Bayesian Parameter Estimation

Name: Rutvik Kolhe
Unity ID: rkolhe

Question 1: **Derive the formula for the posterior distribution of μ**

(Q1.)

To find: Posterior distribution of μ
i.e $P(\mu / X)$

∴ using joint probability

$$P(\mu, X) = P(\mu | X) * P(X) \quad - \mathrm{I}$$
$$= P(X | \mu) \times P(\mu) \quad - \mathrm{II}$$

From I and II

$$P(\mu | X) \, P(X) = P(X | \mu) \, P(\mu)$$

$$\therefore \boxed{P(\mu | X) = \frac{P(X | \mu) \, P(\mu)}{P(X)}}$$

Question 2: **Show that the posterior distribution is the Gaussian,** $p(\mu|X) \sim N(\mu_n, \sigma^2_n)$

Q2)

Given:-

The distribution for $p(x)$ is $N(\mu, \sigma^2)$
The distribution for prior $p(\mu)$ is $N(\mu_0, \sigma_0^2)$

From the above info:-

$$p(X|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

We know that:-

$$\text{posterior} \; P(\mu|X) = \frac{P(X|\mu)\,P(\mu)}{P(X)}$$

Hence, the multiplication of prior and likelihood (gaussian distributions) will give us a gaussian distribution.

Thus,

We can conclude that posterior distribution would be gaussian with mean μ_n and variance σ_n^2.

Question 3: **Show the derivation and the final estimate for** μ_n **and** $1/\sigma^2_n$

Q.3)

$$P(\mu|X) = \frac{P(X|\mu)P(\mu)}{P(X)}$$

$$\therefore \hat{\theta}_{MAP} = \underset{\mu}{\operatorname{argmax}}\, P(X|\mu)P(\mu)$$

$$P(X|\mu) = \sum \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\therefore \log P(X|\mu) = \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi\sigma^2}} + \sum_{i=1}^{n} \log e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$= n\left(-\frac{1}{2}\log 2\pi - \log\sigma\right) + \sum_{i=1}^{n} -\frac{(x_i-\mu)^2}{2\sigma^2} \log e$$

$$= -\frac{n}{2}\log 2\pi - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \quad - \text{I}$$

$$P(\mu) \sim N(\mu_0, \sigma_0^2)$$

$$P(\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}$$

$$\therefore \log P(\mu) = \cancel{\frac{-1}{2\sigma_0^2}(\mu-\mu_0)^2 \log \frac{1}{\sqrt{2\pi\sigma_0^2}} \times e}$$

$$= \log \frac{1}{\sqrt{2\pi\sigma_0^2}} + \frac{-1}{2\sigma_0^2}(\mu-\mu_0)^2 \log e$$

$$= -\frac{1}{2}\log 2\pi - \log\sigma_0 - \frac{1}{2\sigma_0^2}(\mu-\mu_0)^2 \quad - \text{II}$$

$\therefore$ To find μ_n.

differentiate I and II partially wrt μ.

$\therefore$ I $\Rightarrow$ $\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - \mu n\right)$

II $\rightarrow$ $\frac{\mu_0 - \mu}{\sigma_0^2}$

$\therefore$ equating it to 0 to find $\hat{\theta}_{MAP}$

$$\therefore \frac{1}{\sigma^2}\left(\sum_{i=0}^{n} x_i - \mu n\right) + \frac{\mu_0 - \mu}{\sigma_0^2} = 0.$$

$$\Rightarrow \frac{\sum x_i}{\sigma^2} - \frac{\mu n}{\sigma^2} + \frac{\mu_0}{\sigma_0^2} - \frac{\mu}{\sigma_0^2} = 0. \quad \text{--- III}$$

$$\frac{\sum x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2} - \mu\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right) = 0$$

$$\therefore \frac{\sum x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2} = \mu\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)$$

Comparing with $P(\mu|\mu_n)$

$$\therefore \mu = \frac{\sum x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2} \Bigg/ \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}$$

$$= \frac{n\bar{x}}{\sigma^2} \times \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2} + \frac{\mu_0}{\sigma_0^2} \times \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2}$$

$$\boxed{\mu_n = \left(\frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}\right)\bar{x} + \left(\frac{\sigma^2}{n\sigma_0^2 + \sigma^2}\right)\mu_0}$$

From III, we get

$$\frac{\Sigma x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2} - \mu\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right) = 0.$$

comparing it with $p(\mu \mid \mu_n)$

$$\frac{1}{\sigma_0^2} = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}$$

$$\therefore \boxed{\frac{1}{\sigma_0^2} = \frac{n\sigma_0^2 + \sigma^2}{\sigma^2\sigma_0^2}}$$

where

$$p(\mu \mid \mu_n) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(\mu-\mu_n)^2}{2\sigma_n^2}}$$

Question 4: **If the mean of the posterior density (which is the MAP estimate), μ_n is written as the weighted average of the prior mean, μ_0, and the sample (likelihood) mean, $\bar{X}$, then what are the formulas for the weights?**

Question 5: **Are the weights in Question #4 directly or inversely proportional to their variances (justify)?**

Q4.) From the above question, we get-

$$\mu_n = \left(\frac{n\sigma_0^2}{n\sigma_0^2+\sigma^2}\right)\bar{X} + \left(\frac{\sigma^2}{n\sigma_0^2+\sigma^2}\right)\mu_0.$$

If we consider:-

$$\mu_n = \omega_1 \mu_0 + \omega_2 \bar{X}$$

$$\boxed{\omega_1 = \frac{\sigma^2}{(n\sigma_0^2+\sigma^2)}, \quad \omega_2 = \frac{n\sigma_0^2}{(n\sigma_0^2+\sigma^2)}}$$

Q5) As we see in the above solution.

ω_1 n $\mu_0 \rightarrow \sigma_0$; ω_2 n $\bar{X} \rightarrow \sigma$

are $\boxed{\text{inversely}}$ proportional to their variances.

Question 6: **Do the weights in Questions #4 sum up to 1 (justify)?**
Question 7:Is each weight between zero and one (justify)?

Q6. From the above solution:

$$w_1 = \frac{\sigma^2}{n\sigma_0^2 + \sigma^2} \qquad w_2 = \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}$$

$$\therefore \boxed{w_1 + w_2 = \frac{\sigma^2 + n\sigma_0^2}{n\sigma_0^2 + \sigma^2}}$$

$$\therefore w_1 + w_2 = 1$$

Q7) Let's consider $w_1 \rightarrow \frac{\sigma^2}{n\sigma_0^2 + \sigma^2}$

Since n is a positive number where n is number of samples

and $\sigma_0^2 \rightarrow$ positive, $[n\sigma_0^2]$ will always be positive

Therefore $\sigma^2 < n\sigma_0^2 + \sigma^2$ always

Hence $0 < w_1 < 1$

Similarly, $w_2 \rightarrow \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}$

Here $\sigma^2 \rightarrow$ positive (always)

$\therefore n\sigma_0^2 < n\sigma_0^2 + \sigma^2$

Hence; $0 < w_2 < 1$

Question 8: **Given your answers for Questions #4-7, what can you say about the value of μ_n w.r.t. the values of μ_0 and X**

Question 9: **If σ^2 is known, then for the new instance x_{new}, show that** $p(x_{new}|X) \sim N(\mu_n, \sigma_n^2+\sigma^2)$

Q8) The value for μ_n will always lie between the theoretical and prior mean.

Bayesian analysis takes in account multiple values of μ_n for a given prior and theoretical mean. It then selects the most appropriate value for μ_n that is closest to the theoretical mean.

Q9) We can estimate x^{new} from the training sample:-

$$p(x^{new}|X) = \frac{p(x^{new}, X)}{p(X)}$$

using marginalization

$$= \frac{\int p(x^{new}, X, \theta)\, d\theta}{p(X)}$$

using joint probability

$$= \int \frac{p(x^{new}|\theta)\, p(X|\theta)\, p(\theta)\, d\theta}{(p(X))}$$

$$= \int p(\theta|X)\, p(x^{new}|\theta)\, d\theta$$

In our question, the value for $\theta = \mu$.

$$\therefore p(x^{new}|X) = \int p(\mu|X)\, p(x^{new}|\mu)\, d\mu$$

From the above info –

$p(\mu|X) \sim N(\mu_n, \sigma_n^2)$

$p(x^{new}|X) \sim N(\mu, \sigma^2)$

using conjugate

$\therefore p(x^{new}/X) = N(x^{new} | \mu_n, \sigma_n^2 + \sigma^2)$

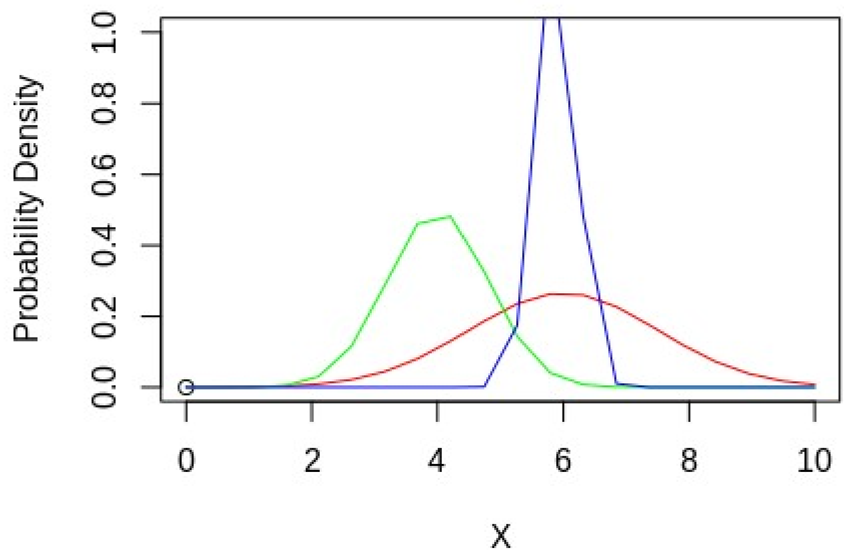
Question 10:

Generate a plot that displays $p(x) \sim N(6,1.5^2)$**, prior** $p(\mu) \sim N(4,0.8^2)$**, and posterior** $p(\mu|X) \sim N(\mu_n,\sigma^2_n)$ **for** $n=20$**sample points. What are the values for** μ_n **and** σ^2_n**?**

R Script:

```
#set variable values
mean_data<- 6
mean_prior<- 4
sd_data<- 1.5
sd_prior<- 0.8
n<- 20

#generate prior and data distribtuions
x<- seq(0, 10, length.out = n)
data<- dnorm(x, mean= mean_data,sd= sd_data)
prior<- dnorm(x, mean= mean_prior, sd= sd_prior)

#calculate mean, sd of posterior based on the formulae

sd_post<- sqrt(((sd_data^2)*(sd_prior^2))/((sd_data^2)+(n*(sd_prior^2))))

#to calculate the mean (x-bar) of the data, generate a random normal distribution
set.seed(123)
random_data<- rnorm(n, mean = mean_data, sd = sd_data)

mu_denom<- (sd_data^2)+(n*(sd_prior^2))
num1<- (sd_data^2)*mean_prior
num2<- n* mean(random_data)*(sd_prior^2)
```

```
mean_post<- (num1+num2)/mu_denom

#generate posterior distribution

posterior<- dnorm(x, mean= mean_post, sd = sd_post)

#plot graphs
plot(0,0, xlim= c(0,10), ylim=c(0,1), xlab='X', ylab='Probability Density')
lines(x, data, type = "l", col = "red")
lines(x, prior, type = "l", col = "green")
lines(x, posterior, type = "l", col = "blue")
```



Note: Green: Prior, Red-Data, Blue-Posterior

The value for mean is **5.8817** and variance is **0.3093^2 = 0.0957**