

CS 2050 Worksheet 3

1.7 Introduction to Proofs

1. Use a direct proof to show that the cube of an odd number is odd.

$P:$ n is odd

$q:$ n^3 is odd

Proof by direct proof

① n is odd

② $n = 2k+1, k \in \mathbb{Z}$ def'n of odd

③ $n^3 = (2k+1)^3$ multiply $n \cdot n \cdot n$

④ $n^3 = 8k^3 + 12k^2 + 6k + 1$ expand

⑤ $n^3 = 2(4k^3 + 6k^2 + 3k) + 1$ factor out 2

⑥ $k' = 4k^3 + 6k^2 + 3k, k' \in \mathbb{Z}$ define new variable

⑦ $n^3 = 2k' + 1$ sub k'

⑧ n^3 is odd def'n of odd

$$\begin{aligned} & (2k+1)(2k+1)(2k+1) \\ & (2k+1)(4k^2 + 4k + 1) \\ & = 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 \\ & = 8k^3 + 12k^2 + 6k + 1 \end{aligned}$$

assume p

2. Use proof by contraposition to prove that, for any integers m and n , if $m+n \geq 15$, then $m \geq 8$ or $n \geq 8$. shown

$P: m+n \geq 15 \rightarrow m+n < 15$

$q: m \geq 8 \text{ or } n \geq 8$

$P \rightarrow q$ by
direct proof.

Proof by contradiction

① $\neg P (m \geq 8 \text{ or } n \geq 8)$

assume $\neg q$

② $m < 8$ and $n < 8$

deMorgan's

you can't add these
w/o equality
sign. Thus
change
 $\rightarrow \leq$

③ $m \leq 7$ and $n \leq 7$

def'n of $<$ and \leq

④ $m+n \leq 14$

add both sides ($m+n$)

⑤ $m+n < 15$

def'n of $<$

Note this is $\neg P (m+n < 15)$

$\neg P: m+n < 15$

Concl: Assuming $\neg q$ leads to $\neg P$ so we have shown $\neg q \rightarrow \neg P$ is true by contraposition. So $P \rightarrow q$ is also true by proof by contraposition.

3. Use proof by contradiction to prove that, for any integer n , if $n^3 + 5$ is odd, then n is even.

p: $n^3 + 5$ is odd

q: n is even

Proof by contradiction

① $p \wedge \neg q$

assume $p \wedge \neg q$

② $\neg q$

Simplifying (#1)

③ $\neg (\text{n is even})$

q

def'n of even/odd + t Margin's

④ n is odd

def'n of odd

⑤ p

Simplify

⑥ $n^3 + 5 = 2k' + 1, k' \in \mathbb{Z}$

def'n of odd

⑦ $(2k+1)^3 + 5 = 2k' + 1$

⑧ $8k^3 + 12k^2 + 6k + 1 + 5 = 2k' + 1 \quad \text{expand}$

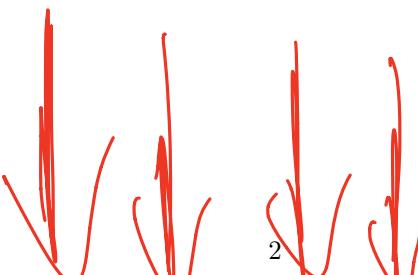
⑨ $8k^3 + 12k^2 + 6k + 6 = 2k' + 1 \quad \text{add}$

⑩ $2(4k^3 + 6k^2 + 3k + 3) = 2k' + 1 \quad \text{factor}$

⑪ $k'' = 4k^3 + 6k^2 + 3k + 3, k'' \in \mathbb{Z} \quad \text{define new variable}$

⑫ $2k'' = 2k' + 1 \quad \text{sub } k''$

⑬



TA METHOD

\neg \vee \sim \neg

p: $n^3 + 5$ is odd

$n \in \mathbb{Z}$

q: n is even

- (1) $n^3 + 5$ is odd
assume p
- (2) n is odd
assume $\neg q$
- (3) $n = 2k+1, k \in \mathbb{Z}$
def'n of odd
- (4) $n^3 = (2k+1)^3$
(cube both sides)
- (5) $n^3 + 5 = 8k^3 + 12k^2 + 6k + 6$
expand
- (6) $n^3 + 5 = 8k^3 + 12k^2 + 6k + 6$
add 5 to both sides
- (7) $n^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3)$
factor out 2
- (8) $k' = 4k^3 + 6k^2 + 3k + 3, k' \in \mathbb{Z}$
define new variable
- (9) $n^3 + 5 = 2k'$
sub k'
- (10) $n^3 + 5$ is even
def'n of even

Note: this contradicts p

Conc: Since assuming $p \rightarrow q$ is false led to a contradiction, I have shown $p \rightarrow q$ is true by proof by contradiction.

4. Use proof by counterexample to disprove that, for any real numbers a and b , if $a^2 - b^2 > 0$ then $a - b > 0$.

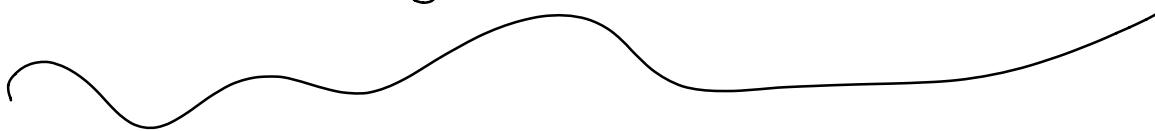
$$a^2 - b^2 > 0 \rightarrow a - b > 0$$

Let's look at $a = -2$ and $b = -1$

$$\begin{aligned} (-2)^2 - (-1)^2 &\stackrel{?}{>} 0 \\ 4 - 1 &> 0 \\ 3 &> 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} (-2) - (-1) &\stackrel{?}{>} 0 \\ -2 + 1 &> 0 \\ -1 &> 0 \quad \times \end{aligned}$$

Since $a - b > 0$ is not true for $a = -2, b = -1$, the statement is false by proof by counter example.



Proof by Cases Example

Prove that $n^2 \geq n$ for $n \in \mathbb{Z}$

Case 1: $n = 0$

$$\begin{aligned} 0^2 &\geq 0 \\ 0 &\geq 0 \quad \checkmark \quad \therefore n^2 \geq n \text{ when } n=0 \end{aligned}$$

Case 2: $n \leq -1$:

n^2 is always pos, because a square number is always positive, so it's $\geq n$

\therefore I have shown if $n \leq -1$, then $n^2 \geq n$

case 3': $n \geq 1$

$$n \cdot n \geq 1 \cdot n$$

assume $n \geq 1$

mult. both sides by n .

$$n^2 \geq n$$

\therefore I have shown if $n \geq 1$, then $n^2 \geq n$.

concl. Since I have shown all cases to be true, $n^2 \geq n$
for all integers by proof by cases.