# Quiz-3 answers and solutions

Coursera. Stochastic Processes

September 8, 2019

## 3 Week quiz

1. Find stationary distribution of Markov chain with the following 1-step transition matrix P:

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

   **Answer:** $(\frac{1}{12}\frac{3}{12}\frac{5}{12}\frac{1}{12}\frac{2}{12})$

   **Solution:**

   One of the properties of the stationary distribution $\vec{\pi}^*$ is that $\vec{\pi}^* \cdot P = \vec{\pi}^*$.

   Let us denote $\vec{\pi}^* = (a; b; c; d; e)$. Therefore, we need to find the vector $(a; b; c; d; e)$ that fit in the system of equations:

$$(a; b; c; d; e) \cdot \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix} = (a; b; c; d; e)$$

$$\begin{cases} \frac{1}{5}c = a \\ \frac{1}{2}(a + d + e) + \frac{1}{5}c = b \\ b + \frac{1}{5}c + \frac{1}{2}e = c \\ \frac{1}{5}c = d \\ \frac{1}{2}(a + d) + \frac{1}{5}c = e \end{cases} \Rightarrow \begin{cases} a = d = \frac{1}{5}c \\ e = \frac{2}{5}c \\ b = \frac{3}{5}c \end{cases} .$$

   And we also need to add the line: $a + b + c + d + e = 1$. Thus, we will obtain the result $(\frac{1}{12}\frac{3}{12}\frac{5}{12}\frac{1}{12}\frac{2}{12})$.

2. Choose all one-step transition matrices which correspond to ergodic Markov chains:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Answer:** none of above

**Solution:** The period of the first Markov chain is equal to 4. The second Markov chain is also not aperiodic.

3. Choose all periodic states of the Markov chain with the following 1-step transition matrix:

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

**Answer:** all states are aperiodic

**Solution:** All states communicate with each other. The first state is aperiodic. Therefore, all states are aperiodic.

4. Let's consider once more the Markov chain from the previous task. How many equivalence classes has this chain?

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

**Answer:** 1

**Solution:**

All points communicate.

5. Assume that there is a series of integer numbers, in which numbers 1,2,..,9 appear randomly and independently of each other with equal probabilities. Let $x_n$ be a quantity of different numbers in $n$ first elements of the series. Find a stationary distribution of this chain.

**Answer:** (0 0 0 0 0 0 0 0 1)

**Solution:** Clearly, $x_1 = 1$. Then, with probability $\frac{1}{9}$ the second picked number will be the same as the first one giving $x_2 = 1$. With probability $\frac{8}{9}$ the second picked number will be different from the first one giving $x_2 = 1$. Therefore, the transition matrix is equal to:

$$P = \begin{pmatrix} \frac{1}{9} & \frac{8}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{9} & \frac{7}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{9} & \frac{6}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & \frac{5}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{9} & \frac{4}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{9} & \frac{3}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{9} & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

6. New: At time moment $t = 0$ a six-dot side of a simple die faces upwards. Each time moment a die randomly flips on one of its sides. Find the $\vec{\pi}^2$ (the distribution of the Markov process after two flips). Note that the total number of dots on opposite sides of a simple die is equal to 7.

   **Answer:** $\vec{\pi}^2 = \vec{\pi}^1 \cdot P = (\frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{4})$

   **Solution:**

   Clearly, $\vec{\pi}^0 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$. To obtain the transition matrix we need to understand that a die can flip to 4 out of all 6 sides, i.e. it cannot flip to a current position and it cannot flip to an opposite side. Therefore,

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}$$

$$\vec{\pi}^1 = \vec{\pi}^0 \cdot P = (0 \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad 0)$$

$$\vec{\pi}^2 = \vec{\pi}^1 \cdot P = (\frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{4})$$

7. New: At time moment $t = 0$ a six-dot side of a simple die faces upwards. Each time moment a die randomly flips on one of its sides. Find all stationary distributions. Note that the total number of dots on opposite sides of a die is equal to 7.

   **Answer:** $\vec{\pi}^2 = \vec{\pi}^1 \cdot P = (\frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{4})$

   **Solution:** The stationary distribution $\vec{\pi}^*$ has the following feature:

   $\vec{\pi}^* \cdot P = \vec{\pi}^*$

$$(a; b; c; d; e; f) \cdot \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix} = (a; b; c; d; e; f)$$

$$\begin{cases} b+c+d+e=4a=4f \\ a+c+d+f=4b=4e \\ a+b+e+f=4c=4d \\ \mathbf{a+b+c+d+e+f=1} \end{cases} \Rightarrow a=b=c=d=e=f=1/6$$

8. New: Jane and Peter participating in a chess championship. For Jane, the probabilities of wining, draw, and losing a game number $t$ are $(w, d, l)$. Peter is slightly more emotional. If he wins the current game, the probabilities of the win, draw, and lose in the next game are equal to $(w+\epsilon, d, l-\epsilon)$; if the current game ends in a draw, then the corresponding probabilities are $(w, d, l)$; if Peter loses - the result of next game is distributed as $(w-\epsilon, d, l+\epsilon)$. Find the condition, which guarantees that the probability of the win is larger then the probability of the loose (separately for Peter and Jane? In other words, under which conditions it is better to be a slightly emotional?

   **Answer:** $l < w$.

   **Solution:** The transition matrices of these two players are:

$$P_{Jane} = \begin{pmatrix} w & d & l \\ w & d & l \\ w & d & l \end{pmatrix}; \quad P_{Peter} = \begin{pmatrix} w+\epsilon & d & l-\epsilon \\ w & d & l \\ w-\epsilon & d & l+\epsilon \end{pmatrix}.$$

   Jane's stationary distribution is $(w \quad d \quad l)$. To calculate that for Peter we need to solve the following system of equations:

$$(x \quad y \quad z) \begin{pmatrix} w+\epsilon & d & l-\epsilon \\ w & d & l \\ w-\epsilon & d & l+\epsilon \end{pmatrix} = (x \quad y \quad z)$$

$$\begin{cases} w(x+y+z)+x\epsilon - z\epsilon = x \\ d(x+y+z)=y \\ l(x+y+z)-x\epsilon+z\epsilon = z \\ \mathbf{x+y+z=1} \end{cases} \Rightarrow x = \frac{w(1-\epsilon)-l\epsilon}{1-2\epsilon}$$

$$\begin{gathered} (\vec{\pi}^*_{Jane})_{win} < (\vec{\pi}^*_{Peter})_{win} \\ w < x \\ w < \frac{w(1-\epsilon)-l\epsilon}{1-2\epsilon} \\ l < w \end{gathered}$$