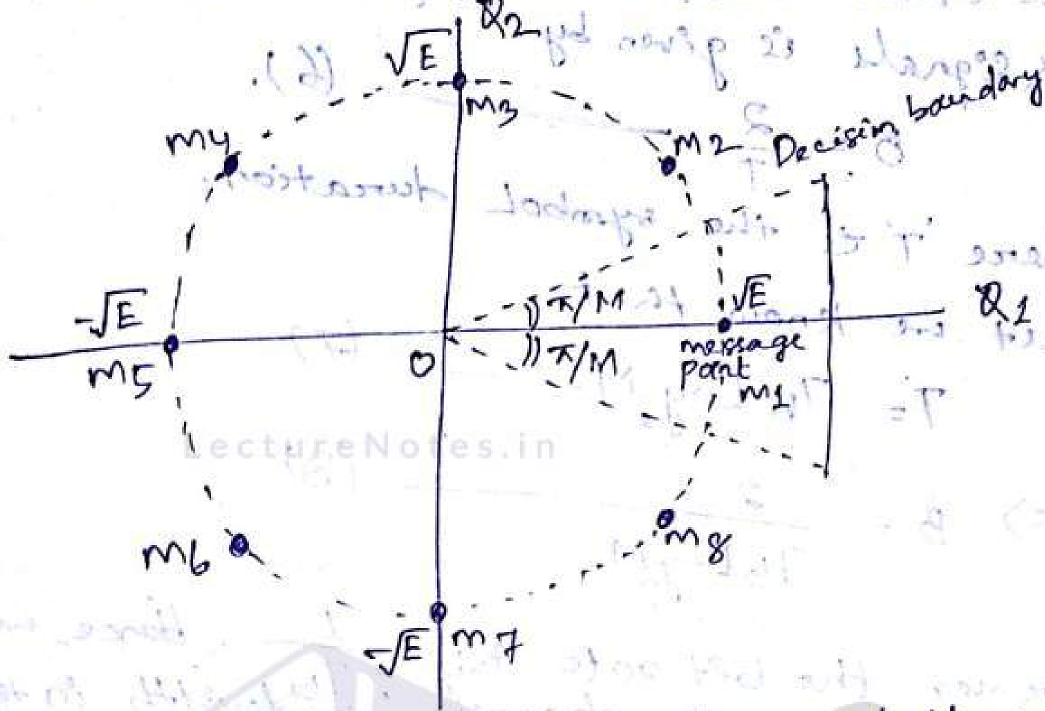
M-ARY PSK:-

QPSK is a special case of M-ary PSK where the phase of the carrier takes on one of M possible values, such as $\theta_i = (2i-1)\pi/M$, where $i = 1, 2, \ldots M$.

During each signaling interval of duration T, one of the M possible signals is

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right) \quad \text{---} (1)$$

Each $S_i(t)$ may be expanded in terms of the same two basis functions $\phi_1(t)$ and $\phi_2(t)$ as defined by QPSK. The signal constellation of M-ary PSK is therefore two-dimensional. The message points are equally spaced on a circle of radius $\sqrt{E}$ and center at the origin is shown below for the case of 8-PSK (M=8).



The Euclidean distance of each of these two points i.e. m_2 and m_8 from m_1 is

$$d_{12} = d_{18} = 2\sqrt{E}\,\sin\left(\frac{\pi}{M}\right) \quad \text{—— (2).}$$

The average probability of symbol error for M-ary PSK is

$$P_e \cong \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\,\sin\left(\frac{\pi}{M}\right)\right) \quad \text{—— (3).}$$

POWER SPECTRA OF M-ARY PSK SIGNALS

The symbol duration of M-ary PSK is given as

$$T = T_b \log_2 M \quad \text{—— (4).}$$

where T_b is the bit duration.

The power spectral density (PSD) of an M-ary PSK signal is

$$S_B(f) = 2E\,\operatorname{sinc}^2(Tf)$$
$$= 2E_b \log_2 M\,\operatorname{sinc}^2(T_b f \log_2 M) \quad \text{—— (5).}$$

BANDWIDTH EFFICIENCY OF M-ARY PSK SIGNALS

The channel bandwidth required to pass M-ary PSK signals is given by

$$B = \frac{2}{T} \quad \text{——— (6).}$$

where 'T' is the symbol duration.

But we know that

$$T = T_b \log_2 M \quad \text{——— (7).}$$

$$\Rightarrow B = \frac{2}{T_b \log_2 M} \quad \text{——— (8).}$$

Moreover the bit rate $R_b = \frac{1}{T_b}$. Hence, we may redefine the channel bandwidth in terms of the bit rate R_b as

$$B = \frac{2}{\frac{1}{R_b} \log_2 M}$$

$$\Rightarrow B = \frac{2R_b}{\log_2 M} \quad \text{——— (9).}$$

Based on this formula, the bandwidth efficiency of M-ary PSK signals is given by

$$\rho = \frac{R_b}{B}$$

$$\Rightarrow \rho = \frac{\log_2 M}{2} \quad \text{——— (10).}$$

M-ARY QUADRATURE AMPLITUDE MODULATION

The M-ary QAM scheme is hybrid in nature, in that the carrier experiences amplitude as well as phase modulation.

M-ary QAM is a two dimensional generalization which involves two orthogonal pass band basis functions, as shown by

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \le t \le T \text{ —— } (1)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \le t \le T \text{ —— } (2)$$

Let the ith message point s_i in the (ϕ_1, ϕ_2) plane be denoted by $(a_i d_{min}/2,\ b_i d_{min}/2)$, where d_{min} is the minimum distance between any two message points in the constellation, a_i and b_i are the integers, $i = 1, 2, \dots M$.

Let $(d_{min}/2) = \sqrt{E_0}$, where E_0 is the energy of the signal with the lowest amplitude.

The transmitted M-ary QAM signal for symbol 'k', is defined as

$$s_k(t) = \sqrt{\frac{2E_0}{T}}\, a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}}\, b_k \sin(2\pi f_c t) \text{ —— } (3)$$

where $0 \le t \le T$ and $k = 0, \pm 1, \pm 2, \dots$

The signal $s_k(t)$ consists of two phase-quadrature carriers with each one being modulated by a set of discrete amplitudes, hence the name QUADRATURE AMPLITUDE MODULATION.

QAM SQUARE CONSTELLATIONS:-

With an even number of bits per symbol, we may write

$$L = \sqrt{M} \text{ —— } (4).$$

Under this condition, an M-ary QAM square constellation can always be viewed as the Cartesian product of a one-dimensional L-ary PAM constellation with itself.

In the case of a QAM square constellation, the ordered pairs of co-ordinates naturally form a square matrix as given below:-

$$\{a_i, b_i\} = \begin{bmatrix} (-L+1, L-1) & (-L+3, L-1) & \cdots & (L-1, L-1) \\ (-L+1, L-3) & (-L+3, L-3) & \cdots & (L-1, L-3) \\ \vdots & & & \vdots \\ (-L+1, -L+1) & (-L+3, -L+1) & \cdots & (L-1, -L+1) \end{bmatrix} \quad (5)$$

Let us consider a 16-QAM whose signal constellation is shown below:-

The encoding of the message points shown above is as follows:-

(1) Two of the four bits, namely the left-most two bits specify the quadrant in which message point lies. The four quadrants are represented by 11, 10, 00, and 01.

(2) The remaining two bits are used to represent one of the four possible symbols lying within each quadrant.

ERROR PROBABILITY OF M-ARY QAM :-

To calculate the probability of symbol error for M-ary QAM, we exploit the property that a QAM square constellation can be factored into the product of the corresponding PAM constellation with itself. We may thus proceed as follows:-

(1) The probability of correct detection for M-ary QAM may be written as

$$P_c = (1 - P_e')^2 \quad \text{——— (6)}$$

where P_e' is the probability of symbol error for the corresponding L-ary PAM with $L = \sqrt{M}$.

(2) The probability of symbol error P_e' is defined by

$$P_e' = \left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \quad \text{——— (7)}$$

(3) The probability of symbol error for M-ary QAM is given by

$$P_e = 1 - P_c = 1 - (1 - P_e')^2 \quad \text{——— (8)}$$

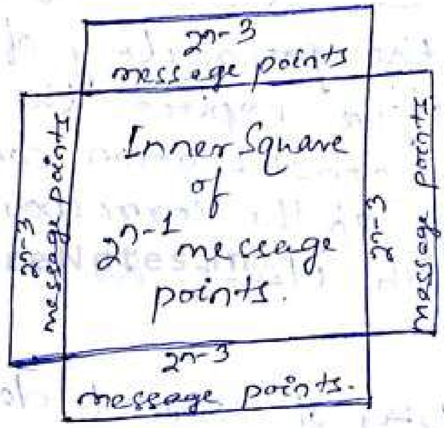
$$\simeq 2P_e'$$

where it is assumed that P_e' is small enough compared to unity to justify ignoring the quadratic term.

$$\therefore P_e \simeq 2\left(1 - \frac{1}{\sqrt{M}}\right)\left[\operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right)\right] \quad \text{——— (9)}$$

QAM CROSS CONSTELLATION :-

To generate an M-ary QAM signal with an odd number of bits per symbol, we require the use of a cross constellation.



Let us consider we want to construct such a signal constellation with n bits per symbol by proceeding as follows:-

(a) Start with a QAM square constellation with $(n-1)$ bits per symbol.

(b) Extend each side of the QAM square constellation by adding 2^{n-3} symbols.

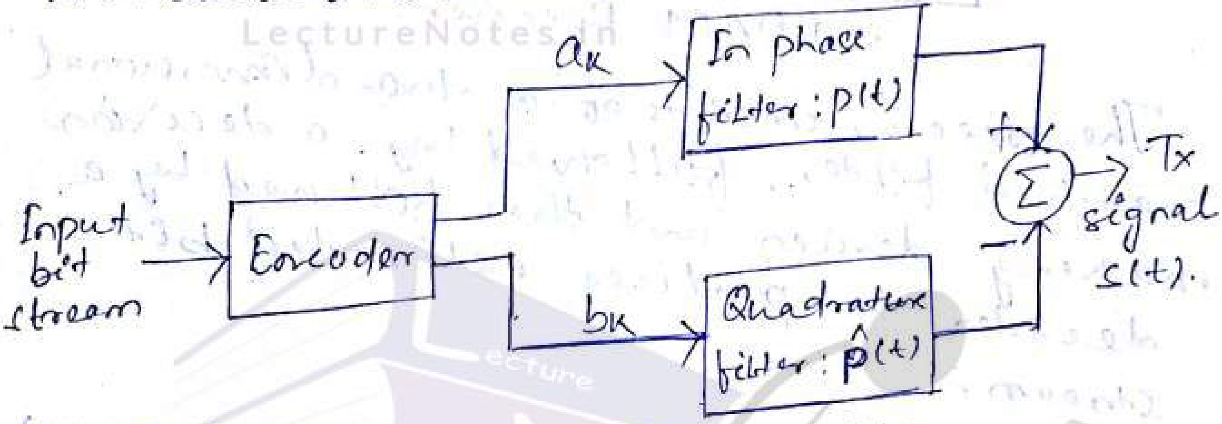
(c) Ignore the corners in the extension.

The inner square represents 2^{n-1} symbols. The four side extensions add $4 \times 2^{n-3} = 2^2 \times 2^{n-3} = 2^{n-1}$ symbols. The total no. of symbols in the cross constellation is therefore $2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$ and therefore represents n bits per symbol as desired.

Unlike QAM square constellation, it is not possible to express a QAM cross constellation as the product of a PAM constellation with itself. The probability of symbol error in this case it is given as

$$P_e \simeq 2\left(1-\frac{1}{\sqrt{2M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \quad (10)$$

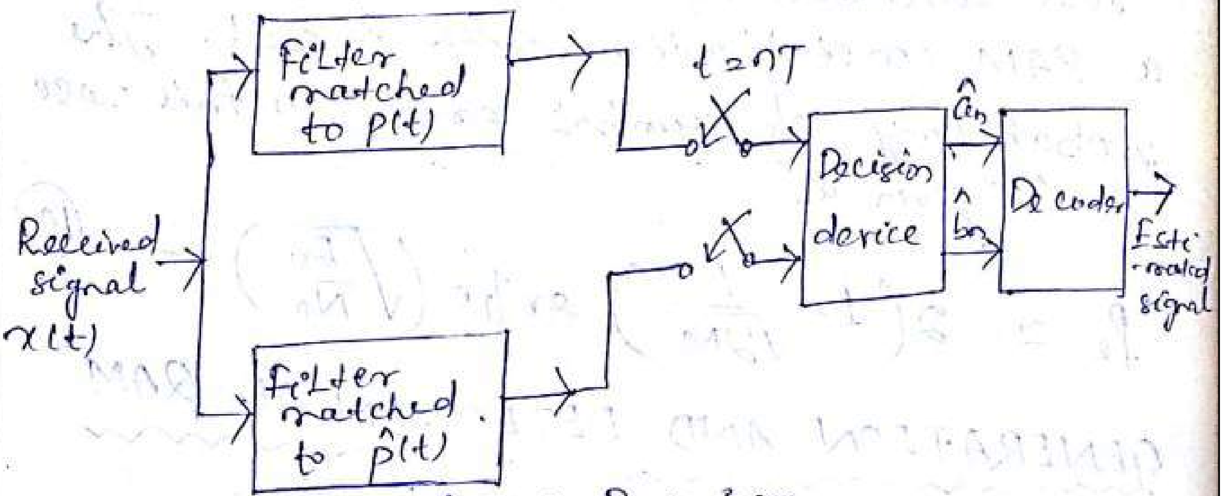
GENERATION AND DETECTION OF QAM



(a) QAM Transmitter.

The transmitter consists of a multilevel encoder and a pair of passband filters.
The multilevel encoder partitions the incoming serial data stream into blocks of n bits each, and then these blocks are mapped into multilevel symbols a_K and b_K where 'K is the symbol period. The passband in phase and quadrature filters process the symbol streams $\{a_K\}$ and $\{b_K\}$ in parallel respectively. The resulting output of these two filters are subtracted to produce the transmitted signal s(t).
The transmitted signal s(t) propagates through a channel of impulse response h(t) and additive noise w(t). So received signal is

$$\therefore x(t) = s(t) * h(t) + w(t) \quad (11)$$



(b) QAM Receiver.

The receiver consists of a two-dimensional matched filter, followed by a decision making devices and then followed by a decoder to produce estimated bit stream.

BINARY FREQUENCY SHIFT KEYING (BFSK)

In a BFSK system, symbols 1 and 0 are distinguished from each other by transmitting one of two sinusoidal waves that differ in frequency by a fixed amount.

A typical pair of sinusoidal waves is described by

$$S_i(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} \cos(2\pi f_i t), & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases} \quad \text{--- (1)}$$

where $i = 1, 2$ and E_b is the transmitted energy per bit.

Thus symbol '1' is represented by $s_1(t)$

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and symbol '0' is by $s_2(t)$. The FSK signal is described here is known as Sunde's FSK.

The orthonormal basis function is

$$\phi_i(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t), & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases} \quad \text{---} (2).$$

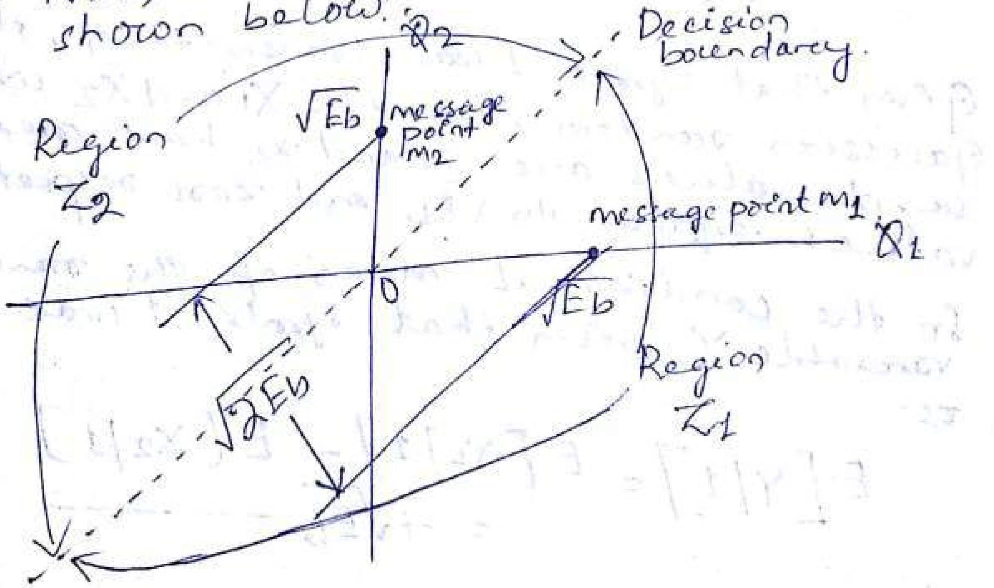
Correspondingly, the co-efficients s_{ij} for $i = 1,2$ and $j = 1,2$ is defined by

$$s_{ij} = \int_0^{T_b} s_i(t)\, \phi_j(t)\, dt$$

$$= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_j t)\, dt$$

$$= \begin{cases} \sqrt{E_b}, & i = j \\ 0, & i \ne j \end{cases} \quad \text{---} (3).$$

Thus BFSK system is characterized by having a signal space that is two-dimensional (i.e. $N=2$) with two message points (i.e. $M=2$) is shown below:



The two message points are defined by the

$$s_{11} = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \quad \text{——— (4).}$$

and

$$s_{22} = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix} \quad \text{——— (5).}$$

with the Euclidean distance between them equal to $\sqrt{2E_b}$.

ERROR PROBABILITY OF BFSK

The observation vector x has two elements x_1 and x_2 which are defined as follows:-

$$x_1 = \int_0^{T_b} x(t)\, \phi_1(t)\, dt \quad \text{——— (6).}$$

and

$$x_2 = \int_0^{T_b} x(t)\, \phi_2(t)\, dt \quad \text{——— (7).}$$

where $x(t)$ is the received signal.

Let us define a new Gaussian random variable Y whose sample value y is equal to the difference between x_1 and x_2, i.e.

$$y = x_1 - x_2 \quad \text{——— (8).}$$

Given that symbol '1' was transmitted, the Gaussian random variables X_1 and X_2 whose sample values are x_1 and x_2 have mean values equal to $\sqrt{E_b}$ and zero respectively.

So the conditional mean of the random variable Y, given that symbol '1' was Tx is

$$E[Y|1] = E[X_1|1] - E[X_2|1]$$
$$= +\sqrt{E_b} \quad \text{——— (9).}$$

On the other hand, given that the symbol '0' was transmitted, the random variables X_1 and X_2 have mean values zero and $\sqrt{E_b}$ respectively.
Correspondingly, the conditional mean of the random variable Y, given that symbol '0' was transmitted is

$$E[Y|0] = E[X_1|0] - E[X_2|0]$$
$$= -\sqrt{E_b} \quad \text{——— (10).}$$

The variance of the random variable Y is independent of which binary symbol was transmitted. The random variables X_1 and X_2 are statistically independent, each with a variance equal to $N_0/2$, so

$$var[Y] = var[X_1] + var[X_2]$$
$$= N_0 \quad \text{——— (11).}$$

Suppose we know that symbol '0' was transmitted. The conditional PDF of the random variable Y is then given by

$$f_Y(y|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(y+\sqrt{E_b})^2}{2N_0}\right] \quad \text{——— (12).}$$

Since the condition $x_1 > x_2$, or equivalently, $y > 0$, corresponds to the receiver making a decision in favour of symbol '1', we deduce that the conditional probability of error, given that symbol '0' was transmitted is

$$P_{10} = P(y > 0 \mid \text{symbol '0' was sent})$$
$$= \int_0^\infty f_Y(y|0)\, dy$$

$$= \frac{1}{\sqrt{2\pi N_0}} \int_0^\infty exp\left[-\frac{(y+\sqrt{E_b})^2}{2N_0}\right] dy \quad \text{(13)}$$

Putting $\frac{y+\sqrt{E_b}}{\sqrt{2N_0}} = z$ — (14)

then changing the variable of integration we can write

$$P_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/2N_0}}^{\infty} exp(-z^2)\, dz$$

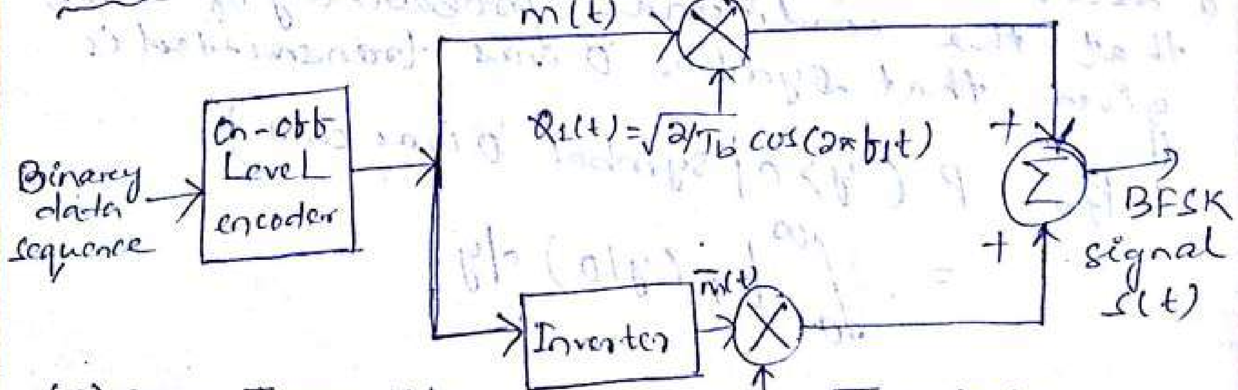
$$\Rightarrow P_{10} = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{2N_0}}\right) \quad \text{(15)}$$

Similarly, we may show that P_{01}, the conditional probability of error given that symbol '1' was transmitted, has the same value as given by equation (15). Hence, BER or average probability of bit error for BFSK is given as

$$P_e = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{2N_0}}\right) \quad \text{(16)}$$

(assuming equiprobable symbols).

GENERATION AND DETECTION OF BFSK

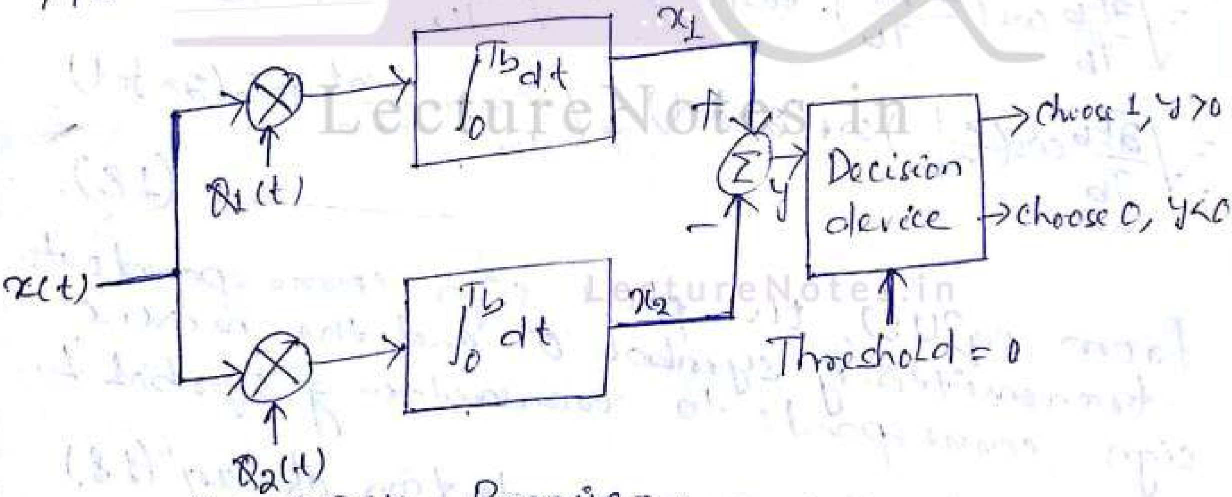


(a) BFSK Transmitter

The incoming binary data sequence is first applied to an on-off level encoder to produce the symbol '1' of constant amplitude of $\sqrt{E_b}$ volts and symbol '0' of amplitude zero volts.

By using an inverter in the lower channel it is ensured that when the symbol '1' at the input, the oscillator with frequency f_1 in the upper channel is switched on while the oscillator with frequency f_2 in the lower channel is switched off, with the result that the frequency f_1 is transmitted.

Conversely, when symbol '0' at the input, the process is reversed with the result that the frequency f_2 is transmitted. The two frequencies f_1 and f_2 are chosen to equal different integer multiples of the bitrate $1/T_b$.



(b) BFSK Receiver.

The BFSK receiver consists of two correlators with a common input $x(t)$ i.e. received signal also supplied with locally generated signals $Q_1(t)$ and $Q_2(t)$. The correlator outputs are subtracted and the resulting difference y is compared with threshold of zero volts.

If $y>0$, the receiver decides in favour of '1'. On the other hand if $y<0$, it decides in favour of '0'. If y is exactly zero, the receiver makes a random guess in favour of '1' or '0'.

POWER SPECTRA OF BFSK SIGNALS:-

Let us consider the BFSK, for which the two transmitted frequencies f_1 and f_2 differ by an amount equal to the bit rate $1/T_b$, and their arithmetic mean equals the nominal carrier frequency f_c. We may express this BFSK as follows:

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_c t \pm \frac{\pi t}{T_b}\right), \quad 0 \le t \le T_b \qquad (17)$$

$$= \sqrt{\frac{2E_b}{T_b}} \cos\left(\pm\frac{\pi t}{T_b}\right) \cdot \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin\left(\pm\frac{\pi t}{T_b}\right) \sin(2\pi f_c t)$$

$$= \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi t}{T_b}\right) \cos(2\pi f_c t) \mp \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{T_b}\right) \sin(2\pi f_c t) \qquad (18)$$

From eqn(18), the plus sign corresponds to transmitting symbol '0' and the minus sign corresponds to transmitting symbol '1'.

Based on the representation of eqn(18) we make the following observations to the in-phase and quadrature components of BFSK signal with continuous phase:

(1) The inphase component is completely independent of the i/p binary wave.

It equals $\sqrt{2E_b/T_b}\cos(\pi t/T_b)$. The PSD of this component therefore consists of two delta functions weighted by the factor $\sqrt{E_b/2T_b}$ and occuring at $f = \pm 1/2T_b$.

(2) The quadrature component is directly related to the input binary wave. During the signaling interval $0 \le t \le T_b$, it equals $-g(t)$ when we have symbol '1', and $+g(t)$ when we have symbol '0'. The symbol shaping function $g(t)$ is defined by

$$g(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}}\sin\left(\dfrac{\pi t}{T_b}\right), & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases} \qquad (19).$$

The ESD of this symbol shaping function is

$$\Psi_g(f) = \frac{8E_bT_b\cos^2(\pi T_b f)}{\pi^2(4T_b^2f^2-1)^2} \qquad (20).$$

and PSD is $S_g(f) = \dfrac{8E_b\cos^2(\pi T_b f)}{\pi^2(4T_b^2f^2-1)^2}$ ——— (20a).

The PSD of the quadrature component equals $\Psi_g(f)/T_b$. It is also apparent that the inphase and quadrature components of BFSK signal are independent of eachother.

Accordingly, the baseband PSD of BFSK signals equals the sum of the power spectral densities of these two components as given below:-

$$S_B(f) = \sqrt{\frac{E_b}{2T_b}}\left[\delta\left(f-\frac{1}{2T_b}\right)+\delta\left(f+\frac{1}{2T_b}\right)\right] + \frac{8E_b\cos^2(\pi T_b f)}{\pi^2(4T_b^2f^2-1)^2} \qquad (21).$$

MINIMUM SHIFT KEYING (MSK)

Let us consider a continuous-phase frequency-shift keying (CPFSK) signal, which is defined for the interval $0 \le t \le T_b$ as follows:-

$$s(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} \cos[2\pi f_1 t + \theta(0)] & \text{for symbol '1'} \\ \sqrt{\dfrac{2E_b}{T_b}} \cos[2\pi f_2 t + \theta(0)] & \text{for symbol '0'} \end{cases} \quad \text{——— (1)}$$

The phase $\theta(0)$, denoting the value of the phase at time $t = 0$, sums up the past history of the modulation process up to time $t = 0$. The frequencies f_1 and f_2 are sent in response to binary symbols 1 and 0 respectively.

Another useful way of representing the CPFSK signal $s(t)$ is to express it in the conventional form of an angle-modulated signal as follows:-

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_c t + \theta(t)] \quad \text{——— (2).}$$

where $\theta(t) = \theta(0) \pm \dfrac{\pi h}{T_b} t$ ——— (3).

where the + sign corresponds to sending symbol '1', and the − sign corresponds to sending symbol '0', and the parameter 'h', which is referred to as deviation ratio.

$$h = T_b(f_1 - f_2) \quad \text{——— (4)}$$

SIGNAL-SPACE DIAGRAM OF MSK :-

Using a well-known trigonometric identity of eqn (2), we may express the CPFSK signal $s(t)$ in terms of its in-phase and quadrature components as follows:-

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] \cos(2\pi f_c t)$$
$$- \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] \sin(2\pi f_c t) \quad \text{——— (5)}$$

Consider first the in-phase component $\sqrt{2E_b/T_b}\cos[\theta(t)]$. With the deviation ratio $h = 1/2$, we have

$$\theta(t) = \theta(0) \pm \frac{\pi}{2T_b} t, \quad 0 \le t \le T_b \quad \text{——— (6)}.$$

where the '+' sign corresponds to symbol '1' and the '−' sign corresponds to symbol '0'.

Since the phase $\theta(0)$ is '0' or 'π', depending on the past history of the modulation process, we find that, in the interval $-T_b \le t \le T_b$, the polarity of $\cos[\theta(t)]$ depends only on $\theta(0)$, regardless of the sequence of 1s and 0s Tx.

Thus, for this time interval, the in-phase component $s_I(t)$ consists of a half-cycle cosine pulse defined as follows:-

$$s_I(t) = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)]$$
$$= \sqrt{\frac{2E_b}{T_b}} \cos[\theta(0)] \cdot \cos\left(\frac{\pi}{2T_b} t\right)$$
$$= \pm \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi}{2T_b} t\right) \quad \text{——— (7)}$$

where the '+' sign corresponds to $\theta(0) = 0$, and the '−' sign corresponds to $\theta(0) = \pi$.

In a similar way, the quadrature component $s_Q(t)$ consists of a half-cycle sine pulse, whose polarity depends only on $\theta(T_b)$ as shown by

$$s_Q(t) = \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)]$$

$$= \sqrt{\frac{2E_b}{T_b}} \sin[\theta(T_b)] \sin\left(\frac{\pi}{2T_b} t\right)$$

$$= \pm\sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right) \qquad (8)$$

where the '+' sign corresponds to $\theta(T_b) = \pi/2$ and the '−' sign corresponds to $\theta(T_b) = -\pi/2$.

Orthonormal basis functions:-

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right) \cos(2\pi f_c t) \qquad (9)$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right) \sin(2\pi f_c t)$$

Correspondingly, we may express the MSK signal in the expanded form as

$$s(t) = s_1 \phi_1(t) + s_2 \phi_2(t) \qquad (10)$$

where the coefficients s_1 and s_2 are related to the phase states $\theta(0)$ and $\theta(T_b)$ respectively.

$$s_1 = \int_{-T_b}^{T_b} s(t)\, \phi_1(t)\, dt$$

$$= \sqrt{E_b} \cos[\theta(0)], \; -T_b \le t \le T_b \qquad (11)$$

$$s_2 = \int_0^{2T_b} s(t)\, \phi_2(t)\, dt$$

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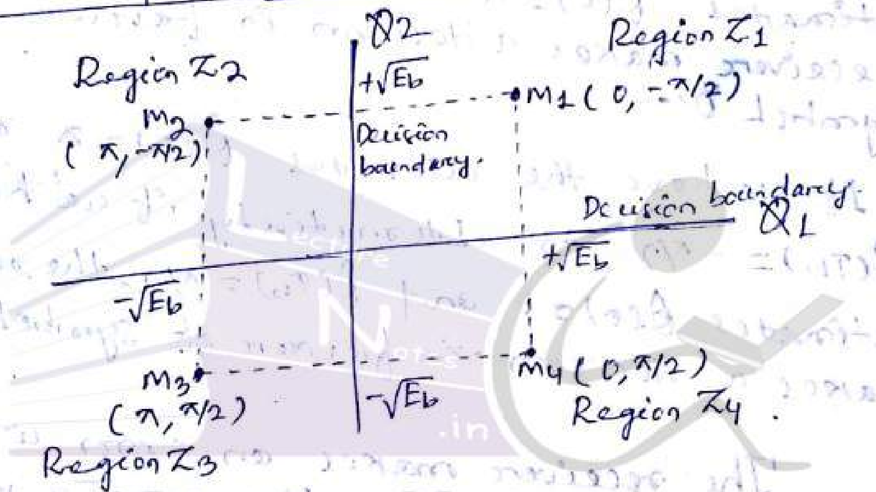
$$\Rightarrow S_2 = -\sqrt{E_b}\,\sin[\theta(T_b)], \quad 0 \leq t \leq 2T_b \quad \text{—— (12).}$$

Transmitted symbol	$\theta(0)$	$\theta(T_b)$	Co-ordinates of message points	
			S_1	S_2
0	0	$-\pi/2$	$\sqrt{E_b}$	$\sqrt{E_b}\,(m_1)$
1	π	$-\pi/2$	$-\sqrt{E_b}$	$\sqrt{E_b}\,(m_2)$
0	π	$\pi/2$	$-\sqrt{E_b}$	$-\sqrt{E_b}\,(m_3)$
1	0	$\pi/2$	$\sqrt{E_b}$	$-\sqrt{E_b}\,(m_4)$



ERROR PROBABILITY OF MSK

In case of an AWGN channel, the received signal is given by

$$x(t) = s(t) + w(t) \quad \text{—— (1).}$$

where $s(t)$ is the transmitted MSK signal and $w(t)$ is the sample function of a white Gaussian noise process of zero mean and PSD of $N_0/2$.

For optimum detection of $\theta(0)$, we first determine the projection of the received signal $x(t)$ onto the $\phi_1(t)$ as

$$x_1 = \int_{-T_b}^{T_b} x(t)\,\phi_1(t)\,dt = s_1 + w_1 \quad \text{—— (2)}$$

$$-T_b \leq t \leq T_b.$$

Similarly, for the optimum detection of $\theta(T_b)$,

$$x_2 = \int_0^{2T_b} x(t)\, \phi_2(t)\, dt \qquad (3).$$
$$= s_2 + w_2$$

To reconstruct the original binary sequence, we have the following observations:

(1) If we have the estimates $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = -\pi/2$ or alternatively if we have the estimates $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = \pi/2$, the receiver makes a decision in favour of symbol '0'.

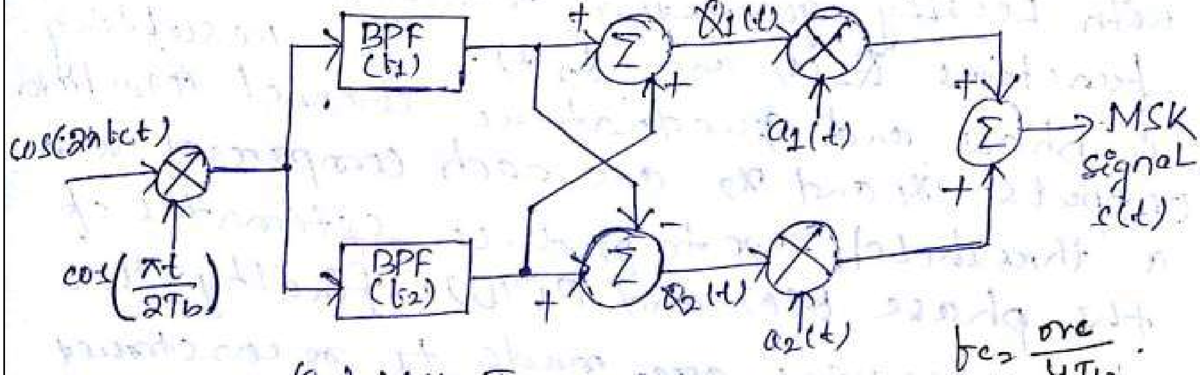
(2) If we have the estimates $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = -\pi/2$, or alternatively if we have the estimates $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = \pi/2$, the receiver makes a decision in favour of symbol '1'.

The receiver makes an error when the I-channel assigns the wrong value to $\theta(0)$ or the Q-channel assigns the wrong value to $\theta(T_b)$. Accordingly, using the statistical characterizations of the product-integrator outputs x_1 and x_2 defined by eq^{n}s (2) and (3) respectively, we readily find that the bit error rate (BER) for MSK is given by

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \qquad (4).$$

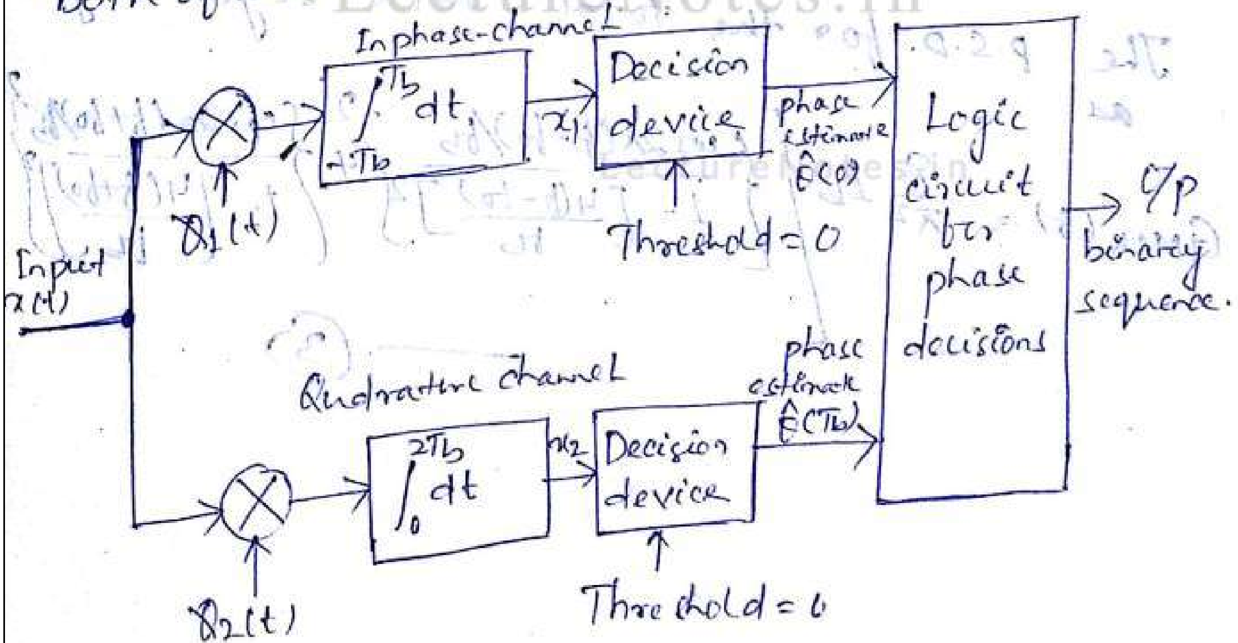
which is exactly the same as that for BPSK and QPSK.

GENERATION AND DETECTION OF MSK SIGNALS



(a) MSK Transmitter.

Two input sinusoidal waves, one of frequency f_c and other of frequency ($1/4T_b$) are applied to a product modulator. This produces two phase-coherent sinusoidal waves at frequencies f_1 and f_2. These two sinusoidal waves are separated from eachother by two narrowband filters, one centered at f_1 and other at f_2. The resulting filter outputs are next linearly combined to produce orthonormal basic functions $\phi_1(t)$ and $\phi_2(t)$. Finally, $\phi_1(t)$ and $\phi_2(t)$ are multiplied with two binary waves $a_1(t)$ and $a_2(t)$ both of which have a bit rate equal to $1/2T_b$.



(b) MSK Receiver.

The received signal $x(t)$ is correlated with locally generated orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$. The resulting in-phase and quadrature channel correlator outputs x_1 and x_2 are each compared with a threshold zero to produce estimates of the phase $\theta(0)$ and $\theta(T_b)$. Finally the phase decisions are made to reconstruct the the original input binary sequence with the minimum average probability of symbol error.

POWER SPECTRAL DENSITY OF MSK

The baseband waveform in MSK is

$$p(t) = \sqrt{2P_s}\, b_0(t) \cdot \cos\frac{\pi}{2} f_b t, \quad -T_b \le t \le T_b \qquad (1)$$

The waveform $p(t)$ has a power spectral density $G_p(f)$ given as

$$G_p(f) = \frac{32 E_b}{\pi^2}\left[\frac{\cos 2\pi f/f_b}{1-\left(\frac{4f}{f_b}\right)^2}\right]^2 \qquad (2).$$

The P.S.D. for the total MSK signal is given as

$$G_{MSK}(f) = \frac{8}{\pi^2} E_b \left[\left\{\frac{\cos 2\pi (f-f_0)/f_b}{1-\left[\frac{4(f-f_0)}{f_b}\right]^2}\right\}^2 + \left\{\frac{\cos 2\pi (f+f_0)/f_b}{1-\left[\frac{4(f+f_0)}{f_b}\right]^2}\right\}^2\right] \qquad (3).$$

BINARY PHASE SHIFT KEYING :- (BPSK)

SPECTRUM OF BPSK SIGNALS :-

→ We know that the waveform $b(t)$ is a NRZ binary waveform.

→ In this waveform, there are rectangular pulses of amplitude $\pm V_b$.

→ If we assume that each pulse is $\pm \frac{T_b}{2}$ around its centre, then it becomes easy to find Fourier Transform of such pulse.

→ The Fourier Transform of this type of pulse is given as

$$X(f) = V_b T_b \frac{\sin(\pi f T_b)}{(\pi f T_b)} \quad \text{--- (1)}$$

→ The PSD $S(f)$ is expressed as

$$S(f) = \frac{|\overline{X(f)}|^2}{T_s} \quad \text{--- (2)}$$

Here, $T_s = T_b$.

→ $\overline{X(f)}$ denotes average value of $X(f)$ due to all the +ve and -ve pulses in $b(t)$, and T_s is symbol duration.

→ Substituting value of $X(f)$ from eq^n (1) in eq^n (2), we get

$$S(f) = \frac{V_b^2 T_b^2}{T_s} \left[\frac{\sin(\pi f T_b)}{(\pi f T_b)} \right]^2 \quad \text{--- (3)}$$

→ Since $T_s = T_b$, so

$$S(f) = V_b^2 T_b \left[\frac{\sin(\pi f T_b)}{(\pi f T_b)} \right]^2 \quad \text{--- (4)}$$

→ The above eq^n (4) gives the PSD of baseband signal $b(t)$ (NRZ baseband signal).

→ The BPSK signal is generated by modulating a carrier by the baseband signal $b(t)$.

→ Due to modulation of carrier of frequency fc, the spectral components are translated from "f" to "fc+f" and "fc−f".

→ The magnitude of these components is divided by half.

→ Therefore, from eq (4), we can write the PSD of BPSK signal as under

$$S_{BPSK}(f) = V_b^2 T_b \left\{ \frac{1}{2}\left[\frac{\sin\pi(f_c-f)T_b}{\pi(f_c-f)T_b}\right]^2 + \frac{1}{2}\left[\frac{\sin\pi(f_c+f)T_b}{\pi(f_c+f)T_b}\right]^2 \right\} \quad \text{--- (5)}$$

→ Let us assume that $\pm V_b = \pm\sqrt{P}$.

→ This means that the NRZ signal is having amplitudes of $+\sqrt{P}$ and $-\sqrt{P}$.

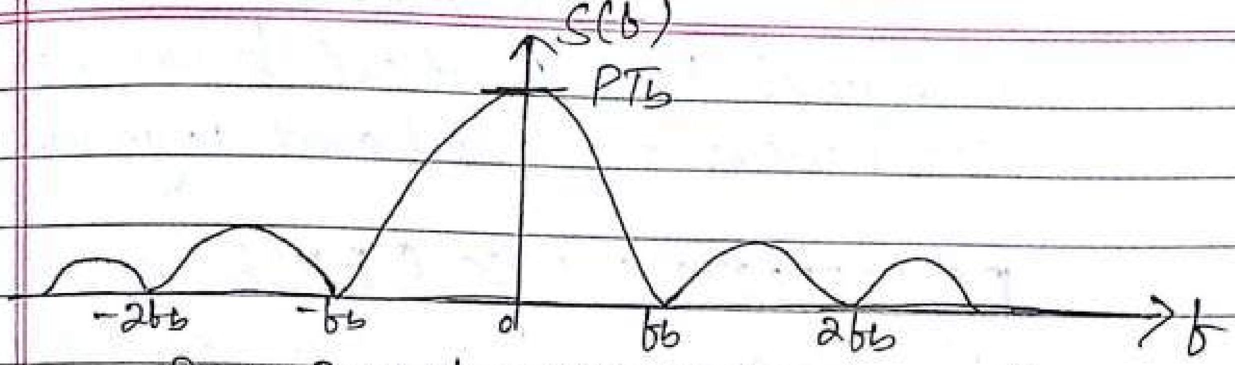
→ By putting this value, the above eq^n (4) and (5) becomes.

$$S(f) = PT_b \left[\frac{\sin(\pi f T_b)}{(\pi f T_b)}\right]^2 \quad \text{--- (6)}$$

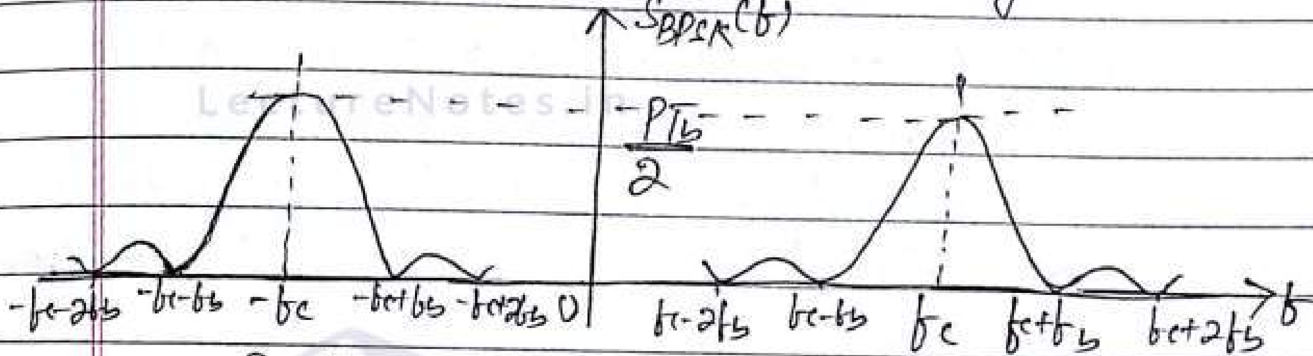
and

$$S_{BPSK}(f) = \frac{PT_b}{2}\left[\left\{\frac{\sin\pi(f_c-f)T_b}{\pi(f_c-f)T_b}\right\}^2 + \left\{\frac{\sin\pi(f_c+f)T_b}{\pi(f_c+f)T_b}\right\}^2\right] \quad \text{--- (7)}$$

→ The power spectral densities of NRZ baseband signal and BPSK signal are shown below.



Fig(a) PSD of NRZ baseband signal.



Fig(b) PSD of BPSK signal

BANDWIDTH FOR BPSK SIGNAL :-

→ As discussed earlier, the spectrum of the BPSK signal is centred around the carrier frequency 'f_c'.

→ If $f_b = 1/T_b$, then for BPSK, from the above figure of PSD of BPSK signal, it is observed that the main lobe is centred around carrier frequency 'f_c' and extends from "$f_c - f_b$" to "$f_c + f_b$".

→ Therefore, Bandwidth of BPSK signal will be BW = Highest frequency - Lowest frequency in the main lobe.

→ $BW = (f_c + f_b) - (f_c - f_b)$

$\Rightarrow BW = 2f_b$.

→ Hence, the minimum bandwidth of BPSK signal

twice of the
is equal to the highest frequency contained in baseband signal.

DIFFERENTIAL PSK (DPSK):-

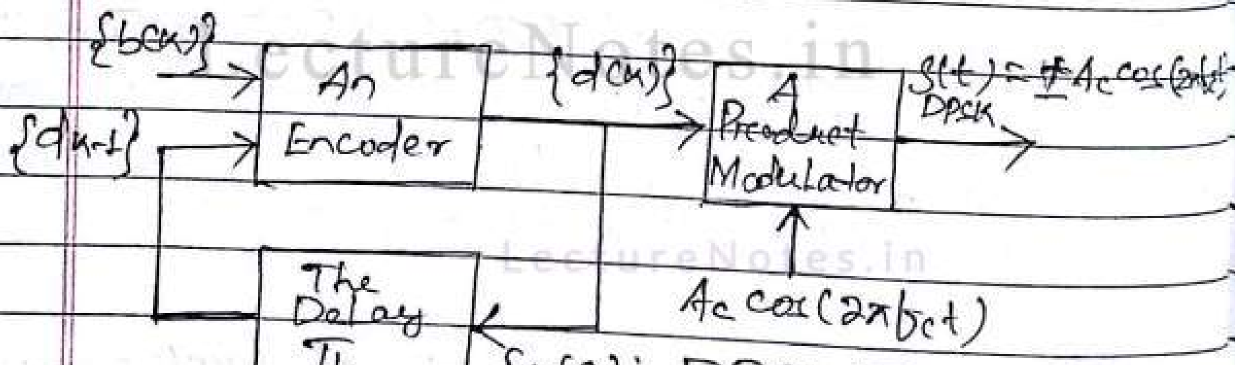
→ We can view differential phase-shift keying as the non-coherent version of the PSK.

→ DPSK doesnot need a synchronous (coherent) carrier at the demodulator.

→ The input sequence of binary bits is modified such that the next bit depends upon the previous bit.

→ Therefore, in the receiver, the previous received bits are used to detect the present bit.

GENERATION OF DPSK:-



fig(a): DPSK Transmitter.

→ In order to eliminate, the need for phase synchronization of coherent receiver, a differential encoding system can be with PSK.

→ The digital information content of the binary data is encoded in terms of signal transitions.

→ As an example, the symbol '0' may be used to represent TRANSITION in a given binary sequence with respect to the previous encoded bit and symbol '1' to indicate NO TRANSITION.

→ This new signaling technique which combines DIFFERENTIAL ENCODING with PSK is known as DIFFERENTIAL PHASE SHIFT KEYING (DPSK).

Table 1. Differentially Encoded sequences with Phase.

Binary data $\{b(k)\}$		0	0	1	0	0	1	0	0	1	1
Differentially encoded data $\{d(k)\}$	1*	0	1	1	0	1	1	0	1	1	1
Phase of DPSK	0	π	0	0	π	0	0	π	0	0	0
Shifted differentially encoded data $\{d_{k-1}\}$		1	0	1	1	0	1	1	0	1	1
Phase of shifted DPSK		0	π	0	0	π	0	0	π	0	0
Phase comparison output		−	−	+	−	−	+	−	−	+	+
Detected binary sequence		0	0	1	0	0	1	0	0	1	1

→ A DPSK Transmitter is shown in fig(a).

→ The data stream $\{b(k)\}$ is applied to the input of the encoder.

→ The output of the encoder is applied to one input of the product modulator.

→ To the other input of this product modulator, a sinusoidal carrier of fixed amplitude and frequency is applied.

→ The relationship between the binary sequence and its differentially encoded version is illustrated in the above Table no.1, for assumed data sequence 0010010011.

→ In this illustration it has been assumed that the encoding has been done in such a way that TRANSITION in the given binary sequence w.r.t. the previous encoded bit is represented by a symbol '0' and NO TRANSITION by symbol '1'.

→ It may be noted that an extra bit (symbol '1') has been arbitrarily added as an initial bit.

→ The phase of the generated DPSK signal has been shown in the third row of Table no.1.

DETECTION OF DPSK :-

→ For the detection of the differentially encoded PSK (i.e. DPSK), we can use the receiver arrangement as shown in fig(b).

→ The received DPSK signal is applied to one input of the multiplier.

→ To the other input of the multiplier, a delayed version of the received DPSK signal by the time interval

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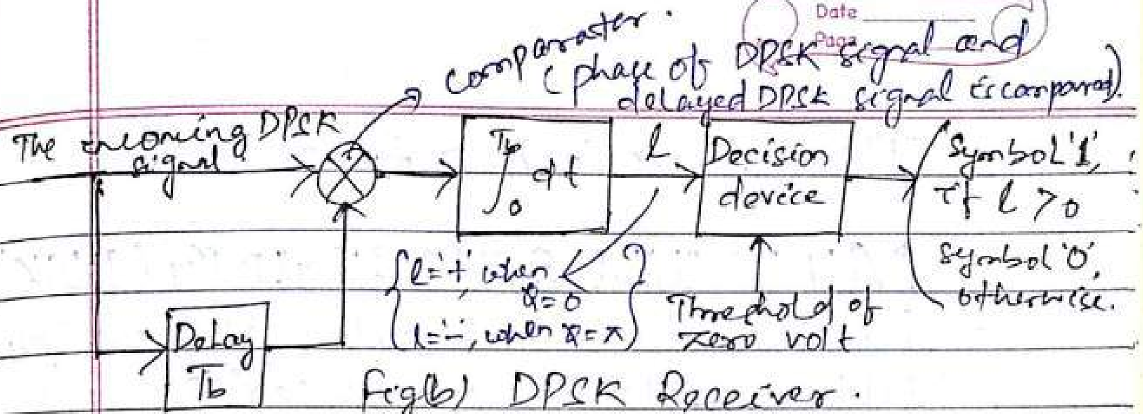


Fig(b) DPSK Receiver.

"T_b" is applied.

→ The delayed version of the received DPSK signal has been shown in the 4th row of the Table no.1.

→ The output of the difference between these two signal produced by the comparator is proportional to the cos (θ), here 'θ' is the difference between the carrier phase angle of the received DPSK signal and its delayed version.

→ The phase angle of the DPSK signal and its delayed version have been shown in 3rd and 5th rows respectively.

→ The phase difference between the two sequence for each bit interval is used to determine the sign of the comparator o/p.

→ When $\theta = 0$, the Integrator output is positive and when $\theta = \pi$, the integrator output is negative.

→ By comparing the integrator output with a decision level of zero volt, the decision device can reconstruct the binary sequence by assigning a symbol "0" for negative output and a symbol

"1" for the positive output.

MINIMUM SHIFT KEYING (MSK)

→ The bandwidth requirement of QPSK is high.
→ Filters and other methods can overcome these problems, but they have other side effects.
→ For example, filters alter the amplitude of the waveform.
→ MSK overcomes these problems. In MSK, the output waveform is continuous in phase, hence there are no abrupt changes in amplitude.
→ Fig 1.1 (a) shows the corresponding NRZ waveform b(t).
→ From b(t), two waveforms are generated for odd and even bits.
→ $b_o(t)$ represents odd bits and $b_e(t)$ represents even bits. Fig 1.1 (b) and (c) shows the waveform of $b_o(t)$ and $b_e(t)$.
→ The duration of each bit in $b_o(t)$ or $b_e(t)$ is $2T_b$, whereas it is T_b in b(t) i.e. $T_s = 2T_b$.
→ The waveforms $b_o(t)$ and $b_e(t)$ have an offset of T_b. This offset is essential in MSK.
→ Two waveforms $\sin 2\pi(t/4T_b)$ and $\cos 2\pi(t/4T_b)$ are generated as shown in fig(d).
→ The symbol duration of $b_o(t)$ consists of complete half cycle of $\cos 2\pi(t/4T_b)$

WAVEFORMS OF MSK MODULATION SCHEME:-

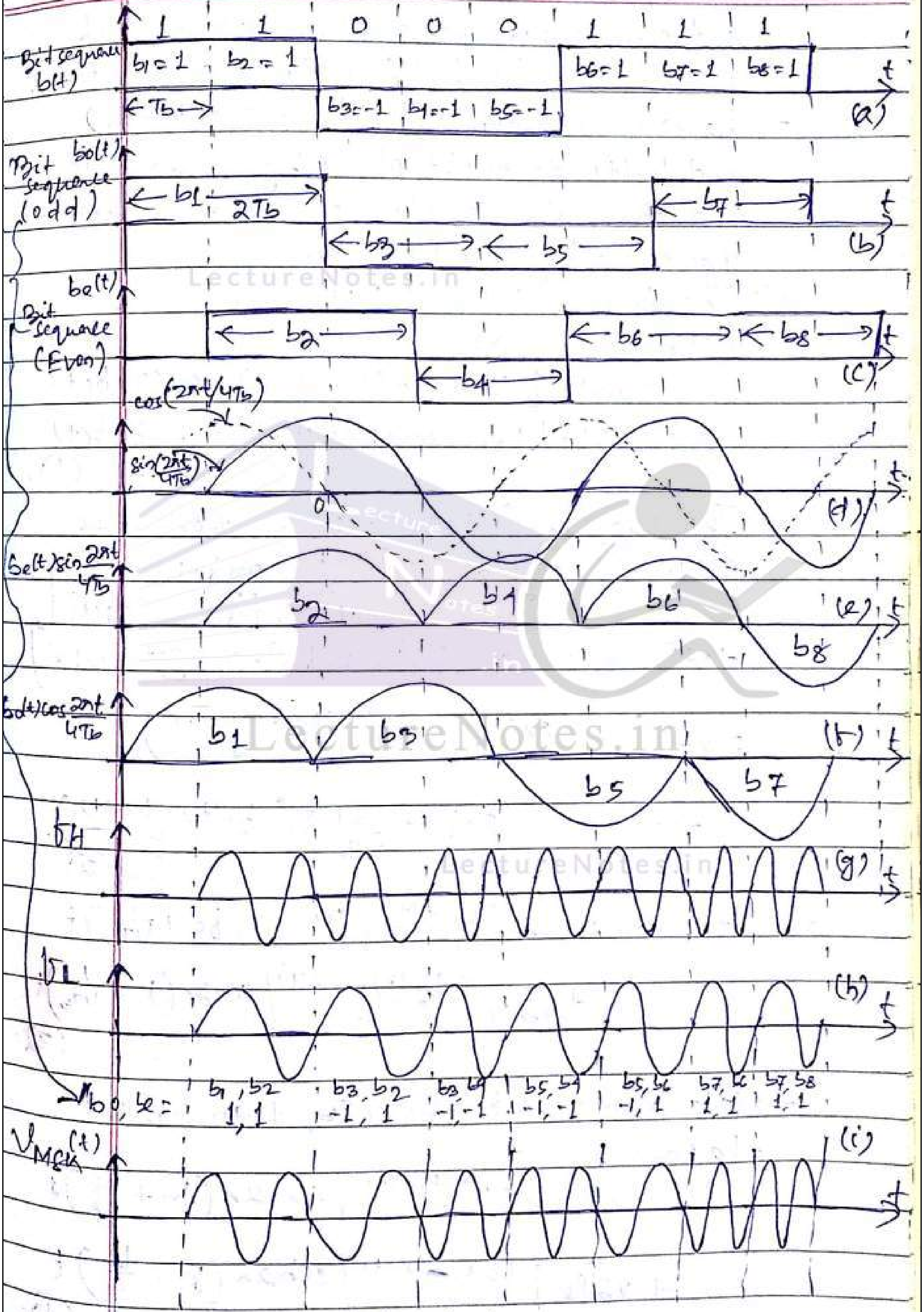


fig 1.1

and similarly one symbol duration of $b_e(t)$ contains half cycle of $\sin 2\pi(t/4T_b)$.

→ $b_e(t)$ is multiplied by $\sin 2\pi(t/4T_b)$ and $b_o(t)$ is multiplied by $\cos 2\pi(t/4T_b)$ and these two product waveforms are shown in fig 1.1 (e) and (f).

→ The transmitted MSK signal is represented as

$$s(t) = \sqrt{2P_s}\left[b_e(t)\sin(2\pi t/4T_b)\right]\cdot\cos(2\pi f_c t) + \sqrt{2P_s}\left[b_o(t)\cos(2\pi t/4T_b)\right]\cdot\sin(2\pi f_c t) \quad \text{—— (1)}$$

→ The above eq^n (1) can also be written as

$$s(t) = \sqrt{2P_s}\,\frac{b_e(t)}{2}\cdot\left[\sin\left\{\frac{2\pi t}{4T_b}+2\pi f_c t\right\} + \sin\left\{\frac{2\pi t}{4T_b}-2\pi f_c t\right\}\right]$$

$$+\sqrt{2P_s}\cdot\frac{b_o(t)}{2}\left[\sin\left\{\frac{2\pi t}{4T_b}+2\pi f_c t\right\} - \sin\left\{\frac{2\pi t}{4T_b}-2\pi f_c t\right\}\right]$$

$$= \sqrt{2P_s}\sin\left(\frac{2\pi t}{4T_b}+2\pi f_c t\right)\left(\frac{b_e(t)+b_o(t)}{2}\right)$$

$$+\sqrt{2P_s}\sin\left(2\pi f_c t-\frac{2\pi t}{4T_b}\right)\left(\frac{b_o(t)-b_e(t)}{2}\right) \quad \text{—— (2)}$$

$$\Rightarrow s(t) = \sqrt{2P_s}\left[\frac{b_o(t)+b_e(t)}{2}\right]\sin 2\pi\left(f_c+\frac{1}{4T_b}\right)t + \sqrt{2P_s}\left[\frac{b_o(t)-b_e(t)}{2}\right]\sin 2\pi\left(f_c-\frac{1}{4T_b}\right)t \quad \text{—— (3)}$$

→ We know that $f_b = 1/T_b$, then eq^n (3) becomes:-

$$s(t) = \sqrt{2P_s}\left[\frac{b_o(t)+b_e(t)}{2}\right]\sin 2\pi\left(f_c+\frac{f_b}{4}\right)t + \sqrt{2P_s}\left[\frac{b_o(t)-b_e(t)}{2}\right]\sin 2\pi\left(f_c-\frac{f_b}{4}\right)t \quad \text{—— (4)}$$

→ Let us substitute

$$C_H(t) = \left[\frac{b_o(t) + b_e(t)}{2}\right]$$

and $$C_L(t) = \left[\frac{b_o(t) - b_e(t)}{2}\right]$$

$$f_H = f_c + \frac{f_b}{4} \text{ and } f_L = f_c - \frac{f_b}{4}. \quad \text{—— (5).}$$

→ Putting these values, eq^n (4) can be written as

$$S(t) = \sqrt{2P_s}\, C_H(t) \sin(2\pi f_H t) + \sqrt{2P_s}\, C_L(t) \cdot \sin(2\pi f_L t) \quad \text{—— (6).}$$

→ The frequencies f_H and f_L are chosen such that $\sin(2\pi f_H t)$ and $\sin(2\pi f_L t)$ are orthogonal over the interval T_b. For orthogonality, the following relation must be satisfied i.e.

$$\int_0^{T_b} \sin(2\pi f_H t) \cdot \sin(2\pi f_L t)\, dt = 0 \quad \text{—— (7)}$$

→ This relation will be satisfied if we have integers 'm' and 'n' such that

$$2\pi (f_H - f_L) T_b = n\pi \quad \text{—— (8)}$$

and $$2\pi (f_H + f_L) T_b = m\pi \quad \text{—— (9).}$$

→ Now putting the values of f_H and f_L we get

$$2\pi \left(f_c + \frac{f_b}{4} - f_c + \frac{f_b}{4}\right) T_b = n\pi$$

$$\Rightarrow f_b T_b = n \Rightarrow f_b \times \frac{1}{f_b} = n \Rightarrow n = 1 \quad \text{—— (10).}$$

→ similarly, $$2\pi \left(f_c + \frac{f_b}{4} + f_c - \frac{f_b}{4}\right) T_b = m\pi$$

$$\Rightarrow 4 f_c T_b = m$$

$$\Rightarrow 4 f_c \times \frac{1}{f_b} = m \Rightarrow f_c = \frac{m}{4} \times f_b \quad \text{—— (11).}$$

→ With $n = 1$ in eq^n (8), we get

$$2\pi (f_H - f_L) T_b = \pi$$

$$\Rightarrow \boxed{(f_H - f_L) = \frac{1}{2T_b} = \frac{f_b}{2}} \quad \text{—— (12).}$$

→ Here $n=1$ means the difference between f_H and f_L is minimum and at the same time, they are orthogonal.

→ Therefore, this modulation technique is called Minimum Shift Keying (MSK).

GEOMETRICAL REPRESENTATION OF MSK:-

→ Eqⁿ (6) can be written as follows:-

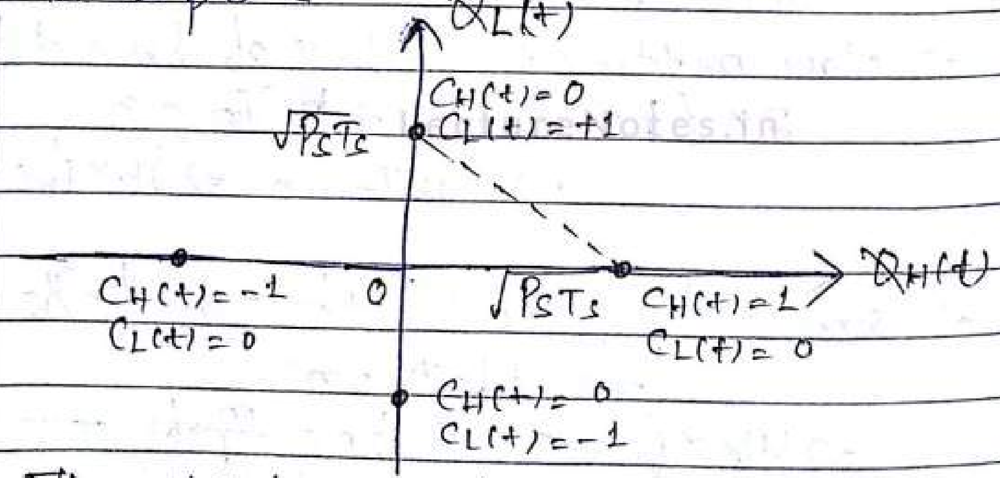
$$s(t) = C_H(t)\sqrt{P_sT_s}\cdot\sqrt{\frac{2}{T_s}}\sin(2\pi f_H t)$$
$$+ C_L(t)\sqrt{P_sT_s}\cdot\sqrt{\frac{2}{T_s}}\sin(2\pi f_L t) \quad \text{—— (13)}$$

→ Here Let $\phi_H(t) = \sqrt{2/T_s}\sin(2\pi f_H t)$ — (14)

and $\phi_L(t) = \sqrt{2/T_s}\sin(2\pi f_L t)$ —— (15).

→ The carriers $\phi_H(t)$ and $\phi_L(t)$ are in quadrature.

→ Depending on the values of $C_H(t)$ and $C_L(t)$, there will be four signal points in $\phi_H\phi_L$ plane.



→ The distance of each signal point from the origin is $\sqrt{P_sT_s}$.

→ Since the points are symmetric, the

distance between any two nearest points is same and it is given as

$$d^2 = (\sqrt{P_s T_s})^2 + (\sqrt{P_s T_s})^2$$

$$\Rightarrow d = \sqrt{2 P_s T_s} \quad \text{—— (16).}$$

$$\Rightarrow d = \sqrt{2E_s} = \sqrt{4E_b} = 2\sqrt{E_b} \quad \text{—— (17)}$$

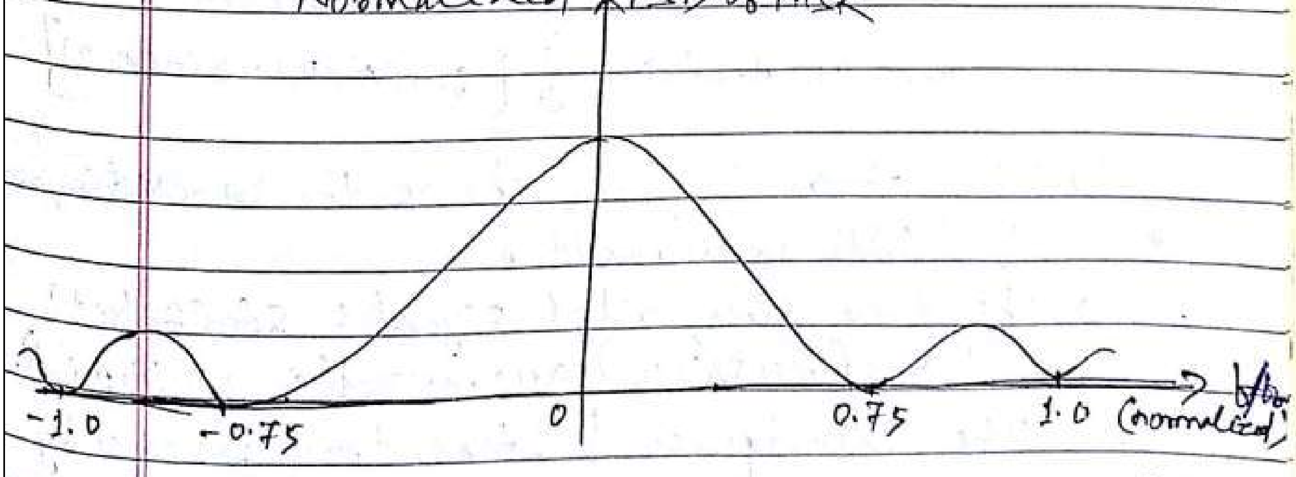
PSD AND BANDWIDTH OF MSK:-

→ The PSD of the baseband signal is expressed as

$$S(f) = \frac{32 E_b}{\pi^2}\left[\frac{\cos(2\pi f T_b)}{1-(4fT_b)}\right]^2 \quad \text{—— (18).}$$

→ When this baseband signal modulates the carrier having frequency 'f_c', then the PSD of MSK signal is given as

$$S_{MSK}(f) = \frac{8E_b}{\pi^2}\left\{\frac{\cos 2\pi(f_c-f)T_b}{1-[4(f_c-f)T_b]}\right\}^2 + \frac{8E_b}{\pi^2}\left\{\frac{\cos 2\pi(f_c+f)T_b}{1-[4(f_c+f)T_b]}\right\}^2 \quad \text{—— (19).}$$



→ Normalization means maximum amplitude

of signals are scaled w.r.t. '1'.

→ From the above figure of PSD of MSK signal, we observe that the width of main lobe in MSK is ± 0.75 i.e.

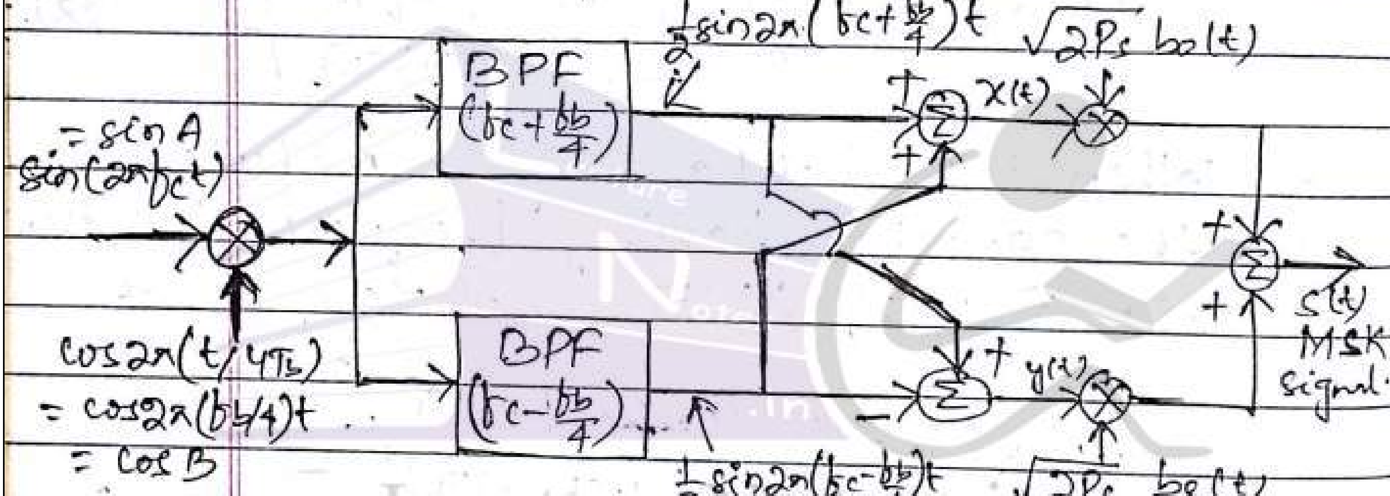
$$f/f_b = \pm 0.75$$

$$\Rightarrow f = \pm 0.75 f_b$$

$$\therefore BW_{(MSK)} = 0.75 f_b - (-0.75 f_b)$$

$$\Rightarrow BW = 1.5 f_b \quad \text{——— (20)}.$$

GENERATION OF MSK SIGNAL:-



Here $x(t) = \sin 2\pi f_c t \cdot \cos(2\pi)(t/4T_b)$

and $y(t) = \sin(2\pi t/4T_b) \cdot \cos(2\pi f_c t)$

NOTE:- $\sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

~~and $\cos A \cdot \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$~~

→ The above figure shows the blockdiagram of MSK Transmitter.

→ The two sinusoidal signals $\sin(2\pi f_c t)$ and $\cos(2\pi t/4T_b)$ are mixed (i.e. multiplied).

→ The Bandpass filters then pass only sum and difference components i.e.

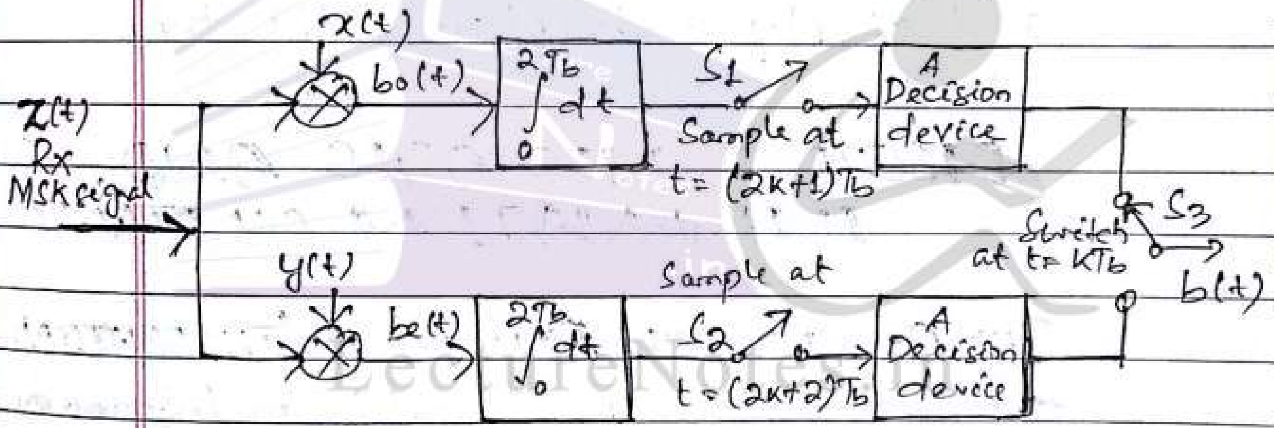
$(f_c + f_b/4)$ and $(f_c - f_b/4)$.

→ The outputs of Bandpass filters are then added and subtracted such that two signals $x(t)$ and $y(t)$ are generated.

→ Signal $x(t)$ is multiplied by $\sqrt{2P_s}\, b_0(t)$ and $y(t)$ is multiplied by $\sqrt{2P_s}\, b_e(t)$.

→ The outputs of the multipliers are then added to give final MSK signal.

DETECTION OF MSK SIGNAL :-



→ The above figure shows the block diagram of MSK Receiver.

→ MSK uses Synchronous detection.

→ The signals $x(t)$ and $y(t)$ are multiplied with the received MSK signal.

→ Here $x(t)$ and $y(t)$ have same values as shown in the transmitter block-diagram.

→ The outputs of the multipliers are $b_0(t)$ and $b_e(t)$.

→ The integrators integrate over the period of $2T_b$.

→ For the upper correlator, the sampling switch(S_1) samples output of the integrator at $t = (2k+1)T_b$.

→ Then the decision device decides whether $b_0(t)$ is +1 or -1. +1 is deciding about binary '1' and -1 deciding about binary '0'.

→ Similarly, Lower correlator output is sampled by the switch S_2.

→ The switch S_3 operates at $t = kT_b$ and simply multiplexes the two decision device outputs.

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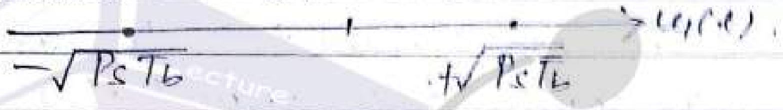
DIGITAL TRANSMISSION AND MODULATION.

Geometrical Representation of BPSK signal

We see that a BPSK signal can be represented in terms of one orthonormal signal i.e. $u_1(t)$ or $Q_1(t) = \sqrt{2/T_b}\cos\omega_0 t$ as follows

$$v_{BPSK}(t) = [\sqrt{P_s T_b}\, b(t)]\sqrt{2/T_b}\cos\omega_0 t$$
$$= [\sqrt{P_s T_b}\, b(t)]\, u_1(t) \quad (1)$$

The binary PSK signal can then be drawn as shown below.



where $E_b = P_s T_b$ is the energy contained in a bit duration.
Note that the distance 'd' between the signals is known as Euclidean distance

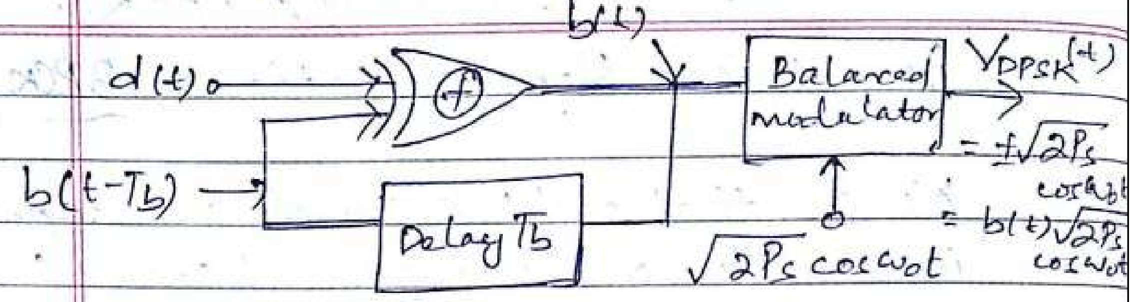
$$d = 2\sqrt{P_s T_b} = 2\sqrt{E_b} \quad (2)$$

DIFFERENTIAL PHASE SHIFT KEYING (DPSK)

→ DPSK is the modification of BPSK which has the merit that it eliminates the ambiguity about whether the demodulated data is or is not inverted.

→ In addition DPSK avoids the need to provide the synchronous carrier required at the demodulator for detecting a BPSK signal.

→ A means for generating a DPSK signal is shown below.-



d(t)	b(t-Tb)	b(t)
0	0	0
0	1	1
1	0	1
1	1	0

→ The data stream to be transmitted, d(t) is applied to one input of an Ex-OR gate.

→ To other gate input is applied the o/p of the Ex-OR gate b(t) delayed by the time T_b allocated to one bit i.e. $b(t-T_b)$.

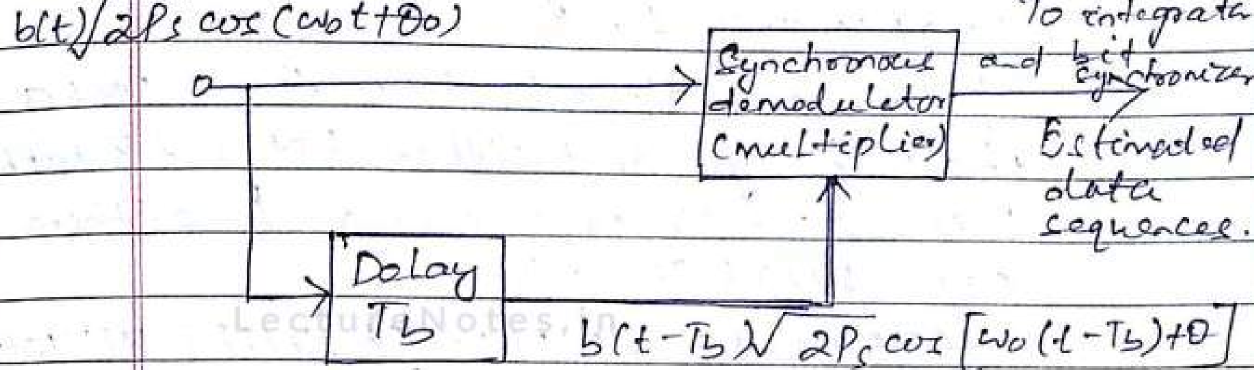
→ As shown in above figure, b(t) is applied to a balanced modulator to which is also applied the carrier $\sqrt{2P_s}\cos\omega_0 t$.

→ The modulator output, which is the transmitted signal is

$$V_{DPSK}(t) = b(t)\sqrt{2P_s}\cos\omega_0 t$$
$$= \pm\sqrt{2P_s}\cos\omega_0 t \quad \text{--- (1)}$$

→ Thus altogether when d(t) = 0, the phase of the carrier does not change at the beginning of the bit interval while when d(t) = 1, there is a phase change of magnitude π.

→ A method for recovering the data bit stream from the DPSK signal is shown below:-



→ Here the received signal and the received signal delayed by the bit time T_b are applied to a multiplier.

→ The multiplier output is

$$b(t)\,b(t-T_b)(2P_s)\cos(\omega_0 t+\theta)\cos[\omega_0(t-T_b)+\theta]$$

$$= b(t)\,b(t-T_b)P_s\left\{\cos\omega_0 T_b + \cos\left[2\omega_0\left(t-\frac{T_b}{2}\right)+2\theta\right]\right\} \quad \text{--- (2).}$$

→ This output is applied to a bit synchronizer and integrator.

→ The first term on the right hand side of eqn (2) is a multiplicative constant, the waveform $b(t)\,b(t-T_b)$ which is the required signal.

→ The output integrator will suppress the double frequency term, and select $\omega_0 T_b$ so that $\omega_0 T_b = 2n\pi$ with 'n' an integer. For this case we shall have $\cos\omega_0 T_b = +1$.

→ The transmitted data bit $d(t)$ can readily be determined from the product $b(t)\,b(t-T_b)$.

→ If $d(t)=0$, then there was no phase change and $b(t)=b(t-T_b)$ both being

+1V or both being -1V. In this case $b(t)\,b(t-T_b) = 1$.

→ If however, $d(t) = 1$, then there was a phase change and either $b(t) = 1V$ with $b(t-T_b) = -1V$ or viceversa. In either case $b(t)\,b(t-T_b) = -1$.

QUADRATURE AMPLITUDE SHIFT KEYING (QASK) Or (QAM)

→ It is a hybrid modulation scheme where the signal vector differ by either in phase or amplitude or both.

→ In this procedure two separate K-bits symbols from information sequence is modulated with two quadrature carrier i.e. $\cos\omega_0 t$ and $\sin\omega_0 t$.

→ Mathematically

$$S_{QASK}(t) = \sqrt{\frac{2E_b}{T_b}}\, a_k \cos\omega_0 t - \sqrt{\frac{2E_b}{T_b}}\, b_k \sin\omega_0 t$$

$u_1(t)$ $\phi_1(t) = \sqrt{2/T_b}\cos\omega_0 t$, $\phi_2(t)$ or $u_2(t) = \sqrt{2/T_b}\sin\omega_0 t$ — (1) — (2)

$0 \le t \le T$.

where a_k = inphase sequence

b_k = quadrature sequence.

→ Representing in polar form :-

$$S_{QASK}(t) = \sqrt{\frac{2E_b}{T_b}}\, r_m \cos(\omega_0 t + \theta_m) \quad (2)$$

where $r_m = \sqrt{a_k^2 + b_k^2}$ — (3)

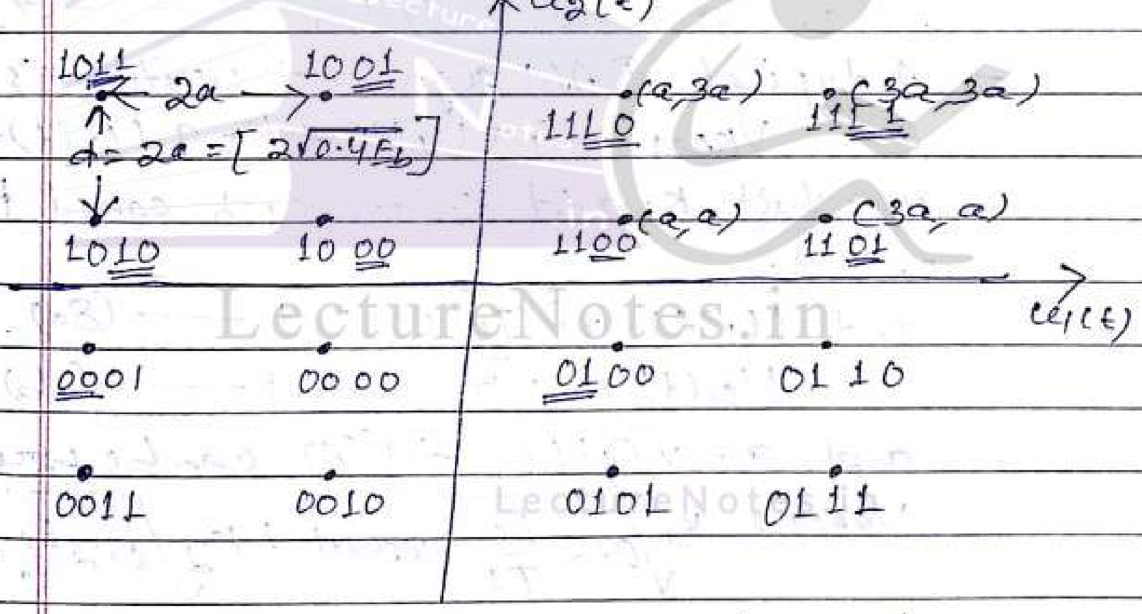
and $\theta_m = \tan^{-1}\left(\frac{b_k}{a_k}\right)$ — (4)

→ So QASK or QAM can be viewed as combined amplitude (r_m) and phase (θ_m) modulation.

→ As an example of a QASK, let us consider that we propose to transmit a symbol for every 4 bits.

→ So there are $2^4 = 16$ different possible symbols and we shall have to be able to generate 16 distinguishable signals.

→ One possible geometrical representation of 16 signals is shown below:-



→ In this configuration each signal point is equally distant from its nearest neighbors, the distance being $d = 2a$.

→ We have placed the points symmetrically about the origin of the signal space to simplify the hardware design of the system.

→ Let us assume that all 16 signals are equally likely. Because of the symmetry

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we can determine the average energy associated with a signal, from the four signals in the first quadrant.

→ The average normalized energy of a signal is :-

$$E_s = \frac{1}{4}\left[(a^2+a^2)+(9a^2+a^2)+(9a^2+9a^2)+(a^2+9a^2)\right]$$

$$\Rightarrow E_s = 10a^2 \Rightarrow a = \sqrt{0.1E_s} \text{ ——— } (5)$$

and $d = 2a = 2\sqrt{0.1E_s}$ ——— (6).

→ In the present case since each symbol represents 4bits, the normalized symbol energy is $E_s = 4E_b$

$$\therefore\ d = 2\sqrt{0.4E_b} \text{ ——— } (7).$$

→ A typical QAM signal is written as

$$V_{QAM} = K_1 a u_1(t) + K_2 a u_2(t) \text{ — } (8).$$

in which K_1 and K_2 are each equal to ± 1 or ± 3.

and $u_1(t) = \sqrt{2/T_s}\cos w_0 t$ ——— (8a).

$u_2(t) = \sqrt{2/T_s}\sin w_0 t$ ——— (8b).

and $a = \sqrt{0.1E_s}$ eq.(8) can be written as

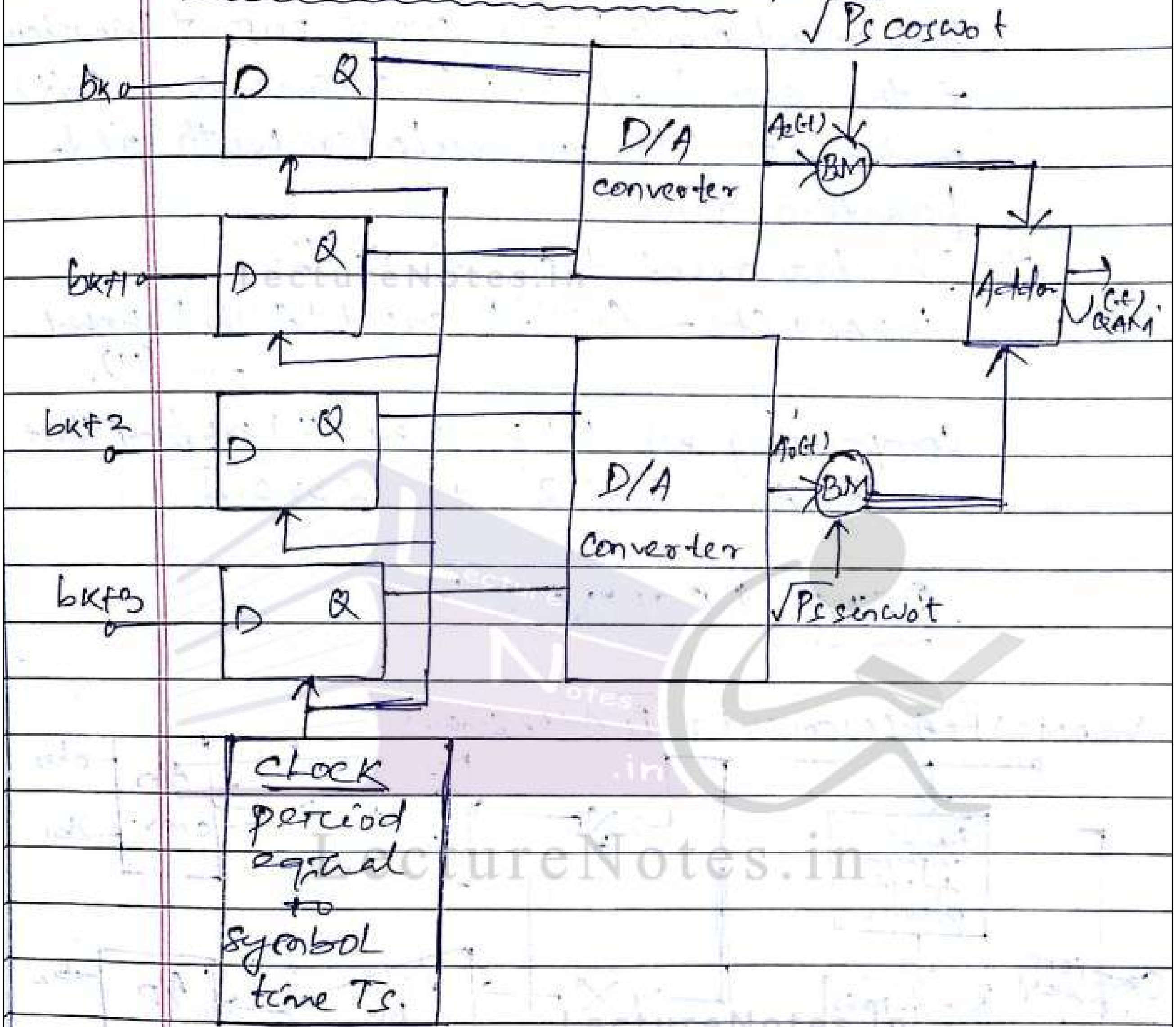
$$V_{QAM} = K_1\sqrt{0.2\frac{E_s}{T_s}}\cos w_0 t + K_2\sqrt{0.2\frac{E_s}{T_s}}\sin w_0 t \text{ ——— } (9)$$

But $E_s/T_s = P_s$

→ So we have

$$V_{QAM} = K_1\sqrt{0.2P_s}\cos w_0 t + K_2\sqrt{0.2P_s}\sin w_0 t \text{ ——— } (10).$$

GENERATION OF QAM :-



→ A generator of a QAM signal for 4-bit symbol is shown above.

→ The 4-bit symbol $b_{k+3}, b_{k+2}, b_{k+1}, b_k$ is stored in the 4-bit register made up of four flip-flops, and the content of the register is correspondingly updated at each active edge of the clock which has a period T_s.

→ Two bits are presented to one D/A converter and two bits to a second converter.

→ The converter output $A_e(t)$ modulates the balanced modulator whose input carrier is the even function $\sqrt{P_s}\cos\omega_0 t$ and $A_o(t)$ modulates the modulator with odd-function carrier.

→ The transmitted signal is then

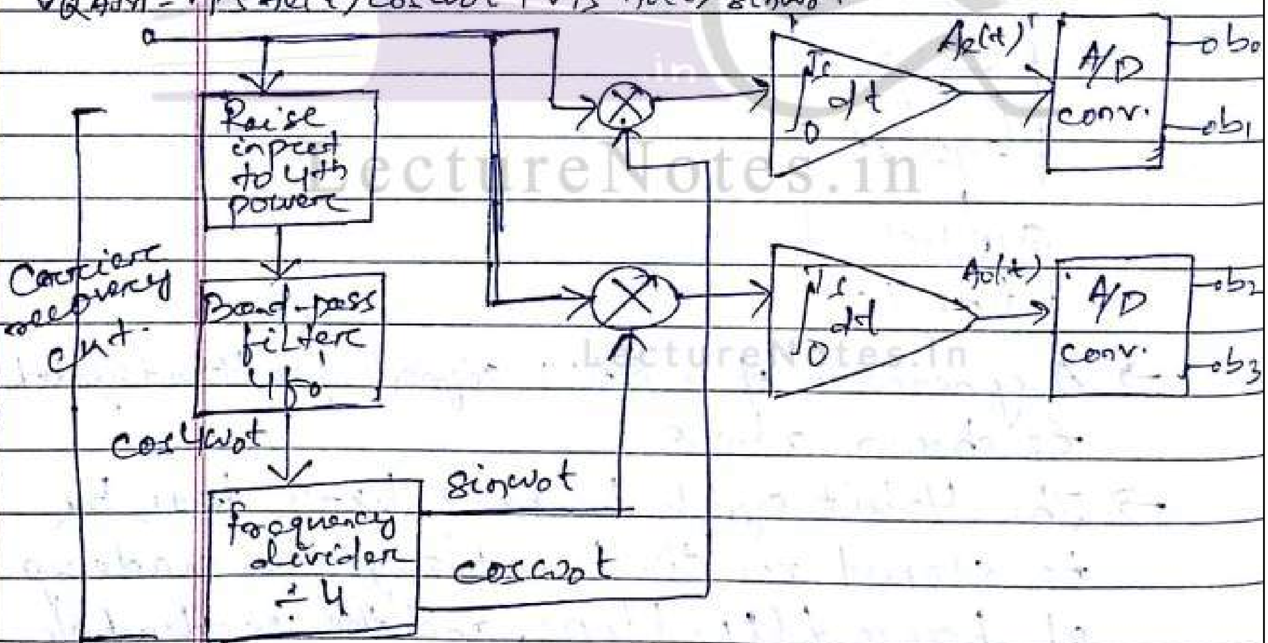
$$V_{QAM}(t) = A_e(t)\sqrt{P_s}\cos\omega_0 t + A_o(t)\sqrt{P_s}\sin\omega_0 t \quad —(11).$$

Comparing eqn (10) with eqn (11) we find that

$$A_e, A_o = \pm\sqrt{0.2} \quad \text{or} \quad \pm 3\sqrt{0.2}.$$

QAM RECEIVER :-

$V_{QAM} = \sqrt{P_s}A_e(t)\cos\omega_0 t + \sqrt{P_s}A_o(t)\sin\omega_0 t$



→ In the present case, it is required that the incoming signal frequency be raised to the fourth power after which filtering recovers a waveform at four times the carrier frequency and finally frequency division by four regenerates the carrier

→ The quadrature carriers being available two balanced modulators are used together with integrators as shown in the above block diagram to recover the signals $A_e(t)$ and $A_o(t)$.

→ The integrators have an integration time equal to the symbol time T_s.

→ Finally, the original input bits are recovered by using A/D converters.

M-ary FSK :-

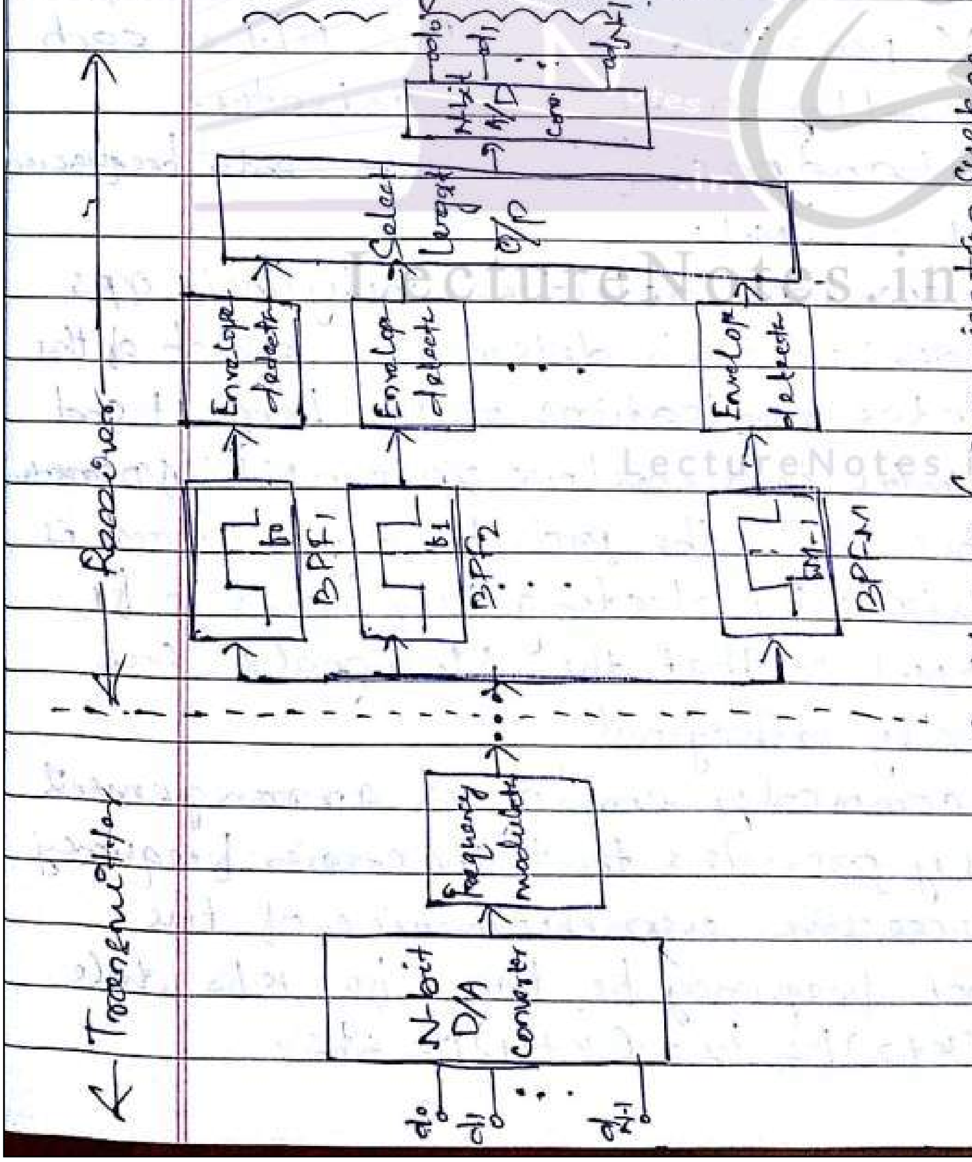


Figure: An M-ary Communication system

→ An M-ary communication system is shown above.
→ It is an obvious extension of a binary FSK system.
→ At the transmitter an N-bit symbol is presented each T_s to an N-bit D/A converter.
→ The converter output is applied to a frequency modulator which generates a carrier waveform.
→ The transmitted signal, for the duration of the symbol interval, is of frequency f_0 or f_1 ... or f_{M-1} with $M = 2^N$.
→ At the receiver the incoming signal is applied to "M" paralleled Band Pass filters each followed by an envelope detector.
→ The bandpass filters have center frequencies $f_0, f_1, \ldots, f_{M-1}$.
→ The envelope detectors apply their o/ps to a device which determines which of the detector indications is the largest and transmits that envelope o/p to an N-bit A/D converter.
→ In this case, the probability of error is minimized by selecting frequencies $f_0, f_1, \ldots, f_{M-1}$ so that the M-signals are mutually orthogonal.
→ One commonly employed arrangement simply provides that the carrier frequency be successive even harmonics of the symbol frequency f_s, thus $f_0 = k f_s$ while $f_1 = (k+2) f_s$, $f_2 = (k+4) f_s$ etc.

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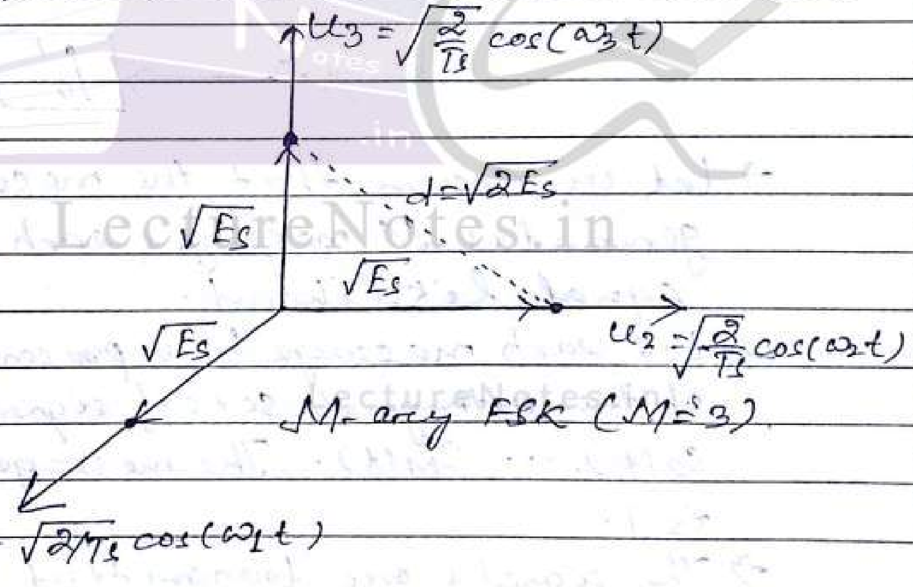
→ We observe that to pass M-ary FSK, the required spectral range is
$$B = 2M f_s \text{ ——— } (1).$$
→ Since $f_s = f_b/N$ and $M = 2^N$ we have
$$B = 2^{N+1} f_b/N \text{ ——— } (2).$$
→ Note that M-ary FSK requires a considerably increased bandwidth in comparison with M-ary PSK, however the probability of error for M-ary FSK decreases as M increases, while for M-ary PSK, the probability of error increases with M.

GEOMETRICAL REPRESENTATION OF M-ary FSK



M-ary FSK (M=3)

→ The geometrical representation of M-ary FSK is shown above with M mutually orthogonal co-ordinate axes.

→ The signal vectors are then parallel to these axes. The best we can do pictorially is the 3-dimensional case shown above.

→ The distance between the signal points is $d = \sqrt{2E_s} = \sqrt{2NE_b}$ ——— (3).

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Digital Communication Techniques

Topic:

Discrete Message And Information Content

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DISCRETE MESSAGES AND INFORMATION CONTENT

THE CONCEPT OF AMOUNT OF INFORMATION:-

→ Few messages produced by an Information source contain more information than others.

→ This can be best understood with the help of following example:-

→ Considere we are planning a tour a city Located in such an area where rain fall is very rare.

→ To know about the weather forecast we will call the weather bureau and may receive one of the following information:

(i) It would be hot and sunny,

(ii) There would scattered rain,

(iii) There would be a cyclone with thunderstorm.

classmate

Date ______
Page ______

→ It may be observed that the amount of information received is clearly different for the three messages.

→ The first message, Just for instance, contains very little information because the weather in a desert city in summer is expected to be hot and sunny for maximum time.

→ The second message forecasting a scattered rain contains some information more as compared to the first one because it is not an event that occurs often.

→ The forecast of a cyclonic storm contains even more information compared to the second message. This is because the third forecast is a rarest event in the city.

→ From the above example, it is concluded that the amount of information received from the knowledge of occurrence of an event may be related to the LIKELIHOOD or PROBABILITY of occurrence of that event.

→ The message associated with the least likelihood event thus consists of max. information

INFORMATION SOURCES :-

→ An information source may be viewed as an object which produces an event.

→ A practical information source in a communication system is a device which produces messages, and it can be either analog or discrete.

→ Here, we deal mainly the discrete sources.

→ A discrete information is a source which has only a finite set of symbols as possible outputs.

→ The set of source symbols is called the source alphabet, and the elements of the set are called symbols or letters.

→ A discrete information source can be classified in two types as having memory or being memoryless.

→ A source with memory is one for which a current symbol depends on the previous symbols.

→ A memoryless source is one for which each symbol is independent of previous symbol.

→ A Discrete Memoryless Source (DMS) can be characterized by the list of symbols, the probability assignment to those symbols and the specification of the rate of generating these symbols by the source.

INFORMATION CONTENT OF A DMS :-

→ The amount of information contained in an event is closely related to its uncertainty.

→ Messages containing knowledge of high probability of occurrence convey relatively little information.

→ If the event is certain (i.e., the event occurs with probability 1), it conveys zero information.

INFORMATION CONTENT OF A SYMBOL

→ Information content of a symbol can be considered as a Logarithmic measure of Information.

→ Let us consider a DMS denoted by X and having alphabet $\{x_1, x_2, \ldots x_m\}$.

→ The information content of a symbol x_i, denoted by $I(x_i)$ is defined as

$$I(x_i) = \log_b \frac{1}{P(x_i)} = -\log_b P(x_i) \quad \text{--- (1)}$$

→ $I(x_i)$ satisfies the following properties:

(a) $I(x_i) = 0$ for $P(x_i) = 1$ ——— (2).

(b) $I(x_i) \geq 0$ ——— (3).

(c) $I(x_i) > I(x_j)$ if $P(x_i) < P(x_j)$ ——— (4)

(d) $I(x_i, x_j) = I(x_i) + I(x_j)$ if x_i and x_j are independent ——— (5).

→ The unit of $I(x_i)$ is the "bit" if $b = 2$, "Hartely" or "decit" if $b = 10$, and 'nat' (natural) if $b = e$. It is standard to use $b = 2$.

ENTROPY (AVERAGE INFORMATION)

→ In a practical communication system, we usually transmit long sequences of symbols from an information source.

→ Thus, we are more interested in the average information that a source produces than the information content of a single symbol.

→ Thus, for quantitative representation of average information per symbol, we make the following assumptions:-

(i) The source is stationary so that the probabilities may remain constant with time.

(ii) The successive symbols are statistically independent and come from the source at a rate of "r" symbols per second.

→ The mean value of $I(x_i)$ of a source 'X' with 'm' different symbols is given by

$$H(X) = E[I(x_i)] = \sum_{i=1}^{m} P(x_i) I(x_i)$$

$$\Rightarrow H(X) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \text{ bits/symbol} \quad \text{——— (6).}$$

→ The quantity $H(X)$ is called the ENTROPY of source X. It is a measure of the average information content per source symbol.

→ It may be noted that for a binary source X which generates independent symbols 0 and 1 with equal probability, the source entropy $H(X)$ is

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1 \text{ bit/symbol} \quad \text{——— (7).}$$

→ The source entropy $H(X)$ satisfies the following relation: $0 \leq H(X) \leq \log_2 m$ — (8) where 'm' is the size i.e. no. of symbols of the source X.

INFORMATION RATE :-

→ If the time rate at which source 'X' emits symbols is 'r' (symbols), the information rate "R" of the source is given by $R = r\,H(X)$ bits/sec ——— (9) where R is the information rate, H(X) is Entropy or average information, and 'r' is rate at which symbols are generated.

→ Information rate 'R' is represented in average number of bits of information per second. It is calculated as under:

$$R = \left[r \text{ in } \frac{\text{symbols}}{\text{second}}\right] \times \left[H(X) \text{ in } \frac{\text{Information bits}}{\text{symbol}}\right] \quad \text{—— (10)}$$

$\therefore$ R = Information bits/second.

NUMERICAL:-

Q.(1) Verify the following expression:-

$0 \le H(X) \le \log_2 m$, where m is the size of the alphabet of X ?

Solution:- Proof of the Lower bound:

Since $0 \le P(x_i) \le 1$,

$\Rightarrow 1/P(x_i) \geq 1 \Rightarrow \log_2(1/P(x_i)) \geq 0$

Then, it follows that

$P(x_i).\log_2(1/P(x_i)) \geq 0$

$\Rightarrow H(X) \geq 0$ or $0 \le H(X)$ (Bound) —— ①

Proof of the Upper bound:

Let us consider two probability distributions $[P(x_i) = P_i]$ and $[Q(x_i) = Q_i]$ on the alphabet $\{x_i\}$, $i = 1, 2, 3, 4 \cdots m$, such that

$$\sum_{i=1}^{m} P_i = 1 \text{ and } \sum_{i=1}^{m} Q_i = 1 \quad \text{—— ②}$$

We know that

$$\sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} = \frac{1}{\ln 2} \sum_{i=1}^{m} P_i \ln \frac{Q_i}{P_i} \quad \text{--- (3)}$$

$$\left[\because \log_2 a = \frac{\ln a}{\ln 2}\right].$$

Next, using the inequality

$$\ln \alpha \leq \alpha - 1, \quad \alpha \geq 0 \quad \text{--- (4)}$$

Applying eqn (4) in eqn (3), we will get

$$\frac{1}{\ln 2} \sum_{i=1}^{m} P_i \ln \frac{Q_i}{P_i} \leq \frac{1}{\ln 2} \sum_{i=1}^{m} P_i \left[\frac{Q_i}{P_i} - 1\right]$$

$$\Rightarrow \frac{1}{\ln 2} \sum_{i=1}^{m} P_i \ln \frac{Q_i}{P_i} \leq \frac{1}{\ln 2} \left[\sum_{i=1}^{m} Q_i - \sum_{i=1}^{m} P_i\right]$$

$$\Rightarrow \sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} \leq \frac{1}{\ln 2} [1 - 1]$$

$$\Rightarrow \sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} \leq 0 \quad \text{--- (5)}$$

Where the equality holds only if $Q_i = P_i$ for all i

Setting $Q_i = \frac{1}{m}$, $i = 1, 2, \cdots m$ --- (6)

We get

$$\sum_{i=1}^{m} P_i \log_2 \frac{1}{P_i m} = -\sum_{i=1}^{m} P_i \log_2 P_i - \sum_{i=1}^{m} P_i \log_2 m$$

$$= H(X) - \log_2 m \sum_{i=1}^{m} P_i$$

$$= H(X) - \log_2 m \times 1$$

$$= H(X) - \log_2 m \quad \text{--- (7)}$$

According to eqn (5), the eqn (7) becomes

$$H(X) - \log_2 m \leq 0 \Rightarrow H(X) \leq \log_2 m$$

(Proved)

Finally $0 \leq H(X) \leq \log_2 m$ (Proved)

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DISCRETE MESSAGES AND INFORMATION CONTENT

Introduction to Information Theory

CONCEPT OF INFORMATION:-

Mathematical models for information source:-

Discrete information source :- It has only a finite set of symbols as possible output. Information source can be classified as having memory and memoryless.
A memoryless source is the source for which each symbol produced is independent of previous source.

Discrete memoryless source (DMS) can be characterized by

(i) List of symbols (Source alphabet).
(ii) the probability assignment to the symbol.
(iii) rate of generating these symbols by source.

If the alphabet is given by $\{x_1, x_2 \cdots x_L\}$ has a given probability of occurance is P_K for $K = 1, 2, \cdots L$

$$P_K = P[X = x_K], \quad 1 \le K \le L.$$

$$\sum_{K=1}^{L} P_K = 1.$$

Logarithimic Measure of information.

Information contained by a source is inversly proportional to its probability of occurance.

When $P \to 1, \; I \to 0$
and $P \to 0, \; I \to \infty$.

Information content of a symbol x_i is given by

$$\boxed{I(x_i) = \log_b \frac{1}{P(x_i)} = -\log_b P(x_i).}$$

where $P(x_i)$ is the probability of occurance of symbol x_i.

Unit of information: if $b = 2$, unit: Bits.
if $b = 10$, unit: Decit
if $b = e$, unit: Nat.

Information has following properties.

(i) $I(x_i) = 0$ for $P(x_i) = 1$.

(ii) $I(x_i) \geq 0$.

(iii) If $P(x_i) < P(x_j)$ then $I(x_i) > I(x_j)$.

(iv) If x_i and x_j are independent

$$I(x_i, x_j) = I(x_i) + I(x_j).$$

Ex A source produces one of four possible symbols during each interval having probabilities $P(x_1) = \frac{1}{2}$, $P(x_2) = \frac{1}{4}$, $P(x_3) = P(x_4) = \frac{1}{8}$. Obtain the information contained in each symbol.

A

$$I(x_i) = \log_2 \frac{1}{P(x_i)}.$$

$I(x_1) = \log_2 \frac{1}{1/2} = \log_2(2) = 1$ bit

$I(x_2) = \log_2 \frac{1}{1/4} = \log_2(4) = 2$ bits.

$I(x_3) = I(x_4) = \log_2 \frac{1}{1/8} = \log_2 8 = 3$ bits

Entropy (average information):- In practical communication system, we usually transmit long sequence of symbols from an information source. So we are more interested in the average information that a source produces than the information content of a single symbol.

The mean value of $I(x_i)$ over the alphabet of source X with 'm' different symbol is given by

$$\boxed{\begin{aligned} H(X) &= \sum_{i=1}^{m} P(x_i) I(x_i). \\ &= -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \end{aligned}}$$ bits/symbol.

H(X) is called "entropy" of source X. It is a measure of average amount of uncertainity with source X or average information content per source symbol.

$$\boxed{0 \le H(X) \le \log_2 m}$$

where 'm' is the number of symbol in source alphabet X.

Ex find the entropy of the source given in previous problem.

A $H(X) = \sum_{i=1}^{m} P(x_i)\, I(x_i)$.

$= P(x_1) I(x_1) + P(x_2) I(x_2) + P(x_3) I(x_3) + P(x_4) I(x_4)$

$= \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{8} \times 3$

$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8} = \frac{4+4+3+3}{8} = 1.75$ bits/symbol.

Information rate:- If a source 'X' is emitting 'r' symbols/sec (symbol rate or baud rate) then the information rate R of the source is given by

$$\boxed{R = r H(X)} \quad \text{bits/sec}$$

unit of R $= \frac{\text{symbols}}{\text{sec}} \times \frac{\text{bits}}{\text{symbols}} =$ bits/sec.

Ex: A ~~BPRSobas~~ high-resolution black-and white TV picture consists of about 2×10^6 picture elements and 16 different brightness levels. Pictures are repeated at the rate of 32 per second. All picture elements are assumed to be independent and all levels have equally likehood of occurance. Calculate the average rate of information conveyed by this TV source.

A. $H(x) = -\sum_{i=1}^{16} p(x_i)\log_2[P(x_i)]$.

As all are equal likely

$$H(x) = -\sum_{i=1}^{16} \frac{1}{16}\log_2\left(\frac{1}{16}\right)$$

$$= -16 \times \frac{1}{16} \times (-4) = 4 \text{ bit / element}.$$

$$r = 2\times10^6 \times 32 = 64\times10^6 \text{ elements/sec}.$$

$$R = r H(x) = 64\times10^6 \times 4 = 256 \text{ Mbps}.$$

Ex: Consider binary memoryless source X with two symbols x_1 and x_2. Prove that H(x) is maximum when both x_1 and x_2 are equiprobable.

A. Let $P(x_1) = \alpha$ and $P(x_2) = 1-\alpha$.

So $H(x) = -\alpha\log_2\alpha - (1-\alpha)\log_2(1-\alpha)$.

$$\Rightarrow \frac{d\,H(x)}{d\alpha} = \frac{d}{d\alpha}\left[-\alpha\log_2\alpha - (1-\alpha)\log_2(1-\alpha)\right].$$

$$\left[\frac{d}{dx}\log_b y = \frac{1}{y}\log_b \frac{dy}{dx}\right].$$

$$\frac{d}{d\alpha}(-\alpha\log_2\alpha) = -\cancel{\alpha}\cdot\frac{1}{\cancel{\alpha}} - \log_2\alpha = -(1+\log_2\alpha).$$

$$\frac{d}{d\alpha}(1-\alpha)\log_2(1-\alpha) = (1-\alpha)\cdot\frac{-1}{1-\alpha} + (-1)\log_2(1-\alpha).$$

$$= -[1+\log_2(1-\alpha)]$$

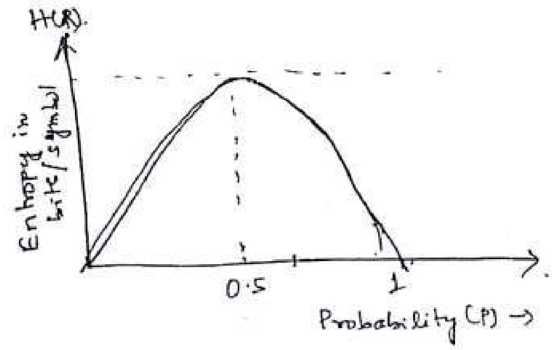
So $\frac{d\,H(x)}{d\alpha} = 1+\log_2(1-\alpha) - 1 - \log_2\alpha = \log_2\left(\frac{1-\alpha}{\alpha}\right)$.

For maximum value $\frac{d\,H(x)}{d\alpha} = 0$

$$\Rightarrow \log_2\frac{1-\alpha}{\alpha} = 0 \Rightarrow \frac{1-\alpha}{\alpha} = 1$$

$$\Rightarrow \alpha = 1/2.$$

i.e. $P(x_1) = P(x_2) = 1/2$.

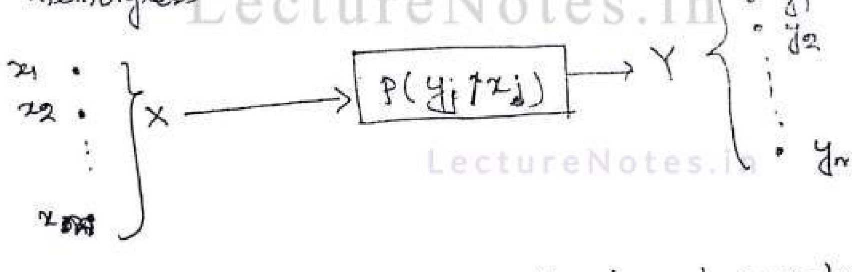


$$\boxed{H(x) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)}$$

is called binary entropy function.

Discrete memoryless channel (DMC) :-

A communication channel is the path or medium through which the symbol flows from transmitter to receiver. DMC is a statistical model with an i/p 'X' and o/p 'Y'. If both i/p and o/p has finite number of symbol it is called discrete. If the current o/p depends on only current i/p not any other previous i/p then it is called memoryless.



During each signalling interval the channel accepts an input symbol from 'X' and in responce it generates an output symbol from Y.

Each possible i/p to o/p path is given by conditional probability $P(y_j/x_i)$ which is called 'channel transsition probability'.

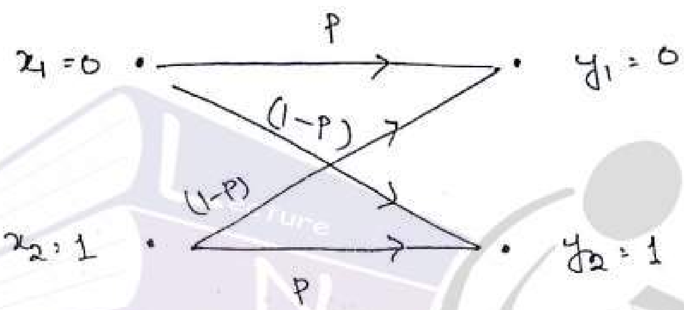
$P(y_j/x_i)$ indecates the probability of occurance of y_j when i/p is only x_i

Channel matrix :- It is the complete set of transmission probabilities.

$$P(Y/X) = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) & \cdots & P(y_n/x_1) \\ P(y_1/x_2) & P(y_2/x_2) & \cdots & P(y_n/x_2) \\ \vdots & & & \\ P(y_1/x_m) & P(y_2/x_m) & \cdots & P(y_n/x_m) \end{bmatrix}$$

Binary Symmetric channel (BSC) :- It is given by the channel matrix as

$$P(Y/X) = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix}$$



Source coding:- The process of efficent representation of data generated by discrete source is called source encoding.

A source coder can generate an efficent source code by exploiting the statistics of the source. The symbols having heigher probability with having lesser information are represented by shorter code whereas lesser probability source are represented by longer code. So an efficent code is a variable length code.

→ Code efficency

$$\boxed{\eta = \frac{H(m)}{\overline{L}}}$$

where $H(m)$ = entropy of the source.

$\overline{L}$ = average code length.

$= \sum_{i=1}^{K} P(x_i)\, L(x_i)$ bits/symbol.

$L(x_i)$ = Length of x_i source.

Redundancy $\gamma = 1 - \eta$.

→ Kraft inequality - A necessary and sufficent condition for existance of instantaneous (separable) binary code is (i.e. whether it is decodable or not).

$$\boxed{K = \sum_{i=1}^{m} 2^{-L(x_i)} \leq 1}$$

If $k = 1$, then it is known as Optimum coding.

If $K > 1$, the code will loss its separability.

Ex. for a DMS 'X' with two symbols x_1 and x_2 with $P(x_1) = 0.9$ & $P(x_2) = 0.1$ with the codes '0' and '1' respectively. Find the efficency 'η' and redundancy 'γ' of the code.

A. $\overline{L} = \sum_{i=1}^{2} P(x_i)\, L(x_i) = 0.9 \times 1 + 0.1 \times 1 = 1$ bit/symbol.

$H(x) = -\sum_{i=1}^{2} P(x_i) \log_2 P(x_i)$

$= -0.9\log_2 0.9 - 0.1\log_2 0.1 = 0.469$ bit/symbol.

NOTE: $\log_2 x = 3.32 \times \log_{10} x$

Code efficency

$$\eta = \frac{H(x)}{L} = \frac{0.469}{1} = 46.9\%.$$

Code redundancy $\gamma = 1-\eta = 1-0.469 = 0.531$
$= 53.1\%.$

Note: Equal Length code has Lesser efficency.

Ex.

Symbol	Probability	Code-1	Code-2	Code-3	Code-4
A	1/2	00	0	0	0
B	1/4	01	1	01	10
C	1/8	10	10	011	110
D	1/8	11	11	0111	111.

Find code efficency and check the code separability.

A. $H(X) = -\sum_{i=1}^{4} P(x_i)\log_2 P(x_i)$

$$= \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{8}\log_2 8$$

$$= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{3}{8} = 1.75 \text{ bits/symbol}.$$

Code-1.

$$L = \sum_{i=1}^{4} P(x_i) L(x_i)$$

$$= \frac{1}{2}\times 2 + \frac{1}{4}\times 2 + \frac{1}{8}\times 2 + \frac{1}{8}\times 2 = 2 \text{ bits/symbol}.$$

$$\eta = \frac{1.75}{2} = 87.5\%.$$

$$K = \sum_{i=1}^{4} 2^{-L(x_i)} = 2^{-2}\times 4 = 1$$

Optimum Coding.

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Code-2

$L = \frac{1}{2}\times 1 + \frac{1}{4}\times 1 + \frac{1}{8}\times 4 + \frac{1}{8}\times 4$

$= 1.25$

$\eta = \frac{1.75}{1.25} > 1.$ (1.4) more than 100%.

$K = 2^{-1} + 2^{-1} + 2^{-2} + 2^{-2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.5 > 1.$

So it is not separalele. So it is a lossy code.

Code-3.

$L = \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{8}\times 4 = 1.875.$

$\eta = \frac{1.75}{1.875} = \frac{12}{13} = 93\%.$

$K = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} < 1.$

$\therefore$ 9t is Separable n decodable coding.

Code-4.

$L = \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{8}\times 3.$

$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8} = \frac{14}{8} = 1.75.$

$\eta = \frac{1.75}{1.75} = 100\%$

$K = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1.$

It is called optimum code.

Huffmann Coding:-

It is the most efficent coding. It follows following steps.

1- List the source symbols in order of decreasing probability.

2- Combine the probabilities of two symbols having lowest probabilities ~~of two symbols~~ and record the ~~lowest~~ resultant probabilities ~~ba~~. This step is called reduction-1. The same procedure is repeated until two order probabilities remaning.

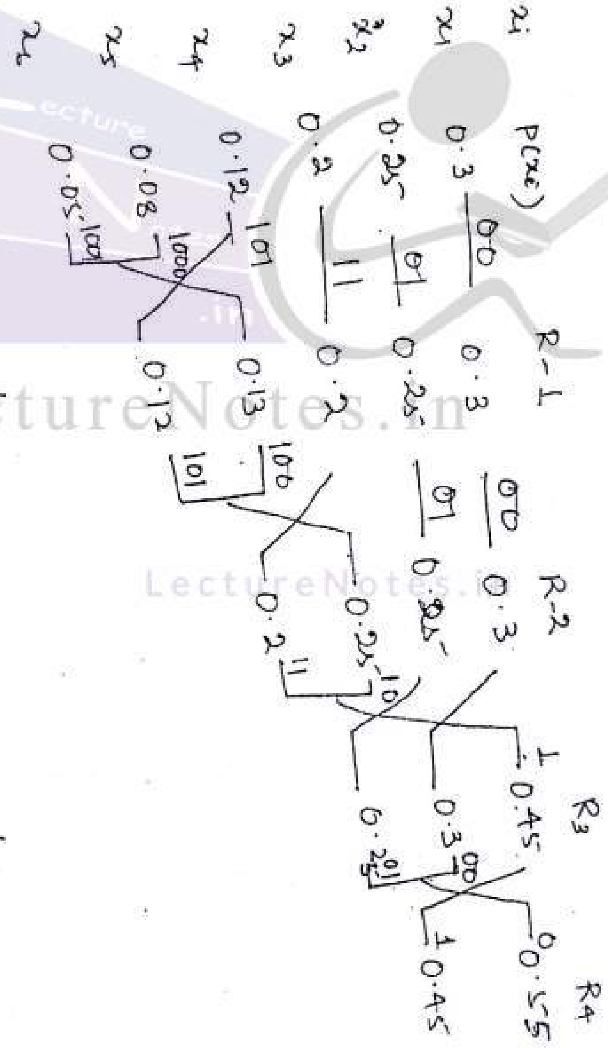
3. Start encoding with last reduction which consists of exactly two orderd probabilities. Assign '0' as the ~~for~~ 1st digit of the code for 1st probability and '1' for 2nd probability.

4- Now go back and assign '0' and '1' to second digit for the probabilities that were combine in previous reduction step, remaining all assignments mode in step-3.

5. Keep representing in this way until 1st col^n is reached.

Ex

x_i	x_1	x_2	x_3	x_4	x_5	x_6
$p(x_i)$	0.3	0.25	0.2	0.12	0.08	0.05

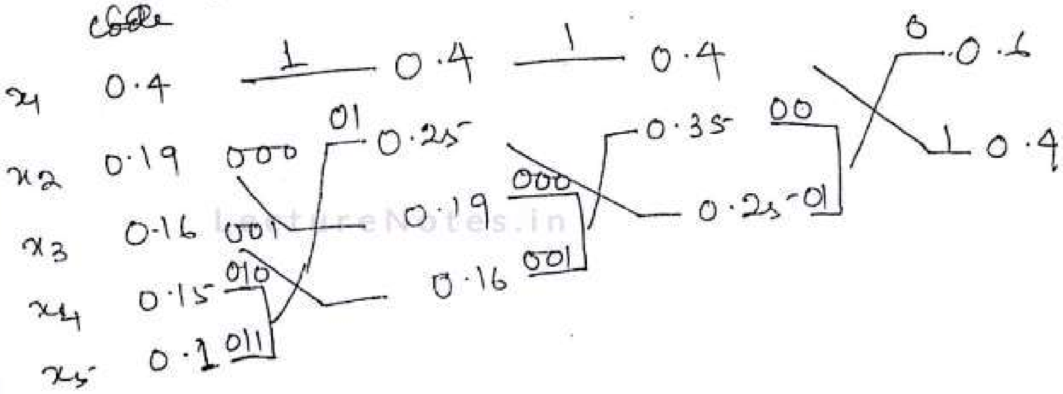


x_i	$p(x_i)$	Code
x_1	0.3	00
x_2	0.25	01
x_3	0.2	11
x_4	0.12	101
x_5	0.08	1000
x_6	0.05	1001

Ex:

x_i	x_1	x_2	x_3	x_4	x_5
$p(x_i)$	0.4	0.19	0.16	0.15	0.1

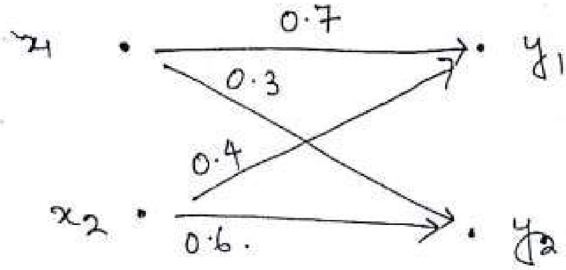
Code



x_i	x_1	x_2	x_3	x_4	x_5
Code	1	000	001	010	011

Ex. $P(x_1) = 0.5$

$P(x_2) = 0.5$



Find $P(X)$, $P(Y)$, $P(XY)$, $P(Y/X)$, $P(X/Y)$.

A. $P(X) = [0.5 \quad 0.5]$.

$P(y_1) = P(y_1/x_1)\,P(x_1) + P(y_1/x_2)\cdot P(x_2)$.

$= 0.7 \times 0.5 + (0.4) \times 0.5 = 0.55$

$P(y_2) = P(y_2/x_1)\,P(x_1) + P(y_2/x_2)\,P(x_2)$:

$= 0.3 \times 0.5 + 0.6 \times 0.5 = 0.45$

$P(Y) = [0.55 \quad 0.45]$

$$P(XY) = \begin{bmatrix} P(x_1 y_1) & P(x_1 y_2) \\ P(x_2 y_1) & P(x_2 y_2) \end{bmatrix}.$$

$$= \begin{bmatrix} P(y_1/x_1)P(x_1) & P(y_2/x_1)\,P(x_1) \\ P(y_1/x_2)\,P(x_2) & P(y_2/x_2)\,P(x_2) \end{bmatrix}.$$

$$= \begin{bmatrix} 0.35 & 0.15 \\ 0.2 & 0.3 \end{bmatrix}.$$

$$P(X/Y) = \frac{P(XY)}{P(Y)} = \begin{bmatrix} \frac{0.35}{0.55} & \frac{0.15}{0.45} \\ \frac{0.2}{0.55} & \frac{0.3}{0.45} \end{bmatrix} = \begin{bmatrix} 0.636 & 0.33 \\ 0.364 & 0.67 \end{bmatrix}$$

$\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32}$
$= \frac{4+4+2+2+2+2+1+1}{32}$
$= \frac{16}{32} = \frac{1}{2}$

Joint and conditional Entropy.

Entropy of i/p $H(X) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i)$

Entropy of o/p $H(Y) = -\sum_{j=1}^{n} P(y_j) \log_2 P(y_j)$.

Entropy of joint probability

$$\boxed{H(XY) = -\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 P(x_i y_j)}$$

$$\Rightarrow H(XY) = -\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 [P(x_i/y_j) P(y_j)].$$

$$= -\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 P(x_i/y_j)$$

$$- \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 P(y_j).$$

conditional entropy is conditional entropy of 'X' is the average uncertainty of X when $Y = y_K$ is received.

$$\boxed{H(X/y_K) = -\sum_{j=1}^{m} P(x_j/y_K) \log_2 [P(x_j/y_K)]}$$

$$\Rightarrow H(X/Y) = -\sum_{i=0}^{n} \sum_{j=1}^{m} P(x_j/y_K) P(y_K) \log_2 [P(x_j/y_K)].$$

$$\boxed{H(X/Y) = -\sum_{i=0}^{n} \sum_{j=1}^{m} P(x_j y_K) \log_2 P(x_j/y_K)}$$

It is defined as amount of uncertainty remaining about the channel input after the channel output has been observed.

Conditional probability:

$P(B/A)$ = Probability of B when A is certain.

$= \frac{P(AB)}{P(A)}$ where $P(AB)$ = joint probability.

$P(A/B) = \frac{P(AB)}{P(B)} \Rightarrow P(AB) = P(A/B)\, P(B)$.

$$\Rightarrow \boxed{P(B/A) = \frac{P(A/B)\, P(B)}{P(A)}} \rightarrow \text{Bayes Rule.}$$

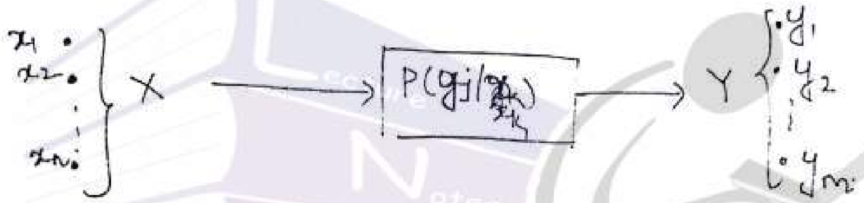
Discrete memoryless channel (DMC)

→ A discrete memoryless channel is a statistical model with an input 'X' and an output 'Y' that is a noisy version of X. Here both X and Y are random variable.

→ Every unit of time, the channel accepts an input symbol x from an alphabet X and in response, it produces an O/p symbol 'y' from an alphabet Y.

→ The channel is said to be 'discrete' when both alphabet 'X' and 'Y' has finite sizes.

→ It is said to be "memoryless" when the current O/p symbol depends on only current I/p symbol, not any other previous symbol.



Let $X = \{x_1, x_2 \ldots x_n\}$.

$Y = \{y_1, y_2, \ldots y_m\}$

The transition probability is a conditional probability given by

$P(y_j/x_k)$ is the probability of getting y_j ~~when~~ at O/p when the I/p is only x_k

The complete set of transmission probabilities are called as channel matrix.

$$P[Y/X] = \begin{bmatrix} P(y_0/x_0) & P(y_1/x_0) & \cdots & P(y_{m-1}/x_0) \\ P(y_0/x_1) & P(y_1/x_1) & \cdots & P(y_{m-1}/x_1) \\ \vdots & \vdots & & \vdots \\ P(y_0/x_{n-1}) & P(y_1/x_{n-1}) & \cdots & P(y_{m-1}/x_{n-1}) \end{bmatrix}$$

Sum of element along any row is one

$$\therefore \sum_{j=0}^{m-1} P(y_j/x_i) = 1.$$

Conditional Entropy -

The conditional entropy is the amount of uncertainity remaining about the channel input after the channel o/p has been observed.

Let at the o/p y_K is observed. Then the uncertainity about I/p alphabet X is given by

$$H(X/y_K) = -\sum_{i=0}^{n-1} P(x_i/y_K) \log_2[P(x_i/y_K)]$$

The mean entropy over the output alphabet Y is given by

$$H(X/Y) = \sum_{K=0}^{m-1} H(X/y_K)\, P(y_K).$$

$$\cancel{= -\sum_{K=0}^{m-1}\sum_{i=0}^{n-1} P(x_i/y_K) \log_2[P(x_i/y_K)]}$$

$$= -\sum_{K=0}^{m-1}\sum_{i=0}^{n-1} P(x_i/y_K)\, P(y_K) \log_2[P(x_i/y_K)].$$

$$\Rightarrow \boxed{H(X/Y) = -\sum_{K=0}^{m-1}\sum_{i=0}^{n-1} P(x_i\, y_K) \log_2[P(x_i/y_K)].}$$

Mutual information and channel capacity.

→ The uncertainty about channel o/p before observing the o/p is entropy $H(X)$.

→ The uncertainty about channel o/p after observing the channel o/p is conditional entropy i.e., $H(X|Y)$.

→ The average mutual information is the difference between them

$$\boxed{I(X;Y) = H(X) - H(X/Y).}$$

It represents uncertainty about the channel o/p that is resolved by observing the channel o/p

Mutual Information

$I(x_i; y_j) = I(x_i) - I(x_i/y_j)$

$= \log\frac{1}{P(x_i)} - \log\frac{1}{P(x_i/y_j)}$

$= \log\left[\frac{P(x_i/y_j)}{P(x_i)}\right]$

$= \log\left[\frac{P(x_i/y_j).P(y_j)}{P(x_i)P(y_j)}\right]$

$= \log\left[\frac{P(x_i, y_j)}{P(x_i)P(y_j)}\right]$

$= \log \frac{P(y_j/x_i)\cdot P(x_i)}{P(x_i)P(y_j)}$

$= \log\left[\frac{P(y_j/x_i)}{P(y_j)}\right] = I(Y,X)$

$$\rightarrow H(X) = \sum_{i=0}^{n-1} P(x_i) \log_2 \frac{1}{P(x_i)}.$$

~~Mutual Information~~ $I(x_i;y_j) = $ [Initial uncertainty about $\{x_i\}$] − [Final uncertainty about $\{x_i\}$ after reception of $\{y_j\}$].

$$= \sum_{i=0}^{n-1} P(x_i) \log_2 \frac{1}{P(x_i)} \cdot \sum_{j=0}^{m-1} P(y_j/x_i)$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} P(y_j/x_i)\, P(x_i) \log_2\left[\frac{1}{P(x_i)}\right]$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)}\right].$$

So ~~I(X;Y) = Σ~~

$$\boxed{\begin{aligned} I(X;Y) &= H(X) - H(X/Y). \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} \end{aligned}}$$

Mutual Information

$$I(x_i; y_j) = \log \left[\frac{P(x_i/y_j)}{P(x_i)}\right]$$

Average Mutual Information

$$I(X,Y) = \sum \left[I(x_i, y_j) \right]$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} P(x_i, y_j) I(x_i, y_j)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} P(x_i, y_j) \log \frac{P(x_i/y_j)}{P(x_i)}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} P(x_i, y_j) \log \frac{1}{P(x_i)} - \sum_{j=1}^{n} \sum_{i=1}^{m} P(x_i, y_j) \log \frac{1}{P(x_i/y_j)}$$

$$= H(X) - H(X/Y)$$

CHANNEL CAPACITY PER SYMBOL (C_s):-

Channel capacity is defined as the maximum value of Average-mutual information.

$$\boxed{C_s = \max [I(X;Y)]}$$

The unit of C_s are bits per transmission or bits per symbol (same as entropy) or bits per channel use.

CHANNEL CAPACITY PER SECOND (C):-

If a symbol enters the channel every T_s second the channel capacity is $C = C_s/T_s$ and unit is bits/s

$$\Rightarrow \boxed{C = rC_s \text{ bits/sec}} \left[\because \frac{\text{Symbols}}{\text{sec}} \times \frac{\text{bits}}{\text{symbols}}\right]$$

where r = symbol rate or Baud rate.

Shannon's theorem for noisy channel coding.

" Reliable communication over a discrete memoryless channel is possible if the communication rate 'R' satisfies $R < C$, where C is the channel capacity. At rates higher than capacity, reliable communication is impossible".

A system performing near capacity is a near optimal system and doesn't have much room for improvement.

Channel capacity of a Lossless channel:-

For lossless channel $H(X/Y) = 0$

So $I(X;Y) = H(X) - H(X/Y) = H(X)$.

So $C_s = \max[I(X;Y)] = \max[H(X)]$.

Entropy is maximum when the symbols are equiprobable.

i.e. $P(x_i) = \frac{1}{m}$.

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$$\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i,y_j)\left\{\log_2 \frac{1}{P(x_i)} + \log_2 P(x_i/y_j)\right\}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i,y_j)\left[-\log_2 P(x_i) + \log_2 P(x_i/y_j)\right]$$

$$= -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i,y_j)\log_2 P(x_i) - \left\{-\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i,y_j)\log_2 P(x_i/y_j)\right\}$$

$$= H(X) - H(X/Y)$$

$$\Rightarrow C_s = \log_2 m \quad \left[\because C_s = \max[H(X)] = \sum_{i=1}^{m} P(x_i)\log_2 \frac{1}{P(x_i)} = m\times\frac{1}{m}\log_2\frac{1}{\frac{1}{m}} = \log_2 m\right]$$

where m = no. of symbol in X.

Channel capacity in bits/sec.

$$C = r C_s = 2BC_s$$

$\because r$ = Nyquist rate = Symbol rate

$$\Rightarrow \boxed{C = 2B\log_2 m}$$

, B = Bandwidth of the channel.

Channel capacity of AWGN channel (Shannon-Hartley Theorem):-

In an additive white gaussian noise (AWGN) channel the channel o/p Y is given by.

$$Y = X + n.$$

Where X is the channel i/p and n is an additive band-limited white gaussian noise with zero mean and variance $\sigma^2 = N_0B = \eta B = N$ (Noise Power) → P.S.D.

$$\Rightarrow \sigma^2 = \frac{N_0}{2}\times 2B \Rightarrow \sigma^2 = N_0B = \eta B.$$

Then the channel capacity is

$$\boxed{C = B\log_2(1 + S/N)} \text{ bit/s}$$

$$\Rightarrow C = B\log_2\left(1+\frac{S}{\sigma^2}\right) \Rightarrow C = B\log_2\left(1+\frac{S}{\eta B}\right) \Rightarrow C = B\log_2\left(1+\frac{S}{N_0B}\right) \text{ bits/sec}$$

Where 'B' is the channel B.W.

'S' is the signal power

'N' or 'σ^2' is the noise power.

This equation is called Shannon-Hartley theorem.

$$\Rightarrow \boxed{C = B\log_2\left(1+\frac{S}{\eta B}\right)} \text{ bit/s}$$

$\Rightarrow N_0 = \eta$ = Noise Power Spectral density.

Bandwidth - S/N tradeoff.

With a fixed signal power and inpresence of white gaussian noise the channel capacity increases with increase of bandwidth. But it also allows the noise to increase. Hence we require to make a trade-off between S/N and BW. The upper limit of capacitity for $B\to\infty$.

$$C = B\log_2\left(1+\frac{S}{\eta B}\right).$$

$$= \frac{S}{\eta}\cdot\frac{\eta}{S} B\log_2\left(1+\frac{S}{\eta B}\right)$$

$$= \frac{S}{\eta}\log_2\left(1+\frac{S}{\eta B}\right)^{\eta B/S}$$

$$C_\infty = \lim_{B\to\infty} \frac{S}{\eta} \log_2\left(1+\frac{S}{\eta B}\right)^{\eta B/S}.$$

$$= \frac{S}{\eta} \lim_{B\to\infty} \log_2\left(1+\frac{S}{\eta B}\right)^{\eta B/S}.$$

Put $x = \frac{\eta B}{S}$.

$$C_\infty = \frac{S}{\eta} \lim_{x\to\infty} \log_2\left(1+\frac{1}{x}\right)^{x}.$$

$$= \frac{S}{\eta} \log_2 \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x}.$$

But $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$.

So $C_\infty = \frac{S}{\eta}\log_2 e$

$$\Rightarrow \boxed{C_\infty = 1.44\,\frac{S}{\eta}.}$$

Shannon - Fano Coding :-

1- List the source symbols in order of decreasing probability.

2 - Partition the set into two sets that are as close to equiprobable as possible and assign '0' to upper set and '1' to lower set.

3 - Continue the step-2, each time partitioning the sets with as nearly equal probabilities as possible until further partition is not possible.

Ex.

x_i	$P(x_i)$	Step-1	Step-2	Step-3	Step-4	Code
x_1	0.3	0 (0.55)	0			00
x_2	0.25	0	1			01
x_3	0.2	1 (0.45)	0 (0.2)			10
x_4	0.12	1	1	0 (0.12)		110
x_5	0.08	1	1 (0.25)	1	0	1110
x_6	0.05	1	1	1 (0.13)	1	1111

x_i	$P(x_i)$	Step-1	Step 2	Step-3	Step-4.	Step-5	Code
m_1	1/2 (1/2)	0					0
m_2	1/8	1 $\rightarrow \frac{1}{4}$	0	0			100
m_3	1/8	1	0	1			101
m_4	1/16 (1/2)	1	1	0	0		1100
m_5	1/16	1	1 $\rightarrow$ 1/8	0	1		1101.
m_6	1/16	1 $\rightarrow$ 1/4	1	1 $\rightarrow$ 1/16	0		1110
m_7	1/32	1	1 $\rightarrow$ 1/8	1 $\rightarrow$ 1/16	1	0	11110
m_8	1/32	1	1	1	1	1	11111

$$\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32} = \frac{2+2+2+1+1}{32}$$

$$= \frac{8}{32} = \frac{1}{4}$$

$$\frac{1}{16}+\frac{1}{32}+\frac{1}{32} = \frac{2+1+1}{32} = \frac{4}{32} = \frac{1}{8}$$

Ex: Consider an AWGN channel with 4-KHz b.w. and noise power spectral density $\eta/2 = 10^{-12}$ W/Hz. The signal power required at the receiver is 0.1 mW. Calculate the capacity of this channel.

A: $B = 4000$ Hz.

$S = 0.1 \times 10^{-3}$ W., $N = \eta B = 2 \times 10^{-12} \times 4000 = 8 \times 10^{-9}$ W.

$= 10^{-4}$

$$\frac{S}{N} = \frac{10^{-4}}{8 \times 10^{-9}} = 1.25 \times 10^{+4}.$$

$$C = B\log_2\left(1 + \frac{S}{N}\right)$$

$$= 4000 \log_2(1 + 1.25 \times 10^{+4}) = \underline{54.44 \text{ Kbps}}.$$

Ex An analog signal having 4KHz b.w. is sampled at 1.25 times the Nyquist rate and each sample is quantized into one of the 256 equally likely levels. Assume that the successive samples are statistically independent.

(i) What is the information rate of the source?

(ii) Can the o/p of the source be transmitted without error over awgn channel with a b.w. of 10KHz and an S/N ratio of 20dB.

(iii) Find the S/N ratio required for error free transmission of part (ii) having B.W = 10KHz.

(iv) Find the b.w. required for an AWGN channel for error-free transmission of the o/p of this source if the S/N ratio is 20dB.

A. (i) $f_M = 4 \times 10^3$ Hz

Nyquist rate $= 2f_M = 2 \times 4 \times 10^3 = 8 \times 10^3$ samples/s.

$r = 8 \times 10^3 \times 1.25 = 10^4$ samples/s.

$H(X) = \log_2 \frac{1}{256} \times 256 \log_2 256 = 8$ bit/sample.

$R = r H(X) = 8 \times 10^4 = 80$ Kbps.

(ii) $C = B\log_2\left(1 + \frac{S}{N}\right)$.

$= 10^4 \log_2(1 + 10^2) = 66.6$ Kbps.

Since $R > C$, error free transmission is not possible.

(iii) Required S/N ratio can be found by

$$C = 10^4 \log_2\left(1 + \frac{S}{N}\right) \geq 8 \times 10^4.$$

$$\Rightarrow \log_2\left(1 + \frac{S}{N}\right) \geq 8$$

$$\Rightarrow 1 + S/N \geq 2^8 = 256$$

$$\Rightarrow S/N \geq 255 = 24.1 \text{ dB}.$$

(iv)

$$C = B\log_2\left(1 + \frac{S}{N}\right) \geq 8 \times 10^4$$

$$\Rightarrow B \geq \frac{8 \times 10^4}{\log_2(101)} = 1.2 \times 10^4 = 12 \text{ kHz}.$$

$$10 \log_{10} \frac{S}{N} = 20 \text{ dB}$$

$$\Rightarrow \log_{10} S/N = 2$$

$$\Rightarrow S/N = 10^2$$

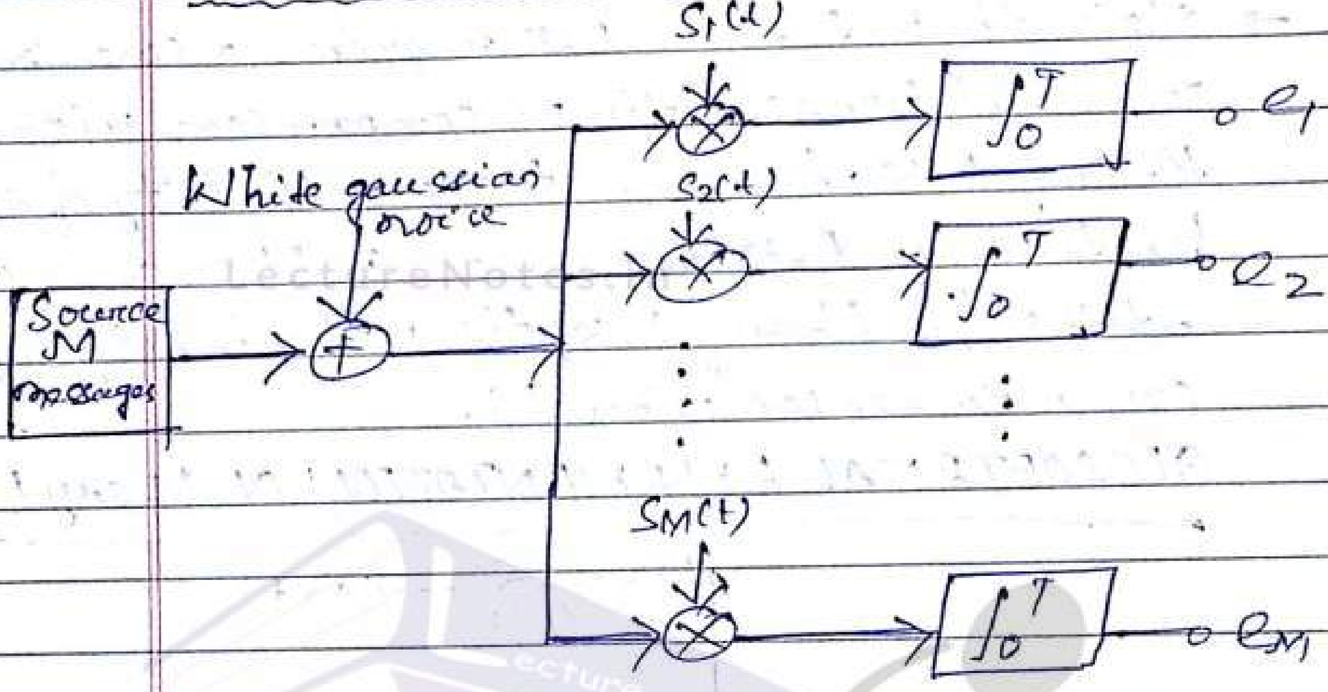
Bandwidth - S/N Tradeoff

The Shannon-Hartley Theorem indicates that a noiseless gaussian channel ($S/N = \infty$) has an infinite capacity. On the other hand, while the channel capacity does increase, it does not become infinite as the bandwidth becomes infinite. It is because with the increase in bandwidth the noise power also increases.

Thus for a fixed signal power and in the presence of white gaussian noise the channel capacity approaches an upper limit with increasing bandwidth. We now calculate that limit.

INFORMATION THEORY AND CODING

MATCHED-FILTER RECEPTION :-



→ Let us assume that the message source generates M messages each with equal likelihood.

→ Let each message be represented by one of the orthogonal sets of signals $S_1(t)$, $S_2(t)$, ... $S_M(t)$. The message interval is T.

→ The signals are transmitted over a communications channel where they are corrupted by additive white gaussian noise

→ As shown in the above block diagram, at the receiver a determination of which message has been transmitted is made through the use of M matched filters that is correlators, which consist of a multiplier followed by an integrator.

→ The local inputs to the multipliers are the signals $S_i(t)$.

→ Suppose then, in the absence of noise, the signal $S_i(t)$ is transmitted, and the output of each integrator is sampled at the end of a message interval.

→ Then, because of the orthogonality condition, all integrators will have zero output, except that the ith integrator output will be E_s as follows

$$\int_0^T S_i^2(t)\,dt = E_s \quad \text{——} \quad (1)$$

→ In the presence of an additive noise waveform $n(t)$, the output of l^{th} correlator $(l \neq i)$ will be

$$e_l = \int_0^T n(t)\,S_l(t)\,dt \equiv n_l \quad \text{——} \quad (2)$$

→ This quantity n_l is a random variable, with zero mean value and has a variance $\sigma^2 = \eta E_s/2$ —— (3)

→ The correlator corresponding to the transmitted message $S_i(t)$ will have an output $l_i = \int_0^T [S_i(t) + n(t)]\,S_i(t)\,dt$ —— (4)

$$= \int_0^T S_i^2(t)\,dt + \int_0^T n(t)\,S_i(t)\,dt$$

$$\Rightarrow l_i = E_s + n_i \quad \text{——} \quad (5)$$

USE OF ORTHOGONAL SIGNALS TO ATTAIN SHANON'S LIMIT

→ Orthogonal Signals:-
The system we are to describe involves the use of a set $S_1(t), S_2(t), \ldots$ of orthogonal signals. Such signals $S_i(t)$, defined as being orthogonal over the interval 0 to T, have the property that

$$\int_0^T S_1(t)\, S_2(t)\, dt = 0 \quad \text{for } i \neq j \quad \text{--- (1)}$$

→ A familiar example of a set of orthogonal signals is M-ary FSK.

→ We shall assume that the amplitudes of the signals have been adjusted so that in every case

$$\int_0^T S_i^2(t)\, dt = E_s \quad \text{--- (2)}$$

that is each signal has the same energy E_s in the interval "T" and also the same power $P_s = E_s/T$.

CALCULATION OF ERROR PROBABILITY

→ To determine which message has been transmitted we shall compare the matched-filter outputs $e_1, e_2, \ldots, e_M$.

→ The signal $S_i(t)$ has been transmitted and if the corresponding e_i is Larger

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than the output of any other filter, then this considered as an error.

→ The probability that some arbitrarily selected output l_l is less than the o/p l_i is given as

$$p(l_l < l_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{l_i} e^{-l_l^2/2\sigma^2} \, dl_l \quad \text{—(1)}$$

→ From the above eq^n (1), it was observed that $p(l_l < l_i)$ depends only on l_i and doesnot depend on which output l_l has been selected for comparison with l_i.

→ The probability that, say l_1 and l_2 are both smaller than l_i is

$$p(l_1 < l_i \text{ and } l_2 < l_i)$$
$$= p(l_1 < l_i)\, p(l_2 < l_i)$$
$$= [p(l_1 < l_i)]^2 = [p(l_2 < l_i)]^2 \quad \text{—— (2).}$$

Since l_1 and l_2 are independent.

→ Hence, the probability P_L that l_i is the Largest of the outputs is

$$P_L = P(l_i > l_1, l_2, \ldots, l_{i-1}, l_{i+1}, \ldots, l_M)$$
$$= \left[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{l_i} e^{-l_l^2/2\sigma^2} \, dl_l \right]^{M-1}$$
$$= \left[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{E_s + n_i} e^{-l_l^2/2\sigma^2} \, dl_l \right]^{M-1} \quad \text{—— (3).}$$

→ Let us consider $x = e_1/\sqrt{2}\,\sigma$, then

$$\Rightarrow \frac{dx}{de_1} = \frac{1}{\sqrt{2}\,\sigma} \Rightarrow de_1 = \sqrt{2}\,\sigma\, dx \quad \text{--- (4).}$$

when $e_1 \to \infty$, $x \to \infty$.

when $e_1 = E_s + n_i$, then $x = \frac{E_s + n_i}{\sqrt{2}\,\sigma}$

→ So the eqⁿ (3) becomes

$$P_L = \left[\frac{1}{\sqrt{2}\sqrt{\pi}\,\sigma}\int_{-\infty}^{\frac{E_s+n_i}{\sqrt{2}\sigma}} e^{-x^2}\sqrt{2}\,\sigma\, dx\right]^{M-1}$$

$$\Rightarrow P_L = \left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{E_s/\eta}+\frac{n_i}{\sqrt{2}\sigma}} e^{-x^2}\, dx\right)^{M-1}$$

P_L depends on E_s/η, M and $n_i/\sqrt{2}\sigma$. $\left[\because \frac{E_s}{\sqrt{2}\sigma} = \sqrt{\frac{E_s}{\eta}}\right]$

$$\Rightarrow P_L = P_L\left(\frac{E_s}{\eta}, M, \frac{n_i}{\sqrt{2}\sigma}\right) \quad \text{--- (5).}$$

→ To find the probability that e_i is the largest output, we need the average of the probability P_L over all possible values of n_i. This average is the probability that we shall be correct in deciding that the transmitted signal corresponds to the correlator which yields the largest output.

→ Let us consider this probability as P_c.

→ The probability of an error is then $P_e = 1 - P_c$.

→ We have seen that n_i is a gaussian random variable with zero mean and variance σ^2.

→ Hence $P_c = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} P_L\left(E_s/\eta, M, \frac{n_i}{\sqrt{2}\sigma}\right) e^{-n_i^2/2\sigma^2} dn_i$ — (6).

→ Let us substitute $y \equiv \frac{n_i}{\sqrt{2}\sigma}$ — (7)

Eqn (6) becomes

$$\Rightarrow P_c = \left(\frac{1}{\sqrt{\pi}}\right)^M \int_{-\infty}^{\infty} e^{-y^2} \left(\int_{-\infty}^{\sqrt{E_s/\eta}+y} e^{-x^2} dx \right)^{M-1} dy \quad \text{— (8)}$$

So Probability of error is

$$\rightarrow P_e = 1 - P_c = 1 - \left(\frac{1}{\sqrt{\pi}}\right)^M \int_{-\infty}^{\infty} e^{-y^2} \left(\int_{-\infty}^{\sqrt{E_s/\eta}+y} e^{-x^2} dx \right)^{M-1} dy \quad \text{— (9)}$$

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Digital Communication Techniques

Topic:

Optimal Reception Of Digital Signals

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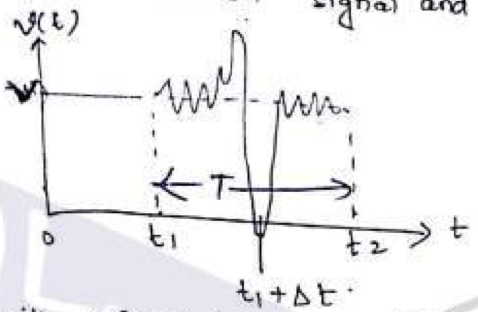
Optimal Reception of Digital Signal

A baseband signal receiver (Integrate and dump filter)

The binary-encoded signal consists of a time sequence of voltage levels +V or −V. But the received signal is the combination of signal and noise which produces sample values generally differ from ±V.

Let a transmitted bit is represented by voltage +V which is sustained over an interval T from t_1 to t_2. Noise has been superimposed on the level +V so that $v(t)$ represents received signal and noise.



So error will definitely occure if the sampling is done at $t = t_1 + \Delta t$.

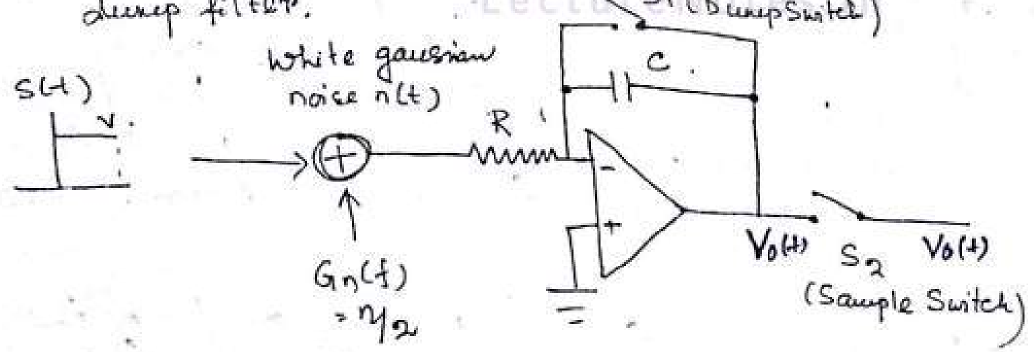
→ Hence the criteria of detection (i.e. Optimal Reception) :-

(i) It should amplify the signal and attenuate the noise i.e. increase SNR.

(ii) The detector will take dicision when the SNR is heigher.

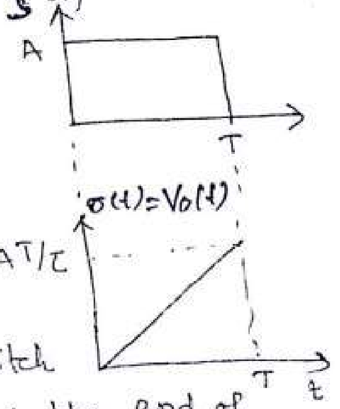
(iii) It should have lowest probability of error.

→ All criterias are satisfied by integrator & dump filter.



NOTE :- The term "DUMP" referring to the abrupt discharge of the capacitor after each sampling.

→ The integrator integrates the noisy i/p signal over one bit period. For square pulse i/p, the o/p of the integrator would be triangular pulse.



→ At the end of one bit period i.e. $t = T$, the magnitude of $r(t)$ attains a maximum amplitude. Hence the value of $r(t)$ is sampled at the end of bit period by closing the sample switch S_2 as the SNR is maximum at the end of bit period.

→ The dump switch closes at the end of every bit period to discharge the capacitor as the o/p bit doesn't depends upon the perivious bit.

Peak Signal to RMS noise o/p voltage ratio :-

For an integrator

$$V_o = \frac{1}{RC}\int_0^T v_i \, dt = \frac{1}{RC}\int_0^T A\, dt = \frac{AT}{RC} = \frac{AT}{\tau}.$$

where $\tau = RC$ = time constant.

So signal power $\boxed{S_o = \frac{A^2 T^2}{\tau^2}}$ — (i)

Noise power calculation :-

i/p noise power spectral density $= \eta/2$.

o/p —do— at the o/p of the integrator $=$ i/p PSD $\times |H(\omega)|^2$.

where $|H(\omega)|$ = transfer function of integrator.

The transfer function of integrator $= \frac{1}{j\omega RC}$.

A delay by an interval 'T' may be represented by a factor $e^{j\omega T}$ in frequency domain. So

Hence a network which performs an integration over an interval T may be represented by a N/w whose T.F. is

$$H(\omega) = \frac{1}{j\omega RC} - \frac{e^{-j\omega T}}{j\omega RC} = \frac{1-e^{-j\omega T}}{j\omega RC}.$$

f

For $\omega = 2\pi f$ and $RC = \tau$

$$H(f) = \frac{1-e^{-j2\pi fT}}{j2\pi f\tau}.$$

Sr

$$= \frac{1-[\cos(2\pi fT) - j\sin(2\pi fT)]}{j2\pi f\tau} = \frac{J\sin(2\pi fT) + 1 - \cos(2\pi fT)}{J2\pi f\tau}$$ Hence

$$= \frac{\sin(2\pi fT)}{2\pi f\tau} - J\frac{(1-\cos 2\pi fT)}{2\pi f\tau}.$$

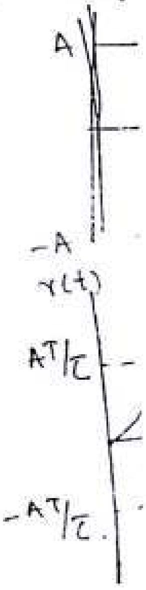
$$= \frac{\sin(2\pi fT)}{2\pi f\tau} - j\frac{1-\cos(2\pi fT)}{2\pi f\tau}.$$

$$\Rightarrow |H(f)|^2 = \frac{\sin^2(2\pi fT) + 1 - 2\cos(2\pi fT) + \cos^2(2\pi fT)}{(2\pi f\tau)^2}$$

$\because \sin^2(\pi fT) = \frac{1-\cos(2\pi fT)}{2}$ Probab

$$= \frac{2[1-\cos(2\pi fT)]}{(2\pi f\tau)^2}$$

$\Rightarrow 1-\cos(2\pi fT) = 2\sin^2(\pi fT)$ Wave $x(t)$



$$\Rightarrow |H(f)|^2 = \left(\frac{T}{\tau}\right)^2 \frac{\sin^2(\pi fT)}{(\pi fT)^2} = \left(\frac{T}{\tau}\right)^2 \mathrm{sinc}^2(\pi fT)$$

$$= \left(\frac{T}{\tau}\right)^2 Sa^2(\pi fT).$$

$\Rightarrow$ So o/p PSD of white noise

$$n_o = \frac{\eta}{2}\left(\frac{T}{\tau}\right)^2 \mathrm{sinc}^2(\pi fT)$$

Noise power $N_o = \int_{-\infty}^{\infty} \text{o/p psd} \times df.$

$$= \int_{-\infty}^{\infty} \frac{\eta}{2}\left(\frac{T}{\tau}\right)^2 \mathrm{sinc}^2(\pi fT)\, df.$$

Let $\pi fT = x$

$$\Rightarrow \pi T\, df = dx \Rightarrow df = \frac{1}{\pi T}\, dx$$

So

$$N_0 = \frac{\eta}{2}\left(\frac{T}{\tau}\right)^2 \int_{-\infty}^{\infty} \text{sinc}^2 x \cdot \frac{1}{\pi T}\, dx .$$

$$= \frac{\eta}{2}\left(\frac{T}{\tau}\right)^2 \cdot \frac{1}{\pi T} \int_{-\infty}^{\infty} \text{sinc}^2 x\, dx$$

Put $\int_{-\infty}^{\infty} \text{sinc}^2 x\, dx = \pi = \int_{-\infty}^{\infty} Sa^2(x)\, dx = \pi .$

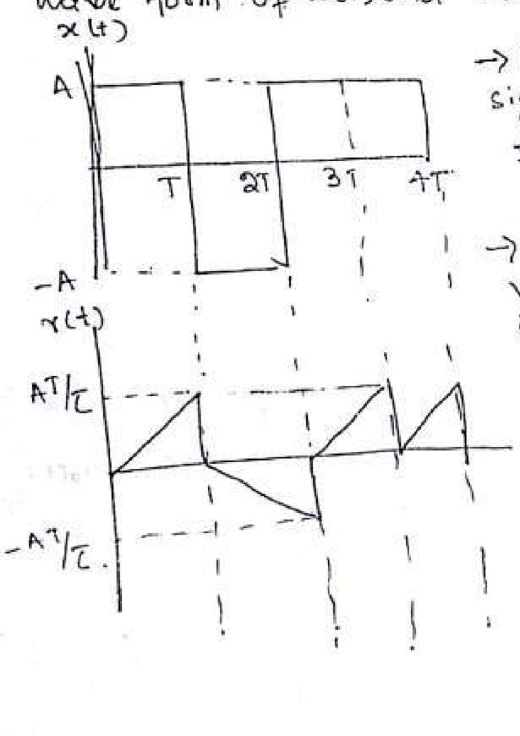
So $N_0 = \frac{\eta}{2}\frac{T}{\tau^2}$ —(ii)

Hence $SNR = \frac{S_0}{N_0} = \frac{\frac{A^2T^2}{\tau^2}}{\frac{\eta}{2}\frac{T}{\tau^2}} = \frac{A^2T}{\eta/2}.$

$$\boxed{SNR = \frac{A^2T}{\eta/2} = \frac{P_T\, T}{\eta/2} = \frac{E_b}{\eta/2}}$$

Probability of error (due to integrator and dump filter):

Wave form of $x(t)$ and $r(t)$ is as shown.



→ It may be observed that the o/p signal voltage reaches the value of $\pm\frac{AT}{\tau}$ at the sampling instant.

→ As the noise has zero average value and Gaussian distribution it doesn't increases in same proportion.

→ So SNR is maximum at the sampling instant.

Probability of error

Let us assume the channel is free space and noise is white Gaussian. Probability density funct^n of any Gaussian Signal is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2} \quad \text{——(i)}$$

Where σ = S.D and m = mean.

Mean of noise signal is zero i.e. $m = 0$.

$$f_X[n_o(t)] = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-n_o^2(t)/2\sigma^2} \quad \text{——(ii)}$$

The S.D. (σ) is expressed as

$$\sigma = [\text{mean square value} - \text{square of mean value}]^{1/2}$$

$$\sigma_X = [\overline{X^2} - m_x^2]^{1/2}$$

For $m_x = 0$ $\rightarrow$ noise power = No

$\overline{X^2} = \frac{N_o T}{2\tau^2}$. So $\sigma_X = \sqrt{\frac{N_o T}{2\tau^2}}$. ——(iii)

So

$$f_X[n_o(t)] = \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_o T}{2\tau^2}}} e^{-n_o^2(t)/2\left(\frac{N_o T}{2\tau^2}\right)} \quad \text{——(iv)}$$

$$= \cancel{\frac{\tau}{\sqrt{\pi N_o T}} e^{[n_o(t)]^2/(N_o T)}}.$$

The o/p of integrator is

$$y(t) = r(t) + n_o(t). \quad \text{——(v)}$$

The maximum value of $r(t)$ is $\frac{AT}{\tau}$ for input A and ~~minimum~~ $-\frac{AT}{\tau}$ for input $-A$.

$$\Rightarrow y(t) = \begin{cases} \frac{AT}{\tau} + n_o(t) & \text{for } x(t) = A \\ -\frac{AT}{\tau} + n_o(t) & \text{for } x(t) = -A \end{cases} \quad \text{——(vi)}$$

So error will occure only in two cases.

(i) When I/p is A, O/p is $\frac{AT}{\tau}$. ——(vii).

Error occures when $n_o(t) < -\frac{AT}{\tau}$.

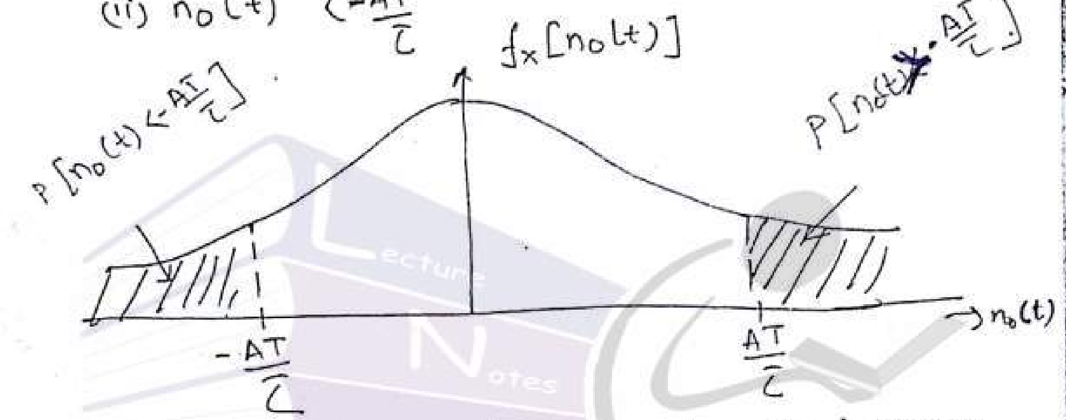
(ii) When i/p is $-A$, o/p is $-\frac{AT}{\tau}$.

Error occures when $n_o(t) > \frac{AT}{\tau}$.

So probability of error can be calculated by calculating the probability that

i) $n_o(t) > \frac{AT}{\tau}$

(ii) $n_o(t) < -\frac{AT}{\tau}$



As Gaussian distribution is symmetric about mean

$$P\left[n_o(t) > \frac{AT}{\tau}\right] = P_o\left[n_o(t) < -\frac{AT}{\tau}\right].$$

As occurance of $+A$ or $-A$ is mutually exclusive, P_e is given by either of two above eqn.

So

$$P_e = P\left[n_o(t) > \frac{AT}{\tau}\right] = \int_{\frac{AT}{\tau}}^{\infty} f_x[n_o(t)]\, d[n_o(t)]$$

$$= \int_{\frac{AT}{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\frac{\eta_0 T}{2\tau^2}}} \cdot e^{-n_o^2(t) / \frac{\eta_0 T}{\tau^2}}\, d\, n_o(t). \quad \text{—(v)}$$

Put $y^2 = \frac{n_o^2(t)}{\eta_0 T / \tau^2}$.

$$\Rightarrow y = \frac{n_o(t)}{\sqrt{\eta_0 T} / \tau}.$$

$$\Rightarrow d[n_0(t)] = \frac{\sqrt{\eta_0 T}}{\tau} dy.$$

When $n_0(t) \to \infty$ then $y \to \infty$.

When $n_0(t) \to \frac{A T}{\tau}$

then $y = \frac{AT/\tau}{\sqrt{\eta_0 T}/\tau} = \sqrt{\frac{A^2 T}{\eta_0}}$.

$$\Rightarrow y = \frac{AT}{\cancel{\tau}} \times \frac{\cancel{\tau}}{\sqrt{\eta_0 T}} = \frac{AT}{\sqrt{\eta_0 T}} = \frac{\sqrt{A^2 T}}{\sqrt{\eta_0}} = \sqrt{\frac{A^2 T}{\eta_0}} \rightarrow$$

Substituting all in eq^n (v).

$$\cancel{P_e = \int_{\sqrt{\frac{A^2 T}{\eta}}}^{\infty} \frac{\tau}{\sqrt{\pi \eta T}} e^{-y^2}}.$$

$$P_e = \int_{\sqrt{\frac{A^2 T}{\eta_0}}}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\frac{\eta_0 T}{2\tau^2}}} \cdot e^{-y^2} \cdot \sqrt{\frac{\eta_0 T}{\tau}} \cdot dy.$$

$$= \int_{\sqrt{\frac{A^2 T}{\eta_0}}}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\frac{\eta_0 T}{2\tau^2}}} \cdot e^{-y^2} \frac{\sqrt{\eta_0 T}}{\tau} dy$$

$$= \int_{\sqrt{\frac{A^2 T}{\eta_0}}}^{\infty} \frac{1}{\sqrt{2}\cdot\sqrt{\pi} \times \frac{1}{\sqrt{2}\cdot\tau}} e^{-y^2} \cdot \frac{1}{\tau} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2 T}{\eta_0}}}^{\infty} e^{-y^2} dy.$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2 T}{\eta_0}}}^{\infty} e^{-y^2} dy.$$

But $\boxed{\frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-y^2} dy = \operatorname{erfc}(u)}$

where erfc $\to$ complimentary error fu^n.

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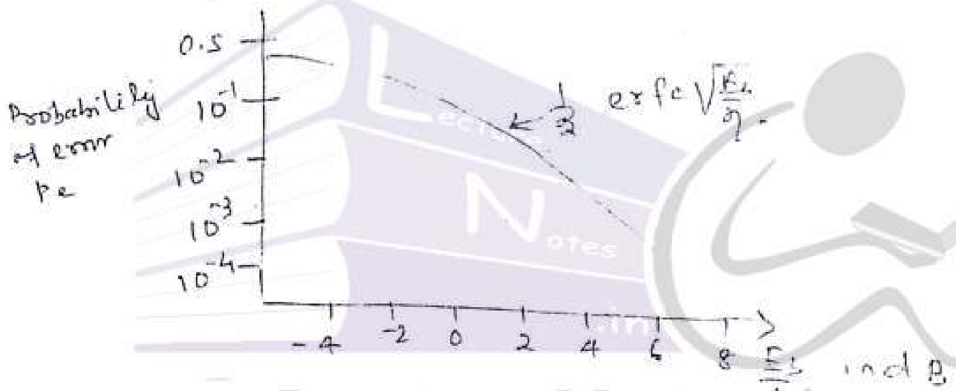
$$\Rightarrow P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{A^2 T}{\eta_0}}$$

$$\Rightarrow \boxed{P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{\eta_0}}}$$

→ P_e decreases rapidly as $\frac{E_b}{\eta_0}$ increases. The maximum value of P_e is $1/2$. Thus even if the signal is entirely lost in noise, the receiver can't be wrong more than half of the time on the average.



Optimal threshold :- Maximum Likelihood Detector and Bayes Receiver

→ When binary data '1' and '0' are associated with +V and −V respectively then the decision threshold is set at 0V inorder to make the probability of error minimum. But this good for the case where both symbols are equally probable and probability density function of noise is symmetrical like Gaussian.

→ But selecting optimal threshold when apriori probabilities are not equal is a difficult task.

→ Let a symbol S_1 with apriori probability $P(S_1)$ is sent, the probability of receiving voltage v is $P(V/S_1)$.

→ For symbol S_2 with a priori probability $P(S_2)$ it is $P(V/S_2)$.

→ The decision threshold 'λ' is such that

for $v > \lambda$, symbol S_1 is selected

for $v < \lambda$, symbol S_2 is selected

Then probability of error

$$P_e = P(S_1) \int_{v<\lambda} P(v/S_1)\,dv + P(S_2) \int_{v>\lambda} P(v/S_2)\,dv.$$

Let us consider a probability i.e. $P(v/S_1)$.

and As total probability is unity

$$\int_{v>\lambda} P(v/S_1)\,dv + \int_{v<\lambda} P(v/S_1)\,dv = 1.$$

$$\Rightarrow \int_{v<\lambda} P(v/S_1)\,dv = 1 - \int_{v>\lambda} P(v/S_1)\,dv.$$

So

$$P_e = P(S_1)\left[1 - \int_{v>\lambda} P(v/S_1)\,dv\right] + P(S_2) \int_{v>\lambda} P(v/S_2)\,dv$$

$$= P(S_1) + \int_{v>\lambda} \left[P(S_2)\,P(v/S_2) - P(S_1)\,P(v/S_1)\right] dv$$

The probability of error is minimum if for ever $v > \lambda$.

$$\boxed{P(S_1)P(v/S_1) > P(S_2)P(v/S_2)} \quad \text{--- (1)}$$

$$\Rightarrow \boxed{\frac{P(v/S_1)}{P(v/S_2)} > \frac{P(S_2)}{P(S_1)}} \quad \text{--- (2)}$$

Hence at decision boundary $v = \lambda$

$$\boxed{\frac{P(\lambda/S_1)}{P(\lambda/S_2)} = \frac{P(S_2)}{P(S_1)}} \quad \text{--- (3)}$$

So we plot $P(S_1)P(v/S_1)$ and $P(S_2)P(v/S_2)$ separately and their intersection gives the threshold. The below figure shows a pictorial representation of this with symmetrical Gaussian distribution of conditional probabilities. The eq^n (2) can be used as maximum likelihood detector.

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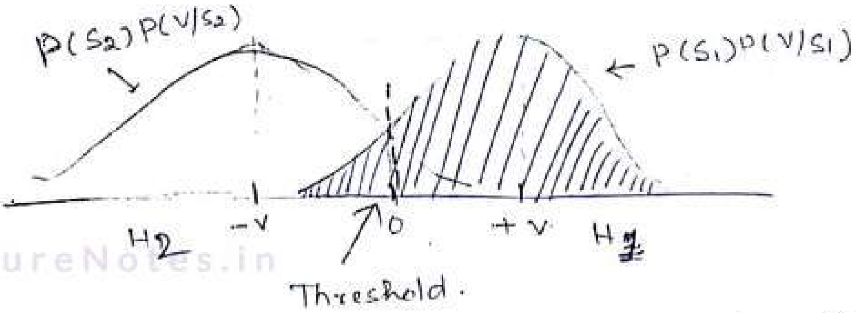
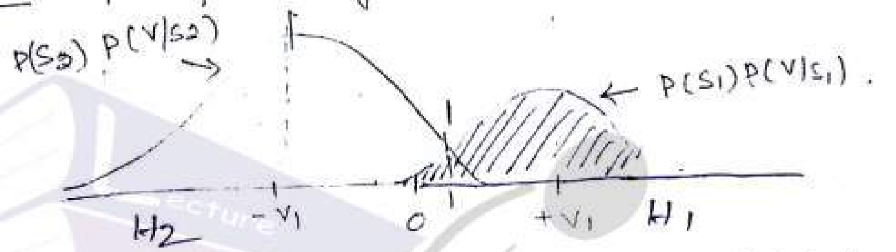
Case-1 Aprion probability is equal i.e. $P(S_1) = P(S_2)$



case-2 Aprion probability is not equal i.e. $P(S_1) < P(S_2)$



Eqn-(1) can be used as maximum likehood detector.
For received voltage 'V'

Hypothesis $H_1 \rightarrow$

If $P(S_1)\,P(V/S_1) > P(S_2)\,P(V/S_2)$

~~Then code received one~~
The symbol sent is most likely to be S_1.

Hypothesis $H_2 \rightarrow$

If $P(S_2)\,P(V/S_2) > P(S_1)\,P(V/S_1)$

The symbol sent is most likely to be S_2.

It is represented in the form of given eqn.

$$\boxed{\frac{P(V/S_1)}{P(V/S_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(S_2)}{P(S_1)}} \quad \text{——— } (4)$$

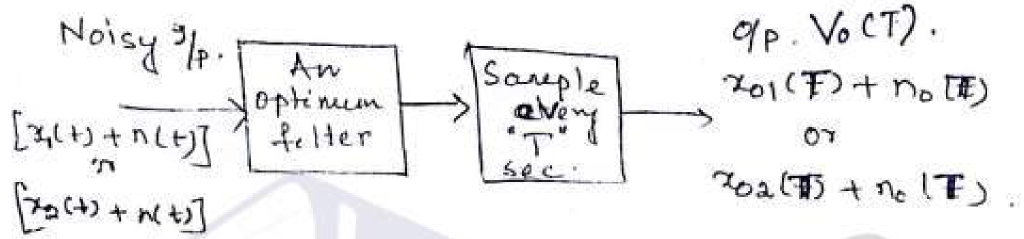
This means that if likelihood ratio exceeds ratio of apriori probabilities, hypothesis H1 is chosen, if not then hypothesis H2 is chosen. If they are equal any of H1 or H2 can be chosen and it will contribute equally to probability of error.

Optimum Receiver for both Baseband and Passband:-

Optimum filter and ~~Pe~~ Expression of Pe for optimum receiving filters:

Optimum filter ~~or matched filter~~:- It is the filter is being used to receive the signal with minimum value of Pe and maximum value of SNR.

Block diagram ~~for~~ of receiver for binary coded signal is as shown.



Let received signal is bipolar NRZ.

for 1, $x_1(t) = +A$

for 0, $x_2(t) = -A$.

The noise $n(t)$ is added to the channel during the transmission. So the i/p to the optimum filter is $[x(t) + n(t)]$.

We note that in the absence of noise the o/p sample would be $V_0(T) = x_{01}(T)$ or $x_{02}(T)$. But when noise is present, to minimize Pe one should assume that $x_1(t)$ has been Tx if $V_0(T)$ is closer to $x_{01}(T)$. Similarly, we assume $x_2(t)$ has been Tx if $V_0(T)$ is closer to $x_{02}(T)$. So Decision boundary is midway between $x_{01}(T)$ and $x_{02}(T)$. Here $s_{01}(T) = x_{01}(T)$ and $s_{02}(T) = x_{02}(T)$.

So O/p from the receiver $= x_{01}(T) + n_0(T)$

or $x_{02}(T) + n_0(T)$

Decision boundary $= \dfrac{x_{01}(T) + x_{02}(T)}{2} = V_0(T)$

Let $x_{01}(T) > x_{02}(T)$. $\Rightarrow x_{01}(T) + x_{02}(T) > x_{02}(T) + x_{02}(T)$

$\Rightarrow x_{01}(T) + x_{02}(T) > 2x_{02}(T)$

So

then $\dfrac{x_{01}(T) + x_{02}(T)}{2} > x_{02}(T)$.

So error will occur if

$$n_0(T) \geq \frac{x_{01}(T) + x_{02}(T)}{2} - x_{02}(T).$$

$$\Rightarrow n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}$$

So the probability of error

$$P_e = P\left[n_0(t) \geq \frac{x_{01}(T) - x_{02}(T)}{2}\right]$$

$$= \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} f_x[n_0(t)]\, d n_0(t).$$

$$= \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-[n_0(t)]^2/2\sigma^2}\, d[n_0(t)].$$

Put $\frac{n_0^2(t)}{2\sigma^2} = y^2.$

$$\Rightarrow y = \frac{n_0(t)}{\sqrt{2}\,\sigma}$$

$$\Rightarrow dy = \frac{1}{\sqrt{2}\,\sigma}\, d n_0(t) \Rightarrow d n_0(t) = \sqrt{2}\,\sigma\, dy$$

When $n_0(t) \to \infty$ then $y \to \infty$

When $n_0(t) = \frac{x_{01}(T) - x_{02}(T)}{2}$

then $y = \frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}$

$$P_e = \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-y^2} \cdot \sqrt{2}\,\sigma\, dy.$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{\pi}} \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} e^{-y^2} \cdot dy$$

$$\Rightarrow \boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right]}$$

→ So probability of error (P_e) decreases with
(i) as the difference $[x_{01}(t) - x_{02}(t)]$ increases
(ii) r.m.s noise voltage σ becomes smaller

→ Hence the optimum filter is the filter which maximizes the ratio

$$\boxed{\gamma = \frac{x_{01}(T) - x_{02}(T)}{\sigma}}$$

Calculation of the optimum-filter transfer function $H(f)$:-

→ The prime function of optimum-filter is to maximize the value of γ.

Let the input difference signal ~~to the optimum filter~~ is

$$p(t) = s_1(t) - s_2(t) = \text{difference signal.}$$

The corresponding o/p signal of the filter is

$$p_o(t) = s_{01}(t) - s_{02}(t)$$

Let $p(t) \longleftrightarrow P(f)$

and $p_o(t) \longleftrightarrow P_o(f)$

If $H(f)$ is the transfer function of the filter then

$$P_o(f) = H(f)\,P(f).$$

$$\Rightarrow p_o(T) = \int_{-\infty}^{\infty} P_o(f)\,e^{j2\pi fT}\,df$$

$$= \int_{-\infty}^{\infty} H(f)\,P(f)\,e^{j2\pi fT}\,df. \quad \text{—(i)}$$

Let input noise to the optimum filter is $n(t)$ and o/p noise $n_o(t)$ with psde of $G_n(f)$ and $G_{no}(f)$ respectively.

So $G_{no}(f) = |H(f)|^2 G_n(f)$.

Hence the noise variance (normalized o/p noise power)

$$\sigma_o^2 = \int_{-\infty}^{\infty} G_{no}(f)\, df = \int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df \quad \text{—(ii)}$$

So $$\gamma^2 = \frac{P_o^2(T)}{\sigma^2} = \frac{\left|\int_{-\infty}^{\infty} H(f) P(f) e^{j2\pi f T} df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df} \quad \text{—(iii)}$$

NOTE :-

Schwarz inequality :- Given arbitrary complex function $X(f)$ and $Y(f)$ of a common variable f, then

$$\left|\int_{-\infty}^{\infty} X(f) Y(f)\, df\right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^2 df \int_{-\infty}^{\infty} |Y(f)|^2 df.$$

The equal sign applies when $\boxed{X(f) = K Y^*(f)}$

where K = arbitrary const.

$Y^*(f)$ - complex conjugate of $Y(f)$

$(e^{j2\pi f T} = 1)$

Now put $X(f) = \sqrt{G_n(f)}\, H(f)$

and $Y(f) = \frac{1}{\sqrt{G_n(f)}} P(f) \cdot e^{j2\pi f T}$.

So $$\frac{P_o^2(T)}{\sigma^2} = \frac{\left|\int_{-\infty}^{\infty} X(f) Y(f)\, df\right|^2}{\int_{-\infty}^{\infty} |X(f)|^2 df} \leq \int_{-\infty}^{\infty} |Y(f)|^2 df.$$

$$\Rightarrow \gamma^2 = \frac{P_o^2(T)}{\sigma^2} \leq \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)}\, df.$$

The ratio $\gamma^2 = \frac{P_0^2(T)}{\sigma^2}$ will attain its maximum value when ~~the equates~~ the 'equal' sign is employed.

i.e. $X(f) = K\,Y^*(f)$.

$$\Rightarrow \sqrt{G_n(f)}\,H(f) = K \cdot \frac{P^*(f)}{\sqrt{G_n(f)}}\,e^{-j2\pi f T}$$

$$\Rightarrow \boxed{H(f) = K\,\frac{P^*(f)}{G_n(f)}\,e^{-j2\pi f T}}$$

and the corresponding value of γ^2

$$\gamma^2_{max} = \left[\frac{P_0^2(T)}{\sigma^2}\right]_{max} = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)}\,df.$$

Optimum filter realization using Matched filter

An optimum filter which produces a maximum ratio $\gamma^2 = \frac{P_0^2(t)}{\sigma^2}$, when the input noise is white is called matched filter.

So $G_n(f) = \eta/2$.

$$\text{So}\quad H(f) = K\,\frac{P^*(f)}{\eta/2}\,e^{-j2\pi f T}.$$

Hence the impulse response of the filter

$$h(t) = F^{-1}[H(f)].$$

$$= \frac{2K}{\eta}\int_{-\infty}^{\infty} P^*(f)\,e^{-j2\pi f T}\,e^{j2\pi f t}\,df$$

$$= \frac{2K}{\eta}\int_{-\infty}^{\infty} P^*(f)\,e^{j2\pi f(t-T)}\,df$$

$$\Rightarrow h^*(t) = \frac{2K}{\eta}\int_{-\infty}^{\infty} P(f)\,e^{j2\pi f(T-t)}\,df$$

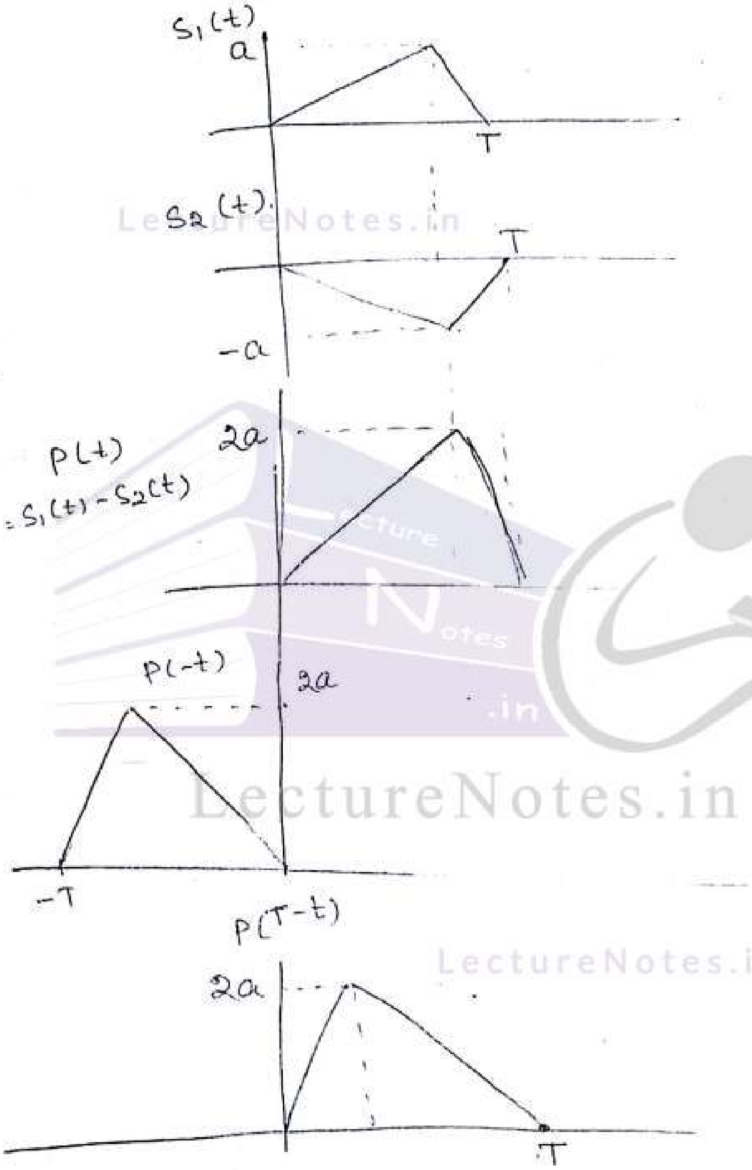
NOTE: A physically realizable filter will have an impulse response which is real, i.e. not complex.

Therefore $h(t) = h^*(t)$.

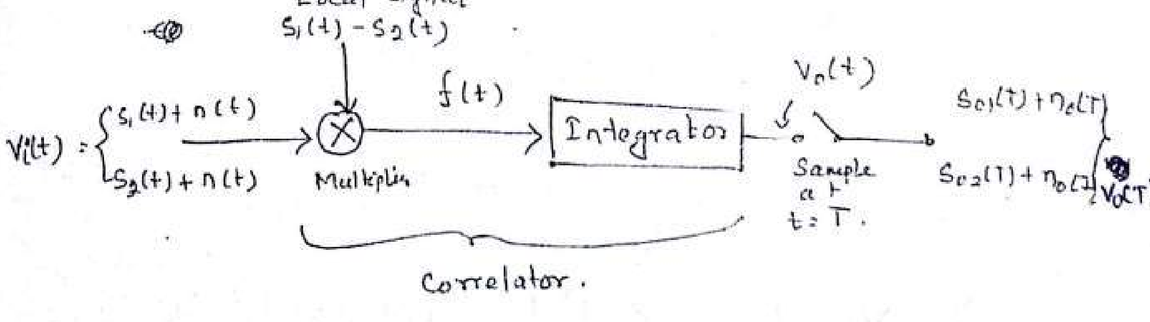
value $\Rightarrow \boxed{h(t) = \frac{2K}{\eta} p(T-t)}$

Scince $p(t) = s_1(t) - s_2(t)$.

So $\boxed{h(t) = \frac{2K}{\eta}\left[s_1(T-t) - s_2(T-t)\right]}$



Optimum filter realizing using correlator :-



→ The received signal plus noise $v_i(t)$ is multiplied by a locally generated waveform $s_1(t) - s_2(t)$. The o/p of the multiplier is passed through an integrator, whose o/p is sampled at $t = T$.

→ After sampling and begining of each new bit interval, the capacitors are discharges.

→ This type of receiver is called correlator as we are correlating the received signal and noise with waveform $s_1(t) - s_2(t)$.

→ The o/p signal of the correlator

$$s_o(T) = \frac{1}{\tau}\int_0^T s_i(t)\left[s_1(t) - s_2(t)\right] dt \quad \text{——— (1)}$$

where $s_i(t)$ is either $s_1(t)$ or $s_2(t)$.

The o/p noise of the correlator where "τ" is the time constant of the integrator.

$$n_o(T) = \frac{1}{\tau}\int_0^T n(t)\left[s_1(t) - s_2(t)\right] dt \quad \text{——— (2).}$$

We now compare these outputs with the matched filter outputs. If $h(t)$ is the impulse response of the matched filter, then

→ Let the o/p of the matched filter is $v_o(t)$.

$$v_o(t) = \int_{-\infty}^{\infty} v_i(\lambda)\, h(t-\lambda)\, d\lambda.$$

$$= \int_0^T v_i(\lambda)\, h(t-\lambda)\, d\lambda.$$

But $h(t) = \frac{2K}{\eta}\left[s_1(T-t) - s_2(T-t)\right]$.

$$\Rightarrow h(t-\lambda) = \frac{2K}{\eta}\left[s_1(T-t+\lambda) - s_2(T-t+\lambda)\right].$$

So $$v_o(t) = \frac{2K}{\eta}\int_0^T v_i(\lambda)\left[s_1(T-t+\lambda) - s_2(T-t+\lambda)\right] d\lambda.$$

As $v_i(\lambda) = \left[s_i(\lambda) + n(\lambda)\right]$ and $v_o(t) = s_o(t) + n(t)$

setting $t = T$, produces $\Rightarrow v_o(T) = s_o(T) + n_o(T)$

$$\Rightarrow v_o(T) = \frac{2K}{\eta}\int_0^T v_i(\lambda)\left[s_1(\lambda) - s_2(\lambda)\right] d\lambda$$

$$\Rightarrow v_o(T) = \frac{2K}{\eta}\int_0^T \left[s_i(\lambda) + n(\lambda)\right]\left[s_1(\lambda) - s_2(\lambda)\right] d\lambda \quad \text{——— (3)}$$

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Calculation of Probability of Error for Matched Filter:-

→ To evaluate the probability of error for the Matched filter, let us consider the error probability of Optimum filter as $P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\sigma}\right]$ ——— (1).

→ In this equation, we have

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df \text{ ——— (2).}$$

→ Let us substitute $S_{ni}(f) = N_0/2$.

Hence $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{(N_0/2)}\, df$

$$= \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2 df \text{ ——— (3).}$$

→ As per Parseval's Theorem

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_0^T x^2(t)\, dt \text{ ——— (4)}$$

In the above equation (4), we have taken limit from '0' to 'T' because $x(t)$ exists from 0 to T only.

→ We also know that $x(t) = x_1(t) - x_2(t)$

→ Hence the eqⁿ (4) becomes

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_0^T [x_1(t) - x_2(t)]^2 dt$$

$$\Rightarrow \int_{-\infty}^{\infty} |X(f)|^2 df = \int_0^T x_1^2(t)\,dt + \int_0^T x_2^2(t)\,dt - 2\int_0^T x_1(t)\cdot x_2(t)\,dt \text{ —— (5)}$$

→ As we know that $\int_0^T x_1^2(t)\,dt = E_1$, and $\int_0^T x_2^2(t)\,dt = E_2$

and $\int_0^T x_1(t)\cdot x_2(t)\,dt = E_{12}$

→ Now, if we choose, $x_1(t) = -x_2(t)$, then we have

$$E_1 = E_2 = -E_{12} = E \text{ (say) ——— (6)}$$

→ Substituting the eq^n (6) in eq^n (5), we have

$$\int_{-\infty}^{\infty} |X(f)|^2 df = [E+E-2(-E)] = 4E \quad \text{—— (7)}$$

→ Now substitute eq^n (7) in eq^n (3), we have

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\cdot 4E = \frac{8E}{N_0} \quad \text{—— (8)}$$

$$\therefore \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{max} = 2\sqrt{2}\sqrt{\frac{E}{N_0}} \quad \text{—— (9)}$$

$$\Rightarrow \left[\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\,\sigma}\right] = \sqrt{\frac{E}{N_0}} \quad \text{—— (10)}$$

→ Now substitute eq^n (10) in eq^n (1), we have

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E}{N_0}}\right) \quad \text{—— (11)}$$

→ Thus, Minimum error probability of Matched filter is

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E}{N_0}}\right)}$$

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Digital Communication Techniques

Topic:

Principle Of Digital Data Transmission

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Principle of Digital Data Transmission

Components of digital communication system:

1 - Source :- It is provided as i/p to digital communication system. It may be ~~o/p from~~ data set o/p from computer, a digitized voice signal (PCM or DM), a digital ~~fa~~ T.V or telemetry equipment.

2 - Multiplexer :- The practical channel has much heigher ~~rate~~ data rate handling capacity than the data rate of individual sources. Hence inorder to utilize the capacity efficently we combine several sources by using time division multiplexing.

3 - Line coder :- The o/p of a multiplexer is coded into electrical pulses or waveforms for the purpose of transmission over the channel. This is called line coding.

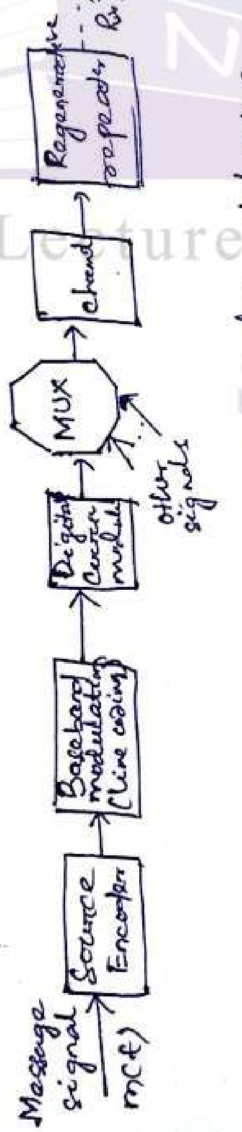


fig: fundamental building blocks of digital communication system.

4- ~~Repeater~~ Regerative Repeater:- ~~These~~ are used at regularly spaced intervals along a digital transmission line to detect incoming digital signal and regenerate new clean pulse for further transmission along the line. This process periodically eliminates the accumulation of noise and signal distortion along the transmission path.

(1) Line Coding:

Properties of good line code:-

(i) Transmission bandwidth of the signal should be as small as possible.

Data transmission rate : R_b bits/sec $= nf_s$

where n = no of bits/sample

f_s = Sampling frequency.

Minimum b.w. required $= \frac{1}{2} R_b$.

(2) Power efficiency:- For a given b.w. and specified detection error probability, the transmitted power must be as small as possible.

(3) Favorable power spectral density:- It is desirable to have zero PSD at $\omega = 0$ (d.c.), as a.c. coupling and transformers are used at the receiver

(4) Error detection and correction capabilities:- The code should have the ability to detect the error and correct the error.

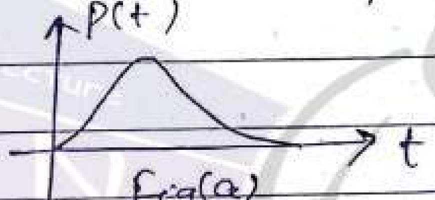
(5) Adequate timing content:- It should be possible to extract timing or clock information from the signal.

(6) Transparency:- It should be possible to transmit a digital signal correctly regardless of pattern of 1's and 0's.

(2) PSD OF VARIOUS LINE CODES

→ Let us consider a pulse $p(t)$ whose corresponding F.T. is $P(f)$. We can denote the line code symbol at time "k" as "a_k".

→ When the transmission rate is $R_b = 1/T_b$ pulses/sec, the line code generates a pulse train constructed from the basic pulse $p(t)$ with amplitude a_k at time $t = kT_b$, in other words, the k^{th} symbol is transmitted as $a_k\, p(t - kT_b)$.



Fig(a)

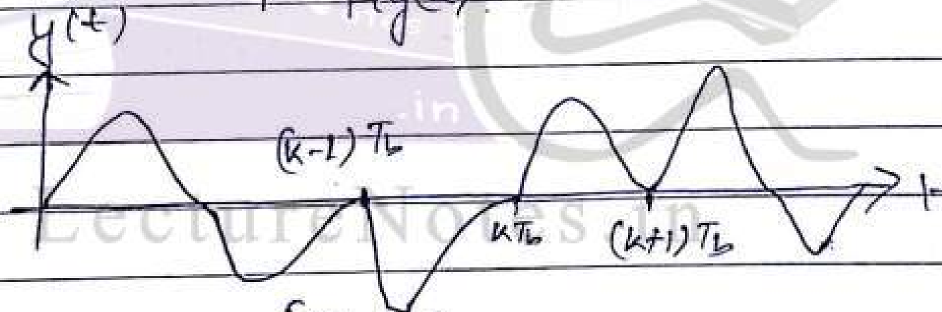


Fig (b).

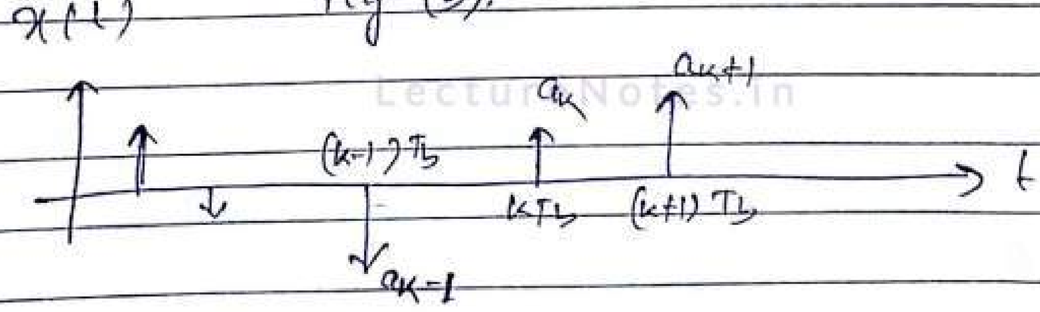


Fig (c).

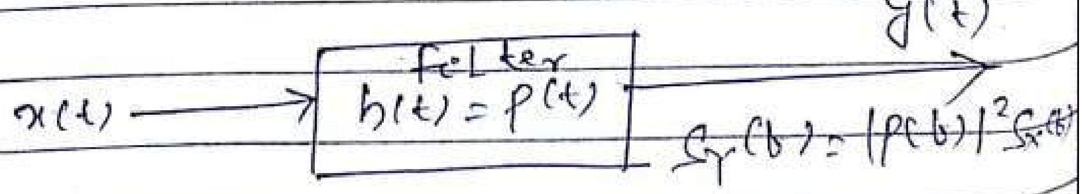


Fig (d).

→ The above fig(a) provides an illustration of a special pulse p(t), whereas fig(b). shows the corresponding pulse train generated by the Line coder at baseband.

So $y(t) = \sum a_k \, p(t-kT_b)$ —— (1).

→ The values a_k are random and depend on the line coder input and the line code itself.

→ y(t) is a pulse-amplitude-modulated (PAM) signal.

→ The on-off, polar, and bipolar line codes are all special cases of this pulse train y(t), where a_k takes on values 0, 1, or -1 randomly.

→ The PSD of y(t) depends on both a_k and p(t).

→ If the pulse shape p(t) changes, we may have to derive the PSD all over again.

→ This difficulty can be overcome by the simple selection of a PAM signal x(t) that uses a unit impulse for the basic pulse p(t) as shown in fig(c).

→ If x(t) is applied to the input of a filter that has a unit impulse response h(t) = p(t) shown in fig(d), the output will be the pulse train y(t) and PSD of y(t) is

$S_y(f) = |P(f)|^2 S_x(f)$ —— (2).

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(4) POLAR SIGNALING :-

→ In polar signaling, '1' is transmitted by a pulse p(t) and '0' is represented by −p(t). In this case, a_k is equally likely to be 1 or −1, and a_k^2 is always 1.
Hence,

$$R_0 = \lim_{N\to\infty} \frac{1}{N} \sum_k a_k^2 \quad \text{—— (1).}$$

→ There are "N" pulses and $a_k^2 = 1$ for each one, the summation on R.H.S. of above eq^n (1) is N. Hence

$$R_0 = \lim_{N\to\infty} \frac{1}{N}(N) = 1 \quad \text{—— (2).}$$

→ Moreover, both a_k and a_{k+1} are either '1' or '−1'. Hence, $a_k a_{k+1}$ is either 1 or −1.

→ Because the pulse amplitude a_k is equally likely to be 1 and −1 on the average, out of N terms the product $a_k \cdot a_{k+1}$ is equal to 1 for N/2 terms and is equal to −1 for the remaining N/2 terms. Therefore,

$$R_1 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{2}(1) + \frac{N}{2}(-1)\right] = 0 \quad \text{—— (3).}$$

Possible Values of $a_k a_{k+1}$

a_{k+1} \ a_k	−1	+1
−1	1	−1
+1	−1	1

$\tau \, \text{sinc} \frac{\omega \tau}{2}$

$\frac{T_b}{2} \text{sinc} \left[\frac{2\pi f \cdot (\frac{T_b}{2})}{2} \right.$

$\frac{1}{2} f.$

Arguing this, we see that the product $a_k a_{k+n}$ is also equally likely to be 1 or -1. Hence

$$R_n = 0, \quad n \geq 1. \quad \text{——— (4)}$$

[What is R_0, R_1 and R_n ??] ↓

NOTE

→ Let us consider a term $R_x(\tau)$ which represents the time autocorrelation function of the impulse train $x(t)$.

→ This can be conveniently done by considering the impulses as a limiting form of the rectangular pulses. Each pulse has a width $\epsilon \to 0$, and the k^{th} pulse height, $h_k = \frac{a_k}{\epsilon} \to \infty$.

→ This way, we guarantee that the strength of the k^{th} impulse is a_k, or $\epsilon h_k = a_k$.

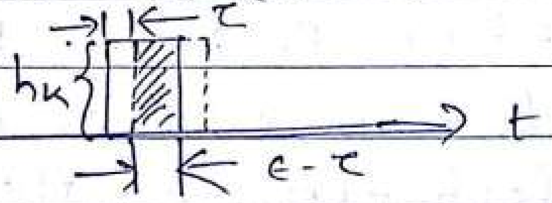
→ If we designate the corresponding rectangular pulse train by $\hat{x}(t)$, then by definition,

$$R_{\hat{x}}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} \hat{x}(t)\, \hat{x}(t-\tau)\, dt \quad \text{——— (5)}.$$

→ As $R_{\hat{x}}(\tau)$ is an even function of τ, we need to consider only positive τ.

→ Let us consider this case where the delay τ is less ϵ i.e. $\tau < \epsilon$. In this case, the integral in eq(5) is the area under the signal $\hat{x}(t)$ is multiplied by $\hat{x}(t)$ delayed by τ $(\tau < \epsilon)$.

→ So the area associated with the k^{th} pulse is $h_k^2(\epsilon-\tau)$



→ So, eqⁿ(5) becomes

$$R_{\hat{x}} = \lim_{T\to\infty} \frac{1}{T} \sum_k h_k^2 (\epsilon-\tau)$$

$$\Rightarrow R_{\hat{x}} = \lim_{T\to\infty} \frac{1}{T} \sum a_k^2 \left(\frac{\epsilon-\tau}{\epsilon^2}\right)$$

$$\Rightarrow R_{\hat{x}} = \frac{R_0}{\epsilon T_b}\left(1-\frac{\tau}{\epsilon}\right) \quad \text{--- (6)}$$

where $R_0 = \lim_{T\to\infty} \frac{T_b}{T} \sum_k a_k^2$ — (7).

→ During the averaging interval T $(T\to\infty)$, there are "N" pulses $(N\to\infty)$ where $N = T/T_b$. — (8).

So eqⁿ (7) becomes

$$R_0 = \lim_{N\to\infty} \frac{1}{N} \sum_k a_k^2 \text{ — (8)}.$$

$= \widetilde{a_k^2}$ (Time average of a_k^2)

→ similarly

$$R_1 = \lim_{T\to\infty} \frac{T_b}{T} \sum_k a_k a_{k+1} \quad \text{— (9)}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_k a_k a_{k+1}$$

$$= \widetilde{a_k a_{k+1}} \quad \text{— (10)}.$$

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→ Observe that R_1 is obtained by multiplying every pulse strength (a_k) by the strength of its immediate neighbor pulse (a_{k+1}), adding all these products, and then dividing this by the total number of pulses.

→ In general

$$R_n = \lim_{T \to \infty} \frac{T_b}{T} \sum_k a_k a_{k+n} \quad \text{--- (11).}$$

$$\Rightarrow R_n = \lim_{N \to \infty} \frac{1}{N} \sum_k a_k a_{k+n} \quad \text{--- (12).}$$

$$\Rightarrow R_n = \widetilde{a_k a_{k+n}} \quad \text{--- (13).}$$

where R_n is essentially the discrete autocorrelation function of the line code symbols $\{a_k\}$.

→ The PSD $S_x(f)$ is the Fourier Transform of $R_x(\tau)$ where

$$R_x(\tau) = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n \, \delta(\tau - nT_b) \quad \text{--- (14).}$$

$$\therefore S_x(f) = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n e^{-jn2\pi f T_b} \quad \text{--- (15).}$$

→ Now eqn (2) becomes

$$S_y(f) = |P(f)|^2 S_x(f) \quad \text{--- (16)}$$

$$\Rightarrow S_y(f) = \frac{|P(f)|^2}{T_b} \left[\sum_{n=-\infty}^{\infty} R_n e^{-jn2\pi f T_b} \right]$$

Recognizing that $R_{-n} = R_n$, we have

$$\Rightarrow S_y(f) = \frac{|P(f)|^2}{T_b} \left[R_0 + 2 \sum_{n=1}^{\infty} R_n \cos n2\pi f T_b \right] \quad \text{--- (17)}$$

From eq^n (17) it is clear that

$$S_Y(f) = \frac{|P(f)|^2}{T_b} R_0$$

$$\Rightarrow S_Y(f) = \frac{|P(f)|^2}{T_b} \quad \text{— (18)} \quad [\because R_0 = 1]$$

→ Let us consider a specific pulse shape. Let p(t) be a rectangular pulse of width $T_b/2$ (half-width rectangular pulse)

i.e. $p(t) = \Pi\left(\frac{t}{T_b/2}\right) = \Pi\left(\frac{2t}{T_b}\right)$

and $P(f) = \frac{T_b}{2} \operatorname{sinc}\left(\frac{\pi f T_b}{2}\right)$ — (19)

→ Therefore the PSD of polar signaling is $S_Y(f) = \frac{T_b}{4} \operatorname{sinc}^2\left(\frac{\pi f T_b}{2}\right)$ — (20).

CONSTRUCTING A DC NULL IN PSD BY PULSE SHAPING:-

→ The PSD has a dc null, which is desirable for ac coupling
→ Because in $S_Y(f)$, the PSD of a line code contains a factor $|P(f)|^2$, we can force the PSD to have a dc null by selecting a pulse p(t) such that P(f) is zero at dc (f=0).

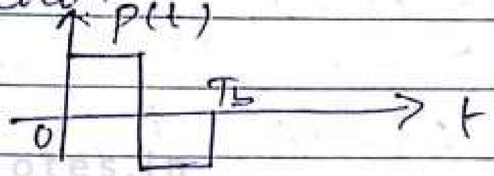
→ Because $P(f) = \int_{-\infty}^{\infty} p(t) e^{-j2\pi f t} dt$

and we have

$$P(0) = \int_{-\infty}^{\infty} p(t)\, dt.$$

→ Hence, if the area under $p(t)$ is made zero, $P(0)$ is zero, and we have a dc null in the PSD.

→ For a rectangular pulse, one possible shape of $p(t)$ to accomplish this is shown below:-



→ When we use this pulse with polar line coding, the resulting signal is known as Manchester code, or split-phase or twinned-binary signal.

ON-OFF SIGNALING:-

→ In on-off signaling, a '1' is transmitted by a pulse $p(t)$ and a '0' is transmitted by no pulse.

→ Hence, a pulse strength a_k is equally likely to be 1 or 0. Out of N pulses in the interval of T seconds, a_k is 1 for N/2 pulses and is 0 for the remaining N/2 pulses on the average.

→ Hence $R_0 = \lim_{N \to \infty} \frac{1}{N} \left[\frac{N}{2}(1)^2 + \frac{N}{2}(0)^2 \right]$

$$= \frac{1}{2} \quad -(1)$$

→ To compute R_n we need to consider the product $a_k a_{k+n}$. Since a_k and a_{k+n}

are equally likely to be 1 or 0, the product $a_n a_{k+n}$ is equally likely to be 1×1, 1×0, 0×1 & 0×0, that is 1, 0, 0, 0.

→ Therefore on the average, the product $a_n a_{k+n}$ is equal to '1' for $N/4$ terms and '0' for $3N/4$ terms and

$$R_n = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{4}(1) + \frac{3N}{4}(0)\right] = 1/4 \quad \text{——(2)}$$

for $n \geq 1$

→ So $S_x(f) = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n e^{-jn2\pi f T_b}$

As per the value of R_0 and R_n

$$S_x(f) = \frac{1}{2T_b} + \frac{1}{4T_b} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-jn2\pi f T_b} \quad \text{——(3)}$$

$$= \frac{1}{4T_b} + \frac{1}{4T_b} + \frac{1}{4T_b} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-jn2\pi f T_b}$$

$$\Rightarrow S_x(f) = \frac{1}{4T_b} + \frac{1}{4T_b} \sum_{n=-\infty}^{\infty} e^{-jn2\pi f T_b} \quad \text{——(4)}$$

Eqn (4) is obtained by splitting the term $1/2T_b$ corresponding to R_0 into two: $1/4T_b$ outside the summation and $1/4T_b$ inside the summation (correspond to $n=0$).

→ We now use the formula

$$\sum_{n=-\infty}^{\infty} e^{-jn2\pi f T_b} = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b}\right) \quad \text{——(5)}$$

NOTE: As we know that fouriere series for the impulse train is

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_b) = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} e^{-jn2\pi R_b t} \quad \text{--- (5a)}.$$

Taking F.T. both sides of eq^n 5(a) we will get

$$\sum_{n=-\infty}^{\infty} e^{-jn2\pi f T_b} = \frac{1}{T_b}\sum_{n=-\infty}^{\infty} \delta(f - n/T_b) \quad \text{--- (5b)}.$$

→ Applying eq^n(5) in eq^n(4), we will get

$$S_x(f) = \frac{1}{4T_b} + \frac{1}{4T_b^2}\sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{T_b}\right) \quad \text{--- (6)}.$$

→ The desired PSD of the on-off waveform $y(t)$ is

$$S_y(f) = \frac{|P(f)|^2}{4T_b}\left[1+\frac{1}{T_b}\sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{T_b}\right)\right] \quad \text{--- (7)}.$$

→ For the example case of a half-width rectangular pulse

i.e. $P(f) = \frac{T_b}{2}\,\mathrm{sinc}\left(\frac{\pi f T_b}{2}\right)$

$$S_y(f) = \frac{T_b}{16}\,\mathrm{sinc}^2\left(\frac{\pi f T_b}{2}\right)\left[1+\frac{1}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_b}\right)\right] \quad \text{--- (8)}.$$

BIPOLAR SIGNALING :- (AMI).

→ The signaling scheme used in PCM for telephone networks is called bipolar signaling.

→ A '0' is transmitted by no pulse, and and a '1' is transmitted by a pulse $p(t)$ or $-p(t)$.

→ Bipolar signaling actually uses three symbols [$p(t)$, 0, and $-p(t)$], and hence it is in reality ternary rather than binary signaling.

→ To calculate the PSD, we have

$$R_0 = \lim_{N\to\infty}\frac{1}{N}\sum_k a_k^2 \quad \text{--- (1)}$$

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→ On the average, half of the a_ks are '0', and the remaining halfs are either 1 or -1, with $a_k^2 = 1$.

→ Therefore,

$$R_0 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{2}(\pm 1)^2 + \frac{N}{2}(0)^2\right] = \frac{1}{2} \quad (2)$$

→ To compute R_1, we have to consider the pulse strength product $a_k a_{k+1}$. There are four equally likely sequences of two bits: 11, 10, 01, 00.

→ Since bit '0' is encoded by no pulse ($a_k = 0$), the product $a_k a_{k+1}$ is zero for the last three of these sequences.

→ This means that on the average, $3N/4$ combinations have $a_k a_{k+1} = 0$ and only $N/4$ combinations have non-zero $a_k a_{k+1}$.

→ Because of the bipolar rule, the bit sequence '11' can be encoded only by two consecutive pulses of opposite polarity.

→ This means that the product $a_k a_{k+1} = -1$ for the $N/4$ combinations.

→ Therefore,

$$R_1 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{4}(-1) + \frac{3N}{4}(0)\right] = -\frac{1}{4} \quad (3)$$

→ To compute R_2 in a similar way, we need to observe the product $a_k a_{k+2}$.

→ For this, we need to consider all possible combinations of three bits in sequence. There are eight equally likely combinations: 111, 101, 110, 100, 011, 010, 001, 000.

→ The last six combinations have either the first/or the last bit '0'. Hence $a_k a_{k+2} = 0$ for all these six combinations.

→ The first two combinations are the only ones that yield nonzero $a_k a_{k+2}$.

→ From the bipolar rule, the 1st and the 3rd pulses in the combination 111 are of the same polarity (∵ according to alternate mark inversion code), yielding $a_k a_{k+2} = 1$.

→ But for 101, the first and the third pulse are of opposite polarity, yielding $a_k a_{k+2} = -1$.

→ Thus, on the average, $a_k a_{k+2} = 1$ for $N/8$ terms, -1 for $N/8$ terms and 0 for $3N/4$ terms.

→ Hence, $R_2 = \lim_{N \to \infty} \frac{1}{N}\left[\frac{N}{8}(1) + \frac{N}{8}(-1) + \frac{3N}{4}(0)\right]$
$= 0$ —— (4).

→ In general $R_n = \lim_{N \to \infty} \frac{1}{N} \sum_k a_k a_{k+n}$

→ By finding the values of R_n for $n > 2$, we will get $R_n = 0$. Thus

$$R_n = 0, \text{ for } n > 1 \quad \text{—— (5).}$$

→ So $S_y(f) = \frac{|P(f)|^2}{T_b}\left[R_0 + 2\sum_{n=1}^{\infty} R_n \cos n 2\pi f T_b\right]$ —— (6)

→ By putting the values of R_0, R_1 and R_n for $n > 2$, we will get

$$S_Y(f) = \frac{|P(f)|^2}{T_b}\left[\frac{1}{2} + 2\times\left(-\frac{1}{4}\right)\cos 2\pi f T_b\right]$$

$$\Rightarrow S_Y(f) = \frac{|P(f)|^2}{T_b}\left[\frac{1}{2} - \frac{1}{2}\cos 2\pi f T_b\right]$$

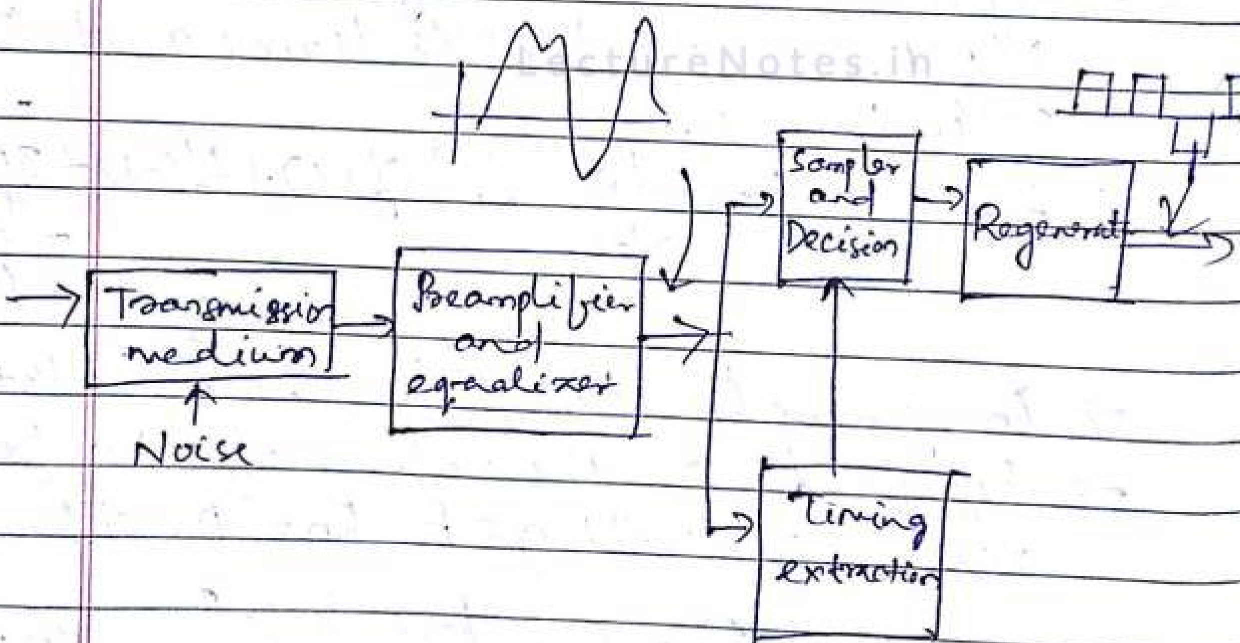
$$\Rightarrow S_Y(f) = \frac{|P(f)|^2}{2T_b}\left[1 - \cos 2\pi f T_b\right]$$

$$\Rightarrow S_Y(f) = \frac{|P(f)|^2}{T_b}\sin^2(\pi f T_b) \quad \text{---} (7).$$

→ For the half-width pulse, the PSD is

$$S_Y(f) = \frac{T_b}{4}\,\text{sinc}^2\left(\frac{\pi f T_b}{2}\right)\sin^2(\pi f T_b) \quad \text{---} (8).$$

DIGITAL RECEIVERS AND REGENERATIVE REPEATERS



Mindfire is a software service provider,
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(1) reshaping incoming pulses by means of an equalizer.
(2) extracting the timing information required to sample incoming pulses at optimum instants.
(3). making symbol detection decisions based on the pulse samples.

TIMING EXTRACTION :-

→ The received digital signal needs to be sampled at precise instants.
→ This requires a clock signal at the receiver in synchronism with the clock signal at the transmitter which is known as symbol or bit synchronization.
→ Three general methods of synchronization exist:
(1) Derivation from a primary or a secondary standard (e.g. transmitter and receiver slaved to a master timing source).
(2). Transmitting a separate synchronizing signal (pilot clock).
(3). Self-synchronization, where the timing information is extracted from the received signal itself.

TIMING JITTER:-

→ Small random variations or deviations of the incoming pulses from their ideal location can cause timing jitter.

→ This results from several causes some of which are dependent on the pulse pattern being transmitted, whereas others are not.

→ Random forms of jitter are caused by noise, interference, and mistuning of the clock circuits.

→ Jitter accumulation over a digital link may be reduced by buffering the link with an elastic store and clocking out the digit stream under the control of a highly stable phase-locked loop.

→ Jitter reduction is reasonably about 200 miles is very much necessary in a long digital link to keep the maximum jitter within limits.

—— x ——

Pulse shaping and intersymbol interference (ISI) :-

PSD of line code

$S_Y(f) = |P(f)|^2 S_x(f)$ ——— (1)

or $S_{psd}(f) = \frac{1}{T_b} |G(f)|^2 S_A(f)$ ——— (2).

Where $G(f)$ = F.T. of $g(t)$, $S_A(f)$ = PSD of random data.

So the shape of the pulse plays a major role for desirable psd of line code. Improper choice of shape of the pulse leads intersymbol interference (ISI).

→ ISI occures when the symbol-1 leaks into the time period of symbol-2.

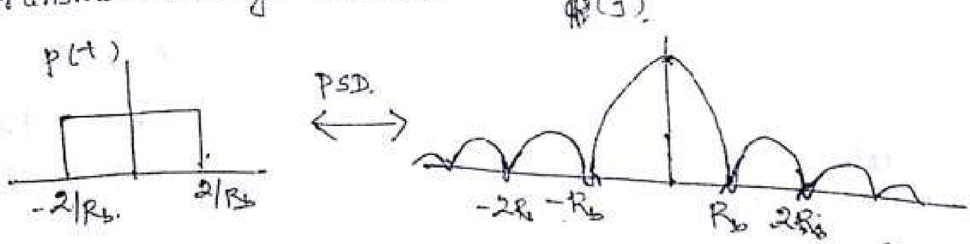
The ~~main ISI~~ major cause of ISI is

$\boxed{\text{signal B.W.} > \text{channel B.W.}}$

→ The B.W. (ideal) of a pulse is infinite. But the essential b.w. is finite. For the bipolar signal the signal contained within the essential band of 0 to R_b Hz. But the PSD is small but still non zero in the range of $f > R_b$ Hz.

→ If the pulse completely is being transmitted then the signal bw > channel b.w.

→ But we can reduce the bw. of the signal within R_b by sacrifising only 2% of the signal to transmit through channel.



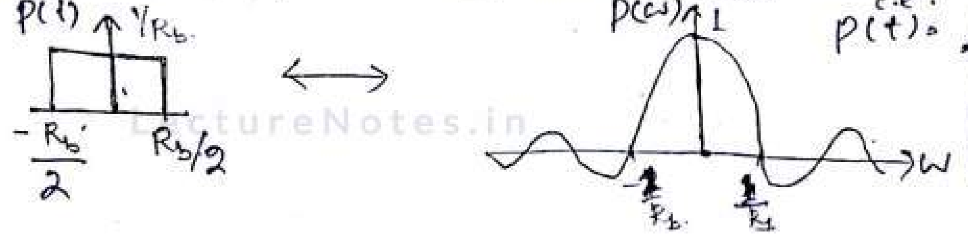
→ But according to the properties of F.T. supresing the bw. in frequency domain causes spreading in time domain.

→ So if we suppress the b.w. of pulse inorder to make it less than channel b.w, the pulse will spread. Spreading the pulse beyond the interval $T_b = \frac{1}{R_b}$

will cause interference with neighboring signal causing ISI.

Nequist Criteria for zero ISI.

Inorder to have zero ISI the pulse shape should be such that, it would be fit in band limitted channel and by choosing a pulse shape that has a nonzero amplitude at its center (say t=0) and zero amplitude at $t=\pm nT_b$. The time domain and frequency domain representation of the channel (pulse) shown below



i.e. $p(t) = \begin{cases} 1, & t=0 \\ 0, & t=\pm nT_b. \end{cases}$

So if we generate a sinc function that can be easily passed through a bandlimitted channel. But there are two major difficulties arises in this case.

(i) Generation of sinc function is very difficult as it is everlasting function.

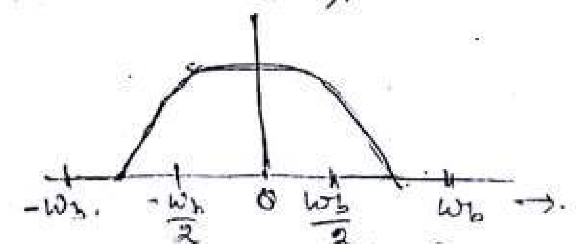
(ii) It will cause ISI because the tail of 1st pulse will interfere with the leading edge of 2nd pulse.

Hence the bandlimitted signal is not time limitted and the time limitted signal is not band limitted.

Let p(t) is the pulse of shape for of Nyquist criterion for zero ISI. So $p(t) = \begin{cases} 1, & t=0 \\ 0, & t=\pm nT_b \end{cases}$ restricted

$p(t) \xleftrightarrow{F.T} P(\omega)$.

Where the B.W. of the pulse $P(\omega)$ is in the range $(\omega_b/2, \omega_b)$ as shown. where $\omega_b = R_b(2\pi)$ $\Rightarrow 2\pi R_b = \omega_b$.



→ Now transmission of R_b bits/sec requires a theoretical minimum bandwidth $R_b/2$ Hz. It would be nice if a pulse satisfying Nyquist's criterion had this minimum B.W. $R_b/2$ Hz. Can we find such a pulse? There exists one pulse which meets Nyquist's criterion and has a bandwidth $R_b/2$ Hz. This pulse, $p(t) = \text{sinc}(\pi R_b t)$

l causing t

is the desired pulse satisfies the given eq^n

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$$\text{sinc}(\pi R_b t) = p(t) = \begin{cases} 1 & , t = 0 \\ 0 & , t = \pm n T_b \quad (T_b = \frac{1}{R_b}) \end{cases} \quad \text{—— (1)}$$

$$\Rightarrow p(t) \leftrightarrow P(f) = \frac{1}{R_b} \Pi (f/R_b) \quad \text{—— (2)}.$$

If we sample $p(t)$ every T_b seconds by multiplying $p(t)$ by an impulse train $\delta_{T_b}(t)$.

cy domain

So the sampled signal

$$\bar{p}(t) = p(t)\,\delta_{T_b}(t) = \delta(t).$$

Hence the spectrum of sampled signal $\bar{p}(t)$ is the spectrum of $P(t)$ repeating periodically at intervals of sampling frequency $\omega_b = \frac{2\pi}{T_b} = 2\pi R_b$. So F.T. of $\bar{p}(t)$ is given by

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$$\frac{1}{T_b} \sum_{n=-\infty}^{\infty} P(\omega - n\omega_b) = 1.$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} P(\omega - n\omega_b) = T_b.$$

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So the sum of the spectra formed by repeating $P(\omega)$ every ω_b is a constant T_b.

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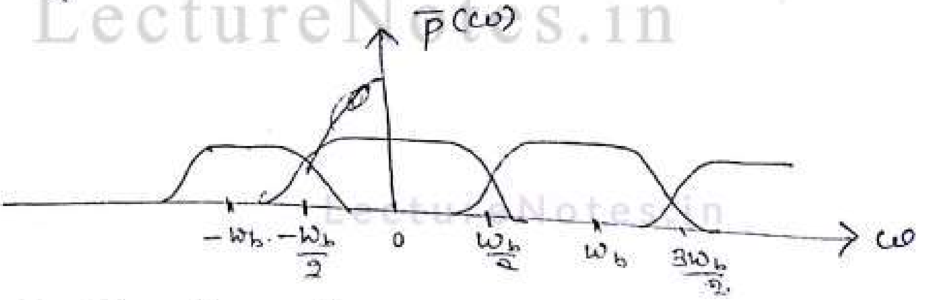

consider the spectrum over the range $0 < \omega < \omega_b$.

Over this range only two terms $P(\omega)$ and $P(\omega - \omega_b)$ exist in the summation.

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Hence

$$P(\omega) + P(\omega - \omega_b) = T_b \qquad 0 < \omega < \omega_b.$$

Put $\omega = x + \frac{\omega_b}{2}$ where $x = f - \frac{R_b}{2}$ or $\omega - \frac{\omega_b}{2}$

So $P(x + \frac{\omega_b}{2}) + P(x - \frac{\omega_b}{2}) = T_b \qquad |x| < \frac{\omega_b}{2}.$

Applying conjugate symmetry property

i.e. $G(-\omega) = G^*(\omega)$ we have

$\Rightarrow G(\omega) = G^*(-\omega)$.

$$P\left(\frac{\omega_b}{2}+x\right) + P^*\left(\frac{\omega_b}{2}-x\right) = T_b.$$

$\Rightarrow$ Taking the amplitude part only

$$\left|P\left(\frac{\omega_b}{2}+x\right)\right| + \left|P\left(\frac{\omega_b}{2}-x\right)\right| = T_b.$$

So $|P(\omega)|$ should be in form of as shown below

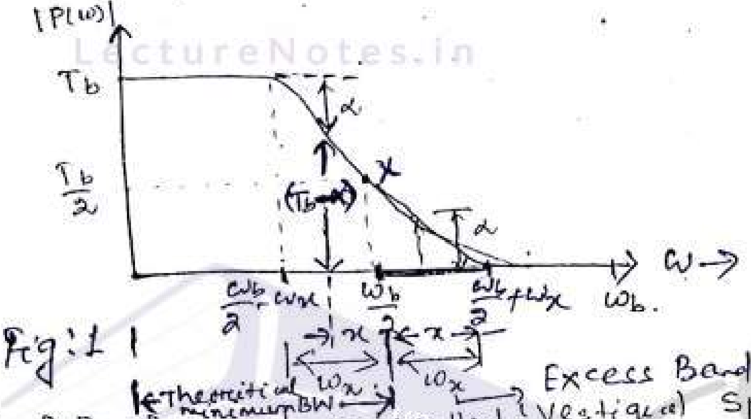


Fig: 1

Excess Bandwidth.

This type of spectrum is called (Vestigial Spectrum)

This curve has an odd symmetry about the set of axis intersecting at point x $\frac{\omega_b}{2}$

The b.w. of $P(\omega)$ is $\frac{\omega_b}{2} + \omega_x$ where ω_x is the b.w. in excess of the theoritical minimum b.w.

Let 'r' be the ratio of the excess b.w. ω_x to the theoritical minimum b.w. $\omega_b/2$.

$$\text{Roll off factor } (r) = \frac{\text{Excess b.w.}}{\text{theoritical minimum b.w.}} = \frac{\omega_x}{\omega_b/2}$$

$$= \frac{2\omega_x}{\omega_b}, \quad 0 \leq r \leq 1.$$

$0 \leq r \leq 1.$

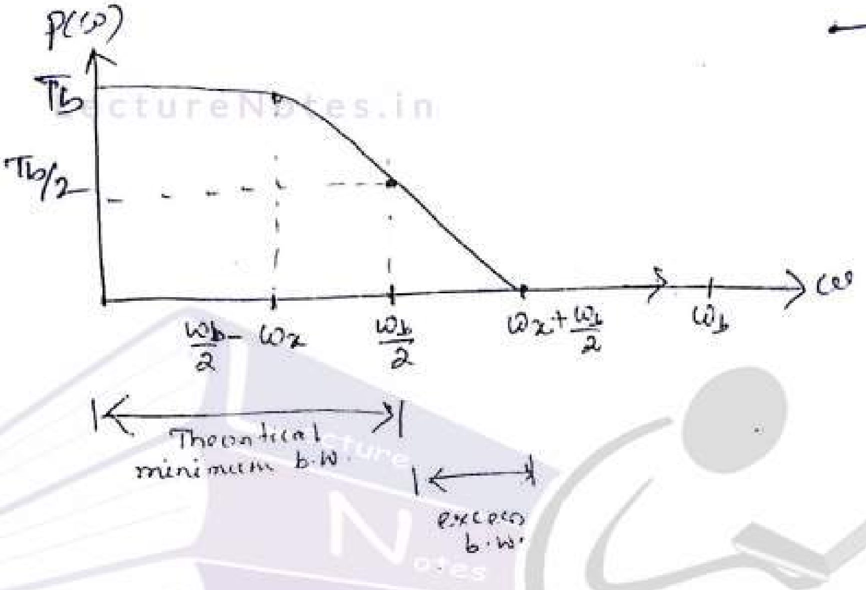
$\frac{1}{T_s} = 2W.$

$\Rightarrow f_s = 2f_m.$

However, the vestigial roll-off characteristics is gradual and it can be more closely approximated by a practical filter. One family of spectra that satisfies Nyquist's criterion is given by eqn (1) as follows:-

This type of spectrum is called 'raised cosine spectrum' and given by the expression

$$P(\omega) = \begin{cases} \frac{1}{2}\left\{1 - \sin\left(\frac{\pi(\omega - (\omega_b/2))}{2\omega_x}\right)\right\} & |\omega - \frac{\omega_b}{2}| < \omega_x \\ 0 & |\omega| > \frac{\omega_b}{2} + \omega_x \\ 1 & |\omega| < \frac{\omega_b}{2} - \omega_x \end{cases}$$



Roll off factor $\gamma = \frac{\text{Excess b.w.}}{\text{Theoritical min. b.w.}}$

$$= \frac{\omega_x}{\omega_b/2} = \frac{2\omega_x}{\omega_b}.$$

Response for different rolloff factor is shown below.

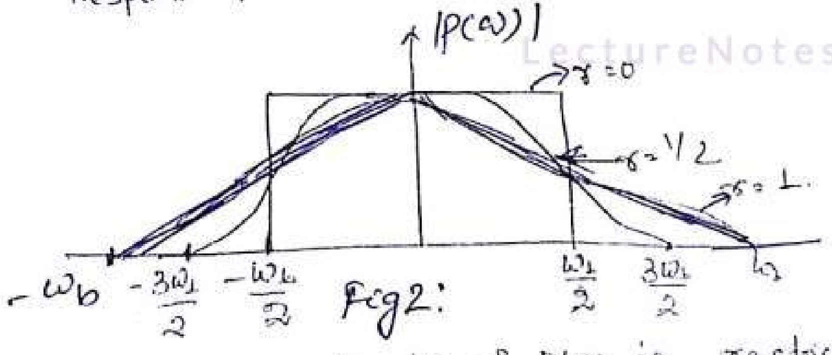


Fig2:

As $0 \leq \gamma \leq 1$. the bw of $P(\omega)$ is restricted to the range of $\frac{\omega_b}{2}$ to ω_b Hz.

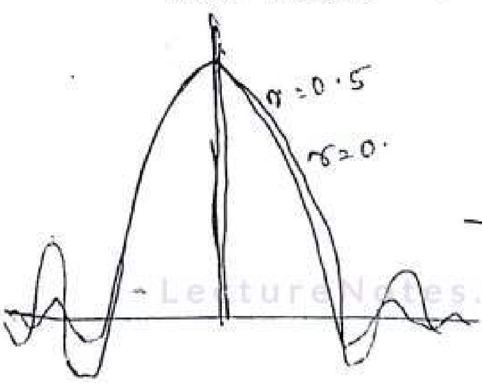
The inverse Fourier transform of this spectrum gives the shape of pulse with allmost zero ISI is given by

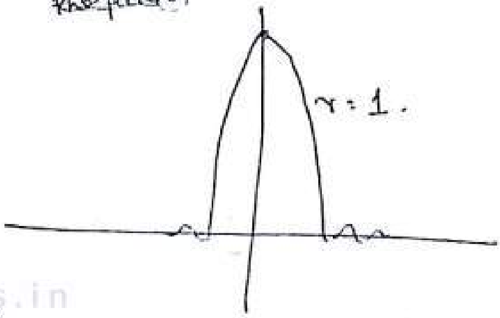
$$\boxed{p(t) = \omega_b \frac{\cos \pi R_b t}{1 - 4R_b^2 t^2} \operatorname{sinc}(\pi R_b t)}$$

The time response $p(t)$ consists of product of two factors

(i) sinc function characterizing the ideal Nyquist channel.

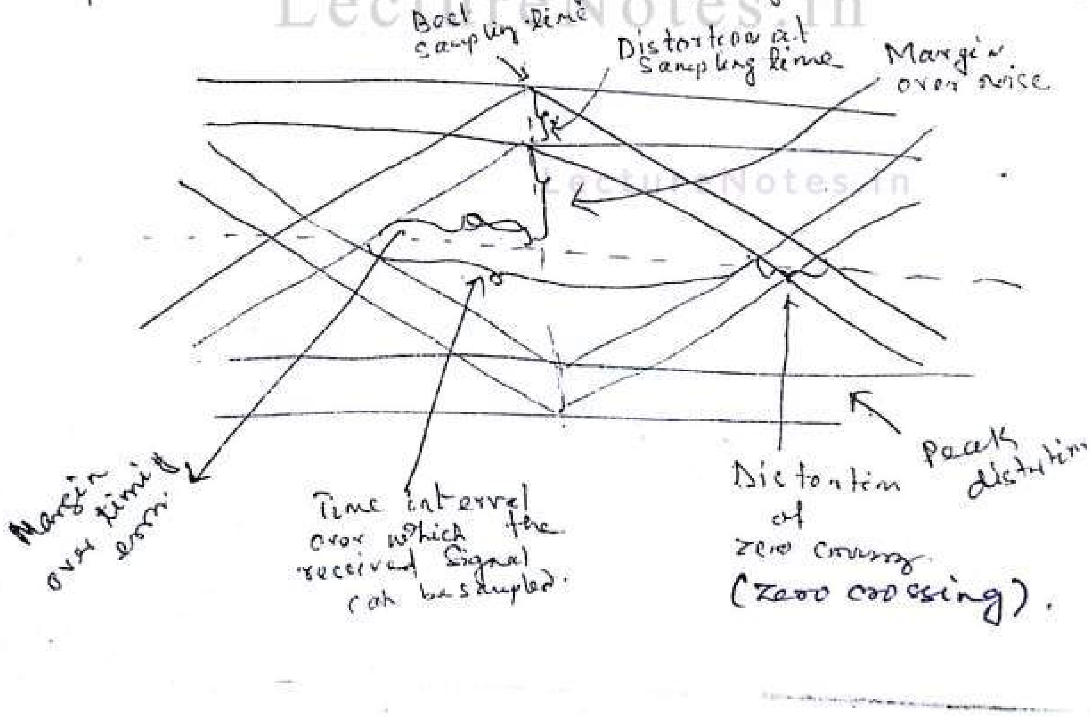
(ii) $\frac{1}{|t|^2}$ term which decreases the amplitude with increase of $|t|$. This factor reduces the tails of the pulses.





the pulse considerably below that obtained from the ideal Nyquist channel, so that the transmission of binary waves using such pulse is relatively insensitive to sampling time errors.

Eye diagram (Pattern): It is defined as the synchronized superposition of all possible realization of the signal of interest veiwed within a particular signalling interval. The interior region of the eye pattern is called as eye opening.



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Regenerative Repeater.

It performs following functions

1- reshaping incoming pulses by means of an equilizer.

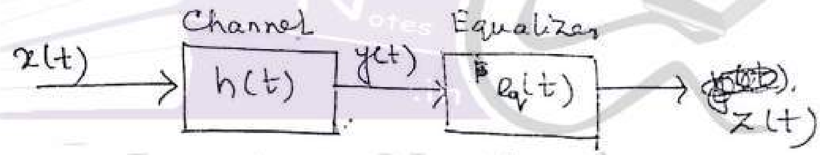
2- the extraction of timing information required to sample incoming pulses at optimum instants.

3- decision making based on pulse samples.

Equalizer:- Equalizer does the inverse ~~function~~ frequency characteristics that the transmission channel ~~does~~ has.

→ In general the transmission channel behaves as low pass filter surpressing the high frequency componets resulting dispersion in pulse which leads to ISI.

→ In this case the frequency charecterization of the channel is equivalent to high pass filter.



The ~~received~~ channel O/p

$$y(t) = x(t) * h(t).$$

The O/p of equalizer

$$Z(t) = y(t) * eq(t) = x(t) * h(t) * eq(t).$$

For perfect equalization

$$x(t) = Z(t).$$

i.e. $\boxed{h(t) * eq(t) = \delta(t)}$

or $\boxed{Eq(\omega) = H^{-1}(\omega)}$

This type of equalizer is called zero forcing equalizer which forces the filtering effect of channel to zero.

→ How ever this type of equalizer ignores the channel noise and it allows the noises present in high frequency.

→ The design of an optimum equalizer requries a trade off between reducing ISI and reducing channel noise.

Ex A com. channel of b.w. 75KHz is requeried to transmit binary data at a rate of 0.1 Mb/s using raised cosine pulses. Determine the roll off factor α.

A.
$$T_b = \frac{1}{0.1(10^6)} = 10^{-5} s.$$

$$f_B = 75KHz = 75(10^3) Hz.$$

Theoritical min. b.w: $f_T = \frac{1}{2} \times \frac{1}{T_b} = 0.5 \times 10^{5} Hz.$

Excess b.w. $= f_B - f_T$

$= 75000 - 50000$

$= 25000$.

$$r = \frac{\text{Excess b.w.}}{\text{Theoritical bw}} = \frac{25000}{50000} = 0.5.$$

Ex. In a certain telemetry system, eight message signals having 2KHz bw each are time-division multiplex using binary PCM. The error in sampling amplitude can't be greater than 1% of the peak amplitude. Determine the minimum transmission b.w. required if raised-cosine pulse with roll-off factor $r = 0.2$ are used. The sampling rate must be at least 25% above the Nyquist rate.

A. $\epsilon_{max} \leq 0.01 mp. \Rightarrow q \geq 100$ ie. $q = 128 = 2^7$.

No of bits / sample : 7.

$f_M = 2KHz$. $f_N = 2 f_M = 4000$ Samples/sec

$f_s = 1.25 \times f_N = 5000$ sam/sec

for 8 signals $= 8 \times 5000 = 40$ Ksample/sec. $= 40K \times 7 = 280 Kb/s$.

Min b.w $= \frac{280}{2} = 140KHz$. Requd bw $= (1+r) 140KHz = 168$ KHz.

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