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Random variables

Descrete Random variables:- These are variables which can take on only finite number of values in a finite observation interval.

for eg:- Let us consider a sample space where we toss 3 coins simultaneously

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

let's suppose x is a random variable that determines the number of tails

$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$
$X_R = \{0, 1, 1, 1, 2, 2, 2, 3\}$

Random variables are classified into
- descrete Random variable
- Continious Random variable

Continious Random variable:- variables that take infinite number of values is known as continious Random variable.

for eg:- Noise voltage that is generated by an electronic amplifier has a continious amplitude. Therefore sample space (s) and random variable (x) both are continious

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Cumulative-Distribution Function (CDF)

For eg :- 1 $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

X is a random variable shows no. of tails

$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$

$X_R = \{0, 1, 1, 1, 2, 2, 2, 3\}$

What is C.D.F ?

The cumulative Distribution function (CDF) of a random variable 'X' may be defined as the probability that the random variable 'X' takes a value "less than or equal to x"

Mathematically C.D.F [$F_X(x)$] may be defined as

$$\boxed{\text{C.D.F } F_X(x) = P(X \le x)}$$

Other names of C.D.F

- probability distribution function of a random variable
- Distribution function of random variable
- cumulative probability distribution function.

$\boxed{\text{properties of C.D.F}}$

property 1: As C.D.F is defined as
$F_X(x) = P(X \leq x)$, so its value is always between '0' & '1'

$\Rightarrow \quad 0 \leq F_X(x) \leq 1$

property 2: $F_X(-\infty) = 0$ &
$F_X(\infty) = 1$

property 3: $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$
F_X i.e C.D.F is monotone decreasing

C.D.F for Discrete Random variables

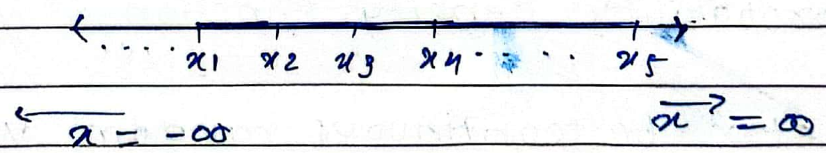
'X' is a discrete random variable, then it takes on values at discrete points.

$\therefore$ CDF can be defined for this case as

we know that

C.D.F $\quad F_X(x) = P(X \leq x)$

suppose $\quad X = \{x_1, x_2, x_3, x_4 \dots x_n\}$



so the C.D.F for a discrete random variable for complete range of x can be defined as

$$F_X(x) = 0 \quad \text{for } -\infty \leq x \leq x_1$$
$$= \sum_{j=1}^{f} P(X = x_j) \quad \text{for } x_1 \leq x \leq x_n$$
$$= 1 \quad \text{for } x_n < x < \infty$$

C.D.F of a ~~de~~ discrete variable at any certain event is equal to the summation of probabilities of random variable upto that certain event.

problem:-

1. suppose a random variable x assumes the value 0, 1, 2 with the probabilities 1/3, 1/6 and 1/2 respectively. find the C.D.F

x	0	1	2
p(x)	1/3	1/6	1/2

$F(x) = P(X = x)$

$x < 0 \qquad F(x) = 0$

$0 \le x < 1 \qquad F(x) = 0 + 1/3 = 1/3$

$1 \le x < 2 \qquad F(x) = 0 + 1/3 + 1/6 = 1/2$

$x \geq 2 \qquad F(x) = 0 + 1/3 + 1/6 + 1/2 = 1$

$$F(x) = \begin{cases} 0 \\ 1/3 \\ 1/2 \\ 1 \end{cases}$$

2. probability Density Function :

Let x be continuous random variable if there exists a function $F(x)$ is called the P.D.F of x satisfies

$f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

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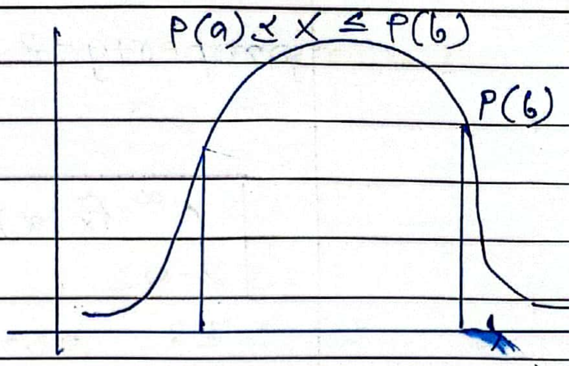
1. $P(a \le x \le b) = \int_a^b f(x)\,dx$

2. $P(x \geq a) = \int_a^\infty f(x)\,dx$

3. $p(x \le a) = \int_{-\infty}^{a} F(x)\,dx$



X. is a continuous random variable with P.D.F

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

C.D.F

probability Density function (PDF)

The derivative of cumulative distribution function (CDF) w.r.t to some dummy variable is P.D.F

P.D.F : $f_x(x) = \frac{d}{dx} F_x(x)$

Property of PDF with derivation

property 1: P.D.F is always non zero for all values of x

$$\boxed{f_x(x) \geq 0 \quad \forall x}$$

proof:- C.D.F increases monotonically
derivative of C.D.F is +ve always

$$\boxed{P.D.F = \frac{d}{dx}(C.D.F)}$$ defination.

$f(x) \geq 0$

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property-2 : The area under the P.D.F curve is always equal to unity

$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1.}$$

proof :- P.D.F : $f_x(x) = \frac{d}{dx} F_X(x)$

on integrating both sides

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_{-\infty}^{\infty} \left[\frac{d}{dx} F_X(x)\right] dx$$

$$= \left[F_X(x)\right]_{-\infty}^{\infty}$$

$$F_X(\infty) - F_X(-\infty) = 1 - 0$$
$$= 1$$

property-3 : It is possible to get C.D.F by integrating P.D.F

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx}$$

property-4 : probability of event $(x_1 < x \le x_2)$ is given by the area under the P.d.f curve in $x_1 < x \le x_2$ range

$$P(x_1 < x \le x_2) = \int_{x_1}^{x_2} f_x(x)\,dx$$

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3. Let X be a continuous random variable with probability density function (P.D.F)

f(x) defined by:

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x < 2 \\ -ax+3a & 2 \le x < 3 \\ 0 & \text{elsewher} \end{cases}$$

Value of a? (ii) C.D.F of X

as C.D.F is defined $F(x) = \int_{-\infty}^{\infty} f(x)$

$$= \int_0^1 ax + \int_1^2 a + \int_2^3 -ax+3a = 1$$

$$\left.\frac{ax^2}{2}\right]_0^1 + \left.ax\right]_1^2 + -\left[\left[\frac{ax^2}{2}\right]_2^3\right] + \left.3ax\right]_2^3 = 1$$

$$\frac{a}{2} + (2a-a) + -\left[+\frac{9a}{2} - \frac{4a}{2}\right] + (9a-6a) =$$

$$\frac{a}{2} + a - \frac{5a}{2} + 3a$$

$$-\frac{4a}{2} + 4a = 1$$

$$\frac{-4a+8a}{2} = 1$$

$$4a = 2$$

$$a = 2/4 = 1/2$$

∴ value of a is 1/2

Now that we know value of a is 1/2

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we can find C.D.F

$$f(x) = \begin{cases} ax = \frac{x}{2} & 0 \le x \le 1 \\ \frac{1}{2} & 1 \le x \le 2 \\ -\frac{1}{2}x + 3\left(\frac{1}{2}\right) & 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 1 \\ \frac{1}{2} & 1 \le x \le 2 \\ \frac{3-x}{2} & 2 \le x \le 3 \\ 0 & \text{elsewhere.} \end{cases}$$

$$f(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 1 \\ \frac{1}{2} & 1 \le x \le 2 \\ \frac{3-x}{2} & 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

0.33

(iii) $P(1/3 \le x \le 2/3)$

$$P(1/3 \le x \le 2/3) = \int_{1/3}^{2/3} \frac{x}{2} = \frac{1}{2}\int_{1/3}^{2/3} x = \frac{1}{2}\left[\frac{x^2}{2}\right]_{1/3}^{2/3}$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]$$

$$= \frac{1}{4}\left[\frac{4}{9} - \frac{1}{9}\right]$$

$$\frac{1}{4}\left[\frac{3}{9}\right] = \frac{3}{36} \quad 1/12$$

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Homework A coin is known to come up 3 times head as often as tail. The coin is tossed 3 time. Let X denote the number of heads that appear. Write the Probability distribution of X & find C.D.F of x.

$P(X) =$

Let X be a continious random variable with P.D.F

$$f(x) = \begin{cases} Kx^4 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{P.D.F} = \frac{d}{dx}\,\text{C.D.F}$$

(i) K?

$$\text{C.D.F} = F(x) = \int_{-\infty}^{\infty} f(x) \quad \text{according to def}^n$$

$$F(X) = \int_0^1 Kx^4 = 1$$

$$K\int_0^1 x^4 \Rightarrow K\frac{x^5}{5}\Big]_0^1 = \left(\frac{K}{5}\right) = 1 \qquad K = 1/5$$

$$f(x) = \begin{cases} \frac{x^4}{5} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(1/4 < X < 3/4) = \int_{1/4}^{3/4} \left(\frac{x^4}{5}\right) = \frac{1}{5}\left(\frac{x^5}{5}\right)_{1/4}^{3/4} \qquad 242/25600$$

$$\frac{1}{25}\Big[(x^5)\Big]_{1/4}^{3/4}$$

$$\frac{1}{25}\left(\frac{3}{4}\right)^5 - \left(\frac{1}{4}\right)^5 = \frac{243-1}{1024} \quad \frac{242}{1024}$$

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Expectation, mean & variance

Mathematical Expectation

Let x be any random variable and $\phi(x)$ be any function of x. Then expectation of $\phi(x)$ is denoted by $E(\phi(x))$ is defined by

$$E(\phi(x))$$

D.R.V: $\sum \phi(x) f(x)$

$$\int_{-\infty}^{\infty} \phi(x) f(x) d(x)$$

if $\phi(x) = x$

$E(X)$

DRV $\longrightarrow$ $E(X) = \sum_{x} x\,P(X)$

C.R.V

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$$

$$\text{variance} = E(X - \overline{X})^2$$
$$= E(X^2) - [E(X)]^2$$

1) Find the mean and variance of the probability distribution given by the following table.

x	1	2	3	4	5
$P(x)$	0.2	0.35	0.25	0.15	0.05

$E(X) = \sum x.P(X) = 1\times 0.2 + 2\times 0.35 + 3\times 0.25 +$
$4\times 0.15 + 5\times 0.05$

$E(X) = 2.5$

Mean = 2.5

$E(X^2) = \sum x^2.P(x) = (1)^2 0.2 + (2)^2 0.35 + (3)^2 0.25 +$
$16\times 0.15 + 25\times 0.05$
$= 7.5$

$var(X) = E(X^2) - [E(X)]^2$
$= 7.5 - (2.5)^2$
$= 7.5 - 6.25$
$= 1.25$

Thirteen cards are drawn simultaneously from a pack of 52 cards if ace count 1, Face cards 10 and are according to their donomination. find Expectation of total 13 cards

X	1	2	3	4	5	6	7	8	9	10	10	10	10
P(X)	1/13	1/13	1/13	-	"			"					1/13

$E(X) = \frac{1}{13}(1+2+3+4+5+6+7+8+9+10+10+10+10)$

$$P.D.F = \frac{d}{dx} C.D.F$$

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A continuous Random variable x has density function given by (P.D.F)

$$f(x) \begin{cases} 2e^{-2x} & x > 0 \\ 0 & o/w \end{cases}$$

find expected value & variance of x

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{0}^{\infty} x\,(2e^{-2x})\,dx$$

$$= 2 \cdot \int_{0}^{\infty} x\, e^{-2x}\, dx$$

$$\int_{0}^{\infty} x^n e^{-ax}\, dx = \frac{\Gamma n}{a^n} \quad \text{gamma fn formula}$$

$$\text{Mean} = \cancel{2} \left[\frac{\Gamma 2}{\cancel{2}} \right] = \frac{1}{2}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{0}^{\infty} x^2 (2e^{-2x})\, dx$$

$$= 2 \int_{0}^{\infty} x^2 e^{-2x}\, dx$$

$$\cancel{2x} \quad \frac{2\,\Gamma 3}{2^3} = \frac{2(2 \cdot 1)}{8} = 1/2$$

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Let x be a continuous random variable with p.d.f.

$$f_x(x) = \begin{cases} 2x^{-2} & \text{for } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

find $E(X)$ and $var(X)$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_x(X)\, dx = \int_{1}^{2} 2x^{-2} \times x\, dx$$

$$E(X) = \frac{2}{x^2} \times x\, dx = \frac{2}{x} = \int_{1}^{2} 2x^{-1}\, dx$$

$$E(X) = 2\log(x)\Big]_1^2$$

$$= 2\log(2) - 2\log(1)$$

$$2\log(2)$$

$$Var(x) = E(x^2) - \{E(x)\}^2$$

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Mean, Median and mode

X is a continious random variable then

Median M $\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x) = \frac{1}{2}$

1) given a P.D.F

$$f(x) = \begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

find mean, median mode

$$\text{Mean } E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 6x(1-x) \cdot x \cdot dx$$

$$\int_0^1 6x^2(1-x)\,dx$$

$$\int_0^1 6x^2 dx - \int_0^1 6x^3 dx$$

$$\left.\frac{6x^3}{3}\right]_0^1 - \left.\frac{6x^4}{4}\right]_0^1$$

$$\left[\frac{\cancel{6}(1)^3}{\cancel{3}} - \frac{6(0)^3}{3}\right] - \left[\frac{6(1)^4}{4} - \frac{6(0)^4}{4}\right]$$

$$2 - \left(\frac{\cancel{6}^{3}}{\cancel{4}_{2}}\right)$$

$$E(X) = \frac{4-3}{2} = \frac{1}{2}$$

$E(X)$ is $1/2$ or we can say
Mean of the given P.D.F is $1/2$

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$\mathbb{E}(x) = \frac{1}{2}$ variance $= \mathbb{E}(x^2) - \{\mathbb{E}(x)\}^2$

$\mathbb{E}(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot 6x(1-x)\,dx$

$\mathbb{E}(x^2) = \int_0^1 6x^3(1-x)\,dx$

$\mathbb{E}(x^2) = \int_0^1 6x^3\,dx - \int_0^1 6x^4\,dx$

$\mathbb{E}(x^2) = \left.\frac{6x^4}{4}\right]_0^1 dx - \left.\frac{6x^5}{5}\right]_0^1 dx.$

$\left(\frac{6}{4}\right) - \left(\frac{6}{5}\right)$

$\mathbb{E}(x^2) = \frac{30 - 24}{20}$

$\mathbb{E}(x^2) = 6/20 = 3/10$

$\mathrm{Var}(x) = \mathbb{E}(x^2) - \{\mathbb{E}(x)\}^2$

$= \frac{3}{10} - \left(\frac{1}{2}\right)^2$

$= \frac{3 \times 2}{10 \times 2} - \frac{1 \times 5}{4 \times 5}$

$= \frac{6}{20} - \frac{5}{20} = \frac{1}{20}$

$$\text{Median} = \frac{1}{2}$$

$$\text{Mode} = 3M - 2E(X)$$
$$= 3\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)$$

$$3 \times 1/2 - 2(1/2$$
$$\text{Mode} = \frac{3}{2} - \frac{2}{2} = \frac{1}{2}$$

$$\text{Median} \int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$$

$$\int_{0}^{M} 6x(1-x)\,dx = \frac{1}{2}$$

$$\int_{0}^{M} 6x - 6x^2 = \frac{1}{2}$$

$$\int_{0}^{M} \frac{6x^2 dx}{2}\Big] - \frac{6x^3}{3}\Big]_{0}^{M} = \frac{1}{2}$$

$$3x^2\Big]_{0}^{M} - 2x^3\Big]_{0}^{M} = \frac{1}{2}$$

$$3M^2 - 2M^3 = \frac{1}{2}$$

$$6M^2 - \cancel{4}MM^3 - 1 = 0$$
$$4M^3 - 6M^2 + 1 = 0$$
$$2M^2(2M - 3) + 1 = 0$$
$$(2M^2 \cancel{+1})\cancel{(2M - 3)} =$$

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Relationship between Join P.D.F & probability for statistically independent random variables X & Y

condition for statistically independent

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_X(x)\cdot f_Y(y)\,dx\,dy$$

$$P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2) = \int_{y_1}^{y_2} f_Y(y)\,dy + \int_{x_1}^{x_2} f_X(x)\,dx.$$

For only statistically independent variables.

Joint Probability mass function

if (x, y) is a two dimensional descrete random variable where $X = x_i$ and $Y = y_j$ then $P(X = x_i, Y = y_i) = P_{ij}$ is called probability mass function of (X, Y)

(i) $P_{ij} \geq 0$

(ii) $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$

Marginal Probability mass function of X

$P(X = x_i) = \sum p_{ij} = P_{11} + P_{12} + \cdots$

collection of pair (x_i, p_j) is called marginal pmf of X.

Two balls are selected at random from a box containing three ~~green~~ red, two green and four white. If X & Y are the number of red balls and green balls respectively included among the ~~red~~ two balls drawn from the box

$\boxed{\begin{array}{l} 3R \\ 2G \\ 4W \end{array}}$

$X = 0, 1, 2, 3$

$Y - 0, 1, 2$

Find Joint probability of X & Y

(2) Marginal P. of X & Y

(3) Conditional distn at X

3R
2G
4W

X and Y are probabilities

$X = 0, 1, 2, 3$

$Y = 0, 1, 2$

X \ Y	0	1	2	Total
0	1/6	2/9	1/36	15/36
1	1/3	1/6	0	18/36
2	1/12	0	0	3/36
3	0	0	0	0
Total	21/36	14/36	1/36	$\boxed{1}$

← Joint probability $P_{XY}(x, y)$

$$P(0,0) = \frac{4C_2}{9C_2} = \frac{(4\times3)/1\times2}{(9\times8)/1\times2}$$

$P(x, y)$

$$P(0,1) = \frac{4C_1 \cdot 2C_1}{9C_2}$$

$P(x, y)$

$$P(0,2) = \frac{2C_2 \cdot 4C_0}{9C_2} = \frac{1}{9\times8/1\times2} =$$

To (ii) Marginal P.m.f of X

X	$P_X(x)$
0	15/36
1	18/36
2	3/36
3	0

(iii) conditional P.m.f of X.

$P\left(\frac{X}{Y=1}\right)$

X	$P(X/Y=1)$
0	8/1[illegible]
1	
2	
3	

Suppose that 3 balls are randomly selected From urn containing 3 red, 4 white, 5 black balls. If X and Y denotes the number of red balls and the number of white balls chosen find the joint probability distribution of X and Y

3R	$\rightarrow X = 0, 1, 2, 3$
4W	$\rightarrow Y = 0, 1, 2, 3, 4$
5 BLK	

$$P(0,0,0) = \frac{5C_3}{12C_3} = \frac{1}{22}$$

P

$$P(0,0,1) = \cancel{5C_1 \cdot 5C_1} \cdot \frac{5C_2 \cdot 4C_1}{12C_3} = \frac{40}{220}$$

$$= \frac{4}{22}$$

$$P(0, \cancel{2}, 2) = \cancel{5C_2}\ \frac{5C_1 \cdot 4C_2}{12C_3} = \frac{30}{220}$$

$$P(0,3) = \frac{5C_1 \cdot 4C_3}{12C_3} = \frac{\cancel{5} 4 \times 4}{220}$$

$$\frac{4}{220}$$

$$P(1,0,0) = \frac{5C_2 \cdot 3C_1}{12C_3} = \frac{30}{220}$$

$$P(2,0) = \frac{5C_1 \cdot 3C_2}{12C_3}$$

$$P(1,1,0) = \frac{3C_1 \cdot 4C_1 \cdot 5C_1}{12C_3} = \frac{60}{220}$$

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X \ Y	0	1	2	3
0	10/220	40/220	30/220	4/220
1	30/220	60/220	18/220	0
2	15/220	12/220	0	0
3	1/220	0	0	0

problems:-
suppose that the following table represent join probability distribution of discrete random variable

$\longrightarrow$ Marginal for Y

Y \ X	1	2	3	Total
1	$\frac{1}{12}$	$\frac{1}{6}$	0	$\cancel{\frac{8}{12}}$ $\frac{1}{4}$
2	0	$\frac{1}{9}$	$\frac{1}{5}$	14/45
3	1/18	1/4	2/15	474/1080
Total	30/216	114/216	25/75	1

$\frac{1}{12}$

$\downarrow$ marginal for X

$\frac{1}{5}+\frac{2}{15}$

$\frac{15+10}{75}$

$\frac{1}{9}+\frac{1}{5}=\frac{5+9}{9\times5}$ $\quad - \quad \frac{1}{18}+\frac{1}{4}+\frac{2}{15}$

$\frac{15+8}{15\cdot4}+\frac{1}{18}$

$\frac{1}{12}+\frac{1}{18}$

$\frac{(15+8)18+15\cdot4}{15\cdot4\cdot18}$

$=\frac{18+12}{216}=\frac{30}{216}$

$\frac{1}{6}+\frac{1}{9}+\frac{1}{4}$

$\frac{9+6}{54}+\frac{1}{4}$

$\frac{15}{54}+\frac{1}{4}=\frac{114}{216}$ $\quad 15/54+1/4$

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suppose that the two-dimensional random variable (X, Y) has joint pdf

$$f(x,y) = kx(x-y) \quad 0 < x < 2 \quad -x < y < x$$
$$= 0 \quad \text{elsewhere.}$$

evaluate k

marginal pdf of X?

marginal pdf of Y?

$$g(x) = \int_{-x}^{x} kx(x-y)\,dy$$

$$h(y) = \int_{0}^{2} kx(x-y)\,dx$$

$$g(x) = \int_{-x}^{x} kx(x-y)\,dy \qquad kx^2 - kxy\Big]_2^0$$

$$\int_{-a}^{a} f(x)\,dx =$$

$$h(y) = \int_0^2 kx(x-y)\,de$$

$$h(y) = \int_0^2 kx(x-y)\,dx$$

$$h(y) = k\int_0^2 (x^2 - xy)\,dx$$

$$k\left[\frac{x^3}{3}\right]_0^2 - \frac{x^2 y}{2}\Big]_0^2$$

$$\frac{8}{3} - \frac{4y}{2}$$

$$\boxed{h(y) = (8/3 - 2y)k}$$

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Expectation of 2D R.V

For discrete R.V

$$E(X) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i P(x_i, y_i)$$

$$= \sum_{i=1}^{\infty} x_i \{ \sum_{j=1}^{\infty} P(x_i, y_i) \}$$

For continious R.V

$$E(X) = \int_{-\infty}^{\infty} x \cdot g(x)\, dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, \underbrace{h(y)}_{\text{marginal p.d.f}}\, dy$$

$$E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f(x,y)\, dx\, dy$$

$$E(X^2) = \int_{-\infty}^{\infty} \int_{-\infty} x^2 \cdot f(x,y)\, dx\, dy$$

property 1: Let (X, Y) be a two-dimensional random variable and suppose that X and Y are independent

$$E(XY) = E(X) \cdot E(Y)$$

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Theorem 2: if (X, Y) is a two-dimensional random variable if X and Y are independent

$$\boxed{V(X+Y) = V(X) + V(Y)}$$

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot g(x)\,dx \qquad g(x) = \int_0^2 x^2 + \frac{xy}{3}$$

$$g(x) = \int_0^1 x^2 + \frac{xy}{3}\,dy$$

$$\cancel{3x^2 + x}$$

$$\left. x^2 y + \frac{xy^2}{6} \right]_0^1 \qquad \frac{x^3}{3} + \frac{x^2 y}{3}$$

$$x^2 + \frac{x}{6} \qquad \frac{8}{3} + \frac{8y}{3}$$

$$6x^2 + x \qquad \frac{8(1+y)}{3}$$

$$x(6x+1)$$

__/__/__

properties of two 2D R.V

property:- Let (X, Y) be a two-dimensional random variable and suppose that X and Y are independent.

$$E(XY) = E(X) \cdot E(Y)$$

property If (X, Y) is a two-dimensional random variable, and if X and Y are independent then

$$V(X+Y) = V(X) + V(Y)$$

Conditional Expectation

(a) If (X, Y) is a two-dimensional continuous random variable we define the conditional expectation of X for given $Y=y$ as

$$E(X/y) = \int_{-\infty}^{+\infty} x\, g(x/y)\, dx$$

$$\Sigma(X/y_i) = \sum_{i=1}^{\infty} x_i\, p(x_i/y_i)$$

$$\rho = \frac{E(XY) - E(X)\cdot E(Y)}{\sqrt{V(X)\cdot V(Y)}}$$

marginal p.m.f of X

X \ Y	0	1	2	Total
0	$\frac{1}{6}$	$\boxed{\frac{2}{9}}$	$\frac{1}{36}$	$\frac{15}{36}$
1	$\frac{1}{3}$	$\frac{1}{6}$	6	$\frac{12}{36}$
2	$\frac{1}{12}$	0	0	$\frac{3}{36}$
3	0	0	0	0
Total	$\frac{21}{36}$	$\boxed{\frac{14}{36}}$	$\frac{1}{36}$	1

Marginal. p.m.f of X

X	$P_X(X)$
0	15/36
1	18/36
2	3/36
3	0

marginal p.m.f of Y

~~0~~	21/36
1	14/36
2	1/36
3	

conditional P.m.f of X given Y

X	$P\left(\frac{X}{Y=1}\right)$
0	2/9 / 14/36
1	
2	
3	

Continue

Binomial Distributions

i) All the trials are independent
ii) Number n of trial is finite
iii) The probability p of success is same of each trial

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
we can solve questions of small events
Doing this for 10 coins is time-consuming

$P(x) = {}^nC_x p^x q^{n-x}$

show that $P(x)$ is PMF ① $P(x) \geq 0$

$$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$$

$$q^n + {}^nC_1 p^1 q^{n-1} +$$
$$(p+q)^n$$
$$(1)^n = 1$$

Moment generating function (M.G.F)

$M_x(t) = (pe^t + q)^m$

~~charac~~ characteristic function (C.G.F)

$$\phi(t) = E(e^{ixt})$$
$$= (pe^{it} + q)^n$$

probability Generating function

$Z_x(t) = (zp + q)^n$

Mean and variance of Binomial Distribution

$p(x) = nC_x \, p^x q^{n-x}$

Mean $E(x) = np$ variance $= E(x^2) - [E(x)]^2$

$\sigma_x^2 = npq$

problem based on Binomial distribution

The probability that man aged 60 will live to 70 is 0.65 out of 10 men now aged 60 find

(i) at least 7 will upto 70

(ii) Exactly 9 will live upto 70

$P(x) = nC_x \, p^x q^{n-x}$

$p = 0.65 \quad q = 1 - 0.65 = 0.35 \quad n = 10$

$10C_x (0.65)^x (0.35)^{10-x}$

$p(X \geq 7) = p(7) + p(8) + p(9) + p(10)$

$10C_7 (0.65)^7 (0.35)^3 + 10C_8 (0.65)^8 (0.35)^2$
$+ 10C_9 (0.65)^9 (0.35)^1 + 10C_{10} (0.65)^{10} (0.35)^0$
$= 0.5189$

$p(X = 9) = 10C_9 (0.65)^9 (0.35)^1$

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Do Later!

out of 800 families with 5 children each How many families could be expected to have (i) 3 boys (ii) 5 girls (iii) Either 2 or 3 boys (iv) At least 2 girls.

$N = 800 \quad n = 5 \quad p = \frac{1}{2}, q = \frac{1}{2}$

$P(x) = {}^nC_x \, p^x q^{n-x}$

$5C \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$

$= 5C_x \left(\frac{1}{2}\right)^5 = 5C_x \left(\frac{1}{32}\right)$

$P(3) = 5C_3 \left(\frac{1}{32}\right) = \frac{5 \times 4}{1 \times 2} \times \frac{1}{3} = \frac{5}{16}$

4 coins are tossed 100 times and following were obtained. Fit a Binomial distribution for data and calculate Theoretical Frequency

No. of Head x	Frequency f	$x f$
0	5	0
1	29	29
2	36	72
3	25	75
4	5	20
	$\Sigma f = 100$	$\Sigma x f = 196$

$np = \bar{x} = \frac{\Sigma x f}{\Sigma f} = \frac{196}{100} = 1.96 \quad \boxed{n = 4}$

$4p = 1.96 \quad \boxed{p = 1.96/4 = 0.49}$

$q = 1 - 0.49 = 0.51$

$$P(X) = {}^{n}p_{x} \; {}^{n}C_{x} \, p^{x} q^{(n-x)}$$

$$P(X) = 4C_x \, (0.49)^x (0.51)^{4-x}$$

$$P(0) = 4C_0 \, (0.49)^0 (0.51)^4$$

$$P(0) = \frac{4!}{4! \times 0!} (0.51)^4 = 0.0676$$

$$P(1) = 4C_1 \, (0.49)^1 (0.51)^3 = 0.2594$$

$$P(2) = 4C_2 \, (0.49)^2 (0.51)^3 = 0.3747$$

$$P(3) = 4C_3 \, (0.49)^3 (0.51)^2 = 0.2900$$

$$P(4) = 4C_3 \, (0.49)^4 (0.51)^0 = \cancel{2.900} \; 0.05765$$

2. Poisson's Distribution

$$P(X) = \frac{e^{-\alpha} \alpha^{k}}{k!} \quad k = 0, 1, 2 \cdots \; \alpha > 0$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$$

_/_/_

Mean and variance of poisson Distribution

$$P(x) = \cancel{\alpha^x} \frac{\alpha^k e^{-\alpha}}{k!}$$

Mean $E(x) = \alpha$ $\quad V(x) = E(x^2) - [E(x)]^2 = \alpha$

Example Question:-

Given that 2% of the fuses manufactured by a firm are defective. Find probability that a box containing 200 fuses has:-

① At least 1 defective fuses.

$n = 200 \quad p = 0.02 \quad q = 1 - 0.2 = 0.8$

$\alpha = np$

$= 200 \times 0.02\cancel{6} = 4$

$$P(x) = \frac{\alpha^k e^{-\alpha}}{k!} = \frac{(4)^k e^{-4}}{\cancel{x!}\, k!}$$

$P(x \geq 1) = 1 - P(x < 1)$

$= 1 - P(0)$

$$P(x \geq 1) = 1 - \frac{4^0 e^{-4}}{0!} = 1 - e^{-4}$$

$P(x \geq 1) = 1 - e^{-4}$

$$\frac{\quad}{k!}$$

__/__/____

$n = 10 \quad P = 0.002 \quad q = 1 - 0.002$
$N = 10{,}000 \quad = 9.998$

$np = \alpha$
$10 \times 0.002 = 0.02 = \alpha$

$$P(x) = \frac{(0.02)^k e^{-0.02}}{k!}$$

$$P(0) = \frac{(0.02)^0 e^{-0.02}}{0!}$$

$P(0) = e^{-0.02}$
$P(0) = 0.9802$

No. of packets containing zero defective $10{,}000\, P(0) = 9802$ packets

$P = 0.01 \quad N = 200 \quad P(X > 2) = ?$
$q = 0.99$

$np = \alpha = 200 \times 0.01$
$\alpha = 2$

$$P(x) = \frac{\alpha^k e^{-\alpha}}{k!} = \frac{(2)^k e^{-2}}{k!}$$

$P(X > 2) = 1 - P(X \leq 2)$
$1 - (P(0) + P(1) + P(2))$

$$1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} \right]$$

$\underline{1 - 5e^{-2}}$

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Uniform Distribution

A continuous Random variable x is said to follow a continuous uniform distribution over a interval (a, b) if its p.d.f is given by

$$f(x) = \begin{cases} K & a < x < b \\ 0 & \text{elsewhere} \end{cases}$$

Here f(x) is P.D.F $\int_{-\infty}^{\infty} f(x)$

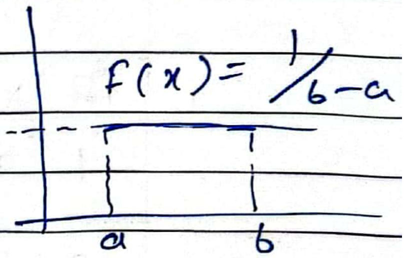
$$\int_a^b K\,dx = 1$$

$K[x]_a^b = 1$

$K(b-a) = 1$

$K = 1/b-a$

$$f(x) = \begin{cases} 1/b-a & a < x < b \\ 0 & \text{elsewhere} \end{cases}$$



$E(x) = \int_{-\infty}^{\infty} x.f(x)\,dx$ $f(x) \begin{cases} 1/b-a & a<x<b \\ 0 & \text{elsewhere} \end{cases}$

$$\frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{1}{\cancel{b-a}}\left(\frac{\cancel{(b-a)}(b+a)}{2}\right)$$

$$\boxed{\text{mean } E(x) = \frac{a+b}{2}.}$$

$E(x^2) = \frac{b^2+ab+a^2}{3}$

$\text{variance}(x) = E(x^2) - [E(x)]^2$

$$\frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$\boxed{\text{var}(x) = \frac{(a-b)^2}{12}}$$

Moment generating function of uniform distribution

$$M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$$

if x is uniformly distributed with mean 1 and variance 4/3 find $P(x<0)$

$\frac{a+b}{2} = 1$ $\quad$ variance $= \frac{(a-b)^2}{12} = \frac{4}{3}$

$a+b = 2$ $\quad$ $(a-b)^2 = \frac{4 \times \cancel{12}^{4}}{\cancel{3}}$

$(a-b)^2 = 16$

$a-b = \pm 4$

$$\begin{array}{l} a - \cancel{b} = 4 \\ a + \cancel{b} = 2 \\ \hline 2a = 6 \\ a = 3 \end{array} \qquad \begin{array}{l} a - \cancel{b} = -4 \\ + \; a + \cancel{b} = 2 \\ \hline 2a = -2 \\ a = -1 \end{array}$$

$\underline{a = -1, 3}$

Mean $E(X) = \int_a^b x f(x)\,dx = \frac{a+b}{2}$

Variance $= E(X^2) - [E(X)]^2 = \frac{(b-a)^2}{12}$

$$f(x) = \begin{cases} 1/b-a & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) \begin{cases} \frac{1}{b-a} & -1 < x < 3 \\ 0 & o/w \end{cases}$$

$$f(x) \begin{cases} \frac{1}{3-(-1)} \\ 0 \end{cases} = \begin{cases} 1/4 & -1 < x < 3 \\ 0 \end{cases}$$

$$P(x<0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-1}^{0} \frac{1}{4}\,dx$$

$$\frac{1}{4}(x)_{-1}^{0} = 1/4\,(0-(-1))$$

$$\boxed{P(X<0) = 1/4}$$

Normal - Distribution

A continous Random variable which has the following P.D.F

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0$$

$$X \sim N(\mu, \sigma)$$

Mean & variance of Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $\frac{x-\mu}{\sigma} = z$

$\boxed{E(x) = \mu}$ — derive by your own

variance

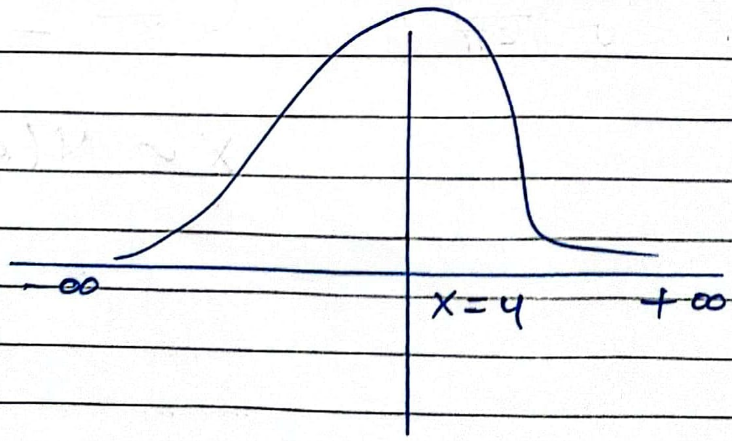
$\boxed{E(x^2) = \mu^2 + \sigma^2}$

$\boxed{\text{var}(x) = \sigma^2}$

Normal Distributions.

AREA UNDER THE CURVE

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

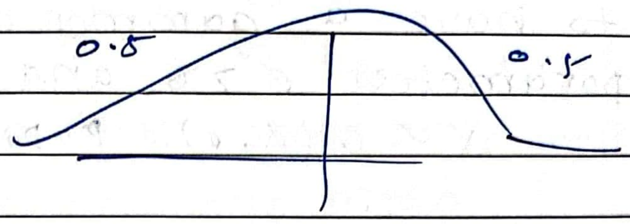


for S.D and mean Related Q

$\mu = 0$ $\quad f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$\sigma = 1$

$z = \frac{x-\mu}{\sigma}$ $\quad -\infty < z < \infty$



Normal distribution

Q) If the height of 300 students are normally distributed with mean 64.5 inches and standard deviation 3.3 inch How many students have height ① Less than 5 feet 9 inches

Solution $\mu = 64.5$ $\quad \sigma = 3.3$

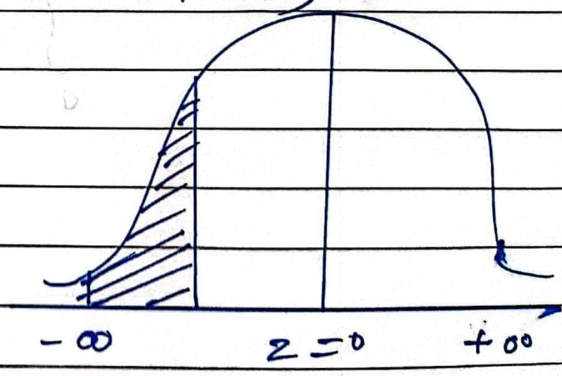
$$z = \frac{x-\mu}{\sigma} = \frac{x - 64.5}{3.3}$$

① $P(x < 60 \text{ inch}) = P\left(z < \frac{60 - 64.5}{3.3}\right)$

$P(z < -1.86)$ 3.3

$P(z < -1.86)$

$0.5 - P(-1.86 < z < 0)$



for Rate and avg determination Question

7. Gamma Distributions

A continious random variable X is said to have a gamma distribution with parameter $\alpha > 0$ and $r > 0$

$X \sim G(\alpha, r)$ P.D.F is given By:-

$$f(x) = \begin{cases} \dfrac{x^{r-1} e^{-\alpha x} \alpha^{r}}{\Gamma r} & x > 0, \; r > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Mean $E(X) = \dfrac{r}{\alpha}$ and $V(X) = \dfrac{r}{\alpha^2}$

$E(X) = 6 = \dfrac{r}{\alpha}$ and $V(X) = \dfrac{r}{\alpha^2} = 18$

P(

$6\alpha = r \qquad 18\alpha^2 = r$

$\alpha = \dfrac{r}{6} \qquad \alpha^2 = \dfrac{r^2}{18} \qquad \dfrac{\sqrt{2}}{\sqrt{r}}$

$r = 2 \quad \alpha = 1/3 \quad r = \dfrac{1}{}$

$$\int_0^8 (1/3) \frac{e^{-(1/3x)} (1/3x)}{\Gamma 2}$$

The chebyshev inequality

(i) Random variable x, with finite mean μ and variance σ^2

(ii) If variance is small, then x is unlikely to be too far from the mean.

chebyshev inequality $P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$

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Basic Definations and Related formulas

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$\int_0^{2a} f(x) = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ & \text{if } f(2a-x) - f(x) \end{cases}$$

$$\int_{-a}^{a} f(x) = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$$

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Joint Probability Density function or Join P.D.F

Join P.D.F is simply the P.D.F of two or more random variables.

The join Probability Density function of any two random variables X & Y can be defined as the partial Derivative of Join cummulative Distribution function w.r.t dummy variables X & Y

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

Properties of Joint PDF

property 1: The join P.D.F is non-negative

$$f_{XY}(x,y) \geq 0$$

Join PDF is a derivative of Join C.D.F which is also a non-negative function. Join P.D.F is always positive.

property 2: The joint P.D.F is continious everywhere as the joint C.D.F is continious & we know that it is derivative of join C.D.F

property 3: The total volume under the surface of join P.D.F is equal to unity

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1.$$

For two statistically independent random variables X & Y

$$\boxed{f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

proof :- $f_{XY}(x,y) = \dfrac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$

if the random variables X & Y are statistically independent then :-

$$f_{XY}(x,y) = \frac{d}{dx} F_{XY}(x,y) \frac{d}{dy} F_{XY}(x,y)$$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

Relation b/w probability & Join P.D.F

$$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$$

on ~~extre~~ extending this relation to the two random variables X & Y :-

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} F_{XY}(x,y)\,dx\,dy$$

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Moment Generating function (MGF)

$$M_X(t) = E(e^{tx}) \begin{cases} \sum_{i=0}^{\infty} e^{tx} p(x) & \text{Descrete Random Real variable} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx \end{cases}$$

$$M_X(t) = E(e^{tx}) = E\left[1 + \frac{(tx)}{1!} + \frac{(tx)^2}{2!} + \cdots\right]$$

$$= 1 + \boxed{E(X)} + \frac{t^2}{2}\boxed{E(X^2)} + \cdots$$

$E(X) \rightarrow$ coefficient of 't' in $M_X(t)$

$E(X^2) \rightarrow$ " " $\frac{t^2}{2}$ " "

$$M_X(t) = 0 + E(x) + \frac{2t}{2!}E(x^2) + \frac{3t^2}{3!}E(x^3) + \cdots$$

$$M_X'(t=0) = E(x)$$

$$M_X^2(t) = 0 + E(x^2) + \frac{3}{3!} 2t\, E(x^3) + \cdots$$

$$M_X^2(t=0) = E(x^2)$$

$E(X)$ is coeff

$E(X^2)$ " $\frac{t^2}{2!}$

$E(X^n)$ " $\frac{t^n}{n!}$

1. Suppose that x has p.d.f $f(x) = \frac{e^{-|x|}}{2}$, $-\infty < x < \infty$ then find $E(X)$ and $V(X)$ using m.g.f

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{e^{-|x|}}{2}\, dx$$

$$= \int_{-\infty}^{0} e^{tx} \frac{e^{x}}{2}\, dx + \int_{0}^{\infty} \frac{e^{tx} e^{-x}}{2}\, dx$$

$$= \int_{-\infty}^{0} \frac{e^{(1+t)x}}{2}\, dx + \int_{0}^{\infty} \frac{e^{-(1-t)x}}{2}\, dx$$

$$M_X(t) = \frac{1}{1+t^2} \quad -1 < x < 1$$

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2. Let x be a random variable taking the values 0, 1, 2 and $f(x) = ab^x$, $a, b > 0$ & $a + b = 1$ find mgf of x. If $E(X) = m_1$, $E(X^2) = m_2$ then S.T $m_2 = m_1(2m_1 + 1)$

$$M_X(t) = \sum_0^\infty ab^x e^{tx} = a\sum_0^\infty (be^t) = a\frac{1}{1-be^t} = \frac{a}{1-be^t}$$

WKT $\left(\frac{1}{1-x} = 1 + x + x^2 + \dots\right)$

$$E(X) = M'_X(0) = \frac{ab}{(1-b)^2}$$

$$E(X^2) = M^2_X(0) = \frac{(1+b)ab}{(1-b)^3}$$

Given:-

$E(X) = m_1$, $E(X^2) = m_2$

prove, $m_2 = m_1(2m_1 + 1)$

$$m_1(2m_1 + 1) = \frac{ab}{(1-b)^2}\left(2 \cdot \frac{ab}{(1-b)^2} + 1\right) = \frac{ab}{(1-b)^2}\left(\frac{2ab + 1 + b^2 - 2b}{(1-b)^2}\right)$$

$$= \frac{ab}{(1-b)^4}(2b(a-1) + 1 + b^2)$$

$$= \frac{ab}{(1-b)^4}(2b(-b) + 1 + b^2) = m_2$$

$$\frac{ab}{(1-1)^4}(-2b^2 + 1 + b^2) = m_2$$

$$\frac{ab}{\cancel{(1-b)^2}(1+b^2)}\cancel{(1-b^2)}$$

$$= \frac{ab}{(1+b^2)} = m_2$$

Moment generating function for binomial distribution.

Let X have the binomial distribution with probability distribution,

$$b(x/n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$
$$\forall\ x \in 0, 1, \dots n$$

Show that:-

a) $M(t) = (1-p+pe^{t})^n$

$$M_X(t) = \sum_{i=0}^{n} e^{tx} p(X)$$

$$= \sum_{i=0}^{n} e^{tx}\ {}^nC_x\ p^x q^{n-x}$$

$$= \sum_{i=0}^{n} {}^nC_x\ (pe^t)^x q^{n-x}$$

$$= (pe^t+q)^n$$

Moment generating function for normal distribution

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\boxed{-(x-\mu)^2/2\sigma^2}}$$

has $M(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

$$M_X(t) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{tx} \exp\left(-\frac{1}{2}\left[\frac{x-u}{\sigma}\right]^2\right)dx$$

let $(x-u)/\sigma = s$ thus $x = \sigma s + u$ and

$dx = \sigma ds$ Therefore

$$M_X(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp[t(\sigma s + u)]\, e^{-s^2/2} ds$$

$$= e^{tu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}[s^2 - 2\sigma t s]\right)ds$$

$$= e^{tu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}\left[(s-\sigma t)^2 - \sigma^2 t^2\right]\right\} ds$$

$$\cancel{M_X(t) = e^{tu+\sigma^2 t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}[s-\sigma t]^2\right)ds}$$

let $s - \sigma t = v$ then $ds = dv$

$$\frac{e^{tu}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}[-2t\sigma s + s^2]}ds$$

$$= \frac{e^{tu}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}[\{s - t\sigma\}^2 - (t\sigma)^2]}ds$$

$$= e^{tu + \frac{1}{2}(t\sigma)^2}\int_{-\infty}^{+\infty} e^{-1/2[s-t\sigma]^2} ds$$

$$= \frac{e^{tu + \frac{(t\sigma)^2}{2}}}{\sqrt{2\pi}} \times 2\int_0^{\infty} e^{-1/2 v^2} dv$$

sub $1/2\, v^2 = x$ $\quad v^2 = 2x$

$v = \sqrt{2x}$

$dv = \sqrt{2}\,\frac{1}{2\sqrt{x}}\, dx$

$$\frac{e^{tu + \frac{(t\sigma)^2}{2}}}{\sqrt{2\pi}} \times \cancel{2}\int_0^{\infty} e^{-x}\frac{\sqrt{2}}{2}x^{-1/2}dx$$

$$\boxed{M_X(t) = e^{tu + \frac{(t\sigma)^2}{2}}} = \frac{e^{tu + \frac{(t\sigma)^2}{2}}}{\sqrt{\pi}}$$

properties of moment generating function.

$Y = \alpha X + \beta$

$M_Y(t) = E(e^{Yt})$

$= E(e^{(\alpha X + \beta)t})$

$= E(e^{(\alpha t)X + \beta t})$

$E(CX) = E(e^{(\alpha t)X} + e^{\beta t})$

$= e^{\beta t} E(e^{(\alpha t)X})$

$= e^{\beta t} M_X(\alpha t)$

$\underbrace{\quad}_{\text{independent}}$

$Z = X + Y$

$M_Z(t) = M_X(t) M_Y(t)$

$\underbrace{\quad}_{\text{independent}}$

$Z = \alpha X + \beta Y$

$M_Z(t) = M_X(\alpha t) \cancel{*} M_Y(\beta t)$

if $\beta = 0$

i.e $Y = \alpha X$

$M_Y(t) = e^{\alpha X t + \frac{1}{2}\alpha^2 \sigma^2}$

$M_Z(t) = e^{(\alpha u_1 + \beta u_2)t + \frac{1}{2}(\alpha^2 \sigma_1^2 + \beta^2 \sigma_2^2)t^2}$

$Z = \underbrace{X + Y}_{\text{Indept}}$ $\qquad$ $Z = \underbrace{X + Y}_{\text{Indept}}$

$X, Y \sim P(\alpha)$ $\qquad$ $X, Y \sim \chi^2(n)$

$M_Z(t) = M_X(t) . M_Y(t)$

$= e^{\alpha_1(e^t - 1)} . e^{\alpha_2(e^t - 1)}$

$= e^{(\alpha_1 + \alpha_2)(e^t - 1)}$

$M_Z(t) = M_X(t) M_Y(t)$

$= (1-2t)^{-n_1/2} (1-2t)^{-n_2/2}$

$= (1-2t)^{-1/2(n_1 + n_2)}$

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1. Find the mgf of random variable X which uniformly distributed with an interval $(-a, a)$ and hence find $E(X^{2n})$

Solution :-

we know, $M_X(t) = \frac{1}{t(b-a)} (e^{bt} - e^{at})$

$$X \sim U(-a, a) = \frac{e^{at} - e^{-at}}{(a+a)t}$$

$$= \frac{e^{at} - e^{-at}}{(2a)t}$$ by expanding

$$E(X^{2n}) = \text{coefficient of } \frac{t^{2n}}{(2n)!} = \frac{a^{2n}}{(2n+1)}$$

$$M_X(t) = \frac{1}{2at}\left[\left\{1 + (at) + \frac{(at)^2}{2!} + \dots\right\} + \left\{1 - \frac{(at)^2}{2!} - \frac{(at)^3}{3!} - \right.\right.$$

$$= \frac{1}{2at}\left[2a + \frac{2(at)^3}{3!} + \frac{2(at)^5}{5!} + \dots\right]$$

$$1 + \frac{(at)^3}{3!} + \frac{(at)^4}{5!} + \dots + \dots$$

$$\text{coeff of } (t^{2n}) \rightarrow \frac{a^{2n}}{(2n+4)}$$

2. If X is normally distributed with mean 4 and variance, σ^2 then show that $E(X-4)^{2n} = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}$

$$M_Y(t) = 1 + \left(\frac{1}{2}\sigma^2 t^2\right) + \frac{1}{2!}\left(\frac{1}{2}\sigma^2 t^2\right)^2 +$$

$$\dots + \frac{1}{k!}\left(\frac{1}{2}\sigma^2 t^2\right)^k + \dots$$

_/_/_

coefficient of t^{2n} $\rightarrow \frac{1}{n!}\left(\frac{1}{2}\sigma^{2}\right)^{n}$

Coefficient of $\frac{t^{2n}}{(2n)!} \rightarrow \frac{(2n)!}{n!} \times \frac{1}{2^{n}} \times \sigma^{2n}$

$$\frac{(2n)(2n-1)(2n-2)\cdots 3.2.1}{2^{n}\,(2n-2)(2n-4)\cdots 6.4.2} * \sigma^{2n}$$

$$= 1.3.5\ldots\ldots(2n-5)(2n-3)(2n-1) * \sigma^{2n}$$

1. Let $X_1 \sim \chi^2_2(3)$, $X_2 \sim \chi^2(5)$ and $Z = X_1 + X_2$ where X_1 and X_2 independent random variables. Find $M_Z(t)$ and $V(t)$ and pdf of Z.

Solution

$X_1 \sim \chi^2(3)$ gives $M_{X_1}(t) = (1-2t)^{-3/2}$

$X_2 \sim \chi^2(5)$ gives $M_{X_2}(t) = (1-2t)^{-5/2}$

$M_Z(t) = M_{X_1}(t)\, M_{X_2}(t) = (1-2t)^{-8/2} \sim \chi^2(8)$

Therefore, $V(Z) = 2n = 2(8) = 16$ where $n = 8$

Chi-square distribution $X \sim \chi^2(n)$	$f(x) = \begin{cases} \dfrac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{\Gamma(n/2)\, 2^{n/2}} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

$$f(z) = \begin{cases} \dfrac{z^{n/2-1} e^{-z/2}}{2^{n/2}\,\Gamma(n/2)}, & z > 0 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} \dfrac{z^{3} e^{-\frac{z}{2}}}{2^{4}\,\Gamma 4}, & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

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