LINEAR ALGEBRA

27/10/22
Thursday

BINARY OPERATION:
For a non-empty set G, a binary operation $*$ on G is a mapping from $G \times G$ into G.

GROUP:
A non-empty set G, together with a binary operation $*$ is called a group if the algebraic system $(G, *)$ satisfies the following axioms:

1. **Associative Axiom**: $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
2. **Identity Axiom**: There exists an element $e \in G$ such that
$$a * e = e * a \quad \forall a \in G$$
3. **Inverse Axiom**: For every element $a \in G$, there corresponds an element $a^{-1} \in G$ such that
$$a * a^{-1} = a^{-1} * a = e$$

Moreover, G is said to be an **albelian group** if $a * b = b * a$ $\forall a, b \in G$

Examples:
1. Set of all integers w.r.t addition is an abelion group
2. Set of all 2×2 integers with real or complex entries is a group w.r.t matrix addition.

28/10/22
Friday

VECTOR:
Vector is a physical quantity which has both direction and magnitude
Vector may originate at any point in space and terminate at any point.

Scalar Multiplication:
Let a be any vector such that $a = (a_1, a_2)$ and α be any Scalar, then $\alpha a = (\alpha a_1, \alpha a_2)$

Example ①: If $a = (2,3)$ $\quad 6a = (12, 18)$
② : If $a = (1,-1)$ $\quad -4a = (-4, 4)$

Graphical Illustration
The rule used in obtaining magnitude and direction of a single force which replaces the original two forces states that

"If a, b are the original two forces, then the single force c, which we shall call the sum of a, b, is the diagonal of the parallelogram with sides a and b."

$a = (a_1, a_2) \quad b = (b_1, b_2)$
$c = a + b = (a_1 + b_1, a_2 + b_2)$

UNIT VECTOR:

→ The vector $(1,0)$ lies along the x_1-axis and has length one.
→ The vector $(0,1)$ lies along the x_2-axis and has length one.
→ The vectors $(1,0)$ and $(0,1)$ are called unit vectors and will be denoted by e_1, e_2 respectively.

$e_1 = (1,0,0)$ $e_2 = (0,1,0)$ $e_3 = (0,0,1)$

e_i is defined with unity as value of its i^{th} component and all other components zero

VECTOR SPACE

Vector space is a collection of vectors which is closed under the operations of addition and multiplication by a scalar. It is denoted by V_n.

COMPONENTS OF VECTOR:

An n-component vector **a** is an ordered n-tuple of numbers written as a row $(a_1, \ldots, a_n)$ or as a column $(a_1, \ldots, a_n)^t$ where a_i, $i = 1, 2, \ldots n$ are assumed to be real numbers and are called the components of the vector.

NULL VECTOR:

Null vector is a vector all of whose components are zero.

SUM VECTOR:

The sum vector is a column vector having all n elements equal to one.

Let a and b be two n-component vectors. Then a and b are equal if and only if $a_i = b_i$ for each i

Given two n-component vectors a, b then $a \geq b$ means $a_i \geq b_i$ for $i = 1, 2, \ldots, n$ and $a \leq b$ means $a_i \leq b_i$ for $i = 1, 2, \ldots, n$.

SCALAR PRODUCT:

If two vectors $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$ are multiplied (scalar product); then the magnitude of the product is $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$a = a_1 i + a_2 j + a_3 k$$
$$b = b_1 i + b_2 j + b_3 k$$
$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

FIELD

A non-empty set F is said to be a field if ther exists two binary operations $+$ and $\cdot$ on F such that

1. $(F,+)$ is an abelian group
2. $(F-\{0\},\cdot)$ is a multiplicative group
   ↳ abelion group w.r.t multiplication
3. For any $a,b,c \in F$,
   $(a+b)c = ac+bc$
   $a(b+c) = ab+ac$

Examples => ①: Set of real numbers with usual addition and multiplication is a field.

②: Set of all complex numbers with addition and multiplication of complex numbers is also field.

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VECTOR SPACE :

A non-empty set V is said to be a vector space over a field F if it is satisfying the following:

1. $(V,+)$ is an abelian group
2. V is closed under scalar multiplication (ie. for every $\alpha \in F$; $v \in V$ we have $\alpha v \in V$) and also this scalar multiplication satisfies the following conditions:

$u, v, w \in V$ (vector space)
$\alpha, \beta \in F$ (field)

- $\alpha(v+w) = \alpha v + \alpha w$
- $(\alpha+\beta)v = \alpha v + \beta v$
- $\alpha(\beta v) = (\alpha\beta)v$
- $1 \cdot v = v$ for all $\alpha, \beta \in F$ (here 1 is the identity of F w.r.t multiplication)

SUBSPACE :

Let V be a vector space over F and $W \subseteq V$. Then W is called a subspace of V if W is a vector space over F under the same operation.

Theorem: Let V be a vector space over F and $W \subseteq V$. Then the following two conditions are equivalent.

1. W is a subspace of V
2. $\alpha, \beta \in F$ and $w_1, w_2 \in W \implies \alpha w_1 + \beta w_2 \in W$

LINEAR COMBINATION :

Suppose V is a vector space over F $V_i \in V$ and $\alpha_i \in F$ for $1 \le i \le n$

then $\alpha_1, \alpha_2 \ldots \alpha_n$ is the linear combination over F of $V_1, V_2 \ldots\ldots V_n$

Ex: Let $a = (2, 3, 4, 7)$ $b = (0, 0, 0, 1)$ $c = (1, 0, 1, 0)$

then the vector $d = (5, 3, 7, 9)$ is a linear combination of the vectors a, b, c as follows

$$(5, 3, 7, 9) = 1(2, 3, 4, 7)$$
$$= 2(0, 0, 0, 1)$$
$$= 3(1, 0, 1, 0)$$

EUCLEDIAN SPACE :

Let the field k be the set R of real numbers and let the vector space V be a Eucledian space R^3

Consider the vectors $e_1 = (1, 0, 0)$ $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$

then any vector in R^3 is a linear combination of e_1, e_2, e_3

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LINEARLY INDEPENDENT SETS :

Let $S = \{V_1, V_2, \ldots\ldots, V_n\}$ be the set of all vectors in V, then $V_1, V_2 \ldots\ldots V_n$ are said to be linearly independent if

$c_1V_1 + c_2V_2 + \cdots + c_nV_n = 0$ implies $c_1 = c_2 = \cdots = c_n = 0$

LINEARLY DEPENDENT

Let $S = \{V_1, V_2 \ldots\ldots, V_n\}$ be the set of all vectors in V (vector space) $V_1, V_2, \ldots\ldots, V_n$ are said to be linearly dependent if there exist scalar $c_1, c_2, \ldots\ldots, c_n$ not all zeroes in F, such that

$$c_1V_1 + c_2V_2 + \cdots + c_nV_n = 0$$

Q) Consider $S = \{(1,0), (1,1), (0,1)\}$. Prove if S is linearly independent or linearly dependent.

$V_1 = (1,0)$ $V_2 = (1,1)$ $V_3 = (0,1)$

$$c_1V_1 + c_2V_2 + c_3V_3 = 0$$

$$(1)(1,0) + (-1)(1,1) + (1)(0,1) = 0$$

$c_1 = 1$ $c_2 = -1$ $c_3 = 1$

$\therefore$ Linearly dependent

Q) Test whether the set of vectors $S = \{(1,2), (2,5)\}$ is linearly independent in R^2 or not.

Let $S = \{\underset{v_1}{(1,2)}, \underset{v_2}{(2,5)}\}$

$c_1 v_1 + c_2 v_2 = 0$
$c_1, c_2 \in R$

$c_1(1,2) + c_2(2,5) = (0,0)$

$(c_1, 2c_1) + (2c_2, 5c_2) = (0,0)$

$c_1 + 2c_2 = 0 \qquad 2c_1 + 5c_2 = 0$

$|\text{co-eff of matrix}| \qquad \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 \neq 0$ in R^2

$\therefore$ Dependent

Q) Test if the set of vectors $S = \{(1,0,1), (1,2,5), (1,-1,1)\}$ is linearly independent in R^3 or not

$S = \{(1,0,1), (1,2,5), (1,-1,1)\}$

$c_1(1,0,1) + c_2(1,2,5) + c_3(1,-1,1) = 0$

$c_1 + c_2 + c_3 = 0$

$2c_2 - c_3 = 0$

$c_1 + 5c_2 + c_3 = 0$

$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 5 & 1 \end{vmatrix} = 1(7) - 1(1) + 1(-2) = 3 \neq 0$

$\therefore$ Linearly Dependent

SPANNING SET [$L(S)$]:

A set of vectors $S = \{v_1, v_2, \dots v_n\}$ is said to be spanning set of V if any vector in V can be expressed as a linear combination of elements of S

The spanning set of S is denoted by $L(S)$.

Theorem:

Let V be a vector space over the field F and $S \subset V$ then $L(S)$ is a subspace of V

7/11/22 **BASIS**:

A set of vectors $S = \{v_1, v_2, \dots, v_n\}$ in V is said to be basis of V if:

1. S is linearly independent
2. S spans V spans = spanning set of

DIMENSION:

The number of elements in the basis of a vector space V is called the dimension of vector space. It is denoted by $\dim(V)$.

Q) Prove that $S = \{(1,1),(2,3)\}$ form a basis of R^2.

Step ①: Proving that S is linearly independent

Let $V_1 = (1,1)$ and $V_2 = (2,3)$

$(x, y) \in R^2$ We know that R^2 is a vector space ove R

$\in R$

$c_1 V_1 + c_2 V_2 = \vec{0} = (0,0)$

$c_1(1,1) + c_2(2,3) = (0,0)$

$(c_1, c_1) + (2c_2, 3c_2) = (0,0)$

$|\text{co-eff of matrix } *| = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$\therefore$ S is linearly independent

Step ②: Proving that S spans V

Let $(x,y) \in R^2$

$\in R$

$(x,y) = c_1 V_1 + c_2 V_2$

$= c_1(1,1) + c_2(2,3)$

$= (c_1, c_1) + (2c_2 - 3c_2)$

$x = c_1 + 2c_2 \rightarrow ①$

$y = c_1 + 3c_2 \rightarrow ②$

② − ①

$y - x = c_1 + 3c_2 - c_1 - 2c_2 = c_2$

$\therefore c_2 = y - x \in R$

c_2 in ① $\Rightarrow$ $x = c_1 + 2(y-x)$

$x = c_1 + 2y - 2x$

$c_1 = 3x - 2y \in R$

Q) Prove that $S = \{(1,1,1), (1,1,0), (1,0,0)\} \subseteq R^3$ form a basis for R^3. Express $(2,-3,5)$ interms of basis elements in S.

Step ①: Proving that S is linearly independent.

Let $V_1 = (1,1,1)$ $V_2 = (1,1,0)$ $V_3 = (1,0,0)$

We know that R^3 is a vector space over R.

$c_1 V_1 + c_2 V_2 + c_3 V_3 = \vec{0} = (0,0,0)$

$c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0) = (0,0,0)$

$c_1 + c_2 + c_3 = 0$

$c_1 + c_2 = 0$

$c_1 = 0$

$\left|\begin{matrix}\text{co-eff of} \\ \text{matrix } *\end{matrix}\right| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1(0) - 1(0) + 1(0-1)$

$= -1$

$\therefore$ S is linearly independent

Step ②: Verifying if S spans R^3 or not

Let $(x,y,z) \in R^3$

$\in R$

$(x,y,z) = c_1 v_1 + c_2 v_2 + c_3 v_3$
$= c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$x = c_1 + c_2 + c_3$ $\quad y - z = c_2$ $\quad c_3 = x - y$
$y = c_1 + c_2$ $\quad \therefore c_2 \in R$ $\quad \therefore c_3 \in R$
$z = c_1 \in R$

$\therefore (x,y,z) = z v_1 + (y-z) v_2 + (x-y) v_3$
$\therefore S$ spans R^3
$\therefore S$ is basis of R

Step ③: Expressing $(2,-3,5)$ in terms of basis elements ins.
$(2,-3,5) \in R^3$
$(2,-3,5) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$
$= 5(1,1,1) + (-8)(1,1,0) + 5(1,0,0)$
$c_1 = 5 \quad c_2 = -8 \quad c_3 = 5$

Q) We know that R^n is the vector space over R. Let $S = \{e_1, e_2 \dots e_n\}$ (ie) $S = \{e_1 = (1,0,\dots 0), e_2 = (0,1,0 \dots 0) \dots\dots e_n = (0,0,\dots,1)\}$ Find if S will form a basis for R.

$e_1 = (1,0,\dots,0)$ $\quad e_2 = (0,1,\dots,0)$ $\quad e_n = (0,0,\dots 1)$

$c_1 e_1 + c_2 e_2 + \dots + c_n e_n = \vec{0} = (0,0,\dots,0)$
$c_1(1,0,0,\dots 0) + c_2(0,1,0,0,\dots 0) + \dots + c_n(0,0,\dots 1) = (0,\dots 0)$
$\therefore c_1 = c_2 = c_3 = \dots = c_n = 0$
$\therefore S$ is linearly independent

Let $(\underbrace{x_1, x_2, \dots, x_n}_{\in R}) \in R^n$

Q) Test if vectors $S = \{(1,1,2), (1,2,3), (0,-1,1)\}$ forms a basis for R^3 or not. If so, express vector $(1,1,1)$ in terms of basis elements.

Let $(x,y,z) \in R^3$, $x,y,z \in R$

$(x,y,z) = c_1 v_1 + c_2 v_2 + c_3 v_3$
$= c_1(1,1,2) + c_2(1,2,3) + c_3(0,-1,1)$

$x = c_1 + c_2$
$y = c_1 + 2c_2 - c_3$
$z = 2c_1 + 3c_2 + c_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix} = |A|$$

$|A| = 2$

By using cramers rule,

$$C_1 = \frac{1}{|A|}\begin{vmatrix} x & 1 & 0 \\ y & 2 & -1 \\ z & 3 & 1 \end{vmatrix} = \frac{5x-y-z}{2}$$

$$C_2 = \frac{1}{2}\begin{vmatrix} 1 & x & 0 \\ 1 & y & -1 \\ 2 & z & 1 \end{vmatrix} = \frac{y+z-3x}{2}$$

$$C_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & x \\ 1 & 2 & y \\ 2 & 3 & z \end{vmatrix} = 1(2z-3y) - 1(z-2y) + x(3-4) = z-y-x$$

$$= 2z - 3y - z + 2y + (-x)$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix} = 2$$

$\therefore$ S forms a basis for R^3

$(1,1,1) = c_1(1,1,2) + c_2(1,2,3) + c_3(0,-1,1)$

$c_1 = \frac{5-1-1}{2} = \frac{3}{2}$ $\quad c_2 = \frac{1+1-3}{2} = \frac{-1}{2}$ $\quad c_3 = (1-1-1) = -1$

$\therefore c_1 = \frac{3}{2}$ $\quad c_2 = \frac{-1}{2}$ $\quad c_3 = -1$

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Test if the set of vectors $S = \{(1,1,1), (1,0,1), (1,1,0)\}$ form a basis of R^3 or not. If so, express the vector $(1,2,3)$ in terms of basis elements.

Let $(x,y,z) \in R^3$

$\in R$

$(x,y,z) = c_1V_1 + c_2V_2 + c_3V_3$

$(x,y,z) = c_1(1,1,1) + c_2(1,0,1) + c_3(1,1,0)$

$x = c_1 + c_2 + c_3$ $\quad c_1 = y+z-x$ $\quad \therefore c_1 \in R$

$y = c_1 + c_3$ $\quad c_2 = x-y$ $\quad \therefore c_2 \in R$

$z = c_1 + c_2$ $\quad c_3 = x-z$ $\quad \therefore c_3 \in R$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} =$$

$(1,2,3) = c_1(1,1,1) + c_2(1,0,1) + c_3(1,1,0)$

$1 = c_1 + c_2 + c_3$ $\quad c_1 = 2+3-1 = 4$

$2 = c_1 + c_3$ $\quad c_2 = 1-2 = -1$

$3 = c_1 + c_2$ $\quad c_3 = 1-3 = -2$

$\therefore c_1 = 4$ $\quad c_2 = -1$ $\quad c_3 = -2$

INNER PRODUCT:

Let V be a vector space. An inner product on V is a rule that assigns to each pair $v, w \in V$ a real number $\langle v, w \rangle$ such that, for all $u, v, w \in V$ and $\alpha \in R$

1. $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$
2. $\langle v, w \rangle = \langle w, v \rangle$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$

ORTHOGONAL VECTORS:

The set of n-dimensional vectors $\{v_1, v_2, \dots v_n\}$ is said to be orthogonal if $\langle v_i, v_j \rangle = v_i \cdot v_j = 0$ if $i \neq j$ (ie)

$$S = \{v_1, v_2, \dots, v_n\}$$
$$\langle v_i, v_j \rangle = 0 \text{ if } i \neq j$$
$$\langle v_i, v_j \rangle = \langle v_j, v_i \rangle$$

ORTHONORMAL VECTORS:

The set of n-dimensional vectors $\{v, v_2, \dots, v_n\}$ is said to be orthonormal if $\langle v_i, v_j \rangle = v_i \cdot v_j = 0$ if $i \neq j$ and $\langle v_i, v_i \rangle = ||v_i||^2 = 1$ (ie)

1. S is orthogonal
2. $\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Note: For any n-dimensional vector v_1, we have, the norm of v_1 is defined as $||v_1|| = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{v_1 \cdot v_1}$

Q) Show that the set of vectors $B = \{v_1 = (3, 0, 4), v_2 = (-4, 0, 3), v_3 = (0, 1, 0)\}$ is an orthogonal set.

$$B = \{\underset{v_1}{(3, 0, 4)}, \underset{v_2}{(-4, 0, 3)}, \underset{v_3}{(0, 1, 0)}\}$$

$\langle v_1, v_2 \rangle = v_1 \cdot v_2 = -12 + 0 + 12 = 0 = \langle v_2, v_1 \rangle$

$\langle v_2, v_3 \rangle = v_2 \cdot v_3 = 0 + 0 + 0 = 0 = \langle v_3, v_2 \rangle$

$\langle v_1, v_3 \rangle = v_1 \cdot v_3 = 0 + 0 + 0 = 0 = \langle v_3, v_1 \rangle$

Here B is an orthogonal set
$\langle V_1, V_1 \rangle = 9+0+16 = 25 \neq 1$
$\therefore$ B is not orthonormal.

Q) Show that the set of vectors $B = \{ V_1 = (\frac{3}{5}, 0, \frac{4}{5}),\ V_2 = (\frac{-4}{5}, 0, \frac{3}{5}),$
$V_3 = (0,1,0) \}$

$\langle V_1, V_2 \rangle = V_1 \cdot V_2 = \frac{-12}{25} + 0 + \frac{12}{25} = 0 = \langle V_2, V_1 \rangle$
$\langle V_2, V_3 \rangle = V_2 \cdot V_3 = 0+0+0 = 0 = \langle V_3, V_2 \rangle$
$\langle V_1, V_3 \rangle = V_1 \cdot V_3 = 0+0+0 = 0 = \langle V_3, V_1 \rangle$
$\therefore$ B is orthogonal
$\langle V_1, V_1 \rangle = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$
$\langle V_2, V_2 \rangle = \frac{16}{25} + 0 + \frac{9}{25} = \frac{25}{25} = 1$
$\langle V_3, V_3 \rangle = 0+1+0 = 1$
$\therefore$ B is orthonormal

GRAM - SCHMIDT ORTHOGONALIZATION PROCESS:

Construction of orthonormal set from a linearly independent set of vectors

Consider a linearly independent set of n-dimensional vectors
$S = \{a_1, a_2, \ldots\ldots, a_n\}$

1. Take $V_1 = a_1$ then $u_1 = \frac{V_1}{\|V_1\|}$ where $\|V_1\| = \sqrt{\langle V_1, V_1 \rangle} = \sqrt{V_1 \cdot V_1}$
2. $V_2 = a_2 - \langle a_2, u_1 \rangle u_1$ then $u_2 = \frac{V_2}{\|V_2\|}$
3. $V_3 = a_3 - \langle a_3, u_1 \rangle u_1 - \langle a_3, u_2 \rangle u_2$

By continuing the same process, we will get
Orthonormal set $= \{u_1, u_2, \ldots\ldots, u_n\}$

Q) Construct an orthonormal set of vectors from the given set of linear independent vectors of $S = \{(1,2), (2,3)\} \subseteq R^2$

$S = \{ \underset{a_1}{(1,2)}, \underset{a_2}{(2,3)} \}$

$V_1 = a_1 \quad u_1 = \frac{(1,2)}{\|(1,2)\|} = \frac{(1,2)}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \quad \therefore u_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$V_2 = a_2 - \langle a_2, u_1 \rangle u_1 = (2,3) - \langle (2,3), (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) \rangle (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = (2,3) - (\frac{8}{5}, \ldots$
$= \left(\frac{2}{5}, \frac{-1}{5}\right) \quad u_2 = \frac{V_2}{\|V_2\|} = \frac{(\frac{2}{5}, -\frac{1}{5})}{1/\sqrt{5}} = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

Q) Using Gram-Schmidt process, construct an orthonormal set of vectors from the given L.I set of vectors $B = \{(1,1,1), (-1,0,-1), (-1,2,3)\}$

$$B = \{\underset{a_1}{(1,1,1)},\ \underset{a_2}{(-1,0,-1)},\ \underset{a_3}{(-1,2,3)}\}$$

$$V_1 = a_1 = (1,1,1) \qquad u_1 = \frac{(1,1,1)}{\|(1,1,1)\|} = \frac{(1,1,1)}{\sqrt{1+1+1}} = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$$

$$V_2 = a_2 - \langle a_2, u_1\rangle u_1 \qquad V_2 = (-1,0,-1) - \langle(-1,0,-1), (\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3})\rangle (\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3})$$

$$V_2 = (-1,0,-1) - \left(-\frac{2}{\sqrt3}\right)\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) = (-1,0,-1) + \left(\frac23, \frac23, \frac23\right)$$

$$V_2 = \left(-\frac13, \frac23, -\frac13\right) \qquad u_2 = \frac{\left(-\frac13, \frac23, -\frac13\right)}{\sqrt{\frac{1+4+1}{9}}} = \left(\frac{-1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{-1}{\sqrt6}\right)$$

$$V_3 = a_3 - \langle a_3, u_1\rangle u_1 - \langle a_3, u_2\rangle u_2$$

$$V_3 = (-1,2,3) - \langle(-1,2,3), (\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3})\rangle \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3} - \langle(-1,2,3), (\tfrac{-1}{\sqrt6}, \tfrac{2}{\sqrt6}, \tfrac{-1}{\sqrt6})\rangle (u_2)$$

$$V_3 = (-1,2,3) - \left(\frac{4}{\sqrt3}\right)\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) - \left(\frac{2}{\sqrt6}\right)\left(\frac{-1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{-1}{\sqrt6}\right)$$

Construct an orthonormal basis from the following vectors using Gram-Schmidt process. $S = \{a_1, a_2, \ldots, a_n\}$

If magnitude of each vector is unity, then the basis is said to be orthonormal

Construction of orthonormal set:

$$S = \{a_1, a_2, \ldots, a_n\} \to L$$

$$V_1 = a_1 \qquad u_1 = \frac{V_1}{\|V_1\|}$$

$$V_2 = a_2 - \langle a_2, u_1\rangle u_1 \qquad u_2 = \frac{V_2}{\|V_2\|}$$

$\vdots$

Q1	Q2	Q3
$a_1 = (1,0,3)$	$(1,2,1)$	$(0,0,-1,0)$
$a_2 = (2,2,0)$	$(1,0,1)$	$(1,0,1,0)$
$a_3 = (3,1,2)$	$(3,1,0)$	$(1,1,1,1)$

$$S = \{\underbrace{a_1, a_2, \ldots a_n}_{\text{Basis}}\}$$

Basis: LI, spans V

$\therefore \{u_1, u_2, \ldots, u_n\}$ forms orthonormal basis

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Q) Construct an orthonormal basis/set from the following vectors using Gram-Schmidt process

$$a_1 = (1,1,1) \qquad a_2 = (2,-1,2) \qquad a_3 = (1,2,3)$$

$$S = \{a_1, a_2, a_3\} \subseteq R^3$$

Step ①: Prove that S is Linearly independent and forms basis

Step ②: $V_1 = a_1 = (1,1,1) \qquad u_1 = \frac{V_1}{\|V_1\|} = \frac{(1,1,1)}{\sqrt3} = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$

Let $V_2 = a_2 - \langle a_2, u_1 \rangle u_1$
$= (2,-1,2) - \langle (2,-1,2), (\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}) \rangle (\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3})$
$= (2,-1,2) - \langle \frac{2}{\sqrt3} - \frac{1}{\sqrt3} + \frac{2}{\sqrt3} \rangle (\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3})$
$= (2,-1,2) - (1,1,1)$ $\quad V_2 = (1,-2,1)$

$u_2 = \frac{V_2}{\|V_2\|}$ $\quad u_2 = \frac{(1,-2,1)}{\sqrt{1+4+1}} = \left(\frac{1}{\sqrt6}, \frac{-2}{6}, \frac{1}{\sqrt6}\right)$

$V_3 =$

$u_3 = \frac{V_3}{\|V_3\|}$

Theorem ①: If $S = \{V_1, V_2, \ldots, V_n\}$ is a linearly independent set in V, then every element in the linear span of S has a unique representation of the form

$\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_2$ where $\alpha_i \in F$ $1 \le i \le n$

Proof:
Given $S = \{V_1, V_2, \ldots, V_n\}$ then
$L(S) = \{\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \ldots + \alpha_n V_n \mid \alpha_i \in F, 1 \le i \le n\}$

Let $V \in L(S)$ then
$V = \alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n \longrightarrow ①$

Assume that $V = \beta_1 V_1 + \beta_2 V_2 + \ldots + \beta_n V_n \longrightarrow ②$
where $\beta_i \in F$ $1 \le i \le n$

$① - ② \Rightarrow 0 = (\alpha_1 - \beta_1)V_1 + (\alpha_2 - \beta_2)V_2 + \ldots + (\alpha_n - \beta_n)V_n$

$\alpha_1 - \beta_1 = 0$ $\quad \because S$ is linearly independent
$\alpha_2 - \beta_2 = 0$ $\quad \therefore \alpha_1 = \beta_1$; $\alpha_2 = \beta_2$, $\ldots$, $\alpha_n = \beta_n$
$\vdots$
$\alpha_n - \beta_n = 0$

$\therefore$ The representation in $L(S)$ is unique

Theorem: If $S = \{V_1, V_2, \ldots, V_n\}$ in a basis for the vector sp[ace] V, then the representation of any element in V in terms of basis element is unique.

(i) S is linearly independent
(ii) S spans V (ie) $L(S) = V$

Since the representation of elements in $L(S)$ is unique, we [...] representation of elements in V in terms of basis elements is u[nique]

MINIMAL SPANNING SET :

Let V be a vector space over F. Let $S \subseteq V$ then S is said to be a minimal spanning set

(i) If S is a spanning set for V

(ii) $S \setminus \{v\}$ does not span V for any $v \in S$

MAXIMAL LINEARLY INDEPENDENT SET

Let V be a vector space over F. Let $S \subseteq V$ then S is said to be a maximal linearly independent set if

(i) S is linearly independent

(ii) $S \cup \{v\}$ is linearly dependent for any $v \in V \setminus S$

Theorem :

In a vector space V, a maximal spanning set of vectors forms a basis.

Let $S = \{V_1, V_2, \ldots\ldots, V_n\}$ be a minimal spanning set for V.

Then S spans V ie $L(S) = V$

In order to prove that S is a basis, we must prove that S is linearly independent.

We will prove that by contradiction

Assume that S is linearly dependent then it exhibits V_j (for some j, $1 \leq j \leq n$) is a linear combination of its preceeding ones

$$V_j = \alpha_1 V_1 + \alpha_2 V_2 + \cdots\cdots + \alpha_{j-1} V_{j-1}$$

where $\alpha_1, \alpha_2, \ldots\ldots, \alpha_{j-1} \in F$

then $L\{V_1, V_2, \ldots\ldots, V_{j-1}, V_{j+1}, \ldots\ldots V_n\} = L(S) = V$

then $L\{V_1, V_2, \ldots\ldots, V_{j-1}, V_{j+1}, \ldots\ldots V_n\}$ spans V

$\Rightarrow \{V_1, V_2, \ldots\ldots, V_{j-1}, V_{j+1}, \ldots\ldots V_n\}$

$\Rightarrow S \setminus \{V_j\}$ spans V, contradiction to S is a minimal spanning set.

Hence our assumption is wrong

$\therefore$ Only possibility is S is linearly independent

$\therefore$ S forms a basis for V

Theorem (2) :

In a vector space V, a maximal linearly independent set forms a basis

Proof:

Let $S = \{v_1, v_2, \ldots, v_n\}$ be maximal linearly independent for V

To prove that S forms a basis, we must prove that S spans

Take $v \in V$

Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha v = 0$

Case-1: When $\alpha = 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

$\alpha_1 = \alpha_2 = \ldots + \alpha_n = 0$ [$\because$ S is linearly independent]

$S \cup \{v\}$ is linearly independent, which is a contradiction to maximality of S.

$\therefore$ The only possibility is $\alpha \neq 0$

$\alpha v = -\alpha_1 v_1 + (-\alpha_2) v_2 + \ldots + (-\alpha_n) v_n$

$$v = \underbrace{\left(\frac{-\alpha_1}{\alpha}\right)}_{\in F} + \underbrace{\left(\frac{-\alpha_2}{\alpha}\right)}_{\in F} v_2 + \ldots + \underbrace{\left(\frac{-\alpha_n}{\alpha}\right)}_{\in F} v_n$$

$\Rightarrow$ v is a linear combination of elements from S

$\Rightarrow$ S spans V

$\therefore$ S forms a basis for V

Theorem (2):

Let S be a linearly independent subset of V and $T \subset S$, t T is linearly independent.

Every subset of a linearly independent set in V is linearly independent.

Proof:

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a linearly independent subs of V

Let $T = \{v_1, v_2, \ldots, v_k\}$ where $1 \leq k < n$

Then $T \subset S$

We need to prove that, T is linearly independent.

By using the method of contradiction:

Let us assume that T is linearly dependent.

If scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ (not all zeros) in F such that

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + 0 v_{k+1} + 0 v_{k+2} + \cdots + 0 v_n = 0$
where some of the a_i's are non-zero
which is a contradiction to S is linearly independent
$\therefore$ Our assumption is wrong.
$\therefore$ T is linearly independent

Theorem (3):
The subset of a vector space is either linearly independent or one of the vectors can be expressed as a linear combination of preceeding vectors.

Proof:
Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of V.
If $v_1, v_2, \ldots, v_n$ are linearly independent, then there is nothing to prove.
Suppose that $v_1, v_2, \ldots, v_n$ are linearly dependent.
Then there exist scalars α_i, $1 \le i \le n$ (not all zeroes)
such that
$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \longrightarrow (1)$$

Since all α_i are not zero, there exists a largest positive integer k such that $\alpha_k \neq 0$
Then $\alpha_{k+1} = 0, \ldots, \alpha_n = 0$
$(1) \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_k v_k = 0$
$\alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 \cdots - \alpha_{k-1} v_{k-1}$
$v_k = \left(\frac{-\alpha_1}{\alpha_k}\right) v_1 + \left(\frac{-\alpha_2}{\alpha_k}\right) v_2 + \cdots \left(\frac{-\alpha_{k-1}}{\alpha_k}\right) v_{k-1}$

where $\left(\frac{-\alpha_i}{\alpha_k}\right) \in F$, $1 \le i \le k-1$

Hence v_k is a linear combination of v_i, $1 \le i \le k-1$

Lemma:
If S and T are subsets of a vector space V, then $S \subseteq T$
$L(S) \subseteq L(T)$

Proof:
Let v be an arbitary element of $L(S)$
Then $v \in L(S)$ $\quad v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$
where $\alpha_i \in F$ and $v_i \in S$ $\quad 1 \le i \le n$

Since $S \subseteq T$, we have $V_i \in T \quad 1 \le i \le n$

$\therefore \alpha$ is also the linear combination of finite elements of T

$\Rightarrow \alpha \in L(T)$ Thus $L(S) \subseteq L(T)$

Theorem (5):

In a vector space V, a minimal spanning set of vectors form a basis

Proof:

Let $S = \{V_1, V_2, \ldots, V_n\}$ be a minimal spanning set for V

Then S spans V (ie) $L(S) = V$

In order to prove that S is a basis for V, it is enough to show that S is linearly independent

Let us assume that S is linearly dependent

Then there exists V_j (for some j, $1 \le j \le n$) is a linear combination of its preceeding ones. (by Theorem (3)).

$$\text{ie} \quad V_j = \alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_{j-1} V_{j-1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{j-1} \in F$

Clearly, $L(\{V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots V_n\})$

$\subseteq L(\{V_i \mid 1 \le i \le n\})$ [$\because$ By Lemma]

$= L(S)$

On the other hand, take $x \in L(S)$

Then $x = \beta_1 V_1 + \beta_2 V_2 + \cdots + \beta_n V_n$, for some $\beta_i \in F$, $1 \le i \le n$

$$x = \beta_1 V_1 + \beta_2 V_2 + \cdots + \beta_{j-1} V_{j-1} + \beta_j V_j + \beta_{j+1} V_{j+1} + \cdots + \beta_n V_n$$

$$= \beta_1 V_1 + \beta_2 V_2 + \cdots + \beta_{j-1} V_{j-1} + \beta_j (\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_{j-1} V_{j-1}) + \beta_{j+1} V_{j+1} + \cdots \beta_n V_n$$

$$= (\beta_1 + \beta_j \alpha_1) V_1 + (\beta_2 + \alpha_2 \beta_j) V_2 + \cdots (\beta_{j-1} + \beta_j \alpha_{j-1}) V_{j-1} + \beta_{j+1} V_{j+1} + \cdots$$

$$\in L(\{V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots V_n\})$$

$\therefore L\{V_1, \ldots, V_{j-1}, V_{j+1}, \ldots V_n\} = L(S) = V$

$\Rightarrow \{V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots V_n\}$ spans V

$\Rightarrow S \setminus \{V_j\}$ spans V, a contradiction to the minimal spanning set

$\therefore$ Our assumption is wrong $\therefore S$ is linearly independent

$\therefore S$ forms a basis for V

Theorem 6:

If the vector V has a basis B with n elements, then

(i) Any set with $(n+1)$ vectors is linearly dependent

(ii) Any set with $(n-1)$ vectors do not span V

Proof:

(i) $\longrightarrow$ follows from Theorem 4

(ii) $\longrightarrow$ follows from Theorem 5

Theorem 7

Prove that an orthogonal set of non-zero vectors is linearly independent.

Proof:

Let $A = \{V_1, V_2, \ldots, V_n\}$ be an orthogonal set of non-zero vectors.

TPT : A is linearly independent

Let $\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n = \vec{0}$

then $\langle \vec{0}, V_k \rangle = 0 \quad \forall V_k \in A$

$\Rightarrow \langle \alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n , V_k \rangle = 0$

$\langle \alpha_1 V_1 , V_k \rangle + \langle \alpha_2 V_2 , V_k \rangle + \ldots + \langle \alpha_k V_k , V_k \rangle + \ldots \langle \alpha_n V_n , V_k \rangle = 0$

$$\alpha_1 \underbrace{\langle V_1, k_k \rangle}_{0} + \alpha_2 \underbrace{\langle V_2, V_k \rangle}_{0} + \ldots + \alpha_k \underbrace{\langle V_k, V_k \rangle}_{\|V_k\|^2} + \ldots + \alpha_n \underbrace{\langle V_n, V_k \rangle}_{0} = 0 \quad (1)$$

Since A is orthogonal,

$$\langle V_i, V_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \|V_i\|^2 & \text{if } i = j \end{cases}$$

$\therefore$ Eqn (1) becomes $\alpha_k \|V_k\|^2 = 0$, $\forall k = 1, 2, \ldots, n.$

Either $\alpha_k = 0$ or $\|V_k\|^2 = 0 \quad \forall k = 1, 2, \ldots, n$

Since V_k's are non-zero vectors, $\|V_k\|^2 = 0$ is impossible.

$\therefore$ The only possibility is

$\alpha_k = 0 \quad \forall k = 1, 2, \ldots, n$

$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$

$\therefore$ A is linearly independent.

Theorem 8:

An orthonormal set of non-zero vectors is linearly independent.

Proof:

Let $A = \{V_1, V_2, \ldots, V_n\}$ be an orthonormal set of non-zero vectors.

TPT: A is linearly independent

Let $\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n = \vec{0}$

Then $\langle \vec{0}, V_k \rangle = 0, \forall V_k \in A$

$\Rightarrow \langle \alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n, V_k \rangle = 0$

$\langle \alpha_1 V_1, V_k \rangle + \langle \alpha_2 V_2, V_k \rangle + \ldots + \langle \alpha_k V_k, V_k \rangle + \ldots + \langle \alpha_n V_n, V_k \rangle = 0$

$$\alpha_1 \underbrace{\langle V_1, V_k \rangle}_{0} + \alpha_2 \underbrace{\langle V_2, V_k \rangle}_{0} + \ldots + \alpha_k \underbrace{\langle V_k, V_k \rangle}_{1} + \ldots + \alpha_n \underbrace{\langle V_n, V_k \rangle}_{0} = 0 \rightarrow (1)$$

Since A is orthonormal,

$$\langle V_i, V_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Eqn (1) becomes $\alpha_k . 1 = 0$

$\alpha_k = 0 \quad \forall k = 1, 2, \ldots, n$

$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$

$\therefore$ A is linearly independent.

PRACTICE QUESTIONS :

Construct an orthonormal basis from the following vectors using Gram-Schmidt process.

1. $S = \{(1,0,3)\ (2,2,0)\ (3,1,2)\}$
2. $S = \{(1,2,1)\ (1,0,1)\ (3,1,0)\}$
3. $S = \{(0,0,1,0)\ ([illegible],0,1,4)\ (1,1,1,1)\}$