Literature Review: Part I

Project Title: GHz Low Noise Amplifier Design for Optical Shot Noise Measurements
Week 1: 4^{th} to 9^{th} June 2018
IISER Summer Student Programme

Project Guide: Dr. Shouvik Datta **Student name:** Rutwik Narendra Jain

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1 Photon Statistics [1]

The photon flux incident on a photon counting device or detector depends on the intensity of the light beam I and its wavelength.

$$\phi = \frac{IA}{\hbar\omega} \tag{1}$$

For a detector having quantum efficiency η, the photon count in a time interval T, and the photon counting rate $\mathcal{R}$ can be evaluated as:

$$\mathcal{N}(T) = \eta\phi T = \frac{\eta PT}{\hbar\omega} \text{ photons} \tag{2}$$

$$\mathcal{R} = \frac{\mathcal{N}(T)}{T} = \eta\phi = \frac{\eta P}{\hbar\omega} \text{ photons s}^{-1} \tag{3}$$

Quantum efficiency and dead time of the detectors limit the maximum possible count rate $\mathcal{R}$, and hance extremely low power beams need to be used for photon counting detectors such as avalanche photodiodes.

However, the flux in equation 1 is an average value, and so are the photon count and count rate. The actual count of photons hitting the detector in any time interval Δt fluctuates about the mean value given by the above expressions.

Because of the random variation in the time interval between successive pulses, the observed average pulse rate will show ever larger fluctuations as the averaging time interval Δt gets shorter.

The photon number on **short time scales** fluctuates due to the **discrete nature of photons**. These fluctuations are described by **photon statistics** of light.

Perfectly coherent light is light which has constant or time-invariant intensity $I(t)$ (more on coherence later). It is shown to obey Poissonian statistics i.e. the photon count follows Poisson distribution, and the proof is not reproduced here [1].

Poisson Distribution
The discrete Poisson distribution for a random variable n that takes only non-negative integer values ($n \in \mathbb{Z}^+$) is given by the probability distribution

$$\mathcal{P}(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}} = \frac{\mu^n}{n!} e^{-\mu} \tag{4}$$

where:

$$\mu = \bar{n} = \text{Average or mean of the distribution}$$

This can be simplified as follows to obtain a recursive definition for $\mathcal{P}(n)$

$$\begin{aligned} \mathcal{P}(n) =& \frac{\bar{n}^n}{n!} e^{-\bar{n}} = \frac{\bar{n}}{n} \frac{\bar{n}^{(n-1)}}{(n-1)!} e^{-\bar{n}} \\ =& \frac{\bar{n}}{n} \mathcal{P}(n-1) \end{aligned}$$

If $n < \bar{n}$, $\mathcal{P}(n) > \mathcal{P}(n-1) \Rightarrow$ P(n) increases with n.
If $n > \bar{n}$, $\mathcal{P}(n) < \mathcal{P}(n-1) \Rightarrow$ P(n) decreases with n.

Thus, the distribution traces a bell curve if $\mathcal{P}$(n) is plotted against n, with the maximum value occuring at the mean $\bar{n}$ of the distribution.

It can be easily shown that the mean, variance and standard deviation of the distribution given by equation 4 are as follows:

$$\textbf{Mean}: \ \langle n \rangle = \bar{n} = \sum_{n=0}^{\infty} n\mathcal{P}(n) \tag{5}$$

$$\textbf{Variance}: \ Var(n) = \sigma^2 = \sum_{n=0}^{\infty} (n-\bar{n})^2 \mathcal{P}(n) = \bar{n} \tag{6}$$

$$\textbf{Std Deviation}: \ \Delta n = \sigma = \sqrt{\bar{n}} \tag{7}$$

Light can be classified on the basis of photon statistics as Poissonian ($\Delta n = \sqrt{\bar{n}}$), Super-Poissonian ($\Delta n > \sqrt{\bar{n}}$) and Sub-Poissonian ($\Delta n < \sqrt{\bar{n}}$).

- Super-Possonian light has a greater deviation, which means the distribution is more spread

Table 1: Classification of light according to photon statistics

Photon Statistics	*Classical Equivalent*	*Intensity $I(t)$*	Δn
Super-Poissonian	Partially coherent or thermal light	Time variant	$> \sqrt{\bar{n}}$
Poissonian	Perfectly coherent light	Constant	$= \sqrt{\bar{n}}$
Sub-Poissonian	No classical eqv.	Constant	$< \sqrt{\bar{n}}$

out. Thus, super-poissonian light has greater fluctuations in photon counts, and therefore allows for fluctuations in intensity, as opposed to the case with constant intensity, which we know to be Poissonian. Thus all classical light beams with time-varying intensities will have Super-Poissonian photon count distribution. Light from black-body radiation and partially coherent (or chaotic) light are examples.

- Sub-Poissonian has a narrower distribution that the Poissonian case, and thus has no classical equivalent. It can be concluded that Sub-Poissonian light is more stable than perfectly coherent light, and hence detection in an experiment is problematic.

Due to inefficiency of the photo-detectors, losses occur, which leads to degradation of these photon statistics. Beam collection inefficiency, losses due to absorption/scattering/reflection, imperfect quantum efficiency can all reduce the efficiency of photon-counting experiments.

2 Noise [3]

- Noise is defined as any unwanted disturbance that intereferes with a signal of interest. Noise is classified as shown in figure 1.
- The Signal-to-Noise ratio (SNR) specifies the quality of a signal in presence of noise.

$$\mathrm{SNR} = 10 \times \log_{10}\left(\frac{P_s}{P_n}\right) = 10 \times \log_{10}\left(\frac{X_s^2}{X_n^2}\right) \tag{8}$$

where:

$$\begin{aligned} P_s &: \text{Signal Power} \\ X_s &: \text{RMS value of signal} \\ P_n &: \text{Noise Power} \\ X_n &: \text{RMS value of noise} \end{aligned}$$

- Since noise is a random process, the instantaneous value of a noise variable is unpredictable. We have to deal with noise using statistical parameters.
- **RMS value** of a noise voltage or current is given by:

$$X_n = \left(\frac{1}{T}\int_0^T x_n^2(t)\,dt\right)^{\frac{1}{2}} \tag{9}$$

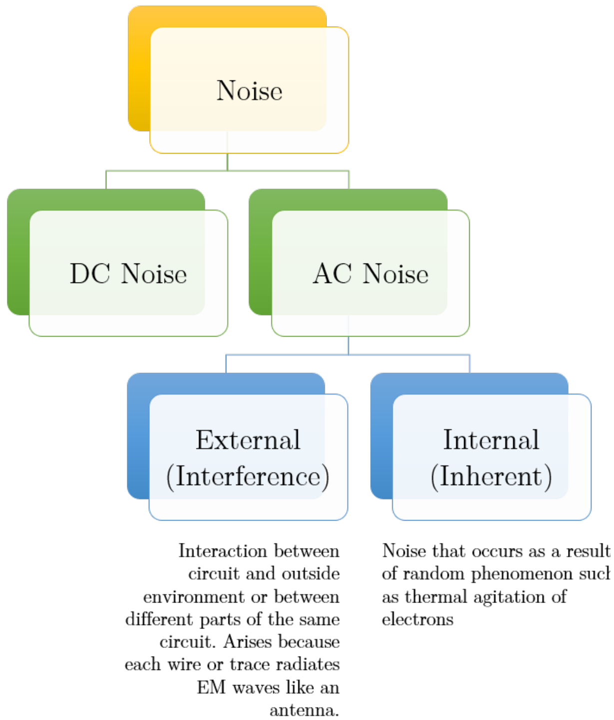


Figure 1: Noise Classification

- The **crest factor (CF)** is defined as the ratio of the peak value to the rms value of noise.
- Voltage noise can be readily observed with an oscilloscope of adequate sensitivity.
- Noise can also be measured with a multimeter. AC meters can be categorised into two types: true-rms meters and averaging-type meters. True RMS meters can accurately measure both sinusoidal as well as non-sinusoidal waveforms, while averaging-type meters calculate the rms value by implementing formulae appropriate only for sinusoidal waveform. Averaging-type meters first rectify the signal and compute its average. For sinusoidal signals this would be $(1/2\pi) \times (1/\sqrt{2})\times$ average value of rectified waveform. The noise readings for averaging-type meters are thus incorrect for non-sinusoidal signals.

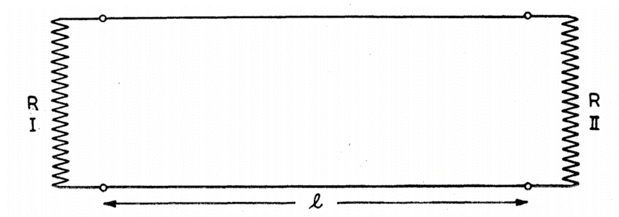
2.1 Sources of Noise

2.1.1 Thermal Noise [4], [5], [6]

Also called Johnson-Nyquist noise, or Johnson noise, it is the universal and unavoidable voltage noise across a resistor at finite temperature, due to thermal agitation in the resistor, whose magnitude is given by the Nyquist theorem.

H. Nyquist showed [6] that the problem of determining the amplitude of the noise is equivalent to summing the energy in the normal modes of electrical oscillation along a shorted transmission line connected to two resistors of resistance R each ($R_{\mathrm{I}} = R_{\mathrm{II}} = R$).

From a circuit analysis approach, it can be derived that,



$$d\langle P_{II}\rangle = \frac{d\langle V_{II}^2\rangle}{R} \tag{10}$$

From a thermodynamic analysis, the following relation is obtained for an infinitesimally small frequency range $d\nu$,

$$d\langle P_{II}\rangle = kTd\nu \tag{11}$$

Combining the two expressions,

$$d\langle V^2\rangle = 4RkTd\nu \tag{12}$$

The experimental setup for Johnson noise measurement is shown in figure 2. The setup till the pre-amplifier front-end can be modelled as an RC circuit. The voltage at the pre-amplifier front end, in terms of Johnson noise (using the model, as shown in figure 3) is a simple low-pass filter transfer function

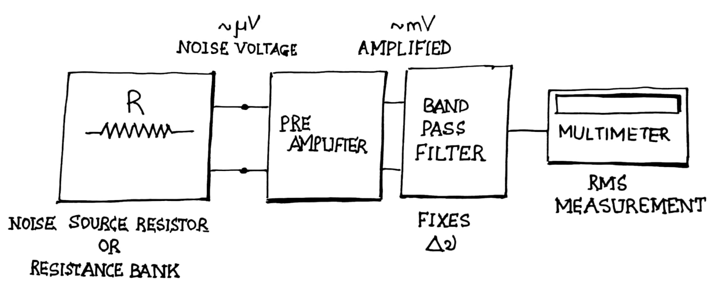


Figure 2: Johnson Noise Measurement Setup: Block Diagram

$$d\langle v_{AB}^2\rangle = \frac{1}{1+(2\pi\nu RC)^2} d\langle v_{\text{Johnson}}^2\rangle \tag{13}$$

The voltage at the pre-amp is simply the Johnson voltage or noise voltage scaled by a frequency dependent factor, which attenutates higher frequencies. The second part of the setup consists of the pre-amplifier itself which will have a gain $g(\nu)$ that varies with frequency. The band pass filter limits the frequency range captured. Combining equations 12 and 13 with the gain term,

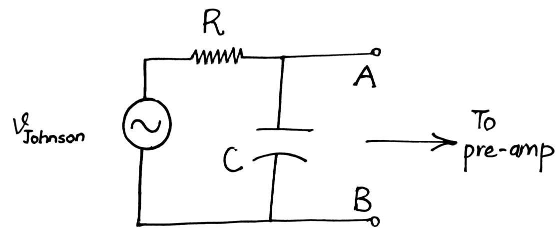


Figure 3: Equivalent RC circuit for the setup till the front-end of the pre-amplifier

$$\begin{aligned}\langle V^2\rangle d\nu =& \frac{g(\nu)}{1+(2\pi\nu RC)^2}\langle v^2_{\text{Johnson}}\rangle d\nu \\ =& \frac{g(\nu)}{1+(2\pi\nu RC)^2} 4RkT d\nu\end{aligned}$$

Integrating over the frequency spectrum,

$$V^2 = 4RkT \int_0^\infty \frac{g(\nu)}{1+(2\pi\nu RC)^2} d\nu \tag{14}$$

From a numerical integration, we write the factor $\int_0^\infty \frac{g(\nu)}{1+(2\pi\nu RC)^2}\, d\nu = \mathcal{G}$, and the output voltage is obtained in terms of Boltzmann constant, k.

$$V^2 = 4R\mathcal{G}kT \tag{15}$$

Thus a plot of $V^2/4\mathcal{G}T$ versus R for different resistances gives a straight line with a slope k, and can be used as a method to determine the Boltzmann constant using only macroscopic quantities such as temperature and voltage. Another way to obtain k is to vary temerature, keeping R fixed. The circuit is calibrated by shorting the resistor out of the circuit first and measuring voltage level present and subtracting that uncorrelated voltage from the actual noise voltage measurement on the multimeter.

2.1.2 Shot Noise in Photodiodes [1]

Photodiode

- A photodiode is a **reverse-biased** p-n junction that consumes light energy to generate electric current.

- Different types - PN photodiode (not common), PIN photodiode (commonly used), Avalanche photodiode (high-speed, electron multiplication applications).
- A p-n junction in equilbrium (with no external bias or excitation) has net diffusion current (due to concentration gradient) = net drift current of minority carriers (due to depletion space-charge region at the junction). The depletion width W has immobile dopant ions, creating a built-in potential V_0 (Figure 4).
- When external light falls on the photodiode, the valence electrons in the depletion region gain energy. If $\hbar\omega \geq E_G$, valence electron moves to conduction band, leading to the creation of electron-hole pairs in the depletion region.
- Free electrons in the depletion region move towards n-region under influence of the field $\mathcal{E}_{depletion}$ (due to depletion region space charge), while free holes move along $\mathcal{E}_{depletion}$ to p-region
- The free electrons are attracted to the positive terminal of the battery, connected to reverse-bias the junction. As shown in figure 5, due to the reverse bias applied to the junction, minority carrier drift increases. Thus, a net current is generated due to photo-excitation at depletion region.
- When no light is applied to the reverse biased photo-diode, it still carries a small reverse current due to the external reverse-bias voltage. This is called **dark current**.
- In a photodiode, the reverse current depends prominently on intensity of incident light.
- PIN (p type - intrinsic - n type) photodiodes operate similarly. Because of different energy band and potential gradient though, pin photodiodes have **wider bandwidth, higher quantum efficiency and high response speed**.
- **Avalanche photodiodes** make use of avalanche breakdown of a p-n junction. A reverse bias voltage is applied sufficient enough to obtain avalanche multiplication of carriers.
- The performance parameters of photo-diodes are:
  - Quantum Efficiency $\eta = \frac{\text{no. of photoelectrons generated in circuit}}{\text{no. of incident photons}}$
  - Responsivity $\frac{i}{P} = \frac{\eta e}{\hbar\omega}$ A W^{-1}

The principle behind using photo-diode detectors to study the statistical properties of light is that the photocurrent generated will fluctuate because of the fluctuations in the impinging photon number [1].

For a constant intensity beam, the photocurrent will also have Poissonian distribution.

$$i(t) = \langle i \rangle + \Delta i(t) \tag{16}$$

The average value of the time varying fluctuations is 0 i.e. $\langle \Delta i(t) \rangle = 0$. However, the average value of the square of current (given by the variance of poisson distribution) is non-zero and so is the noise power.

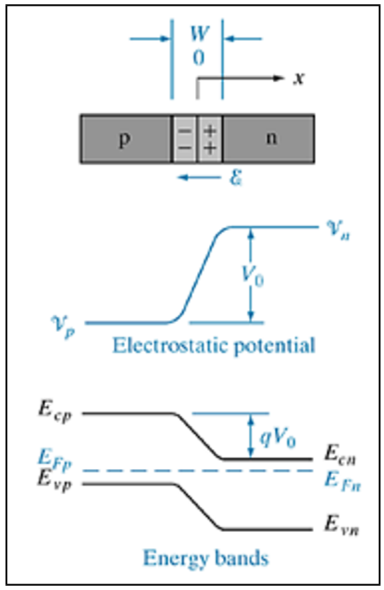


Figure 4: PN Junction in equilibrium (under no external bias, or photoexcitation)

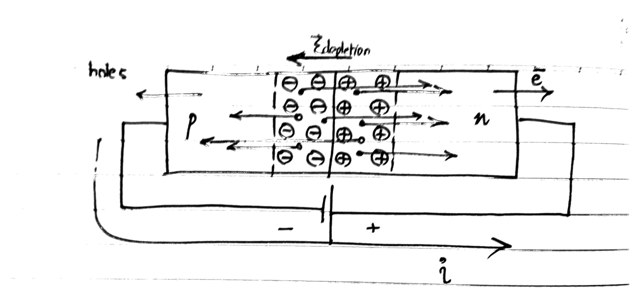


Figure 5: Under reverse bias and illumination

$$P_{\text{noise}}(t) = (\Delta i(t))^2 R_L \tag{17}$$

and

$$(\Delta i(t))^2 \propto \langle i \rangle \tag{18}$$

For a frequency bandwidth of Δf, the noise power variation with frequency is given by:

$$P_{\text{noise}}(f) = 2eR_L \Delta f \langle i \rangle \tag{19}$$

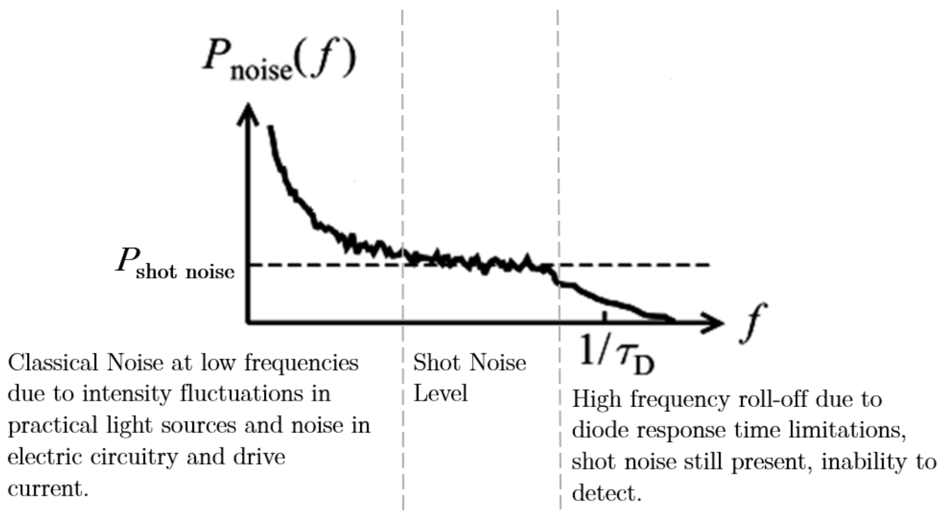


Figure 6: Noise Power Spectral density

Thus, there exists a linear relationship between the shot noise and the average photocurrent. A typical spectrum for the shot noise power spectral density is shown in figure 6, with classical noise at low frequencies and detector constraints causing high-frequency roll-off.

Q. Prove that Johnson-Nyquist Noise is smaller than the shot noise for the same average current value provided that the voltage dropped across the resistor is greater than $2k_BT/e$.

A.

$$\begin{aligned}\frac{P_{\text{Shot}}}{P_{\text{Johnson}}} =& \frac{\langle(\Delta i_{\text{Shot}})^2\rangle R}{\langle(\Delta i_{\text{Johnson}})^2\rangle R} \\ =& \frac{2eR\cancel{\Delta f}\langle i\rangle}{4k_BT\cancel{\Delta f}} \\ =& \frac{eR\langle i\rangle}{2k_BT} \\ >& 1 \qquad \text{if} \\ e\langle i\rangle R >& 2k_BT \\ \langle i\rangle R >& \frac{2k_BT}{e}\end{aligned}$$

$$\boxed{V_R > \frac{2k_BT}{e}}$$

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3 Op Amp Noise

Operational Amplifier Noise is characterised by three noise sources at the op-amp input, a voltage source with noise spectral density e_n, and two current sources i_{nn} and i_{np}, where the subscript n stands for noise. The opamp noise model is shown in figure 7.

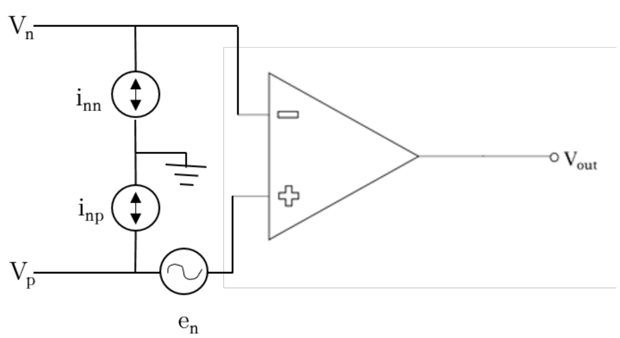

Figure 7: Op Amp Noise Model

For voltage feedback amplifiers (VFAs), datasheets generally mention only a single noise curent spectral density i_n. For current feedback amplifiers (CFAs), the inputs are asymmetric due to presence of an input buffer, and consequently i_{nn} and i_{np} are graphed separately.

Since broadband noise is proportional to the square root of bandwidth, noise filtering can be acheived using a passive RC based filter with a resistance R small enough to not add appreciable thermal noise. Figure 8 shows the circuit that filters both voltages as well as currents.

Applying KCL analysis at the inverting input node and at the output node,

$$\frac{v_0 - v_i}{mR} + (v_1 - v_i)nsC = I_i \tag{20}$$

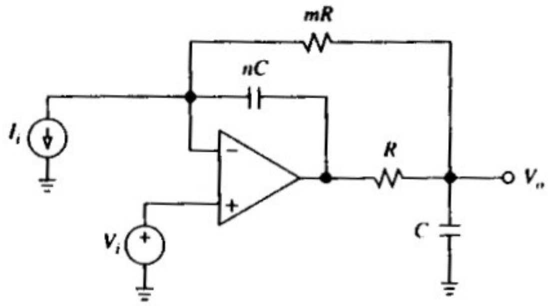

Figure 8: Low-Pass Noise Filter Circuit

$$\frac{v_1 - v_0}{R} + \frac{v_i - v_0}{mR} = v_0 sC \tag{21}$$

From equations 20 and 21, we obtain

$$V_0 = H_{\mathrm{LP}} mRI_i + (H_{\mathrm{LP}} + H_{\mathrm{BP}})V_i$$

where:

$$H_{\mathrm{LP}} = \frac{1}{s^2 + s\left(\frac{m+1}{mRC}\right) + \left(\frac{1}{mnR^2C^2}\right)}, \text{ and}$$
$$H_{\mathrm{BP}} = \frac{s(mnRC)}{s^2 + s\left(\frac{m+1}{mRC}\right) + \left(\frac{1}{mnR^2C^2}\right)}$$

A second order low-pass transfer function is of the form $K/(s^2 + (\omega_0/Q)s + \omega_0^2)$ where ω_0 is the cut-off frequency and Q is the Quality factor of the circuit.

For the noise filtering circuit,

$$f_0 = \frac{1}{2\pi\sqrt{mn}RC} \qquad Q = \frac{\sqrt{m/n}}{m+1} \tag{22}$$

The filter finds application in voltage-reference and photodiode-amplifier noise reduction.

References

[1] Mark Fox, Quantum Optics: An Introduction, Oxford Publications.

[2] X. T. Zou, L. Mandel, Photon-anti-bunching and sub-Poissonian photon statistics, Physical Review A, Volume 41, Number 1, 1 January 1990.

[3] Sergio Franco, Design with Operational Amplifiers and Analog Integrated Circuits, (Chapter 7, Noise), Third Edition, Tata McGraw Hill Publishing Company Ltd.

[4] Dennis V. Perepilitsa, Johnson Noise and Shot Noise, MIT, Department of Physics, November 27, 2006.

[5] Johnson Noise and Shot Noise: The Determination of the Boltzmann Constant, Absolute Zero Temperature and the Charge of the Electron, MIT Department of Physics, September 3, 2013.

[6] H. Nyquist, Thermal Agitation of Electric Charge in Conductors, Physical Review, Volume 32, July 1928.

[7] Frank Rice, A frequency-domain derivation of shot-noise, Am. J. Phys. 84 (1), pp. 44-51, January 2016.

[8] 1 MHz, Single-Supply, Photodiode Amplifier Reference Design, Texas Instruments.

[9] Carlo Beenakker and Christian Schonenberger,Quantum Shot Noise: Fluctuations in the flow of electrons signal the transition from particle to wave behavior. Physics Today, page 37, May 2003.