Quiz-8 answers and solutions

Coursera. Stochastic Processes

September 11, 2020

8 week quiz

1. $X_t = bt + \sigma W_t + cN_t$, where W_t is a Brownian Motion, N_t is a Poisson process with intensity λ, and W_t, N_t are independent; $b, c \in \mathbb{R}$, $\sigma \geq 0$. Find the characteristic function of this process.

   Answer: $\exp\left\{iubt + \lambda t(e^{icu} - 1) - \dfrac{t(\sigma u)^2}{2}\right\}$

   Solution:

$$
\begin{aligned}
\mathbb{E}\exp\{iuX_t\} &= \mathbb{E}\exp\{iu(bt + \sigma W_t + cN_t)\} \\
&= \mathbb{E}\left[\exp\{iubt\}\exp\{iu\sigma W_t\}\exp\{iucN_t\}\right] \\
&= \exp\{iubt\}\mathbb{E}\exp\{iu\sigma W_t\}\mathbb{E}\exp\{iucN_t\} \\
&= \exp\{iubt\}\int_{\mathbb{R}} \exp\{iu\sigma x\}\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\{-\frac{x^2}{2\sigma^2 t}\}\,dx \times \\
&\qquad\qquad \times \sum_{k=0}^{+\infty}\exp\{iuck\}\frac{(\lambda c)^k}{k!}\exp\{-\lambda c\} \\
&= \exp\left\{iubt - \frac{t(\sigma u)^2}{2} + \lambda t(e^{icu} - 1)\right\}
\end{aligned}
$$

2. Consider the previous process $X_t = bt + \sigma W_t + cN_t$, where W_t is a Brownian Motion, N_t is a Poisson process with intensity λ, and W_t, N_t are independent; $b, c \in \mathbb{R}$, $\sigma \geq 0$. What are the mean, variance and covariance function of this process?

   Answer: $\mathbb{E}[X_t] = t(b + c\lambda)$, $Var(X_t) = t(\sigma^2 + c^2\lambda)$, $K(t, s) = (c^2\lambda + \sigma^2)\min(t, s)$

   Solution: $\mathbb{E}X_t = \mathbb{E}\{bt + \sigma W_t + cN_t\} = bt + 0 + c\lambda t$.

   For $t > s$,

$$
\begin{aligned}
K(t, s) &= cov(bt + \sigma W_t + cN_t; bs + \sigma W_s + cN_s) \\
&= cov(\sigma W_t; \sigma W_s) + cov(cN_t; cN_s) \\
&= \sigma^2 cov(W_t - W_s + W_s; W_s) + c^2 cov(N_t - N_s + N_s; N_s) \\
&= (c^2\lambda + \sigma^2)\min(t, s).
\end{aligned}
$$

3. Consider the previous process $X_t = bt + \sigma W_t + cN_t$, where W_t is a Brownian Motion, N_t is a Poisson process with intensity λ and W_t, N_t are independent; $b, c \in \mathbb{R}$, $\sigma \geq 0$. Denote the Lévy measure of this process by ν. What is measure ν of a Borel set B ?

   Answer: $\nu(B) = \lambda$, if $c \in B$ and 0 otherwise

   Solution:

   Since the Brownian motion is continuous, jumps occur only due to the Poisson process. The size of all possible jumps is exactly equal to c. Thus, if c is not comprised by B, then the Lévy measure is equal to zero. On the other hand, when $c \in B$, the Lévy measure is equal to the expected value of the number of jumps occurring between time moments 0 and 1, that is, $\nu(B) = \mathbb{E}N_1 = \lambda$.

4. Let X_t be a Levy process. What is the correct expression for $Var(X_t)$ in terms of characteristic exponent ψ?

   Answer: $Var(X_t) = -t\psi''(0)$

   Solution: According to the Lévy-Khinchine theorem, for any Lévy process a characteristic exponent is equal to:

$$\psi(u) = iu\mu - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x| < 1\})\nu(dx)$$

$$\psi''(u) = -\sigma^2$$

$$Var(X_t) = \sigma^2 t = (-\psi''(u))t$$

5. Let X_t be a Lévy process. Assuming that $X_1 \sim N(0, 1)$, find the mean and the variance of X_t:

   Answer: $\mathbb{E}[X_t] = 0$, $Var(X_t) = t$

   Solution:

   The characteristic exponent of X_1 is $\psi_{X_1}(u) = -\frac{1}{2}u^2$. Since X_t is a Lévy process, then $\psi_{X_t}(u) = -\frac{1}{2}u^2 t$. Hence, X_t is a Brownian motion. Consequently, $Var(X_t) = t$.

6. Let $X_t = bt + N_t$, where N_t is a Poisson process with intensity λ and $b \in \mathbb{R}$. Find the Lévy triplet of this process under truncation function $h(x) = \mathbb{1}\{|x| < 1\}$.

   Answer: $(b, 0, \nu)$, where $\nu(B) = \lambda\mathbb{I}\{1 \in B\}$ for any Borel set B.

   Solution:

Under this truncation the characteristic exponent $\psi(u)$ of the process X_t can be represented as

$$\begin{aligned}
\psi(u) &= iub + \lambda(e^{iu} - 1) = iub + \int_{\mathbb{R}} (e^{iux} - 1)\, \nu(dx) \\
&= iub + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x| < 1\} + iux\mathbb{1}\{|x| < 1\})\, \nu(dx) \\
&= iub + iu \int_{\mathbb{R}} x\mathbb{1}\{|x| < 1\}\, \nu(dx) \\
&\qquad\qquad + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x| < 1\}\, \nu(dx) \\
&= iub + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x| < 1\}\, \nu(dx)
\end{aligned}$$

with the Lévy measure as in question 3. From this we can see that $\mu = b$ and $\sigma = 0$.

7. Let $T_a = \min\{s : B_s \geq a\}$, where B_s is a Brownian motion. Find the distribution function of the process T_a.

   Hint: $\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t) = \mathbb{P}(B_t - B_{T_a} > 0)$. It follows from the fact that for the Brownian motion and all other Lévy processes the increments are independent.

   Answer: $2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right)$

   Solution:

$$\begin{aligned}
\mathbb{P}(B_t > a) &= \mathbb{P}(T_a < t, B_t > a) \\
&= \mathbb{P}(T_a < t)\mathbb{P}(B_t - a > 0 | T_a < t) \\
&= \mathbb{P}(T_a < t)\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t)
\end{aligned}$$

   The conditional probability $\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t)$ is equal to the unconditional one, because the condition $(T_a < t)$ gives an information on the BM before T_a, which is, literally, the **time** by which BM has reached a . The increment $B_t - B_{T_a}$ is independent from that type of information.

   Thus, $\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t) = \mathbb{P}(B_t - B_{T_a} > 0) = \mathbb{P}(B_{t-T_a} > 0) = 1/2$.

   Therefore,

$$\begin{aligned}
\mathbb{P}(T_a < t) &= \frac{\mathbb{P}(B_t > a)}{\mathbb{P}(T_a < t, B_t > a)} \\
&= \frac{1 - \Phi(a/\sqrt{t})}{1/2} \\
&= 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right).
\end{aligned}$$

8. Let L_t be a Lévy process. Choose the equality, which can serve as a proof of its infinite divisibility.

   Answer:

$$L_t = L_{t/n} + (L_{2t/n} - L_{t/n}) + \cdots + (L_t - L_{(n-1)t/n}), \qquad \forall t > 0,\ \forall n \in \mathbb{N}$$

Solution: Since Lévy processes have stationary and independent increments, this equality shows that L_t can be represented as a sum of n independent identically distributed random variables.

Other options:

$$L_t = L_{t/n} + L_{t/n} + \cdots + L_{t/n}, \qquad \forall t > 0,\ \forall n \in \mathbb{N}$$

$$L_t = L_t/n + L_t/n + \cdots + L_t/n, \qquad \forall t > 0,\ \forall n \in \mathbb{N}$$