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The Hopf Map in Magnetohydrodynamics

bachelor's thesis

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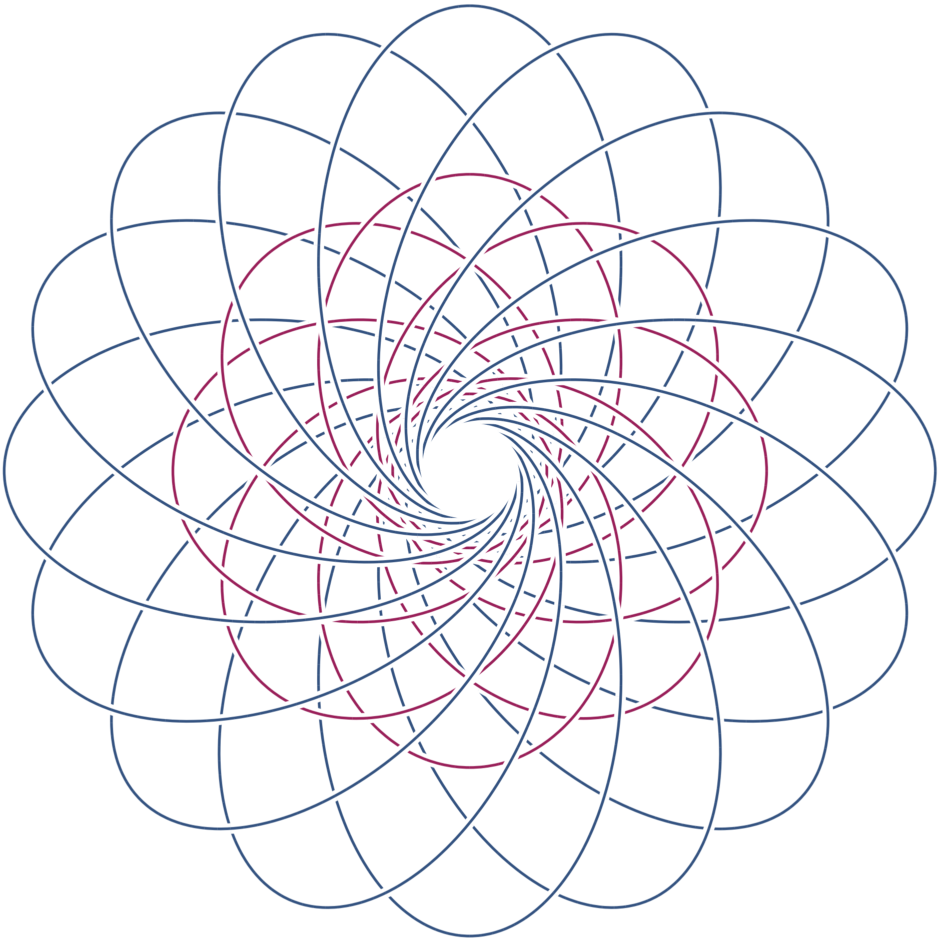
ABSTRACT

In this thesis we will investigate the Hopf map, a differentiable map from the three-sphere to the two-sphere. Its fibres, the inverse images of points on the sphere, are circles that are all linked with every other fibre. Based on the Hopf map we will construct divergenceless vector fields that have a physical interpretation as the magnetic field in the theory of magnetohydrodynamics. The concept of linking relates to helicity in this theory, a quantity that will be used to exhibit self-stable configurations of plasma.



Fibres of the Hopf map, visualised through stereographic projection. Inspired by the cover of [Hatcher 2002].

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CHAPTER 1

Introduction

One of the problems in plasma physics is the issue of plasma confinement. How does one confine a dense plasma to a reactor vessel for a sustained period of time? Plasmas are extremely hot, so any contact with the walls of a reactor would be fatal. Solving this problem is an important step towards nuclear fusion, a sustainable energy source that unlike nuclear fission does not produce radioactive byproducts. Current efforts focus on repelling the plasma from the walls of the reactor with intense magnetic fields, although other options might be feasible. One approach is that of self-stability, where the magnetic field of the plasma prevents it from deforming too much. In this thesis we will investigate how self-stability can arise, and we will construct a few magnetic fields with desirable properties.

As we will see, linking of the field lines is important for these magnetic fields. This leads us to the Hopf map, a differentiable function from S^3 to S^2 of which the fibres, the inverse images of points on S^2, are linked. Before we can define the Hopf map, we will recall some of the theory involved in chapter 2, and we will investigate a few useful group actions. In chapter 3 we will turn to the Hopf map itself. Via stereographic projection we can visualise the fibres in $\mathbb{R}^3$, and with ideas from topology we can quantify linking of the fibres. To construct a vector field with field lines based on the fibres of the Hopf map, we use tools from differential geometry developed in chapter 4. Finally we make the link to magnetohydrodynamics in chapter 5. The Hopf invariant, a quantity that appears purely algebraic at first sight, will turn out to have a direct physical interpretation as the helicity of a field, a conserved quantity that plays a role in the stability of plasmas.

Chapter 2 through 4 are mathematical in nature. For physicists who are not familiar with the formalism, or who care about results instead of proofs, a paragraph with “physical interpretation” has been added after every section whenever possible. When the theory does not admit a direct physical interpretation, a paragraph “informal summary” has been added instead.

CHAPTER 2

Preliminaries

There are several topological spaces that play a key role when describing properties of the Hopf map $h : S^3 \to S^2$. These include of course the domain S^3 and codomain S^2, which are traditionally defined as subspaces of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. As we will see later, it is useful to consider S^3 and S^2 as quotient spaces of $\mathbb{C}^2 \setminus \{0\}$ or subspaces of the quaternion algebra $\mathbb{H}$ instead. Both $\mathbb{C}^2$ and $\mathbb{H}$ are isomorphic to $\mathbb{R}^4$ as a real vector space, but their additional structure sheds light on various properties of the Hopf map. We will therefore briefly examine these spaces before defining the Hopf map. In later chapters we will explore the differentiable structure of these spaces, but for now we focus on their topological properties. Furthermore, group actions are used extensively throughout this chapter, so we quickly recall some of the terms involved.

2.1 Definitions

Definition 2.1 · Let X be a set with additional structure, such as a vector space or a topological space. The *automorphism group* of X, denoted $\mathrm{Aut}(X)$ is the group of bijections $X \to X$ that preserve its structure. More formally we can say that X is an object in a concrete category $\mathcal{A}$, a category equipped with a faithful functor $\mathcal{A} \to \mathbf{Sets}$, the forgetful functor. The automorphism group is the group of invertible morphisms $X \to X$ in this category.

Example 2.2 · For a group G, $\mathrm{Aut}(G)$ is the group of group isomorphisms $G \to G$. For a module M over a ring R, or for a vector space V over a field F, the automorphism group consists of R-linear bijections $M \to M$ and F-linear bijections $V \to V$ respectively. For a topological space X, $\mathrm{Aut}(X)$ is the group of homeomorphisms $X \to X$. For sets, the automorphism group is simply the group of bijections from the set to itself.

Definition 2.3 · Let G be a group and X a set with additional structure. A *group action* of G on X is a group homomorphism $\phi : G \to \mathrm{Aut}(X)$. If X is an object in the category

$\mathcal{A}$ (e.g. the category of vector spaces, topological spaces, etc.), we say G acts on X in this category. Given an element $x \in X$ and $g \in G$, we will often write $g \cdot x$ for the element $\phi(g)(x)$.

In the definition above, the group G acts *from the left* on X. For $g, h \in G$ and $x \in X$, we have $(gh) \cdot x = g \cdot (h \cdot x)$. Sometimes we encounter a natural *antihomomorphism* $G \to \mathrm{Aut}(X)$. In this case, we say that G acts *from the right* on X in the category $\mathcal{A}$, and to make the notation more natural we write $x \cdot g$ instead of $g \cdot x$, so that $x \cdot (gh) = (x \cdot g) \cdot h$.

Definition 2.4 · Let G and X be as before, and $x \in X$. The *orbit* of x is the set

$$Gx = \{g \cdot x \mid g \in G\}$$

Having the same orbit defines an equivalence relation on X, and the quotient with respect to this relation is called the *orbit space*, written X/G.

Definition 2.5 · Let G and X be as before, and $x \in X$. The *stabiliser* of x is the subgroup

$$G_x = \{g \in G \mid g \cdot x = x\}$$

Proposition 2.6 · Let X be a topological space, G a group that acts on X. When X/G is endowed with the quotient topology, the quotient map $q : X \twoheadrightarrow X/G$ is an open map.

Proof: Let $U \subseteq X$ be open, define $V = q(U)$. V is open if and only if $q^{-1}(V)$ is open by definition of the quotient topology. We have

$$q^{-1}(V) = \bigcup_{g \in G} g \cdot U$$

Because the automorphisms of the group action are homeomorphisms, they are open maps, so $g \cdot U$ is open for all $g \in G$. Hence, $q^{-1}(V)$ is the union of open sets, so it is open. □

Definition 2.7 · A *topological group* is a group G that is also a Hausdorff space, such that the map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$ is continuous when $G \times G$ is endowed with the product topology. This is equivalent to the statement that multiplication and inversion are continuous; see for example [Bourbaki 1971, ch. III, § 1.1] or [Szekeres 2004, p. 276].

Definition 2.8 · Let G be a topological group and X a topological space, such that G acts on X in the category of sets. The action is said to be *continuous* if

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

is a continuous map. It follows immediately from this definition that $x \mapsto g \cdot x$ is a homeomorphism for all $g \in G$ when G acts continuously on X, therefore G acts on X in the category of topological spaces; the group homomorphism $G \to \mathrm{Aut}(X)$ is actually a group homomorphism $G \to \mathrm{Homeo}(X)$.

Theorem 2.9 · Universal property of the quotient topology · Let X and Y be topological spaces and $\sim$ an equivalence relation on X. Denote by $q : X \twoheadrightarrow X/\sim$ the quotient map. Let $f : X \to Y$ be a continuous map such that for all $x, y \in X$ it holds that $x \sim y$ implies $f(x) = f(y)$. (Such f is said to be *compatible* with the equivalence relation.) Then there exists a unique continuous map $g : X/\sim \to Y$ that makes the following diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "q"', two heads] \arrow[dr, "f"] & \\ X/\sim \arrow[rr, dashed, "\exists! g"'] & & Y \end{tikzcd}$$

Proof: See for example [Bourbaki 1971, ch. I, § 3.4].

Proposition 2.10 · Let G_1 and G_2 be groups, and X a set. Suppose that $G_1 \times G_2$ acts on X in the category of sets. This implies that the subgroups G_1 and G_2 act on X individually as well. Then the following holds:

i There is a natural action of G_1 on X/G_2 in the category of sets.

ii If X is a topological space and $G_1 \times G_2$ acts in the category of topological spaces, G_1 acts on X/G_2 in this category.

iii If G_1 and G_2 are topological groups such that $G_1 \times G_2$ acts continuously on X, then G_1 acts continuously on X/G_2.

Proof: Let $x \in X$ such that $[x] \in X/G_2$, and $g \in G_1$. Define $g \cdot [x] = [g \cdot x]$. We have to show that this action is well-defined. Suppose that $[x] = [y]$ for some $y \in X$. Then there exists an $h \in G_2$ such that $x = h \cdot y$. Because h and g commute in $G_1 \times G_2$, we have

$$g \cdot x = g \cdot (h \cdot y) = (g, h) \cdot y = h \cdot (g \cdot y)$$

Thus, we find $[g \cdot x] = [g \cdot y]$. That this defines a homomorphism $G_1 \to \mathrm{Aut}(X/G_2)$ follows from the fact that $G_1 \to \mathrm{Aut}(X)$ is a homomorphism. This proves statement **i**.

Suppose that X is a topological space and $G_1 \times G_2$ acts on X in the category of topological spaces. Let $g \in G_1$, then g induces a homeomorphism $\phi : X \to X$, and a bijection $\psi : X/G_2 \to X/G_2$. Denote by $q : X \twoheadrightarrow X/G_2$ the quotient map, then $q \circ \phi$ is a continuous map $X \to X/G_2$ that satisfies $\psi \circ q = q \circ \phi$ due to statement **i**. This means that $q \circ \phi$ is compatible with the quotient map, so by the universal property of the quotient topology (theorem 2.9), there exists a unique continuous map ψ' such that $q \circ \phi = \psi' \circ q$. Uniqueness implies that $\psi' = \psi$, therefore ψ is continuous. The same argument applies to ψ^{-1}, hence ψ is a homeomorphism. This shows that G_1 acts on X/G_2 in the category of topological spaces, which proves statement **ii**.

To prove statement **iii**, we will use the following diagram:

$$\begin{tikzcd}
& X \arrow[dr, two heads, "q"] & \\
G_1 \times X \arrow[ur, "a"] \arrow[r, "f"] \arrow[dr, "r"'] & G_1 \times (X/G_2) \arrow[r, "T"] & X/G_2 \\
& (G_1 \times X)/G_2 \arrow[u, dashed, "\exists ! \phi"] \arrow[ur, dashed, "\exists ! \psi"'] &
\end{tikzcd}$$

The map $a : G_1 \times X \to X$, $(g, x) \mapsto g \cdot x$ is continuous because it is the restriction of $G_1 \times G_2 \times X \to X$ that is continuous by assumption. Let $q : X \twoheadrightarrow X/G_2$ denote the quotient map. Define $f : G_1 \times X \to G_1 \times (X/G_2)$ as $f = (\mathrm{id}, q)$. This map is continuous because both of its coordinates are. (See proposition 1 of [Bourbaki 1971, ch. I, § 4.1].) Furthermore f is open, for id and q are open. (See proposition 2.6.) Let G_2 act on $G_1 \times X$ by $h \cdot (g, x) = (g, h \cdot x)$ where $g \in G_1, h \in G_2, x \in X$, and let r denote the quotient map. f is compatible with r, so by the universal property of the quotient topology (theorem 2.9), there exists a unique continuous map ϕ that makes the bottom left triangle of the diagram commute. The map is given by $[g, x] \mapsto (g, [x])$ and its inverse is given by $(g, [x]) \mapsto [g, x]$. An open set in $(G_1 \times X)/G_2$ is the image under r of an open set in $G_1 \times X$, so from commutativity it follows that ϕ is an open map. Hence, ϕ is a homeomorphism.

On the top of the diagram, we have the map $q \circ a : G_1 \times X \to X/G_2$, given by $(g, x) \mapsto [g \cdot x]$. As the composition of continuous maps it is continuous, and it is compatible with r. Thus, by the universal property of the quotient topology, there exists a unique continuous map ψ such that $\psi \circ r = q \circ a$. Composing with ϕ^{-1}, we find that the map

$$T : G_1 \times (X/G_2) \longrightarrow X/G_2, \quad (g, [x]) \longmapsto [g \cdot x]$$

is continuous, which proves claim **iii**. Furthermore, the above diagram commutes. □

Theorem 2.11 · Let G_1 and G_2 be groups and X a topological space such that $G_1 \times G_2$ acts on X. Then $X/(G_1 \times G_2)$ is canonically homeomorphic to $(X/G_1)/G_2$. In particular, the quotient map $X \twoheadrightarrow X/(G_1 \times G_2)$ factors over X/G_1.

Proof: Let $q_1 : X \twoheadrightarrow X/G_1$, $q_2 : (X/G_1) \twoheadrightarrow (X/G_1)/G_2$, and $q_{12} : X \twoheadrightarrow X/(G_1 \times G_2)$ denote the quotient maps. Then we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, two heads, "q_1"] \arrow[d, two heads, "q_{12}"] & X/G_1 \arrow[d, two heads, "q_2"] \arrow[dl, dashed, "\phi_1"'] \\
X/(G_1 \times G_2) \arrow[r, dashed, bend left, "\phi_{12}"] & (X/G_1)/G_2 \arrow[l, dashed, bend left, "\phi_2"]
\end{tikzcd}$$

The map $q_2 \circ q_1$ is continuous and compatible with q_{12}, so by the universal property of the quotient topology (theorem 2.9) there exists a unique continuous map ϕ_{12} that makes the diagram commute. Because q_{12} is compatible with q_1, there exists a unique continuous map ϕ_1 such that $q_{12} = \phi_1 \circ q_1$. It follows that ϕ_1 is compatible with q_2, so there exists a unique continuous map ϕ_2 that makes the diagram commute. Now we see that ϕ_{12} and ϕ_2 are continuous inverses of one another, hence $X/(G_1 \times G_2)$ and $(X/G_1)/G_2$ are homeomorphic. □

Physical interpretation

Groups are prevalent in mathematics. In physics, groups are often encountered in the context of symmetries. In that case one may think of a group as a set of transformations of a system, transformations under which a certain property is invariant. For instance, angular momentum is invariant under rotation of space, and four-momentum is invariant under Lorentz transformations. A *group action* generalises this idea. Elements of the group induce a transformation of a system. By applying all possible transformations to a point, we obtain the *orbit* of a point. For instance, when we let the Lorentz group act on Minkowski space, the orbit of a timelike vector is all of the light cone (past and future). Often, a group encodes transformations that we are *not* interested in. The *orbit space* is what remains if we consider points that differ by such a transformation to be equal. For example, the orbit space of the Lorentz group action on Minkowski space consists of four elements: the origin, the class of null (or light-like) vectors, the class of timelike vectors, and the class of spacelike vectors. The *stabiliser* of a point is the subgroup of transformations under which the point is invariant.

Topology is the branch of mathematics that studies abstract properties of space. It gives us the tools to study properties that do not depend on exact distances, but rather on overall shape. For instance, one would like to think of a garden hose as a one dimensional system where water can move back and forth, regardless of how the hose is bent or twisted. Topology allows us to ignore the bending and twisting. Virtually all spaces that occur in physics are topological spaces: $\mathbb{R}^3$, Minkowski space, Hilbert spaces, etc. Often these spaces have additional structure such as a metric or inner product, but many properties can be derived from the topology alone. An important example of such a property is *continuity* of a map between topological spaces, a notion that is prevalent throughout physics. Many of the groups encountered in this thesis happen to have a natural topology as well. In this case, an action on another topological space can be *continuous*. The definition given in this section codifies our intuition: if two group elements that are near act on a point, the resulting points should be near as well.

2.2 Projective space

It is possible to identify $\mathbb{R}^4$ and $\mathbb{C}^2$ as four-dimensional real vector spaces, by identifying the standard basis (e_1, e_2, e_3, e_4) with the basis $((1,0),(i,0),(0,1),(0,i))$. The space $\mathbb{C}^2 \setminus \{0\}$ will be prevalent in the rest of this section, so we introduce a shorthand notation. Furthermore, we embed S^1 in $\mathbb{C}$.

Definition 2.12 · $\mathbb{C}^2_\circ = \mathbb{C}^2 \setminus \{0\}$.

Definition 2.13 · The *unit circle* is defined by

$$S^1 = \{\, z \in \mathbb{C} \mid 1 = |z| \,\}$$

This is a group under multiplication.

Definition 2.14 · The *three-sphere* is defined by

$$S^3 = \{\, x \in \mathbb{R}^4 \mid 1 = \|x\|^2 \}$$

Here $\|\cdot\|$ denotes the regular Euclidean norm. By identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ as above, we can consider S^3 to be a subset of $\mathbb{C}^2_\circ$.

Consider the multiplicative group $\mathbb{R}_{>0}$ of positive real numbers. It acts continuously on $\mathbb{C}^2$ (in the category of real vector spaces) by scalar multiplication, and this action can be restricted to $\mathbb{C}^2_\circ$ (in the category of sets). This allows us to give an alternative definition of S^3 as a quotient:

Definition 2.15 · $S^3_\mathbb{C}$ is the orbit space of $\mathbb{C}^2_\circ$ with respect to the $\mathbb{R}_{>0}$ action. Denote by $r : \mathbb{C}^2_\circ \twoheadrightarrow S^3_\mathbb{C}$ the quotient map. $\mathbb{C}^2$ is endowed with its regular topology induced by the Euclidean metric, and $S^3_\mathbb{C}$ is endowed with the quotient topology.

Intuitively, this definition is not that different from definition 2.14. Every point p at the three-sphere defines a ray from the origin through p. This ray, except for the origin, is the orbit of p under the $\mathbb{R}_{>0}$ action. In other words, every orbit can be represented by a point at unit distance from the origin. The quotient map r corresponds to projection onto the sphere.

Proposition 2.16 · S^3 and $S^3_\mathbb{C}$ as defined in definition 2.14 and 2.15 are homeomorphic.

Proof: Write $\mathbb{R}^4 \setminus \{0\} = \mathbb{R}^4_\circ$. Let $i : S^3 \to \mathbb{R}^4_\circ$ be the inclusion, and let $\phi : \mathbb{R}^4 \to \mathbb{C}^2$ be the vector space isomorphism induced by the identification of the bases given earlier in this section. The inclusion i is continuous, and the restriction $\phi|_{\mathbb{R}^4_\circ} = \phi_\circ$ is a homeomorphism. Therefore, the composition $\psi = r \circ \phi_\circ \circ i : S^3 \to S^3_\mathbb{C}$ is continuous. Consider the map

$$\mathbb{C}^2_\circ \longrightarrow S^3, \quad x \longmapsto \phi_\circ^{-1}\left(\frac{x}{\|x\|}\right)$$

which is continuous and compatible with r. By the universal property of the quotient topology (theorem 2.9), this map induces a unique continuous map $\psi^{-1} : S^3_{\mathbb{C}} \to S^3$ that is the inverse of ψ. Thus, ψ is a homeomorphism. □

Consider the multiplicative group $\mathbb{C}^*$ (the complex plane minus the origin). It acts continuously on $\mathbb{C}^2$ (in the category of complex vector spaces) by scalar multiplication, and this action can be restricted to $\mathbb{C}^2_\circ$. This allows us to define the projective space:

Definition 2.17 · The *complex projective line* $\mathbb{P}^1(\mathbb{C})$ is the orbit space of $\mathbb{C}^2_\circ$ with respect to the $\mathbb{C}^*$ action. Denote by $q : \mathbb{C}^2_\circ \twoheadrightarrow \mathbb{P}^1(\mathbb{C})$ the quotient map. $\mathbb{P}^1(\mathbb{C})$ is endowed with the quotient topology.

Elements of $\mathbb{P}^1(\mathbb{C})$ are indicated by *homogeneous coordinates*: if $(z_1, z_2) \in \mathbb{C}^2$ is nonzero, then we write $(z_1 : z_2)$ for $q(z_1, z_2)$. We can embed $\mathbb{C}$ in $\mathbb{P}^1(\mathbb{C})$ via $z \mapsto (z : 1)$. The only point that is not reached in this manner is $(1 : 0)$.

Theorem 2.18 · There exists a homeomorphism between S^2 and $\mathbb{P}^1(\mathbb{C})$.

Proof: We will postpone the proof until section 3.2, and prove this with the aid of quaternions in theorem 3.4. For an alternative proof, see [Bourbaki 1974, ch. VIII, § 4.3].

The general linear group $\mathrm{GL}_2(\mathbb{C})$ of invertible complex 2×2 matrices acts on $\mathbb{C}^2$ by matrix multiplication. This induces a group action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2_\circ$. Furthermore, the groups $\mathbb{R}_{>0}$, S^1, and $\mathbb{C}^*$ are isomorphic to subgroups of $\mathrm{GL}_2(\mathbb{C})$: given an element $z \in \mathbb{C}^*$, we can identify it with the matrix

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

in the centre of $\mathrm{GL}_2(\mathbb{C})$. $\mathbb{C}^*$ is isomorphic to the direct product $\mathbb{R}_{>0} \times S^1$: this is the decomposition of a complex number into its modulus and argument. It follows that S^1 and $\mathbb{R}_{>0}$ are central in $\mathrm{GL}_2(\mathbb{C})$, because their elements correspond to scalar matrices. Consequently, S^1 and $\mathbb{R}_{>0}$ are normal in $\mathrm{GL}_2(\mathbb{C})$.

Informal summary

The *projective space* $\mathbb{P}^1(\mathbb{C})$ is a construction with several interpretations. For starters, $\mathbb{P}^1(\mathbb{C})$ can be thought of as $\mathbb{C}$ with one extra point, a point "at infinity". This allows us to talk about z_1/z_2 even when z_2 is zero. Instead of z_1/z_2, we write $(z_1 : z_2)$, called *homogeneous coordinates*. Secondly, theorem 2.18 tells us that $\mathbb{P}^1(\mathbb{C})$ can be thought of as the unit sphere S^2. (In fact, $\mathbb{P}^1(\mathbb{C})$ is sometimes called the *Riemann sphere*.) A *homeomorphism* between two spaces is a function, both one-to-one and onto, that preserves all topological properties. From a topological point of view, $\mathbb{P}^1(\mathbb{C})$ and S^2 are the same space. This means that when we formulate the Hopf map later on — a function from S^3 to S^2 — we can express it as a function to $\mathbb{P}^1(\mathbb{C})$. This expression is significantly simpler than the one involving Cartesian coordinates on S^2.

2.3 Quaternions

Definition 2.19 · The *quaternion algebra* $\mathbb{H}$ is the real noncommutative algebra with basis $(1, i, j, k)$. Multiplication is given by the identities

$$i^2 = j^2 = k^2 = -1, \quad ij = k,\ jk = i,\ ki = j, \quad ji = -k,\ kj = -i,\ ik = -j$$

and 1 commutes with all elements. In particular, $\mathbb{H}$ is a ring and a four-dimensional real vector space. Analogously to complex numbers, this algebra has an involution $\overline{\cdot}$ called *conjugation* that flips the sign of the i, j, and k components.

Definition 2.20 · The *trace* is the map $\mathrm{Tr} : \mathbb{H} \to \mathbb{R},\ q \mapsto q + \overline{q}$. Because the imaginary parts cancel, the trace of a quaternion is real. Furthermore, the trace is $\mathbb{R}$-linear.

The reals commute with all quaternions, so $\mathrm{Tr}(q)$ commutes with q for all $q \in \mathbb{H}$. Because $\overline{q} = \mathrm{Tr}(q) - q$, it follows that q and $\overline{q}$ commute.

Definition 2.21 · The standard inner product on $\mathbb{H}$ is given by

$$\langle \cdot , \cdot \rangle : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}, \quad (p, q) \longmapsto \tfrac{1}{2}\,\mathrm{Tr}(p\overline{q}) = \tfrac{1}{2}(p\overline{q} + q\overline{p})$$

Symmetry is clear from the definition, and bilinearity follows from the linearity of the trace. For positive definiteness, remark that for $q = a + bi + cj + dk$, we have $q\overline{q} = a^2 + b^2 + c^2 + d^2$. Therefore $\langle q, q\rangle \geq 0$, and $\langle q, q\rangle = 0 \Rightarrow a = b = c = d = 0 \Rightarrow q = 0$.

Definition 2.22 · The *norm* of $q \in \mathbb{H}$ is given by $\|q\|^2 = q\overline{q}$. Because q and $\overline{q}$ commute, $q\overline{q} = \frac{1}{2}(q\overline{q} + \overline{q}q)$, so the norm is induced by the inner product. This norm coincides with the Euclidean norm on $\mathbb{H}$ as real vector space with orthonormal basis $(1, i, j, k)$. Therefore, $\mathbb{H}$ with the topology induced by the norm is homeomorphic to $\mathbb{R}^4$.

Because conjugation reverses the order of multiplication, the norm is multiplicative: for $p, q \in \mathbb{H}$, we have

$$\|pq\|^2 = (pq)\overline{(pq)} = p\, q\overline{q}\,\overline{p} = p\|q\|^2\,\overline{p} = \|q\|^2 p\overline{p} = \|q\|^2\|p\|^2$$

Because $q\overline{q} = \|q\|^2$, we have $q^{-1} = \overline{q}\,\|q\|^{-2}$ for $\|q\| \neq 0$. Therefore, $\mathbb{H}$ is a division algebra: every nonzero element has an inverse.

Proposition 2.23 · $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$ is a topological group.

Proof: Multiplication is continuous, because for $p, q \in \mathbb{H}^*$, the components of the product pq can be written as a polynomial in the components of p and q. Inversion is continuous, because the components of q^{-1} are rational functions of the components of q, which do not vanish because $\|q\|^2 \neq 0$. See also [Bourbaki 1974, ch. VIII, § 1.4]. □

With this machinery, we can give a quaternionic definition of S^3 and S^2. Whereas the definitions in section 2.2 emphasise how S^3 and S^2 are quotients with respect to a group action, the quaternionic definitions emphasise the group structure on the three-sphere itself, and the action of S^3 on S^2.

Let us revisit the three-sphere as defined in definition 2.14. By identifying $\mathbb{H}$ with $\mathbb{R}^4$ as a normed real vector space via the basis given earlier in this section, we can consider S^3 to be a subset of $\mathbb{H}$, the set of quaternions with unit norm:

$$S^3 = \{q \in \mathbb{H} \mid 1 = \|q\|^2\}$$

This set is closed under multiplication due to the multiplicativity of the norm, and it contains 1. Therefore, this is a subgroup of $\mathbb{H}^*$. We can embed S^2 in S^3, but in $\mathbb{R}^4$ there is no preferred way of doing so. For quaternions, there is one natural choice:

Definition 2.24 · The *two-sphere* $S^2 = \{q \in S^3 \mid \mathrm{Tr}(q) = 0\}$, the set of pure imaginary quaternions with unit norm. This definition coincides with the conventional definition of S^2 when $\mathbb{R}^3$ is identified with the subspace of $\mathbb{H}$ spanned by i, j, and k. S^2 may alternatively be written as $\{q \in S^3 \mid \langle 1, q\rangle = 0\} = 1^\perp \cap S^3$.

The group $\mathbb{H}^*$ acts on $\mathbb{H}$ in the category of $\mathbb{R}$-algebras via the following homomorphism:

$$\phi : \mathbb{H}^* \longrightarrow \mathrm{Aut}(\mathbb{H}), \quad p \longmapsto (q \mapsto pqp^{-1})$$

Because quaternion multiplication is continuous, this is a continuous action. By restriction to the subgroup S^3, we get a continuous action of S^3 on $\mathbb{H}$.

Proposition 2.25 · The inner product on $\mathbb{H}$ is invariant under the action of $\mathbb{H}^*$.

Proof: Let $p \in \mathbb{H}^*$, $q_1, q_2 \in \mathbb{H}$, then we have

$$\begin{aligned}
2\langle p \cdot q_1, p \cdot q_2\rangle &= pq_1p^{-1}\overline{pq_2p^{-1}} + pq_2p^{-1}\overline{pq_1p^{-1}} \\
&= pq_1\|p^{-1}\|^2\overline{q_2}\,\overline{p} + pq_2\|p^{-1}\|^2\overline{q_1}\,\overline{p} \\
&= \|p^{-1}\|^2 p(q_1\overline{q_2} + q_2\overline{q_1})\overline{p} \\
&= \|p^{-1}\|^2\,\|p\|^2\, 2\langle q_1, q_2\rangle \\
&= 2\langle q_1, q_2\rangle
\end{aligned}$$

□

Corollary 2.26 · Identify $\mathbb{R}^3$ with the subspace of $\mathbb{H}$ spanned by i, j, and k. Then $\mathbb{R}^3 = 1^\perp$ and $S^2 = 1^\perp \cap S^3$ are invariant under the action of $\mathbb{H}^*$, which means $\mathbb{H}^*$ and its subgroup S^3 act continuously on $\mathbb{R}^3$ and S^2.

$\mathbb{C}$ is a commutative subring of $\mathbb{H}$. As real vector spaces with bases $(1, i)$ and $(1, i, j, k)$, $\mathbb{C}$ can be identified with the subspace of $\mathbb{H}$ spanned by 1 and i. The stabiliser of $i \in \mathbb{H}$ consists of the nonzero elements that commute with i. These elements are linear combinations of 1 and i, so we have $\mathbb{H}_i^* = \mathbb{C}^*$ and $S_i^3 = S^1$.

Proposition 2.27 · S^3 is isomorphic to SU_2, the group of unitary 2×2 matrices with determinant 1.

Proof: Define the unitary matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices are sometimes called the *Pauli spin matrices*. Let $\phi : \mathbb{H} \to \mathrm{Mat}(2 \times 2, \mathbb{C})$ be the $\mathbb{R}$-linear extension of

$$1 \longmapsto I, \quad i \longmapsto i\sigma_1, \quad j \longmapsto i\sigma_2, \quad k \longmapsto i\sigma_3$$

Let ψ be the restriction of ϕ to S^3. All matrices in the image of ψ are unitary, and a little computation shows that for $q \in S^3$, $\det \psi(q) = 1$. The matrices $I, i\sigma_1, i\sigma_2, i\sigma_3$ satisfy the same multiplication rules as $1, i, j, k$. That is, $i\sigma_1 i\sigma_2 = i\sigma_3$, etc. Therefore, ψ is a group homomorphism $S^3 \to \mathrm{SU}_2$. This homomorphism is surjective (see [Szekeres 2004, p. 173]), and 1 is the only element in its kernel. Therefore, ψ is an isomorphism. □

Theorem 2.28 · The map

$$\phi : S^3 \longrightarrow \mathrm{SO}_3(\mathbb{R}), \qquad q \longmapsto (x \mapsto q \cdot x)$$

is a surjective group homomorphism with kernel $\{\pm 1\}$. Here $x \in \mathbb{R}^3 \cong \mathrm{Span}(i, j, k)$.

Proof: The map $x \mapsto q \cdot x$ is linear, and orthogonality follows from the fact that the inner product is invariant under the action, as shown in proposition 2.25. To show that $x \mapsto q \cdot x$ is not a reflection, note that $\det : \mathrm{O}_3(\mathbb{R}) \to \mathbb{R}$ is a continuous map (see for example [Hatcher 2002, p. 281]). We can express ϕ as a polynomial on all coordinates when elements of $\mathrm{SO}_3(\mathbb{R})$ are written as matrices, so ϕ is continuous. By composition we get a continuous map $S^3 \to \{\pm 1\}$. Because S^3 is connected, this map must be constant. The determinant of id is 1, so all $q \in S^3$ induce an orthogonal map with positive determinant.

To show that the kernel of ϕ is $\{\pm 1\}$, suppose that $q \in S^3$ is such that $q \cdot x = qxq^{-1} = x$ for all $x \in \mathbb{R}^3$. Then q commutes with all $x \in \mathbb{R}^3$, so q must be real. Because $\|q\|^2 = 1$, it follows that $q = 1$ or $q = -1$.

To prove surjectivity, suppose that $\rho \in \mathrm{SO}_3(\mathbb{R})$ is an anticlockwise rotation of α radians about an axis spanned by $u \in \mathbb{R}^3$, where $\|u\|^2 = 1$. Then the quaternion $q = \cos(\frac{1}{2}\alpha) + u \sin(\frac{1}{2}\alpha)$ will map to ρ. To see this, note that all points on the axis of rotation are fixed points, for u commutes with q. Furthermore, suppose that $v \in \mathbb{R}^3$ is such that $\langle u, v \rangle = 0$.

Set $q_0 = \cos(\frac{1}{2}\alpha)$ and $\vec{q} = u\sin(\frac{1}{2}\alpha)$. By using identities from [Szekeres 2004, p. 157], we find

$$\begin{aligned}
q \cdot v &= (q_0 + \vec{q})v(q_0 - \vec{q}) \\
&= (q_0 + \vec{q})(q_0 v - v \times \vec{q}) \\
&= -\langle \vec{q}, q_0 v - v \times \vec{q} \rangle + q_0(q_0 v - v \times \vec{q}) + \vec{q} \times (q_0 v - v \times \vec{q}) \\
&= q_0^2 v - q_0 v \times \vec{q} + q_0 \vec{q} \times v - \vec{q} \times (v \times \vec{q}) \\
&= q_0^2 v - 2q_0 v \times \vec{q} - v\langle \vec{q}, \vec{q} \rangle + \vec{q}\langle \vec{q}, v \rangle \\
&= (\cos^2(\tfrac{1}{2}\alpha) - \sin^2(\tfrac{1}{2}\alpha))v - 2\cos(\tfrac{1}{2}\alpha)\sin(\tfrac{1}{2}\alpha)\, v \times u \\
&= \cos(\alpha)v + \sin(\alpha)\, u \times v
\end{aligned}$$

This demonstrates that q rotates v anticlockwise by α radians about u. We saw already that $x \mapsto q \cdot x$ is an orthogonal map with determinant 1. Therefore, q maps to ρ. □

Corollary 2.29 · S^3 acts transitively on S^2, for every point on S^2 can be mapped into any other point on S^2 by a rotation of the sphere.

The proof of theorem 2.28 gives us a way to explicitly get a $q \in S^3$ such that $q \cdot i = p$ for any $p \in S^2$: we rotate i onto p with a rotation of $\mathbb{R}^3$. If $p = -i$, $q = j$ will suffice, so suppose $p \neq -i$. Then an axis that we can rotate about is the one spanned by $i + p$, which bisects the angle between i and p, so we need to rotate by π radians. We find

$$q = \frac{i + p}{\|i + p\|} \tag{2.30}$$

To verify that this works, note that for $p \in S^2$ we have $p\overline{p} = 1$ and $\overline{p} = -p$, so $p^2 = -1$. It then follows that

$$p^2 = -1 \quad \Rightarrow \quad p(i + p) = (i + p)i \quad \Rightarrow \quad p(i + p)\overline{(i + p)} = (i + p)i\overline{(i + p)}$$

Multiplying by $\|i + p\|^{-2}$ on both sides then yields $p = qiq^{-1}$.

Physical interpretation

Just like complex numbers are an extension of the real numbers, quaternions are an extension of the complex numbers. These extensions come at a cost: when going from $\mathbb{R}$ to $\mathbb{C}$, you have to give up the ordering. When going from $\mathbb{C}$ to $\mathbb{H}$, you have to give up commutativity. Apart from their rich structure that is interesting in its own right, quaternions have many useful applications. By considering S^3 as a subset of $\mathbb{H}$, it inherits a group structure. Theorem 2.28 tells us that this group is in a sense twice $\mathrm{SO}_3(\mathbb{R})$: every rotation of $\mathbb{R}^3$ is represented by two antipodal quaternions. When traversing a great circle through 1 in S^3, the points 1 and -1 both correspond to the identity in $\mathrm{SO}_3(\mathbb{R})$.

This path in S^3 corresponds to a 4π rotation of $\mathbb{R}^3$, and after a 2π rotation we will have moved from $1 \in S^3$ to $-1 \in S^3$. This property is reminiscent of *spinors*, and indeed proposition 2.27 links the unit quaternions to the Pauli spin matrices.

CHAPTER 3

The Hopf map

The different definitions of S^3 and S^2 that were explored in the previous chapter go along with different definitions of the Hopf map. In this chapter we will give those definitions, and show that they are equivalent in the following sense: if h_{proj} and h_{quat} represent the projective and quaternionic definition of the Hopf map respectively, then the following diagram commutes:

$$\begin{tikzcd} S^3_{\mathbb{C}} \arrow[r, two heads, "h_{\text{proj}}"] \arrow[d, leftrightarrow] & \mathbb{P}^1(\mathbb{C}) \arrow[d, leftrightarrow] \\ S^3 \arrow[r, two heads, "h_{\text{quat}}"] & S^2 \end{tikzcd}$$

S^3 and $S^3_{\mathbb{C}}$ were shown to be homeomorphic in proposition 2.16. In theorem 2.18 it was stated that $\mathbb{P}^1(\mathbb{C})$ and S^2 are homeomorphic, which we will be able to prove at last. Finally, the group actions explored in the previous chapter will be used to examine the fibres of the Hopf map.

3.1 The projective Hopf map

As we saw in section 2.2, $\mathbb{P}^1(\mathbb{C})$ can be defined as a quotient of $\mathbb{C}^2_\circ$ with respect to the action of $\mathbb{C}^*$. Because $\mathbb{C}^* \cong \mathbb{R}_{>0} \times S^1$, the quotient map factors over $S^3_{\mathbb{C}}$ by theorem 2.11, where $S^3_{\mathbb{C}} = \mathbb{C}^2_\circ \,/\, \mathbb{R}_{>0}$ as in definition 2.15. This allows us to define the Hopf map:

Definition 3.1 · The *Hopf map* is the unique continuous map $h : S^3_{\mathbb{C}} \twoheadrightarrow \mathbb{P}^1(\mathbb{C})$ that makes the following diagram commute:

$$\begin{tikzcd} & \mathbb{C}^2_\circ \arrow[dl, "r"'] \arrow[dr, "q"] & \\ S^3_{\mathbb{C}} \arrow[rr, two heads, "h"] & & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

By theorem 2.11, the Hopf map is the quotient map of the S^1-action on $S^3_{\mathbb{C}}$.

This definition tells a lot about the Hopf map already. It shows that its *fibres* — the inverse images of points in $\mathbb{P}^1(\mathbb{C})$ — are orbits of the S^1-action; the fibres can be parametrised by S^1. In section 3.3 we will explore the geometry of the fibres, which will turn out to be great circles on S^3. Furthermore, because h is the quotient map of a group action, it is surjective, continuous, and open by proposition 2.6.

3.2 The quaternionic Hopf map

In section 2.3 we defined S^3 and S^2 as subsets of $\mathbb{H}$, with S^3 acting on S^2. With this action, we can define the Hopf map as follows:

Definition 3.2 · The *Hopf map* is the map

$$h : S^3 \longrightarrow S^2, \quad q \longmapsto q^{-1} \cdot i$$

Recall that '$\cdot$' denotes the action, $q^{-1} \cdot i = q^{-1}iq$. For quaternion multiplication we will simply use juxtaposition. The reason that we choose $q^{-1} \cdot i$ here instead of $q \cdot i$, will become clear in theorem 3.4. An other way to think of this, is that h is the map $q \mapsto i \cdot q = q^{-1}iq$, where $\mathbb{H}^*$ acts *from the right* on $\mathbb{H}$. The quaternionic definition is more suitable for doing computations than the projective definition, because it allows us to work with Cartesian coordinates on S^2. The image of the quaternion $q = a + bi + cj + dk \in S^3$ under the Hopf map is given by

$$\begin{aligned} h(q) &= q^{-1} \cdot i = q^{-1}\, i\, q = \overline{q}\, i\, q \\ &= (a - bi - cj - dk)\, i\, (a + bi + cj + dk) \\ &= (a^2 + b^2 - c^2 - d^2)i + 2(bc - ad)j + 2(ac + bd)k \end{aligned} \tag{3.3}$$

Because h is given by polynomial equations on every coordinate, it is continuous. Because S^3 acts transitively on S^2 by corollary 2.29, h is surjective.

The projective definition of the Hopf map in section 3.1 emphasises that the fibres of the Hopf map are orbits of a group action. The quaternionic definition given here, instead emphasises *stabilisers* of a group action. The fibre above i consists of all $q \in S^3$ with $q^{-1} \cdot i = i$, the stabiliser S^3_i of i. The fibre above $p \in S^2$ is a right coset of the stabiliser. Because S^3 acts transitively on S^2 by corollary 2.29, there exists an $x \in S^3$ such that $p = x \cdot i$. The fibre above p consists of all $q \in S^3$ such that $q^{-1} \cdot i = p$. It follows that $q \cdot x \cdot i = q \cdot p = i$, thus qx stabilises i and $h^{-1}(p) = S^3_i x^{-1}$.

At present, it is not at all obvious that the Hopf map as defined in definition 3.1 is related to the Hopf map as defined in definition 3.2. On the contrary: we have not even proven that the codomains $\mathbb{P}^1(\mathbb{C})$ and S^2 are homeomorphic. Fortunately, we can prove both statements at once.

Theorem 3.4 · Denote by $S^3_{\mathbb{C}}$ and S^3 the three-sphere as defined by definition 2.15 and 2.14 respectively. Let $\Psi : \mathbb{C}^2_\circ \to \mathbb{H}^*$ be the restriction of the $\mathbb{R}$-linear isomorphism $(z_1, z_2) \mapsto z_1 + z_2 j$. From proposition 2.16 it follows that Ψ descends to $\psi : S^3_{\mathbb{C}} \to S^3$. Denote by r the quotient map, by s projection onto S^3, and by h_{proj} and h_{quat} the Hopf map as defined in definition 3.1 and 3.2 respectively. Then there exists a unique homeomorphism $\phi : \mathbb{P}^1(\mathbb{C}) \to S^2$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathbb{C}^2_\circ \arrow[r, two heads, "r"] \arrow[d, "\Psi"] & S^3_{\mathbb{C}} \arrow[r, two heads, "h_{\text{proj}}"] \arrow[d, "\psi"] & \mathbb{P}^1(\mathbb{C}) \arrow[d, dashed, "\exists!\phi"] \\
\mathbb{H}^* \arrow[r, two heads, "s"] & S^3 \arrow[r, two heads, "h_{\text{quat}}"] & S^2
\end{tikzcd}$$

Proof: We will show that $\Phi = h_{\text{quat}} \circ s \circ \Psi$ is compatible with the quotient map $\mathbb{C}^2_\circ \twoheadrightarrow \mathbb{P}^1(\mathbb{C})$. Let $(z_1, z_2) \in \mathbb{C}^2_\circ$ and $z \in \mathbb{C}^*$, such that $(z_1 : z_2) = (zz_1 : zz_2)$. Because $\mathbb{C}^*$ stabilises $i \in S^2$, we have

$$\Phi(zz_1, zz_2) = (zz_1 + zz_2 j)^{-1} \cdot i = (z_1 + z_2 j)^{-1} \cdot (z^{-1} \cdot i) = (z_1 + z_2 j)^{-1} \cdot i = \Phi(z_1, z_2)$$

By the universal property of the quotient topology (theorem 2.9), there exists a unique continuous map ϕ that makes the diagram commute.

To give the inverse of ϕ, let a point $p \in S^2$ be given. We saw before that there exists an $x \in S^3$ with $p = x \cdot i$, such that $h^{-1}_{\text{quat}}(p) = S^3_i x^{-1}$. Because $S^3_i = S^1 \subseteq \mathbb{C}^*$, we can write every point in S^3 that maps to p as zx^{-1} for some $z \in S^1$, and we can write x^{-1} as $z_1 + z_2 j$ for some $z_1, z_2 \in \mathbb{C}$. Therefore, all points in the fibre above p map to $(z_1 : z_2)$ under $h_{\text{proj}} \circ \psi^{-1}$. To show that this does not depend on the choice of x, note that if we had $y \in S^3$ with $p = y \cdot i$, then $x = yz^{-1}$ for some $z \in S^1$, so $x^{-1} = zy^{-1}$.

Recall that $S^3_{\mathbb{C}}$ is compact by proposition 2.16, so $\mathbb{P}^1(\mathbb{C})$ is compact, for it is the continuous image of a compact space. S^2 is Hausdorff because it is a subspace of $\mathbb{H}$, which is Hausdorff. Therefore, ϕ is a continuous bijection from a compact space to a Hausdorff space. It follows that ϕ is a homeomorphism. (See for instance theorem 3.3.11 of [Runde 2005, p. 81].) □

Using equation 3.3, we can give an explicit expression for ϕ. Using equation 2.30, we can give an explicit expression for ϕ^{-1}:

$$\begin{aligned}
(a + bi : c + di) &\longmapsto (a^2 + b^2 - c^2 - d^2)i + 2(bc - ad)j + 2(ac + bd)k \\
(i + \alpha i : \beta + \gamma i) &\longleftarrow \alpha i + \beta j + \gamma k
\end{aligned} \tag{3.5}$$

It is assumed here that $a^2 + b^2 + c^2 + d^2 = 1$ and $\alpha^2 + \beta^2 + \gamma^2 = 1$, with $\alpha \neq -1$. For $\alpha = -1$, we have $\phi^{-1}(-i) = (0 : 1)$. Note that it is impossible to give a globally valid expression for ϕ^{-1}: the map $\phi \circ h : S^3_{\mathbb{C}} \twoheadrightarrow S^2$ does not admit a global continuous section. If it did, this would imply that $S^3 = S^2 \times S^1$, which is not the case.

Informal summary

In this section and in the previous section, we have given two definitions of the Hopf map. The projective definition as given in definition 3.1 can be written as $h : S^3_{\mathbb{C}} \twoheadrightarrow \mathbb{P}^1(\mathbb{C})$, $(z_1, z_2) \mapsto (z_1 : z_2)$. Recall that if z_2 is nonzero, $(z_1 : z_2)$ may be thought of as z_1/z_2. This shows that multiplying z_1 and z_2 by e^{it} for any $t \in \mathbb{R}$ does not change the image under the Hopf map. It follows that the fibres of the Hopf map are circular, a feature that will be explored further in the next section. Definition 3.2 gives an alternative definition of the Hopf map based on quaternions. This definition is useful for doing computations, because it allows us to work with Cartesian coordinates on S^2. Theorem 3.4 shows that both definitions are equivalent. This theorem also shows that $\mathbb{P}^1(\mathbb{C})$ and S^2 are *homeomorphic*, meaning that for topological purposes they are indistinguishable.

3.3 Fibres

The Hopf map, a surjective, continuous map from S^3 to S^2, is interesting for many reasons. The primary reason that we are interested in it here, are its fibres. Those are circles in S^3 that — as we will see in section 3.5 — are linked, like keyrings can be linked. Moreover, *all* fibres are linked with *every* other fibre. Before we can study linking however, we will first introduce the tools for studying the fibres.

In section 3.2, we saw already that the fibre above $p \in S^2$ is given by $S^1 x^{-1}$, where $x \in S^3$ is such that $x \cdot i = p$. In combination with equation 2.30 (an expression for x), this allows us to explicitly parametrise the fibres of the Hopf map. While such an expression is useful for computations, it does not give us any geometrical insight. Therefore, we will study the fibes of the Hopf map in a different way. For this, we will first revisit the $\mathrm{GL}_2(\mathbb{C})$ action on $\mathbb{C}^2_\circ$.

In section 2.2 we saw how $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{C}^2_\circ$. Every element of $\mathrm{GL}_2(\mathbb{C})$ induces a homeomorphism $\mathbb{C}^2_\circ \to \mathbb{C}^2_\circ$. These homeomorphisms are restrictions of $\mathbb{C}$-linear (and thereby also $\mathbb{R}$-linear) automorphisms $\mathbb{C}^2 \to \mathbb{C}^2$, which means the action descends to $S^3_{\mathbb{C}}$ and $\mathbb{P}^1(\mathbb{C})$.

Proposition 3.6 · Let $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ be a $\mathbb{C}$-linear automorphism. Then there exist unique homeomorphisms β, γ, δ that make the following diagram commute:

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[d, "\alpha"] & \mathbb{C}^2_\circ \arrow[l, hook', "i"'] \arrow[r, two heads, "r"] \arrow[d, dashed, "\exists! \beta"] \arrow[rr, bend left, two heads, "q"] & S^3_{\mathbb{C}} \arrow[r, two heads, "h"] \arrow[d, dashed, "\exists! \gamma"] & \mathbb{P}^1(\mathbb{C}) \arrow[d, dashed, "\exists! \delta"] \\
\mathbb{C}^2 & \mathbb{C}^2_\circ \arrow[l, hook', "i"'] \arrow[r, two heads, "r"] \arrow[rr, bend right, two heads, "q"'] & S^3_{\mathbb{C}} \arrow[r, two heads, "h"] & \mathbb{P}^1(\mathbb{C})
\end{tikzcd}$$

Here q and r denote the quotient maps, i denotes the inclusion, and h denotes the Hopf map.

Proof: The map β is the restriction of α to $\mathbb{C}^2_\circ$. The map $r \circ \beta$ is compatible with r because β is $\mathbb{R}$-linear; if two elements are equivalent in $\mathbb{C}^2_\circ$, then their images under β are also equivalent. By the universal property of the quotient topology (theorem 2.9), we get a unique continuous map γ. By applying this argument to β^{-1}, we find a unique continuous map γ^{-1} which is the inverse of γ, so γ is a homeomorphism. Similarly, the map $q \circ \beta$ is compatible with q, because if two elements in $\mathbb{C}^2_\circ$ differ by a factor $\lambda \in \mathbb{C}^*$, then their images under β differ by a factor λ, as β is $\mathbb{C}$-linear. By the universal property of the quotient topology we get a unique homeomorphism δ that makes the diagram commute. □

This proposition tells us that the action of $\mathrm{GL}_2(\mathbb{C})$ descends naturally to $S^3_{\mathbb{C}}$ and $\mathbb{P}^1(\mathbb{C})$ by letting $g \in \mathrm{GL}_2(\mathbb{C})$ act on a representative. Beware that although $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{C}^2_\circ$ by linear automorphisms, the induced automorphisms are *not* linear; in general they are not linear automorphisms of $\mathbb{R}^4$ and $\mathbb{R}^3$ restricted to S^3 and S^2.

The general linear group $\mathrm{GL}_4(\mathbb{R})$ also acts on $\mathbb{C}^2$, and by restriction on $\mathbb{C}^2_\circ$ when $\mathbb{C}^2$ is considered a four-dimensional real vector space. This action induces an action of $\mathrm{GL}_4(\mathbb{R})$ on $S^3_{\mathbb{C}}$; the same argument as before holds. However, this action does *not* induce an action on $\mathbb{P}^1(\mathbb{C})$, because elements of $\mathrm{GL}_4(\mathbb{R})$ are not $\mathbb{C}$-linear automorphisms in general.

We saw already that the fibres of the projective hopf map are the orbits of the S^1-action on $\mathbb{C}^2_\circ$. This allows us to parametrise fibres easily. For $(z_1 : z_2) \in \mathbb{P}^1(\mathbb{C})$, if we assume that $|z_1|^2 + |z_2|^2 = 1$, and if we work with S^3 instead of $S^3_{\mathbb{C}}$ by taking representatives of unit length, we have:

$$h^{-1}(z_1 : z_2) = \{(e^{it}z_1, e^{it}z_2) \in S^3 \mid t \in \mathbb{R}\}$$

For the fibres above $(1 : 0)$ and $(0 : 1)$, the geometrical picture is clear: the fibres are unit circles in the planes spanned by $(1, 0)$ and $(i, 0)$, and $(0, 1)$ and $(0, i)$ respectively. Because the circles have unit radius, they are great circles on S^3. An other way to state this, is that the fibres are precisely the intersections of S^3 with complex linear subspaces of dimension one (in particular those are intersections of S^3 with real linear subspaces of dimension two). For other points on $\mathbb{P}^1(\mathbb{C})$ however, it is not immediately clear what the fibres look like. This is where the $\mathrm{GL}_2(\mathbb{C})$-action is useful. Via the homeomorphism $S^3_{\mathbb{C}} \to S^3$ from proposition 2.16, $\mathrm{GL}_2(\mathbb{C})$ acts on S^3. If $g \in \mathrm{GL}_2(\mathbb{C})$ maps $(0:1)$ to $(z_1 : z_2)$, then commutativity of the diagram in proposition 3.6 means that g maps the fibre above $(0:1)$ into the fibre above $(z_1 : z_2)$. The fibre above $(0:1)$ is the intersection of a linear subspace with S^3, and because g is linear, the fibre above $(z_1 : z_2)$ is also the intersection of a linear subspace with S^3. Therefore it is a great circle as well.

Proposition 3.7 · $\mathrm{GL}_2(\mathbb{C})$ acts transitively on $\mathbb{P}^1(\mathbb{C})$.

Proof: $\mathrm{GL}_2(\mathbb{C})$ acts transitively on $\mathbb{C}^2_\circ$. Because $\mathbb{P}^1(\mathbb{C})$ is a quotient space of $\mathbb{C}^2_\circ$, every element can be represented by an element of $\mathbb{C}^2_\circ$, and because $\mathrm{GL}_2(\mathbb{C})$ acts transitively, every representative can be reached from e.g. $(0,1)$. □

Corollary 3.8 · All fibres of the Hopf map are great circles on S^3. We saw already that for all $g \in \mathrm{GL}_2(\mathbb{C})$, the fibre above $g \cdot (0:1)$ is a great circle, and because $\mathrm{GL}_2(\mathbb{C})$ acts transitively, every point in $\mathbb{P}^1(\mathbb{C})$ is of this form.

3.4 Stereographic projection

The fibres of the Hopf map that we investigated in the previous section are subsets of S^3. But S^3 can be hard to visualise; it is a subset of a four-dimensional space. Furthermore, with the goal of constructing a vector field on $\mathbb{R}^3$ in mind, we somehow have to get to $\mathbb{R}^3$. The way we can move between $\mathbb{R}^3$ and S^3 is by stereographic projection. Given that S^3 and $\mathbb{R}^3$ are not homeomorphic, we will have to make some concessions. Fortunately S^3 is the one-point compactification of $\mathbb{R}^3$, so if we want to go from S^3 to $\mathbb{R}^3$ we only loose a single point. Nevertheless, this discrepancy will turn out to introduce some artefacts, but these will in turn help to understand the geometry of S^3.

Definition 3.9 · The *stereographic projection* from S^n onto $\mathbb{R}^n$, with projection point $p \in S^n$ is given by

$$\pi : S^n \setminus \{p\} \longrightarrow \mathbb{R}^n, \quad x \longmapsto \overline{px} \cap \mathbb{R}^n$$

Here we embed $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as $p^\perp$, and $\overline{px}$ denotes the line connecting p and x.

We will use coordinates $x_0, \dots, x_n$ on $\mathbb{R}^{n+1}$ and coordinates $x_1, \dots, x_n$ on $\mathbb{R}^n$. Setting $p = (1, 0, \dots, 0)$ fixes the embedding of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ via $(x_1, \dots, x_n) \mapsto (0, x_1, \dots, x_n)$. The connection line $\overline{px}$ can then be parametrised as $p + \lambda(x - p)$ for $\lambda \in \mathbb{R}$. Intersecting this line with the hyperplane $x_0 = 0$ by setting $p_0 + \lambda(x_0 - p_0) = 0$ yields $\lambda = -\frac{1}{x_0 - 1}$, where the subscript zero denotes the first coordinate. Substituting λ, we find the projection of x:

$$\pi(x_0, \dots, x_n) = \left(-\frac{x_1}{x_0 - 1}, \dots, -\frac{x_n}{x_0 - 1}\right) = \left(\frac{x_1}{1 - x_0}, \dots, \frac{x_n}{1 - x_0}\right) \tag{3.10}$$

Conversely, if we have a point $x \in \mathbb{R}^n$, then we can embed it in $\mathbb{R}^{n+1}$ and parametrise the line through x and p as $p + \lambda(x - p)$ for $\lambda \in \mathbb{R}$. This time we want to find the intersection with S^n, so we must solve $\|p + \lambda(x - p)\|^2 = 1$. This yields $\lambda = \frac{2}{\|x\|^2+1}$. Substituting λ, we find the inverse image of x:

$$\pi^{-1}(x_1, \ldots, x_n) = \frac{1}{\|x\|^2 + 1}\left(\|x\|^2 - 1,\ 2x_1,\ \ldots,\ 2x_n\right) \tag{3.11}$$

From equation 3.10 and 3.11 it is clear that π and π^{-1} are continuous. It follows that π is a homeomorphism between $S^n \setminus \{p\}$ and $\mathbb{R}^n$.

It turns out that the stereographic projection of a circle on S^n is a circle in $\mathbb{R}^n$, a property that will be useful when studying the fibres of $h \circ \pi^{-1}$. To see why this is the case, we will first study the more general mapping of spheres.

Definition 3.12 · A *sphere* in S^n is the intersection of S^n with a hyperplane given by $\langle x, \hat{n}\rangle = t$, where $\hat{n} \in S^n$ is a normal vector of the hyperplane, and $t \in (-1, 1)$ is its offset to the origin. We choose $|\,t\,| < 1$ such that the intersection is not empty or finite.

Example 3.13 · A sphere in S^2 is simply a circle. If $t = 0$, it is a great circle.

Definition 3.14 · A *sphere* in $\mathbb{R}^n$ with centre $x_c \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_{>0}$ is the set

$$\{x \in \mathbb{R}^n \mid r^2 = \|x - x_c\|^2\}$$

Note that a sphere in S^n is the intersection of a sphere in $\mathbb{R}^{n+1}$ with S^n. By expanding the square, we may alternatively write a sphere in $\mathbb{R}^n$ with centre x_c and radius r as

$$\{x \in \mathbb{R}^n \mid r^2 - \|x_c\|^2 = \|x\|^2 - 2\langle x, x_c\rangle\} \tag{3.15}$$

Now we can turn to the relation between spheres in S^n and in $\mathbb{R}^n$.

Proposition 3.16 · Let $B \subseteq S^n$ be a sphere defined by the normal vector $\hat{n} \in S^n$ and offset $t \in (-1, 1)$. Then for its image under the stereographic projection $\pi : S^n \setminus \{p\} \to \mathbb{R}^n$, the following holds:

i If $p \notin B$, $\pi(B)$ is a sphere in $\mathbb{R}^n$.

ii If $p \in B$, $\pi(B \setminus \{p\})$ is a hyperplane in $\mathbb{R}^n$.

Proof: The image of B or $B \setminus \{p\}$ when $p \in B$, is given by the set of $x \in \mathbb{R}^n$ such that $\pi^{-1}(x) \in B$. Embed $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as the hyperplane $x_0 = 0$. Then we can write

$$\pi(B) \;=\; \{x \in \mathbb{R}^n \mid \langle \pi^{-1}(x), \hat{n}\rangle = t\} \;=\; \{x \in \mathbb{R}^n \mid n_0(\|x\|^2 - 1) + \langle x, \hat{n}\rangle = t\}$$

Here n_0 denotes the first coordinate of $\hat{n}$. If $n_0 \neq 0$, we recognise equation 3.15, so $\pi(B)$ is a sphere in $\mathbb{R}^n$. If $n_0 = 0$, then the predicate reduces to $\langle x, \hat{n}\rangle = t$, which is the equation

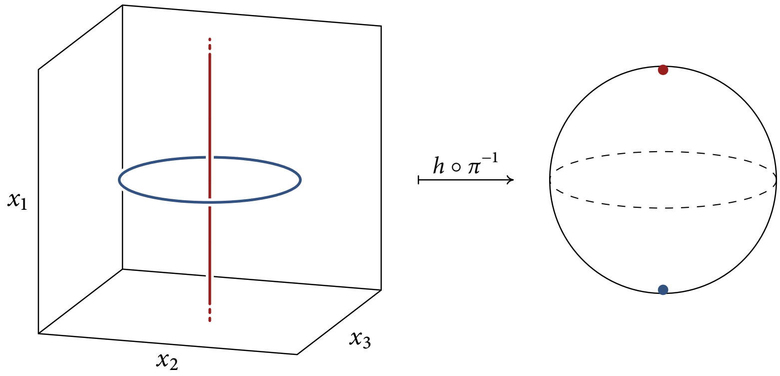

Figure 3.1 · Fibres of the Hopf map visualised through stereographic projection; the fibre above $i \in S^2$ is the x_1-axis (coloured •), the fibre above $-i \in S^2$ is the unit circle in the plane $x_1 = 0$ (coloured •). See also figure 3.4.

for a hyperplane in $\mathbb{R}^n$ with normal $\hat{n}$ and offset t to the origin. Furthermore, $n_0 = 0$ if and only if $p \in B$. □

Proposition 3.17 · Let $n \geq 2$ and let $C \subseteq S^n$ be a circle, the nonempty intersection of $n-1$ spheres defined by hyperplanes with linearly independent normal vectors. Then for its image under the stereographic projection $\pi : S^n \setminus \{p\} \to \mathbb{R}^n$, the following holds:

i If $p \notin C$, $\pi(C)$ is a circle in $\mathbb{R}^n$.

ii If $p \in C$, $\pi(C \setminus \{p\})$ is a line in $\mathbb{R}^n$.

Proof: We use the fact that for $U, V \subseteq S^n$, we have $\pi(U \cap V) = \pi(U) \cap \pi(V)$. As C is the intersection of $n-1$ spheres, its image is the intersection of $n-1$ spheres or hyperplanes. These all lie in distinct hyperplanes with linearly independent normal vectors, so the image is a subset of a two-dimensional plane in $\mathbb{R}^n$. If $p \notin C$ then at least one of the images will be a sphere, so $\pi(C)$ is a circle. If $p \in C$ then all of the images will be distinct hyperplanes, the intersection of which is a line. □

Corollary 3.18 · The fibres of $h \circ \pi^{-1}$ are all circles in $\mathbb{R}^3$, except for the fibre above $(1:0)$ which is the x_1-axis. Furthermore, from equation 3.10 it is clear that the fibre above $(0:1)$ is the unit circle in the x_2x_3-plane. This has been visualised in figure 3.1.

3.5 Linking

The fibres of the Hopf map — circles, as shown in the previous section — possess an interesting property: they are all linked with every other fibre. In this section we will give a formal definition of linking, and prove linkedness of the fibres. In section 5.2 we will explore a physical application of linking.

Linking is a property of a pair of closed curves that is not intrinsic to the curves as topological spaces themselves, but rather to their embedding in a surrounding space. Considering this, it makes sense to look at the complement of the curves. By studying the fundamental group of the complement, we can tell different situations apart. For instance, the fundamental group of the complement of two linked circles in $\mathbb{R}^3$ is the free abelian group on two generators, whereas the fundamental group of the complement of two unlinked circles is the free *nonabelian* group on two generators. (See [Hatcher 2002, p. 46]. Incidentally, Hatcher introduces linking as one of the main motivations for studying the fundamental group.)

Definition 3.19 · Let X be a topological space. An *n-link* in X is an ordered collection of n continuous maps $\sigma_i : S^1 \to X$, such that the images of $\sigma_1, \ldots, \sigma_n$ are disjoint. The link is called *proper* if every σ_i is a homeomorphism onto its image.

This definition is based on [Milnor 1954]. Because in a proper link every σ_i is a homeomorphism onto its image, the components of the link do not self-intersect. Because the images are disjoint, they do not intersect eachother.

Definition 3.20 · Two n-links $(\sigma_1, \ldots, \sigma_n)$ and $(\tau_1, \ldots, \tau_n)$ are said to be *homotopic* if there exist homotopies $H_i : [0,1] \times S^1 \to X$ from σ_i to τ_i, such that for all $t \in [0,1]$, the images of $H_i(t, \cdot)$ are disjoint. The links are said to be *properly homotopic* if for all $t \in (0,1)$ the maps $H_i(t, \cdot)$ are homeomorphisms onto their images. Homotopy and proper homotopy define two equivalence relations on the set of n-links. We call a proper n-link *trivial* if it is properly homotopic to an n-link of n distinct constant functions.

A homotopy between links captures the idea of links being "the same". Homotopy enables us to tell apart many different types of links, but there is one caveat: a homotopy from one link into another might have self-intersecting components at some point in time. For instance, the Whitehead link is homotopic to two unlinked circles, but it can only be unlinked if the components are allowed to self-intersect. With the notion of proper homotopy we can also differentiate between the Whitehead link and and unlinked circles: the unlinked circles are trivial, but the Whitehead link is not. These types of links are beyond the scope of this thesis though; for the fibres of the Hopf map the notion of homotopy will be sufficient. To determine whether two closed curves are linked, we will examine the fundamental group of the complement of *one* curve. The other curve then determines an element of the fundamental group. If the fundamental group happens to be $\mathbb{Z}$, we can quantify linking with an integer.

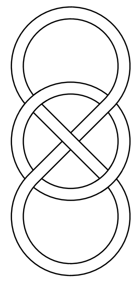

Figure 3.2 · The Whitehead link.

Definition 3.21 · Let (σ_1, σ_2) be a proper two-link in a topological space X. Suppose that $G_1 = \pi_1(X \setminus \operatorname{im} \sigma_1, \sigma_2(0)) \cong \mathbb{Z}$. Then $[\sigma_2]$ is an element of G_1, so under an isomorphism $G_1 \to \mathbb{Z}$ it maps to an integer n. Its absolute value $|n|$ is independent of the choice of isomorphism. This $|n|$ is the *linking number* of σ_2 with σ_1.

This definition of linking number is not symmetric with regard to σ_1 and σ_2: we require only the complement of σ_1 to have a fundamental group isomorphic to $\mathbb{Z}$. Even in $\mathbb{R}^3$, the fundamental group of such a complement can be quite surprising. For example, the fundamental group of the complement of an (m, n) torus knot is shown in [Hatcher 2002, p. 47] to be the quotient group of the free group with generators a and b, where a^m and b^n are identified. This means that the linking number of a nontrivial torus knot with a circle is well-defined, but the linking number of the circle with the knot is not. This problem can be alleviated by considering the first homology group instead of the fundamental group, an approach that is taken in [Rolfsen 2003, p. 132]. Rolfsen also relates the linking number as defined here to other definitions, such as the *Gauss linking integral*. In the remainder of this section, we will only consider curves of which the fundamental group of the complement is isomorphic to $\mathbb{Z}$. For curves in $\mathbb{R}^3$ or S^3, it is shown in theorem 6 of [Rolfsen 2003, p. 135] that the linking number does not depend on the order of σ_1 and σ_2, nor on their orientation. This means that the we can quantify the linking of $\{\operatorname{im}\sigma_1, \operatorname{im}\sigma_2\}$ with a unique nonnegative integer. A nonzero linking number implies that two curves are linked, but the converse does not hold: the Whitehead link has linking number zero, but it is not trivial. In any case, the linking number suffices to show that the fibres of the Hopf map are linked. Before we prove the general case we will demonstrate linkedness of two particular fibres. By using the action of $\mathrm{GL}_2(\mathbb{C})$ this proof can be extended to the general case.

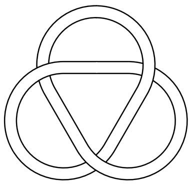

Figure 3.3 · A 2,3 torus knot, also called a *trefoil knot*.

As shown in section 3.3, the fibres of the Hopf map above $(1 : 0)$ and $(0 : 1)$ may be parametrised as

$$\sigma_1 : [0,1] \longrightarrow S^3, \quad t \longmapsto (e^{2\pi i t}, 0) \quad \text{and} \quad \sigma_2 : [0,1] \longrightarrow S^3, \quad t \longmapsto (0, e^{2\pi i t})$$

In corollary 3.18 we saw that under stereographic projection, σ_1 maps to the x_1 axis and σ_2 maps to the unit circle in the x_2x_3-plane.

Proposition 3.22 · Let σ_1 and σ_2 be as introduced above. Then σ_2 is linked once with σ_1 in S^3.

Proof: Restricted to $S^3 \setminus \operatorname{im}\sigma_1$, the stereographic projection $\pi : S^3 \setminus \{p\} \to \mathbb{R}^3$ is a homeomorphism onto $\mathbb{R}^3 \setminus \pi(\operatorname{im}\sigma_1)$, because the projection point $p = (1, 0)$ lies on the image of σ_1. Therefore, it induces an isomorphism $\pi_1(S^3 \setminus \operatorname{im}\sigma_1, \sigma_2(0)) \to \pi_1(\mathbb{R}^3 \setminus \pi(\operatorname{im}\sigma_1), \pi(\sigma_2(0)))$ on fundamental groups. As we saw before, $\pi(\operatorname{im}\sigma_1)$ is the x_1-axis in $\mathbb{R}^3$, so the space $\mathbb{R}^3 \setminus \pi(\operatorname{im}\sigma_1)$ deformation retracts onto $\mathbb{R}^2$ minus the origin by projecting on the x_2x_3-plane. This induces an isomorphism $\pi_1(S^3 \setminus \operatorname{im}\sigma_1, \sigma_2(0)) \to \pi_1(\mathbb{R}^2 \setminus \{0\}, (1, 0))$. (See for example proposition 1.17 of [Hatcher 2002, p. 31].) Because the image of $\pi \circ \sigma_2$ lies in the x_2x_3 plane, $[\pi \circ \sigma_2]$ is an element of $\pi_1(\mathbb{R}^2 \setminus \{0\}, (1, 0))$. This fundamental group is of course isomorphic to $\mathbb{Z}$, and $\pi \circ \sigma_2$ is a curve that goes around the origin once, so it is a generator of the fundamental group. It follows that σ_2 is linked once with σ_1. □

To demonstrate that any two fibres are linked, we will improve upon the result of pro-

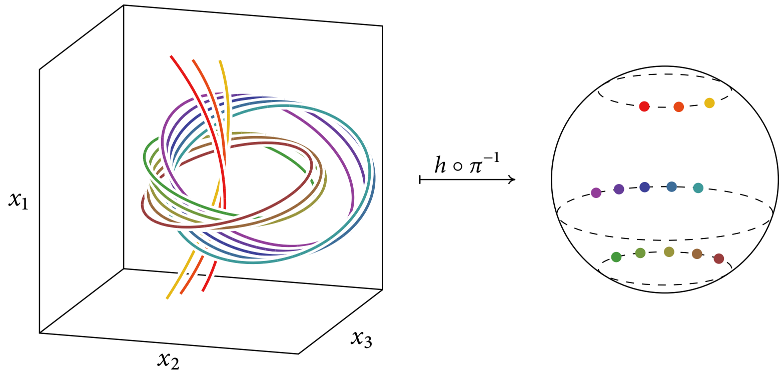

Figure 3.4 · Linked fibres of the Hopf map visualised through stereographic projection. Fibres above points near $i \in S^2$ (the north pole) are circles with a large radius in $\mathbb{R}^3$, close to the x_1-axis (truncated here). Fibres above points near $-i \in S^2$ (the south pole) are circles close to the unit circle in the x_2x_3-plane.

position 3.7, which stated that $\mathrm{GL}_2(\mathbb{C})$ acts transitively on $\mathbb{P}^1(\mathbb{C})$. In fact, the stabiliser $\mathrm{GL}_2(\mathbb{C})_p$ of a point $p \in \mathbb{P}^1(\mathbb{C})$ still acts transitively on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{p\}$.

Proposition 3.23 · Let $(z_1 : z_2)$ and $(v_1 : v_2) \in \mathbb{P}^1(\mathbb{C})$ be distinct points. Then there exists a $g \in \mathrm{GL}_2(\mathbb{C})$, such that $g \cdot (1 : 0) = (z_1 : z_2)$ and $g \cdot (0 : 1) = (v_1 : v_2)$.

Proof: Consider the matrix

$$g = \begin{pmatrix} z_1 & v_1 \\ z_2 & v_2 \end{pmatrix}$$

The columns of this matrix are linearly independent by assumption, so its determinant is nonzero. It follows that $g \in \mathrm{GL}_2(\mathbb{C})$, and clearly $g \cdot (1 : 0) = (z_1 : z_2)$ and $g \cdot (0 : 1) = (v_1 : v_2)$. □

Corollary 3.24 · Any two fibres of the Hopf map are linked in S^3: proposition 3.23 tells us that the situation of any two fibres can be transformed into the situation of $(1 : 0)$ and $(0 : 1)$ by a homeomorphism, and the linking number is invariant under such a homeomorphism.

Because the stereographic projection is a homeomorphism, the projection of any two fibres in S^3 that do not pass through the projection point will be a set of two linked circles in $\mathbb{R}^3$. Even if one of the fibres passes through the projection point (and thus projects to the x_1-axis), there is a sense of linkedness in $\mathbb{R}^3$: the fibre that does not pass through the projection point will project to a circle around the x_1-axis. A few of the fibres have been visualised in figure 3.4.

Informal summary

In this section we used topology to quantify linkedness. An *n-link* is a collection of non-intersecting closed curves, and for a *proper n-link* the curves cannot be self-intersecting either. If two links can be defomed into one another by bending and twisting but not intersecting, the links are called *homotopic*. With homotopy we allow the curves to self-intersect in the process, but for a *proper homotopy* even this is disallowed. If we want to know whether e.g. a collection of rubber bands can be unlinked, we must ask whether the corresponding link is *trivial*. If it is, it is possible to separate all of the bands. Because homotopy does not allow us to quantify linkedness, we turn to another quantity: the *linking number*. The linking number of two closed curves σ_1 and σ_2 counts how many times σ_2 winds around σ_1, a concept that can be made precise by using the fundamental group. Finally, we showed that any two fibres of the Hopf map have linking number one in S^3. Using stereographic projection, we can see that the fibres are linked in $\mathbb{R}^3$ as well.

CHAPTER 4

Differential forms

Up to now, we have investigated the Hopf map and its fibres. Though interesting in its own right, we eventually want to use this map to construct a magnetic field, a vector field on $\mathbb{R}^3$. Differential geometry gives us the tools to do so. First, we will recall some of the terms involved. Next, we will outline under which conditions several important vector spaces are isomorphic. These isomorphisms will allow us to identify vector fields with differential forms. Furthermore, we will apply this theory to the Hopf map, and derive a vector field on $\mathbb{R}^3$ with various desirable properties. Finally, we will define the *Hopf invariant* of a differential form, a quantity that will turn out to have an important physical interpretation.

4.1 Manifolds and the exterior algebra

For this chapter, a little background in differential geometry is assumed. Many concepts in differential geometry can be defined in various different — but equivalent — ways. Most applications in this chapter do not depend on technical details of one particular definition. Therefore we will mostly introduce notation here, and we will restate a few definitions for convenience. The definitions can all be found in chapter 1, 2 and 4 of [Warner 1971]. Alternatively, one may refer to section 6.4 and chapter 15 and 16 of [Szekeres 2004]. In the following sections of this thesis, we will assume *differentiable* to mean C^∞, i.e. infinitely differentiable.

Definition 4.1 · A *differentiable manifold of dimension n* is a nonempty second countable Hausdorff space M for which each point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, together with a *differentiable structure* $\mathcal{A}$ of class C^∞. (Beware that [Szekeres 2004] does not require a manifold to be second countable.) A pair $(U, \phi) \in \mathcal{A}$ of an open subset $U \subseteq M$, and a continuous map $\phi : U \to \mathbb{R}^n$ that is a homeomorphism onto its image, is called a *coordinate chart*. When there is no ambiguity, we will refer to

M simply as a manifold.

Notation 4.2 · Let M be a manifold of dimension n, let $p \in M$. The *tangent space to M at p* is written T_pM. It is a real vector space of dimension n. The *cotangent space to M at p*, the dual space of T_pM, is written T_p^*M. A coordinate chart (U, ϕ) around $p_0 \in M$ induces a basis $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ on T_pM for all $p \in U$, and thereby a dual basis $(dx_1, \ldots, dx_n)$ on T_p^*M.

Definition 4.3 · Let M be a manifold. Its *tangent bundle* is defined as

$$TM = \coprod_{p \in M} T_pM$$

The tangent bundle comes with a natural projection map $\pi : TM \to M$ that sends a tangent vector $v \in T_pM$ to p.

Definition 4.4 · Let V be an n-dimensional vector space over a field F and $k \geq 0$ an integer. The *k-th exterior power of V* is the unique (up to isomorphism) vector space $\bigwedge^k V$ with a linear map $\wedge : V^k \to \bigwedge^k V$ such that, for every alternating k-linear map $f : V^k \to W$ to an F-vector space W, there exists a unique $g : \bigwedge^k V \to W$ that makes the following diagram commute:

$$\begin{tikzcd}
& V^k \arrow[dl, "\wedge"'] \arrow[dr, "f"] & \\
\bigwedge^k V \arrow[rr, dashed, "\exists! g"'] & & W
\end{tikzcd}$$

Elements of $\bigwedge^k V$ can be written as sums of wedge products $v_1 \wedge \cdots \wedge v_k$ of k elements of V, and the map $\wedge$ is then given by $(v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k$. It follows that $\bigwedge^1 V = V$. By convention, $\bigwedge^0 V = F$. One can show that $\dim(\bigwedge^n V) = 1$ and $\dim(\bigwedge^k V) = 0$ for all $k > n$. $\bigwedge^k V$ is a subspace of $\bigwedge V$, the *exterior algebra* or *Grassmann algebra* of V, a graded-commutative F-algebra.

Notation 4.5 · Let M be a manifold and $k \geq 0$ an integer. The *space of differentiable k-forms*, written $\Omega^k M$, is a subspace of the real vector space

$$\prod_{p \in M} \bigwedge^k (T_p^*M)$$

An element $\omega \in \Omega^k M$ is called a *differentiable differential k-form*. When there is no ambiguity, we will refer to ω simply as a k-form. $\Omega^k M$ is a subspace of ΩM, the *exterior algebra* of M. Elements of $\Omega^0 M$ may be identified with *differentiable functions $M \to \mathbb{R}$*.

Definition 4.6 · Let M be a manifold of dimension n. A *vector field* on M is a function $X : M \to TM$ that satisfies $\pi \circ X = \mathrm{id}_M$. The vector field is called *differentiable* if, locally in a coordinate chart (U, ϕ), it can be written as

$$\sum_{i=1}^{n} f_i \cdot \frac{\partial}{\partial x_i}$$

where the f_i are differentiable functions $U \to \mathbb{R}$. The set of differentiable vector fields on M is a vector space, denoted ΥM.

Definition 4.7 · Let V be a real vector space of dimension n. An *orientation* on V is an element of

$$\{\omega \in \textstyle\bigwedge^n V \mid \omega \neq 0\}/\sim$$

where $\omega_1 \sim \omega_2$ if and only if $\omega_1 = \lambda\omega_2$ for some $\lambda \in \mathbb{R}_{>0}$. Note that an ordered basis $(v_1, \ldots, v_n)$ for V induces an orientation $[v_1 \wedge \cdots \wedge v_n]$ on V.

Definition 4.8 · Let M be a manifold of dimension n. An *orientation* on M is an element of

$$\{\omega \in \Omega^n M \mid \forall p \in M : \omega(p) \neq 0\}/\sim$$

where $\omega_1 \sim \omega_2$ if and only if $\omega_1 = f \cdot \omega_2$ for some differentiable function $f : M \to \mathbb{R}$.

Theorem 4.9 · Let R be a commutative ring. Taking the exterior algebra is a functor from the category of R-modules to the category of graded R-algebras.

Proof: See proposition 2 of [Bourbaki 1970, ch. III, § 7.2].

Corollary 4.10 · Let k be a nonnegative integer. Then taking the k-th exterior power is an endofunctor of the category of finite dimensional vector spaces over a field F. If V is a vector space over F and $f : V \to V$ a linear endomorphism, then theorem 4.9 yields a unique F-algebra endomorphism of $\bigwedge V$, which restricts to a linear map $\bigwedge^k V \to \bigwedge^k V$. This map is given by the linear extension of

$$v_1 \wedge \cdots \wedge v_k \longmapsto f(v_1) \wedge \cdots \wedge f(v_k)$$

See also equation 4 of [Bourbaki 1970, ch. III, § 7.2]. For $k = 0$ the induced map is the identity map.

4.2 Vector space isomorphisms

For a general vector space, there is little to say about its relation to other spaces. When the vector space has additional properties however, such as an inner product or an orientation, several canonical identifications can be made. These will play an important role in identifying differential forms with differentiable functions and vector fields, which we will do at the end of this section.

Proposition 4.11 · Let V be a finite dimensional vector space over a field F. Then V is canonically isomorphic to its double-dual V^{**} via the following linear isomorphism:

$$\alpha : V \longrightarrow V^{**}, \quad v \longmapsto (u \mapsto u(v))$$

Proof: It follows immediately that $u \mapsto u(v)$ is the zero function if and only if $v = 0$. Therefore, α is injective, for its kernel is trivial. Because V is finite dimensional, its dual space has the same dimension. Consequently, α is surjective. □

Definition 4.12 · Let V be a finite dimensional real vector space. A bilinear form $B : V \times V \to \mathbb{R}$ is said to be *nondegenerate* if the map

$$\chi : V \longrightarrow V^{*}, \quad v \longmapsto (u \mapsto B(v, u))$$

is an isomorphism.

In general, a finite dimensional real vector space V will not be canonically isomorphic to its dual space V^*. A nondegenerate bilinear form (of which an inner product is an example) establishes a canonical isomorphism between V and V^*. On a pseudo-Riemannian manifold, every tangent space is equipped with a nondegenerate bilinear form.

Definition 4.13 · A *pseudo-Riemannian manifold* is a pair (M, B) of a manifold M and a nondegenerate symmetric bilinear form $B_p : T_pM \times T_pM \to \mathbb{R}$ on every tangent space T_pM that satifies the following condition: for all X, Y differentiable vector fields on M, the map

$$B(X, Y) : M \longrightarrow \mathbb{R}, \quad p \longmapsto B_p(X(p), Y(p))$$

is a differentiable function on M.

Lemma 4.14 · Let V be a real vector space of dimension n, and let k, l be integers such that $k + l = n$. Then for all nonzero $\eta \in \bigwedge^k V$, there exists an $\omega \in \bigwedge^l V$ such that $\eta \wedge \omega \neq 0$.

Proof: Let $v_1, \ldots, v_n$ be a basis for V and suppose that $\eta \in \bigwedge^k V$ is nonzero. It can be written as

$$\eta = \sum_{1 \le i_1 \le \cdots \le i_k \le n} \lambda_{i_1 \cdots i_k} v_{i_1} \wedge \cdots \wedge v_{i_k}$$

with $\lambda_{i_1 \cdots i_k} \in \mathbb{R}$. Because $\eta \neq 0$, there exists a set of indices $i_1, \ldots, i_k$ such that $\lambda_{i_1 \cdots i_k} \neq 0$. Let $j_1, \ldots, j_l$ be the subsequence of $(1, 2, \ldots, n)$ with $i_1, \cdots, i_k$ removed. Define $\omega = v_{j_1} \wedge \cdots \wedge v_{j_l} \in \bigwedge^l V$. In all terms of $\eta \wedge \omega$ except the one with coefficient $\lambda_{i_1 \cdots i_k}$, at least one vector v_i will occur twice in the wedge product so these terms vanish. Because $\lambda_{i_1 \cdots i_k} \neq 0$, it follows that $\eta \wedge \omega \neq 0$. □

Proposition 4.15 · Let V be a finite dimensional real vector space of dimension n, and suppose that $\omega_0 \in \bigwedge^n V$ is given, with $\omega_0 \neq 0$. Then ω_0 induces a linear isomorphism $\bigwedge^{n-1} V \to V^*$.

Proof: First, recall that $\bigwedge^n V$ has dimension 1, so every $\omega \in \bigwedge^n V$ can be written uniquely as $\lambda\omega_0$ for some $\lambda \in \mathbb{R}$. Thus, we have a linear isomorphism

$$\rho : \bigwedge^n V \longrightarrow \mathbb{R}, \quad \lambda\omega_0 \longmapsto \lambda$$

This allows us to define the following linear map:

$$\psi : \bigwedge^{n-1} V \longrightarrow V^*, \quad \omega \longmapsto (v \mapsto \rho(v \wedge \omega))$$

Here $v \in V$. Injectivity of ψ follows from lemma 4.14. Because $\dim V^* = \dim \bigwedge^{n-1} V$, ψ is an isomorphism. □

The form ω_0 induces an orientation on V, and an orientation on V determines ω_0 modulo a positive real factor. When V is equipped with a symmetric nondegenerate bilinear form B as well, ω_0 can be fixed by requiring that it is a wedge product of an orthonormal basis for V. An orientation on a manifold induces an orientation on its cotangent spaces.

Proposition 4.16 · Let V be a real oriented finite dimensional vector space equipped with a symmetric nondegenerate bilinear form $B : V \times V \to \mathbb{R}$. Then there exists a unique $\omega_0 = v_1 \wedge \cdots \wedge v_n \in \bigwedge^n V$ such that $(v_1, \ldots, v_n)$ is a positively oriented orthonormal basis for V.

Proof: First of all, note that we can still talk about an orthonormal basis even when B is not positive-definite: for a basis $(v_1, \ldots, v_n)$ for V we require that $|B(v_i, v_j)| = \delta_{ij}$. With the Gram-Schmidt orthonormalisation process it is always possible to construct an orthonormal basis for V. See for instance theorem 5.2 of [Szekeres 2004, p. 129]. (Note that Szekeres does not require an inner product to be positive definite, so his proof is applicable in our situation.) If required, we reorder this basis to obtain a positively oriented orthonormal basis $(v_1, \ldots, v_n)$ for V. Finally, define $\omega_0 = v_1 \wedge \cdots \wedge v_n$.

Suppose that we have a different positively oriented orthonormal basis $(v_1', \ldots, v_n')$, and $\omega_0' = v_1' \wedge \cdots \wedge v_n'$. Then exists a linear isomorphism $f : V \to V$ that maps $(v_1, \ldots, v_n)$ onto $(v_1', \ldots, v_n')$. Because both bases are positively oriented and orthonormal, $\det(f) = 1$. By corollary 4.10, f induces a unique linear map $\bigwedge^n f : \bigwedge^n V \to \bigwedge^n V$ that maps ω_0 to ω_0'. Because $\bigwedge^n V$ has dimension 1, $\bigwedge^n f$ is given by $\omega \mapsto \lambda\omega$, where $\lambda = \det(f)$. (See definition 1 of [Bourbaki 1970, ch. III, § 8.1].) It follows that f induces the identity on $\bigwedge^n V$, so $\omega_0 = \omega_0'$. □

[Szekeres 2004, p. 218] provides an alternative proof of proposition 4.16 that uses Levi-Civita symbols instead of universal constructions.

Putting together the isomorphisms earlier in this section for a real vector space V of dimension n, we obtain the commutative diagram below. The dashed arrows are induced by a nondegenerate bilinear form on V, whereas the dotted arrows are induced by the choice of an element $\omega_0 \in \bigwedge^n(V^*)$.

$$
\begin{tikzcd}
V \arrow[r, dashed, "\chi"] \arrow[rr, bend left, "\alpha"] & V^* \arrow[r, dashed] & V^{**} \\
& \bigwedge^{n-1}(V^*) \arrow[ul, dotted] \arrow[ur, dotted, "\psi"'] &
\end{tikzcd}
$$

If V is an oriented semi-inner product space (a vector space with a symmetric nondegenerate bilinear form), this establishes a canonical isomorphism between $\bigwedge^1(V^*)$ and $\bigwedge^{n-1}(V^*)$.

The above isomorphisms can help to give an intuitive description of the various forms of the wedge product. For a real n-dimensional vector space V, we can make the following identifications:

- Taking the wedge product with $v \in \bigwedge^0 V$ is simply scalar multiplication.

- If V is an oriented semi-inner product space with bilinear form B, the wedge product corresponds to applying the bilinear form, in the sense that the following diagram commutes:

$$
\begin{tikzcd}
V \times V \arrow[r, "B"] \arrow[d, "{(\chi,\, \psi^{-1} \circ \alpha)}"] & \mathbb{R} \\
\bigwedge^1(V^*) \times \bigwedge^{n-1}(V^*) \arrow[r, "\wedge"] & \bigwedge^n(V^*) \arrow[u, "\rho \circ \bigwedge^n \chi^{-1}"']
\end{tikzcd}
$$

- If V is an oriented semi-inner product space and $n = 3$, we have the following commutative diagram:

$$
\begin{tikzcd}
V \times V \arrow[r] \arrow[d, "{(\chi,\, \chi)}"] & V \\
\bigwedge^1(V^*) \times \bigwedge^1(V^*) \arrow[r, "\wedge"] & \bigwedge^2(V^*) \arrow[u, "\alpha^{-1} \circ \psi"']
\end{tikzcd}
$$

  For $V = \mathbb{R}^3$ with its standard orientation and inner product, the map $V \times V \to V$ is the cross product.

With the isomorphisms of vector spaces in this section, we can identify spaces of differential forms with the spaces of differentiable functions and vector fields. For an n-dimensional manifold M, we will need a nowhere-vanishing $\omega_0 \in \Omega^n M$ to make these identifications. If M is an oriented pseudo-Riemannian manifold, there is one natural choice for ω_0: the *volume form*.

Theorem 4.17 · Let (M, B) be a pseudo-Riemannian manifold of dimension n. Then

$$\phi : \Upsilon M \longrightarrow \Omega^1 M, \quad X \longmapsto B(X, \cdot\,)$$

is a linear isomorphism between the spaces of differentiable vector fields and differentiable 1-forms on M.

Proof: A coordinate chart on an open neighbourhood U of $p_0 \in M$ induces a basis $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ on T_pM and $(dx_1, \ldots, dx_n)$ on T_p^*M for every $p \in U$. We can express $B(X, \cdot\,)$ locally as

$$\sum_{i=1}^{n} B(X, \partial/\partial x_i)\, dx_i$$

The vector field $p \mapsto \partial/\partial x_i$ defined on U is differentiable for all $1 \leq i \leq n$, so by definition 4.13, $B(X, \partial/\partial x_i)$ is differentiable. Therefore, $B(X, \cdot\,)$ is a differentiable 1-form. Linearity of ϕ follows from bilinearity of B_p for all $p \in M$. Since by definition 4.12 the map $v \mapsto B_p(v, \cdot\,)$ is a bijection from T_pM to T_p^*M, and $\bigwedge^1(T_p^*M) = T_p^*M$, we find that ϕ is an isomorphism. □

Theorem 4.18 · Let (M, B) be an n-dimensional oriented pseudo-Riemannian manifold. Then there exists a unique nowhere-vanishing $\omega_0 \in \Omega^n M$, such that for all $p \in M$, $\omega_0(p)$ is a wedge product of an orthonormal basis, and $[\omega_0]$ is the orientation on M. This ω_0 is called the *volume form*.

Proof: The pseudo-Riemannian structure of M induces a nondegenerate symmetric bilinear form on T_pM for every $p \in M$. By definition 4.12 T_pM is isomorphic to T_p^*M, so we can transport the bilinear form to T_p^*M. The orientation on M induces an orientation on T_p^*M, so proposition 4.16 will give us a unique nonzero $\xi_p \in \bigwedge^n(T_p^*M)$. This we can use to define

$$\omega_0 : M \longrightarrow \coprod_{p \in M} \textstyle\bigwedge^n(T_p^*M), \quad p \longmapsto \xi_p$$

By construction ω_0 is unique and nowhere-vanishing. For every $p \in H$, $\omega_0(p)$ is a wedge product of an orthonormal basis for T_p^*M, and $[\omega_0]$ is the orientation on M. The crux of this theorem however, is that ω_0 is differentiable. This follows from the way in which an orthonormal basis for T_pM is constructed: a coordinate chart on an open neighbourhood U around $p_0 \in M$ induces a basis $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ on T_pM for all $p \in U$, so $p \mapsto \partial/\partial x_1$ is a differentiable vector field on U. By definition 4.13, the function $\|\partial/\partial x_1\| = |B(\partial/\partial x_1, \partial/\partial x_1)|$ is differentiable. Next we apply the Gram-Schmidt process, but with differentiable vector fields on U. This process involves taking sums of differentiable vector fields and dividing by their norm. If B is not positive-definite, the norm could vanish even if the vector field does not, but by shrinking U if necessary, we can ensure that the norm does not vanish on U, so the process results in differentiable vector fields. Theorem 4.17 then yields a set of n differentiable 1-forms $\eta_i \in \Omega^1 M$, $1 \leq i \leq n$, such that $(\eta_1(p), \ldots, \eta_n(p))$ is an orthonormal basis for T_p^*M at every $p \in U$. We can write $\omega_0 = \eta_1 \wedge \cdots \wedge \eta_n$, so it follows that ω_0 is a differentiable n-form. □

Theorem 4.19 · Let M be a manifold of dimension n, and let $\omega_0 \in \Omega^n M$ be a nowhere-vanishing n-form. (From theorem 4.18, it follows that an oriented pseudo-Riemannian manifold is naturally equipped with such a form.) Then ω_0 induces a natural linear isomorphism

$$\phi : \Omega^{n-1}M \longrightarrow \Upsilon M$$

Proof: Let $\eta \in \Omega^{n-1}M$ be given. By proposition 4.15 and definition 4.12 we have an isomorphism $\phi_p : \bigwedge^{n-1}(T_p^*M) \to T_pM$ for every $p \in M$, so we can define $\phi(\eta) = p \mapsto \phi_p(\eta(p))$. Say $\phi(\eta) = X$, then we must show that X is differentiable. It suffices to verify this locally in an open neighbourhood U around $p_0 \in M$. A coordinate chart on U induces a basis $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ on T_pM and a basis $(dx_1, \dots, dx_n)$ on T_p^*M for all $p \in U$, so $p \mapsto dx_i$ is a differentiable one-form on U for all $1 \le i \le n$. We can then write $dx_i \wedge \eta$ as $f_i \cdot \omega_0$ for differentiable functions $f_i : U \to \mathbb{R}$, because ω_0 is nowhere-vanishing. The vector field $X : M \to TM$ is then locally given by

$$X = \sum_{i=1}^{n} f_i \cdot \frac{\partial}{\partial x_i}$$

It follows that X is differentiable. Because every ϕ_p is a linear isomorphism, ϕ is itself a linear isomorphism. □

Example 4.20 · For all positive integers n, $\mathbb{R}^n$ is a differentiable manifold of dimension n. If we take the standard orientation and Euclidean inner product, then $\mathbb{R}^n$ is an oriented Riemannian manifold. Cartesian coordinates induce a basis $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ on every tangent space, which we can identify with the standard basis $(e_1, \dots, e_n)$. The isomorphism from theorem 4.17 maps $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ to a basis for the cotangent space, which for the standard inner product on $\mathbb{R}^n$ coincides with the dual basis $(dx_1, \dots, dx_n)$. Theorem 4.19 tells us that we can identify elements of $\Omega^{n-1}\mathbb{R}^n$ with vector fields via the identification

$$dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n \longmapsto (-1)^{i+1} e_i$$

The wedge product of all basis vectors except dx_i corresponds to e_i. Furthermore, via the volume form $dx_1 \wedge \cdots \wedge dx_n$, we can identify an n-form $f \cdot dx_1 \wedge \cdots \wedge dx_n$ with the differentiable function f on $\mathbb{R}^n$. To summarise: in $\mathbb{R}^n$, we can identify one-forms and $(n-1)$-forms with differentiable vector fields, and we can identify zero-forms and n-forms with differentiable functions.

4.3 Constructing a vector field

With the tools of the previous section we can identify vector fields and k-forms on manifolds. In particular, we are interested in a divergenceless vector field on $\mathbb{R}^3$ that we will interpret as a magnetic field in section 5.3. The operator that allows us to express divergence is the *exterior derivative* that we will introduce in this section. Furthermore, we will recall how differentiable functions between manifolds introduce a *pullback* on their exterior algebras. Finally, we will combine the two to construct a divergenceless vector field.

Notation 4.21 · Let M be a manifold and $f \in \Omega^0 M$ a differentiable function. Then its *differential*, written df, is an element of $\Omega^1 M$. It is a differentiable function $TM \to \mathbb{R}$ that is linear on every tangent space. For a definition one may refer to definition 1.22 of [Warner 1971, p. 16] or [Szekeres 2004, p. 423].

For a manifold M of dimension n and $f \in \Omega^0 M$, locally in a coordinate chart (U, ϕ) the differential df can be expressed as

$$df = \sum_{i=1}^{n} \frac{\partial(\phi^{-1} \circ f)}{\partial x_i} \, dx_i \tag{4.22}$$

Note that this notation is compatible with the usage of dx_i as a cotangent vector: in a coordinate chart (U, ϕ) the function $x_i : U \to \mathbb{R}$ that sends a point p to the i-th coordinate of $\phi(p)$ has differential dx_i.

Theorem 4.23 · Let M be a manifold. Then for each integer $k \geq 0$, there exists a unique linear operator $d : \Omega^k M \to \Omega^{k+1} M$, called the *exterior derivative*, that satisfies the following properties:

- For $f \in \Omega^0 M$, df is the differential of f.
- $d^2 = 0$.
- For $\omega \in \Omega^i M$ and $\eta \in \Omega^j M$ it holds that $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^i \omega \wedge d\eta$. An operator that satisfies this property is called an *anti-derivation*.

Proof: See theorem 2.20 of [Warner 1971, p. 65] or section 16.1 of [Szekeres 2004, p. 448].

The anti-derivation property of d is the analogue of the product rule for differentiation. It allows us to reduce computations to the case of equation 4.22. A k-form $\omega \in \Omega^k M$ is said to be *closed* if $d\omega = 0$. It is said to be *exact* if there exists an $\alpha \in \Omega^{k-1} M$ such that $d\alpha = \omega$.

Theorem 4.24 · Poincaré's lemma · Let M be a contractible manifold, $k > 1$ an integer. Then every closed k-form on M is exact.

Proof: See for instance corollary A of [Warner 1971, p. 156] (although its preconditions are slightly different) or corollary 4.1.2.1 of [Bott and Tu 1982, p. 36].

Poincaré's lemma is in fact a corollary of a more general result involving de Rham cohomology. We will not go in depth here, although there are some interesting consequences for S^3. We highlight one particular result:

Proposition 4.25 · Let $\omega \in \Omega^2 S^3$ be closed. Then ω is exact.

Proof: This follows from the fact that $H^2(S^3) = 0$, where $H^2(S^3)$ is the second de Rham cohomology group, defined as

$$H^2(S^3) = \frac{\ker(d : \Omega^2 S^3 \to \Omega^3 S^3)}{\operatorname{im}(d : \Omega^1 S^3 \to \Omega^2 S^3)}$$

See also [Bott and Tu 1982, p. 36]. □

Example 4.26 · For $\mathbb{R}^3$ with standard orientation and inner product, the d operator has well-known interpretations. By Poincaré's lemma the following sequence of vector spaces is exact:

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0\mathbb{R}^3 \xrightarrow{d_0} \Omega^1\mathbb{R}^3 \xrightarrow{d_1} \Omega^2\mathbb{R}^3 \xrightarrow{d_2} \Omega^3\mathbb{R}^3 \longrightarrow 0$$

As we saw in the previous section, we can identify vector fields with 1-forms and 2-forms and differentiable functions with 0-forms and 3-forms on $\mathbb{R}^3$. Under these identifications, we have the following correspondences:

- d_0 corresponds to taking the gradient of a function.
- d_1 corresponds to taking the curl of a vector field.
- d_2 corresponds to taking the divergence of a vector field.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and $F : \mathbb{R}^3 \to \mathbb{R}^3$ a differentiable vector field. Because $d^2 = 0$, we get the following identities for free:

$$\nabla \times (\nabla f) = 0 \qquad\qquad \nabla \cdot (\nabla \times F) = 0$$

More generally, in $\mathbb{R}^n$ with standard orientation and inner product, the operator d_0 corresponds to the gradient and d_{n-1} corresponds to the divergence.

Apart from the exterior derivative, we will need another concept for the construction of a divergenceless vector field: the pullback. Recall that an linear map $f : V \to W$ between vector spaces induces a dual map (sometimes called transpose map) $f^* : W^* \to V^*$ between the dual spaces. Similarly, a function between tangent bundles of manifolds induces a dual map, and by functorality of the exterior algebra, this allows us to *pull back* k-forms. Definitions of the concepts below can be found in chapter 1 of [Warner 1971, p. 16] and section 15.4 of [Szekeres 2004, p. 426].

Notation 4.27 · Let M, N be manifolds and $f : M \to N$ a differentiable function. For every $p \in M$, f induces a *tangent map* in p, a linear map

$$f_*(p) : T_p M \longrightarrow T_{f(p)} N$$

The tangent map of f is sometimes called the *differential* of f, and in fact the differential dg of a differentiable function $g : M \to \mathbb{R}$ as introduced in notation 4.21 is the tangent map g_* when $T_p\mathbb{R}$ is identified with $\mathbb{R}$ for every $p \in \mathbb{R}$.

Definition 4.28 · Let M, N be manifolds and $f : M \to N$ a differentiable function. By theorem 4.9, the dual maps of the tangent maps induce a map

$$f^* : \Omega N \longrightarrow \Omega M$$

As shown in proposition 2.23A of [Warner 1971, p. 68], f^* is an algebra homomorphism. For $\omega \in \Omega^k N$, the element $f^*(\omega) \in \Omega^k M$ is called the *pullback* of ω by f.

Theorem 4.29 · The pullback commutes with the exterior derivative. In other words, for manifolds M, N and a differentiable function $f : M \to N$ it holds that for all $\omega \in \Omega N$ we have $f^*(d\omega) = d(f^*(\omega))$.

Proof: See proposition 2.23B of [Warner 1971, p. 68] or theorem 16.2 of [Szekeres 2004, p. 451].

Commutativity of the pullback and exterior derivative is what allows us to construct divergenceless vector fields. Suppose that we have $\xi \in \Omega^n M$ on an n-dimensional manifold M. Then $\Omega^{n+1}M = 0$, so $d\xi = 0$. If we now take a different manifold N of larger dimension, then $\Omega^{n+1}N \neq 0$. If we have a differentiable function $f : N \to M$, then $d(f^*(\xi)) = f^*(d\xi) = 0$; the n-form $f^*(\xi)$ is closed. We will apply this procedure to the Hopf map and the stereographic projection, but first we have to verify some technicalities. First of all, we need a nowhere-vanishing two-form on S^2. We have not equipped S^2 with a pseudo-Riemannian structure, so we cannot apply theorem 4.18 here. While we could embed the tangent spaces of S^2 in $\mathbb{R}^3$ and restrict the Euclidean inner product of $\mathbb{R}^3$ to the tangent spaces, we will take a different route that has the same result. There is one preferred choice of two-form: because S^2 is invariant under rotation, we require the two-form to be $\mathrm{SO}_3(\mathbb{R})$-invariant.

Lemma 4.30 · There exists a nowhere-vanishing two-form $\omega_0 \in \Omega^2 S^2$ which is invariant under $\mathrm{SO}_3(\mathbb{R})$.

Proof: Define in $\mathbb{R}^3$ the two-form

$$\omega = x_1 \cdot dx_2 \wedge dx_3 \ + \ x_2 \cdot dx_3 \wedge dx_1 \ + \ x_3 \cdot dx_1 \wedge dx_2$$

Then define $\omega_0 = i^*(\omega)$, where $i : S^2 \hookrightarrow \mathbb{R}^3$ is the inclusion. Under the isomorphism $\Omega^2\mathbb{R}^3 \to \Upsilon\mathbb{R}^3$ from theorem 4.19, ω corresponds to a vector field pointing radially outward, the identity function on $\mathbb{R}^3$ in fact. This field is $\mathrm{SO}_3(\mathbb{R})$-invariant. Because elements of $\mathrm{SO}_3(\mathbb{R})$ are orientation-preserving and orthogonal, it follows that ω and thereby ω_0 are invariant under $\mathrm{SO}_3(\mathbb{R})$. Furthermore, because ω_0 does not vanish on e.g. $(0,1,0)$, it follows from invariance that it vanishes nowhere. □

Lemma 4.31 · The inverse stereographic projection $\pi^{-1} : \mathbb{R}^3 \to S^3$ and the Hopf map $h : S^3 \twoheadrightarrow S^2$ are differentiable maps.

Proof: As we can see in equation 3.11 and 3.3, π^{-1} is given by a rational function of polynomials on every coordinate (with a denominator that is positive everywhere), and h is given by a polynomial expression on every coordinate. It follows that both maps are differentiable. □

Example 4.32 · Consider $\mathbb{R}^3$, S^3, and S^2. We have the maps $h : S^3 \to S^2$ and $\phi = h \circ \pi^{-1} : \mathbb{R}^3 \to S^2$. To a differentiable function $f : S^2 \to \mathbb{R}$ we can assign the two-form $\xi = f \cdot \omega_0$ with ω_0 as in lemma 4.30. Because S^2 has dimension two, we have $d\xi = 0$. By pulling back ξ we obtain the closed two-forms

$$\beta = \phi^*(\xi) \in \Omega^2\mathbb{R}^3, \quad d\beta = 0 \qquad \text{and} \qquad \gamma = h^*(\xi) \in \Omega^2 S^3, \quad d\gamma = 0$$

As we saw before, we can identify two-forms on $\mathbb{R}^3$ with vector fields, and the exterior derivative then corresponds to taking the divergence. We can identify $\beta \in \Omega^2\mathbb{R}^3$ with $B : \mathbb{R}^3 \to \mathbb{R}^3$. Closedness, $d\beta = 0$, then translates to $\nabla \cdot B = 0$. Hence, we have constructed a divergenceless vector field on $\mathbb{R}^3$ from a differentiable function $f : S^2 \to \mathbb{R}$. The function ϕ is constant on field lines of B.

We can even say more about β and γ. By Poincaré's lemma (theorem 4.24), there exists an $\alpha \in \Omega^1\mathbb{R}^3$ such that $d\alpha = \beta$. Translated to vector fields, this means that there exists a vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\nabla \times A = B$. In physics, this field is called a *vector potential*, and it will play an important role in chapter 5. Moreover, by proposition 4.25 α is the pullback of a one-form on S^3, an analogue of a vector potential on the three-sphere.

For an explicit computation of the field induced on $\mathbb{R}^3$ by $\omega_0 \in \Omega^2 S^2$, recall that we have a sequence of differentiable maps:

$$\mathbb{R}^3 \xhookrightarrow{\pi^{-1}} S^3 \xrightarrow{h} S^2 \xhookrightarrow{i} \mathbb{R}^3$$

We define the function $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ to be the composition of these maps. As the composition of differentiable functions ϕ is differentiable, so we can compute the pullback by ϕ of ω as defined in lemma 4.30. This will be the pullback of ω_0 by $h \circ \pi^{-1}$. Using equation 3.11 and 3.3 we can express ϕ as

$$\phi(x_1, x_2, x_3) = \frac{4}{(\|x\|^2 + 1)^2} \begin{pmatrix} \frac{1}{4}(\|x\|^2 - 1)^2 - \|x\|^2 + 2x_1^2 \\ 2x_1x_2 - (\|x\|^2 - 1)x_3 \\ 2x_1x_3 + (\|x\|^2 - 1)x_2 \end{pmatrix} \tag{4.33}$$

Here $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$. Before we compute the pullback of ω by ϕ, we will compute the pullback by a general differentiable function $f : \mathbb{R}^3 \to \mathbb{R}^3$ with component functions

f_1, f_2, f_3. Observe that for $i = 1, 2, 3$,

$$f^*(dx_i) = d(f^*(x_i)) = d(x_i \circ f) = df_i$$

The pullback of ω by f is given by

$$f^*(\omega) = \sum_{(i,j,k)} \left(\sum_{(p,q,r)} f_p \cdot \left(\frac{\partial f_q}{\partial x_j} \frac{\partial f_r}{\partial x_k} - \frac{\partial f_q}{\partial x_k} \frac{\partial f_r}{\partial x_j} \right) \right) dx_j \wedge dx_k \tag{4.34}$$

where (i, j, k) and (p, q, r) run over $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. Via the isomorphisms in the previous section we obtain a vector field, its i-th coordinate given by the term $dx_j \wedge dx_k$. For $f = \phi$, we find the field

$$H(x_1, x_2, x_3) = \frac{16}{(\|x\|^2 + 1)^3} \begin{pmatrix} 1 + x_1^2 - x_2^2 - x_3^2 \\ 2(x_1 x_2 + x_3) \\ 2(x_1 x_3 - x_2) \end{pmatrix} \tag{4.35}$$

This field coincides with the field in [Dalhuisen 2014, p. 31] exept for a factor -1 and a point reflection $x \mapsto -x$. (Such a point reflection is called a *parity transformation* in physics.) This is because the projection point of the stereographic projection differs.

Now that we have an expression for the field, consider a general two-form $\xi \in \Omega^2 S^2$. It can be written as $f \cdot \omega_0$ for some differentiable function $f : S^2 \to \mathbb{R}$, so for the pullback it holds that

$$\phi^*(\xi) = \phi^*(f \cdot \omega_0) = (f \circ \phi) \cdot \phi^*(\omega_0)$$

Consequently, the pullback of ξ will correspond to the product of the vector field H with the real-valued function $f \circ \phi$.

We can summarise the main result of this section with the following statement:

> Given a differentiable function $\phi : \mathbb{R}^3 \to M$, where M is an oriented pseudo-Riemannian manifold of dimension two, and a differentiable function $f : M \to \mathbb{R}$, we can construct a divergenceless vector field on $\mathbb{R}^3$. The function ϕ determines the structure of the field; ϕ is constant along field lines. The function f determines the magnitude of the field; multiplying f by a differentiable function $g : M \to \mathbb{R}$ will change the field magnitude by a factor $g \circ \phi$.

The mathematics in this chapter do admit a physical interpretation, but rather than giving a brief physical summary here, we will defer the physics to the next chapter, section 5.3.

4.4 The Hopf invariant

We have seen before that the fibres of the Hopf map are linked, and that we can construct a vector field with linked field lines. In order to quantify the amount of linking in a vector field, we will define the *Hopf invariant*, a quantity that is invariant under orientation-preserving diffeomorphisms. Although its definition is purely algebraic, it has an important physical interpretation as the *helicity* of a field. This connection and the relation to fluid flow will be explored briefly. In section 5.2 we will make the connection to magnetohydrodynamics.

The definition of the Hopf invariant given below follows [Arnold 1974], because it most closely aligns with its applications in chapter 5. It differs from the more common definition of the Hopf invariant (given in [Bott and Tu 1982, p. 228], for example), in the sense that we define the Hopf invariant as a property of a two-form, not as a property of a map between spheres. Nevertheless, both definitions involve the same integral.

Definition 4.36 · Let M be an oriented three-dimensional manifold. Let $\xi \in \Omega^2 M$ be a compactly supported two-form that is exact, i.e. there exists an $\alpha \in \Omega^1 M$ such that $d\alpha = \xi$. Then the *Hopf invariant* of ξ is defined to be

$$H(\xi) = \int_M \alpha \wedge d\alpha$$

To show that this definition is independent of the choice of α, suppose that $\beta \in \Omega^1 M$ also satisfies $d\beta = \xi$. Then we have

$$\int_M \alpha \wedge d\alpha - \int_M \beta \wedge d\beta = \int_M (\alpha - \beta) \wedge d\alpha$$

Because $d((\alpha - \beta) \wedge \alpha) = d(\alpha - \beta) \wedge \alpha - (\alpha - \beta) \wedge d\alpha$, we find

$$= \int_M d(\alpha - \beta) \wedge \alpha - d((\alpha - \beta) \wedge \alpha)$$

The first term vanishes because $d(\alpha - \beta) = \xi - \xi = 0$, so we are left with

$$= \int_M d((\beta - \alpha) \wedge \alpha)$$

Using Stokes' theorem, we can write this as an integral over ∂M which vanishes because $\partial M = \varnothing$. (See also [Warner 1971, p. 148].) Therefore, the Hopf invariant is well-defined.

The requirement that ξ has compact support prevents us from defining the Hopf invariant for a nowhere-vanishing two-form on $\mathbb{R}^3$. If the two-form happens to be the pullback of a two-form on S^3 as discussed in the previous section, we can compute the Hopf invariant on S^3 instead. For subsets of a manifold M we can also define a Hopf invariant, but the argument that $\partial M = \varnothing$ cannot be used anymore to ensure that such an invariant is well-defined. To resolve this issue, we must require that the subset is well-behaved with respect to ξ.

Definition 4.37 · Let M be a manifold of dimension n and $D \subseteq M$. D is called a *regular domain* if for every $p \in M$ one of the following conditions holds:

- There exists an open neigbourhood of p contained in $M \setminus D$.
- There exists an open neighbourhood of p contained in D.
- There exists a centered chart (U, ϕ) about p such that $\phi(U \cap D) = \phi(U) \cap H^n$ with $H^n = \{x \in \mathbb{R}^n \mid x_0 \geq 0\}$.

The domain D is said to have a *smooth boundary* ∂D. Regular domains are subsets of manifolds that we can integrate n-forms over and where we can apply Stokes' theorem. (See also [Warner 1971, p. 145].)

Definition 4.38 · Let M be a three-dimensional manifold. Let $U \subseteq M$ be open in M with smooth boundary ∂U, such that its closure $\overline{U}$ in M is compact. Denote by $i : \partial U \hookrightarrow M$ the inclusion. Let $\xi \in \Omega^2 M$ be an exact two-form. U is said to be *compatible with* ξ if $i^*(\xi) = 0$.

In $\mathbb{R}^3$, where ξ can be identified with a vector field, this definition has a geometric interpretation: U is compatible with ξ if ξ is tangent to ∂U.

Definition 4.39 · Let M be an oriented three-dimensional contractible manifold. Let $\xi \in \Omega^2 M$ be an exact two-form, i.e. there exists an $\alpha \in \Omega^1 M$ such that $d\alpha = \xi$. Let $U \subseteq M$ be open in M with smooth boundary ∂U, such that its closure $\overline{U}$ in M is compact, and such that U is compatible with ξ. Then the *Hopf invariant of ξ restricted to U* is defined to be

$$H(\xi|_U) = \int_U \alpha \wedge d\alpha$$

Again, we have to show that this definition does not depend on the choice of α. Suppose that $\beta \in \Omega^1 M$ satisfies $d\beta = \xi$, then $\alpha - \beta$ is closed. Because M is contractible, it follows from Poincaré's lemma (theorem 4.24) that β can be written as $\alpha + df$ for some $f \in \Omega^0 M$. Exploiting this, we find

$$\int_U \alpha \wedge d\alpha - \int_U \beta \wedge d\beta = \int_U df \wedge \xi$$

Because $d(f \cdot \xi) = df \wedge \xi + f \cdot d\xi$ we can write this as

$$= \int_U d(f \cdot \xi) - f \cdot d\xi$$

The factor $d\xi$ vanishes because $d\xi = d^2\alpha = 0$, so we may apply Stokes' theorem

$$= \int_{\partial U} (f \circ i) \cdot i^*(\xi)$$

Here $i : \partial U \hookrightarrow M$ denotes the inclusion. Because ξ is compatible with U, $i^*(\xi) = 0$, so the integral evaluates to zero. Therefore, the Hopf invariant restricted to U is well-defined.

Definition 4.40 · Let M be an oriented n-dimensional manifold with orientation $[\omega_0]$ for some nowhere-vanishing $\omega_0 \in \Omega^n M$. Then a diffeomorphism $f : M \to M$ is said to be *orientation-preserving* if $[f^*(\omega_0)] = [\omega_0]$. This does not depend on the choice of ω_0, for if $\omega_0 = g \cdot \omega_0'$ for a differentiable function $g : M \to \mathbb{R}$ and $\omega_0' \in \Omega^n M$ nowhere-vanishing, then $f^*(\omega_0) = (g \circ f) \cdot f^*(\omega_0')$. Because f is a diffeomorphism, $f^*(\omega_0)$ will be nowhere-vanishing, so $[f^*(\omega_0)]$ is well-defined.

Proposition 4.41 · Let M be an oriented n-dimensional manifold and let $f : M \to M$ be an orientation-preserving diffeomorphism. Suppose $U \subseteq M$ has a smooth boundary. Let $\xi \in \Omega^n M$ be such that $\mathrm{Supp}(\xi) \cap \overline{f(U)}$ is compact. Then it holds that

$$\int_U f^*(\xi) = \int_{f(U)} \xi$$

Proof: See [Warner 1971, p. 148].

From this proposition it follows that the Hopf invariant is actually an invariant: it is invariant under orientation-preserving diffeomorphisms of M.

Corollary 4.42 · Let M and ξ be as in definition 4.36, and let $f : M \to M$ be an orientation-preserving diffeomorphism. Then it holds that

$$H(\xi) = H(f^*(\xi))$$

Corollary 4.43 · Let M, ξ and U be as in definition 4.39. Let $f : M \to M$ be a orientation-preserving diffeomorphism. Then it holds that

$$H(f^*(\xi)|_U) = H(\xi|_{f(U)})$$

So far, we have shown that the Hopf invariant is indeed an invariant, but we have yet to show how it relates to linking. When ξ corresponds to a vector field, we can give a heuristic argument involving involving flux tubes, which we will do in section 5.2. A more formal relation between the Hopf invariant and linking number is established in [Arnold 1974], and the correspondence with several alternative definitions of linking number is given in [Bott and Tu 1982, pp. 229–234].

Physical interpretation

The formalism in this section admits an almost 1:1 translation to a physical situation. The exact two-form will be replaced by the divergenceless vector field **B**, and *helicity* will play the role of the Hopf invariant.

In definition 4.36 and 4.39 we defined the Hopf invariant of an exact two-form ξ. Exactness means that there exists a one-form α such that $d\alpha = \xi$, and because $d^2 = 0$ this implies that $d\xi = 0$. As we saw in example 4.20 and 4.26, in $\mathbb{R}^3$ we can identify both one-forms and two-forms with vector fields, and the d operator corresponds to the curl and divergence. The two-form ξ would correspond to a vector field $\mathbf{B}$ with $\nabla \cdot \mathbf{B} = 0$, and α would correspond to a vector potential $\mathbf{A}$ with $\nabla \times \mathbf{A} = \mathbf{B}$. In this case the wedge product corresponds to the dot product, and the Hopf invariant of $\mathbf{B}$ restricted to a volume $U \subseteq \mathbb{R}^3$ can be expressed as the *helicity* H of $\mathbf{B}$ in U:

$$H = \iiint_U \mathbf{A} \cdot \mathbf{B}\, dx^3$$

The choice of vector potential is not unique. We may apply a gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla \cdot f$ for some differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$, and this will satisfy $\nabla \times (\mathbf{A} + \nabla \cdot f) = \mathbf{B}$ as well. It is not obvious that helicity is gauge invariant, and in fact it only is if U satisfies some requirements. The statement that U is compatible with ξ as defined in definition 4.38, corresponds in $\mathbb{R}^3$ with the statment that $\mathbf{B}$ is tangent to the boundary of U everywhere. We may take U to be a bounded *flux tube*, a volume enclosed by a surface of field lines of $\mathbf{B}$, such that no field line penetrates the boundary of U. The check of well-definedness of the Hopf invariant in definition 4.39 can be translated directly into a proof of the gauge invariance of the helicity. In the context of electrodynamics, the application of Stokes' theorem in the proof is often called *Gauss' law*.

In corollary 4.43 we have shown that the Hopf invariant is invariant under an orientation-preserving diffeomorphism. An orientation-preserving diffeomorphism of $\mathbb{R}^3$ is a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}^3$ that does not invert parity. An example of such a function can be constructed from a fluid flow. If the position of a test particle as a function of time is given by $\mathbf{r}(t)$, then the map $\mathbf{r}(0) \mapsto \mathbf{r}(t)$ is an orientation-preserving function. If the field lines of $\mathbf{B}$ move along with this flow, then corollary 4.43 tells us that the helicity of $\mathbf{B}$ in a volume U does not change as U moves along with the flow.

CHAPTER 5

Magnetohydrodynamics

Magnetohydrodynamics (henceforth abbreviated MHD) is a combination of the theories of fluid dynamics, governed by the Navier-Stokes equations, and the theory of electrodynamics, governed by Maxwell's equations. The field of MHD studies electrically conducting fluids, the prime example of which are plasmas.

Plasma physics has promising applications such as nuclear fusion, a source of energy that unlike nuclear fission does not produce radioactive byproducts. A big problem here is the issue of *plasma confinement*: for controlled fusion, the plasma must be contained in a reactor vessel. However, the temperature required for nuclear fusion is so high (above 150 million °C), that no known material is able to withstand this amount of heat. The plasma will melt the walls of the reactor if it makes contact with them. To avoid this, present-day reactors employ intense magnetic fields to confine the plasma.

Alternatives have been proposed, where the plasma has an inherent stability due to the structure of its magnetic field. In order to understand this, we will first give a bit of background about MHD. As we will see, the magnetic field is a key ingredient of this theory, but the electric field plays a secondary role at best. Next, we will show how knots and links can be used to provide stability. We formalise this concept with the idea of *helicity*. Finally, we construct several magnetic fields with high helicity.

5.1 Ideal magnetohydrodynamics

Ideal MHD describes the dynamics of a conducting fluid with no net charge. The quantities that play a role here are:

- The mass density ρ
- The fluid velocity $\mathbf{v}$

- The pressure p
- The magnetic field $\mathbf{B}$

Vector quantities have been set in boldface. Note that there is no electric field here; in MHD the electric field is fully determined by $\mathbf{v}$ and $\mathbf{B}$. In ideal MHD, the electric resistivity of the fluid is assumed to be zero. That is, the fluid is a perfect conductor. This gives us a first hint about why the electric field may be neglected: in electrostatics, the electric field inside a perfect conductor is zero. However, MHD is not a static theory. The reason that the electric field is secondary nonetheless, is that the magnitude of the electric field is of the order $|\mathbf{v}||\mathbf{B}|$, as will be shown below. Whereas the time derivative of the electric field does play a role in electromagnetic waves, where it has a magnitude of the order $c\,|\mathbf{B}|$, its contribution is negligible in MHD when the velocity $\mathbf{v}$ is nonrelativistic.

The evolution of a system in ideal MHD, ignoring the effects of gravity, is given by the following equations (see also [Goedbloed and Poedts 2004, p. 133]):

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \tag{5.1}$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p \tag{5.2}$$

$$\frac{\partial p}{\partial t} = -\mathbf{v} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{v} \tag{5.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{5.4}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.5}$$

Equation 5.1, the *continuity equation* embodies conservation of mass: if the mass density changes, the fluid must have flowed somewhere else. Equation 5.2, the *momentum equation*, describes the forces acting on the fluid. The left-hand side represents the change in momentum, the right hand side has a Lorentz force term ($\mathbf{j} \times \mathbf{B}$, where $\mathbf{j} = \mu_0^{-1} \nabla \times \mathbf{B}$ with μ_0 the magnetic permeability of the vacuum) and a pressure term. Equation 5.3 concerns the *internal energy* of the fluid. A flow in the direction of the pressure gradient will reinforce the gradient (the first term), and if there is a net influx of fluid into a volume, pressure will build up in this volume (the second term). Here the constant γ is the *adiabatic index*, the ratio of the heat capacity at constant pressure and the heat capacity at constant volume. Equation 5.4 represents Faraday's law, $\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}$. The expression $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ is Ohm's law for a perfect conductor. Because of the infinite conductance of the fluid, any electric field will vanish in the reference frame of a test particle moving with the fluid. In the lab frame, we then find an electric field $-\mathbf{v} \times \mathbf{B}$. Finally, the magnetic field has no charges (equation 5.5).

5.2 Linked and knotted fields

Linking and knotting provide a promising way of creating stable plasmas, because in ideal MHD they are preserved. This imposes constraints on the evolution of the system. Consequently, plasmas with linked or knotted field lines might not be able to relax to a state of global minimum energy. In this manner, topological properties of the field can provide stability.

In ideal MHD, field lines of the magnetic field are said to be *frozen in* in the fluid. This idea, which is encoded in equation 5.4, was hinted at in [Alfvén 1942] and is sometimes called *Alfvén's theorem*. It states that the magnetic flux through a surface does not change as the surface moves along with the fluid flow. From this principle it can be derived that points connected by a magnetic field line will remain connected by the same field line as they move with the fluid. In particular, field lines cannot pass through one another. Linked field lines will stay linked throughout the evolution of the system. A thorough derivation of these effects can be found in [Stern 1966].

To study the dynamics of a conducting fluid, consider a circular flux tube (a surface of magnetic field lines) with high flux inside the tube, and zero flux outside. We assume that $V \approx 2\pi r \Theta$ is a good approximation of the volume V of this tube, where $2\pi r$ is the length of the tube, and Θ the surface area of its cross section. Recall that the magnetic energy is given by

$$E_B = \frac{1}{2} \iiint |\mathbf{B}|^2 \, dx^3$$

If we assume a magnetic field of constant magnitude B inside the tube, then the flux Φ through a cross section perpendicular to the field is simply $B\Theta$, and conversely $B = \Phi/\Theta$. We find that

$$E_B \approx \pi r \Theta B^2 = \pi r \Theta^{-1} \Phi^2$$

By the frozen-in principle, the flux Φ through a cross section of the tube is constant in time. Scaling the area Θ by a factor a while keeping Φ constant, will change the energy by a factor a^{-1}. Thickening the tube will therefore decrease its magnetic energy. On the other hand, scaling the length of the tube by a factor b while keeping Φ constant will change the energy by a factor b. Contracting the tube will decrease its magnetic energy. From this we can conclude that an unconstrained flux tube will contract and thicken as it relaxes.

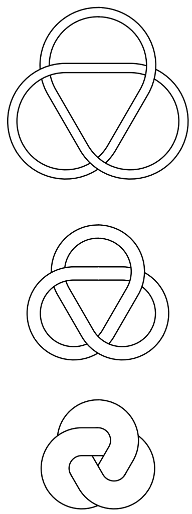

Figure 5.1 · Like linked flux tubes, a knotted flux tube cannot collapse onto itself.

The single flux tube helps us understand why linking is important for stability. Suppose that instead of a single flux tube, we have two linked tubes. Because the field lines cannot pass through eachother, linking prevents the tubes from collapsing onto theirselves. In order for one tube to contract, the other tube must become thinner — increasing its flux density and thereby its magnetic energy. This is the way in which linking provides stability. A more rigorous discussion can be found in [Moffatt 1969] and [Arnold 1974].

The degree of stability that different types of links and knots provide is an area of active research, but this is beyond the scope of this thesis.

To formalise the concept of linked field lines, we introduce a new quantity.

Definition 5.1 · The *magnetic helicity* of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ in a bounded volume V, such that $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$ on the boundary of V where $\hat{\mathbf{n}}$ is a unit length normal vector of the boundary, is defined by

$$H_V = \iiint_V \mathbf{A} \cdot \mathbf{B} \, dx^3$$

An example of such a volume V is a *flux tube*, a volume formed by all of the field lines that intersect a certain surface. See also [Goedbloed and Poedts 2004, p. 156].

The definition above is a special case of the Hopf invariant as defined in definition 4.39. In section 4.4 we proved the gauge invariance of H_V, a property that is not obvious from its definition. Furthermore, we showed that H_V is invariant when both V and the field lines of $\mathbf{B}$ are transformed by an orientation-preserving function. The flow of a fluid is an example of such a function, and because of the frozen-in principle, the magnetic field does move along with the fluid flow. It follows that in ideal MHD

$$\frac{dH_V}{dt} = 0$$

The conservation of this quantity was discovered by [Woltjer 1958]. An alternative derivation can be found in [Goedbloed and Poedts 2004, p. 157]. Besides the helicity in V it is possible to define a global helicity by integrating over all space, but this requires restrictions on the field if the helicity is to be gauge invariant.

To show the relation between linking and helicity, consider again the circular flux tube for which $V \approx 2\pi r \Theta$ is a good approximation of its volume. Here Θ is the surface area of its cross section. Suppose that $\mathbf{B}$ inside the tube has constant magnitude B, and zero magnitude outside of the tube. Assume that this flux tube is linked once with an identical flux tube, rotated by 90 degrees with respect to the first one. We will denote the first tube by T_1 and the second tube by T_2. To compute the magnetic helicity inside T_1, we can factor the integral into a part along the field, and a part perpendicular to the field. Denote by σ a field line inside T_1, and let D be the disk of which σ is the boundary. Let dl be a line element along σ, and dS a surface element of D. Note that dl is parallel to $\mathbf{B}$. We find

$$H_{T_1} = \iiint_{T_1} \mathbf{A} \cdot \mathbf{B} \, dx^3 \approx \Theta B \oint_\sigma \mathbf{A} \cdot dl = \Theta B \iint_D (\nabla \times \mathbf{A}) \cdot dS = \Theta B \iint_D \mathbf{B} \cdot dS \approx (\Theta B)^2$$

Here we used Stokes' theorem to write the integral as an integral of $\mathbf{B}$ over D, which picks up a factor ΘB from T_2 passing through it once. The field inside T_1 does not contribute, because $\mathbf{B}$ is perpendicular to the surface normal of D there. Beware that although we are computing the helicity *inside* T_1, it depends on $\mathbf{A}$ and $\mathbf{B}$ *outside* of T_1. If T_2 would be wound around T_1 twice instead of once, we would get an extra factor 2. In general, when T_2 and T_1 are linked n times, the helicity in T_1 (and by symmetry, in T_2) will be given by $n(\Theta B)^2$. The factor n is how topology enters into MHD.

When resistivity of the fluid is incorporated (non-ideal, resistive, or dissipative MHD), the frozen-in principle no longer holds. Among others, an extra term must be added to equation 5.4. [Goedbloed and Poedts 2004, p. 162] It follows that magnetic flux is no longer conserved, and field lines may break and recombine. The tools of topology break down here: continuity is at the heart of topology, so if field lines can break, we cannot meaningfully speak about linking. Fortunately, for suitable boundary conditions the helicity remains a well-defined quantity, and as argued in [Taylor 1974], helicity is *approximately conserved*, meaning that it changes at timescales much larger than typical timescales of fluid dynamics. The extent to which helicity still provides stability in resistive MHD is beyond the scope of this thesis, but work is being done in this area.

5.3 Constructing a magnetic field

In the previous sections we showed that we can construct self-stable plasma configurations in MHD by giving a vector field with high helicity. The field derived in section 4.3 comes to mind: its field lines are the fibres of the Hopf map projected stereographically onto $\mathbb{R}^3$, so all of the field lines are linked with every other field line.

The approach taken in section 5.3 was also used in [Kamchatnov 1982] to construct a magnetic field. Kamchatnov only considered the pullback of ω_0, not of a general two-form. In his case the vector potential was found in a deus ex machina manner, but obviously this approach does not generalise to different two-forms on S^2. By Poincaré's lemma the vector potential always exists, but an explicit computation can get quite involved. It was shown by Kamchatnov that the configuration obtained from the Hopf map is a magnetohydrodynamic *soliton* — a wave that preserves its shape while propagating.

By the principle given at the end of section 4.3, there are two ways to generalise the field given in equation 4.35 which we now take to be the magnetic field. Firstly, we can pull back a different two-form on the sphere. As this affects the magnitude of the field but not its field lines, this allows us to control the *energy density* of the magnetic energy of the field. (See figure 5.2.) Secondly, we may pull back by a different function, or even from a different manifold altogether. The convenient property of linked field lines is a consequence of the Hopf map, so this we do not change. Instead, we will intersperse a differentiable function $g : S^3 \to S^3$, and pull back by the composition

$$\mathbb{R}^3 \xhookrightarrow{\pi^{-1}} S^3 \xrightarrow{g} S^3 \xrightarrow{h} S^2 \xhookrightarrow{i} \mathbb{R}^3$$

Many functions g could potentially be interesting here and perhaps future research can be done in this area. For instance, by considering S^3 as a subgroup of $\mathbb{H}$ as in section 2.3, the map

$$S^3 \longrightarrow S^3, \quad q \longmapsto q^n$$

is a differentiable function for all $n \in \mathbb{Z}$. If we take $S^3 \subseteq \mathbb{C}^2_\circ$ instead, the following map is interesting:

$$S^3 \longrightarrow S^3, \quad (z_1, z_2) \longmapsto \tau(z_1^n, z_2^m) \tag{5.2}$$

Here $m, n \in \mathbb{Z}$ and $\tau : \mathbb{C}^2_\circ \twoheadrightarrow S^3$ denotes projection onto the sphere. The above map is differentiable because it is the composition of τ with a polynomial. For $m, n \notin \mathbb{Z}$ the map is not differentiable; it is not even continuous, so it does not make sense to compute the pullback by such a map. It turns out that for coprime m, n the field lines of the field induced by this map form torus knots. In this way we can produce not only linked field

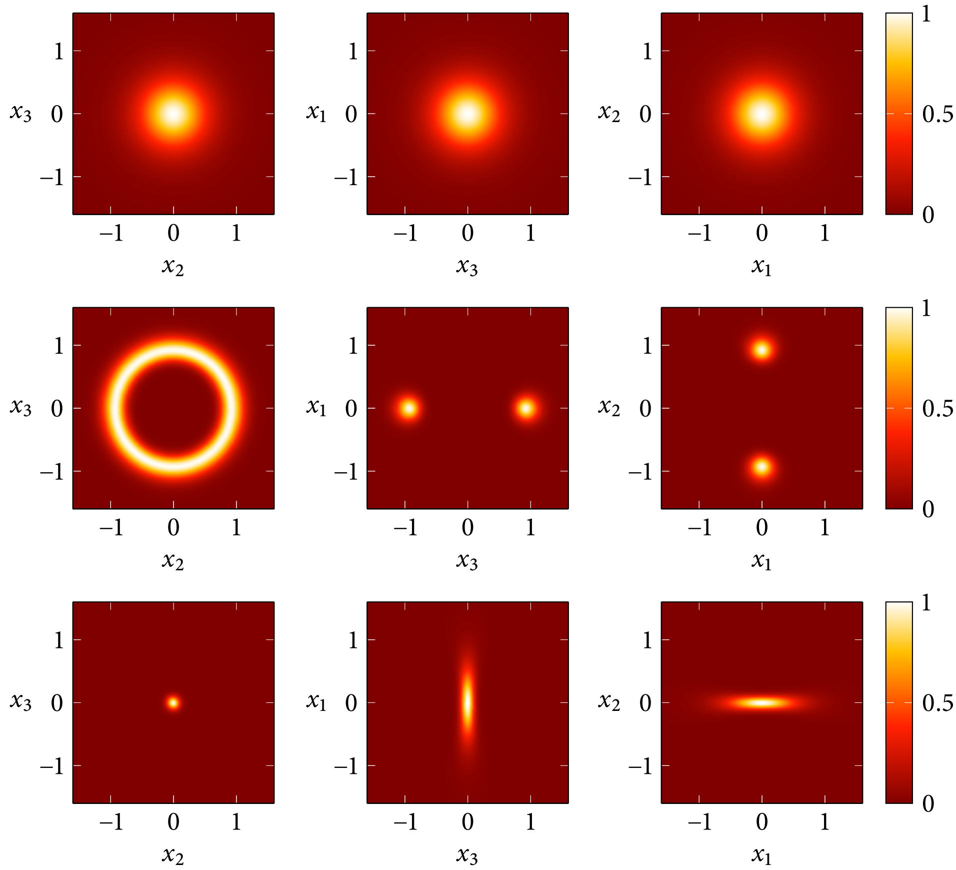

Figure 5.2 · Magnetic energy density $\|\mathbf{B}\|^2$ in the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ for the pullback by $h \circ \pi^{-1}$ of the following functions on S^2: $f(x) = 1$ for the top row, $f(x) = \exp(-3\|x + i\|^2)$ for the middle row, and $f(x) = \exp(-3\|x - i\|^2)$ for the bottom row. Intensity has been normalised per row.

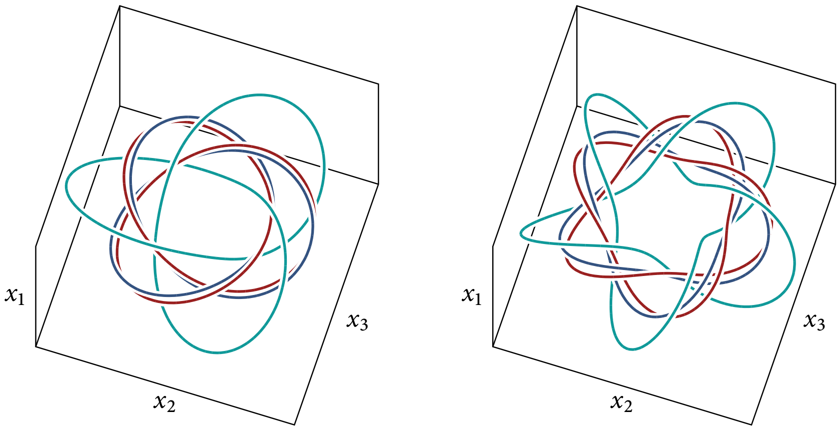

Figure 5.3 · A few field lines of fields where $g \neq \text{id}$; the function from equation 5.2 has been interspersed. On the left, $m = 3$ and on the right $m = 5$. In both cases $n = 2$. The field lines form torus knots, knotted themselves and linked with eachother.

lines, but also knotted field lines. See also figure 5.3. A slightly different map,

$$S^3 \longrightarrow S^3, \quad (z_1, z_2) \longmapsto (z_1^{(n)}, z_2^{(m)})$$

can occasionally be found in literature. Here the map $z \mapsto z^{(n)}$ denotes multiplying the argument of z with n. Unfortunately the map $z \mapsto z^{(n)}$ is not differentiable in 0, so the above function is not differentiable. It has been used nevertheless in [Arrayás and Trueba 2012], albeit in a different construction.

More generally we could consider the map

$$S^3 \longrightarrow S^3, \quad (z_1, z_2) \longmapsto \tau(p(z_1, z_2), q(z_1, z_2))$$

where $p, q \in \mathbb{C}[Z_1, Z_2]$ are polynomials that have no common roots except for $(0, 0)$. For polynomials with a common root other than $(0, 0)$ the function would map some $(z_1, z_2) \in S^3$ to $(0, 0)$, but this is not an element of $\mathbb{C}^2_\circ$; there is no way to project the origin onto the three-sphere.

The construction used in this thesis to produce magnetic fields is not limited to $\mathbb{R}^3$ or S^2, and a generalisation of this procedure to electrodynamics could potentially be interesting for future research. Minkowski space $\mathcal{M}$ is a four-dimensional pseudo-Riemannian manifold, where the bilinear form is given by the Lorentzian metric. By mapping $\mathcal{M}$ to a two-dimensional manifold via a differentiable function, we can construct a two-form $\omega \in \Omega^2\mathcal{M}$ that satisfies $d\omega = 0$. Maxwell's source-free equations can be expressed neatly in the language of differential geometry as

$$d\xi = 0 \qquad \text{and} \qquad d{*}\xi = 0$$

Here $\xi \in \Omega^2\mathcal{M}$ can be identified with the electromagnetic field tensor (sometimes called the Faraday tensor) and $*\xi$ denotes the *Hodge dual* of ξ. See [Szekeres 2004, p. 502] for further information on expressing Maxwell's equations in this form. With the construction in this thesis we can trivially satisfy $d\xi = 0$, which corresponds to solving the two homogeneous equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

The two-form ξ will not automatically satisfy $d{*}\xi = 0$ in general though. It would be interesting to investigate whether functions $\mathcal{M} \to N$ exists for a two-dimensional manifold N such that the pullback does satisfy $d{*}\xi = 0$ trivially.

CHAPTER 6

Conclusion

In this thesis we have given two equivalent definitions of the Hopf map, and with these we parametrised its fibres and showed that they are all linked with one another. We have given a procedure for constructing a divergenceless vector field from a differentiable function from $\mathbb{R}^3$ to a two-dimensional manifold, and applied this to the Hopf map composed with stereographic projection. We explored how variations of the field can be constructed by pulling back different two-forms or by altering the differentiable function $\mathbb{R}^3 \to S^2$. Finally, we interpreted the divergenceless vector field obtained from the Hopf map as the magnetic field in MHD, and we gave a heuristic argument as to why this field exhibits a form of self-stability. Areas of future research could be quantifying the degree of stability and exploring extensions of the given procedure to electromagnetism.

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