# Final Exam answers and solutions

Coursera. Stochastic Processes

October 29, 2019

## Final exam

1. Let $N_t$ be a counting process of a renewal process $S_n = S_{n-1} + \xi_n$ such that the i.i.d. random variables $\xi_1, \xi_2, ...$ have a probability density function $p_\xi(x) = \begin{cases} \frac{1}{3}e^{-x}(x+2), & x \geq 0 \\ 0, & x < 0. \end{cases}$ Find the mean value of $N_t$ .

   **Answer:** $-\dfrac{1}{16} + \dfrac{3}{4}t + \dfrac{1}{16}e^{-(4/3)t}$

   **Solution:**

   step 1. $p \to \mathcal{L}_p$

$$
\begin{aligned}
\mathcal{L}_p(s) &= \int_0^\infty e^{-sx} \cdot \frac{1}{3}e^{-x}(x+2)dx \\
&= -\frac{1}{3}\int_0^\infty \frac{x+2}{s+1} d(e^{-(s+1)x}) \\
&= \frac{1}{3(s+1)}\int_0^\infty e^{-(s+1)x}dx - \frac{x+2}{3(s+1)}e^{-(s+1)x}|_0^\infty \\
&= \left(-\frac{1}{3(s+1)^2}e^{-(s+1)x} - \frac{x+2}{3(s+1)}e^{-(s+1)x}\right)|_0^\infty \\
&= \frac{1}{3(s+1)^2} + \frac{2}{3(s+1)} \\
&= \frac{2s+3}{3(s+1)^2}.
\end{aligned}
$$

   step 2. $\mathcal{L}_p \to \mathcal{L}_u$

$$
\mathcal{L}_u = \frac{\frac{2s+3}{3(s+1)^2}}{s\left(1 - \frac{2s+3}{3(s+1)^2}\right)} = \frac{2s+3}{s^2(3s+4)}.
$$

   step 3. $\mathcal{L}_u \to u$

$$
\begin{aligned}
\frac{2s+3}{s^2(3s+4)} &= \frac{A}{s}+\frac{B}{s^2}+\frac{C}{3s+4}\\
&= \frac{A\cdot s(3s+4)+B\cdot(3s+4)+C\cdot s^2}{s^2(3s+4)}\\
&= \frac{s^2(3A+C)+s(4A+3B)+4B}{s^2(2s+3)}.
\end{aligned}
$$

$$
\begin{cases} 3A+C=0\\ 4A+3B=2\\ 4B=3 \end{cases} \to \begin{cases} C=3/16\\ A=-1/16\\ B=3/4 \end{cases}
$$

$$
L_u = -\frac{1}{16s}+\frac{3}{4s^2}+\frac{1}{16(s+\frac{4}{3})}
$$

Therefore,

$$
\mathbb{E}N_t = u(t) = -\frac{1}{16}+\frac{3}{4}t+\frac{1}{16}e^{-(4/3)t}.
$$

2. Purchases in a shop are modelled by a non-homogeneous Poisson process. It is known that $30t^{5/4}$ purchases are made on average during $t$ hours after the opening of the shop. Find the probability that the interval between $k$ and $k+1$ purchases will be more than 2 minutes, but less than 4 minutes, given that the purchase number $k$ was in the time moment $s$ :

   **Answer:** $e^{-30(s+1/30)^{5/4}+30s^{5/4}} - e^{-30(s+1/15)^{5/4}+30s^{5/4}}$

   **Solution:**

$$
\begin{aligned}
\mathbb{P}(2<S_{k+1}-S_k<4|N_s=k) &= \mathbb{P}(S_{k+1}-S_k<4|N_s=k)-\mathbb{P}(S_{k+1}-S_k\le 2|N_s=k)\\
&= \mathbb{P}(N_{s+4}-N_s\ge 1|N_s=k)-\mathbb{P}(N_{s+2}-N_s\ge 1|N_s=k)\\
&= \mathbb{P}(N_{s+4}-N_s\ge 1)-\mathbb{P}(N_{s+2}-N_s\ge 1)\\
&= (1-\mathbb{P}(N_{s+4}-N_s=0))-(1-\mathbb{P}(N_{s+2}-N_s=0))\\
&= e^{-30(s+1/30)^{5/4}+30s^{5/4}} - e^{-30(s+1/15)^{5/4}+30s^{5/4}}.
\end{aligned}
$$

3. Find stationary distribution of Markov chain with the following 1-step transition matrix P:

$$
P = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0\\ 1/2 & 1/2 & 0 & 0\\ 1/4 & 1/4 & 0 & 1/2\\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}
$$

   **Answer:** $(\frac{6}{17}\ \frac{7}{17}\ \frac{2}{17}\ \frac{2}{17})$

   **Solution:**

   If $\vec{\pi}^* = (\pi_1,\pi_2,\pi_3,\pi_4)$ is the stationary distribution, then $\vec{\pi}^*P=\vec{\pi}^*$.

$$
\begin{cases} \frac{1}{3}\pi_1+\frac{1}{2}\pi_2+\frac{1}{4}\pi_3=\pi_1\\ \frac{1}{3}\pi_1+\frac{1}{2}\pi_2+\frac{1}{4}\pi_3+\frac{1}{2}\pi_4=\pi_2\\ \frac{1}{3}\pi_1=\pi_3\\ \frac{1}{2}\pi_3+\frac{1}{2}\pi_4=\pi_4 \end{cases}.
$$

4. Let $X_t = e^{2W_t}$ . Find mathematical expectation and covariance function of this process (in the answers below $t > s \geq 0$ ).

**Answer:** none of above

**Solution:**

$$\begin{aligned}
\mathbb{E}(X_t) &= \mathbb{E}\left(e^{2W_t}\right) \\
&= \int_{-\infty}^{+\infty} e^{2x} \cdot \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}\, dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2t}+2x} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x}{\sqrt{2t}} - \sqrt{2t}\right)^2 + 2t\right) dx \\
&= \frac{e^{2t}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x}{\sqrt{2t}} - \sqrt{2t}\right)^2\right) dx \\
&= e^{2t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\left(\frac{x}{\sqrt{2t}} - \sqrt{2t}\right)^2\right) dx \\
&= e^{2t}
\end{aligned}$$

$$\begin{aligned}
\operatorname{cov}(X_t, X_s) &= \mathbb{E}(X_t X_s) - \mathbb{E}(X_t)\mathbb{E}(X_s) \\
&= \mathbb{E}\left(e^{W_t + W_s}\right) - e^{2t+2s} \\
&= \mathbb{E}\left(e^{W_t - W_s + 2W_s}\right) - e^{2t+2s} \\
&= \mathbb{E}\left(e^{W_t - W_s + 2(W_s - W_0)}\right) - e^{2t+2s} \\
&= \mathbb{E}\left(e^{W_t - W_s}\right) \mathbb{E}\left(e^{2W_s}\right) - e^{2t+2s} \\
&= e^{2(t-s)} e^{8s} - e^{2t+2s}
\end{aligned}$$

5. Let $Y_t$ be a stochastic process which is defined as follows: $\mathbb{E}[Y_t] = \alpha + \beta t$, $cov(Y_t, Y_{t+h}) = e^{-\lambda h}$ , $\forall t > 0$ , $h \geq 0$ , where $\lambda > 0$ and $\alpha$ , $\beta$ are some constants. Find the mathematical expectation and the covariance function of process $X_t = Y_{t+1} - Y_t$ .

**Answer:** none of above

**Solution:**

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[Y_{t+1} - Y_t] \\
&= \alpha + \beta(t+1) - \alpha - \beta t \\
&= \beta
\end{aligned}$$

$$\begin{aligned}
cov(X_t; X_s) &= cov(Y_{t+1} - Y_t; Y_{s+1} - Y_s) \\
&= cov(Y_{t+1}; Y_{s+1}) - cov(Y_{t+1}; Y_s) - cov(Y_t; Y_{s+1}) + cov(Y_t; Y_s) \\
&= e^{-\lambda(t-s)} - e^{-\lambda(t+1-s)} - e^{-\lambda|t-s-1|} + e^{-\lambda(t-s)}.
\end{aligned}$$

6. Let $X_t = \varepsilon_t + \xi \cos(\pi t/24)$ , $t = 1, 2, ...$ , where $\xi, \varepsilon_1, \varepsilon_2, ...$ are i.i.d. standard normal random variables. Is the process $X_t$ stationary and ergodic?

   **Answer:** $X_t$ is not weakly stationary but it is ergodic

   **Solution:**

   The mean of the process is, obviously, nil, however, its covariance function is equal to: $K(t,s) = \mathbb{1}\{t = s\}Var\xi_t + cov(\xi cos(\pi t/24); \xi cos(\pi s/24)) = \mathbb{1}\{t - s = 0\} + cos(\pi t/24)cos(\pi s/24)$, which cannot be presented as a function on $(t - s)$. So it is not stationary. Let us show that it is ergodic. In fact, the limit of the sequence of Gaussian random variables

   $$M_T := \frac{1}{T}\sum_{t=0}^{T} X_t$$

   have normal distribution. Moreover, $\mathbb{E}X_T = 0$ for any $T > 0$, and

   $$VarM_T = \frac{1}{T} + \frac{T}{T^2}\sum_{t=0}^{T}(\cos(\pi t/24)) \leq \frac{13}{T} \to 0.$$

   Therefore, $M_T \to 0$ as $T \to \infty$, and the process is ergodic.

7. Find the equivalent expressions for the stochastic integral $\int_0^T 3W_t^2 dW_t$ , where $W_t$ is a Brownian motion.

   **Answer:** $W_T^3 - 3\int_0^T W_s ds$

   **Solution:**

   $f(t,x) = W_t^3, f_2(t,x) = 3W_t^2, f_1(t,x) = 0, f_{2,2}(t,x) = 6W_t$.

   $W_T^3 = 0 + 0 + \int_0^T 3W_t^2 dW_t + \frac{1}{2}\int_0^T 6W_s \sigma_s^2 ds$, where $\sigma_s^2 = 1$ which can be derived by applying the definition of the Itô process to the Brownian motion. From that equation we obtain the answer:

   $$\int_0^T 3W_t^2 dW_t = W_T^3 - \int_0^T 3W_s ds$$

8. Let $X_t = bt + CPP_t$ , where $b$ is a constant and $CPP_t$ is a compound Poisson process defined as $CPP_t = \sum_{k=0}^{N_t} \xi_k$ , $N_t$ is a Poisson process with intensity $\lambda$ and $\xi_1, \xi_2, ...$ is a sequence of i.i.d. random variables independent of $N_t$ . Find the Levy measure of the process $X_t$ (in the answers below $B$ is a Borel subset).

   **Answer:** $\nu(B) = \lambda\mathbb{P}\{\xi_1 \in B\}$

   **Solution:**

   Since the first component is continuous, jumps occur only due to the Compound Poisson process. The Lévy measure is equal to the expected value of the number of jumps occurring between time moments 0 and 1, that is, $\nu(B) = \mathbb{E}CPP_1 = \lambda\mathbb{P}\{\xi_1 \in B\} + 0 \cdot \mathbb{P}\{\xi_1 \notin B\}$.

9. New: Let $X_t$ be a stochastic process with the following dynamics: $dX_t = \alpha X_t dt + \sigma X_t dW_t$. Find $dY_t$, where $Y_t = e^{r(T-t)}X_t$.

   **Answer:** $(\alpha - r)Y_t dt + \sigma Y_t dW_t$

**Solution:**

$$
\begin{aligned}
dY_t &= (Y_t)'_t dt + (Y_t)'_{X_t} dX_t + \frac{1}{2}(Y_t)''_{X_t} dX_t dX_t \\
&= -rY_t dt + \frac{Y_t}{X_t} dX_t + \frac{1}{2} \cdot 0 \\
&= -rY_t dt + \frac{Y_t}{X_t}(\alpha X_t dt + \sigma X_t dW_t) \\
&= (\alpha - r)Y_t dt + \sigma Y_t dW_t.
\end{aligned}
$$