Higher-ranked Exception Types

Ruud Koot

March 17, 2015

1 The $\lambda^{\cup}$-calculus

Types

$$\begin{array}{rcll} \tau \in \mathbf{Ty} & ::= & \mathcal{P} & \text{(base type)} \\ & | & \tau_1 \to \tau_2 & \text{(function type)} \end{array}$$

Terms

$$\begin{array}{rcll} t \in \mathbf{Tm} & ::= & x, y, ... & \text{(variable)} \\ & | & \lambda x : \tau . t & \text{(abstraction)} \\ & | & t_1\ t_2 & \text{(application)} \\ & | & \emptyset & \text{(empty)} \\ & | & \{c\} & \text{(singleton)} \\ & | & t_1 \cup t_2 & \text{(union)} \end{array}$$

Values Values v are terms of the form

$$\lambda x_1 : \tau_1. \cdots \lambda x_i : \tau_i. \{c_1\} \cup (\cdots \cup (\{c_j\} \cup (x_1\ v_{11} \cdots v_{1m} \cup (\cdots \cup x_k\ v_{k1} \cdots v_{kn}))))$$

Environments

$$\Gamma \in \mathbf{Env} ::= \quad \cdot \quad | \quad \Gamma, x : \tau$$

1.1 Typing relation

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ [\text{T-Var}] \qquad \frac{\Gamma, x : \tau_1 \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . t : \tau_1 \to \tau_2}\ [\text{T-Abs}] \qquad \frac{\Gamma \vdash t_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\ t_2 : \tau_2}\ [\text{T-App}]$$

$$\frac{}{\Gamma \vdash \emptyset : \mathcal{P}}\ [\text{T-Empty}] \qquad \frac{}{\Gamma \vdash \{c\} : \mathcal{P}}\ [\text{T-Con}] \qquad \frac{\Gamma \vdash t_1 : \tau \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash t_1 \cup t_2 : \tau}\ [\text{T-Union}]$$

1.2 Semantics

1.3 Reduction relation

Definition 1. *Let $\prec$ be a strict total order on* $\mathbf{Con} \cup \mathbf{Var}$*, with $c \prec x$ for all $c \in$* $\mathbf{Con}$ *and $x \in$* $\mathbf{Var}$.

$$
\begin{array}{rcllr}
(\lambda x : \tau . t_1)\ \ t_2 & \longrightarrow & [t_2/x]\, t_1 & & (\beta\text{-reduction}) \\
(t_1 \cup t_2)\ \ t_3 & \longrightarrow & t_1\ t_3 \cup t_2\ t_3 & & \\
(\lambda x : \tau . t_1) \cup (\lambda x : \tau . t_2) & \longrightarrow & \lambda x : \tau .\, (t_1 \cup t_2) & & (\text{congruences}) \\
x\ t_1 \cdots t_n \cup x'\ t'_1 \cdots t'_n & \longrightarrow & x\ \ (t_1 \cup t'_1) \cdots (t_n \cup t'_n) & & \\
(t_1 \cup t_2) \cup t_3 & \longrightarrow & t_1 \cup (t_2 \cup t_3) & & (\text{associativity}) \\
\varnothing \cup t & \longrightarrow & t & & \\
t \cup \varnothing & \longrightarrow & t & & (\text{unit}) \\
x \cup x & \longrightarrow & x & & \\
x \cup (x \cup t) & \longrightarrow & x \cup t & & \\
\{c\} \cup \{c\} & \longrightarrow & \{c\} & & (\text{idempotence}) \\
\{c\} \cup (\{c\} \cup t) & \longrightarrow & \{c\} \cup t & & \\
x\ t_1 \cdots t_n \cup \{c\} & \longrightarrow & \{c\} \cup x\ t_1 \cdots t_n & & (1) \\
x\ t_1 \cdots t_n \cup (\{c\} \cup t) & \longrightarrow & \{c\} \cup (x\ t_1 \cdots t_n \cup t) & & (2) \\
x\ t_1 \cdots t_n \cup x'\ t'_1 \cdots t'_n & \longrightarrow & x'\ t'_1 \cdots t'_n \cup x\ t_1 \cdots t_n & \text{if } x' \prec x & \\
 & & & & (3) \\
x\ t_1 \cdots t_n \cup \left(x'\ t'_1 \cdots t'_n \cup t\right) & \longrightarrow & x'\ t'_1 \cdots t'_n \cup (x\ t_1 \cdots t_n \cup t) & \text{if } x' \prec x & (4) \\
\{c\} \cup \{c'\} & \longrightarrow & \{c'\} \cup \{c\} & \text{if } c' \prec c & (5) \\
\{c\} \cup \left(\{c'\} \cup t\right) & \longrightarrow & \{c'\} \cup (\{c\} \cup t) & \text{if } c' \prec c & (6)
\end{array}
$$

Conjecture 1. *The reduction relation $\longrightarrow$ preserves meaning.*

Conjecture 2. *The reduction relation $\longrightarrow$ is strongly normalizing.*

Conjecture 3. *The reduction relation $\longrightarrow$ is locally confluent.*

Corollary 1. *The reduction relation $\longrightarrow$ is confluent.*

Proof. Follows from SN, LC and Newman's Lemma. □

Corollary 2. *The $\lambda^{\cup}$-calculus has unique normal forms.*

Corollary 3. *Equality of $\lambda^{\cup}$-terms can be decided by normalization.*

2 Completion

$$
\begin{array}{lcll}
\kappa \in \mathbf{Kind} & ::= & \textsc{e} & (\text{exception}) \\
 & | & \kappa_1 \Rightarrow \kappa_2 & (\text{exception operator})
\end{array}
$$

$$
\begin{array}{llll}
\varphi \in \mathbf{Exn} & ::= & e & \text{(exception variables)} \\
 & \mid & \lambda e : \kappa.\varphi & \text{(exception abstraction)}
\end{array}
$$

$$
\begin{array}{llll}
\widehat{\tau} \in \mathbf{ExnTy} & ::= & \forall e :: \kappa.\widehat{\tau} & \text{(exception quantification)} \\
 & \mid & \mathrm{b}\widehat{\mathrm{oo}}\mathrm{l} & \text{(boolean type)} \\
 & \mid & [\widehat{\tau}\langle\varphi\rangle] & \text{(list type)} \\
 & \mid & \widehat{\tau}_1\langle\varphi_1\rangle \to \widehat{\tau}_2\langle\varphi_2\rangle & \text{(function type)}
\end{array}
$$

The completion procedure as a set of inference rules:

$$
\frac{}{\overline{e_i :: \kappa_i} \vdash \mathbf{bool} : \mathrm{b}\widehat{\mathrm{oo}}\mathrm{l} \ \&\ e\ \overline{e_i} \triangleright e :: \overline{\kappa_i \Rightarrow} \mathrm{E}}\ [\text{C-Bool}]
$$

$$
\frac{\overline{e_i :: \kappa_i} \vdash \tau : \widehat{\tau} \ \&\ \varphi \triangleright \overline{e_j :: \kappa_j}}{\overline{e_i :: \kappa_i} \vdash [\tau] : [\widehat{\tau}\langle\varphi\rangle] \ \&\ e\ \overline{e_i} \triangleright e :: \overline{\kappa_i \Rightarrow} \mathrm{E}, \overline{e_j :: \kappa_j}}\ [\text{C-List}]
$$

$$
\frac{\vdash \tau_1 : \widehat{\tau}_1 \ \&\ \varphi_1 \triangleright \overline{e_j :: \kappa_j} \quad \overline{e_i :: \kappa_i}, \overline{e_j :: \kappa_j} \vdash \tau_2 : \widehat{\tau}_2 \ \&\ \varphi_2 \triangleright \overline{e_j :: \kappa_j}}{\overline{e_i :: \kappa_i} \vdash \tau_1 \to \tau_2 : \forall \overline{e_j :: \kappa_j}.\,(\widehat{\tau}_1\langle\varphi_1\rangle \to \widehat{\tau}_2\langle\varphi_2\rangle) \ \&\ e\ \overline{e_i} \triangleright e :: \overline{\kappa_j \Rightarrow} \mathrm{E}, \overline{e_k :: \kappa_k}}\ [\text{C-Arr}]
$$

Figure 1: Type completion ($\Gamma \vdash \tau : \widehat{\tau} \ \&\ \varphi \triangleright \Gamma'$)

The completion procedure as an algorithm:

```
$complete :: \mathbf{Env} \times \mathbf{Ty} \to \mathbf{ExnTy} \times \mathbf{Exn} \times \mathbf{Env}$
$complete\ \overline{e_i :: \kappa_i}\ \mathbf{bool} =$
  $\mathbf{let}\ e\ be\ fresh$
  $\mathbf{in}\ \ \langle \mathrm{b}\widehat{\mathrm{oo}}\mathrm{l}; e\ \overline{e_i}; e :: \overline{\kappa_i \Rightarrow}\ \mathrm{E} \rangle$
```

3 Type system

3.1 Terms

$$
\begin{array}{llll}
t \in \mathbf{Tm} & ::= & x & \text{(term variable)} \\
& \mid & c_\tau & \text{(term constant)} \\
& \mid & \lambda x : \tau . t & \text{(term abstraction)} \\
& \mid & t_1\ t_2 & \text{(term application)} \\
& \mid & t_1 \oplus t_2 & \text{(operator)} \\
& \mid & \mathbf{if}\ t_1\ \mathbf{then}\ t_2\ \mathbf{else}\ t_3 & \text{(conditional)} \\
& \mid & \lightning^{\ell}_{\tau} & \text{(exception constant)} \\
& \mid & t_1\ \mathbf{seq}\ t_2 & \text{(forcing)} \\
& \mid & \mathbf{fix}\ t & \text{(anonymous fixpoint)} \\
& \mid & []_\tau & \text{(nil constructor)} \\
& \mid & t_1 :: t_2 & \text{(cons constructor)} \\
& \mid & \mathbf{case}\ t_1\ \mathbf{of}\ \{[] \mapsto t_2; x_1 :: x_2 \mapsto t_3\} & \text{(list eliminator)}
\end{array}
$$

3.2 Underlying type system

$$
\frac{}{\Gamma, x : \tau \vdash x : \tau}\ [\text{T-}\textsc{Var}]
\qquad
\frac{}{\Gamma \vdash c_\tau : \tau}\ [\text{T-}\textsc{Con}]
\qquad
\frac{}{\Gamma \vdash \lightning^{\ell}_{\tau} : \tau}\ [\text{T-}\textsc{Crash}]
$$

$$
\frac{\Gamma, x : \tau_1 \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . t : \tau_1 \to \tau_2}\ [\text{T-}\textsc{Abs}]
\qquad
\frac{\Gamma \vdash t_1 : \tau_2 \to \tau \quad \Gamma \vdash t_2 : \tau_2}{\Gamma \vdash t_1\ t_2 : \tau}\ [\text{T-}\textsc{App}]
$$

$$
\frac{\Gamma \vdash t : \tau \to \tau}{\Gamma \vdash \mathbf{fix}\ t : \tau}\ [\text{T-}\textsc{Fix}]
$$

$$
\frac{\Gamma \vdash t_1 : \mathbf{int} \quad \Gamma \vdash t_2 : \mathbf{int}}{\Gamma \vdash t_1 \oplus t_2 : \mathbf{bool}}\ [\text{T-}\textsc{Op}]
\qquad
\frac{\Gamma \vdash t_1 : \tau_1 \quad \Gamma \vdash t_2 : \tau_2}{\Gamma \vdash t_1\ \mathbf{seq}\ t_2 : \tau_2}\ [\text{T-}\textsc{Seq}]
$$

$$
\frac{\Gamma \vdash t_1 : \mathbf{bool} \quad \Gamma \vdash t_2 : \tau \quad \Gamma \vdash t_3 : \tau}{\Gamma \vdash \mathbf{if}\ t_1\ \mathbf{then}\ t_2\ \mathbf{else}\ t_3 : \tau}\ [\text{T-}\textsc{If}]
$$

$$
\frac{}{\Gamma \vdash []_\tau : [\tau]}\ [\text{T-}\textsc{Nil}]
\qquad
\frac{\Gamma \vdash t_1 : \tau \quad \Gamma \vdash t_2 : [\tau]}{\Gamma \vdash t_1 :: t_2 : [\tau]}\ [\text{T-}\textsc{Cons}]
$$

$$
\frac{\Gamma \vdash t_1 : [\tau_1] \quad \Gamma \vdash t_2 : \tau \quad \Gamma, x_1 : \tau_1, x_2 : [\tau_1] \vdash t_3 : \tau}{\Gamma \vdash \mathbf{case}\ t_1\ \mathbf{of}\ \{[] \mapsto t_2; x_1 :: x_2 \mapsto t_3\} : \tau}\ [\text{T-}\textsc{Case}]
$$

Figure 2: Underlying type system ($\Gamma \vdash t : \tau$)

3.3 Declarative exception type system

- In T-Abs and T-AnnAbs, should the term-level term-abstraction also have an explicit effect annotation?
- In T-AnnAbs, might need a side condition stating that e is not free in Δ.
- In T-App, note the double occurence of φ when typing t_1. Is subeffecting sufficient here? Also note that we do *not* expect an exception variable in the left-hand side annotation of the function space constructor.
- In T-AnnApp, note the substitution. We will need a substitution lemma for annotations.
- In T-Fix, the might be some universal quantifiers in our way. Do annotation applications in t take care of this, already? Perhaps we do need to change **fix** t into a binding construct to resolve this? Also, there is some implicit subeffecting going on between the annotations and effect.
- In T-Case, note the use of explicit subeffecting. Can this be done using implicit subeffecting?
- For T-Sub, should we introduce a term-level coercion, as in Dussart–Henglein–Mossin? We now do shape-conformant subtyping, is subeffecting sufficient?
- Do we need additional kinding judgements in some of the rules? Can we merge the kinding judgement with the subtyping and/or -effecting judgement? Kind-preserving substitutions.

3.4 Type elaboration system

- For T-Fix: how would a binding fixpoint construct work?

3.5 Type inference algorithm

- In R-App and R-Fix: check that the fresh variables generated by $\mathcal{I}$ are subsituted away by the substitution θ created by $\mathcal{M}$.
- In R-Fix we could get rid of the auxillary underlying type function if the fixpoint construct was replaced with a binding variant with an explicit type annotation.
- For R-Fix, make sure the way we handle fixpoints of exceptional value in a manner that is sound w.r.t. to the operational semantics we are going to give to this.
- Note that we do not construct the elaborated term, as it is not useful other than for metatheoretic purposes.
- Simplification does not exactly match the prototype (update the latter).
- Lemma: The algorithm maintains the invariant that exception types and exceptions are in normal form.

3.6 Subtyping

- Possibly useful lemma: $\widehat{\tau}_1 = \widehat{\tau}_2 \iff \widehat{\tau}_1 \leqslant \widehat{\tau}_2 \wedge \widehat{\tau}_2 \leqslant \widehat{\tau}_1$.

4 Interesting observations

- Exception types are not invariant under η-reduction.

$$\frac{}{\Gamma, x : \widehat{\tau}\ \&\ \varphi; \Delta \vdash x : \widehat{\tau}\ \&\ \varphi}\ [\text{T-}\textsc{Var}]$$

$$\frac{}{\Gamma; \Delta \vdash c_\tau : \bot_\tau\ \&\ \emptyset}\ [\text{T-}\textsc{Con}] \qquad \frac{}{\Gamma; \Delta \vdash \lightning^{\ell}_{\tau} : \bot_\tau\ \&\ \{\ell\}}\ [\text{T-}\textsc{Crash}]$$

$$\frac{\Gamma, x : \widehat{\tau}_1\ \&\ \varphi_1; \Delta \vdash t : \widehat{\tau}_2\ \&\ \varphi_2}{\Gamma; \Delta \vdash \lambda x : \widehat{\tau}_1\ \&\ \varphi_1 . t : \widehat{\tau}_1\langle\varphi_1\rangle \to \widehat{\tau}_2\langle\varphi_2\rangle\ \&\ \emptyset}\ [\text{T-}\textsc{Abs}]$$

$$\frac{\Gamma; \Delta, e : \kappa \vdash t : \widehat{\tau}\ \&\ \varphi}{\Gamma; \Delta \vdash \Lambda e : \kappa . t : \forall e : \kappa . \widehat{\tau}\ \&\ \varphi}\ [\text{T-}\textsc{AnnAbs}]$$

$$\frac{\Gamma; \Delta \vdash t_1 : \widehat{\tau}_2\langle\varphi_2\rangle \to \widehat{\tau}\langle\varphi\rangle\ \&\ \varphi \quad \Gamma; \Delta \vdash t_2 : \widehat{\tau}_2\ \&\ \varphi_2}{\Gamma; \Delta \vdash t_1\ t_2 : \widehat{\tau}\ \&\ \varphi}\ [\text{T-}\textsc{App}]$$

$$\frac{\Gamma; \Delta \vdash t_1 : \forall e : \kappa . \widehat{\tau}\ \&\ \varphi \quad \Delta \vdash \varphi_2 : \kappa}{\Gamma; \Delta \vdash t_1\ \langle\varphi_2\rangle : [\varphi_2 / e]\,\widehat{\tau}\ \&\ \varphi}\ [\text{T-}\textsc{AnnApp}]$$

$$\frac{\Gamma; \Delta \vdash t : \widehat{\tau}\langle\varphi\rangle \to \widehat{\tau}\langle\varphi\rangle\ \&\ \varphi}{\Gamma; \Delta \vdash \mathbf{fix}\ t : \widehat{\tau}\ \&\ \varphi}\ [\text{T-}\textsc{Fix}]$$

$$\frac{\Gamma; \Delta \vdash t_1 : \widehat{\mathbf{int}}\ \&\ \varphi \quad \Gamma; \Delta \vdash t_2 : \widehat{\mathbf{int}}\ \&\ \varphi}{\Gamma; \Delta \vdash t_1 \oplus t_2 : \widehat{\mathbf{bool}}\ \&\ \varphi}\ [\text{T-}\textsc{Op}]$$

$$\frac{\Gamma; \Delta \vdash t_1 : \widehat{\tau}_1\ \&\ \varphi \quad \Gamma; \Delta \vdash t_2 : \widehat{\tau}_2\ \&\ \varphi}{\Gamma; \Delta \vdash t_1\ \mathbf{seq}\ t_2 : \widehat{\tau}_2\ \&\ \varphi}\ [\text{T-}\textsc{Seq}]$$

$$\frac{\Gamma; \Delta \vdash t_1 : \widehat{\mathbf{bool}}\ \&\ \varphi \quad \Gamma; \Delta \vdash t_2 : \widehat{\tau}\ \&\ \varphi \quad \Gamma; \Delta \vdash t_3 : \widehat{\tau}\ \&\ \varphi}{\Gamma; \Delta \vdash \mathbf{if}\ t_1\ \mathbf{then}\ t_2\ \mathbf{else}\ t_3 : \widehat{\tau}\ \&\ \varphi}\ [\text{T-}\textsc{If}]$$

$$\frac{}{\Gamma; \Delta \vdash []_\tau : [\bot_\tau\langle\emptyset\rangle]\ \&\ \emptyset}\ [\text{T-}\textsc{Nil}]$$

$$\frac{\Gamma; \Delta \vdash t_1 : \widehat{\tau}\ \&\ \varphi_1 \quad \Gamma; \Delta \vdash t_2 : [\widehat{\tau}\langle\varphi_1\rangle]\ \&\ \varphi_2}{\Gamma; \Delta \vdash t_1 :: t_2 : [\widehat{\tau}\langle\varphi_1\rangle]\ \&\ \varphi_2}\ [\text{T-}\textsc{Cons}]$$

$$\frac{\begin{array}{c}\Gamma; \Delta \vdash t_1 : [\widehat{\tau}_1\langle\varphi_1\rangle]\ \&\ \varphi' \quad \Delta \vdash \varphi' \leqslant \varphi \quad \Gamma; \Delta \vdash t_2 : \widehat{\tau}\ \&\ \varphi \\ \Gamma, x_1 : \widehat{\tau}_1\ \&\ \varphi_1, x_2 : [\widehat{\tau}_1\langle\varphi_1\rangle]\ \&\ \varphi'; \Delta \vdash t_3 : \widehat{\tau}\ \&\ \varphi\end{array}}{\Gamma; \Delta \vdash \mathbf{case}\ t_1\ \mathbf{of}\ \{[] \mapsto t_2; x_1 :: x_2 \mapsto t_3\} : \widehat{\tau}\ \&\ \varphi}\ [\text{T-}\textsc{Case}]$$

$$\frac{\Gamma; \Delta \vdash t : \widehat{\tau}\ \&\ \varphi \quad \Delta \vdash \widehat{\tau} \leqslant \widehat{\tau}' \quad \Delta \vdash \varphi \leqslant \varphi'}{\Gamma; \Delta \vdash t : \widehat{\tau}'\ \&\ \varphi'}\ [\text{T-}\textsc{Sub}]$$

Figure 3: Declarative type system ($\Gamma; \Delta \vdash t : \widehat{\tau}\ \&\ \varphi$)

$$\frac{}{\Gamma, x : \widehat{\tau}\ \&\ \varphi; \Delta \vdash x \rightsquigarrow x : \widehat{\tau}\ \&\ \varphi}\ \text{[T-Var]}$$

$$\frac{}{\Gamma; \Delta \vdash c_\tau \rightsquigarrow c_\tau : \tau\ \&\ \emptyset}\ \text{[T-Con]} \qquad \frac{}{\Gamma; \Delta \vdash \lightning^{\ell}_{\tau} \rightsquigarrow \lightning^{\ell}_{\tau} : \bot_\tau\ \&\ \{\ell\}}\ \text{[T-Crash]}$$

$$\frac{\begin{array}{c} \Delta, \overline{e_i : \kappa_i} \vdash \widehat{\tau}_1 \triangleright \tau_1 \quad \Delta, \overline{e_i : \kappa_i} \vdash \varphi_1 : \textsc{e} \\ \Gamma, x : \widehat{\tau}_1\ \&\ \varphi_1; \Delta, \overline{e_i : \kappa_i} \vdash t \rightsquigarrow t' : \widehat{\tau}_2\ \&\ \varphi_2 \end{array}}{\Gamma; \Delta \vdash \lambda x : \tau_1.t \rightsquigarrow \Lambda\overline{e_i : \kappa_i}.\lambda x : \widehat{\tau}_1\ \&\ \varphi_1.t' : \forall\overline{e_i : \kappa_i}.\widehat{\tau}_1\langle\varphi_1\rangle \rightarrow \widehat{\tau}_2\langle\varphi_2\rangle\ \&\ \emptyset}\ \text{[T-Abs]}$$

$$\frac{\begin{array}{c} \Delta \vdash \widehat{\tau}_2 \leqslant [\overline{\varphi_i}/\overline{e_i}]\,\widehat{\tau} \quad \Delta \vdash \varphi_2 \leqslant [\overline{\varphi_i}/\overline{e_i}]\,\varphi \quad \overline{\Delta \vdash \varphi_i : \kappa_i} \\ \Gamma; \Delta \vdash t_1 \rightsquigarrow t_1' : \forall\overline{e_i : \kappa_i}.\widehat{\tau}_1\langle\varphi_1\rangle \rightarrow \widehat{\tau}\langle\varphi\rangle\ \&\ \varphi' \quad \Gamma; \Delta \vdash t_2 \rightsquigarrow t_2' : \widehat{\tau}_2\ \&\ \varphi_2 \end{array}}{\Gamma; \Delta \vdash t_1\ t_2 \rightsquigarrow t_1'\ \langle\overline{\varphi_i}\rangle\ t_2' : [\overline{\varphi_i}/\overline{e_i}]\,\widehat{\tau}\ \&\ [\overline{\varphi_i}/\overline{e_i}]\,\varphi \cup \varphi'}\ \text{[T-App]}$$

$$\frac{\begin{array}{c} \Gamma; \Delta \vdash t \rightsquigarrow t' : \forall\overline{e_i : \kappa_i}.\widehat{\tau}\langle\varphi\rangle \rightarrow \widehat{\tau}'\langle\varphi'\rangle\ \&\ \varphi'' \\ \Delta \vdash [\overline{\varphi_i}/\overline{e_i}]\,\widehat{\tau}' \leqslant [\overline{\varphi_i}/\overline{e_i}]\,\widehat{\tau} \quad \Delta \vdash [\overline{\varphi_i}/\overline{e_i}]\,\varphi' \leqslant [\overline{\varphi_i}/\overline{e_i}]\,\varphi \quad \overline{\Delta \vdash \varphi_i : \kappa_i} \end{array}}{\Gamma; \Delta \vdash \mathbf{fix}\ t \rightsquigarrow \mathbf{fix}\ t'\ \langle\overline{\varphi_i}\rangle : [\overline{\varphi_i}/\overline{e_i}]\,\widehat{\tau}\ \&\ [\overline{\varphi_i}/\overline{e_i}]\,\varphi \cup \varphi''}\ \text{[T-Fix]}$$

$$\frac{\Gamma; \Delta \vdash t_1 \rightsquigarrow t_1' : \widehat{\mathbf{int}}\ \&\ \varphi_1 \quad \Gamma; \Delta \vdash t_2 \rightsquigarrow t_2' : \widehat{\mathbf{int}}\ \&\ \varphi_2}{\Gamma; \Delta \vdash t_1 \oplus t_2 \rightsquigarrow t_1' \oplus t_2' : \widehat{\mathbf{bool}}\ \&\ \varphi_1 \cup \varphi_2}\ \text{[T-Op]}$$

$$\frac{\Gamma; \Delta \vdash t_1 \rightsquigarrow t_1' : \widehat{\tau}_1\ \&\ \varphi_1 \quad \Gamma; \Delta \vdash t_2 \rightsquigarrow t_2' : \widehat{\tau}_2\ \&\ \varphi_2}{\Gamma; \Delta \vdash t_1\ \mathbf{seq}\ t_2 \rightsquigarrow t_1'\ \mathbf{seq}\ t_2' : \widehat{\tau}_2\ \&\ \varphi_1 \cup \varphi_2}\ \text{[T-Seq]}$$

$$\frac{\Gamma; \Delta \vdash t_1 \rightsquigarrow t_1' : \widehat{\mathbf{bool}}\ \&\ \varphi_1 \quad \Gamma; \Delta \vdash t_2 \rightsquigarrow t_2' : \widehat{\tau}_2\ \&\ \varphi_2 \quad \Gamma; \Delta \vdash t_3 \rightsquigarrow t_3' : \widehat{\tau}_3\ \&\ \varphi_3}{\Gamma; \Delta \vdash \mathbf{if}\ t_1\ \mathbf{then}\ t_2\ \mathbf{else}\ t_3 \rightsquigarrow \mathbf{if}\ t_1'\ \mathbf{then}\ t_2'\ \mathbf{else}\ t_3' : \widehat{\tau}_2 \sqcup \widehat{\tau}_3\ \&\ \varphi_1 \cup \varphi_2 \cup \varphi_3}\ \text{[T-If]}$$

$$\frac{}{\Gamma; \Delta \vdash []_\tau \rightsquigarrow []_\tau : \bot_\tau\ \&\ \emptyset}\ \text{[T-Nil]}$$

$$\frac{\Gamma; \Delta \vdash t_1 \rightsquigarrow t_1' : \widehat{\tau}_1\ \&\ \varphi_1 \quad \Gamma; \Delta \vdash t_2 \rightsquigarrow t_2' : \left[\widehat{\tau}_1'\langle\varphi_1'\rangle\right]\ \&\ \varphi_2}{\Gamma; \Delta \vdash t_1 :: t_2 \rightsquigarrow t_1' :: t_2' : \left[\widehat{\tau}_1 \sqcup \widehat{\tau}_1'\langle\varphi_1 \cup \varphi_1'\rangle\right]\ \&\ \varphi_2}\ \text{[T-Cons]}$$

$$\frac{\begin{array}{c} \Gamma; \Delta \vdash t_1 \rightsquigarrow t_1' : [\tau_1\langle\varphi_1\rangle]\ \&\ \varphi_1' \quad \Gamma; \Delta \vdash t_2 \rightsquigarrow t_2' : \widehat{\tau}_2\ \&\ \varphi_2 \\ \Gamma, x_1 : \widehat{\tau}_1\ \&\ \varphi_1, x_2 : [\tau_1\langle\varphi_1\rangle]\ \&\ \varphi_1'; \Delta \vdash t_3 \rightsquigarrow t_3' : \widehat{\tau}_3\ \&\ \varphi_3 \end{array}}{\begin{array}{l} \Gamma; \Delta \vdash \mathbf{case}\ t_1\ \mathbf{of}\ \{[] \mapsto t_2; x_1 :: x_2 \mapsto t_3\} \rightsquigarrow \\ \qquad\qquad \mathbf{case}\ t_1'\ \mathbf{of}\ \{[] \mapsto t_2'; x_1 :: x_2 \mapsto t_3'\} : \widehat{\tau}_2 \sqcup \widehat{\tau}_3\ \&\ \varphi_1' \cup \varphi_2 \cup \varphi_3 \end{array}}\ \text{[T-Case]}$$

Figure 4: Syntax-directed type elaboration system ($\Gamma; \Delta \vdash t \rightsquigarrow t' : \widehat{\tau}\ \&\ \varphi$)

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathcal{R} : \mathbf{TyEnv} \times \mathbf{KiEnv} \times \mathbf{Tm} \to \mathbf{ExnTy} \times \mathbf{Exn}} \\
\mathcal{R}\ \Gamma\ \Delta\ x & = & \Gamma_x \\
\mathcal{R}\ \Gamma\ \Delta\ c_\tau & = & \langle \bot_\tau ; \emptyset \rangle \\
\mathcal{R}\ \Gamma\ \Delta\ \lightning^{\ell}_{\tau} & = & \langle \bot_\tau ; \{\ell\} \rangle \\
\mathcal{R}\ \Gamma\ \Delta\ (\lambda x : \tau . t) & = & \mathbf{let}\ \langle \widehat{\tau}_1 ; e_1 ; \overline{e_i : \kappa_i} \rangle = \mathcal{C}\ \emptyset\ \tau \\
 & & \qquad \langle \widehat{\tau}_2 ; \varphi_2 \rangle = \mathcal{R}\ (\Gamma, x : \widehat{\tau}_1\ \&\ e_1)\ (\Delta, \overline{e_i : \kappa_i})\ t \\
 & & \mathbf{in}\ \langle \forall \overline{e_i : \kappa_i} . \widehat{\tau}_1 \langle e_1 \rangle \to \widehat{\tau}_2 \langle \varphi_2 \rangle ; \emptyset \rangle \\
\mathcal{R}\ \Gamma\ \Delta\ (t_1\ t_2) & = & \mathbf{let}\ \langle \widehat{\tau}_1 ; \varphi_1 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_1 \\
 & & \qquad \langle \widehat{\tau}_2 ; \varphi_2 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_2 \\
 & & \qquad \langle \widehat{\tau}_2' \langle e_2' \rangle \to \widehat{\tau}' \langle \varphi' \rangle ; \overline{e_i : \kappa_i} \rangle = \mathcal{I}\ \widehat{\tau}_1 \\
 & & \qquad \theta = [e_2' \mapsto \varphi_2] \circ \mathcal{M}\ \emptyset\ \widehat{\tau}_2\ \widehat{\tau}_2' \\
 & & \mathbf{in}\ \langle \llbracket \theta \widehat{\tau}' \rrbracket_\Delta ; \llbracket \theta \varphi' \cup \varphi_1 \rrbracket_\Delta \rangle \\
\mathcal{R}\ \Gamma\ \Delta\ (\mathbf{fix}\ t) & = & \mathbf{let}\ \langle \widehat{\tau} ; \varphi \rangle = \mathcal{R}\ \Gamma\ \Delta\ t \\
 & & \qquad \langle \widehat{\tau}' \langle e' \rangle \to \widehat{\tau}'' \langle \varphi'' \rangle ; \overline{e_i : \kappa_i} \rangle = \mathcal{I}\ \widehat{\tau} \\
 & & \mathbf{in}\ \langle \widehat{\tau}_0 ; \varphi_0 ; i \rangle \leftarrow \langle \bot_{\lfloor \widehat{\tau}' \rfloor} ; \emptyset ; 0 \rangle \\
 & & \qquad \mathbf{do}\ \theta \leftarrow [e' \mapsto \varphi_i] \circ \mathcal{M}\ \emptyset\ \widehat{\tau}_i\ \widehat{\tau}' \\
 & & \qquad\qquad \langle \widehat{\tau}_{i+1} ; \varphi_{i+1} ; i \rangle \leftarrow \langle \llbracket \theta \widehat{\tau}'' \rrbracket_\Delta ; \llbracket \theta \varphi'' \rrbracket_\Delta ; i + 1 \rangle \\
 & & \qquad \mathbf{until}\ \langle \widehat{\tau}_i ; \varphi_i \rangle \equiv \langle \widehat{\tau}_{i-1} ; \varphi_{i-1} \rangle \\
 & & \qquad \mathbf{return}\ \langle \widehat{\tau}_i ; \llbracket \varphi \cup \varphi_i \rrbracket_\Delta \rangle \\
\mathcal{R}\ \Gamma\ \Delta\ (t_1 \oplus t_2) & = & \mathbf{let}\ \langle \widehat{\mathbf{int}} ; \varphi_1 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_1 \\
 & & \qquad \langle \widehat{\mathbf{int}} ; \varphi_2 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_2 \\
 & & \mathbf{in}\ \langle \widehat{\mathbf{bool}} ; \llbracket \varphi_1 \cup \varphi_2 \rrbracket_\Delta \rangle \\
\multicolumn{3}{l}{\mathcal{R}\ \Gamma\ \Delta\ (t_1\ \mathbf{seq}\ t_2)} \\
 & = & \mathbf{let}\ \langle \widehat{\tau}_1 ; \varphi_1 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_1 \\
 & & \qquad \langle \widehat{\tau}_2 ; \varphi_2 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_2 \\
 & & \mathbf{in}\ \langle \widehat{\tau}_2 ; \llbracket \varphi_1 \cup \varphi_2 \rrbracket_\Delta \rangle \\
\multicolumn{3}{l}{\mathcal{R}\ \Gamma\ \Delta\ (\mathbf{if}\ t_1\ \mathbf{then}\ t_2\ \mathbf{else}\ t_3)} \\
 & = & \mathbf{let}\ \langle \widehat{\mathbf{bool}} ; \varphi_1 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_1 \\
 & & \qquad \langle \widehat{\tau}_2 ; \varphi_2 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_2 \\
 & & \qquad \langle \widehat{\tau}_3 ; \varphi_3 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_3 \\
 & & \mathbf{in}\ \langle \llbracket \widehat{\tau}_2 \sqcup \widehat{\tau}_3 \rrbracket_\Delta ; \llbracket \varphi_1 \cup \varphi_2 \cup \varphi_3 \rrbracket_\Delta \rangle \\
\mathcal{R}\ \Gamma\ \Delta\ []_\tau & = & \langle [\bot_\tau \langle \emptyset \rangle] ; \emptyset \rangle \\
\mathcal{R}\ \Gamma\ \Delta\ (t_1 :: t_2) & = & \mathbf{let}\ \langle \widehat{\tau}_1 ; \varphi_1 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_1 \\
 & & \qquad \langle [\widehat{\tau}_2 \langle \varphi_2' \rangle] ; \varphi_2 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_2 \\
 & & \mathbf{in}\ \langle \llbracket [(\widehat{\tau}_1 \sqcup \widehat{\tau}_2) \langle (\varphi_1 \cup \varphi_2') \rangle] \rrbracket_\Delta ; \varphi_2 \rangle \\
\multicolumn{3}{l}{\mathcal{R}\ \Gamma\ \Delta\ (\mathbf{case}\ t_1\ \mathbf{of}\ \{[] \mapsto t_2 ; x_1 :: x_2 \mapsto t_3\})} \\
 & = & \mathbf{let}\ \langle [\widehat{\tau}_1 \langle \varphi_1' \rangle] ; \varphi_1 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_1 \\
 & & \qquad \langle \widehat{\tau}_2 ; \varphi_2 \rangle = \mathcal{R}\ (\Gamma, x_1 : \widehat{\tau}_1\ \&\ \varphi_1', x_2 : [\widehat{\tau}_1 \langle \varphi_1' \rangle]\ \&\ \varphi_1)\ \Delta\ t_2 \\
 & & \qquad \langle \widehat{\tau}_3 ; \varphi_3 \rangle = \mathcal{R}\ \Gamma\ \Delta\ t_3 \\
 & & \mathbf{in}\ \langle \llbracket \widehat{\tau}_2 \sqcup \widehat{\tau}_3 \rrbracket_\Delta ; \llbracket \varphi_1 \cup \varphi_2 \cup \varphi_3 \rrbracket_\Delta \rangle
\end{array}
$$

Figure 5: Type inference algorithm

$$\frac{}{\Delta \vdash \widehat{\mathbf{bool}} \leqslant \widehat{\mathbf{bool}}}\ [\text{S-}\textsc{Bool}] \qquad \frac{}{\Delta \vdash \widehat{\mathbf{int}} \leqslant \widehat{\mathbf{int}}}\ [\text{S-}\textsc{Int}]$$

$$\frac{\Delta \vdash \widehat{\tau}'_1 \leqslant \widehat{\tau}_1 \quad \Delta \vdash \varphi'_1 \leqslant \varphi_1 \quad \Delta \vdash \widehat{\tau}_2 \leqslant \widehat{\tau}'_2 \quad \Delta \vdash \varphi_2 \leqslant \varphi'_2}{\Delta \vdash \widehat{\tau}_1\langle\varphi_1\rangle \to \widehat{\tau}_2\langle\varphi_2\rangle \leqslant \widehat{\tau}'_1\langle\varphi'_1\rangle \to \widehat{\tau}'_2\langle\varphi'_2\rangle}\ [\text{S-}\textsc{Arr}]$$

$$\frac{\Delta \vdash \widehat{\tau} \leqslant \widehat{\tau}' \quad \Delta \vdash \varphi \leqslant \varphi'}{\Delta \vdash [\widehat{\tau}\langle\varphi\rangle] \leqslant [\widehat{\tau}'\langle\varphi'\rangle]}\ [\text{S-}\textsc{List}] \qquad \frac{\Delta, e : \kappa \vdash \widehat{\tau}_1 \leqslant \widehat{\tau}_2}{\Delta \vdash \forall e : \kappa.\widehat{\tau}_1 \leqslant \forall e : \kappa.\widehat{\tau}_2}\ [\text{S-}\textsc{Forall}]$$

Figure 6: Subtyping