Practice Problems (Higher Order Differential Equations)

Concepts

Linear Differential Equations

A general linear D.E of nth order is

$$P_0(x)\frac{d^ny}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = R(x) \quad \text{--- (1)}$$

where $P_0(x), P_1(x), \ldots P_{n-1}(x), P_n(x), R(x)$ are functions of x.

Note 1) If $R(x) = 0 \rightarrow$ Linear Homogeneous D.E

2) If $R(x) \neq 0 \rightarrow$ Linear non-Homogeneous D.E.

Linear D.E with constant coefficients

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = R(x) \quad \text{--- (2)}$$

where $P_0, P_1, \ldots P_n$ are constants.

Linear Homogeneous D.E. with constant coefficients

The D.E of the form

$$f(D)\,y = 0 \quad \text{--- (3)}$$

where $f(D) = P_0D^n + P_1D^{n-1} + \cdots + P_n$

Solution of Linear Homogeneous D.E with constant coefficients

If $y = e^{ax}$ is a sol^n of (3), then

$$f(D)\,e^{ax} = 0$$

$$e^{ax} f(a) = 0 \Rightarrow f(a) = 0 \quad \text{--- (4)}$$

[$\because$ Property of Differential operator]

$\therefore$ Eq^n (4) is called the Auxillary Equation.

Properties of Differential operator

1) $f(D)\,e^{ax} = e^{ax} f(a)$; 2) $f(D)\,e^{ax}y = e^{ax} f(D+a)\,y$

3) $(D-a)^k e^{ax} x^j = \begin{cases} 0, & j = 0,1,2,\ldots k-1 \\ e^{ak}\,k!, & j = k \end{cases}$

→ Depending on the nature of the roots of A.E we have following cases.

Case 1 → If the roots of A.E are real and distinct, then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

is the general solution.

Case 2 → If the roots are real and equal, say, $m_1 = m_2 = \dots = m_n = m$

Then, $y = (c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1}) e^{mx}$

is the general solution.

Case 3 → If the roots are complex say $\alpha = a+ib, \alpha_2 = a-ib$

then $y = c_1 y_1 + c_2 y_2$

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

is the general solution.

Practice Problems

Q.1) $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, given $x(0) = 0$, $\frac{dx}{dt}(0) = 15$

Soln.. $x = 15(e^{-2t} - e^{-3t})$

Q.2) $(D^2 + 6D + 9)x = 0$

Soln.. $x = (c_1 + c_2 t)e^{-3t}$

Q.3) $(D^3 + D^2 + 4D + 4)y = 0$

Soln.. $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$

Q.4) $(D^2 - 2D + 4)^2 y = 0$

Soln.. $y = e^x((c_1 + c_2 x)\cos\sqrt{3}x + (c_3 + c_4 x)\sin\sqrt{3}x)$

Q.5) $(D+1)^3 y = 0$

Soln.. $y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$

Q.6) $(D^4+4)y=0$

Sol^n.. $y = e^{-t}[c_1\cos t + c_2\sin t] + e^{t}[c_3\cos t + c_4\sin t]$

Q.7) $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$

Sol^n.. $y = (c_1 + c_2 x)e^{x} + (c_3\cos x + c_4\sin x)$

Solution of Linear Non-Homogeneous D.E. with constant Coefficients

Consider the linear non-homogeneous D.E

$$f(D)y = R(x) \quad \text{---} \; (1)$$

where $f(D) = P_0 D^n + P_1 D^{n-1} + \dots + P_n$

Sol^n to Eq. (1) is $y = y_c + y_p$, where

y_c is complementary function is the sol^n $f(D)y = 0$

& y_p is particular solution is found by one of the method described below.

Inverse Differential operator

If $f(D)y = R(x)$, then we define the inverse Differential operator denoted by $\dfrac{1}{f(D)}$ as $\dfrac{1}{f(D)}R(x) = y$.

Propertiy of inverse differential operator

1) $\dfrac{1}{D}R(x) = \int R(x)\,dx$

2) $\dfrac{1}{D-a}R(x) = e^{ax}\int R(x)\,e^{-ax}\,dx.$

To determine P.I of a linear non-homogeneous D.E

If $f(D)y = R(x)$, then $y_p = \frac{1}{f(D)} R(x)$ is the P.I

Case 1 If $R(x) = e^{ax}$

$$y_p = \frac{1}{f(D)} e^{ax}$$

$$= \frac{e^{ax}}{f(a)}, \text{ if } f(a) \neq 0$$

$$= x. \frac{e^{ax}}{f'(a)}, \text{ if } f(a) = 0 \ \& \ f'(a) \neq 0$$

$$= x^2. \frac{e^{ax}}{f''(a)}, \text{ if } f(a) = f'(a) = 0 \ \& \ f''(a) \neq 0$$

⋮

goes on.

Case 2 If $R(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$y_p = \frac{1}{f(D)} \sin(ax+b)$$

If the differential operator can be written as $f(D^2)$ then $y_p = \frac{1}{f(D^2)} \sin(ax+b)$

$$= \frac{1}{f(-a^2)} \sin(ax+b) \text{ if } f(-a^2) \neq 0$$

$$= x. \frac{1}{f'(-a^2)} \sin(ax+b) \text{ if } f(-a^2) = 0, f'(-a^2) \neq 0$$

$$= x^2. \frac{1}{f''(-a^2)} \sin(ax+b) \text{ if } f(-a^2) = f'(-a^2) = 0 \ \& \ f''(-a^2) \neq 0.$$

⋮

goes on.

Similarly for $\cos(ax+b)$.

Case 3 If $R(x) = x^m$

Then $y_p = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$

Expand $[f(D)]^{-1}$ in ascending powers of D till D^m & operate on x^m term by term. Since $(m+1)^{th}$ & higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Case 4 If $R(x) = e^{ax}.V$, where V is function of x.

Then $y_p = \frac{1}{f(D)} e^{ax} V$

$= e^{ax} \frac{1}{f(D+a)} . V.$

Case 5 If R(x) is any function of x, then resolve $\frac{1}{f(D)}$ into partial fractions and proceed.

Problems on $R(x) = e^{ax}$ (case 1)

Q.8) $y'' - 6y' + 9y = e^x$

Soln. $y = (c_1 + c_2 x) e^{3x} + e^x/4$

Q.9) $y'' - 5y' + 6y = e^{2x}$

Soln. $y = c_1 e^{2x} + c_2 e^{3x} - x e^{2x}$

Q.10) Find the P.I of $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Soln. $y_p = \frac{x e^{-2x}}{9} + \frac{x^2 e^x}{6} - \frac{e^{-x}}{4}$

Hint. $\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$

Q.11) $(D^2+D+1)y = (1-e^x)^2$

Soln. $y = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + 1 - \frac{2e^x}{3} + \frac{e^{2x}}{7}$

Q.12) $(D^2-4)y = \cosh(2x-1) + 1$

Soln. $y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4}\cosh(2x-) - \frac{1}{4}$

Problems on $R(x) = \sin(ax+b) / \cos(ax+b)$ (case II)

Q.13) $(D^2-6D+10)y = \cos 2x + e^{-3x}$

Soln. $y = e^{3x}(c_1\cos x + c_2 \sin x) + \frac{1}{30}(-2\sin 2x + \cos 2x) + \frac{e^{-3x}}{37}$

Q.14) $y'' + y = \cos x$

Soln. $y = c_1\cos x + c_2 \sin x + \frac{x\sin x}{2}$

Q.15) $y'' - y = 10\sin^2 x$

Soln. $y = c_1 e^x + c_2 e^{-x} - 5 + \cos 2x$

Hint → $\sin^2 x = \frac{1-\cos 2x}{2}$

Q.16) $(D^2-4D+3)y = \sin 3x \cos 2x$

Soln. $y = c_1 e^x + c_2 e^{3x} + \frac{1}{1768}(20\cos 5x - 22\sin 5x)$

$+ \frac{1}{40}(4\cos x + 2\sin x)$

Hint → $\sin 3x \cdot \cos 2x = \frac{1}{2}\left[\sin(3x+2x) + \sin(3x-2x)\right]$

Q. 17) $(D^3+1)y = \cos(2x-1)$

Sol^n.. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

$+ \frac{\cos(2x-1)}{65} - \frac{8}{65}\sin(2x-1)$

Q.18) $(D^4+18D^2+81)y = \cos^3 x$

Sol^n.. $(c_1+c_2x)\cos 3x + (c_3+c_4x)\sin 3x + \frac{3}{256}\cos x$

$- \frac{x^2 \cos 3x}{288}$

Hint : $\cos 3A = 4\cos^3 A - 3\cos A$

Note

$\frac{1}{1-D} = 1 + D + D^2 + D^3 + \cdots$

$\frac{1}{1+D} = 1 - D + D^2 - D^3 + \cdots$

$\frac{1}{(1-D)^2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

$\frac{1}{(1+D)^2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

Problems on $R(x) = x^m$ (case III)

Q.19) $y'' + 4y = x^2$

Sol^n.. $y = c_1\cos 2x + c_2 \sin 2x + \frac{1}{4}(x^2 - 1/2)$

Q.20) $(D^2+D)y = x^2+2x+4$

Sol^n.. $y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x.$

Q. 21) $(D^2+4)y = x^2 + \cos 2x$

Sol^n.. $y = c_1\cos 2x + c_2 \sin 2x + \frac{x^2}{4} + \frac{x}{4}\sin 2x - \frac{1}{8}$

Q.22) $(D^2-6D+25)y = e^{2x} + \sin x + x$

Sol^n.. $y = e^{3x}(c_1\cos 4x + c_2 \sin 4x) + \frac{e^{2x}}{17} + \frac{4\sin x + \cos x}{102}$

$$+ \frac{2}{25} + \frac{6}{625}$$

Q.23) $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

Sol^n.. $y = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$

Problems on $R(x) = e^{ax}.V$ $(\cos, \sin V)$

Q.24) $(D^2-2D+4)y = e^x \cos x$

Sol^n.. $y = e^x(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x) + \frac{e^x}{2}\cos x$

Q.25) $(D^2-3D+2)y = xe^{3x} + \sin 2x$

Sol^n.. $y = c_1 e^{2x} + c_2 e^{x} + \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}\left(3\cos 2x - \sin 2x\right)$

Q.26) $(D^2-1)y = x\sin 3x + \cos x$

Sol^n.. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{10}\left[\frac{6\cos 3x}{10} + x\sin 3x\right] - \frac{\cos x}{2}$

Q.27) $(D^4+2D^2+1)y = x^2\cos x$

Sol^n.. $y = (c_1+c_2x)\cos x + (c_1+c_2x)\sin x + \frac{\cos x}{4}\left(\frac{3x^2}{4} - \frac{x^4}{12}\right)$

$$+ \frac{x^3 \sin x}{12}$$

Q.28) $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x.$

Sol^n.. $y = (c_1 + c_2 x)e^{2x} + 8e^{2x}\left[-\frac{x^2\sin 2x}{4} - \frac{x\cos 2x}{2} + \frac{3\sin 2x}{8}\right]$

Hint: Use the property of Inverse Differential operator to simplify.

Problems on $R(x)$ = any function of x (case v)

Q.29) $(D^2+a^2)y = \sec ax$

Soln. $y = C_1\cos ax + C_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log\cos ax.$

Method of variation of parameters

This method is applicable to eqn of the form

$$y'' + py' + qy = R(x) \quad --(1)$$

where p, q, and $R(x)$ are functions of x.

Then $P.I = -y_1\int \frac{y_2 X}{W}dx + y_2\int \frac{y_1 X}{W}dx$

where y_1 and y_2 are the solution of $y''+py'+qy=0$.

& $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called wronskian of y_1, y_2.

Problems on method of variation of parameters

Q.30) $(D^2+4)y = \tan 2x$

Soln. $y = C_1\cos 2x + C_2\sin 2x - \frac{\cos 2x}{4}\log(\sec 2x + \tan 2x)$

Q.31) $y'' - y = \frac{2}{1+e^x}$

Soln. $y = C_1e^x + C_2e^{-x} + e^x\log(1+e^{-x}) - e^{-x}\log(1+e^x) - 1$

Q.32) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

Soln. $y = (C_1 + C_2x)e^{3x} - (1+\log x)e^{3x}$

Q.33) $y'' + y = \sec^2 x$

Soln. $y = C_1\cos x + C_2\sin x - \sec x\cos x + \sin x\log(\sec x + \tan x)$

Q.34) $y'' + y' - 2y = \dfrac{1}{1-e^x}$

Soln. $y = -\dfrac{e^x}{3}\left(e^{-x} + \log(e^{-x}+1)\right) + \dfrac{e^{-2x}}{3}\left[\log(1-e^x) - 1 + e^x\right]$

Cauchy Homogeneous Linear Equation

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_{n-1}\, x\frac{dy}{dx} + k_n y = R(x) \quad \text{--- (1)}$$

where $R(x)$ is a function of x

k_i's $(i = 1,2,3 \ldots n)$ are constants, is called Cauchy homogeneous linear equations.

Eq^n (1) can be converted/reduced to linear differential equations with constant coefficients as follows.

Take. $x = e^t$

i.e $t = \log x$

Then $\dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx} = \dfrac{dy}{dt}\cdot\dfrac{1}{x}$.

$$x.\frac{dy}{dx} = \frac{dy}{dt} = Dy \quad \because D = \frac{d}{dt}$$

Similarly

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\cdot\frac{1}{x}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dt}\right)\cdot\frac{1}{x} + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}\cdot\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dt} \quad \left[\because \frac{dt}{dx} = \frac{1}{x}\right]$$

$$= \frac{d^2y}{dt^2}\cdot\frac{1}{x^2} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

i.e $$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left[D^2y - Dy\right]$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

In the same way

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \text{ and so on.}$$

On substituting these values in (1), it reduces to linear equations with constant coefficients.

Problems on Cauchy Homogeneous Linear Equations

Q.35) $x^2y'' - 2xy' - 4y = x^4$

Sol^n.. $y = C_1x^4 + \frac{C_2}{x} + \frac{x^4 \log x}{5}$

Q.36) $x^2y'' - xy' + y = \log x, \quad y(1) = y'(1) = 0$

Sol^n.. $y = (-2 + \log x)x + \log x + 2$

Q.37) $x^2y'' - xy' + 2y = x\sin(\log x)$

Sol^n.. $y = x\left[\{C_1\cos(\log x) + C_2\sin(\log x)\} - \frac{\cos(\log x)\log x}{2}\right]$

Q.38) $x^2y'' + 3xy' + y = \frac{1}{(1-x)^2}$

Sol^n.. $y = \frac{1}{x}\left[C_1 + C_2\log x + \log\frac{x}{x-1}\right]$

Q.39) $2xy'' + 3y' - \frac{y}{x} = 5 - \frac{1}{2}\frac{\sin(\log x)}{2}$

Sol^n.. $C_1\sqrt{x} + \frac{C_2}{x} + \frac{5x}{2} + \frac{\cos(\log x) + 3\sin(\log x)}{20}$

Hint. multiply $x/2$ to D.E to get the standard form

Legendre's Differential Equation

A Legendre differential equation is of the form

$$(ax+b)^n \frac{d^ny}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_{n-1}(ax+b)\frac{dy}{dx} + k_n y = R(x) \quad \text{--- (1)}$$

Soln for Eq. (1)

let $ax+b = e^t$

$t = \log(ax+b)$

$$\frac{dt}{dx} = \frac{a}{ax+b}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a}{ax+b} Dy$$

i.e $(ax+b)\frac{dy}{dx} = aDy \quad \because D = \frac{d}{dt}$

Similarly $(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$

substituting these in (1), we get linear D.E with constant coefficients.

Q.40) $(x+2)^2 y'' - (x+2)y' + y = 3x+4$

Soln.. $y = (c_1 + c_2 \log(x+2))(x+2) + \frac{3}{2}\log(x+2)(\log(x+2)-1)(x+2) - 2$

Q.41) $(2x-1)^3 y''' + (2x-1)y' - 2y = 0$

Soln.. $y = c_1(2x-1) + c_2(2x-1)^{\frac{1+\sqrt{3}}{2}} + c_3(2x-1)^{\frac{1-\sqrt{3}}{2}}$