35. By Fermat's theorem we have $n^p \equiv n \pmod p$ while Wilson's theorem gives $(p-1)! \equiv -1 \pmod p$. So, we find

$$n^p + n \cdot (p-1)! \equiv n + n \cdot (-1) \equiv 0 \pmod p.$$

36. We observe that $10^k \equiv 4 \pmod 6$ for $k \geq 1$ as $10^1 \equiv 4 \pmod 7, 10^2 \equiv 4^2 \equiv 4 \pmod 7, \ldots$

From Fermat's theorem, we have $10^{6k+4} \equiv \left(10^6\right)^k \cdot 10^4 \equiv 10^4 \equiv 4 \pmod 7$, so we have

$$10^{10^0} + 10^{10^1} + 10^{10^2} + \cdots + 10^{10^{10}} \equiv 3 + 4 + 4 + 4 + \cdots + 4 \equiv 43 \equiv 1 \pmod 7.$$

37. Let's write $x = \overline{a_1 a_2 \cdots a_{2021}}$, then we have

$$\overline{a_1 a_2 \cdots a_{2022}} = 10x + a_{2022}$$

and

$$\overline{a_{2022} a_1 a_2 \cdots a_{2021}} = 10^{2021} \cdot a_{2022} + x.$$

If 7 divides $\overline{a_1 a_2 \cdots a_{2022}}$, then we have

$$10x + a_{2022} \equiv 0 \pmod 7 \Longrightarrow a_{2022} \equiv -10x = 4x \pmod 7.$$

Now,

$$10^{2021} \cdot a_{2022} + x \equiv \left(10^6\right)^{336} \cdot 10^5 \cdot 4x + x \equiv 1 \cdot 5 \cdot 4x + x \equiv 21x \equiv 0 \pmod 7,$$

i.e. $\overline{a_{2022} a_1 a_2 \cdots a_{2021}}$ is also divisible by 7.

38. We first observe that $n \not\equiv 0 \pmod{11}$ because this would require $5^n \equiv 0 \pmod{11}$, which is clearly not possible.

We then have $n^5 \equiv \pm 1 \pmod{11}$ because $\left(n^5\right)^2 \equiv n^{10} \equiv 1 \pmod{11}$ by Fermat's theorem (and we know that $x^2 \equiv 1 \pmod p$ has two solutions $x \equiv \pm 1 \pmod p$ for prime p).

So, we should also have $5^n \equiv \pm 1 \pmod{11}$. We observe $5^{5k} \equiv \left(5^5\right)^k \equiv 1 \pmod{11}$ and hence

$$\begin{aligned} 5^{5k+1} &\equiv 5 \pmod{11} \\ 5^{5k+2} &\equiv 3 \pmod{11} \\ 5^{5k+3} &\equiv 4 \pmod{11} \\ 5^{5k+4} &\equiv 9 \pmod{11}. \end{aligned}$$

That means $5^n \equiv -1 \pmod{11}$ is not possible and for $5^n \equiv 1 \pmod{11}$ we must have $n \equiv 0 \pmod 5$.

We should also have $n^5 \equiv -1 \pmod{11}$ now. Writing down all possible values of n modulo 11,i.e.

$$1^5 \equiv 1, 2^5 \equiv -1, 3^5 \equiv 1, 4^5 \equiv 1, 5^5 \equiv 1, 6^5 \equiv -1, 7^5 \equiv -1, 8^5 \equiv -1, 9^5 \equiv 1, (10)^5 \equiv -1 \pmod{11},$$

we find out that we must have $n \equiv 2, 6, 7, 8,$ or $10 \pmod{11}$.

We can combine the possible values of n modulo 11 with $n \equiv 0 \pmod 5$ with the Chinese Remainder Theorem and we find

$$n \equiv 10, 30, 35, 40, 50 \pmod{55}.$$

39. Using Wilson's theorem, we have

$$\begin{aligned} (p-1)! \equiv -1 \pmod p &\Longrightarrow (p-1) \cdot (p-2)! \equiv -1 \pmod p \\ &\Longrightarrow (-1) \cdot (p-2)! \equiv -1 \pmod p \\ &\Longrightarrow (p-2)! \equiv 1 \pmod p \end{aligned}$$

and

$$\begin{aligned} (p-2)! \equiv 1 \pmod p &\Longrightarrow (p-2) \cdot (p-3)! \equiv 1 \pmod p \\ &\Longrightarrow (-2) \cdot (p-3)! \equiv 1 \pmod p \\ &\Longrightarrow \frac{p-1}{2} \cdot (-2) \cdot (p-3)! \equiv \frac{p-1}{2} \pmod p \\ &\Longrightarrow (1-p) \cdot (p-3)! \equiv \frac{p-1}{2} \pmod p \\ &\Longrightarrow (p-3)! \equiv \frac{p-1}{2} \pmod p. \end{aligned}$$

40. We know that $7 \cdot 23 \cdot q$ is a Carmichael number if and only if

$$\begin{aligned} 7-1 &\mid 7 \cdot 23 \cdot q - 1 \\ 23-1 &\mid 7 \cdot 23 \cdot q - 1 \\ q-1 &\mid 7 \cdot 23 \cdot q - 1 \end{aligned}$$

The last condition $q-1 \mid 7 \cdot 23 \cdot q - 1$, i.e. $7 \cdot 23 \cdot q - 1 \equiv 0 \pmod{q-1}$ gives

$$7 \cdot 23 - 1 \equiv 0 \pmod{q-1} \Longrightarrow 160 \equiv 0 \pmod{q-1}.$$

The primes q satisfying the last congruence are only $q = 2, 5, 11, 17, 41$, and 161. Checking the other two divisibility conditions with these values of q, we see that only $q = 41$ satisfies them both.

41. We have $7^2 \equiv 49 \equiv -1 \pmod{25}$ and therefore, $7^{25} \equiv \left(7^2\right)^{12} \cdot 7 \equiv (-1)^{12} \cdot 7 \equiv 7 \pmod{25}$.

42. Since n passes the base a-test and the base b-test, we have $a^n \equiv a \pmod{a}$ and $b^n \equiv b \pmod{n}$. Therefore, we have

$$(ab)^n \equiv a^n \cdot b^n \equiv ab \pmod{n},$$

i.e. n passes the base ab-test.

43. We prove the first part by contradiction. Assume n passes the base a-test and the base ab-test while failing the base b-test. We have

$$(ab)^n \equiv ab \pmod{n} \Longrightarrow a^n \cdot b^n \equiv ab \pmod{n}$$

from the base ab-test. Since n is passing the base a-test, we can replace a^n with a in the congruence above and we get

$$a \cdot b^n \equiv ab \pmod{n}.$$

Cancelling out a from both sides of the congruence, we get

$$b^n \equiv b \pmod{n/\gcd(n, a)} \Longrightarrow b^n \equiv b \pmod{n}$$

for $\gcd(a, n) = 1$, i.e. n passes the base b-test, a contradiction.

This is not necessarily true when $\gcd(a, n) \neq 1$. A counter-example is $a = 4, b = 3, n = 4$.

44. Let's write $f(x) = x^3 + 3x^2 + x + 3$. We should first consider the congruence $f(x) \equiv 0 \pmod{5}$. We have

$$\begin{aligned} f(0) &\equiv 3 \pmod{5} \\ f(1) &\equiv 3 \pmod{5} \\ f(2) &\equiv 0 \pmod{5} \\ f(3) &\equiv 0 \pmod{5} \\ f(4) &\equiv 4 \pmod{5}. \end{aligned}$$

So, the only solutions are $x \equiv 2, 3 \pmod{5}$.

Since $f'(x) = 3x^2 + 6x + 1$ and $f'(2) \equiv 0 \pmod{5}$, either all possible lifts of $2 \pmod{5}$ to $\mathbb{Z}_{25}$ are solutions to the original congruence or none of them is a solution. We just check $x = 2$ to see that it is a solution, i.e. $f(2) \equiv 0 \pmod{25}$. So, all the other lifts will be solutions as well. We have $x \equiv 2, 7, 12, 17, 22 \pmod{25}$.

As $f'(3) \not\equiv 0 \pmod{5}$, a unique lift of $3 \pmod{5}$ will be a solution to the original congruence. We compute $f(18) \equiv 0 \pmod{25}$, so that should be the unique lift.

To summarize, the solutions are $x \equiv 2, 7, 12, 17, 18, 22 \pmod{25}$.

45. There are many ways to prove this. One way is to show that $\phi(n)$ is even when $n > 2$ using the formula for $\phi(n)$. Here we prove using the different argument.

If u is a unit in $\mathbb{Z}_n$, then $\gcd(u, n) = 1 \Longrightarrow \gcd(-u, n) = 1$, i.e. $-u$ is a unit as well. Pairing up u with $-u \equiv n - u \pmod{n}$, we get even number of units. The only problem can occur when we pair up the same element, i.e. when $u \equiv -u \pmod{n}$. However, this means $u \equiv n/2 \pmod{n}$ and $\gcd(n, n/2) = 1$ only when $n = 2$.

To demonstrate this, here is an example: when $n = 28$, we write the units as

$$(1, 27), (3, 25), (5, 23), (9, 19), (11, 17), (13, 15).$$