Quiz-6 answers and solutions

Coursera. Stochastic Processes

October 31, 2020

1 6 week quiz

1. Let W_t be a Brownian Motion considered at integer time points $t = 0, 1, 2, ...$ Choose the ergodic processes:

   $X_t = Ct + W_t$, where C is a non-zero constant

   $X_t = \xi t + W_t$, where $\xi \sim N(0,1)$ and ξ is independent of W_t.

   Answer: none of above.

   Solution: In the first case, $\mathbb{E}(\frac{1}{T}\sum\limits_{t=1}^{T}(Ct + W_t)) = \frac{T+1}{2}C \to \infty$ as $T \to \infty$. Hence, the first process is not ergodic.

   In the second case, despite $\mathbb{E}(\frac{1}{T}\sum\limits_{t=1}^{T}(\xi t + W_t)) = \frac{T+1}{2}\mathbb{E}\xi = 0$, we need to check whether its variance converges to 0 as $T \to \infty$:
   $\mathrm{Var}\left(\frac{1}{T}\sum\limits_{t=1}^{T}(\xi t + W_t)\right) = \frac{(T+1)^2 T^2}{4T^2} + \frac{1}{T^2}\mathrm{Var}\sum\limits_{t=1}^{T} W_t \to \infty$. Hence, it is also not ergodic.

2. Let $X_t = \cos(\omega t + \theta)$ be a stochastic process and $\theta \sim \mathrm{Unif}[0, 2\pi]$, $\omega = \pi/10$. Is this process ergodic? Is it stationary?

   Answer: It is ergodic and weakly stationary.

   Solution:

   Since the distribution of X_t is symmetric, its mean is 0.

   $$\begin{aligned} K(t,s) &= \mathbb{E}(cos(wt+\theta)cos(ws+\theta)) \\ &= \frac{1}{2}\mathbb{E}cos(w(t-s)) + \frac{1}{2}\mathbb{E}cos(w(t+s)+2\theta) \\ &= \gamma(t-s) + \mathbb{E}cos(w(t+s))\mathbb{E}cos(2\theta) - \mathbb{E}sin(w(t+s))\mathbb{E}sin(2\theta) \\ &= \gamma(t-s), \end{aligned}$$

   because the means of $cos(2\theta)$ and $sin(2\theta)$ are equal to 0. Consequently, this process is weakly stationary.

   To prove that it is also ergodic, we need to look at the following:

   $$\begin{aligned} \frac{1}{T}\sum_{r=0}^{T}\gamma(r) &= \frac{1}{2T}(cos0 + \cdots + cos\frac{Tw}{10}) \\ &\le \frac{c_w}{2T} \to 0, \end{aligned}$$

where c_w depends on w. For instance, if $w = 10$, then $c_w \geq 5$, just $c_w = 5$ to have a sharp bound.

Therefore, this process is ergodic.

3. Let $X_t = \varepsilon_t + \xi \cos(\pi t/12)$, $t = 1, 2, ...$, where $\xi, \varepsilon_1, \varepsilon_2, ...$ are i.i.d. standard normal random variables. Choose the correct statement.

   Answer: X_t is not weakly stationary, but it is ergodic.

   Solution:

   The mean of the process is, obviously, nil, however, its covariance function is equal to: $K(t, s) = \mathbb{1}\{t = s\} \operatorname{Var} \xi_t + \operatorname{cov}(\xi \cos(\pi t/12), \xi \cos(\pi s/12)) = \mathbb{1}\{t - s = 0\} + \cos(\pi t/12) \cos(\pi s/12)$, which cannot be presented as a function on $(t - s)$. Thus, it is not stationary.

$$\mathbb{E}\frac{1}{T}\sum_{t=0}^{T}(\varepsilon_t + \xi \cos(\pi t/12)) = 0$$

$$\begin{aligned}
\operatorname{Var}\frac{1}{T}\sum_{t=0}^{T}(\varepsilon_t + \xi \cos(\pi t/12)) &= \frac{1}{T^2}\left(T + \sum_{t=1}^{T}\cos^2(\pi t/12)\right) \\
&= \frac{1}{T} + \frac{1}{T^2}\sum_{t=1}^{T}\cos^2(\pi t/12) \\
&\leq \frac{1}{T} + \frac{T}{T^2} = \frac{2}{T} \to 0
\end{aligned}$$

   as $T \to \infty$.

   Therefore, the process is not stationary, but is ergodic, because $\mathbb{E}\frac{1}{T}\sum\limits_{t=0}^{T} X_t \to$ *const*.

4. Assume that for a process X_t it is known that $\mathbb{E}[X_t] = \alpha + \beta t$, $\operatorname{cov}(X_t, X_{t+h}) = e^{-h\lambda}$ for all $h \geq 0$, $t > 0$, and some constants $\lambda > 0$, α, β. Is the process $Y_t = X_{t+1} - X_t$ stationary and ergodic?

   Answer: Y_t is weakly stationary and ergodic.

   Solution:

   $\mathbb{E}[X_{t+1} - X_t] = \beta$ does not depend on time. And, $Y_t = X_{t+1} - X_t$, clearly, has an autocovariance function. Hence, it is weakly stationary. A strict stationarity is not the case, because, for instance, $Y_0 = X_1$ and $Y_{100} = X_{101} - X_{100}$ have different distribution laws:

$$\operatorname{Var} X_1 = 1$$
$$\operatorname{Var}(X_{101} - X_{100}) = 1 + 1 - 2\operatorname{cov}(X_{101}; X_{100}) = 2 - 2e^{-\lambda}.$$

   Additionally,

$$\frac{1}{T}\sum_{t=0}^{T}(Y_t) = \frac{X_{T+1} - X_0}{T} \to const$$

   as $T \to \infty$, so it is ergodic.

5. Let $X_t = \sigma W_t + ct$, where W_t is Brownian motion, σ, $c > 0$. Choose the correct statements about this process:

   Answer: X_t has continuous trajectories.

6. Let the process X_t have an autocovariance function $\gamma(r) = e^{-\alpha r}$. Is $Y_t = X_t + w$ an ergodic process?

   Answer: yes, if w is a constant.

   Solution:

   Since $\gamma(r)$ tends to 0 when $r \to \infty$, then the process X_t is ergodic, which means $\frac{1}{T}\sum\limits_{t=0}^{T} X_t \to const$. As for Y_t,

$$\frac{1}{T}\sum_{t=0}^{T} Y_t = \frac{1}{T}\sum_{t=0}^{T} X_t + w \to const + w.$$

   So, Y_t is ergodic for constant ws and non-ergodic otherwise.