Strong Converses for High-dimensional Statistical Estimation

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Inference set-up

Want to estimate $\theta \in \mathcal{F}$ from data $\mathbf{Y} = (Y_1, \ldots, Y_n)$

Data $\mathbf{Y}$ generated according to $P_\theta(\mathbf{Y})$

How well can we estimate θ as the number of samples grows?

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Density Estimation [Yu '97]

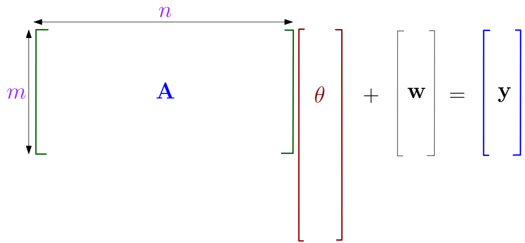
$\mathcal{F}$: smooth densities on $[0, 1]$ with bounded second derivative

For $\theta \in \mathcal{F}$, samples $Y_1, \ldots, Y_n$ drawn i.i.d. $\sim \ \theta$

Measure of performance:

$$d(\theta, \widehat{\theta}) = \int_0^1 \left(\sqrt{\theta(x)} - \sqrt{\widehat{\theta}(x)} \right)^2 dx$$

Compressed sensing



Vector $\theta \in \mathcal{F}$ observed through linear model:

$$\mathbf{y} = \mathbf{A}\,\theta + \text{ noise}$$

$\mathcal{F}$: unit norm vectors in $\mathbb{R}^n$ with at most k non-zeros

How well can we estimate θ?

Measure of performance:

$$\mathrm{M}^*(\mathbf{A}) := \inf_{\widehat{\theta}} \sup_{\theta \in \mathcal{F}} \mathbb{E}\left[\frac{1}{n}\,\|\widehat{\theta}(\mathbf{y}) - \theta\|^2\right],$$

Candès, Davenport, *How well can we estimate a sparse vector?*, 2013

Loss function and Risk

Want to estimate $\theta \in \mathcal{F}$ from data $\mathbf{Y} = (Y_1, \dots, Y_n)$

Data $\mathbf{Y}$ generated according to $P_\theta(\mathbf{Y})$

Performance of an estimator $\widehat{\theta}$ measured via $d(\theta, \widehat{\theta}(\mathbf{Y}))$

Loss function d is a distance or semi-distance

Risk $R(\theta, \hat{\theta}) = \mathbb{E}\left[d(\theta, \widehat{\theta}\,)\right]$

GOAL: Lower bounds on the *minimax risk*

$$\inf_{\widehat{\theta}} \sup_{\theta \in \mathcal{F}} \mathbb{E}\left[d(\theta, \widehat{\theta}\,)\right]$$

Standard approach (see Tsybakov 2009)

For any $\psi_n > 0$,

$$\mathbb{P}\left(d(\theta, \widehat{\theta}) \geq \psi_n\right) \leq \frac{1}{\psi_n}\mathbb{E}\left[d(\theta, \widehat{\theta})\right]$$

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Hence

$$\sup_{\theta \in \mathcal{F}} \mathbb{E}\left[d(\theta, \widehat{\theta})\right] \geq \psi_n \sup_{\theta \in \mathcal{F}} \mathbb{P}\left(d(\theta, \widehat{\theta}\,) \geq \psi_n\right)$$

Want to choose ψ_n such that prob. is bounded below by a constant

Packing set

$$\sup_{\theta \in \mathcal{F}} \mathbb{E}\left[d(\theta, \widehat{\theta})\right] \geq \psi_n \sup_{\theta \in \mathcal{F}} \mathbb{P}\left(d(\theta, \widehat{\theta}) \geq \psi_n\right)$$

Construct a packing set $\{\theta_1, \ldots, \theta_M\} \subseteq \mathcal{F}$ such that

$$d(\theta_i, \theta_j) \geq d_{\min} = 2\psi_n, \qquad \text{for all } i \neq j$$

- Existence of packing set can be generally guaranteed via Gilbert-Varshamov bound or the probabilistic method

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- Existence of packing set can be generally guaranteed via Gilbert-Varshamov bound or the probabilistic method
- Idea: Any estimator $\widehat{\theta}$ defines an M-ary hypothesis test between $\{\theta_1, \ldots, \theta_M\}$

$$\widehat{i} = \arg\min_{1 \leq j \leq M} d(\theta_j, \widehat{\theta})$$

Channel coding interpretation

- Channel $P_{\mathbf{Y}|\theta} := P_\theta(\mathbf{Y})$
- Codebook $\{\theta_1, \ldots, \theta_M\}$
- 'Transmitted' codeword $\theta = \theta_i$
- Channel output $\mathbf{Y} = (Y_1, \ldots, Y_n)$
- Minimum-distance decoder Distance measured via $d(\cdot, \cdot)$
  Decode codeword that is closest to $\widehat{\theta}(\mathbf{Y})$.

Probability of decoding error

Minimum distance between codewords is $d_{\min} = 2\psi_n \Rightarrow$

Decoder makes error only if $d(\theta_i, \widehat{\theta}) \geq \frac{d_{\min}}{2} = \psi_n \ \Rightarrow$

$$\mathbb{P}\left(\widehat{i} \neq i \mid \theta_i \text{ true codeword}\right) \leq \mathbb{P}\left(d(\theta_i, \widehat{\theta}) \geq \psi_n\right)$$

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Therefore

$$\varepsilon_M := \frac{1}{M}\sum_{i=1}^{M} \mathbb{P}\left(\widehat{i} \neq i \mid \theta_i \text{ true codeword}\right) \leq \sup_{\theta \in \mathcal{F}} \mathbb{P}\left(d(\theta, \widehat{\theta}\,) \geq \psi_n\right)$$

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Plugging into our risk lower bound,

$$\sup_{\theta \in \mathcal{F}} \mathbb{E}\left[d(\theta, \widehat{\theta})\right] \geq \psi_n \sup_{\theta \in \mathcal{F}} \mathbb{P}\left(d(\theta, \widehat{\theta}) \geq \psi_n\right) \geq \psi_n \, \varepsilon_M$$

Risk Lower Bound

$$\sup_{\theta \in \mathcal{F}} \mathbb{E}\left[d(\theta, \widehat{\theta}) \right] \geq \psi_n \, \varepsilon_M$$

ε_M is average error probability of codebook with $d_{\min} = 2\psi_n$

Ibragimov and Khasminskii, *Estimation of infinite dimensional parameter in Gaussian white noise*, 1977

Risk Lower Bound

$$\sup_{\theta\in\mathcal{F}} \mathbb{E}\left[d(\theta,\widehat{\theta})\right] \geq \psi_n\,\varepsilon_M$$

ε_M is average error probability of codebook with $d_{\min} = 2\psi_n$

Fano's inequality is a standard way to lower bound ε_M:

$$\varepsilon_M \geq 1 - \frac{\log 2 + \frac{1}{M}\sum_{i=1}^M D(P_{\mathbf{Y}|\theta_i}\|\overline{P}_{\mathbf{Y}})}{\log M}, \qquad \text{where } \overline{P}_{\mathbf{Y}} = \frac{1}{M}\sum_{i=1}^M P_{\mathbf{Y}|\theta_i}$$

If we show that $\frac{1}{M}\sum_{i=1}^M D(P_{\mathbf{Y}|\theta_i}\|\overline{P}_{\mathbf{Y}}) \leq \alpha \log M$, then

$$\varepsilon_M \geq 1 - \alpha - \frac{1}{\log M} > 0,$$

Ibragimov and Khasminskii, *Estimation of infinite dimensional parameter in Gaussian white noise*, 1977

Improving on Fano

Generalized versions of Fano: [Birgé '05], [Sason-Verdú '18]

Other converse techniques:

Sphere-packing bound: [Shannon-Gallager-Berlekamp '67]

Based on information spectrum: [Wolfowitz '68], [Verdú-Han '94]

Based on general f-divergences: [Guntuboyina '11]

$\vdots$

Based on binary hypothesis testing:

[Hayashi, Nagaoka '03]

[Polyanskiy, Poor, Verdú '10] ("Meta-converse")

[Vazquez-Vilar, Tauste Campo, Guillén i Fàbregas, Martinez '16]

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$$\sup_{\theta \in \mathcal{F}} \mathbb{E}\left[d(\theta, \widehat{\theta})\right] \geq \psi_n \, \varepsilon_M$$

$$\varepsilon_M = \frac{1}{M} \sum_{i=1}^{M} \mathbb{P}\left(\widehat{i} \neq i \mid \theta_i \text{ true codeword}\right)$$

What we want is a lower bound for ε_M that:

- is computable for wide range of statistical problems
- with *existing* packing sets
- shows $\varepsilon_M \to 1$ as M grows (strong converse)

Obtaining a tighter lower bound

- Channel $P_{\mathbf{Y}|\theta}$
- Codebook $\{\theta_1, \dots, \theta_M\}$ (equally likely codewords)
- Average error probability ε_M

A channel decoder defines a hypothesis test to distinguish between:

$$H_0 : (\theta, \mathbf{Y}) \sim Q = P_\theta Q_{\mathbf{Y}}$$
$$H_1 : (\theta, \mathbf{Y}) \sim P = P_\theta P_{\mathbf{Y}|\theta}$$

Does the data look like it came from the true generating model ?

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For the channel decoder based test [Polyanskiy, Poor, Verdú '10]:

$$Q[T = 1] = \frac{1}{M}, \qquad P[T = 0] = \varepsilon_M$$

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$$Q[T = 1] = \frac{1}{M}, \qquad P[T = 0] = \varepsilon_M$$

For any randomized hypothesis test T and $\gamma > 0$, we have

$$P[T = 1] - \gamma\, Q[T = 1] \leq P\left[\frac{\mathrm{d}P}{\mathrm{d}Q} > \gamma\right].$$

Hence, in our case, for any $\gamma > 0$

$$\frac{1}{M} \geq \frac{1}{\gamma}\left(1 - \varepsilon_M - P_{\theta\mathbf{Y}}\left[\frac{\mathrm{d}P_{\mathbf{Y}|\theta}}{\mathrm{d}Q_{\mathbf{Y}}} > \gamma\right]\right)$$

- Can bound $P_{\theta\mathbf{Y}}\left[\frac{\mathrm{d}P_{\mathbf{Y}|\theta}}{\mathrm{d}Q_{\mathbf{Y}}} > \gamma\right]$ in terms of Rényi divergences using Markov inequality type argument
- Can optimize over γ to deduce ...

Theorem

For any $\lambda > 0$, and any distribution $Q_{\mathbf{Y}}$ over $\mathcal{Y}$ (satisfying mild absolute continuity condition),

$$\varepsilon_M \geq 1 - \frac{(1+\lambda)}{(\lambda M)^{\frac{\lambda}{1+\lambda}}} \left[\sum_{i=1}^{M} \frac{1}{M} \exp\left(\lambda\, D_{1+\lambda}(P_{\mathbf{Y}|\theta_i} \| Q_{\mathbf{Y}})\right) \right]^{\frac{1}{1+\lambda}} .$$

Here $D_{1+\lambda}(P_{\mathbf{Y}|\theta_i} \| Q_{\mathbf{Y}})$ is the Rényi divergence of order $(1+\lambda)$:

$$D_{1+\lambda}(P_{\mathbf{Y}|\theta_i} \| Q_{\mathbf{Y}}) := \frac{1}{\lambda} \log \left(\int_{\mathcal{Y}} \left(\frac{dP_{\mathbf{Y}|\theta_i}}{dQ_{\mathbf{Y}}} \right)^{1+\lambda} dQ_{\mathbf{Y}} \right) .$$

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- Pick a good $Q_{\mathbf{Y}}$, compute lower bound for ε_M via upper bound for Rényi divergence, e.g., [Sason-Verdú '16]
- Have free choice of λ, often $\lambda = 1$ works well enough

Improved risk lower bounds

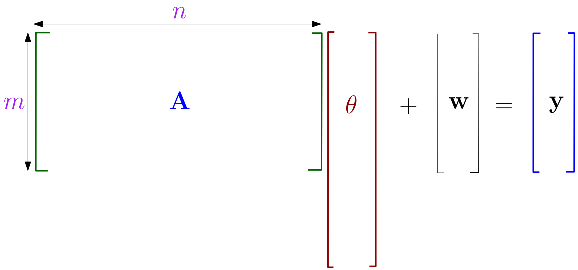
$$\sup_{\theta \in \mathcal{F}} \mathbb{E}\left[d(\theta, \widehat{\theta})\right] \geq \psi_n \, \varepsilon_M$$

In paper we use the result to study three illustrative examples:

1. Compressed sensing
2. Density estimation problem
3. Active learning of a binary classifier. . . see paper.

In each case, get improved bounds with $\varepsilon_M \to 1$ (strong converse), essentially for free.

Application: Compressed Sensing



$$\mathbf{y} = \mathbf{A}\,\theta + \mathbf{w}, \qquad \mathbf{w} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$$

$\mathcal{F}_k$: unit norm vectors θ in $\mathbb{R}^n$ with at most k non-zeros

Want to lower bound

$$\mathrm{M}^*(\mathbf{A}) := \inf_{\widehat{\theta}} \sup_{\theta \in \mathcal{F}_k} \mathbb{E}\left[\frac{1}{n}\,\|\widehat{\theta}(\mathbf{y}) - \theta\|^2\right],$$

Packing set (see [Candès-Davenport '13])

Packing set of vectors $\{\theta_1, \ldots, \theta_M\} \in \mathbb{R}^n$ with:

- $\|\theta_i\|^2 = 1$ for all i
- $\|\theta_i - \theta_j\|^2 \geq \frac{1}{2}$ for $i \neq j$
- $\|\frac{1}{M}\sum_{i=1}^{M} \theta_i \theta_i^T - \frac{1}{n}\mathbf{I}\|_{\mathrm{op}} \leq \frac{\beta}{n}$ for some small $\beta > 0$

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- Size of packing set $M = \left(\frac{n}{k}\right)^{k/4} = \exp\left(\frac{k}{4}\log\left(\frac{n}{k}\right)\right)$

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Computing the Renyi Divergence

$$\varepsilon_M \geq 1 - \frac{(1+\lambda)}{(\lambda M)^{\frac{\lambda}{1+\lambda}}} \left[\sum_{i=1}^{M} \frac{1}{M} \exp\left(\lambda\, D_{1+\lambda}(P_{\mathbf{Y}|\theta_i} \| Q_{\mathbf{Y}})\right) \right]^{\frac{1}{1+\lambda}}$$

Since $\mathbf{y} = \mathbf{A}\,\theta + \mathbf{w}$, $\quad P_{\mathbf{Y}|\theta_i} \sim \mathcal{N}(\mathbf{A}\theta_i, \sigma^2\mathbf{I})$

Choose $Q_{\mathbf{Y}} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$

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Then

$$D_{1+\lambda}(P_{\mathbf{Y}|\theta_i} \| Q_{\mathbf{Y}}) = \frac{(1+\lambda)}{2\sigma^2} \|\mathbf{A}\theta_i\|^2$$

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Then

$$D_{1+\lambda}(P_{\mathbf{Y}|\theta_i} \| Q_{\mathbf{Y}}) = \frac{(1+\lambda)}{2\sigma^2} \|\mathbf{A}\theta_i\|^2$$

We use a subset $\mathcal{P}$ of the Candès-Davenport packing set with $M' = \frac{M}{\log M}$ elements such that

$$\max_{\theta_i \in \mathcal{P}} \|\mathbf{A}\theta_i\|^2 \leq \frac{\|\mathbf{A}\|_F^2}{n}(1+\delta) \quad \text{for some small } \delta > 0$$

Proposition:

For any $\lambda > 0$, $\Delta \in (0,1)$, and $M = (n/k)^{k/4}$, we have

$$\varepsilon_M \geq 1 - (1+\lambda)\left(\frac{(\log M)M^{-\Delta}}{\lambda}\right)^{\lambda/(1+\lambda)},$$

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and

$$\begin{aligned}
\mathrm{M}^*(\mathbf{A}) &= \inf_{\widehat{\theta}} \sup_{\theta \in \mathcal{F}_k} \mathbb{E}\left[\frac{1}{n}\|\widehat{\theta}(\mathbf{y}) - \theta\|^2\right] \\
&\geq \frac{\sigma^2}{4\|\mathbf{A}\|_F^2}\left(\frac{k}{4}\log\frac{n}{k} - 1\right)\frac{(1-\Delta)}{(1+\lambda)}\varepsilon_M,
\end{aligned}$$

For large n we have

$$\mathrm{M}^*(\mathbf{A}) \geq \frac{\sigma^2}{4\|\mathbf{A}\|_F^2} \left(\frac{k}{4} \log \frac{n}{k} \right) (1 - o(1)).$$

- Improvement of factor close to 8 over Fano argument of Candès-Davenport, which gave

$$\mathrm{M}^*(\mathbf{A}) \geq \frac{\sigma^2}{32\|\mathbf{A}\|_F^2 (1+\beta)} \left(\frac{k}{4} \log \frac{n}{k} - 2 \right).$$

- MSE of same order achievable with $\mathbf{A}$ that satisfies RIP $\Rightarrow$ improvement beyond constant factors not possible

Density estimation

- Consider $\mathcal{F}$, set of probability densities θ on $[0,1]$ such that

$$a_0 \leq \theta \leq a_1 \quad \text{and} \quad |\theta''(x)| \leq a_2$$

- We are given $(Y_1, \ldots, Y_n)$ generated IID from θ.
- Wish to estimate density with $\widehat{\theta}_n = \widehat{\theta}_n(Y_1, \ldots, Y_n)$.
- Measure performance by squared Hellinger distance

$$d(\theta, \widehat{\theta}_n) = \int_0^1 \left(\sqrt{\theta(x)} - \sqrt{\widehat{\theta}_n(x)} \right)^2 dx.$$

Wish to obtain lower bound on minimax risk $\inf_{\widehat{\theta}_n} \sup_{\theta \in \mathcal{F}} \mathbb{E} d(\theta, \widehat{\theta}_n)$

Packing set (see Yu '97)

Packing set consists of densities that are small perturbations of uniform density on $[0,1]$

- Fix a smooth, bounded $g(x)$ with

$$\int_0^1 g(x)dx = 0 \quad \text{and} \quad \int_0^1 (g(x))^2\,dx = a.$$

- Partition $[0,1]$ into m subintervals of length $1/m$
- Perturb uniform density in each subinterval by small amount proportional to rescaled version of g
- That is, for some c define

$$g_j(x) = \frac{c}{m^2} g(mx - j)\,\mathbb{I}\left(\frac{j}{m} \le x < \frac{j+1}{m}\right), \quad \text{for } j = 0, \ldots, m-1.$$

Packing set (contd.)

- Hypercube class of 2^m densities

$$\left\{ f_{\boldsymbol{\tau}}(y) = 1 + \sum_{j=0}^{m-1} \tau_j g_j(y) : \boldsymbol{\tau} = (\tau_1, \ldots, \tau_m) \in \{\pm 1\}^m \right\}$$

(In subinterval j, perturb uniform by either g_j or $-g_j$)

- Bandwidth parameter m chosen later to optimize lower bound

Packing set (contd.)

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Pick packing set corresponding to well-separated sequences in $\{-1, 1\}^m$ (guaranteed by Gilbert-Varshamov)

- $\mathcal{A} \subseteq \{-1, 1\}^m$ whose elements have pairwise Hamming distance $\geq m/3$
- Size of $\mathcal{A} \geq \exp(c_0 m)$, where $c_0 \simeq 0.082$
- Resulting packing set $\{f_{\boldsymbol{\tau}} : \boldsymbol{\tau} \in \mathcal{A}\}$ has minimum squared Hellinger distance $d_{\min} = ac^2/(3m^4)$ (see Bin Yu)

Using main theorem

For Q_Y uniform and $\lambda = 1$ in main theorem, Rènyi term is

$$\left[\sum_{\boldsymbol{\tau}\in\mathcal{A}} \frac{1}{M}\int_{[0,1]^n} f_{\boldsymbol{\tau}}^n(\mathbf{y})^2 d\mathbf{y}\right]^{\frac{1}{2}} \leq \exp\left(\frac{c^2 an}{2m^4}\right).$$

Proposition:

With $m = n^{1/5}/\nu$ for any positive constant $\nu < (c_0/(c^2 a))^{1/5}$, the minimax risk satisfies

$$\inf_{\widehat{\theta}_n} \sup_{\theta \in \mathcal{F}} \mathbb{E} d(\widehat{\theta}_n, \theta) \geq \frac{c^2 a \nu^4}{6} n^{-4/5} \varepsilon_M,$$

where

$$\varepsilon_M \geq 1 - 2 \exp\left(\frac{-n^{1/5}}{2\nu} \left(c_0 - \nu^5 c^2 a\right) \right).$$

- Bin Yu method uses same packing set + Fano, but gives ε_M bounded away from zero, not converging to 1

Summary

Lower bounds on minimax risk: packing set + lower bound on ε_M

- Computable via bounding Rényi divergence, gives strong converse
- Other example in paper: active learning of binary classifier
- Improvements over main theorem possible (Baraud `arxiv:1807.05410`)

Further work:

- Can this method give improved minimax rates, rather than just improved constants?
- Extend results to work with global metric entropy features [Yang-Barron '99], [Guntuboyina '11]

Paper in *Electronic Journal of Statistics* (OA), 2018
`doi: 10.1214/18-EJS1419`
`https://arxiv.org/abs/1706.04410`