Quiz-5 answers and solutions

Coursera. Stochastic Processes

June 7, 2019

5 week quiz

1. Let λ be a non-zero constant. Does a stochastic process with the covariance function $K(t,s) = \sin(\lambda(t-s))$ exist?

   No, because the function K is not positively semi-definite.

2. Let Y_n be a stochastic process which is defined as follows: $Y_{n+1} = \alpha Y_n + X_n$, $n = 0, 1, ...$ Assume $Y_0 = 0$, $|\alpha| < 1$ and X_n is a sequence of i.i.d. standard normal random variables for $n = 0, 1, 2, ...$ Determine whether Y_n is stationary and find its mean and variance:

   Answer: Y_n is non-stationary, $\mathbb{E}Y_n = 0$, $VarY_n = \dfrac{1-\alpha^{2n}}{1-\alpha^2}$

   Solution: The key for solution lies in the finding the covariance function:

$$
\begin{aligned}
K(t,s) &= Cov(Y_t; Y_s) \\
&= Cov(\alpha^{t-1}X_0 + \cdots + \alpha^0 X_{t-1}; \alpha^{s-1}X_0 + \cdots + \alpha^0 X_{s-1}) \\
&= \alpha^{t-1}\alpha^{s-1} + \alpha^{t-2}\alpha^{s-2} + \cdots + \alpha^{t-s+1}\alpha + \alpha^{t-s}\alpha \\
&= \alpha^{t-s}(\alpha^{2s-2} + \alpha^{2s-4} + \cdots + 1) \\
&= \alpha^{t-s}\frac{1-\alpha^{2s}}{1-\alpha^2}.
\end{aligned}
$$

3. Let W_t be a Brownian Motion and define $X_t = (1-t)W_{t/(1-t)}$ for $t \in (0,1)$. Choose all correct statements.

   Solution:

   Let's find the covariance function:

$$
\begin{aligned}
K(t,s) &= Cov(X_t; X_s) \\
&= (1-t)(1-s)Cov(W_{t/(1-t)}; W_{s/(1-s)}) \\
&= (1-t)(1-s)Cov(W_{t/(1-t)} - W_{s/(1-s)} + W_{s/(1-s)}; W_{s/(1-s)} - W_0) \\
&= (1-t)(1-s)Var(W_{s/(1-s)}) \\
&= s(1-t) \\
&\neq \gamma(t-s).
\end{aligned}
$$

   Hence, X_t is not weakly (and strictly) stationary.

4. Let W_t be a Brownian Motion and $h > 0$ be a fixed number. Find a covariance function of the process $X_t = W_{t+h} - W_t$.

   Answer: $K(t,s) = \begin{cases} h - |t-s|, & if |t-s| \le h \\ 0, & if |t-s| > h \end{cases}$

   Solution: $K(t,s) = Cov(W_{t+h} - W_t; W_{s+h} - W_s)$=

   $$\begin{cases} 0, & if \quad t > s+h \\ Cov(W_{t+h} - W_{s+h} + W_{s+h} - W_t; W_{s+h} - W_s), & if \quad t \le s+h \end{cases}$$

   $$\begin{cases} 0, & if \quad t > s+h \\ Cov(W_{s+h} - W_t; W_{s+h} - W_t + W_t - W_s), & if \quad t \le s+h \end{cases}$$

   $$\begin{cases} 0, & if \quad t > s+h \\ Var(W_{s+h} - W_t), & if \quad t \le s+h \end{cases}$$

   $$\begin{cases} 0, & if \quad t > s+h \\ s+h-t, & if \quad t \le s+h \end{cases}$$

5. Let X_t is a process with independent and stationary increments and h is a positive constant. Moreover, $\mathbb{E}X_t = 0$ and $\mathbb{E}X_t^2 < \infty$. Is $Y_t = X_{t+h} - X_t$ a wide-sense stationary process ?

   Answer: Yes

   Hint: If increments of the process X_t are stationary, then X_t is also stationary.

6. Let X_t be a wide-sense stationary process with autocovariance function γ, such that $\gamma(0) = 2$, $\gamma(1) = \gamma(-1) = 1$ and $\gamma(n) = 0$ for all other n. Find the spectral density $g_X(u)$ of this process.

   Answer: $g_X(u) = \dfrac{1 + cos(u)}{\pi}$

   Solution: Let the covariance fucnction of some stochastic process X_t be

   $$\gamma_X(u) = \begin{cases} 2, & u = 0 \\ 1, & u = \pm 1 \\ 0, & else \end{cases}.$$

   $g_X(u) = \frac{1}{2\pi}(2 + e^{-iu} + e^{iu}) = \frac{1}{2\pi}(2 + 2cos(u))$.

7. Let the autocovariance function of some stochastic process X_t be $\gamma_X(u) = \begin{cases} 3, & u = 0 \\ 1, & u = \pm 2 \\ 0, & else \end{cases}$. Find the spectral density of $Y_t = 3X_t + 2X_{t-1} + X_{t-2}$.

   Solution:

   $g_X(u) = \frac{1}{2\pi}(3 + e^{-2iu} + e^{2iu}) = \frac{1}{2\pi}(3 + 2cos(2u))$.

   $g_Y(u) = g_X(u)|\mathcal{F}[\rho](u)|^2$,

   where $\rho(h) = \begin{cases} 3, & h = 0 \\ 2, & h = 1 \\ 1, & h = 2 \\ 0, & else \end{cases}$.

Therefore, $\mathcal{F}[\rho](u) = e^{2iu} + 2e^{iu} + 3$ (some complex number).

$|\mathcal{F}[\rho](u)|^2 = \mathcal{F} \times \bar{\mathcal{F}} = (e^{2iu} + 2e^{iu} + 3) \times (e^{-2iu} + 2e^{-iu} + 3) = 9 + 4 + 1 + 8(e^{iu} + e^{-iu}) + 3(e^{2iu} + e^{-2iu}) = 14 + 3 \cdot 2cos(2u) + 8 \cdot 2cos(u)$.

$g_Y(u) = \frac{1}{2\pi}(3 + 2cos(2u))(14 + 3 \cdot 2cos(2u) + 8 \cdot 2cos(u))$