Term Test

Due June 23, 7:00 PM on Crowdmark

Part 0. Before beginning the test, you must do the followings.

- Take your student number and replace all 0's with 1's in this number.
- Call the 3-digit integer formed by the last three digits of this number A.
- Call the 2-digit prime number which is closest to the 2-digit integer formed by the last two digits of this number B. (If there are more than one such prime, then choose whichever you want)
- Make sure to write your student number, A, and B on top of each page of your solutions.

For example, for the student number 10072634107, the number A will be 117 and the number B will be 17 while for the student number 1221254720, we will have $A = 721$ and $B = 19$ or $B = 23$, depending on your choice.

Part A. For the questions in this part, you do not need to justify your answers. (2 points each).

1. If $5^n \| A!$, what is n?

2. What are the two largest positive integers n satisfying $n - 1 \mid n^2 - A$?

3. What is the smallest positive integer n such that $315 \cdot n$ is cube of an integer?

4. What is the smallest positive integer n that is divisible by 17 and both of its last two digits are 1? (Hint: mod 100)

5. Write a set $\{a_1, a_2, \cdots, a_7\}$ which is a complete set of residues modulo 7 such that all of the numbers a_i are even.

Part B. For the questions in this part, you must show all your work and justify your answer to get a full mark (6 points each).

1. Let $p \neq 2, 3$ be a prime. Prove that $p^2 \equiv 1 \pmod{24}$.

2. Find all the possible values of $\gcd(2B + n, 4B + 3n)$ (Don't forget to give an example of n for each possible value).

3. Solve the system of linear congruences

$$\begin{aligned} x &\equiv 1 \pmod{3} \\ x &\equiv 2 \pmod{5} \\ 2x &\equiv -1 \pmod{B}. \end{aligned}$$

   Make sure to show all your steps finding the solution.

4. A positive integer n is called *oddly powerful* if for every prime p dividing n, we have $p^{2a_p+1} \| n$ for some non-negative integer a_p (in other words, all the exponents must be odd in the prime factorisation of n).

   For example, $270 = 2 \cdot 3^3 \cdot 5$ and $210 = 2 \cdot 3 \cdot 5 \cdot 7$ are oddly powerful while $36 = 2^2 \cdot 3^2$ and $350 = 2 \cdot 5^2 \cdot 7$ are not.

   Prove that there are infinitely many positive integers a such that none of the integers $a, a + 1, a + 2$ are oddly powerful.