Quiz-7 answers and solutions

Coursera. Stochastic Processes

September 4, 2020

1. Find the mean of $I(g) = \int_0^1 t^2 dW_t$.

   Answer: 0

   Solution: Let $f(t,x) = xt^2$. Then, $g = f_2' = t^2$. Thus,

$$W_1 \cdot 1^2 = 0 + \int_0^1 2t \cdot W_t \, dt + \int_0^1 t^2 \, dW_t + 0.$$

$$\int_0^1 t^2 \, dW_t = W_1 - \int_0^1 2t \cdot W_t \, dt.$$

$$\mathbb{E}\left(\int_0^1 t^2 \, dW_t\right) = 0 - \int_0^1 2t \cdot \mathbb{E}W_t \, dt = 0.$$

2. Find the variance of $I(g) = \int_0^1 t^2 \, dW_t$.

   Answer: 1/5

   Solution:

$$\begin{aligned}
\mathbb{V}ar\left(\int_0^1 t^2 \, dW_t\right) &= \mathbb{V}arW_1 + \mathbb{V}ar\int_0^1 2t \cdot W_t \, dt - 2cov\left(W_1; \int_0^1 2tW_t \, dt\right) \\
&= 1 + 2\int_0^1 \int_0^t cov(2t \cdot W_t; 2s \cdot W_s) \, ds \, dt - 2\mathbb{E}\left(\int_0^1 2t \cdot W_1 W_t \, dt\right) \\
&= 1 + 2\int_0^1 \int_0^t 4ts^2 \, ds \, dt - 2\int_0^1 2t \cdot \mathbb{E}\left(W_1 W_t\right) \, dt \\
&= 1 + \frac{8}{3}\int_0^1 t^4 \, dt - 4\int_0^1 t^2 \, dt \\
&= 1 + \frac{8}{15} - \frac{4}{3} = \frac{1}{5}.
\end{aligned}$$

3. Let N_t be a Poisson process. Find the mean, covariance function and variance of $X_t = \int_0^t N_s ds$ (in the answers below $t > s \geq 0$).

Answer: $\mathbb{E}[X_t] = \dfrac{\lambda t^2}{2}$, $Var(X_t) = \dfrac{\lambda t^3}{3}$, $K(t,s) = \lambda\left(-\dfrac{s^3}{6} + \dfrac{ts^2}{2}\right)$

Solution:

$$\mathbb{E}[X_t] = \int_0^t \mathbb{E}N_s ds = \int_0^t \lambda s ds = \frac{\lambda t^2}{2}$$

$$\begin{aligned}
K(t,s) &= \int_0^s \int_0^t cov(N_u; N_v)\, dv\, du \\
&= \mathbf{2}\int_0^s \left(\int_0^u cov(N_u; N_v)\, dv\right) du + \int_0^s \left(\int_s^t cov(N_u; N_v)\, dv\right) du \\
&= \mathbf{2}\int_0^s \left(\int_0^u \lambda v\, dv\right) du + \int_0^s \left(\int_s^t \lambda u\, dv\right) du \\
&= \int_0^s \lambda u^2\, du + \int_0^s \lambda u(t-s)\, du \\
&= \frac{\lambda s^3}{3} + \frac{s^2}{2}\cdot \lambda(t-s) \\
&= \lambda\left(-\frac{s^3}{6} + \frac{ts^2}{2}\right).
\end{aligned}$$

4. Let $X_t = \begin{cases} \xi_1, & t \in [0,1), \\ \xi_2, & t \in [1,2), \\ \xi_3, & t \geq 2, \end{cases}$ where ξ_1, ξ_2, ξ_3 - i.i.d. random variables having exponential distribution with parameter λ. Find the mean and the variance of $\int_0^T X_t\, dt$.

Answer:

$$\mathbb{E}\left[\int_0^T X_t\, dt\right] = \frac{T}{\lambda}$$

$$Var\left(\int_0^T X_t\, dt\right) = \begin{cases} \dfrac{T^2}{\lambda^2}, & 1 > T, \\ \dfrac{1}{\lambda^2} + \dfrac{(T-1)^2}{\lambda^2}, & 1 \leq T < 2, \\ \dfrac{2}{\lambda^2} + \dfrac{(T-2)^2}{\lambda^2}, & T \geq 2, \end{cases}$$

Solution:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T X_t\, dt\right] &= \int_0^T \mathbb{E}X_t\, dt \\
&= \int_0^T 1/\lambda\, dt \\
&= \frac{T}{\lambda}
\end{aligned}$$

For $T < 1$,

$$\begin{aligned}
\mathbb{V}ar\left[\int_0^T X_t\,dt\right] &= 2\int_0^T\int_0^t cov(\xi_1;\xi_1)\,ds\,dt \\
&= \int_0^T 2t/\lambda^2\,dt \\
&= \frac{T^2}{\lambda^2}
\end{aligned}$$

For $1 \leq T < 2$,

$$\begin{aligned}
\mathbb{V}ar\left[\int_0^T X_t\,dt\right] &= \mathbb{V}ar\left[\int_0^1 X_t\,dt + \int_1^T X_t\,dt\right] \\
&= \mathbb{V}ar\left[\int_0^1 \xi_1\,dt + \int_1^T \xi_2\,dt\right] \\
&= \mathbb{V}ar\left[\int_0^1 \xi_1\,dt\right] + \mathbb{V}ar\left[\int_1^T \xi_2\,dt\right] \\
&= \frac{1}{\lambda^2} + 2\int_1^T\int_0^t cov(X_t;X_s)\,ds\,dt \\
&= \frac{1}{\lambda^2} + \int_0^T \frac{2t}{\lambda^2}\,dt \\
&= \frac{T^2}{\lambda^2} + \frac{(T-1)^2}{\lambda^2}
\end{aligned}$$

The other case is equivalent.

5. Find the equivalent expression for the stochastic integral $\int_0^T W_t^2\,dW_t$, where W_t is a Brownian motion.

   Answer: $\frac{1}{3}W_T^3 - \int_0^T W_s\,ds$

   Solution: $f(t,x) = W_t^3/3$, $\quad f_2'(t,x) = W_t^2$, $\quad f_1'(t,x) = 0$, $\quad f_{2,2}''(t,x) = 2W_t$.

$$\frac{1}{3}W_T^3 = 0 + 0 + \int_0^T W_t^2\,dW_t + \frac{1}{2}\int_0^T 2W_s\sigma_s^2\,ds,$$

   where $\sigma_s^2 = 1$ which can be derived by applying the definition of the Itô process to the Brownian motion. From that equation we obtain the answer:

$$\int_0^T W_t^2 dW_t = \frac{1}{3}W_T^3 - \int_0^T W_s ds$$

6. Compute the variance of the stochastic integral $\int_0^T W_t\,dW_t$, where W_t is a Brownian motion.

Answer: $\dfrac{T^2}{2}$

Solution:

$f(t,x) = W_t^2/2, \quad f_2'(t,x) = W_t, \quad f_1'(t,x) = 0, \quad f_{2,2}''(t,x) = 1.$

$$\frac{1}{2}W_T^2 = 0 + 0 + \int_0^T W_t\,dW_t + \frac{1}{2}\int_0^T 2\sigma_s^2\,ds,$$

where $\sigma_s^2 = 1$ which can be derived by applying the definition of the Itô process to the Brownian motion. From that equation we obtain:

$$\int_0^T W_t dW_t = \frac{1}{2}W_T^2 - T$$

$$\begin{aligned}
\mathbb{V}ar\left(\int_0^T W_t\,dW_t\right) &= \mathbb{V}ar\left(\frac{1}{2}W_T^2\right) \\
&= \frac{1}{4}(\mathbb{E}W_T^4 - (\mathbb{E}W_T^2)^2) \\
&= \frac{1}{4}(\mathbb{E}W_T^4 - (\mathbb{E}W_T^2)^2) \\
&= \frac{1}{4}(T^2\mathbb{E}N(0;1)^4 - T^2(\mathbb{E}N(0;1)^2)^2) \\
&= \frac{1}{4}(3T^2 - T^2).
\end{aligned}$$

7. Choose the process X_t which satisfies the following property:

$$X_t = X_0 + \int_0^t X_s dW_s + \int_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}}\,ds. \qquad (1)$$

Answer: $X_t = e^{W_t}\sqrt{1+t}$

Solution:

Itô formula:

$$f(t,W_t) = f(0,0) + \int_0^t f_1'(s,W_s)\,ds + \int_0^t g(s,W_s)\,dW_s + \frac{1}{2}\int_0^t g_2'(s,W_s)\,ds,$$

where $g = f_2'$.

Let $X_t = a \cdot e^{W_t}\sqrt{1+t}$ with $a \neq 0$. Then $f(t,x) = g_2'(t,x) = g(t,x) = a \cdot e^x\sqrt{1+t}$ and $f_1'(t,x) = \frac{a\cdot e^x}{2\sqrt{1+t}}$.

Therefore,

$$\begin{aligned}
\int_0^t X_s \, dW_s &= \int_0^t a \cdot e^{W_s} \sqrt{1+s} \, dW_s \\
&= a \cdot \left[e^{W_t} \sqrt{1+t} - 1 - \int_0^t \left(\frac{1}{2} \frac{e^{W_s}}{\sqrt{1+s}} + \frac{1}{2} e^{W_s} \sqrt{1+s} \right) ds \right] \\
&= a \cdot \left[e^{W_t} \sqrt{1+t} - 1 - \int_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}} \, ds \right].
\end{aligned}$$

Substituting the result into (1) gives

$$a \cdot e^{W_t} \sqrt{1+t} = a + a \cdot e^{W_t} \sqrt{1+t} - a - a \int_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}} \, ds + \int_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}} \, ds$$

which is true if and only if $a = 1$.