CS5100: Foundations of Artificial Intelligence

Uncertainty

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Lecture 9

Administrative

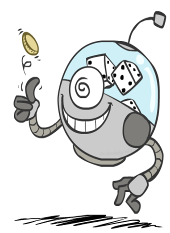
- We have finished the first 2 sections of this class – Search, and Logic
- Final Exam Cancelled!
  - Higher weightage on projects and Assignments
  - More EC assignments
  - Grading:
    - 3 Projects (63% of your grade) (20, 20, 23)
    - In class problem sets (32% of your grade)
    - Participation (5%)

Administrative – Contd.

- Project 2 is Due next week
  - Some people are already done!
- Assignment 4 is graded and scores are returned back to you
- Assignment 5 today!
- Assignment 3 grading – still in progress

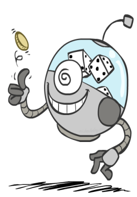
Today

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes' Rule
  - Inference
  - Independence
- You'll need all this stuff A LOT for the next few weeks, so make sure you go over it now!



Random Variables

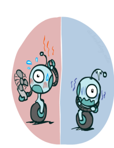
- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters
- Random variables have domains
  - R in {true, false} (often write as {+r, -r})
  - T in {hot, cold}
  - D in [0, ∞)
  - L in possible locations, maybe {(0,0), (0,1), …}



Probability Distributions

- Associate a probability with each value
  AKA – Prior Probability

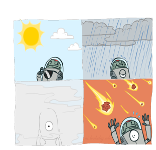
Temperature:



$P(T)$

T	P
hot	0.5
cold	0.5

Weather:



$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- A distribution is a TABLE of probabilities of values
- A probability (lower case value) is a single number
  $P(W = rain) = 0.1$
- Must have:
  $\forall x \;\; P(X = x) \geq 0 \qquad \sum_x P(X = x) = 1$

Shorthand notation:

$P(hot) = P(T = hot),$
$P(cold) = P(T = cold),$
$P(rain) = P(W = rain),$
$\ldots$

OK *if* all domain entries are unique

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):
  $P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$
  $P(x_1, x_2, \ldots x_n)$
  - Must obey:
    $P(x_1, x_2, \ldots x_n) \geq 0$
    $\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Size of distribution if n variables with domain sizes d?
  - For all but the smallest distributions, impractical to write out!

Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized:* sum to 1.0
  - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Events

- An *event* is a set E of outcomes
  $P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$
- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?
- Typically, the events we care about are *partial assignments*, like P(T=hot)

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Events

- P(+x, +y) ?
- P(+x) ?
- P(-y OR +x) ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(t) = \sum_s P(t, s)$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(s) = \sum_t P(t, s)$

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz: Marginal Distributions

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x,y)$$

$P(X)$

X	P
+x	
-x	

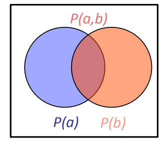
$$P(y) = \sum_x P(x,y)$$

$P(Y)$

Y	P
+y	
-y	

Conditional Probabilities

- Also Known as Posterior Probability



$$P(a|b) = \frac{P(a,b)}{P(b)}$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$
$$= 0.2 + 0.3 = 0.5$$

Quiz: Conditional Probabilities

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W|T)$

$P(W|T=hot)$

W	P
sun	0.8
rain	0.2

$P(W|T=cold)$

W	P
sun	0.4
rain	0.6

Joint Distribution

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence →

$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one) →

$P(W|T = c)$

W	P
sun	0.4
rain	0.6

- Why does this work? Sum of selection is P(evidence)! (P(T=c), here)

$$P(x_1|x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}$$

Quiz: Normalization Trick

- P(X | Y=-y) ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

SELECT the joint probabilities matching the evidence →

NORMALIZE the selection (make it sum to one) →

To Normalize

- (Dictionary) To bring or restore to a normal condition

All entries sum to ONE

- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

Normalize → Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize → Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Inference

- **Probabilistic inference**: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's *beliefs* given the evidence

- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes *beliefs to be updated*

Inference by Enumeration

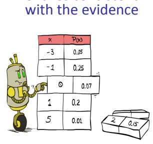
General case:
- Evidence variables: $E_1 \ldots E_k = e_1 \ldots e_k$
- Query* variable: Q
- Hidden variables: $H_1 \ldots H_r$

$\left.\right\} X_1, X_2, \ldots X_n$ *All variables*

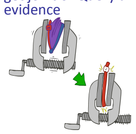
** Works fine with multiple query variables, too*

- We want: $P(Q|e_1 \ldots e_k)$

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \ldots e_k) = \sum_{h_1 \ldots h_r} \underbrace{P(Q, h_1 \ldots h_r, e_1 \ldots e_k)}_{X_1, X_2, \ldots X_n}$$

- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration

- P(W)?

- P(W | winter)?

- P(W | winter, hot)?

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

Inference by Enumeration

- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

Independence

If 2 events (X and Y) are independent then:

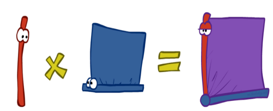
p(X|Y) = p(X)
p(Y|X) = p(Y)
p(X,Y) = p(X) p(Y)

E.g. color of eyes and color of front door are independent

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x,y) \quad \Longleftrightarrow \quad P(x|y) = \frac{P(x,y)}{P(y)}$$



The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$\Longleftrightarrow$

$P(D,W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

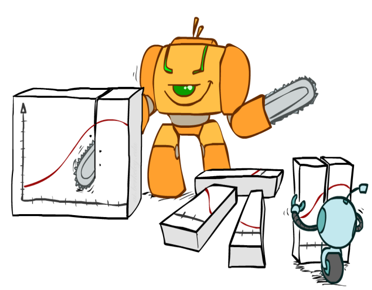
The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

Bayes Rule



Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) \quad = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later
- In the running for most important AI equation!

Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

  - Note: posterior (or conditional) probability of meningitis still very small

Quiz: Bayes' Rule

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

Now:

- In Class Assignment