Persistent homology and groups

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Motivation

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Examples of group invariants:

- order
- number of elements of specific order
- abelian or not
- number of Sylow p-subgroups
- nilpotency class (will introduce it later)

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Examples of group invariants:

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- nilpotency class (will introduce it later)

Goal: Construct a computable group descriptor that encodes useful information about a given group.

Overview

1. Simplicial homology
   - simplices
   - cycles, boundaries, homology groups
   - chain maps and chain complexes
2. Persistent homology
   - formalism
   - examples
3. Groups
   - classifying spaces
   - lower central series
4. Persistent homology on groups
   - procedure
   - one small example: D_4
5. Future directions

①

Homology: Motivation

Strongest notion of equality for topological spaces: homeomorphism.

- Q: Given spaces T_1 and T_2, can we tell if they are homeomorphic or not?

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- Markov (1958): The problem is undecidable for manifolds of dimension 4 or higher.

We are interested in meaningful (though imperfect) topological invariants.
Many options:

- fundamental group (and higher homotopy groups)
  Difficult to compute
- number of connected components
  Weak invariant
- homology groups

Homology: Idea

Idea: associate a sequence of abelian groups to a given space.
In the simplicial setting:

1. start with a (triangulable) space
2. find a triangulation
3. quantify the ‘holes’ in each dimension as homology groups

Simplicial homology

Definition

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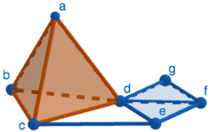
   Call the points that span σ vertices, and each m-simplex $\tau \subseteq \sigma$ an m-face.

2. A simplicial complex K is a countable set of simplices in $\mathbb{R}^m$ which is (1) downward closed, (2) every non-empty intersection of two simplices is a face of both simplices.
   The n-skeleton of K, denoted $K^{(n)}$ is the collection of all simplices of dimension at most n.

Example

a, ad, $def \in K^{(2)}$, but $dce \in K^{(2)}$.



Simplicial homology

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Two more (informal) definitions:

- $|K|$, underlying space of K
- triangulation of a space T: a homeomorphism $h : |K| \to T$

Simplicial homology

Now keep track of orientation of simplices.
Fix a ring R, and let K be a simplicial complex. Denote by m_i the number of i-simplices in K.

Definition

A p-**chain** in K is a finite linear combination $\sum_{i=1}^{m_p} \alpha_i \sigma_i$, where σ_i p-simplices in K, and $\alpha_i \in R$.

Define addition of two p-chains $c = \sum_{i=1}^{m_p} \alpha_i \sigma_i$ and $d = \sum_{i=1}^{m_p} \beta_i \sigma_i$ as

$$c + d := \sum_{i=1}^{m_p} (\alpha_i + \beta_i)\sigma_i.$$

The collection of all p-chains in K: $C_p(K)$, the p-**chain group**
Note: C_p with addition forms an R-module (in particular, C_p is an abelian group).

Claim

The p-simplices in K form a set of generators of minimum cardinality for C_p. Thus, rank $C_p = m_p$, the number of p-simplices.

Simplicial homology

Definition

Let $\sigma = [v_0, .., v_k]$ be a k-dimensional oriented simplex. The **boundary** of σ is defined as

$$\delta_k \sigma := \sum_{i=0}^{k} (-1)^i (v_0, .., \hat{v}_i, .., v_k).$$

Now, extend the map linearly to the chain group C_k: for a k-chain $c = \sum_{i=1}^{m_p} \alpha_i \sigma_i$, we define

$$\delta_k c := \sum_{i=1}^{m_p} \alpha_i (\delta_k \sigma_i).$$

Lemma

$\delta_{k-1} \circ \delta_k = 0$ *for any* k.

Corollary

im $\delta_{k+1} \subseteq \ker \delta_k$.

Simplicial homology

Define

$$\text{im } \delta_{k+1} =: B_k, \text{ the } k\text{-th boundary group}$$
$$\ker \delta_k =: Z_k, \text{ the } k\text{-th cycle group}$$

and

$$Z_k / B_k =: H_k, \text{ the } k\text{-th homology group.}$$

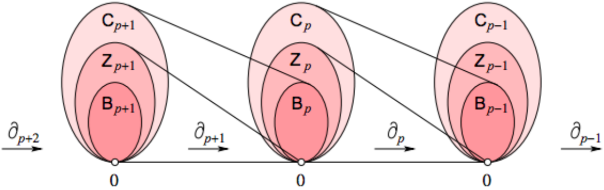


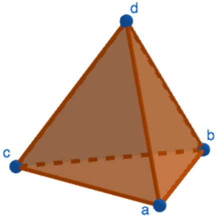
Figure: Boundary, cycle, and chain groups[1]

[1] Source: courses.cs.duke.edu/fall06/cps296.1

Simplicial homology

Example

‘Hollow’ tetrahedron:



Simplicial homology

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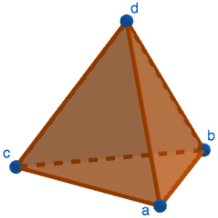

Fix $R = \mathbb{Z}$.

Simplicial homology

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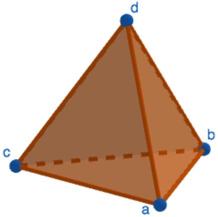

Fix $R = \mathbb{Z}$. The chain complex is:

$$\langle abc, adb, bcd, adc\rangle \xrightarrow{\delta_2} \langle ab, bc, ac, ad, db, dc\rangle \xrightarrow{\delta_1} \langle a, b, c, d\rangle \xrightarrow{\delta_0} 0$$

Simplicial homology

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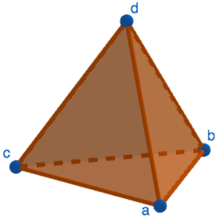

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and so:

$H_0(K) \cong \ker \delta_0/\mathrm{im}\ \delta_1 = \langle a, b, c, d\rangle/\langle a-b, b-c, c-d, d-a\rangle \cong \langle a\rangle \cong \mathbb{Z}$

$H_1(K) \cong \ker \delta_1/\mathrm{im}\delta_2 = 0$

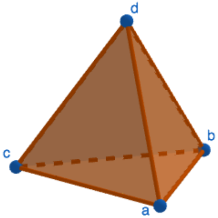
$H_2(K) \cong \ker \delta_2 \cong \langle bcd - acd + abd - abc\rangle \cong \mathbb{Z}$

$H_n(K) = 0$ for any $n \geq 3$.

Simplicial homology

Example

'Filled' tetrahedron:



Simplicial homology

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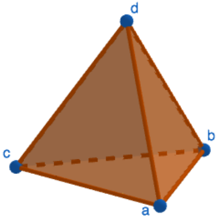

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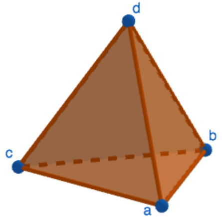

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$$H_1(K) \cong \ker \delta_1 / \mathrm{im} \delta_2 = 0$$
$$H_2(K) \cong \ker \delta_2 / \mathrm{im}\ \delta_3 \cong 0$$
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Simplicial homology

Many (interesting) aspects swept under the rug:

- triangulable spaces? Triangulation conjecture (settled by C. Manolescu)
- different triangulation of the same space - same homology?
- claimed homology is a topological invariant.

Beyond chain groups

There is a slightly more general setting:

Definition

A **chain complex** is a sequence of abelian groups and group homomorphisms called boundary operators

$$\dots \xrightarrow{\delta_{n+1}} A_n \xrightarrow{\delta_n} A_{n-1} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_1} A_0 \xrightarrow{\delta_0} 0$$

with the property $\delta_{n-1} \circ \delta_n = 0$ for all $n \geq 1$. We denote it by $(A_\bullet, \delta_\bullet)$.

One can define the homology groups of a chain complex just as we did for chain groups. If $(A_\bullet, \delta_\bullet)$ is a chain complex, define

$$H_n(A_\bullet) = \ker \delta_n / \mathrm{im}\ \delta_{n+1}$$

Definition

Let $(A_\bullet, \delta_\bullet)$ and $(A'_\bullet, \delta'_\bullet)$ be two chain complexes. A **chain map** $f_\bullet : (A_\bullet, \delta_\bullet) \to (A'_\bullet, \delta'_\bullet)$ is a collection of homomorphisms $f_n : A_n \to A'_n$ such that $f_{n-1}\delta_n = \delta'_n f_n$ for all n.

Beyond chain groups

Proposition (1)

A chain map $f_\bullet : (A_\bullet, \delta_\bullet) \to (A'_\bullet, \delta'_\bullet)$ *induces a homomorphism* $\bar{f}_n : H_n(A_\bullet) \to H_n(A'_\bullet)$ *between homology groups for each* $n \geq 0$.

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Define $\bar{f}_n([c]) := [f_n(c)]$, for $c \in H_n(A_\bullet)$. $\square$

The induced maps respect composition:

Proposition (2)

Given two chain maps $f_\bullet : (A_\bullet, \delta) \to (A'_\bullet, \delta'_\bullet)$, $g_\bullet : (A'_\bullet, \delta') \to (A''_\bullet, \delta''_\bullet)$, *we have*

$$\bar{g}_n \circ \bar{f}_n = (\overline{g \circ f})_n$$

for each $n \geq 0$, *where* $\bar{f}_n : H_n(A_\bullet) \to H_n(A'_\bullet)$, $\bar{g}_n : H_n(A'_\bullet) \to H_n(A''_\bullet)$ *are the induced maps.*

②

Persistent homology

Given a (potentially noisy) sample S of discrete points collected from a topological manifold M, what can we tell about the topology of M?

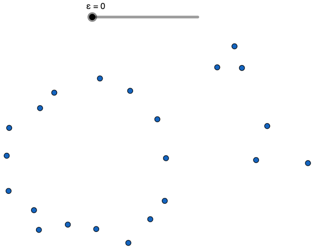

Figure: Initial sample S

Idea: 'Thicken' S and observe what happens with the homological features.

Persistent homology

New features can be created:

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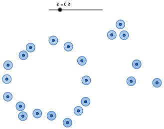

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Persistent homology

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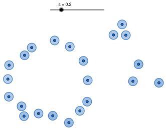

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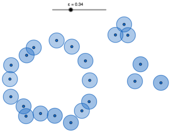

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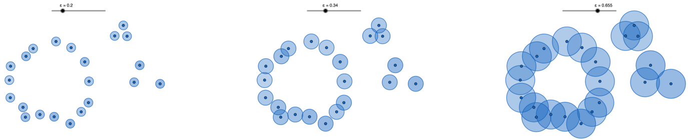

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Figure: Evolution of homological features with increasing ϵ

previous ones can die, and others can persist.

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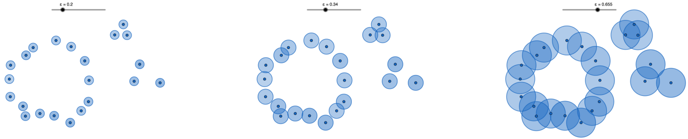

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Figure: Evolution of homological features with increasing ϵ

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Idea: Features that persist longer are more relevant to the global structure of the underlying space.

Persistent homology

Want to work in a discrete setting: many ways to construct meaningful simplicial complexes using the sampled points (e.g., Čech complex).
Assume we are in the simplicial setting.
Call K' a **subcomplex** of K if K' is itself a complex and if $\sigma \in K'$, then $\sigma \in K$.

Definition

A (simplicial) filtration $\mathcal{F}$ is a nested sequence of subcomplexes of K

$$\varnothing = K_0 \subseteq K_1 \subseteq .. \subseteq K_n = K.$$

Persistent homology

A filtration induces a chain map (the inclusion maps commute with the boundary operator). So a filtration $\mathcal{F}$

$$\varnothing = K_0 \hookrightarrow K_1 \hookrightarrow .. \hookrightarrow K_n = K$$

induces a homology module $H_p\mathcal{F}$:

$$\{e\} = H_p(K_0) \xrightarrow{h_p^{0,1}} H_p(K_1) \to .. \xrightarrow{h_p^{n-1,n}} H_p(K_n) = H_p(K)$$

where each morphism $h_p^{i,j}$ is just the composition of the morphisms

$$H_p(\mathcal{C}_i) \xrightarrow{h_p^{i,i+1}} \cdots \xrightarrow{h_p^{j-1,j}} H_p(\mathcal{C}_j).$$

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The p-th persistent homology groups are

$$H_p^{i,j} := \text{im } h_p^{i,j}.$$

Persistent homology

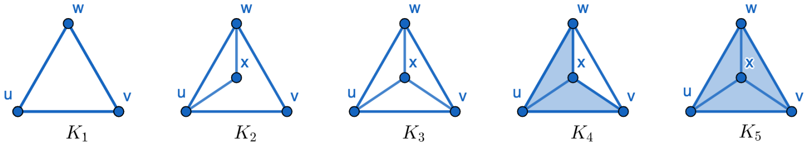


Figure: A simplicial filtration

Persistent homology

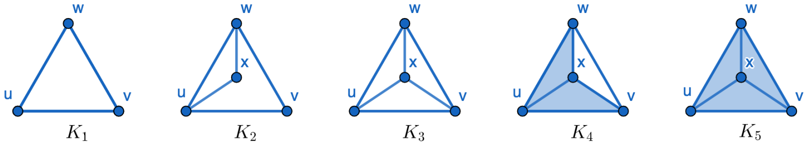


Figure: A simplicial filtration

Fix $R = \mathbb{Z}/2\mathbb{Z}$.

Persistent homology

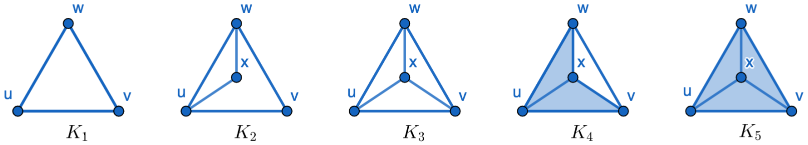


Figure: A simplicial filtration

Fix $R = \mathbb{Z}/2\mathbb{Z}$.

$$
\begin{aligned}
H_1(K_1) &\cong \langle uv + vw + wu \rangle / \{e\} \\
H_1(K_2) &\cong \langle wx + ux + uw, wx + ux + uv + vw \rangle \\
H_1(K_3) &\cong \langle wx + ux + uw, uv + vx + xu, xv + vw + wx \rangle \\
H_1(K_4) &\cong \langle wx + ux + uw, uv + vx + xu, xv + vw + wx \rangle \\
&\qquad / \langle wx + ux + uw, uv + vx + xu \rangle \\
H_1(K_5) &\cong 0
\end{aligned}
$$

Persistent homology

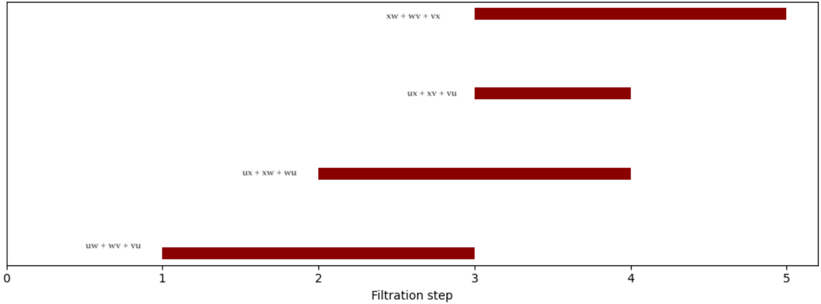


Figure: Persistent barcode for H_1

Persistent algorithms

Main appeal of persistent homology in the simplicial setting: efficient algorithms.
Implemented a version in Haskell.

Persistent algorithms

```haskell
-- Given a simplex and the current list of unpaired positive simplices, find a compatible pair for the
-- negative simple or add it to the list of positive unpaired simplices
pairSimplex :: Int -> [Int] -> Matrix Int -> (Maybe Int, [Int])
pairSimplex s posSimpl matr | candidate `elem` posSimpl = (Just candidate, delete candidate posSimpl)
                            | otherwise  = (Nothing, posSimpl ++ [s])
  where candidate = lowOfCol s matr


makeRowZero :: Int -> Matrix Int -> Matrix Int
makeRowZero r matr = mapRow (\_ r -> 0) r matr

leftSimplify :: Int -> [Int] -> Maybe Int
leftSimplify j lowMatr | lowMatr !! j == -1 = Nothing
                       | null candidates    = Nothing
                       | otherwise          = Just (head candidates)
  where candidates = filter (\j' -> lowMatr !! j' == lowMatr !! j) [1..(j-1)]


pairingAlgorithm :: SimplicialComplex -> [(Int, Int)]
pairingAlgorithm c = pairRec (length c) matr [] []
  where matr = computeBoundaryMatrix c

infinity = 2^30

pairRec :: Int -> Matrix Int -> [Int] -> [(Int, Int)] -> [(Int, Int)]
pairRec 0 matr positiveSimplList pairings = pairings ++ map (\s -> (s, infinity)) positiveSimplList
pairRec j matr positiveSimplList pairings
  | chosenPosSimpl == Nothing  = pairRec (j - 1) matr positiveSimplList' pairings
  | otherwise                  = pairRec (j - 1) (makeRowZero (fromJust chosenPosSimpl) matr)
                                   positiveSimplList' (pairings ++ [(fromJust chosenPosSimpl, i)])
  where (chosenPosSimpl, positiveSimplList') = pairSimplex i positiveSimplList matr
        i = nrows matr - j + 1
```

Persistent algorithms

Main appeal of persistent homology in the simplicial setting: efficient algorithms.
Implemented a version in Haskell. Standard algorithm (SNF-like) and many optimisations in [2].

③

Classifying spaces

Want to associate a meaningful simplicial complex to a given group. A good choice for this: the classifying space of the group.

Prerequisites:

- *a topological group* G is a group endowed with a topology such that the multiplication and taking inverses are continuous functions;
- an action $\cdot : G \times X \to X$ is called *free* if for any $x \in X$

$$gx = x \Rightarrow g = e.$$

Classifying spaces

Definition

The classifying space of G is defined (up to homotopy equivalence) as $BG = EG/G$, where EG is a (sufficiently nice) contractible space on which G acts freely.

Example

1. $G = \mathbb{Z}$. Can choose $E\mathbb{Z} = \mathbb{R}$, and so $B\mathbb{Z} \simeq \mathbb{R}/\mathbb{Z} = S^1$.
2. $G = \mathbb{R}$. Can choose again $E\mathbb{R} = \mathbb{R}$, and so $B\mathbb{R} \simeq \mathbb{R}/\mathbb{R} \simeq \{*\}$
3. $G = \mathbb{Z}/2\mathbb{Z}$. Can choose $EG = \cup_{n=1}^{\infty} S^n = S^\infty$, and $BG \simeq \mathbb{RP}^\infty$

BG encodes a wealth of information:

- classifies principal G-bundles.
- $\pi_1(BG) \cong G$ (what happens when trying to lift a loop in BG to EG?)

The bar construction

However, computing BG seems difficult: can we even always find a suitable EG?

The bar construction

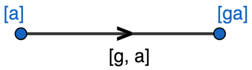
However, computing BG seems difficult: can we even always find a suitable EG? Yes!

The bar construction

G finite group, denote its elemenents by $\{g_1, .., g_n\}$. Construct $E_\bullet G$:

- 0-simplices: elements of g, $[g_1]$, $[g_2]$, .., $[g_n]$
- 1-simplices: $[ga]$ for $g, a \in G$ such that $[ga]$ connects the vertex $[a]$ to the vertex $[ga]$, and

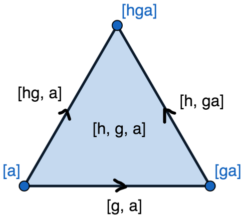
$$\delta[g, a] = [a] - [ga]$$



The bar construction (Cont.)

- 2-simplices: $[h, g, a]$ for $g, h, a \in G$ such that $[h, g, a]$ 'fills' the triangle with edges $[g, a]$, $[h, ga]$, $[hg, a]$

$$\delta[h, g, a] = [g, a] + [h, ga] - [hg, a].$$



- n-simplices are $(n+1)$-tuples $[g_n, .., g_1, a]$ that have boundary

$$\delta[g_n, \ldots, g_1, a] = [g_{n-1}, \ldots, g_1, a] + \sum_{i=1}^{n-1} (-1)^i [g_{n-1}, g_{n-2}, \ldots, g_{i+2}, g_{i+1}g_i, g_{i-1}, \ldots g_1].$$

Continue inductively to infinity to obtain $E_\bullet G$, i.e. $E_\bullet G = E_\bullet G^{(\infty)}$.

The bar construction

For every $n \in \mathbb{N}$, we define the G-action on $E_\bullet G^{(n)}$ to be

$$i[g_n, \dots, g_1, a] := [i g_n i^{-1}, \dots, i g_1 i^{-1}, a].$$

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Let $B_\bullet G = E_\bullet G/G$, and let EG and BG be the geometric realizations of $E_\bullet G$ and $B_\bullet G$.

Claim

The bar construction gives a model of BG.

Proof sketch.

Need to show EG is a contractible space on which G acts freely. We can show this one skeleton at a time. $\square$

The bar construction

Now, the explicit construction for BG:

- one 0-simplex $\{\star\}$ with $\delta(\star) = 0$.
- 1-simplices: $[g]$ for $g \in G$ with $\delta[g] = 0$

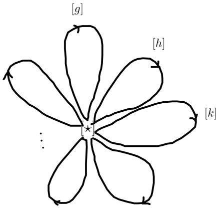

Figure: $B_\bullet G^{(1)}$

- the 2-simplices are $[g, h]$ for $g, h \in G$ with $\delta[g, h] = [h] - [gh] + [g]$.
- the n-simplices are $[g_1, \ldots, g_n]$ with $g_1, \ldots, g_n \in G$., with boundary

$$\delta[g_1, \ldots, g_n] = [g_2, g_3, \ldots, g_n] - [g_1 g_2, g_3, \ldots, g_n] + [g_1, g_2 g_3, g_4, \ldots, g_n] - \\ + (-1)^{n-1}[g_1, g_2, \ldots, g_{n-1} g_n] + (-1)^n [g_1, g_2, \ldots, g_{n-1}].$$

Call $B_\bullet G$ as the **bar complex**.

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G group, $a, b \in G$.
$X, Y \subseteq G$.

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Definition

We define $L_0(G) = G$ and $L_n(G) = [G, L_{n-1}(G)]$ for $n \geq 1$. The sequence $L_0(G) \supseteq L_1(G) \supseteq L_2(G) \supseteq ..$ is called the lower central series of G.

If $n < \infty$, say G has nilpotency class n.

Remark

Not hard to show that $L_n(G) \lhd G$ *for each* $n \geq 1$.

Lower central series

Example

1. $\mathbb{Z}/5\mathbb{Z}$

Lower central series

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Lower central series

Example

1. $\mathbb{Z}/5\mathbb{Z} \supseteq \{e\}$
2. $S_3 \supseteq A_3 \supseteq A_3 \supseteq ..$

The series is either finite or stabilises. Define augmented LCS as stabilised portion of LCS with $\{e\}$ appended at the end if necessary.

Putting it all together

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Claim

G group, and $N_1, N_2 \lhd G$ with $N_1 \subseteq N_2$. The natural map $q : G/N_1 \to G/N_2$ is a surjective homomorphism.

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Theorem

A nested sequence of normal groups

$$G = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_n \supseteq N_{n+1} = \{e\}$$

induces a sequence of well-defined homomorphisms between homology groups:

$$H_k(G) \to H_k(G/N_n) \to H_k(G/N_{n-1}) \to \cdots \to H_k(G/N_1) \to 0$$

for any $k \geq 0$.

Corollary

The augmented LCS of G induces a well-defined sequence of homomorphisms between homology groups

$$H_k(G) \to H_k(G/L_n(G)).. \to H_k(G/L_1(G))$$

for any dimension $k \geq 0$.

Using the procedure for D_4

Consider

$$D_4 = \langle r, s \mid r^4 = s^2 = e, srs^{-1} = r^{-1} \rangle.$$

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$$D_4 = \langle r, s \mid r^4 = s^2 = e, srs^{-1} = r^{-1} \rangle.$$

The (augmented) LCS of D_4 is:

$$D_4 \supseteq C_2 \supseteq \{e\}.$$

Take the quotient of D_4 with each term and reverse the sequence, we get surjections:

$$D_4/\{e\} \to D_4/C_2 \to D_4/D_4$$

i.e.

$$D_4 \to D_2 \to \{e\}.$$

Using the procedure for D_4 (Cont.)

This induces a sequence of surjective chain maps

$$B_\bullet D_4 \to B_\bullet D_2 \to B_\bullet \{e\}$$

which have well-defined homology. We have:

$$H_1(D_4) = \ker \delta_1 / \text{im } \delta_2 = \langle [g] \rangle_{g \in D_4} / ([h] + [g] = [gh]).$$

. We can compute:

$$[r] = [er] = [s^2 r] = [srs] = [r^{-1}].$$

Thus $[e] = [r^2]$, $[r] = [r^3]$, $[s] = [r^2 s]$, $[rs] = [r^3 s]$, and so

$$H_1(D_4) = \{[r], [s], [rs], [e]\}.$$

Similarly, $H_1(D_2) = \{[r], [s], [rs], [e]\}$.

Using the procedure for D_4 (Cont.)

Now,

$$H_2(D_4) = \ker \delta_2 / \mathrm{im}\ \delta_3 = \langle [g,h] \rangle_{g,h \in D_4} / ([h,i] - [gh,i] + [g,hi] - [g,h] = 0\rangle$$

. . . we obtain

$$H_2(D_4) = \{[r,s],[e,e]\}.$$

Similarly, we obtain $H_2(D_2) = \{[r,s],[e,e]\}$.

Using the procedure for D_4 (Cont.)

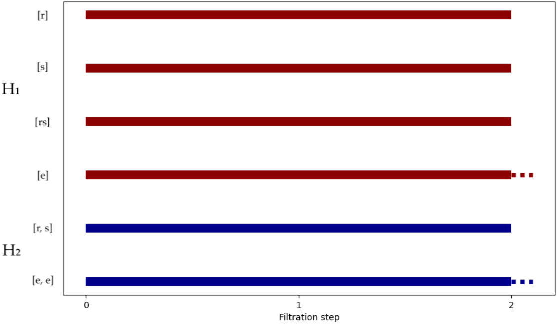

Figure: Barcode for D_4

Other examples

- D_4 vs A_4
- D_4 vs Q_8

Future directions

1. Compute barcodes using other normal series (derived, upper central) to obtain a stronger invariant.
2. Automate the process using the GAP software and the SMALL group database.
3. Use a more economical complex (presentation complex).

References

[1] G. Ellis and S. A. King. Persistent homology of groups. *Journal of Group Theory* 14(4):575–587, 2010.

[2] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. ISBN:0-521-79160-X; 0-521-79540-0.

[3] T. K. Dey and Y. Wang. *Computational topology for data analysis*. Cambridge University Press, Cambridge, 2022.

Thank you!