Truncation function in the Lévy-Khintchine representation

Coursera. Stochastic Processes

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As it is known, the characteristic exponent of any Lévy process can be represented in the form

$$\psi(u) = iu\mu - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{1}\{|x| < 1\}\right) \nu(dx) \tag{1}$$

with $\mu \in \mathbb{R}, \sigma \geq 0$ and ν being a Lévy measure. The triplet (μ, σ, ν) is called the Lévy triplet and uniquely characterises any Lévy process.

More generally, $\psi(u)$ can be represented as

$$\psi(u) = iu\mu(h) - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iuxh(x)\right) \nu(dx),$$

where $h(x)$ is any bounded measurable function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} h(x) = 1 + \bar{o}(|x|) \text{ as } |x| \to 0 \\ h(x) = O\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty. \end{cases}$$

and

$$\mu(h) = \mu + \int_{\mathbb{R}} x(h(x) - \mathbb{1}\{|x| < 1\})\nu(dx).$$

The function $h(x)$ is known as the truncation function. Some examples of $h(x)$ are

$$\begin{aligned} h(x) &= \frac{1}{1+x^2}, \\ h(x) &= \mathbb{1}\{|x| < c\}, \quad c > 0, \\ h(x) &= \mathbb{1}\{|x| \leq c\}, \quad c > 0, \\ h(x) &= \mathbb{1}\{|x| \leq 1\} + \mathbb{1}\{1 < |x| \leq 2\}(2 - |x|), \\ h(x) &= \frac{\sin x}{x}. \end{aligned}$$

Also, if ν is such that $\int_{|x|<1} |x|\nu(dx) < \infty$, one can take $h(x) = 0$ and come to the expression

$$\psi(u) = iu\mu(h) - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} \left(e^{iux} - 1\right) \nu(dx)$$

with

$$\mu(h) = \mu - \int_{\mathbb{R}} x\mathbb{1}\{|x| < 1\}\nu(dx).$$

As can be seen, the drift parameter μ in the Lévy triplet depends on $h(x)$. Thus, it is important to note that the Lévy triplet is unique with respect to the truncation function. The parameters σ and ν, however, remain unchanged.