22. First we check for $p = 5$ that the number $|p^4 - 86| = 539 = 7^2 \cdot 11$ is not a prime, so we can assume $p \neq 5$. Since $1^4 \equiv 2^4 \equiv 3^4 \equiv 4^4 \equiv 1 \pmod 5$, we have $p^4 - 86 \equiv 0 \pmod 5$. For $|p^4 - 86|$ to be a prime, we must have either $p^4 - 86 = 5$ or $p^4 - 86 = -5$ because 5 is the only prime which is 0 modulo 5. The former case is not satisfied by an integer p while the latter cases gives $p = 3$.

23. We observe

$$\begin{aligned} 0^4 \equiv 2^4 \equiv 4^4 \equiv 6^4 \equiv 8^4 \equiv 10^4 \equiv 12^4 \equiv 14^4 \equiv 0 \pmod{16} \\ 1^4 \equiv 3^4 \equiv 5^4 \equiv 7^4 \equiv 9^4 \equiv 11^4 \equiv 13^4 \equiv 15^4 \equiv 1 \pmod{16}, \end{aligned}$$

and hence $a_i^4 \equiv 0$ or $1 \pmod{16}$ for every $i = 1, 2, \cdots, 14$. Then, we clearly have $a_1^4 + a_2^4 + \cdots + a_{14}^4 \not\equiv 15 \pmod{16}$, so there is no integer solution.

24. We claim that the only integer solution is $(0, 0, 0)$. Now suppose there is another solution (x_0, y_0, z_0). Then, clearly (x_1, y_1, z_1) is also a solution where we define $x_1 = \frac{x_0}{\gcd(x_0,y_0,z_0)}$, $y_1 = \frac{y_0}{\gcd(x_0,y_0,z_0)}$, and $z_1 = \frac{z_0}{\gcd(x_0,y_0,z_0)}$. This solution (x_1, y_1, z_1) also satisfies $\gcd(x_1, y_1, z_1) = 1$ sicne we cancelled out the common factors of x_0, y_0, z_0.

First we observe that none of the numbers x_1, y_1, z_1 can be 0 because it would make the other numbers 0 as well and we must have a different solution than $(0, 0, 0)$.

Since $3z_1^2 = x_1^2 + y_1^2$ must be divisible by 3 and a square can only be 0 or 1 modulo 3, we must have $x_1^2 \equiv y_1^2 \equiv 0 \pmod 3$. That means both x_1 and y_1 are divisible by 3 and hence $3z_1^2 = x_1^2 + y_1^2$ must be divisible 9. So, we must also have z_1 divisible by 3 which makes $\gcd(x_1, y_1, z_1) \neq 1$, a contradiction.

25. For $a \equiv b \pmod n$, we can write $a = kn + b$ for an integer k and then we have

$$\gcd(a, b) = \gcd(kn + b, n) = \gcd(kn + b - kn, n) = \gcd(b, n).$$

26. Since $\gcd(a, n) = 1$, we can talk about the inverse $a^{-1} \pmod n$. Multiplying the both sides of the second congruence by $a^{-1} \pmod n$, we get

$$a^k \equiv b^{k+1}a^{-1} \pmod n \Longrightarrow b^k \equiv b^{k+1}a^{-1} \pmod n$$

Also,

$$\gcd(a, n) = 1 \Longrightarrow \gcd(a^k, n) = 1 \Longrightarrow \gcd(b^k, n) = 1$$

means we can divide both sides of last congruence we obtained by b^k. After that we have $1 \equiv ba^{-1} \pmod n$ and this gives $b \equiv a \pmod n$ by multiplying both sides of the congrunce by $a \pmod n$.

This won't be true without $\gcd(a, n) = 1$. A counter-example is given by $n = 9, a = 3, b = 6$, and $k = 2$.

27. It is enough to prove that $2, 3$, and 5 divides $n^5 - n$ for every integer n. We have

$$0^5 - 0 \equiv 1^5 - 1 \equiv 0 \pmod 2,$$

so $2 \mid n^5 - n$ for every n, and

$$0^5 - 0 \equiv 1^5 - 1 \equiv 2^5 - 2 \equiv 0 \pmod 3,$$

so $3 \mid n^5 - n$ for every n, and

$$0^5 - 0 \equiv 1^5 - 1 \equiv 2^5 - 2 \equiv 3^5 - 3 \equiv 4^5 - 4 \equiv 0 \pmod 5,$$

so $5 \mid n^5 - n$ for every n.

28. (This is a little bit difficult and long) We are looking for an n that satisfies $n^2 \equiv 444 \pmod{1000}$. We begin with rewriting this congruence by splitting into two congruences modulo prime powers

$$n^2 \equiv 444 \equiv 69 \pmod{125} \text{ and } n^2 \equiv 444 \equiv 4 \pmod 8.$$

Trying all possible values of $n \in \mathbb{Z}_8$, we find $n \equiv 2$ or $6 \pmod 8$ for the second congruence. We will solve the first congruence step by step: first in $\mathbb{Z}_5$, then in $\mathbb{Z}_{25}$, and finally in $\mathbb{Z}_{125}$.

In $\mathbb{Z}_5$, we need

$$n^2 \equiv 69 \equiv 4 \pmod 5$$

which has the solutions $n \equiv 2 \pmod 5$ and $n \equiv 3 \pmod 5$. Since we were only asked to find one such integer n, we don't have to find all solutions and we can continue with $n \equiv 2 \pmod 5$ to reach a solution in $\mathbb{Z}_{125}$. Now we replace n with $5k+2$ in the first congruence.

In $\mathbb{Z}_{25}$, we need

$$(5k+2)^2 \equiv 25k^2+20k+4 \equiv 69 \equiv 19 \pmod{25} \Longrightarrow 20k \equiv 15 \pmod{25} \Longrightarrow 4k \equiv 3 \pmod 5 \Longrightarrow k \equiv 2 \pmod 5.$$

Now we replace n with $5k+2 = 5(5l+2)+2 = 25l+12$ in the first congruence.

Finally in $\mathbb{Z}_{125}$, we are solving

$$(25l+12)^2 \equiv 625l^2+600l+144 \equiv 69 \pmod{125} \Longrightarrow 100l \equiv 50 \pmod{125} \Longrightarrow 4l \equiv 2 \pmod 5 \Longrightarrow l \equiv 3 \pmod 5$$

and we have $n = 25l+2 = 25(5m+3)+12 = 125m+87$, i.e. $n \equiv 87 \pmod{125}$.

Now, we need to find an n satisfying both of the congruences $x \equiv 87 \pmod{125}$ and $n \equiv 2$ or $6 \pmod 8$. For $n \equiv 6 \pmod 8$, we have $n = 462$ satisfying both congruences. Indeed we have $462^2 = 213444$.

The last four digits cannot be 4 because it would mean

$$n^2 \equiv 4444 \pmod 1)0000 \Longrightarrow n^2 \equiv 4444 \equiv 12 \pmod{16},$$

but a square can never be $12 \pmod{16}$.

29. We can solve the given congruences using the standard techniques, but there is an easier method for this problem. Adding one to both sides of these congruences, we find

$$\begin{aligned} n+1 &\equiv 0 \pmod 3 \\ n+1 &\equiv 0 \pmod 5 \\ n+1 &\equiv 0 \pmod 7 \\ n+1 &\equiv 0 \pmod{11} \end{aligned}$$

which can be solved as $n+1 \equiv 0 \pmod{3 \cdot 5 \cdot 7 \cdot 11}$, i.e. $n \equiv -1 \pmod{1155}$. So, the three smallest positive integers satisfying them wil be $1155-1, 2 \cdot 1155 - 1$, and $3 \cdot 1155 - 1$.

30. We first split the given congruences into three congruences:

$$\begin{aligned} x^3 &\equiv 1 \pmod 3 \\ x^3 &\equiv 1 \pmod 7 \\ x^3 &\equiv 1 \pmod{13}. \end{aligned}$$

As $0^3 \equiv 0 \pmod 3$; $1^3 \equiv 1 \pmod 3$; $2^3 \equiv 2 \pmod 3$, we have only one solution in $\mathbb{Z}_3$.

Since $0^3 \equiv 0 \pmod 7$; $1^3 \equiv 1 \pmod 7$; $2^3 \equiv 1 \pmod 7$; $3^3 \equiv 6 \pmod 7$; $4^3 \equiv 1 \pmod 7$; $5^3 \equiv 6 \pmod 7$; $6^3 \equiv 6 \pmod 7$, we have three solutions in $\mathbb{Z}_7$.

Similarly, we will have three solutions in $\mathbb{Z}_{13}$ as well. By the Chinese Remainder Theorem, there are 9 solutions in $\mathbb{Z}_{273}$.

31. Let's split the congruence classes in $\mathbb{Z}_{2022}$ into 1012 groups:

$$(1, 2021), (2, 2020), (3, 2019), \cdots (1010, 1012), (0), (1011).$$

To not have $a_i \equiv a_j \pmod{2022}$, each a_i must correspond to a different congruence class. Each of the 1013 numbers $a_1, a_2, \cdots, a_{1013}$ correspond a congruence class in one of these 1012 groups. Since $1013 > 1012$, two of the given numbers must correspond to the numbers in the same group but that means these two numbers will satisfy $a_i \equiv -a_j \pmod{2022}$.

32. Since we can replace each number with any representative of its congruence class in $\mathbb{Z}_m$ in a summation modulo m, we have

$$r_0 + r_1 + \cdots + r_{m-1} \equiv 0 + 1 + \cdots + m - 1 \equiv \frac{(m-1)m}{2} \pmod{m}.$$

If $m = 2k+1$ is odd, then we have

$$\frac{(m-1)m}{2} =\equiv \frac{2k(2k+1)}{2} \equiv k(2k+1) \equiv 0 \pmod{2k+1}$$

and if $m = 2k$ is even, then we have

$$\frac{(m-1)m}{2} \equiv \frac{(2k-1)2k}{2} = 2k^2 - k \equiv k \pmod{2k}.$$

33. Clearly, given m numbers form a complete residue system modulo m if and only if they are all distinct modulo m.

If $\gcd(c, m) = 1$, then we have $cr_i \not\equiv cr_j \pmod{m}$ for every $i \neq j$ because $c(r_i - r_j) \equiv 0 \pmod{m}$ requires $r_i - r_j \equiv 0 \pmod{m}$ by cancelling out c from both sides of the congruence and we know that $r_i \not\equiv r_j \pmod{m}$ since $\{r_0, r_1, \cdots, r_{m-1}\}$ is a complete residue system modulo m.

Assume now that $\gcd(c, m) = k \neq 1$. Since $\{r_0, r_1, \cdots, r_{m-1}\}$ is a complete residue system modulo m, we can find some $i \neq j$ such that $r_i \equiv \frac{m}{k} \pmod{m}$ and $r_j \equiv 0 \pmod{m}$. Then we will have $cr_i \equiv cr_j \equiv 0 \pmod{m}$ and therefore $\{cr_0, cr_1, \cdots, cr_{m-1}\}$ cannot be a complete residue system.

34. (a) $\gcd(a, p^k) = 1$ is equivalent to $p \nmid a$. We should count all the values in $\mathbb{Z}_{p^k}$ (there are p^k of them) except $0, p, 2 \cdot p, \cdots, p^{k-1} \cdot p$ (there are p^{k-1} of them), so there are exactly $p^k - p^{k-1} = p^{k-1} \cdot (p-1)$ values of $a \in \mathbb{Z}_{p^k}$ satisfying $\gcd(a, p^k) = 1$.

(b) $\gcd(a, n) = 1$ is equivalent to $\gcd(a, p_i^{\alpha_i}) = 1$ for every $i = 1, 2, \cdots, k$. From the previous part there are $p_i^{\alpha-1} \cdot (p_i - 1)$ values of a in $\mathbb{Z}_{p_i^{\alpha_i}}$ satisfying $\gcd(a, p_i^{\alpha_i}) = 1$. By the Chinese Remainder Theorem, there are $\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1} \cdot (p_i - 1)$ values of a in $\mathbb{Z}_n = \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$ satisfying $\gcd(a, n) = 1$, i.e. $\gcd(a, p_i^{\alpha_i}) = 1$ for every $i = 1, 2, \cdots, k$.