1. We want to find the integers n that satisfy $3n+2 \mid 5n^2+6n+4$ and we have

$$3n+2 \mid 5n^2+6n+4 \Longrightarrow 3n+2 \mid 15n^2+18n+12 \Longrightarrow 3n+2 \mid (5n+3)\cdot(3n+2)-(15n^2+18n+12) \Longrightarrow 3n+2 \mid n-6.$$

To have $3n+2 \mid n-6$, we must have $|3n+2| \leq |n-6|$ and clearly this inequality is not satisfied unless $-4 \leq n \leq 1$. Now, we plug in all the values of n in this interval to check if the expression $\frac{5n^2+6n+4}{3n+2}$ is an integer or not and we see that $\{-4,-2,-1,0,1\}$ are the values of n that makes it an integer.

2. We first prove that the cube of an integer n is always of the form $9k$ or $9k+1$ or $9k+8$. To prove this, we write $n = 3k+r$ for some $r \in \{0,1,2\}$ and we prove our claim for each value of r separately.

$$\begin{aligned}
&(3k)^3 = 27k^3 = 9\cdot(3k^3)\\
&(3k+1)^3 = 27k^3+27k+9k+1 = 9\cdot(3k^3+3k^2+k)+1\\
&(3k+2)^3 = 27k^3+54k+36k+8 = 9\cdot(3k^3+6k^2+4k)+8.
\end{aligned}$$

Next, we observe that the sum of three numbers of the form $9k+1$ or $9k+8$ can never be of the form $9k$ and therefore at least one of the numbers a^3, b^3, and c^3 must be of the form $9k$, which means at least one of the numbers a, b, and c must be of the form $3k$, so abc is divisible by 3.

3. First assume that the integers x, y satisfy $\gcd(x,y) = g$ and $xy = s$, then we can write $x = g\cdot k, y = g\cdot l$ and we have $s = xy = g^2\cdot(kl)$ is divisible by g^2.

Next, assume we have $g^2 \mid s$. Then $s = g^2\cdot k$ for some k and the integers $x = g, y = g\cdot k$ will satisfy $\gcd(x,y) = g, xy = s$ as desired.

4.

$$\begin{aligned}
1378 &= 1\cdot 1259+119\\
1259 &= 10\cdot 119+69\\
119 &= 1\cdot 69+50\\
69 &= 1\cdot 50+19\\
50 &= 2\cdot 19+12\\
19 &= 1\cdot 12+7\\
12 &= 1\cdot 7+5\\
7 &= 1\cdot 5+2\\
5 &= 2\cdot 2+1\\
2 &= 2\cdot 1+0.
\end{aligned}$$

So, we have $\gcd(1378,1259) = 1$ and we have

$$\begin{aligned}
1 &= 5-2\cdot 2\\
&= 5-2\cdot(7-1\cdot 5) = -2\cdot 7+3\cdot 5\\
&= -2\cdot 7+3\cdot(12-1\cdot 7) = 3\cdot 12-5\cdot 7\\
&= 3\cdot 12-5\cdot(19-1\cdot 12) = -5\cdot 19+8\cdot 12\\
&= -5\cdot 19+8\cdot(50-2\cdot 19) = 8\cdot 50-21\cdot 19\\
&= 8\cdot 50-21\cdot(69-1\cdot 50) = -21\cdot 69+29\cdot 50\\
&= -21\cdot 69+29\cdot(119-1\cdot 69) = 29\cdot 119-50\cdot 69\\
&= 29\cdot 119-50\cdot(1259-10\cdot 119) = -50\cdot 1259+529\cdot 119\\
&= -50\cdot 1259+529\cdot(1378-1\cdot 1259) = 529\cdot 1378-579\cdot 1259.
\end{aligned}$$

5.

$$\begin{aligned}
\gcd(2n-1, 9n+4) &= \gcd(2n-1, 9n+4-4\cdot(2n-1))\\
&= \gcd(2n-1, n+8)\\
&= \gcd(2n-1-2\cdot(n+8), n+8)\\
&= \gcd(-17, n+8)
\end{aligned}$$

can be only 1 or 17 and indeed both of these values are possible: $\gcd(2n-1, 9n+4) = 1$ when $n = 1$ and $\gcd(2n-1, 9n+4) = 17$ when $n = 9$.

6.

$$\begin{aligned}
\gcd(2^{19}+1, 2^{96}+1) &= \gcd(2^{19}+1, 2^{96}+1-2^{77}\cdot(2^{19}+1)) \\
&= \gcd(2^{19}+1, 1-2^{77}) \\
&= \gcd(2^{19}+1, 1-2^{77}+2^{58}\cdot(2^{19}+1)) \\
&= \gcd(2^{19}+1, 2^{58}+1) \\
&= \gcd(2^{19}+1, 2^{58}+1-2^{39}\cdot(2^{19}+1)) \\
&= \gcd(2^{19}+1, 1-2^{39}) \\
&= \gcd(2^{19}+1, 1-2^{39}+2^{20}\cdot(2^{19}+1)) \\
&= \gcd(2^{19}+1, 2^{20}+1) \\
&= \gcd(2^{19}+1, 2^{20}+1-2\cdot(2^{19}+1)) \\
&= \gcd(2^{19}+1, -1) \\
&= 1.
\end{aligned}$$

7.

$$\begin{aligned}
\gcd(n!+1, (n+1)!+1) &= \gcd(n!+1, (n+1)!+1-(n+1)\cdot(n!+1)) \\
&= \gcd(n!+1, -n) \\
&= \gcd(n!+1-(n-1)!\cdot n, -n) \\
&= \gcd(1, -n) \\
&= 1.
\end{aligned}$$