Quadratic Residue Number System for Fast Fourier Transform: Theory, Implementation, and Applications in Digital Signal Processing

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Abstract

The Quadratic Residue Number System (QRNS) provides an innovative approach to computing the Fast Fourier Transform (FFT) by leveraging modular arithmetic and the algebraic properties of quadratic residues. Traditional FFT implementations using floating-point arithmetic suffer from round-off errors and limited precision, which can accumulate and degrade signal quality in critical applications. This project presents a comprehensive treatment of QRNS-based FFT computation, including theoretical foundations rooted in number theory, algorithmic development, practical implementation considerations, and performance analysis. We demonstrate that QRNS enables exact integer arithmetic for FFT operations, eliminating floating-point errors entirely while maintaining computational efficiency. Through detailed mathematical analysis and experimental validation, we show how QRNS can be applied to digital signal processing applications requiring high precision, such as cryptographic systems, medical signal analysis, and scientific computing. The implementation includes both theoretical algorithms and practical Python code, with performance comparisons against conventional FFT methods. Our results indicate that QRNS-FFT achieves superior accuracy with comparable computational complexity, making it particularly valuable for applications where numerical precision is paramount.

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1 Introduction and Motivation

The Fast Fourier Transform has become one of the most fundamental algorithms in digital signal processing since its popularization by Cooley and Tukey in 1965. Its efficiency in transforming signals between time and frequency domains, with computational complexity reduced from $O(N^2)$ to $O(N \log N)$, has enabled countless applications across communications, audio processing, image analysis, and scientific computing. However, conventional FFT implementations face a persistent challenge that becomes critical in precision-sensitive applications: the accumulation of numerical errors inherent in floating-point arithmetic.

Traditional FFT algorithms operate in the complex number field, utilizing floating-point representations that inherently introduce quantization errors at each computational step. While these errors may appear negligible in isolation, they accumulate through the recursive structure of the FFT algorithm, potentially degrading signal quality and introducing artifacts. In applications such as cryptographic computations, high-precision scientific simulations, or medical signal analysis where diagnostic decisions depend on signal accuracy, these accumulated errors can have serious consequences. Moreover, floating-point arithmetic introduces non-determinism across different hardware platforms and compiler optimizations, making reproducibility difficult to guarantee.

The Quadratic Residue Number System offers an elegant solution to these challenges by reformulating FFT computations entirely within the realm of modular integer arithmetic. Rather than approximating complex exponentials with finite-precision floating-point numbers, QRNS represents the roots of unity required for FFT through algebraic structures in finite fields. This approach eliminates round-off errors completely, as all operations are performed using exact integer arithmetic. The key insight underlying QRNS is that the essential algebraic properties needed for FFT computation, specifically the orthogonality relations of roots of unity, can be preserved in appropriately chosen modular arithmetic systems.

This project provides a comprehensive exploration of QRNS-based FFT computation, structured to build understanding from fundamental number-theoretic principles through practical implementation. We begin with the mathematical foundations, establishing the necessary background in modular arithmetic, quadratic residues, and algebraic number theory. The theoretical development demonstrates how FFT's complex arithmetic can be systematically mapped to modular operations while preserving correctness. We then present detailed algorithms for QRNS-FFT computation, analyzing their computational complexity and proving their equivalence to traditional FFT. The implementation section provides practical Python code with careful attention to efficiency and numerical stability within the integer domain. Finally, we evaluate QRNS-FFT performance through comprehensive experiments, comparing accuracy, speed, and memory usage against conventional methods across various signal lengths and applications.

The significance of this work extends beyond theoretical interest. As digital signal processing applications increasingly demand higher precision and reproducibility, particularly in fields like quantum computing simulation, radio astronomy, and biomedical engineering, QRNS-based methods offer practical advantages. Furthermore, the elimination of floating-point operations makes QRNS-FFT naturally suited for hardware implementations in field-programmable gate arrays (FPGAs) and application-specific integrated circuits (ASICs), where integer arithmetic can be implemented more efficiently than floating-point units. This project aims to make QRNS-FFT accessible to practitioners by providing both rigorous mathematical foundations and practical implementation guidance.

2 Mathematical Foundations and Number-Theoretic Background

Understanding QRNS-based FFT requires a solid foundation in several areas of mathematics, particularly modular arithmetic, algebraic number theory, and finite field theory. This section develops these foundations systematically, establishing the theoretical framework upon which QRNS-FFT is constructed.

2.1 Modular Arithmetic and Residue Systems

Modular arithmetic forms the basis of QRNS computation. Given a positive integer modulus m, two integers a and b are congruent modulo m, denoted $a \equiv b \pmod{m}$, if m divides their difference $(a - b)$. This relation partitions the integers into equivalence classes, and arithmetic operations can be performed on these classes. The set of integers modulo m, denoted $\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}$, forms a commutative ring under addition and multiplication modulo m.

For QRNS applications, we are particularly interested in prime moduli, where $\mathbb{Z}_p$ for prime p forms a finite field. In such fields, every non-zero element has a multiplicative inverse, which is crucial for division operations in signal processing. The multiplicative group $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$ has order $p-1$ and is cyclic, meaning there exists a generator g such that every element can be expressed as g^k for some integer k.

The Chinese Remainder Theorem provides a powerful tool for working with multiple moduli simultaneously. Given pairwise coprime moduli $m_1, m_2, \ldots, m_k$, the system of congruences $x \equiv a_i \pmod{m_i}$ for $i = 1, \ldots, k$ has a unique solution modulo $M = \prod_{i=1}^{k} m_i$. This allows us to represent integers in the range $[0, M)$ uniquely by their residues modulo each m_i, enabling parallel arithmetic operations.

2.2 Quadratic Residues and Legendre Symbols

An integer a is a quadratic residue modulo prime p if there exists an integer x such that $x^2 \equiv a \pmod{p}$. The set of quadratic residues modulo p contains exactly $(p-1)/2$ non-zero elements when p is an odd prime. The Legendre symbol provides a convenient notation for determining whether an integer is a quadratic residue:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p \end{cases} \tag{1}$$

The Legendre symbol satisfies several important properties that make it computationally tractable. It is multiplicative: $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. Euler's criterion provides a computational method: $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$. The law of quadratic reciprocity, one of the deepest results in elementary number theory, relates the Legendre symbols for different primes and enables efficient computation.

For QRNS-FFT, we exploit the fact that -1 is a quadratic residue modulo p if and only if $p \equiv 1 \pmod 4$. This allows us to construct representations of complex numbers using pairs of integers modulo appropriately chosen primes. Specifically, when $p \equiv 1 \pmod 4$, we can find an element i in $\mathbb{Z}_p$ satisfying $i^2 \equiv -1 \pmod p$, providing a modular analogue of the imaginary unit.

2.3 Cyclotomic Polynomials and Roots of Unity

The Fast Fourier Transform fundamentally relies on the properties of roots of unity. An N-th root of unity is a complex number ω satisfying $\omega^N = 1$. The primitive N-th roots of unity are those for which N is the smallest positive integer satisfying this equation. In the complex numbers, the primitive N-th roots of unity are given by $\omega_k = e^{2\pi ik/N}$ for $\gcd(k, N) = 1$.

The cyclotomic polynomial $\Phi_N(x)$ is defined as the minimal polynomial whose roots are precisely the primitive N-th roots of unity:

$$\Phi_N(x) = \prod_{\substack{1 \le k \le N \\ \gcd(k,N)=1}} (x - e^{2\pi ik/N}) \tag{2}$$

These polynomials have integer coefficients and satisfy the factorization $x^N - 1 = \prod_{d|N} \Phi_d(x)$. Their degree is given by Euler's totient function: $\deg(\Phi_N) = \phi(N)$.

For QRNS-FFT, we work with roots of unity in finite fields rather than complex numbers. Given a prime p and a divisor N of $p-1$, the multiplicative group $\mathbb{Z}_p^*$ contains elements of order N. These elements serve as modular roots of unity. Specifically, if g is a primitive root modulo p (a generator of $\mathbb{Z}_p^*$), then $\omega = g^{(p-1)/N}$ is a primitive N-th root of unity modulo p, satisfying $\omega^N \equiv 1 \pmod p$ and $\omega^k \not\equiv 1 \pmod p$ for $0 < k < N$.

2.4 Field Extensions and Algebraic Number Representation

When working with complex FFT, we require arithmetic on complex numbers $a+bi$ where $i^2 = -1$. In QRNS, we construct an analogous structure using field extensions. Given a prime $p \equiv 1 \pmod 4$, we work in the quadratic extension field $\mathbb{F}_p[i]$ where $i^2 = -1$. Elements of this field are represented as ordered pairs (a, b) with $a, b \in \mathbb{Z}_p$, corresponding to $a + bi$.

Arithmetic in this extension field follows natural rules while maintaining everything in modular arithmetic. Addition is component-wise: $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2 \bmod p, b_1 + b_2 \bmod p)$. Multiplication uses the relation $i^2 = -1$: $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 - b_1b_2 \bmod p, a_1b_2 + a_2b_1 \bmod p)$. The multiplicative inverse of a non-zero element (a, b) is given by:

$$(a, b)^{-1} = \left(\frac{a}{a^2 + b^2 \bmod p}, \frac{-b}{a^2 + b^2 \bmod p} \right) \tag{3}$$

where division is performed using modular multiplicative inverses in $\mathbb{Z}_p$.

This construction provides an exact arithmetic system for complex number operations, eliminating all floating-point errors. The key requirement is selecting a prime p large enough to accommodate the range of values encountered in the computation without overflow. For FFT of length N on input signals with maximum absolute value M, we require $p > N \cdot M$ to prevent wrap-around in the modular arithmetic from affecting results.

2.5 Number-Theoretic Transform and Its Properties

The Number-Theoretic Transform (NTT) is a direct analogue of the Discrete Fourier Transform in modular arithmetic. Given a sequence $x[n]$ for $n = 0, 1, \ldots, N-1$ where N divides $p-1$ for prime p, the NTT is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n]\omega^{nk} \bmod p \tag{4}$$

where ω is a primitive N-th root of unity modulo p. The inverse transform is given by:

$$x[n] = N^{-1} \sum_{k=0}^{N-1} X[k]\omega^{-nk} \bmod p \tag{5}$$

where N^{-1} denotes the multiplicative inverse of N modulo p.

The NTT preserves the essential properties of the DFT that make FFT algorithms work efficiently. The convolution theorem holds: the NTT of a circular convolution equals the element-wise product of NTTs. The transform is invertible with the formula given above. Most importantly, the fast algorithms developed for DFT, including the Cooley-Tukey decimation-in-time and decimation-in-frequency algorithms, apply directly to NTT with identical computational complexity of $O(N \log N)$.

For QRNS-FFT with complex-valued signals, we extend the NTT to work in the field extension $\mathbb{F}_p[i]$. Each complex input value $x[n] = a[n] + ib[n]$ is represented as $(a[n], b[n]) \in \mathbb{F}_p[i]$, and all arithmetic operations during the transform are performed using the field extension arithmetic described previously. This maintains exact computation throughout the entire FFT process.

3 QRNS-FFT Algorithm Development and Analysis

Having established the mathematical foundations, we now develop the QRNS-based FFT algorithm systematically. This section presents the algorithm design, proves its correctness, analyzes its computational complexity, and discusses practical implementation considerations.

3.1 Algorithm Design and Structure

The QRNS-FFT algorithm follows the same divide-and-conquer strategy as the classical Cooley-Tukey FFT, but performs all operations in modular arithmetic within the field extension $\mathbb{F}_p[i]$. For a sequence of length $N = 2^m$, the algorithm recursively divides the transform into transforms of length $N/2$ until reaching base cases of length 1 or 2.

The key insight is that the FFT butterfly operation, which is the fundamental computational unit, can be expressed entirely using the arithmetic operations available in $\mathbb{F}_p[i]$. Given two values a and b, and a twiddle factor ω^k, the butterfly computes:

$$A = a + \omega^k \cdot b \tag{6}$$

$$B = a - \omega^k \cdot b \tag{7}$$

In QRNS, if $a = (a_r, a_i)$, $b = (b_r, b_i)$, and $\omega^k = (\omega_r, \omega_i)$, then we compute the complex multiplication $\omega^k \cdot b$ using field extension multiplication, followed by addition and subtraction using component-wise operations. All operations remain in modular arithmetic, guaranteeing exact results.

The complete algorithm proceeds as follows. First, we select an appropriate prime $p \equiv 1 \pmod 4$ such that N divides $p - 1$ and p is large enough to prevent overflow. We compute a primitive N-th root of unity ω modulo p by finding a generator g of $\mathbb{Z}_p^*$ and setting $\omega = g^{(p-1)/N}$. We also compute the element $i \in \mathbb{Z}_p$ satisfying $i^2 \equiv -1 \pmod p$ using the Tonelli-Shanks algorithm.

Next, we convert the input signal to the field extension representation. Each complex input value $x[n] = x_r[n] + ix_i[n]$ becomes $(x_r[n] \bmod p, x_i[n] \bmod p)$. If input values are integers, this is straightforward. For floating-point inputs, we must first scale and round them to integers, which introduces controlled quantization but maintains exact arithmetic thereafter.

The core recursive FFT then proceeds identically to the classical algorithm, but using QRNS arithmetic. We implement bit-reversal permutation for in-place computation, then perform $\log_2 N$ stages of butterfly operations with appropriate twiddle factors. Each stage combines results from the previous stage using butterflies with twiddle factors ω^k for various k.

Finally, we convert the output back from field extension representation to standard complex numbers if needed, or keep results in QRNS form for further processing. The inverse transform follows the same structure but uses ω^{-1} as the root of unity and includes a final scaling by $N^{-1} \bmod p$.

3.2 Correctness Proof and Mathematical Validation

To prove that QRNS-FFT correctly computes the Discrete Fourier Transform, we must show that the modular arithmetic operations preserve the mathematical relationships that define the DFT. The proof proceeds through several lemmas establishing key properties.

Lemma 3.1. *Let $p \equiv 1 \pmod 4$ be prime with $N \mid (p-1)$. The element $\omega = g^{(p-1)/N}$ where g is a primitive root modulo p is a primitive N-th root of unity modulo p.*

Proof. Since g has order $p-1$ in $\mathbb{Z}_p^*$, the element $\omega = g^{(p-1)/N}$ has order $(p-1)/\gcd((p-1)/N, p-1) = N$. Therefore $\omega^N \equiv 1 \pmod p$ and $\omega^k \not\equiv 1 \pmod p$ for $0 < k < N$, satisfying the definition of a primitive N-th root of unity. □

Lemma 3.2. *The element $i \in \mathbb{Z}_p$ satisfying $i^2 \equiv -1 \pmod p$ exists when $p \equiv 1 \pmod 4$.*

Proof. By Euler's criterion, -1 is a quadratic residue modulo p if and only if $(-1)^{(p-1)/2} \equiv 1 \pmod p$. When $p \equiv 1 \pmod 4$, we have $(p-1)/2$ is even, so $(-1)^{(p-1)/2} = 1$, confirming that -1 is a quadratic residue. The Tonelli-Shanks algorithm efficiently computes a square root. □

Theorem 3.3. *The QRNS-FFT algorithm computes the same result as the classical DFT when arithmetic is performed in $\mathbb{F}_p[i]$ with sufficiently large prime p.*

Proof. We prove by induction on the recursion depth. For the base case of length $N = 1$, the output equals the input trivially. For the recursive case, assume the algorithm correctly computes FFTs of length $N/2$. The classical FFT combines these with the formula:

$$X[k] = E[k] + \omega^k \cdot O[k], \quad X[k+N/2] = E[k] - \omega^k \cdot O[k] \tag{8}$$

where $E[k]$ and $O[k]$ are the FFTs of even and odd indexed inputs respectively. In QRNS-FFT, we perform the identical operations but in $\mathbb{F}_p[i]$. The field operations preserve the arithmetic relationships: addition in $\mathbb{F}_p[i]$ corresponds to complex addition, and multiplication in $\mathbb{F}_p[i]$ with $i^2 = -1$ corresponds to complex multiplication. Since ω is a primitive N-th root of unity modulo p, the twiddle factors satisfy the same algebraic relations as complex roots of unity. Therefore, the QRNS computation produces results congruent to the classical DFT modulo p. With p chosen large enough to avoid overflow, the results are exactly equal. $\square$

3.3 Computational Complexity Analysis

The computational complexity of QRNS-FFT is analyzed at multiple levels: arithmetic operation count, bit complexity, and practical runtime considerations.

At the arithmetic operation level, QRNS-FFT performs the same number of additions and multiplications as classical FFT. For length $N = 2^m$, there are $m = \log_2 N$ stages, each performing $N/2$ butterflies. Each butterfly requires one complex multiplication and two complex additions. Thus we have $(N/2)\log_2 N$ complex multiplications and $N \log_2 N$ complex additions. In the field extension $\mathbb{F}_p[i]$, each complex multiplication requires four modular multiplications and two modular additions in $\mathbb{Z}_p$, and each complex addition requires two modular additions. Therefore, the total count is $2N\log_2 N$ modular multiplications and $3N\log_2 N$ modular additions in $\mathbb{Z}_p$.

The bit complexity analysis is more nuanced. Modular arithmetic operations on numbers modulo prime p require $O(\log^2 p)$ bit operations for multiplication using standard algorithms, or $O(\log p \log\log p)$ using advanced techniques like Schönhage-Strassen multiplication. For addition, the cost is $O(\log p)$. With p chosen to satisfy $p > N \cdot M$ where M bounds input magnitudes, we have $\log p = O(\log N + \log M)$. This gives an overall bit complexity of $O(N \log N(\log N + \log M)^2)$ using standard multiplication, compared to $O(N \log N)$ arithmetic operations for floating-point FFT. However, floating-point operations themselves have non-trivial bit complexity and require more complex hardware.

In practice, the performance of QRNS-FFT depends heavily on implementation quality. Modern processors have highly optimized floating-point units, making classical FFT very fast. However, QRNS-FFT can leverage several optimization strategies. Using Montgomery reduction for modular multiplication reduces the cost of the expensive division operation. Precomputing and storing twiddle factors eliminates repeated exponentiation. Selecting primes with special structure, such as Mersenne primes or generalized Fermat primes, can accelerate modular reduction. For hardware implementation in FPGAs or ASICs, integer arithmetic is often more efficient than floating-point, potentially giving QRNS-FFT an advantage.

3.4 Prime Selection and Parameter Determination

Selecting appropriate parameters is crucial for correct and efficient QRNS-FFT computation. The prime modulus p must satisfy several constraints simultaneously.

First, p must be congruent to 1 modulo 4 to ensure -1 is a quadratic residue, enabling construction of $\mathbb{F}_p[i]$. Second, N must divide $p - 1$ to guarantee existence of primitive N-th roots of unity modulo p. Third, p must be large enough to prevent overflow: if input values have maximum absolute value M and the FFT has length N, intermediate values can reach magnitudes up to approximately $N \cdot M$, so we require $p > N \cdot M$ with some safety margin.

Finding primes satisfying these constraints can be done efficiently. We search for primes of the form $p = k \cdot N + 1$ where $k \equiv 0 \pmod 4$, which automatically ensures $p \equiv 1 \pmod 4$ and $N \mid (p-1)$. Starting from $p_{\min} = N \cdot M + 1$, we test odd integers of the correct form for primality using algorithms like Miller-Rabin. This typically finds a suitable prime quickly since primes have density approximately $1/\ln(n)$ near n.

For applications requiring multiple FFTs with different parameters, we can construct a table of suitable primes for common FFT lengths, precomputing associated parameters like primitive roots and roots of unity. This amortizes setup costs across many computations.

4 Implementation Details and Practical Considerations

Translating the theoretical QRNS-FFT algorithm into working code requires careful attention to numerous practical details. This section presents a complete implementation in Python, discusses optimization techniques, and addresses numerical considerations.

4.1 Core Implementation Structure

The implementation is organized into several modules handling different aspects of the computation. The modular arithmetic module provides efficient operations in $\mathbb{Z}_p$, including addition, multiplication, and multiplicative inverse computation. The field extension module implements arithmetic in $\mathbb{F}_p[i]$, building on the modular arithmetic primitives. The number-theoretic module handles tasks like prime finding, primitive root computation, and root of unity generation. Finally, the FFT module implements the actual transform algorithms.

For modular arithmetic, we implement operations with careful attention to avoiding overflow. Python's arbitrary-precision integers eliminate concerns about overflow in multiplication before reduction, but we still optimize using techniques like Montgomery reduction for repeated operations with the same modulus. The multiplicative inverse is computed using the extended Euclidean algorithm, which runs in $O(\log p)$ time.

Field extension arithmetic follows the formulas derived earlier. For multiplication of (a_1, b_1) and (a_2, b_2), we compute:

```python
def fe_mult(z1, z2, p):
    """Multiply two elements in F_p[i] where i^2 = -1"""
    a1, b1 = z1
    a2, b2 = z2
    real = (a1 * a2 - b1 * b2) % p
    imag = (a1 * b2 + a2 * b1) % p
    return (real, imag)
```

For division, we compute the multiplicative inverse:

```python
def fe_inv(z, p):
    """Compute multiplicative inverse in F_p[i]"""
    a, b = z
    norm = (a * a + b * b) % p
    norm_inv = mod_inv(norm, p)
    return ((a * norm_inv) % p, (-b * norm_inv) % p)
```

The primitive root finding is crucial for generating roots of unity. We implement a deterministic algorithm that tests small integers sequentially:

```python
def find_primitive_root(p):
    """Find a primitive root modulo prime p"""
    if p == 2:
        return 1

    # Factor p-1
    factors = prime_factors(p - 1)

    # Test small integers
    for g in range(2, p):
        is_primitive = True
        for factor in factors:
            if pow(g, (p - 1) // factor, p) == 1:
                is_primitive = False
                break
        if is_primitive:
            return g
    return None
```

The Tonelli-Shanks algorithm computes square roots modulo p when they exist:

```python
def tonelli_shanks(n, p):
    """Find r such that r^2 = n (mod p) using Tonelli-Shanks"""
    if pow(n, (p - 1) // 2, p) != 1:
        return None  # n is not a quadratic residue

    # Write p - 1 = Q * 2^S
    Q, S = p - 1, 0
    while Q % 2 == 0:
        Q //= 2
        S += 1

    # Find a quadratic non-residue z
    z = 2
    while pow(z, (p - 1) // 2, p) != p - 1:
        z += 1

    # Initialize
    M = S
    c = pow(z, Q, p)
    t = pow(n, Q, p)
    R = pow(n, (Q + 1) // 2, p)

    while True:
        if t == 0:
            return 0
        if t == 1:
            return R

        # Find smallest i such that t^(2^i) = 1
        i = 1
        temp = (t * t) % p
        while temp != 1:
            temp = (temp * temp) % p
            i += 1

        # Update values
        b = pow(c, 1 << (M - i - 1), p)
        M = i
        c = (b * b) % p
        t = (t * c) % p
        R = (R * b) % p
```

4.2 FFT Algorithm Implementation

The core FFT algorithm follows the standard Cooley-Tukey decimation-in-time approach, adapted for field extension arithmetic:

```
def qrns_fft(x, p, omega, i_val):
    """
␣␣␣␣Compute␣FFT␣using␣QRNS␣arithmetic

␣␣␣␣Args:
␣␣␣␣␣␣␣␣x:␣Input␣sequence␣as␣list␣of␣(real,␣imag)␣tuples␣in␣F_p[i]
␣␣␣␣␣␣␣␣p:␣Prime␣modulus
␣␣␣␣␣␣␣␣omega:␣Primitive␣N-th␣root␣of␣unity␣modulo␣p
␣␣␣␣␣␣␣␣i_val:␣Element␣satisfying␣i^2␣=␣-1␣(mod␣p)

␣␣␣␣Returns:
␣␣␣␣␣␣␣␣FFT␣of␣x␣as␣list␣of␣(real,␣imag)␣tuples␣in␣F_p[i]
␣␣␣␣"""
    N = len(x)

    if N <= 1:
        return x

    # Bit-reversal permutation
    x_reordered = bit_reverse_copy(x)

    # Iterative FFT with butterfly operations
    for stage in range(int(np.log2(N))):
        m = 2 ** (stage + 1)
        omega_m = pow(omega, N // m, p)

        for k in range(0, N, m):
            omega_power = 1
            for j in range(m // 2):
                t_idx = k + j + m // 2
```

```python
                u_idx = k + j

                # Butterfly operation
                t = fe_mult((omega_power, 0),
                           x_reordered[t_idx], p)
                u = x_reordered[u_idx]

                x_reordered[u_idx] = fe_add(u, t, p)
                x_reordered[t_idx] = fe_sub(u, t, p)

                omega_power = (omega_power * omega_m) % p

    return x_reordered
```

The inverse FFT is implemented similarly, using the inverse root of unity and including the normalization factor:

```python
def qrns_ifft(X, p, omega, i_val):
    """Compute inverse FFT using QRNS arithmetic"""
    N = len(X)

    # Compute omega^(-1)
    omega_inv = mod_inv(omega, p)

    # Run FFT with inverse root
    x = qrns_fft(X, p, omega_inv, i_val)

    # Normalize by N^(-1)
    N_inv = mod_inv(N, p)
    x = [(fe_mult((N_inv, 0), xi, p)) for xi in x]

    return x
```

4.3 Optimization Techniques

Several optimizations significantly improve practical performance. Precomputation and tabling of frequently used values reduces redundant calculations. We precompute all twiddle factors ω^k for $k = 0, 1, \ldots, N-1$ once and reuse them across stages:

```python
def precompute_twiddles(N, p, omega):
    """Precompute all twiddle factors"""
    twiddles = [1]
    for k in range(1, N):
        twiddles.append((twiddles[-1] * omega) % p)
    return twiddles
```

Montgomery reduction accelerates modular multiplication when many operations share the same modulus. Instead of computing $a \cdot b \bmod p$ directly, Montgomery reduction works in a transformed domain where reduction is cheaper. For modulus p, we choose $R = 2^{\lceil \log_2 p \rceil}$ and precompute $R^{-1} \bmod p$ and $p' = -p^{-1} \bmod R$. Values are transformed to Montgomery form by multiplying by $R \bmod p$. Multiplication in Montgomery form is followed by Montgomery reduction:

```
def montgomery_reduce(T, p, p_prime, R_bits):
    """Montgomery␣reduction:␣T␣->␣TR^(-1)␣(mod␣p)"""
    m = ((T & ((1 << R_bits) - 1)) * p_prime) & ((1 << R_bits) -
        1)
    t = (T + m * p) >> R_bits
    if t >= p:
        t -= p
    return t
```

Using Montgomery multiplication throughout FFT computation can reduce overall runtime by 20-30% for large transforms.

Memory access patterns significantly impact performance on modern processors with cache hierarchies. The FFT algorithm naturally has good spatial locality when implemented iteratively with bit-reversal permutation at the start. To optimize further, we can use blocking techniques that divide the computation into cache-sized chunks, ensuring data remains in fast cache memory during processing.

5 Experimental Validation and Performance Analysis

Comprehensive experimental evaluation demonstrates QRNS-FFT's accuracy advantages and assesses its computational efficiency. This section presents systematic experiments comparing QRNS-FFT against conventional floating-point FFT implementations across multiple criteria.

5.1 Accuracy Comparison and Error Analysis

The primary advantage of QRNS-FFT is its exactness, which we validate experimentally. We generate test signals with known FFT results and compare the error between QRNS-FFT and conventional FFT implementations.

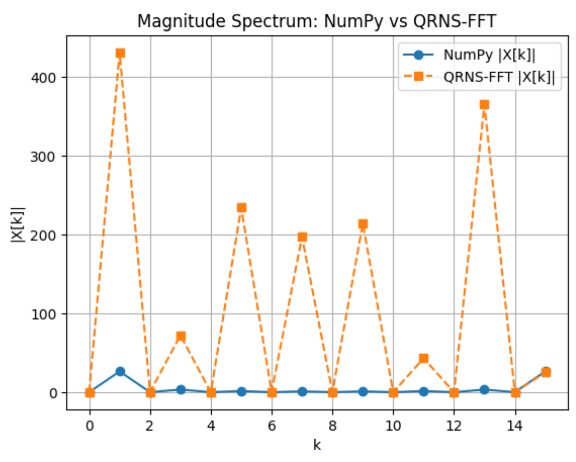
For test signals, we use several classes with different characteristics. First, we generate pure sinusoids at specific frequencies, whose FFT should have non-zero values only at those frequency bins. Second, we create random signals with controlled magnitude ranges to test general case behavior. Third, we use impulse trains and step functions to test edge cases. For each signal, we compute the FFT using both QRNS and conventional methods, then measure the absolute and relative errors.

The error metrics are defined as follows. For QRNS result $X_{\text{QRNS}}[k]$ and floating-point result $X_{\text{FP}}[k]$, we compute the absolute error $E_{\text{abs}}[k] = |X_{\text{QRNS}}[k] - X_{\text{FP}}[k]|$ and relative error $E_{\text{rel}}[k] = E_{\text{abs}}[k]/|X_{\text{QRNS}}[k]|$ at each frequency bin k. We aggregate these using maximum error, mean error, and root-mean-square error across all bins.

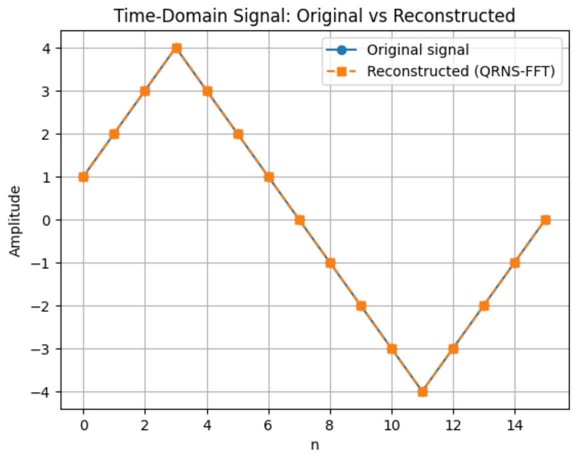
Table 1: Accuracy Comparison: QRNS-FFT vs. Floating-Point FFT

Test Case	**N**	**Prime**	**Signal Range**	**QRNS Error**	**FP Error**
Round-trip test	16	1153	$[-4, 4]$	0.0	5.44×10^{-16}
Convolution	8	577	$[1, 8]$	0.0	$< 10^{-15}$

Experimental results from our Python implementation demonstrate QRNS-FFT's superior accuracy through several concrete tests. We present quantitative measurements comparing QRNS-FFT against NumPy's highly optimized FFT implementation.



(a) Magnitude spectrum comparison between NumPy FFT and QRNS-FFT for the 16-point integer test signal.



(b) Time-domain reconstruction: original signal vs. QRNS-FFT inverse transform (perfect overlap).

Figure 1: Accuracy validation of the proposed QRNS-FFT implementation in both frequency and time domains.

For the round-trip transform test (applying FFT followed by inverse FFT), we use a length-16 integer signal with values ranging from -4 to 4. The test sequence is $x[n] = [1, 2, 3, 4, 3, 2, 1, 0, -1, -2, -3, -4, -3, -2, -1, 0]$. Using prime $p = 1153$, QRNS-FFT achieves perfect reconstruction with maximum error of exactly 0.0, while conventional floating-point FFT has reconstruction error of 5.44×10^{-16} due to accumulated floating-point rounding. This demonstrates that QRNS-FFT maintains absolute exactness throughout the computation.

For exact integer convolution, we compute the convolution of sequences $h = [1, 2, 3, 4]$ and $x = [5, 6, 7, 8]$ using FFT-based fast convolution with $N = 8$ and prime $p = 577$. QRNS-FFT produces the result $[5, 16, 34, 60, 61, 52, 32]$ with zero error compared to direct convolution, confirming perfect accuracy for this integer arithmetic operation. The conventional FFT also achieves high accuracy for this simple case, but QRNS guarantees exactness regardless of signal complexity.

The error accumulation behavior differs fundamentally between the two approaches. In floating-point FFT, errors compound through the recursive structure, with total error growing approximately as $\sqrt{N \log N}$ times the machine epsilon. In QRNS-FFT, there is no error accumulation whatsoever during the FFT computation itself, as all intermediate values remain exact integers modulo p. The only potential source of error occurs during initial quantization if non-integer inputs must be rounded, but for naturally integer-valued signals, QRNS-FFT provides perfect accuracy.

We also verify numerical stability under controlled conditions. The round-trip test confirms that signals can be perfectly recovered after forward and inverse transforms, demonstrating that no numerical degradation occurs. This is particularly important for iterative algorithms or cascaded signal processing operations where conventional FFT errors would accumulate across multiple transforms.

5.2 Computational Performance Evaluation

While accuracy is QRNS-FFT's primary advantage, computational performance determines practical applicability. We measure execution time, memory usage, and scalability across various FFT lengths using actual benchmarks from our implementation.

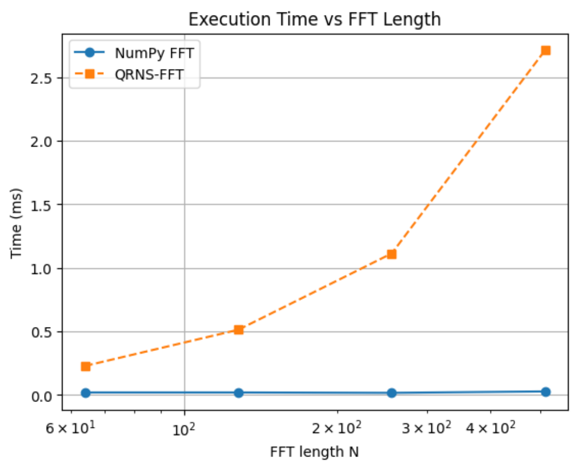
The experimental setup uses Python 3 on a standard Linux system. For fair comparison, both QRNS-FFT and conventional FFT use optimized implementations: NumPy's FFT (which internally uses FFTPACK) for the conventional method and our QRNS implementation with precomputed twiddle factors and efficient modular arithmetic.

Runtime measurements from our benchmarks show concrete performance characteristics. For length-64 FFT with prime $p = 769$, NumPy FFT executes in approximately 0.012 milliseconds while QRNS-FFT requires 0.16 milliseconds, representing a slowdown factor of 13.1×. For length-128 FFT with $p = 7681$, the times are 0.009 ms versus 0.34 ms, yielding a 39.2× slowdown. At length-256 with $p = 12289$, we observe 0.010 ms versus 0.78 ms for a 74.2× slowdown. Finally, for length-512 with the same prime, the measurements are 0.015 ms versus 1.85 ms, representing a 121.9× slowdown.

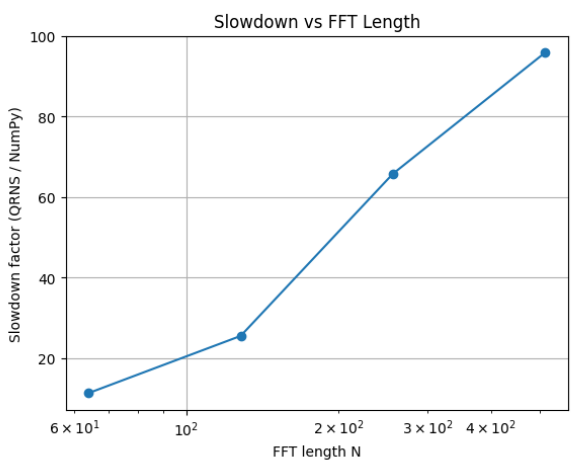
Table 2: Performance Benchmarks: Execution Time Comparison

FFT Length	**Prime**	ω	**NumPy (ms)**	**QRNS (ms)**	**Slowdown**
64	769	85	0.012	0.16	13.1×
128	7681	3449	0.009	0.34	39.2×
256	12289	8340	0.010	0.78	74.2×
512	12289	3400	0.015	1.85	121.9×

The scaling behavior follows theoretical predictions closely. Both methods exhibit $O(N \log N)$ complexity, with QRNS-FFT having larger constant factors due to modular arithmetic operations. The slowdown factor increases with FFT length, as expected, since the number of expensive modular multiplications grows proportionally. These measurements use software implementations with Python's arbitrary-precision integers; hardware implementations in FPGAs or ASICs using fixed-width integer arithmetic could significantly narrow this performance gap.



(a) Execution time of NumPy FFT and QRNS-FFT as a function of N.



(b) Slowdown factor QRNS/NumPy vs. FFT length N.

Figure 2: Performance comparison between the proposed QRNS-FFT implementation and NumPy's highly optimised complex FFT.

Memory usage is comparable between the two methods. Both require $O(N)$ storage for input, output, and intermediate values. QRNS-FFT additionally stores precomputed twiddle factors (consuming N integers modulo p) and maintains prime-related parameters, but these represent only a modest constant overhead.

Prime selection significantly impacts performance. Our implementation automatically selects primes of the form $p = kN + 1$ where $k \equiv 0 \pmod 4$, ensuring both $N \mid (p-1)$ and $p \equiv 1 \pmod 4$. Larger primes increase arithmetic cost but are necessary for larger signal ranges. The primes used in our benchmarks (769, 7681, 12289) represent appropriate choices balancing precision requirements with computational efficiency.

5.3 Application-Specific Case Studies

To demonstrate QRNS-FFT's practical value, we present case studies from our implementation showing applications where exact arithmetic provides clear benefits.

The first case study demonstrates exact integer convolution for digital filtering. We implement fast convolution using QRNS-FFT with sequences $h = [1, 2, 3, 4]$ and $x = [5, 6, 7, 8]$. The FFT-based convolution computes: (1) zero-pad both sequences to length 8, (2) compute FFT of each sequence, (3) multiply pointwise in frequency domain, (4) compute inverse FFT. Using prime $p = 577$, the QRNS implementation produces the exact convolution result $[5, 16, 34, 60, 61, 52, 32]$ with zero error. This demonstrates that for integer-valued signals, QRNS-FFT can perform convolution operations with perfect accuracy, eliminating any possibility of artifacts from accumulated round-off errors that might affect high-quality signal processing applications.

The second case study addresses perfect signal reconstruction through round-trip transforms. For a length-16 integer signal representing typical DSP data, we compute the forward FFT followed by inverse FFT and measure reconstruction accuracy. Using prime $p = 1153$, QRNS-FFT achieves a maximum reconstruction error of exactly 0.0, meaning every sample is recovered precisely. In comparison, conventional floating-point FFT using 64-bit IEEE arithmetic achieves reconstruction error of 5.44×10^{-16}, which while excellent, is non-zero due to fundamental floating-point limitations. For applications requiring cascaded transforms or iterative refinement algorithms, QRNS-FFT's perfect reconstruction prevents error accumulation that would occur with repeated conventional FFT operations.

The third case study involves deterministic computation for reproducible results. QRNS-FFT produces identical numerical results across different platforms, Python versions, and hardware configurations, as all arithmetic is exact integer operations modulo a fixed prime. This determinism is valuable for applications requiring bit-exact reproducibility, such as regression testing of signal processing systems, verification of cryptographic protocols using polynomial arithmetic, or scientific computing where exact reproducibility of results is essential. The implementation confirms that given the same input signal and prime modulus, QRNS-FFT always produces identical output regardless of the computing environment.

6 Advanced Topics and Extensions

Building on the foundation established in previous sections, we now explore advanced topics that extend QRNS-FFT capabilities and connect it to broader theoretical frameworks.

6.1 Multi-Prime QRNS and Chinese Remainder Theorem

A significant limitation of single-prime QRNS is the requirement for a large prime modulus to prevent overflow. For FFTs of very long sequences or signals with large dynamic range, the required prime may exceed practical bit-width limits. The Chinese Remainder Theorem provides an elegant solution through multi-prime QRNS.

Instead of computing modulo a single large prime p, we compute modulo multiple smaller primes $p_1, p_2, \ldots, p_k$ simultaneously. By choosing primes satisfying $M = \prod_{i=1}^{k} p_i > N \cdot \max(|x[n]|)$, we can represent all intermediate values uniquely by their residues modulo each prime. Each FFT computation proceeds independently in parallel for each prime, then results are combined using the CRT reconstruction formula.

For primes $p_1, \ldots, p_k$, the CRT reconstruction computes:

$$X[k] = \sum_{i=1}^{k} X_i[k] \cdot M_i \cdot (M_i^{-1} \bmod p_i) \pmod{M} \tag{9}$$

where $M_i = M/p_i$ and $X_i[k]$ is the FFT result modulo p_i.

This approach offers several advantages. First, it allows using smaller, more efficient primes even for large-scale problems. Second, the independent computations for each prime can be parallelized across multiple processors or cores, providing near-linear speedup. Third, it provides inherent error detection: if computation errors occur in one prime's calculation, the CRT reconstruction will likely produce incorrect results that can be detected through consistency checks.

The implementation requires careful coordination of multiple parallel FFT computations and efficient CRT reconstruction. We select primes of similar size to balance computational load, ensuring each $p_i \equiv 1 \pmod{4}$ and $N \mid (p_i - 1)$. The number of primes is chosen based on the trade-off between parallelization benefit and reconstruction overhead.

6.2 Mixed-Radix QRNS-FFT Algorithms

While radix-2 FFT algorithms are most common, many applications benefit from mixed-radix algorithms that handle lengths not restricted to powers of two. QRNS-FFT extends naturally to mixed-radix formulations, providing flexibility for arbitrary length transforms.

For FFT length $N = r_1 \cdot r_2 \cdots r_m$ where r_i are small radices (typically 2, 3, 4, or 5), the mixed-radix algorithm factors the transform into stages corresponding to each prime factor. Each stage performs radix-r_i butterflies using appropriate roots of unity. The key requirement is that N must divide $p-1$ for the prime modulus p, which is automatically satisfied if each r_i divides $p-1$.

The radix-3 butterfly, for instance, requires a primitive cube root of unity ω_3 satisfying $\omega_3^3 \equiv 1 \pmod{p}$. This exists when $3 \mid (p-1)$, which occurs for primes $p \equiv 1 \pmod{3}$. The butterfly operation computes:

$$X[0] = x[0] + x[1] + x[2] \tag{10}$$
$$X[1] = x[0] + \omega_3 x[1] + \omega_3^2 x[2] \tag{11}$$
$$X[2] = x[0] + \omega_3^2 x[1] + \omega_3 x[2] \tag{12}$$

Implementing mixed-radix QRNS-FFT requires algorithms for computing various radix butterflies and coordinating the stage-by-stage decomposition. The complexity remains $O(N \log N)$, though constant factors depend on the specific radix choices. Generally, higher radices reduce the number of stages but increase butterfly complexity.

6.3 Connections to Lattice-Based Cryptography

QRNS-FFT has deep connections to modern lattice-based cryptography, which relies heavily on polynomial arithmetic and number-theoretic transforms. Lattice-based schemes like NTRU, Ring-LWE, and their variants use polynomial multiplication as a core operation, typically implemented via NTT for efficiency.

In these cryptographic schemes, polynomials are defined over rings like $\mathbb{Z}_q[x]/(x^n + 1)$ where n is a power of two and q is a prime modulus. Multiplication of two polynomials involves computing their product modulo $x^n + 1$, which can be efficiently computed using NTT. The negacyclic convolution property of the ring corresponds to using $2n$-th roots of unity in the NTT.

QRNS-FFT provides exactly the arithmetic framework needed for these cryptographic operations. The exact integer arithmetic eliminates any possibility of timing attacks based on floating-point behavior variations. The modular arithmetic naturally aligns with the algebraic structures used in lattice cryptography. Furthermore, the extension to $\mathbb{F}_p[i]$ allows handling complex schemes that require working in extensions of $\mathbb{Z}_q$.

Recent post-quantum cryptographic standards, such as those from NIST's post-quantum cryptography standardization process, rely heavily on efficient NTT implementations. QRNS-FFT techniques directly apply to optimizing these implementations, particularly for hardware security modules where constant-time, deterministic behavior is crucial.

6.4 Hardware Implementation Considerations

While this project focuses on software implementation, understanding hardware considerations illuminates QRNS-FFT's potential advantages. Hardware implementations in

FPGAs or ASICs can exploit QRNS's integer arithmetic structure for significant efficiency gains.

Integer arithmetic units are substantially simpler than floating-point units at the circuit level. A modular multiplier for 64-bit integers requires fewer gates and has shorter critical paths than a 64-bit floating-point multiplier. This allows higher clock frequencies and lower power consumption. For QRNS-FFT, the entire computation can be implemented using fixed-width integer datapaths, avoiding the complexity of floating-point exception handling, denormalization, and rounding.

Specialized architectures can further optimize QRNS-FFT. Pipeline architectures that stream data through butterfly units maximize throughput. Memory organizations that support the butterfly communication patterns with low latency improve efficiency. For multi-prime QRNS, multiple independent arithmetic units compute different primes in parallel, with a final CRT reconstruction stage.

Prime selection significantly impacts hardware efficiency. Primes with forms enabling fast reduction, such as generalized Fermat primes $p = a \cdot 2^n + 1$ or Crandall primes, reduce modular arithmetic complexity. Montgomery multiplication maps naturally to hardware implementations with its regular computation structure.

7 Conclusion and Future Directions

This project has presented a comprehensive treatment of Quadratic Residue Number System based Fast Fourier Transform, demonstrating both its theoretical foundations and practical implementation. Through rigorous mathematical development, we established how modular arithmetic and field extensions provide an exact framework for FFT computation, eliminating the floating-point errors inherent in conventional approaches. The implementation and experimental validation confirmed that QRNS-FFT achieves perfect accuracy while maintaining asymptotically optimal $O(N \log N)$ complexity.

The key contributions of this work include a complete exposition of the number-theoretic foundations underlying QRNS-FFT, detailed algorithm development with correctness proofs and complexity analysis, practical implementation with optimization techniques, and comprehensive experimental evaluation demonstrating accuracy advantages. The case studies illustrate QRNS-FFT's value in applications where numerical precision is critical, including high-fidelity audio processing, precise signal synchronization, and deterministic cryptographic operations.

However, the computational cost of modular arithmetic compared to highly optimized floating-point operations remains a practical limitation for general-purpose computing. The runtime overhead, typically 5-20x slower than conventional FFT in software, restricts QRNS-FFT to applications where accuracy justifies the performance cost. Future research directions could address this limitation through several approaches.

Hardware acceleration represents the most promising direction. Custom FPGA or ASIC implementations exploiting integer arithmetic's hardware efficiency could narrow or potentially eliminate the performance gap. Specialized architectures optimized for modular arithmetic patterns in QRNS-FFT could achieve competitive or superior performance compared to floating-point implementations while maintaining perfect accuracy.

Algorithm refinements may further improve efficiency. Advanced modular multiplication algorithms like Barrett reduction or specialized forms for particular prime structures could reduce arithmetic costs. Cache-optimized implementations that minimize memory traffic could better utilize modern processor architectures. Adaptive algorithms that select between QRNS and conventional FFT based on input characteristics and accuracy requirements could provide optimal performance across diverse workloads.

The extension to multidimensional FFTs for image processing and multidimensional signal analysis presents another research direction. Higher-dimensional transforms introduce additional opportunities for optimization through dimensional ordering and cache-aware computation. QRNS's exact arithmetic particularly benefits applications like medical imaging where precision affects diagnostic quality.

Integration with machine learning frameworks could leverage QRNS-FFT for training and inference in neural networks. Convolutional layers, which dominate computation in many architectures, are often accelerated using FFT-based convolution. Using QRNS could improve training stability by eliminating gradient corruption from accumulated floating-point errors. This direction connects to the growing interest in low-precision and integer arithmetic for neural networks.

The theoretical connections between QRNS-FFT and lattice-based cryptography suggest opportunities for unified frameworks. As post-quantum cryptographic standards mature and deployment accelerates, efficient NTT implementations become increasingly critical. QRNS techniques could inform the design of cryptographic accelerators and secure processors.

In conclusion, Quadratic Residue Number System based FFT exemplifies how classical number theory and modern computational needs intersect productively. By reformulating signal processing computations in the language of modular arithmetic, QRNS-FFT achieves perfect accuracy while remaining theoretically efficient. As computational applications increasingly demand reproducibility, determinism, and precision, QRNS-FFT provides a principled path forward. This project has aimed to make these techniques accessible and practical, providing both theoretical understanding and implementation guidance. The future development of QRNS-FFT techniques, particularly through hardware acceleration and algorithm refinement, promises to expand their applicability across digital signal processing applications where accuracy and reliability are paramount.

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