Higher-Ranked Exception Types

Work-in-Progress

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June 12, 2014

Motivation

- Types should not lie; we would like to have *checked exceptions* in Haskell:

$$map :: (\alpha \to \beta) \to [\alpha] \to [\beta]\ \mathbf{throws}\ e$$

- What should be the correct value of e?

Motivation

Assigning accurate exception types is complicated by:

Higher-order functions Exceptions raised by higher-order functions depend on the exceptions raised by functional arguments.

$$map :: (\alpha \to \beta\ \mathbf{throws}\ e_1) \to [\alpha] \to [\beta]\ \mathbf{throws}\ (e_1 \cup e_2)$$

Non-strict evaluation Exceptions are embedded inside values.

$$\begin{aligned} map :: {} & (\alpha\ \mathbf{throws}\ e_1 \to \beta)\ \mathbf{throws}\ e_2 \\ & \to [\alpha\ \mathbf{throws}\ e_3]\ \mathbf{throws}\ e_4 \to [\beta\ \mathbf{throws}\ e_5]\ \mathbf{throws}\ e_6 \end{aligned}$$

Motivation

- Instead of τ **throws** e, write τ^e for a type τ that can evaluate to $\bot_\chi$ for some $\chi \in e$.
- The fully annotated exception type for *map* would be:

$$
\begin{aligned}
&map :: (\alpha^{e_1} \to \beta^{(e_1 \,\cup\, e_2)})^{e_3} \to [\alpha^{e_1}]^{e_4} \to [\beta^{(e_1 \,\cup\, e_2 \,\cup\, e_3)}]^{e_4} \\
&map = \lambda f.\lambda xs.\ \mathbf{case}\ xs\ \mathbf{of} \\
&\qquad\qquad\quad \begin{array}{lll} [\,] & \mapsto & [\,] \\ (y : ys) & \mapsto & f\ y : map\ f\ ys \end{array}
\end{aligned}
$$

Motivation

- Instead of τ **throws** e, write τ^e for a type τ that can evaluate to $\bot_\chi$ for some $\chi \in e$.
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\begin{array}{l}
\mathit{map} :: (\alpha^{e_1} \to \beta^{(e_1 \,\cup\, e_2)})^{e_3} \to [\alpha^{e_1}]^{e_4} \to [\beta^{(e_1 \,\cup\, e_2 \,\cup\, e_3)}]^{e_4} \\
\mathit{map} = \lambda f.\lambda xs.\ \mathbf{case}\ xs\ \mathbf{of} \\
\qquad\qquad\qquad [\,] \qquad\quad \mapsto [\,] \\
\qquad\qquad\qquad (y : ys) \mapsto f\ y : \mathit{map}\ f\ ys
\end{array}
$$

- If you want to be pedantic:

$$
\begin{array}{l}
\mathit{map} :: \forall \alpha\ \beta\ e_1\ e_2\ e_3\ e_4. \\
\quad ((\alpha^{e_1} \to \beta^{(e_1 \,\cup\, e_2)})^{e_3} \to ([\alpha^{e_1}]^{e_4} \to [\beta^{(e_1 \,\cup\, e_2 \,\cup\, e_3)}]^{e_4})^{\emptyset})^{\emptyset}
\end{array}
$$

Motivation

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&\qquad\qquad\quad [\,] \qquad\; \mapsto [\,] \\
&\qquad\qquad\quad (y : ys) \mapsto f\ y : map\ f\ ys
\end{aligned}
$$

- If you want to be pedantic:

$$
\begin{aligned}
&map :: \forall \alpha\ \beta\ e_1\ e_2\ e_3\ e_4. \\
&\quad (\alpha^{e_1} \xrightarrow{e_3} \beta^{(e_1 \,\cup\, e_2)}) \xrightarrow{\emptyset} [\alpha^{e_1}]^{e_4} \xrightarrow{\emptyset} [\beta^{(e_1 \,\cup\, e_2 \,\cup\, e_3)}]^{e_4}
\end{aligned}
$$

Motivation

- The exception type

$$map :: (\alpha^{e_1} \xrightarrow{e_3} \beta^{(e_1 \,\cup\, e_2)}) \to [\alpha^{e_1}]^{e_4} \to [\beta^{(e_1 \,\cup\, e_2 \,\cup\, e_3)}]^{e_4}$$

  is not as accurate as we would like.
- Consider the instantiations:

$$\begin{aligned} &map\ id && :: [\alpha^{e_1}]^{e_4} \to [\alpha^{e_1}]^{e_4} \\ &map\ (const\ \bot_{\mathbf{E}}) && :: [\alpha^{e_1}]^{e_4} \to [\beta^{(e_1 \,\cup\, \{\mathbf{E}\})}]^{e_4} \end{aligned}$$

- A more appropriate type for *map* $(const\ \bot_{\mathbf{E}})$ would be

$$map\ (const\ \bot_{\mathbf{E}}) :: [\alpha^{e_1}]^{e_4} \to [\beta^{\{\mathbf{E}\}}]^{e_4}$$

  as it cannot propagate exceptional elements inside the input list to the output list.

Motivation

- The problem is that we have already committed the first argument of *map* to be of type

$$\alpha^{e_1} \xrightarrow{e_3} \beta^{(e_1 \cup e_2)},$$

i.e. it propagates exceptional values from the its input to the output while possibly adding additional exceptional values.
- This is a worst-case scenario: it is sound but inaccurate.

Motivation

- The solution is to move from Hindley–Milner to F_ω, introducing *higher-ranked types* and *type operators*.
  - Recall that System F_ω replicates the *simply typed λ-calculus* on the type level.
- This gives us the expressiveness to state the exception type of *map* as:

$$\begin{aligned}&\forall e_2\ e_3.(\forall e_1.\alpha^{e_1} \xrightarrow{e_3} \beta^{(e_2\ e_1)})\\ &\quad \to (\forall e_4\ e_5.[\alpha^{e_4}]^{e_5} \to [\beta^{(e_2\ e_4\ \cup\ e_3)}]^{e_5})\end{aligned}$$

- Note that e_2 is an *exception operator* of kind $\textsc{exn} \to \textsc{exn}$.

Motivation

- Given the following functions:

$$\begin{aligned}
map \quad & :: \forall e_2\, e_3.(\forall e_1.\alpha^{e_1} \xrightarrow{e_3} \beta^{(e_2\, e_1)}) \\
& \quad \to (\forall e_4\, e_5.[\alpha^{e_4}]^{e_5} \to [\beta^{(e_2\, e_4\, \cup\, e_3)}]^{e_5}) \\
id \quad & :: \forall e.\alpha^{e} \xrightarrow{\emptyset} \alpha^{e} \\
const\ \bot_{\mathbf{E}} & :: \forall e.\alpha^{e} \xrightarrow{\emptyset} \beta^{\{\mathbf{E}\}}
\end{aligned}$$

- Applying *id* or $const\ \bot_{\mathbf{E}}$ to *map* will give rise the the instantiations $e_2 \mapsto \lambda e.e$, respectively $e_2 \mapsto \lambda e.\{\mathbf{E}\}$.
- This gives us the exception types:

$$\begin{aligned}
map\ id \quad & :: \forall e_4\, e_5.[\alpha^{e_4}]^{e_5} \to [\alpha^{e_4}]^{e_5} \\
map\ (const\ \bot_{\mathbf{E}}) & :: \forall e_4\, e_5.[\alpha^{e_4}]^{e_5} \to [\beta^{\{\mathbf{E}\}}]^{e_5}
\end{aligned}$$

as desired.

Intermezzo: Simply-typed λ-calculus

Types

$$
\begin{array}{llll}
\tau \in \mathbf{Ty} & ::= & B & \text{(base type)} \\
 & \mid & \tau_1 \to \tau_2 & \text{(function type)}
\end{array}
$$

Terms

$$
\begin{array}{llll}
t \in \mathbf{Tm} & ::= & x, y, \ldots & \text{(variable)} \\
 & \mid & \lambda x : \tau . t & \text{(abstraction)} \\
 & \mid & t_1 \; t_2 & \text{(application)}
\end{array}
$$

Values

$$
\begin{array}{llll}
v \in \mathbf{Val} & ::= & x, y, \ldots & \text{(free variable)} \\
 & \mid & \lambda x : \tau . v & \text{(abstraction value)}
\end{array}
$$

Intermezzo: Simply-typed λ-calculus

Typing

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ [\textsc{T-Var}] \qquad \frac{\Gamma, x : \tau_1 \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . t : \tau_1 \rightarrow \tau_2}\ [\textsc{T-Abs}]$$

$$\frac{\Gamma \vdash t_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\ t_2 : \tau_2}\ [\textsc{T-App}]$$

Intermezzo: Simply-typed λ-calculus

Evaluation We perform *full β-reduction*, i.e. we also evaluate under binders.

$$\frac{t \longrightarrow t'}{\lambda x : \tau.t \longrightarrow \lambda x : \tau.t'} \text{ [E-ABS]}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \text{ [E-APP}_1\text{]} \qquad \frac{t_2 \longrightarrow t_2'}{t_1\ t_2 \longrightarrow t_1\ t_2'} \text{ [E-APP}_2\text{]}$$

$$\frac{}{(\lambda x : \tau.t_1)\ t_2 \longrightarrow [t_2/x]\, t_1} \text{ [E-BETA]}$$

Intermezzo: Simply-typed λ-calculus

Theorem (Progress)

A term t is either a value v, or we can reduce $t \longrightarrow t'$.

Theorem (Preservation)

If $\Gamma \vdash t : \tau$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : \tau$.

Theorem (Confluence)

If $t \longrightarrow t_1$ and $t \longrightarrow t_2$, then exists a term t' such that $t_1 \longrightarrow^* t'$ and $t_2 \longrightarrow^* t'$.

Theorem (Normalization)

For any term t we have that $t \longrightarrow^* v$ (in a finite number of steps).

Corollary (Uniqueness of normal forms)

If $t \longrightarrow^* v_1$ and $t \longrightarrow^* v_2$, then $v_1 \equiv v_2$.

Intermezzo: The lambda "cube"

- The simply-typed λ-calculus can be extended with *parametric polymorphism*, or *type operators*, or both.

$$\begin{tikzcd} F \arrow[r] & F_\omega \\ \lambda \arrow[u] \arrow[r] & \lambda_\omega \arrow[u] \end{tikzcd}$$

- $id : B \to B$
  $id = \lambda x : B.x$
- $id : \forall \alpha :: *.\alpha \to \alpha$
  $id = \Lambda \alpha : *.\lambda x : \alpha.x$
- $Id :: * \Rightarrow *$
  $Id = \lambda \alpha :: *.\alpha$

  $id : B \to Id\ B$
  $id = \lambda x : B.x$
- $Id :: * \Rightarrow *$
  $Id = \lambda \alpha :: *.\alpha$

  $id : \forall \alpha :: *.\alpha \to Id\ \alpha$
  $id = \Lambda \alpha : *.\lambda x : \alpha.x$

- Omitted: the axis for dependent types.

Intermezzo: System F_ω

Types

$$
\begin{array}{lcll}
\tau \in \mathbf{Ty} & ::= & \alpha & \text{(type variable)} \\
 & | & \tau_1 \to \tau_2 & \text{(function type)} \\
 & | & \forall \alpha :: \kappa . \tau & \text{(universal type)} \\
 & | & \lambda \alpha :: \kappa . \tau & \text{(operator abstraction)} \\
 & | & \tau_1 \; \tau_2 & \text{(operator application)}
\end{array}
$$

Kinds

$$
\begin{array}{lcll}
\kappa \in \mathbf{Kind} & ::= & * & \text{(base kind)} \\
 & | & \kappa_1 \Rightarrow \kappa_2 & \text{(operator kind)}
\end{array}
$$

Intermezzo: System F_ω

Terms

$$
\begin{array}{rcll}
t \in \mathbf{Tm} & ::= & x, y, \ldots & \text{(variable)} \\
& | & \lambda x : \tau . t & \text{(abstraction)} \\
& | & t_1 \ t_2 & \text{(application)} \\
& | & \Lambda \alpha :: \kappa . t & \text{(type abstraction)} \\
& | & t \ \langle \tau \rangle & \text{(type application)}
\end{array}
$$

Values

$$
\begin{array}{rcll}
v \in \mathbf{Val} & ::= & x, y, \ldots & \text{(free variable)} \\
& | & \lambda x : \tau . v & \text{(abstraction value)} \\
& | & \Lambda \alpha : \kappa . v & \text{(type abstraction value)}
\end{array}
$$

Intermezzo: System F_ω

Kinding Note the similarity between the *type* system of the simply typed λ-calculus.

$$\frac{}{\Gamma, \alpha :: \kappa \vdash \alpha :: \kappa}\ [\textsc{K-Var}] \qquad \frac{\Gamma, \alpha :: \kappa_1 \vdash \tau_2 :: \kappa_2}{\Gamma \vdash \lambda\alpha :: \kappa_1 . \tau_2 :: \kappa_1 \Rightarrow \kappa_2}\ [\textsc{K-Abs}]$$

$$\frac{\Gamma \vdash \tau_1 :: \kappa_1 \Rightarrow \kappa_2 \quad \Gamma \vdash \tau_2 :: \kappa_1}{\Gamma \vdash \tau_1\ \tau_2 :: \kappa_2}\ [\textsc{K-App}]$$

$$\frac{\Gamma \vdash \tau_1 :: * \quad \Gamma \vdash \tau_2 :: *}{\Gamma \vdash \tau_1 \to \tau_2 :: *}\ [\textsc{K-Arrow}] \qquad \frac{\Gamma, \alpha :: \kappa \vdash \tau :: *}{\Gamma \vdash \forall\alpha :: \kappa . \tau :: *}\ [\textsc{K-Forall}]$$

Intermezzo: System F_ω

Type equivalence

$$\frac{}{\tau \equiv \tau}\ [\textsc{Q-Refl}] \qquad \frac{\tau_1 \equiv \tau_2}{\tau_2 \equiv \tau_1}\ [\textsc{Q-Symm}]$$

$$\frac{\tau_1 \equiv \tau_2 \quad \tau_2 \equiv \tau_3}{\tau_1 \equiv \tau_3}\ [\textsc{Q-Trans}] \qquad \frac{\tau_1 \equiv \tau_1' \quad \tau_2 \equiv \tau_2'}{\tau_1 \to \tau_2 \equiv \tau_1' \to \tau_2'}\ [\textsc{Q-Arrow}]$$

$$\frac{\tau_1 \equiv \tau_2}{\forall \alpha :: \kappa.\tau_1 \equiv \forall \alpha :: \kappa.\tau_2}\ [\textsc{Q-Forall}]$$

$$\frac{\tau_1 \equiv \tau_2}{\lambda \alpha :: \kappa.\tau_1 \equiv \lambda \alpha :: \kappa.\tau_2}\ [\textsc{Q-Abs}] \qquad \frac{\tau_1 \equiv \tau_1' \quad \tau_2 \equiv \tau_2'}{\tau_1\ \tau_2 \equiv \tau_1'\ \tau_2'}\ [\textsc{Q-App}]$$

$$\frac{}{(\lambda \alpha :: \kappa.\tau_1)\ \tau_2 \equiv [\alpha \mapsto \tau_2]\ \tau_1}\ [\textsc{Q-Beta}]$$

Intermezzo: System F_ω

Typing In addition to the rules for the simply typed λ-calculus:

$$\frac{\Gamma \vdash \tau_1 :: * \quad \Gamma, x : \tau_1 \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . t : \tau_1 \to \tau_2} \text{[T-ABS]}$$

$$\frac{\Gamma, \alpha :: \kappa_1 \vdash t_2 : \tau_2}{\Gamma \vdash \Lambda \alpha :: \kappa_1 . t_2 : \forall \alpha :: \kappa_1 . \tau_2} \text{[T-TYABS]}$$

$$\frac{\Gamma \vdash t_1 : \forall \alpha :: \kappa_1 . \tau_1 \quad \Gamma \vdash \tau_2 :: \kappa_1}{\Gamma \vdash t_1 \langle \tau_2 \rangle : [\alpha \mapsto \tau_2]\, \tau_1} \text{[T-TYAPP]}$$

$$\frac{\Gamma \vdash t : \tau_1 \quad \tau_1 \equiv \tau_2 \quad \Gamma \vdash \tau_2 :: *}{\Gamma \vdash t : \tau_2} \text{[T-EQ]}$$

Intermezzo: System F_ω

Evaluation In addition to the rules for the simply typed λ-calculus:

$$\frac{t \longrightarrow t'}{\Lambda\alpha : \kappa.t \longrightarrow \Lambda\alpha : \kappa.t'} \text{[E-TyAbs]} \qquad \frac{t \longrightarrow t'}{t\ \langle\tau\rangle \longrightarrow t'\ \langle\tau\rangle} \text{[E-TyApp]}$$

$$\frac{}{(\Lambda\alpha : \kappa.t)\ \langle\tau\rangle \longrightarrow [\tau/\alpha]\, t} \text{[E-TyBeta]}$$

Intermezzo: System F_ω

Metatheory

- We still have *progress*, *preservation* and *decidability* (of type checking).
- Proofs rely on the structure of the types and type equivalence relation and thus the properties (especially *normalization* and *uniqueness of normal forms*) of the simply typed λ-calculus.

Technicalities

- Due to their syntactic weight, higher-ranked exception type only seem useful if they can be infered automatically.
- Unlike for HM type inference is undecidable in F_ω.
- However, the exception types are annotations piggybacking on top of an underlying type system.
- Holdermans and Hage [HH10] showed type inference is decidable for a higher-ranked annotated type system with type operators performing control-flow analysis.

Technicalities

1. Perform Hindley–Milner type inference to reconstruct the underlying types.
2. Run a second inference pass to reconstruct the exception types.
    2.1 Collect a set of subtyping constraints.
    2.2 In case of a λ-abstraction $\lambda x : \tau.e$, we *complete* the type τ to an exception type.
    2.3 In case of an application we *match* the types of the formal and actual parameter.
3. Solve the generated subtyping constraints.

Technicalities: Reconstruction (variables)

$$
\begin{array}{l}
\textit{reconstruct}\ \widehat{\Gamma}\ x = \\
\quad \textbf{let}\ (\widehat{\tau}, \chi) = \widehat{\Gamma}\ (x) \\
\quad \phantom{\textbf{let}}\ \ e\ \textit{be fresh} \\
\quad \textbf{in}\ \ (\widehat{\tau}, e, \{\chi \subseteq e\})
\end{array}
$$

Technicalities: Reconstruction (abstractions)

$$
\begin{array}{l}
\mathit{reconstruct}\ \widehat{\Gamma}\ (\lambda x : \tau . t) = \\
\quad \textbf{let}\ (\widehat{\tau}_1, \overline{e_i :: \kappa_i}) = \mathit{complete}\ \tau\ \emptyset \\
\qquad e_1\ \mathit{be\ fresh} \\
\qquad (\widehat{\tau}_2, e_2, C_1)\ = \mathit{reconstruct}\ (\widehat{\Gamma}, x \mapsto (\widehat{\tau}_1, e_1))\ t \\
\qquad X\ = \{e_1\} \cup \{\overline{e_i}\} \cup \mathit{fv}\ \widehat{\Gamma} \\
\qquad \chi_2 = \mathit{solve}\ C_1\ X\ e_2 \\
\qquad \widehat{\tau}\ = \forall e_1 :: \textsc{exn}.\overline{\forall e_i :: \kappa_i}.\ \widehat{\tau}_1^{e_1} \to \widehat{\tau}_2^{\chi_2} \\
\qquad e\ \mathit{be\ fresh} \\
\quad \textbf{in}\ \ (\widehat{\tau}, e, \emptyset)
\end{array}
$$

Technicalities: Completion

- The completion procedure adds as many quantifiers and type operators as possible to a type.

$$\frac{}{\overline{e_i :: \kappa_i} \vdash \mathbf{bool} : \widehat{\mathrm{bool}}\ \&\ e\ \overline{e_i} \rhd e :: \overline{\kappa_i} \Rightarrow \text{EXN}}\ \text{[C-Bool]}$$

$$\frac{\overline{e_i :: \kappa_i} \vdash \tau : \widehat{\tau}\ \&\ \chi \rhd \overline{e_j :: \kappa_j}}{\overline{e_i :: \kappa_i} \vdash [\tau] : [\widehat{\tau}\ \mathbf{throws}\ \chi]\ \ \&\ e\ \overline{e_i} \rhd e :: \overline{\kappa_i} \Rightarrow \text{EXN}, \overline{e_j :: \kappa_j}}\ \text{[C-List]}$$

$$\frac{\vdash \tau_1 : \widehat{\tau}_1\ \&\ \chi_1 \rhd \overline{e_j :: \kappa_j} \qquad \overline{e_i :: \kappa_i}, \overline{e_j :: \kappa_j} \vdash \tau_2 : \widehat{\tau}_2\ \&\ \chi_2 \rhd \overline{e_j :: \kappa_j}}{\overline{e_i :: \kappa_i} \vdash \tau_1 \to \tau_2 : \forall \overline{e_j :: \kappa_j}. (\widehat{\tau}_1\ \mathbf{throws}\ \chi_1 \to \widehat{\tau}_2\ \mathbf{throws}\ \chi_2)\ \ \&\ e\ \overline{e_i} \rhd e :: \overline{\kappa_j} \Rightarrow \text{EXN}, \overline{e_k :: \kappa_k}}\ \text{[C-Arr]}$$

Figure : Type completion ($\Gamma \vdash \tau : \widehat{\tau}\ \&\ \chi \rhd \Gamma'$)

Technicalities: Completion

- $\cdot \vdash \mathbf{bool} : \widehat{\mathrm{bool}} \;\&\; e_1 \triangleright e_1 :: \textsc{exn}$

Technicalities: Completion

- $\cdot \vdash \mathbf{bool} : \widehat{\mathrm{bool}} \ \& \ e_1 \triangleright e_1 :: \textsc{exn}$
- $e_1 :: \textsc{exn} \vdash \mathbf{bool} : \widehat{\mathrm{bool}} \ \& \ e_2\ e_1 \triangleright e_2 :: \textsc{exn} \Rightarrow \textsc{exn}$

Technicalities: Completion

- $\mathbf{bool} \to \mathbf{bool}$
- $\forall e_1 :: \textsc{exn}.\ \widehat{\mathrm{bool}}^{e_1} \to \widehat{\mathrm{bool}}^{(e_2\ e_1)}\ \&\ e_3$
- $e_2 :: \textsc{exn} \Rightarrow \textsc{exn}, e_3 :: \textsc{exn}$

Technicalities: Completion

- $\mathbf{bool} \to \mathbf{bool} \to \mathbf{bool}$
-
$$\forall e_1 :: \textsc{exn}.\ \widehat{\text{bool}}^{e_1} \to \left(\forall e_4 :: \textsc{exn}.\ \widehat{\text{bool}}^{e_4} \xrightarrow{e_2\ e_1} \widehat{\text{bool}}^{(e_5\ e_1\ e_4)}\right) \,\&\, e_3$$
- $e_2 :: \textbf{EXN} \Rightarrow \textbf{EXN}, e_3 :: \textbf{EXN}, e_5 :: \textbf{EXN} \Rightarrow \textbf{EXN} \Rightarrow \textbf{EXN}$

Technicalities: Completion

- $(\mathbf{bool} \to \mathbf{bool}) \to \mathbf{bool}$
-

$$\begin{aligned}&\forall e_2 :: \textsc{exn} \Rightarrow \textsc{exn}.\ \forall e_3 :: \textsc{exn}.\\ &\quad \left(\forall e_1 :: \textsc{exn}.\ \widehat{\mathrm{bool}}^{e_1} \xrightarrow{e_3} \widehat{\mathrm{bool}}^{(e_2\ e_1)}\right) \to \widehat{\mathrm{bool}}^{(e_4\ e_2\ e_3)}\ \&\ e_5\end{aligned}$$

- $e_4 :: (\textsc{exn} \Rightarrow \textsc{exn}) \Rightarrow \textsc{exn} \Rightarrow \textsc{exn}, e_5 :: \textsc{exn}$

Technicalities: Reconstruction (applications)

$$
\begin{array}{l}
\mathit{reconstruct}\ \widehat{\Gamma}\ (t_1\ t_2) = \\
\quad \mathbf{let}\ (\widehat{\tau}_1, e_1, C_1)\ = \mathit{reconstruct}\ \widehat{\Gamma}\ t_1 \\
\qquad\ \ (\widehat{\tau}_2, e_2, C_2)\ = \mathit{reconstruct}\ \widehat{\Gamma}\ t_2 \\
\qquad\ \ \widehat{\tau}_2'^{e_2'} \to \widehat{\tau}'^{\chi'} = \mathit{instantiate}\ \widehat{\tau}_1 \\
\qquad\ \ \theta = [e_2' \mapsto e_2] \circ \mathit{match}\ \emptyset\ \widehat{\tau}_2\ \widehat{\tau}_2' \\
\qquad\ \ e\ \mathit{be\ fresh} \\
\qquad\ \ C = \{...\} \\
\quad \mathbf{in}\ (\widehat{\tau}, e, C)
\end{array}
$$

Technicalities: Matching

$$
\begin{array}{llll}
\mathit{match} :: \mathbf{Env} \to \mathbf{Ty} \to \mathbf{Ty} \to \mathbf{Subst} & & & \\
\mathit{match}\ \Sigma\ \widehat{\mathrm{bool}} & \widehat{\mathrm{bool}} & & = \mathit{Id} \\
\mathit{match}\ \Sigma\ (\forall e :: \kappa.\widehat{\tau}_1) & (\forall e :: \kappa.\widehat{\tau}'_1) & & = \mathit{match}\ (\Sigma, e \mapsto \kappa)\ \widehat{\tau}_1\ \widehat{\tau}'_1 \\
\mathit{match}\ \Sigma\ (\widehat{\tau}_1^{e_1} \to \widehat{\tau}_2^{\chi_2}) & (\widehat{\tau}_1^{e_1} \to \widehat{\tau}_2'^{(e_0\ \overline{e_j})}) & = & \\
\quad \left[e_0 \mapsto (\overline{\lambda e_j :: \Sigma(e_j).}\ \chi_2)\right] \circ \mathit{match}\ \Sigma\ \widehat{\tau}_2\ \widehat{\tau}'_2 & & & \\
\mathit{match}\ \Sigma\ \_ & \_ & & = \mathbf{fail}
\end{array}
$$

Technicalities: Matching — Example

- $\mathit{match}\ [e_1 :: \textsc{exn}, e_2 :: \textsc{exn} \Rightarrow \textsc{exn}, e_3 :: \textsc{exn}]$
  $(\mathrm{b\widehat{oo}l}^{e_1} \to \mathrm{b\widehat{oo}l}^{(e_2\ \ e_1\ \cup\ e_3)})\ (\mathrm{b\widehat{oo}l}^{e_1} \to \mathrm{b\widehat{oo}l}^{(e_0\ \ e_1\ \ e_2\ \ e_3)})$

Technicalities: Matching — Example

- $match\ [e_1 :: \textsc{exn}, e_2 :: \textsc{exn} \Rightarrow \textsc{exn}, e_3 :: \textsc{exn}]$
  $(\widehat{\mathrm{bool}}^{e_1} \to \widehat{\mathrm{bool}}^{(e_2\ e_1\ \cup\ e_3)})\ (\widehat{\mathrm{bool}}^{e_1} \to \widehat{\mathrm{bool}}^{(e_0\ e_1\ e_2\ e_3)})$
- $[e_0 \mapsto \lambda e_1 :: \textsc{exn}.\lambda e_2 :: \textsc{exn} \Rightarrow \textsc{exn}.\lambda e_3 :: \textsc{exn}.e_2\ e_1 \cup e_3])$

Technicalities: Constraint solving

- Solving subtyping constraints can be done using a fixed-point iteration.
- To decide we have reached a fixed point we need an equality on types.
- But types are now a simply typed λ-calculus.

Technicalities: $\lambda^{\cup}$

Types

$$\begin{array}{rcll} \tau \in \mathbf{Ty} & ::= & \mathcal{P} & \text{(base type)} \\ & | & \tau_1 \to \tau_2 & \text{(function type)} \end{array}$$

Terms

$$\begin{array}{rcll} t \in \mathbf{Tm} & ::= & x, y, \ldots & \text{(variable)} \\ & | & \lambda x : \tau . t & \text{(abstraction)} \\ & | & t_1 \ t_2 & \text{(application)} \\ & | & \emptyset & \text{(empty)} \\ & | & \{c\} & \text{(singleton)} \\ & | & t_1 \cup t_2 & \text{(union)} \end{array}$$

Values Values v are terms of the form

$$\lambda x_1{:}\tau_1.\cdots\lambda x_i{:}\tau_i.\{c_1\}\cup(\cdots\cup(\{c_j\}\cup(x_1 \ v_{11}\cdots v_{1m}\cup(\cdots\cup x_k \ v_{k1}\cdots v_{kn}))))$$

Technicalities: $\lambda^{\cup}$

$$
\begin{array}{rcll}
(\lambda x:\tau.t_1)\ t_2 & \longrightarrow & [t_2/x]\,t_1 & (\beta\text{-reduction}) \\
(t_1 \cup t_2)\ t_3 & \longrightarrow & t_1\ t_3 \cup t_2\ t_3 & \\
(\lambda x:\tau.t_1) \cup (\lambda x:\tau.t_2) & \longrightarrow & \lambda x:\tau.\,(t_1 \cup t_2) & (\text{congruences}) \\
x\ t_1 \cdots t_n \cup x'\ t'_1 \cdots t'_n & \longrightarrow & x\ \ (t_1 \cup t'_1) \cdots (t_n \cup t'_n) & \\
(t_1 \cup t_2) \cup t_3 & \longrightarrow & t_1 \cup (t_2 \cup t_3) & (\text{associativity}) \\
\emptyset \cup t & \longrightarrow & t & \\
t \cup \emptyset & \longrightarrow & t & (\text{unit}) \\
x \cup x & \longrightarrow & x & \\
x \cup (x \cup t) & \longrightarrow & x \cup t & \\
\{c\} \cup \{c\} & \longrightarrow & \{c\} & (\text{idempotence}) \\
\{c\} \cup (\{c\} \cup t) & \longrightarrow & \{c\} \cup t & \\
x\ t_1 \cdots t_n \cup \{c\} & \longrightarrow & \{c\} \cup x\ t_1 \cdots t_n & (1) \\
x\ t_1 \cdots t_n \cup (\{c\} \cup t) & \longrightarrow & \{c\} \cup (x\ t_1 \cdots t_n \cup t) & (2) \\
x\ t_1 \cdots t_n \cup x'\ t'_1 \cdots t'_n & \longrightarrow & x'\ t'_1 \cdots t'_n \cup x\ t_1 \cdots t_n \quad \text{if } x' \prec x & (3) \\
x\ t_1 \cdots t_n \cup (x'\ t'_1 \cdots t'_n \cup t) & \longrightarrow & x'\ t'_1 \cdots t'_n \cup (x\ t_1 \cdots t_n \cup t) \quad \text{if } x' \prec x & (4) \\
\{c\} \cup \{c'\} & \longrightarrow & \{c'\} \cup \{c\} \quad \text{if } c' \prec c & (5) \\
\{c\} \cup (\{c'\} \cup t) & \longrightarrow & \{c'\} \cup (\{c\} \cup t) \quad \text{if } c' \prec c & (6)
\end{array}
$$

Technicalities: $\lambda^{\cup}$

Conjecture

The reduction relation $\longrightarrow$ preserves meaning.

Conjecture

The reduction relation $\longrightarrow$ is strongly normalizing.

Conjecture

The reduction relation $\longrightarrow$ is locally confluent.

Corollary

The reduction relation $\longrightarrow$ is confluent.

Corollary

The $\lambda^{\cup}$-calculus has unique normal forms.

Corollary

Equality of $\lambda^{\cup}$-terms can be decided by normalization.

Problems

- Not sound w.r.t. *imprecise exception semantics*.
- Making it sound negates the precision gained by higher-ranked types.
- Need to move to a more powerful constraint language.
  - In previous work we used conditionals/implications and a somewhat ad hoc non-emptyness guard.
  - Now I want to look at *Boolean rings*, which look more well-behaved.
  - May make more sense to use *equational unification* instead of constraints.

Problems: Imprecise exception semantics

- In an optimizing compiler we want the following equality, called the *case-switching transformation*, to hold:

$$
\begin{array}{l}
\forall e_i.\textbf{if } e_1 \textbf{ then} \\
\quad\quad \textbf{if } e_2 \textbf{ then } e_3 \textbf{ else } e_4 \\
\quad \textbf{else} \\
\quad\quad \textbf{if } e_2 \textbf{ then } e_5 \textbf{ else } e_6 \equiv
\begin{array}[t]{l}
\textbf{if } e_2 \textbf{ then} \\
\quad \textbf{if } e_1 \textbf{ then } e_3 \textbf{ else } e_5 \\
\textbf{else} \\
\quad \textbf{if } e_1 \textbf{ then } e_4 \textbf{ else } e_6
\end{array}
\end{array}
$$

- This does not hold if we have observable exceptions and track them precisely.
  - Counterexample: Take $e_1 = \bot_{\mathbf{E_1}}$ and $e_2 = \bot_{\mathbf{E_2}}$.
- Introduce some "imprecision": If the guard can reduce to an exceptional value, then pretend both branches get executed.

Problems: Imprecise exception semantics

- In an optimizing compiler we want the following equality, called the *case-switching transformation*, to hold:

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\forall e_i.\textbf{if } e_1 \textbf{ then} \\
\qquad \textbf{if } e_2 \textbf{ then } e_3 \textbf{ else } e_4 \\
\quad \textbf{else} \\
\qquad \textbf{if } e_2 \textbf{ then } e_5 \textbf{ else } e_6 \equiv \textbf{if } e_2 \textbf{ then} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \textbf{if } e_1 \textbf{ then } e_3 \textbf{ else } e_5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textbf{else} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \textbf{if } e_1 \textbf{ then } e_4 \textbf{ else } e_6
\end{array}
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- This does not hold if we have observable exceptions and track them precisely.
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