Quiz-3 answers and solutions

Coursera. Stochastic Processes

June 7, 2019

4 Week quiz

1. Consider the condition from the Kolmogorov continuity theorem: $\mathbb{E}\left[|X_t - X_s|^\alpha\right] \leq K|t-s|^{1+\beta}, \qquad \forall\, t,s > 0.$

   For which parameters α, K and β this condition holds, if X_t is a Brownian motion?

   Solution: $\mathbb{E}\left[|X_t - X_s|^\alpha\right] \leq K|t-s|^{1+\beta}, \qquad \forall\, t,s > 0.$

   If we take $\alpha = 4$ and keep in mind that $X_t - X_s \sim N(0; t-s)$, then we will get $\mathbb{E}\left[|X_t - X_s|^4\right] = 3(t-s)^2$.

2. Let $X_t = e^{W_t}$, where W_t is a Brownian motion. Find mathematical expectation $\mathbb{E}[X_t]$, variance $Var(X_t)$ and covariance function $K(t,s) = cov(X_t, X_s)$ (in the answers below it is assumed that $t > s \geq 0$).

   Solution:

$$
\begin{aligned}
\mathbb{E}(X_t) &= \mathbb{E}\left(e^{W_t}\right) \\
&= \int_{-\infty}^{+\infty} e^x \cdot \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}\, dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2t}+x} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x}{\sqrt{2t}} - \sqrt{t/2}\right)^2 + t/2\right) dx \\
&= \frac{e^{t/2}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x}{\sqrt{2t}} - \sqrt{t/2}\right)^2\right) dx \\
&= \frac{e^{t/2}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \sqrt{2t} e^{-u^2}\, du \\
&= \frac{e^{t/2}}{\sqrt{2\pi t}} \cdot \sqrt{2t} \cdot \sqrt{\pi} \\
&= e^{t/2}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cov}(X_t, X_s) &= \mathbb{E}(X_t X_s) - \mathbb{E}(X_t)\mathbb{E}(X_s) \\
&= \mathbb{E}\left(e^{W_t+W_s}\right) - e^{t/2+s/2} \\
&= \mathbb{E}\left(e^{W_t-W_s+2W_s}\right) - e^{\frac{t+s}{2}} \\
&= \mathbb{E}\left(e^{W_t-W_s+2(W_s-W_0)}\right) - e^{\frac{t+s}{2}} \\
&= \mathbb{E}\left(e^{W_t-W_s}\right)\mathbb{E}\left(e^{2W_s}\right) - e^{\frac{t+s}{2}} \\
&= e^{(t-s)/2}e^{2s} - e^{\frac{t+s}{2}}
\end{aligned}
$$

3. Let W_t be the Brownian motion. Calculate $\mathbb{P}\{W_1 + W_2 > 2\}$. In the possible answers below Φ is the distribution function of the standard normal distribution.

   Answer: $1 - \Phi\left(\frac{2}{\sqrt{5}}\right)$, where Φ is a normal distribution function

   Solution:

   $E(W_1) = E(W_2) = 0; Var(W_1) = 1; Var(W_2) = 2;$

   $E(W_1+W_2) = 0; Var(W_1+W_2) = Var(W_1)+Var(W_2)+2cov(W_1; W_2) = 1 + 2 + 2min(1; 2) = 5$

4. Let $Y_{n+1} = aY_n+X_n$, where $n = 0, 1, 2, \cdots$. $Y_0 = 0, |a| < 1, X_0, X_1, X_2, \cdots \sim N(0; 1)$. Find $cov(Y_4; Y_3)$.

   Answer: $a^5 + a^3 + a$.

   Solution: $cov(Y_4; Y_3) = cov(a^3X_0+a^2X_1+aX_2+X_3; a^2X_0+aX_1+X_2) = a^5 + a^3 + a$

5. Let X_t be a Brownian motion. Find

$$K(t, s) - Var(X_{min(t;s)}).$$

   Answer: 0

   Solution: $Var(X_{min(t;s)}) = min(t; s)$.