Patchworking Curves

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March 27, 2024

Goal

Definition

A degree-m real algebraic hypersurface in $\mathbb{R}^n$ is a subset given by an equation

$$f_m(x_1, x_2, .., x_n) = 0$$

where f_m is a real polynomial of degree m.

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Let's fix $n = 2$. If

- $m = 1$?

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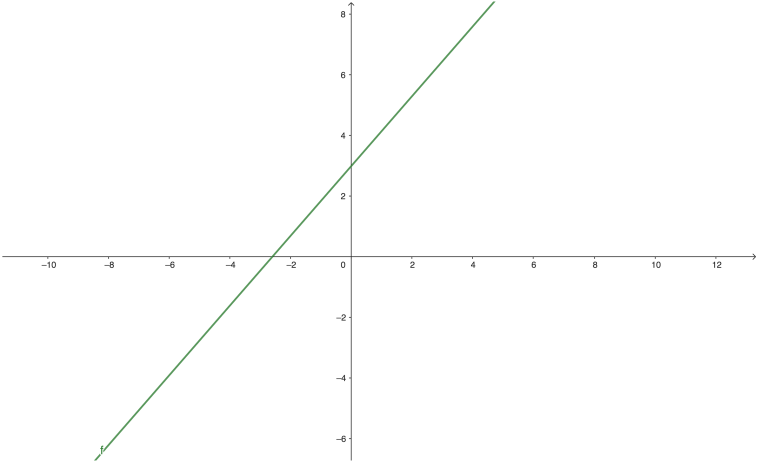

Figure: Graph of $y = 1.15x + 3$

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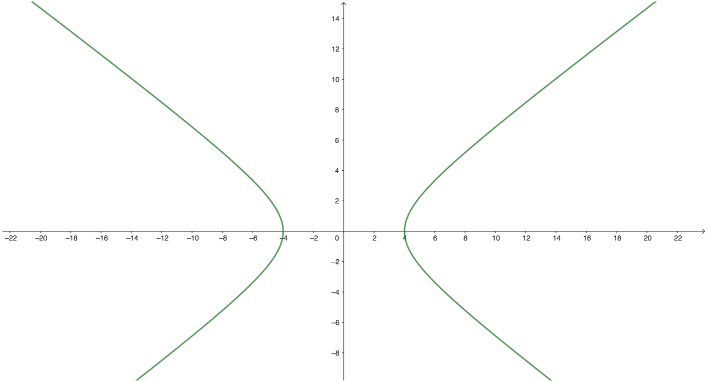

Figure: Graph of $\frac{x^2}{16} - \frac{y^2}{9} = 1$

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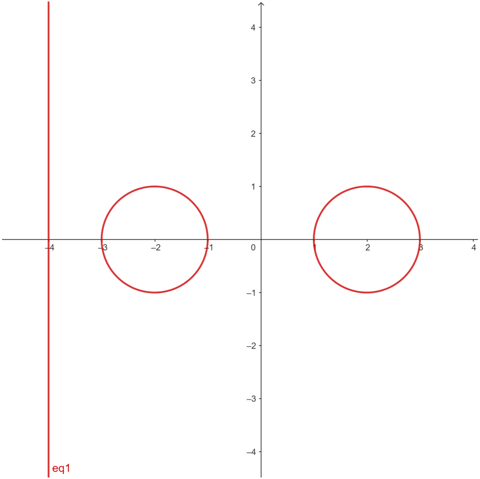

Figure: Graph of $(x+4)((x-2)^2+y^2-1)((x+2)^2+y^2-1)=0$

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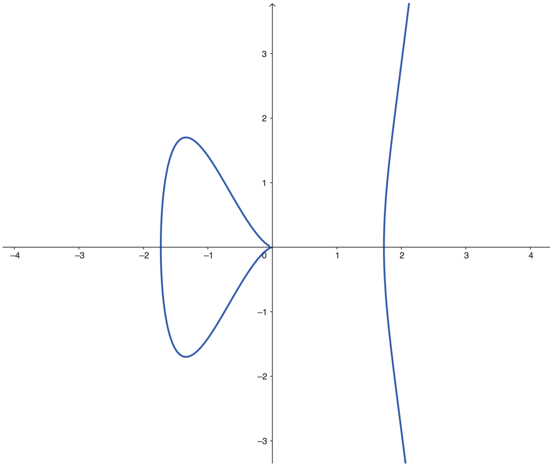

Figure: Graph of $y^2 = x^5 - 3x^3$

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- arbitrary m? not always easy

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- $m = 1$?

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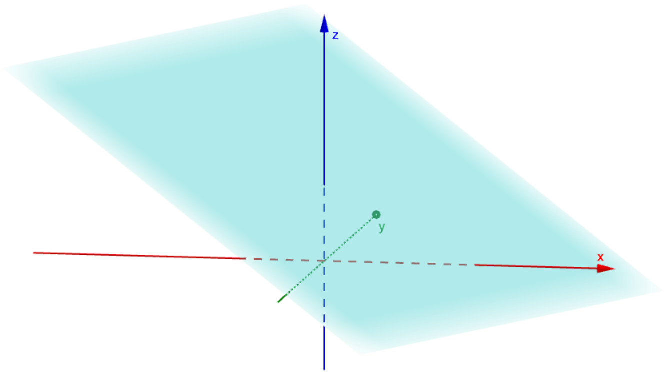

Figure: Graph of $2x - y + 4z = 10$

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$C_{n,m}$:= the maximum number of components achievable by a degree-m curve in $\mathbb{R}^n$

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E.g. $C(2,1) = 1$
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Definition

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$C(2,2) = 2$

Want to answer our initial question for $\mathbb{R}^2$ and $\mathbb{R}^3$, i.e. we want to find good $LB_{1,2}$ and $UB_{1,2}$ such that

$$LB_1(m) \leq C_{2,m} \leq UB_1(m)$$

$$LB_2(m) \leq C_{3,m} \leq UB_2(m).$$

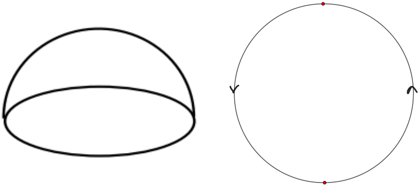
Overview

- Upper bounds
  - Harnack's inequality
  - Milnor-Thom's theorem
- Patchworking
  - Statement of the theorem
  - Example
  - Applications
  - Lower bound on $C_{2,m}$
- 3D Patchworking
  - Statement
  - Lower bound on $C_{3,m}$
- Further work

Harnack's Inequality

Brief aside: The projective plane

Let $\sim$ be the relation on the unit sphere S^2 given by $x \sim -x$.
We can define $\mathbb{RP}^2 = S^2/\sim$.



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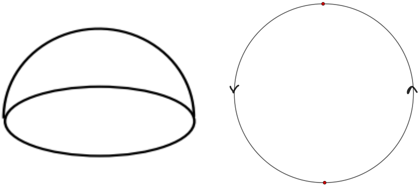


Theorem (Harnack, 1876)

Let C be a degree-m non-singular real plane algebraic curve in $\mathbb{RP}^2$. Then

$$\# \text{ of connected components of } C \leq \frac{(m-1)(m-2)}{2} + 1.$$

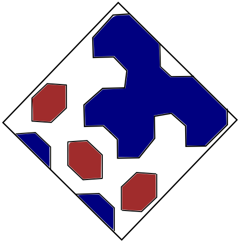
Harnack's inequality

Corollary

Let $m \in \mathbb{N}$. Then

$$C_{2,m} \leq \frac{(m-1)(m-2)}{2} + m.$$

Idea: Think of L_∞ as splitting up the components it intersects.



C_i, the i-th component of C

$$|C \cap \mathbb{R}^2| = |C \setminus (C \cap L_\infty)| \quad = |C \cap L_\infty| + |\{C_i \mid C_i \cap L_\infty = \varnothing, i \in \{1, .., n\}\}|$$

The Milnor-Thom Theorem

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Let $V \subset \mathbb{R}^n$ be a real algebraic hypersurface of degree $m > 0$ i.e.

$$V = \left\{ \sum_{i_1,..,i_n \geq 0} c_{i_1,..,i_n} x_1^{i_1} \dots x_n^{i_n} = 0 \right\} \subseteq \mathbb{R}^n.$$

Then $\#$ of connected components of V is at most $m(2m-1)^{n-1}$.

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Then $\#$ of connected components of V is at most $m(2m-1)^{n-1}$.

i.e. $C_{n,m} \leq m(2m-1)^{n-1}$.

Combinatorial Patchworking

Introduced by Itenberg & Viro in [1].

Initial Data

- $m \in \mathbb{N}$
- a triangle Δ_m in $\mathbb{R}^2$ with vertices $(0,0), (0,m)$ and $(m,0)$
- a *convex* triangulation T of the Δ_m
- a sign distribution ϵ

Procedure

- construct a square $\diamondsuit_m$ by reflecting Δ_m in all four quadrants
- extend T and ϵ to $\diamondsuit_m$
- for each small triangle given by T, draw a midline that separates vertices of opposite signs

We will refer to the union of these midlines as the patchworked curve, denoted L.

Combinatorial Patchworking

Theorem (Patchwork Theorem)

There exists a non-singular real algebraic plane affine curve of degree m and a homeomorphism of the plane $\mathbb{R}^2$ onto the interior of the square $\diamondsuit_m$ which maps the set of real points of this curve onto the patchworked cu

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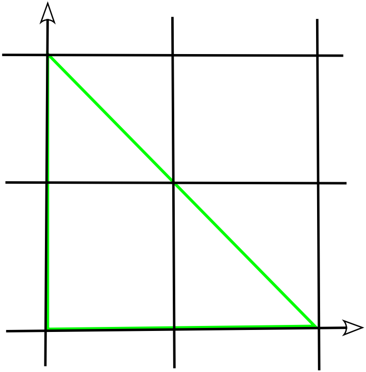
Observation: We can formulate a similar claim about curves in $\mathbb{RP}^2$.

Informal statement

For a choice (m, T, ϵ) as above, there exists a degree m-curve of the same shape as the patchworked curve.

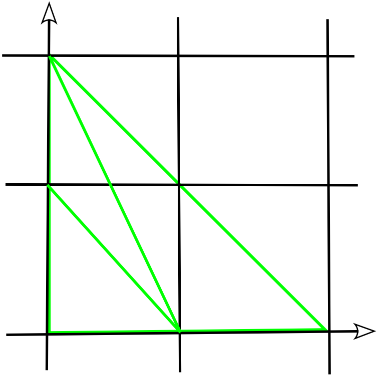
A Simple Example

Fix $m = 2$. We consider the triangle Δ_2 with vertices (0, 0), (2, 0), (0, 2).



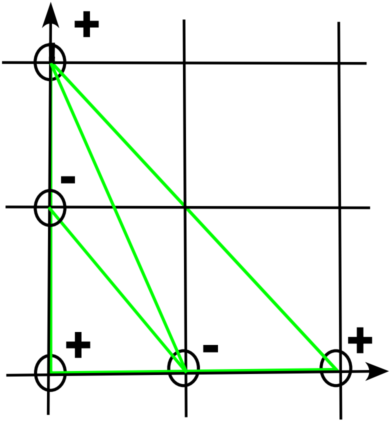
A Simple Example

We now choose a convex triangulation T of Δ_2 such that its vertices have integer coordinates:



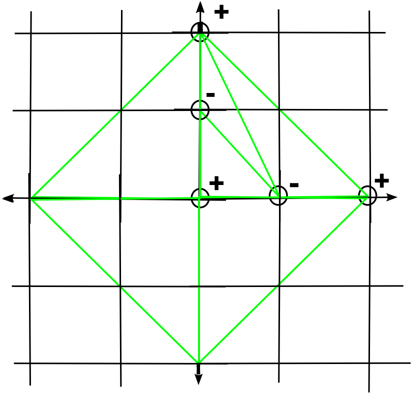
A Simple Example

We now choose a sign distribution ϵ for Vert(T). The shown sign distribution is known as the Harnack sign distribution.



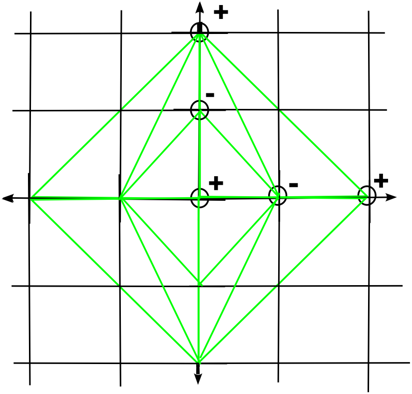
A Simple Example

We construct the square $\diamondsuit_2$.



A Simple Example

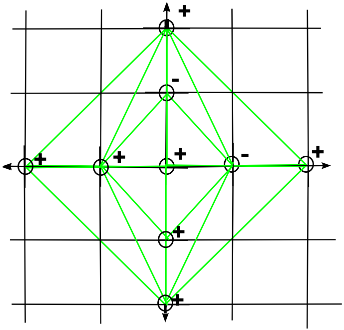
We now extend the triangulation T of Δ_2 to a symmetric triangulation of $\diamondsuit_2$.



A Simple Example

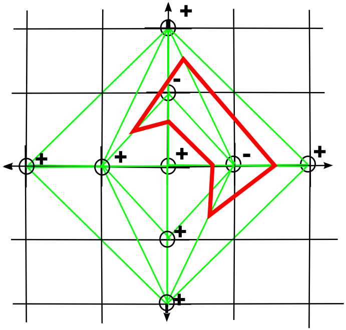
We now extend the sign distribution ϵ to Vert(extended T) by the rule

$$\epsilon((-1)^a i, (-1)^b j) = (-1)^{a+b}\epsilon(i,j).$$



A Simple Example

We draw a midline in the triangles which have vertices endowed with opposite signs. The union of these midlines is our patchworked curve, L.



Patchworking: Intuition

Claim

We can actually give the explicit form of the polynomials which describe the curve mentioned in the theorem above.
We can construct a family of polynomials that tends towards our desired shape. This construction can be found in [1].

Patchworking: applications

We can construct curves via patchworking in order to:

1. show that the upper bound given by Harnack's Inequality is tight (such curves with maximum number of components are called M-curves)

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We can construct curves via patchworking in order to:

1. show that the upper bound given by Harnack's Inequality is tight (such curves with maximum number of components are called M-curves)
2. construct a counterexample to the Ragsdale conjecture (Itenberg & Viro)

Patchworking: finding a LB on $C_{2,m}$

Observation: Any degree-m curve will produce a LB on $C_{2,m}$.

Idea: construct a curve with as many components as possible via patchworking.

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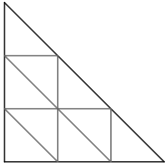

convex? ✓

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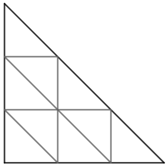


  convex? ✓

- sign distribution ϵ

$$\epsilon(i,j) = \begin{cases} +1, & \text{if } i \text{ is odd and } j \text{ is even} \\ -1, & \text{otherwise.} \end{cases}$$

Finding a LB on $C_{2,m}$

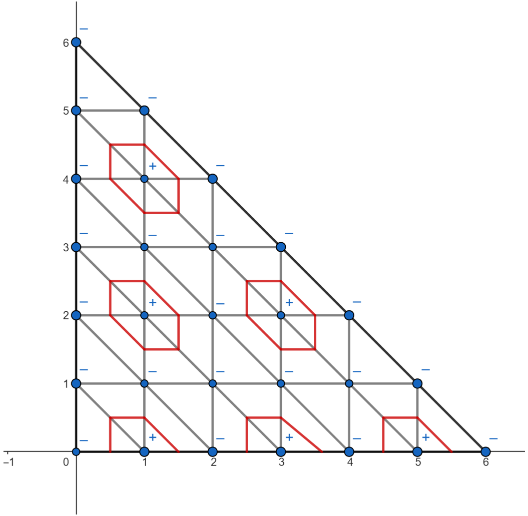
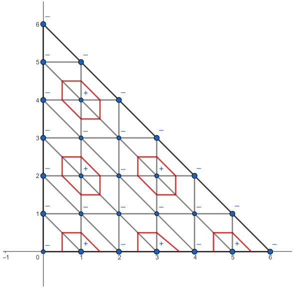


Figure: Our patchworked curve for $m = 6$ ($\mathbb{R}^2_+$ quadrant)

Finding a LB on $C_{2,m}$



The components
are arranged in a triangle with vertices $(1,2)$, $(1,p)$, $(q,2)$, and of side length $\frac{m-2}{2}$
$\Rightarrow \frac{\lfloor\frac{m-2}{2}\rfloor(\lfloor\frac{m-2}{2}\rfloor+1)}{2}$ components,
where
$p :=$ largest even number smaller than m,
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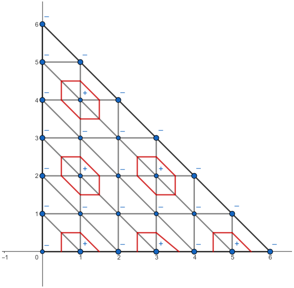

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Putting it all together:

$$\frac{\lfloor \frac{m-2}{2} \rfloor (\lfloor \frac{m-2}{2} \rfloor + 1)}{2} \leq \quad C_{2,m} \overset{Harnack(Cor)}{\leq} \frac{(m-1)(m-2)}{2} + m.$$

Theorem 1

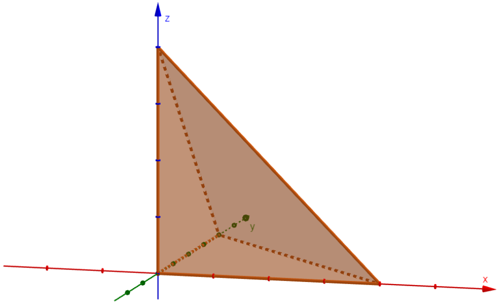
$C_{2,m}$ is a quadratic polynomial in m, for any $m \in \mathbb{N}$.

3D Patchworking

Now, we want to do a similar procedure for patchworking surfaces.

First, we must update our notation. Fix $m \in \mathbb{N}$.

- Δ_m tetrahedron (why?) with vertices $(0,0)$, $(0,m)$, $(m,0)$, (m,m)



3D Patchworking

- T a convex triangulation of Δ_m
- $\epsilon : \mathrm{Vert}(T) \to \{\pm 1\}$ a sign distribution
- s_x, s_y, s_z reflection maps
  Define $s_{xy} = s_x \circ s_y$, $s_{yz} = s_y \circ s_z$, $s_{xz} = s_z \circ s_z$
  $s_{xyz} = s_x \circ s_y \circ s_z$

Now,

- construct octahedron $\diamondsuit_m = \Delta_m \cup s_x(\Delta_m) \cup s_y(\Delta_m) \cup s_z(\Delta_m) \cup s_{xy}(\Delta_m) \cup s_{yz}(\Delta_m) \cup s_{xz}(\Delta_m) \cup s_{xyz}(\Delta_m)$
- Extend T to a symmetric triangulation of $\diamondsuit_m$
- Extend ϵ to Vert(extended T) by symmetry

3D Patchworking

For every small tetrahedron given by the triangulation T of $\diamondsuit_m$, we consider the plane determined by the midpoints of edges whose vertices have opposite signs. Note that this plane separates the '$+$'s from the '-'s.

Our patchworked surface will be the union of these planes.

Informal statement: There exists a real algebraic surface that has the same shape as the patchworked surface.

3D Patchworking

Recall: we want to find a lower bound $LB(m)$ on $C_{3,m}$.
Observation: any surface will give a lower bound. We want to construct a surface with as many components as possible.

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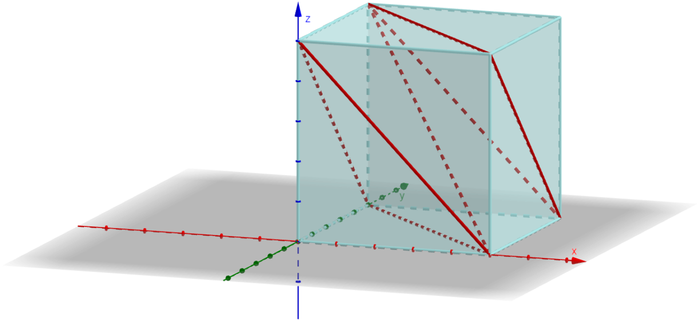
- fix some $m \in \mathbb{N}$
- choose a convex triangulation T - not easy!

3D Patchworking: our triangulation

1. Choose this triangulation for the unit cube:



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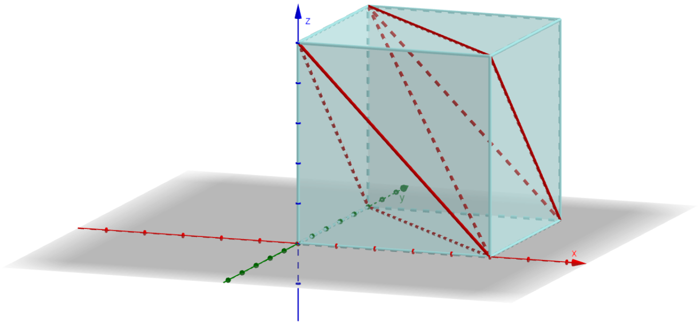

2. Extend it to all of $\mathbb{R}^3$ by requiring it to be invariant under reflection in any plane $\{x = k\}$, $\{y = k\}$, $\{z = k\}$ $(k \in \mathbb{Z})$. Denote this triangulation T.

3. Some tetrahedra are cut by the boundary $x + y + z = m$ of T. Keep only the tetrahedra entirely contained in Δ_m.

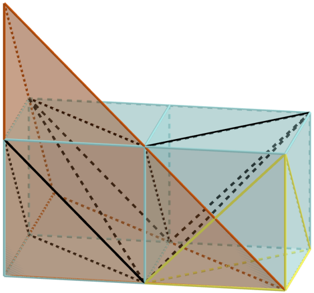

Figure: Δ_2

3D Patchworking: our triangulation

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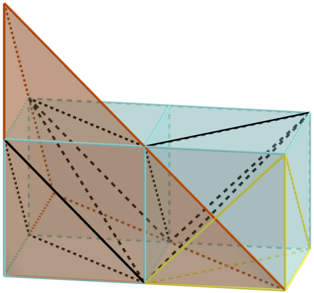

Figure: Δ_2

4. Claim:
   (a) this partial triangulation of Δ_m is convex
   (b) the partial triangulation can be extended to a full one

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Let's follow the patchworking the procedure:

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- choose a convex triangulation T ✓

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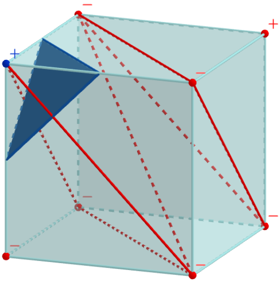
Let's follow the patchworking the procedure:

- fix some $m \in \mathbb{N}$
- choose a convex triangulation T ✓
- choose a sign distribution ϵ

3D Patchworking: sign distribution

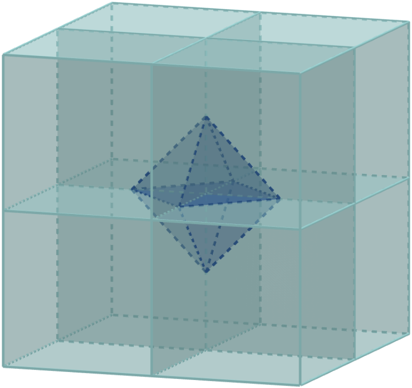
Definition

We say a vertex $v = (a, b, c)$ is a cut-vertex if for all vertices $w = (a', b', c')$ adjacent to v (that is, vertices that are connected to v by an edge of the triangulation T) we have that $\epsilon(v)\epsilon(w) = -1$.



3D Patchworking: sign distribution

Lemma: The patchworked surface has a small spherical component for each* cut vertex.



3D Patchworking: sign distribution

Same strategy as for triangulation: choose ϵ for unit cube in the $+++$ octant, and then extend it by symmetry.

Choose

$$\epsilon(v) = \begin{cases} -1, & \text{if } v = (0,0,0) \text{ or } v = (1,1,1) \\ +1, & \text{otherwise.} \end{cases}$$

(Why? Found using exhaustive search - Python script)

3D Patchworking: counting components

Brief aside: triangular and tetrahedral numbers

$T_n = \frac{n(n+1)}{2}$

$Te_n = \sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{6}$

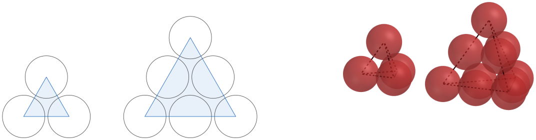


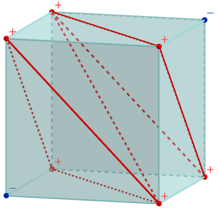
Figure: T_2, T_3, Te_2, Te_3

3D Patchworking: counting components

Strategy: For each octant (1) determine the sign distribution of the unit cube, then (2) count the number of cut vertices.

We will explain the process for the $+++$ octant.

Unit cube in octant $+++$:



Observe that a vertex (i, j, k) is a cut vertex if either

$$(1) \begin{cases} i, j, k \text{ are all even} \\ i, j, k > 0 \\ i + j + k \leq m - 1 \end{cases} \quad \text{or} \quad (2) \begin{cases} i, j, k \text{ are all odd} \\ i, j, k > 0 \\ i + j + k \leq m - 1 \end{cases}$$

3D Patchworking: counting components

Let $A \subseteq \mathbb{Z}^3$ be the set of points that satisfy case (1) and let N be the largest integer such that

$$2N + 2 + 2 \leq m - 1$$

i.e. $N = \left\lfloor \frac{m-5}{2} \right\rfloor$. Then, the points in A form a tetrahedron with vertices $(2, 2, 2)$, $(2, 2, 2N)$, $(2, 2N, 2)$, and $(2, 2, 2N)$. Each side of the tetrahedron contains N points in A. Thus the total number of points is given by the N^{th} tetrahedral number, Te_N.

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Similarly, let $B \subseteq \mathbb{Z}^3$ be the set of points that satisfy (2), and let M be the largest integer such that

$$2M - 1 + 1 + 1 \leq d - 1$$

i.e. $M = \left\lfloor \frac{m-2}{2} \right\rfloor$. Then, the points in B form a tetrahedron with vertices $(1, 1, 1)$, $(1, 1, 2M - 1)$, $(1, 2M - 1, 1)$, $(2M - 1, 1, 1)$. Each side of the tetrahedron has M points in B, and so the total number of points is given by the M^{th} tetrahedral number, Te_M.

3D Patchworking: counting components

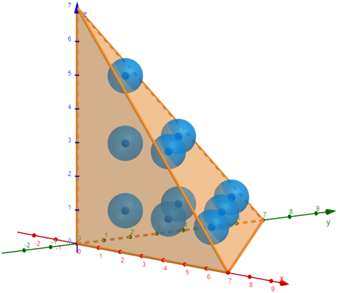

Figure: Set B for Δ_7

So the total number of cut vertices, and thus of spherical components in this octant is

$$Te_N + Te_M = \frac{\lfloor \frac{m-5}{2} \rfloor (\lfloor \frac{m-5}{2} \rfloor + 1)(\lfloor \frac{m-5}{2} \rfloor + 2)}{6} + \frac{\lfloor \frac{m-2}{2} \rfloor (\lfloor \frac{m-2}{2} \rfloor + 1)(\lfloor \frac{m-2}{2} \rfloor + 2)}{6}.$$

3D Patchworking

Theorem

Let $m \in \mathbb{N}$. Then

1. *For $m = 2k$ with $k \in \mathbb{N}$,*

$$C_{3,m} \geq \frac{1}{8}(m^3 - 6m^2 + 20m - 24) + 1$$

2. *For $m = 2k + 1$ with $k \in \mathbb{N}$,*

$$C_{3,m} \geq \frac{1}{8}(m^3 - 6m^2 + 11m - 6) + 1.$$

But we also have

$$C_{3,m} \overset{Milnor-Thom}{\leq} m(2m-1)^2.$$

Corollary

$C_{3,m}$ is a cubic polynomial in m, for any $m \in \mathbb{N}$.

Further work

Possible directions:

- study Betti numbers of real algebraic surfaces (our patchworked surface probably has a very high second Betti number)
- study bounds for $C_{4,m}$

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The pictures were created with Geogebra.

Thank you to our supervisor, Nick Sheridan!