Problem Set 1

Due: Friday, May 20, 2022 11:59 pm (EDT)

Assignment description

To get a full mark you must show all your work and justify your answers. Feel free to discuss the problem set questions and post about them on Ed Discussion. You can provide clarity on the problems when needed and give hints when appropriate. All numbers/variables are integers unless stated otherwise. This problem set covers Lecture 1, Lecture 2, and Chapter 1 of our main textbook.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

For $a, b, c, d \in \mathbb{Z}$, prove that $a - c \mid ab + cd$ if and only if $a - c \mid ad + bc$.

Q2 (10 points)

(a) What are the possible remainders when $n^2 + 16n + 20$ is divided by 11?

(b) Prove for every $n \in \mathbb{Z}$ that $121 \nmid n^2 + 16n + 20$.

Q3 (10 points)

The sum of seven distinct positive integers $a_1, a_2, \cdots, a_7$ is 315. What is the maximum possible value of their greatest common divisor $\gcd(a_1, a_2, \cdots, a_7)$?

Q4 (10 points)

(a) Prove for the non-zero integers a, b that $\gcd(13a + 8b, 21a + 13b) = \gcd(a, b)$.

(b) (not to be graded) Prove the following generalization $\gcd(F_n a + F_{n-1} b, F_{n+1} a + F_n b) = \gcd(a, b)$, where $\{F_n\}$ is the Fibonacci sequence.

Q5 (10 points)

A pair of integers is written on a blackboard. At each step, we are allowed to erase the pair of numbers (m, n) from the board and replace it with one of the following pairs: (n, m), $(m - n, n)$, $(m + n, n)$. If we start with $(2022, 315)$ written on the blackboard, then can we eventually have the pair

(a) $(30, 45)$,

(b) $(222, 15)$?

(c) (not to be graded) Describe the set of all pairs of integers (m, n) that we can eventually reach.

Q6 (10 points)

(a) Is there an integer solution (x, y, z) to the equation $20x + 22y + 33z = 1$ with $x = 1$?

(b) Is there an integer solution (x, y, z) to the equation $20x + 22y + 33z = 1$ with $x = 5$?

(c) For which values of $c \in \mathbb{Z}$, the equation $20x + 22y + cz = 315$ has integer solution(s) (x, y, z)?

Q1 (10 points)

For $a, b, c, d \in \mathbb{Z}$, prove that $a - c \mid ab + cd$ if and only if $a - c \mid ad + bc$.

Let $a, b, c, d \in \mathbb{Z}$

WTS: $a-c \mid ab+cd \Leftrightarrow a-c \mid ad+bc$

1) $\rightarrow$

Assume $a-c \mid ab+cd$

WTS: $a-c \mid ad+bc$

Note that $ab+cd = ab-bc+bc+cd$

$= b(a-c) + c(b+d)$

Thus, $a-c \mid b(a-c)+c(b+d)$

Thus, $a-c \mid c(b+d)$

Also, $ad+bc = ad+cd-cd+bc$

$= d(a-c) + c(d+b)$

Since $a-c \mid c(b+d)$ and $a-c \mid d(a-c)$

we know that $a-c \mid ad+bc$ must be true.

2) $\leftarrow$

Assume $a-c \mid ad+bc$

WTS: $a-c \mid ab+cd$.

Similarly, $a-c \mid ad+bc$

$\rightarrow a-c \mid d(a-c)+c(d+b)$

$$\Rightarrow \quad a-c \mid c(d+b)$$

$$\Rightarrow \quad a-c \mid b(a-c) + c(d+b)$$

$$\Rightarrow \quad a-c \mid ab+cd.$$

$\blacksquare$

Q2 (10 points)

(a) What are the possible remainders when $n^2 + 16n + 20$ is divided by 11?

(b) Prove for every $n \in \mathbb{Z}$ that $121 \nmid n^2 + 16n + 20$.

a) Let $f(n) = n^2 + 16n + 20$

$f(0) = 20 \ \% \ 11 = 9$

$f(1) = 37 \ \% \ 11 = 4$

$f(2) = 56 \ \% \ 11 = 1$

$f(3) = 77 \ \% \ 11 = 0$

$f(4) = 100 \ \% \ 11 = 1$

$f(5) = 125 \ \% \ 11 = 4$

$f(6) = 152 \ \% \ 11 = 9$

$f(7) = 181 \ \% \ 11 = 5$

$f(8) = 212 \ \% \ 11 = 3$

$f(9) = 245 \ \% \ 11 = 3$

$f(10) = 280 \ \% \ 11 = 5$

$f(11) = 317 \ \% \ 11 = 9$

$f(12) = 356 \ \% \ 11 = 4$

$f(13) = 397 \ \% \ 11 = 1$

$f(14) = 440 \ \% \ 11 = 0$

$f(15) = 485 \ \% \ 11 = 1$

$f(16) = 532 \ \% \ 11 = 4$

$f(17) = 581 \ \% \ 11 = 9$

$f(18) = 632 \ \% \ 11 = 5$

$f(19) = 685 \ \% \ 11 = 3$

$f(20) = 740 \ \% \ 11 = 3$

$f(21) = 797 \ \% \ 11 = 5$

We have seen the patterns and can now generalize it.

Let $k \in \mathbb{N}$.

When $n = 11k$. $f(n) = (11k)^2 + 16(11k) + 20$

$$\Rightarrow f(n) \% 11 = 20 \% 11 = 9$$

when $n = 11k+1$ $f(n) = (11k+1)^2 + 16(11k+1) + 20$

$$= (11k)^2 + 22k + 1 + 16(11k) + 16 + 20$$

$$= (11k)^2 + 2(11k) + 16(11k) + 37$$

$$\Rightarrow f(n) \% 11 = 37 \% 11 = 4$$

when $n = 11k+2$ $f(n) = (11k+2)^2 + 16(11k+2) + 20$

$$= (11k)^2 + 4 \times 11k + 4 + 16(11k) + 32 + 20$$

$$\Rightarrow f(n) \% 11 = 56 \% 11 = 1$$

when $n = 11k+3$ $f(n) = (11k+3)^2 + 16(11k+3) + 20$

$$= (11k)^2 + 6(11k) + 9 + 16(11k) + 48 + 20$$

$$\Rightarrow f(n) \% 11 = 77 \% 11 = 0$$

when $n = 11k+4$ $f(n) = (11k)^2 + 8(11k) + 16 + 16(11k) + 84$

$$\Rightarrow f(n) \% 11 = 100 \% 11 = 1$$

When $n=11k+5$ $f(n) = (11k)^2+10(11k)+25+16(11k)+100$

$\Rightarrow f(n) \% 11 = 125 \% 11 = 4$

when $n=11k+6$ $f(n) = (11k)^2+12(11k)+36+16(11k)+116$

$\Rightarrow f(n) \% 11 = 152 \% 11 = 9$

When $n=11k+7$ $f(n) = (11k)^2+14(11k)+49+16(11k)+132$

$\Rightarrow f(n) \% 11 = 181 \% 11 = 5$

when $n=11k+8$ $f(n) = (11k)^2+16(11k)+64+16(11k)+148$

$\Rightarrow f(n) \% 11 = 212 \% 11 = 3$

when $n=11k+9$ $f(n) = (11k)^2+18(11k)+81+16(11k)+164$

$\Rightarrow f(n) \% 11 = 245 \% 11 = 3$

When $n=11k+10$ $f(n) = (11k)^2+18(11k)+100+16(11k)+180$

$\Rightarrow f(n) \% 11 = 280 \% 11 = 5$

Thus, the possible remainders are 0, 1, 3, 4, 5, 9

(b) Prove for every $n \in \mathbb{Z}$ that $121 \nmid n^2 + 16n + 20$.

Let $n \in \mathbb{Z}$.

Assume $121 \mid n^2+16n+20$.

That is, $11 \times 11 \mid n^2+16n+20$

Thus, $n^2+16n+20 \,/\, 11\times 11 = 0$

Thus $n^2+16n+20 \,/\, 11 = 0$.

Thus, we know that $n=11k+3$ must be true.

$(11k+3)^2 + 16(11k+3) + 20$

$= (11k)^2 + 6(11k) + 9 + 16(11k) + 48 + 20$

$= (11k)^2 + 22(11k) + 77$

$= (11k)^2 + 2(11)^2 k + 7\times 11.$

Note that $(11k)^2 + 2(11)^2 k + 7\times 11. \% (11)^2$

$= 77 \% (11)^2$

$= 77 \% 121$

$= 77 \neq 0.$

Thus, reach contradiction.

Thus, $121 \nmid n^2+16n+20$.

Q3 (10 points)

The sum of seven distinct positive integers $a_1, a_2, \cdots, a_7$ is 315. What is the maximum possible value of their greatest common divisor $\gcd(a_1, a_2, \cdots, a_7)$?

$\sum_{i=1}^{7} a_i = 315$ is given.

Let $d = \gcd(a_1, a_2, \dots, a_7)$

Then, $\sum_{i=1}^{7} a_i = (m_1 + m_2 + m_3 + \dots + m_7) d$ where

$m_1, m_2, \dots, m_7$ are distinct as a_i are distinct.

Therefore, $\sum_{i=1}^{7} m_i \geq 1+2+3+4+5+6+7 = 28.$ $(*)$

The prime factorization for 315 is $3^2 \times 5^1 \times 7$.

Thus, the possible gcd are:

$3\times7\times5$, 7×5, 3×7, 3×5, 5×9, 3^2, 3, 5 and 7.

CASE#1: $\gcd = 3\times7\times5 = 105$.

Then, $315 = \left(\sum_{i=1}^{7} m_i\right) 105$

$\Rightarrow \sum_{i=1}^{7} m_i = 3 \leq 28.$

Reach contradiction. (see $*$)

CASE#2: $\gcd = 7\times5 = 35$

Then, $315 = \left(\sum_{i=1}^{7} m_i\right) 35$

$$\Rightarrow \sum_{i=1}^{7} m_i = 9 \leq 28$$

Reach contradiction. (see $*$)

CASE #3: gcd $= 3 \times 7 = 21$

Then, $315 = \left(\sum_{i=1}^{7} m_i\right) 21$

$$\Rightarrow \sum_{i=1}^{7} m_i = 15 \leq 28$$

Reach contradiction. (see $*$)

CASE #4: gcd $= 3 \times 5 = 15$

Then, $315 = \left(\sum_{i=1}^{7} m_i\right) 15$

$$\Rightarrow \sum_{i=1}^{7} m_i = 21 \leq 28$$

Reach contradiction. (see $*$)

CASE #5: gcd $= 5 \times 9 = 45$

Then, $315 = \left(\sum_{i=1}^{7} m_i\right) 45$

$$\Rightarrow \sum_{i=1}^{7} m_i = 7 \leq 28$$

Reach contradiction. (see $*$)

CASE# 6: $\gcd = 3^2 = 9$

Then, $315 = \left(\sum_{i=1}^{7} m_i\right) 9$

$$\Rightarrow \sum_{i=1}^{7} m_i = 35 \geq 28.$$

Thus, 9 is one of the common divisors.

Since 9 is greater than 3, 5 and 7, there is no need to check the cases for 3, 5 and 7.

The gcd we are looking for is 9.

Q4 (10 points)

(a) Prove for the non-zero integers a, b that $\gcd(13a + 8b, 21a + 13b) = \gcd(a, b)$.

(b) (not to be graded) Prove the following generalization $\gcd(F_n a + F_{n-1} b, F_{n+1} a + F_n b) = \gcd(a, b)$, where $\{F_n\}$ is the Fibonacci sequence.

1) WTS: $\gcd(13+8b, 21a+13b) \mid \gcd(a,b)$

Let n be a common divisor of $13+8b$ and $21a+13b$.

Then, $\exists k_1, k_2$ s.t. $\begin{cases} nk_1 = 13a+8b \\ nk_2 = 21a+13b \end{cases}$

Then, $\begin{cases} 13nk_1 = (13)^2 a + (13)8b \\ 8nk_2 = (21)(8)a + (13)(8b) \end{cases}$

Then, $n(13k_1 - 8k_2) = (169-168)a = a$

Thus, $n \mid a$.

AND, $\begin{cases} 21nk_1 = (13)(21)a + (8)(21)b \\ 13nk_2 = (21)(13)a + (13)(13)b \end{cases}$

$\Rightarrow n(21k_1 - 13k_2) = -b$

$\Rightarrow n(13k_2 - 21k_1) = b$

$\Rightarrow n \mid b$.

By lecture 2

$c \mid a$ and $c \mid b \Leftrightarrow c \mid (a,b)$

Thus, $n \mid \gcd(a,b)$

Thus, $\gcd(13+8b, 21a+13b) \mid \gcd(a,b)$ must be true.

2) WTS: $\gcd(a,b) \mid \gcd(13a+8b, 21a+13b)$

Let $n = \gcd(a,b)$.

Then, $\exists t_1, t_2$ s.t. $t_1 n = a$, $t_2 n = b$.

Then, $13a+8b$ AND $21a+13b$

$= 13t_1 n + 8t_2 n \qquad = 21t_1 n + 13t_2 n$

$= n(13t_1 + 8t_2) \qquad = n(21t_1 + 13t_2)$

Thus, $n \mid 13a+8b$ and $n \mid 21a+13b$.

Thus, $n \mid \gcd(13a+8b, 21a+13b)$.

Thus $\gcd(a,b) \mid \gcd(13a+8b, 21a+13b)$

Therefore, we have shown that

$\gcd(a,b) = \gcd(13a+8b, 21a+13b)$

Q5 (10 points)

A pair of integers is written on a blackboard. At each step, we are allowed to erase the pair of numbers (m, n) from the board and replace it with one of the following pairs: (n, m), $(m - n, n)$, $(m + n, n)$. If we start with $(2022, 315)$ written on the blackboard, then can we eventually have the pair

(a) $(30, 45)$,

(b) $(222, 15)$?

(c) (not to be graded) Describe the set of all pairs of integers (m, n) that we can eventually reach.

- $\gcd(m,n) = \gcd(n,m)$
- WTS: $\gcd(m-n, n) = \gcd(m,n)$

  Let $k = \gcd(m,n)$

  Then, $k \mid m$ and $k \mid n$.

  Thus, $k \mid m-n$.

  Thus, $k \mid \gcd(m-n,n)$

  Let $k' = \gcd(m-n,n)$

  Thus, $k \mid k'$

  Thus, $k' \mid m-n$ and $k' \mid n$.

  Thus $k' \mid m-n+n$

  Thus $k' \mid m$.

  Thus, $k' \mid \gcd(m,n)$

  $\Rightarrow k' \mid k$

  Thus, $k = k'$.

  Thus, $\gcd(m-n, n) = \gcd(m,n)$

- WTS: $\gcd(m+n, n) = \gcd(m,n)$

Let $k = \gcd(m, n)$

Then, $k | m$ and $k | n$.

Thus, $k | m+n$.

Thus, $k | \gcd(m+n, n)$

Let $k' = \gcd(m+n, n)$

Thus, $k | k'$

Thus, $k' | m+n$ and $k' | n$.

Thus $k' | m-n+n$

Thus $k' | m$.

Thus, $k' | \gcd(m, n)$

$\Rightarrow k' | k$

Thus, $k = k'$.

Notice how after all replacements, the gcd should stay the same.

a) $\gcd(30, 45) = 15 \neq \gcd(2022, 315) = 3$

Thus, we cannot eventually have the pair $(30, 45)$

b) $\gcd(222, 15) = 3 = \gcd(2022, 315)$

$(2022, 315) \rightarrow (1707, 315) \rightarrow (1392, 315) \rightarrow (1077, 315)$

$\downarrow$

$$
\begin{tikzcd}
(315,132) \arrow[d] & (132,315) \arrow[l] & (447,315) \arrow[l] & (762,315) \arrow[l] \\
(183,132) \arrow[r] & (51,132) \arrow[r] & (132,51) \arrow[r] & (81,51) \arrow[d] \\
(30,21) \arrow[d] & (21,30) \arrow[l] & (51,30) \arrow[l] & (30,51) \arrow[l] \\
(9,21) \arrow[r] & (21,9) \arrow[r] & (12,9) \arrow[r] & (3,9) \arrow[d] \\
(15,12) \arrow[d] & (3,12) \arrow[l] & (12,3) \arrow[l] & (9,3) \arrow[l] \\
(12,15) \arrow[r, "12+15\times 14"] & (222,15) & &
\end{tikzcd}
$$

Q6 (10 points)

(a) Is there an integer solution (x, y, z) to the equation $20x + 22y + 33z = 1$ with $x = 1$?

(b) Is there an integer solution (x, y, z) to the equation $20x + 22y + 33z = 1$ with $x = 5$?

(c) For which values of $c \in \mathbb{Z}$, the equation $20x + 22y + cz = 315$ has integer solution(s) (x, y, z)?

a) Let $x=1$

Then, $20+22y+33z=1$

$22y+33z=-19$

From lecture 2:

corollary: $ax+by=c$ iff $(a,b) \mid c$

$\gcd(22,33)=11$ and $11 \nmid -19$.

Thus, by corollary, there is no integer solution (x,y,z) to the equation $20x+22y+33z=1$ with $x=1$.

b) Let $x=5$.

$20\times5+22y+33z=1$

$22y+33z=-99$

$22\times3-(33\times5) = 66-165=-99$

$(5,3,-5)$ is therefore a integer solution.

c) For which values of $c \in \mathbb{Z}$, the equation $20x + 22y + cz = 315$ has integer solution(s) (x, y, z)?

$$\gcd(20, 22, c) = \gcd((20, 22), c)$$
$$= \gcd(2, c)$$

$\gcd(2, c) \mid 315$ is required for the equation to have integer solutions

If c is odd, then $\gcd(2, c) = 1 \mid 315$ is true.
If c is even, then $\gcd(2, c) = 2 \nmid 315$ is false

Thus, for odd c, there are integer solutions. For even c, there aren't.