**15.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of $n$, with $p_i$ being distinct primes and $\alpha_i$ positive integers. If $n$ is powerful, then we must have $\alpha_i \geq 2$ for every $1 \leq i \leq k$. From Q10, we know that $\alpha_i = 2a_i + 3b_i$ has non-negative integer solutions $a_i, b_i$ because $\alpha_i \geq 2 = (2-1)\cdot(3-1)$ for every $i$. Now, if we choose $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$, then they will satisfy $n = a^2 b^3$.

**16.** Let's denote the prime factorisations of $a, b,$ and $c$ with $\prod_p p^{a_p}, \prod_p p^{b_p}$, and $\prod_p p^{c_p}$ where $p$ ranges through prime numbers. In other words, $a_p, b_p,$ and $c_p$ are the largest powers of the prime $p$ dividing the numbers $a, b,$ and $c$. Then, we have

$$\gcd(a,b,c)\cdot \mathrm{lcm}[ab,bc,ca] = \prod_p p^{\min(a_p,b_p,c_p)} \cdot \prod_p p^{\max(a_p+b_p,b_p+c_p,c_p+a_p)} = \prod_p p^{\min(a_p,b_p,c_p)+\max(a_p+b_p,b_p+c_p,c_p+a_p)}$$

and

$$abc = \prod_p p^{a_p+b_p+c_p}.$$

To prove the given equality, we just need to show that the powers of each prime $p$ are equal on both sides of the equation, i.e.

$$\min(a_p,b_p,c_p) + \max(a_p+b_p,b_p+c_p,c_p+a_p) = a_p+b_p+c_p.$$

The last equality follows immediately from the fact that $\max(a_p+b_p, b_p+c_p, c_p+a_p)$ is the sum of the largest two of the numbers $a_p, b_p,$ and $c_p$.

**17.** By Dirichlet's Theorem, there are infinitely many primes of the form $ak+b$, where $a = 10^n$ and $b = 10^n - 1$ for an arbitrary positive integer $n$ (Note that $\gcd(a,b) = \gcd(a-b,b) = \gcd(1,b) = 1$). Clearly, every number of the form $ak+b$ has 9 in their last $n$ digits. Therefore, there are infinitely many primes with the sum of digits at least $9n$.

**18.** Suppose the last four digits of $2^n$ are equal. Since $2^n$ has at least four digits, we must have $n \geq 10$. Let $\overline{a_k a_{k-1} \cdots a_4 aaaa}$ be the decimal representation of $2^n$ for some integer $0 \leq a \leq 9$. Since both $2^n$ and $\overline{a_k a_{k-1} \cdots a_4 0000} = 10^4 \cdot \overline{a_k a_{k-1} \cdots a_4}$ are divisible by 16, their difference $\overline{aaaa}$ should also be divisible by 16. However,

$$16 \mid \overline{aaaa} \Longrightarrow 16 \mid a \cdot \overline{1111} \Longrightarrow 16 \mid a \Longrightarrow a = 0,$$

but the last digit of $2^n$ can never be 0 because it is not divisible by 10, a contradiction.

**19.** We will prove by induction on $n$.

For the base case $n = 0$, we have $5^{2^0} - 1 = 4$ is fully divisible by $2^{0+2} = 4$.

Now, assume for an integer $n \geq 0$ that $2^{n+2}$ fully divides $5^{2^n} - 1$ and we want to prove the same result for $n+1$. We have $5^{2^{n+1}} - 1 = (5^{2^n} - 1)\cdot(5^{2^n}+1)$ and $5^{2^n}+1$ is an even integer which is not divisible by 4 because $5^{2^n} + 1 \equiv 1^{2^n} + 1 \equiv 2 \pmod 4$. Now, $2^n \| 5^{2^n} - 1$ and $2^1 \| 5^{2^n} + 1$ gives $2^{n+1} \| 5^{2^{n+1}} - 1$ as desired.

**20.** (a) From the lecture notes, we have $a \equiv b \pmod n \Longrightarrow p(a) \equiv p(b) \pmod n$. Choosing $n = |a-b|$, we get $p(a) \equiv p(b) \mod |a-b|$ and hence $a - b \mid p(a) - p(b)$.

(b) Suppose an integer $n$ satisfies $p(n) = 0$, then we have $a - n \mid p(a) - p(n) = 1, b - n \mid p(b) - p(n) = 1$, and $c - n \mid p(c) - p(n) = 1$ from the previous part. Three distinct integers $a-n, b-n,$ and $c-n$ must divide 1, but there are only two numbers dividing 1: 1 itself and $-1$, a contradiction.

**21.** (from Wikipedia) Since $n!$ is the product of the integers from 1 to $n$, we obtain at least one factor of $p$ in $n!$ for each of the $\left\lfloor \frac{n}{p} \right\rfloor$ multiples of $p$ in $\{1, 2, \cdots, n\}$. Each multiple of $p^2$ contributes an additional factor of $p$, each multiple of $p^3$ contributes yet another factor of $p$, etc. Adding them all up, we obtain

$$e = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots.$$