8. First, we should find some integers x an y such that $1990x + 173y = \gcd(1990, 173)$ using Euclidean algorithm.

$$\begin{aligned} 1990 &= 11 \cdot 173 + 87 \\ 173 &= 1 \cdot 87 + 86 \\ 87 &= 1 \cdot 86 + 1 \\ 86 &= 86 \cdot 1 + 0. \end{aligned}$$

So, we have $\gcd(1990, 173) = 1$ and we have

$$\begin{aligned} 1 &= 87 - 1 \cdot 86 \\ &= 87 - (173 - 1 \cdot 87) = 2 \cdot 87 - 1 \cdot 173 \\ &= 2 \cdot (1990 - 11 \cdot 173) - 1 \cdot 173 = 2 \cdot 1990 - 23 \cdot 173. \end{aligned}$$

Now, one solution to the equation is given by $x_0 = 11 \cdot 2 = 22$, and $y_0 = 11 \cdot (-23) = -253$, and the set of all solutions will be $\{(x, y) = (22 + 173m, -253 - 1990m) : m \in \mathbb{Z}\}$.

9. For the existence of a solution to $6x+10y = 1+15z$, we must have $\gcd(6, 10) \mid 1+15z$, which means z must be odd. Let's write $z = 2k+1$ for an integer k. Then, $1+15z = 30k+16$ and we need to solve $6x+10y = 30k+16$. One solution to this diophantine equation is $(x_0, y_0) = (5k+1, 1)$ and therefore the all solutions will be $x = 5k+1+5m, y = 1-3m$ for some integer m.

So, the set of all integer solutions to the diophantine equation $6x + 10y - 15z = 1$ is

$$\{(x, y, z) = (5k + 5m + 1, 1 - 3m, 2k + 1) : k, m \in \mathbb{Z}\}.$$

10. Since $\gcd(a, b) = 1$, there are infinitely many solutions to the equation $ax + by = c$. Let (x_0, y_0) be the solution with the minimum non-negative x_0 among all solutions. Since $(x_0 - b, y_0 + a)$ is also a solution and $x_0 - b < x_0$ and x_0 is the minimum among all non-negative values of x, we must have $x_0 - b \leq -1$, i.e. $x_0 \leq b - 1$. Then, we have

$$ab - a - b + 1 = (a - 1)(b - 1) \leq c = ax_0 + by_0 \leq a(b - 1) + by_0 = ab - a + by_0 \implies -b + 1 \leq by_0 \implies -1 < y_0$$

which means y_0 is non-negative and (x_0, y_0) is a solution in the non-negative integers.

11. For $n = 3$, both n and $n^2 + 2$ are primes, assume $n \neq 3$ now. For n to be prime other than 3, it shouldn't be divisible by 3. So, n is either of the form $3k + 1$ or $3k + 2$.

If $n = 3k + 1$, then $n^2 + 2 = (3k + 1)^2 + 2 = 9k^2 + 6k + 3 = 3 \cdot (3k^2 + 2k + 1)$ is divisible by 3. The only prime divisible by 3 is 3 itself, but $n^2 + 2 = 3$ gives $n = \pm 1$ which is not a prime.

If $n = 3k + 2$, then $n^2 + 2 = (3k + 2)^2 + 2 = 9k^2 + 12k + 6 = 3 \cdot (3k^2 + 4k + 2)$ is divisible by 3. The only prime divisible by 3 is 3 itself, but $n^2 + 2 = 3$ gives $n = \pm 1$ which is not a prime.

12. Since 2 divides the left side of the equation, it must divide the right side $3b^3$ as well. So, b must be even and we can write $b = 2b_1$ for a positive integer b_1.

Similarly, 3 must divide the left side $2a^2$. So, a must be divisible by 3 and we can write $a = 3a_1$ for a positive integer a_1.

Replacing b with $2b_1$ and a with $3a_1$, we get

$$2(3a_1)^2 = 3(2b_1)^3 \implies 18a_1^2 = 24b_1^3 \implies 3a_1^2 = 4b_1^3.$$

Using the same idea again, we can write $a_1 = 2a_2$ and $b_1 = 3b_2$ for some positive integers a_2 and b_2. Replacing a_1 and b_1, we get

$$3(2a_2)^2 = 4(3b_2)^3 \implies 12a_2^2 = 108b_2^3 \implies a_2^2 = 9b_2^3.$$

It can be easily seen that the smallest positive integer solution to the last equation is $a_2 = 3, b_2 = 1$. This gives $a_1 = 6, b_1 = 3$ and $a = 18, b = 6$.

13. First, we observe that $\gcd(n, n+17) = \gcd(n, 17)$ is either 1 or 17.

If $\gcd(n, n+17) = 1$, then they are coprime and we know that the product of two positive coprime integers is a square if and only if both of them are squares. That means both n and $n+17$ must be squares (or both $-n$ and $-n-17$ must be squares if n is negative). For $a^2 = n$ and $b^2 = n+17$, we have $17 = b^2 - a^2 = (b-a)(b+a)$. In this case,we must have $b - a = 1, b + a = 17$ which gives $a = 8, b = 9$ and we find $n = 64$. For $a^2 = -n$ and $b^2 = -n - 17$, similarly we find $n = -81$.

If $\gcd(n, n+17) = 17$, then n is divisible by 17 and we can write $n = 17k$ for an integer k. Now we write

$$n(n+17) = (17k) \cdot (17k+7) = 17^2 \cdot k(k+1)$$

and $17^2 \cdot k(k+1)$ is a square if and only if $k(k+1)$ is a square. Similar to the previous case, it is to see that $k = -1$ and $k = 0$ are the only possible values that make $k(k+1)$ a square. So, we have $n = -17$ and $n = 0$ from this case.

14. This is very similar to the infinitude of the primes of the form $4k+3$.

Suppose there are finitely many of them, say $p_1, p_2, \cdots, p_n$. Then, we consider $m = 3p_1p_2\cdots p_n - 1$. Clearly m is of the form $3k+2$ and an integer of the form $3k+2$ must have a prime divisor of the form $3k+2$ since we can never have a number of the form $3k+2$ by multiplying some numbers of the form $3k$ and $3k+1$. So, m must have a prime divisor of the form $3k+2$, but $p_1, p_2, \cdots, p_n$ are all such primes and they don't divide m which is a contradiction.