CS 2050 Worksheet 3

1.7 Introduction to Proofs

1. Use a direct proof to show that the cube of an odd number is odd.

p: n is odd

q: n^3 is odd

$(2k+1)(2k+1)(2k+1)$

$(2k+1)(4k^2+4k+1)$

$= 8k^3+8k^2+2k+4k^2+4k+1$

$= 8k^3+12k^2+6k+1$

Proof by direct proof

(1) n is odd — assume p

(2) $n = 2k+1, k \in \mathbb{Z}$ — def'n of odd

(3) $n^3 = (2k+1)^3$ — multiply $n \cdot n \cdot n$

(4) $n^3 = 8k^3+12k^2+6k+1$ — expand

(5) $n^3 = 2(4k^3+6k^2+3k)+1$ — factor out 2

(6) $k' = 4k^3+6k^2+3k, k' \in \mathbb{Z}$ — Define new variable

(7) $n^3 = 2k'+1$ — sub k'

(8) n^3 is odd — def'n of odd

Note: this is q. Conc: Assuming p leads to q thus I have shown $p \to q$ by direct proof.

2. Use proof by contraposition to prove that, for any integers m and n, if $m + n \geq 15$, then $m \geq 8$ or $n \geq 8$.

p: $m+n \geq 15$ $\rightarrow$ $m+n < 15$

q: $m \geq 8$ or $n \geq 8$

Proof by contradiction

(1) $\neg(m \geq 8$ or $n \geq 8)$ — assume $\neg q$

(2) $m < 8$ and $n < 8$ — deMorgan's

(3) $m \leq 7$ and $n \leq 7$ — def'n of < and ≤

(4) $m+n \leq 14$ — add both sides (m+n)

(5) $m+n < 15$ — def'n of <

you can't add these w/o equality sign. Thus change to ≤

Note this is $\neg p$ $(m+n < 15)$

$\neg p: m+n < 15$

Conc: assuming $\neg q$ leads to $\neg p$ so I have shown $\neg q \to \neg p$ is true by contraposition. So $p \to q$ is also true by proof by contraposition.

3. Use proof by contradiction to prove that, for any integer n, if $n^3 + 5$ is odd, then n is even.

p: n^3+5 is odd
q: n is even

proof by contradiction

1	$p \wedge \neg q$	assume $p \wedge \neg q$
2	$\neg q$	simplify (#1)
3	$\neg$ (n is even)	q
4	n is odd	def'n of even/odd + deMorgan's
5	$n = 2k+1, k \in \mathbb{Z}$	def'n of odd
6	p	simplify
7	$n^3+5 = 2k'+1, k' \in \mathbb{Z}$	def'n of odd
8	$(2k+1)^3+5 = 2k'+1$	
9	$8k^3+12k^2+6k+1+5 = 2k'+1$	expand
10	$8k^3+12k^2+6k+6 = 2k'+1$	add
11	$2(4k^3+6k^2+3k+3) = 2k'+1$	factor
12	$k'' = 4k^3+6k^2+3k+3, k'' \in \mathbb{Z}$	define new variable
13	$2k'' = 2k'+1$	sub k''
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TA METHOD

p: n^3+5 is odd $\qquad n \in \mathbb{Z}$

q: n is even

1.	n^3+5 is odd	assume p
2.	n is odd	assume $\neg q$
3.	$n = 2k+1, k \in \mathbb{Z}$	def'n of odd
4.	$n^3 = (2k+1)^3$	cube both sides
5.	$n^3 = 8k^3 + 12k^2 + 6k + 1$	expand
6.	$n^3+5 = 8k^3+12k^2+6k+6$	add 5 to both sides
7.	$n^3+5 = 2(4k^3+6k^2+3k+3)$	factor out 2
8.	$k' = 4k^3+6k^2+3k+3, k' \in \mathbb{Z}$	define new variable
9.	$n^3+5 = 2k'$	sub k'
10.	n^3+5 is even	def'n of even

note: this contradicts p

Conc: Since assuming $p \to q$ is false led to a contradiction, I have shown $p \to q$ is true by proof by contradiction.

4. Use proof by counterexample to disprove that, for any real numbers a and b, if $a^2 - b^2 > 0$ then $a - b > 0$.

$a^2 - b^2 > 0 \rightarrow a - b > 0$

Let's look at $a = -2$ and $b = -1$

$(-2)^2 - (-1)^2 \overset{?}{>} 0$

$4 - 1 > 0$

$3 > 0$ ✓

$(-2) - (-1) \overset{?}{>} 0$

$-2 + 1 > 0$

$-1 > 0$ ✗

Since $a - b > 0$ is not true for $a = -2, b = -1$, the statement is false by proof by counter example

Proof by cases example

prove that $n^2 \geq n$ for $n \in \mathbb{Z}$

Case 1: $n = 0$

$0^2 \geq 0$

$0 \geq 0$ ✓ $\therefore n^2 \geq n$ when $n = 0$

case 2: $n \leq -1$:

n^2 is always pos, because a squared number is always positive, so $n^2 \geq n$ pos $\geq$ neg

$\therefore$ I have shown if $n \leq -1$, then $n^2 \geq n$

case 3: $n \geq 1$ assume $n \geq 1$

$n \cdot n \geq 1 \cdot n$ mult. both sides by n.

$n^2 \geq n$

$\therefore$ I have shown if $n \geq 1$, then $n^2 \geq n$.

conc: Since I have shown all cases to be true, $n^2 \geq n$ for all integers by proof by cases.