TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Morse Theory: The Existence of Morse Functions on Smooth Manifold Triads

Recall that we now have the following lemmas:

Lemma 1.1. *There exists a smooth function $f : W \to [0,1]$ with $f^{-1}(0) = V_0$ and $f^{-1}(1) = V_1$, such that f has no critical point in a neighborhood of the boundary W.*

Lemma 1.2. *Lemma A: If $f : U \to \mathbb{R}$ with U open is a C^2 mapping, then, for almost all linear mappings $L : \mathbb{R}^n \to R$, the function $f + L$ has only non-degenerate critical points.*

Lemma 1.3. *Lemma B: Let K be a compact subset of an open set U in $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is C^2 and has only non-degenerate critical points in K, then there is a number $\delta > 0$ such that if $g : U \to \mathbb{R}$ is C^2 and at all points of K satisfies*

$$(1)\ \left|\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}\right| < \delta,\ (2)\ \left|\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 g}{\partial x_i \partial x_j}\right| < \delta$$

$i, j = 1, ..., n$, then g likewise has only non-degenerate critical points.

Lemma 1.4. *Lemma C: Suppose $h : U \to U'$ is a diffeomorphism and carries a compact subset $K \subset U$ onto $K' \subset U'$. Given a number $\varepsilon > 0$, there is a number $\delta > 0$ such that if a smooth map $f : U' \to \mathbb{R}$ satisfies*

$$|f| < \delta, |\frac{\partial f}{\partial x_i}| < \delta, |\frac{\partial^2 f}{\partial x_i \partial x_j}| < \delta, i, j = 1, ..., n$$

At all points of $K' \subset U'$, then $f \circ h$ satisfies

$$|f \circ h| < \varepsilon, |\frac{\partial f \circ h}{\partial x_i}| < \varepsilon, |\frac{\partial^2 f \circ h}{\partial x_i \partial x_j}| < \varepsilon, i, j = 1, ..., n$$

at all points of K.

The following theorem is proven for Morse functions on $(M, \emptyset, \emptyset)$.

Theorem 1.5. *If M is a compact manifold without boundary, the morse functions form an open dense subset of $F(M, \mathbb{R})$ in the C^2 topology.*

Proof. Let $(U_1, h_1), ..., (U_k, h_k)$ be a finite covering of M by coordinate neighborhoods. We can easily find compact sets $C_i \subset U_i$ such that $C_1, ..., C_k$ cover M. We say f is "good" on $S \subset M$ if f has no degenerate critical points on S.

I) The set of Morse functions is open in $C^2(M)$. Let $f : M \to \mathbb{R}$ be a Morse function. By Lemma B, there is a neighborhood N_i of f in $C^2(M)$ such that every

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function is good in C_i. Thus we take $N = \bigcap_{i=1}^{k} N_i$ which is an open neighborhood of f of functions that are good on $\bigcup_{i=1}^{k} C_i = M$.

II) The set of Morse functions is dense. Let N be a neighborhood of $f \in C^2(\mathbb{R})$. Let $\lambda : M \to [0,1]$ be a smooth function such that $\lambda = 1$ in some neighborhood of C_1 and $\lambda = 0$ in a neighborhood of $M - U_1$. By Lemma A, almost all choices of linear map $L : \mathbb{R}^n \to R$, the function $f_1(p) = f(p) + \lambda(p)L(h_1(p))$ will be good on $C_1 \subset U_1$.

We note that $f_1 - f = \lambda(p)L(h_1(p))$ so f_1 differs from f on the compact set $K = Supp(\lambda) \subset U_1$. We can write $L(x)$ as

$$L(x_1, ..., x_n) = \sum l_i x_i$$

and then we see that

$$f_1 \circ h_1^{-1} - f \circ h_1^{-1} = f \circ h_1^{-1} + (\lambda \circ h_1^{-1}(x)) \sum l_i x_i - f \circ h_1^{-1} = (\lambda \circ h_1^{-1}(x)) \sum l_i x_i$$

For all $x \in h_1(K)$. Now choose l_i sufficiently small and we guarantee that all the conditions of Lemma C are satisfied for any $\varepsilon > 0$ throughout $h(K)$. If $\varepsilon > 0$ is small enough, we can guarantee that $f_1 \in N$. Thus we know f_1 is good on C_1. We apply lemma B and obtain a neighborhood of $f_1, N_1 \subset N$ of functions good on C_1.

Continue the above process with f_1 and N_1 and obtain a function f_2 good on C_2, etc.

After k steps, we have a function $f_k \in N_k \subset N_{k-1} \subset ... \subset N_1 \subset N$ which is good on $\bigcup_{i=1}^{k} C_i = M.3$ □

Definition 1.6. A topological space X is normal if every two disjoint closed sets of X have disjoint open neighborhoods.

Theorem 1.7. *A space is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.*

Theorem 1.8. *Let $\{U_i\}$ be a finite open covering of a compact manifold M then there exists a closed refinement $\{C_i\}$ of $\{U_i\}$.*

Theorem 1.9. *There exists a Morse function on any smooth manifold triad (W, V_0, V_1).*

Proof. The first lemma above gives us a function $f : W \to [0,1]$ such that

1. $f^{-1}(0) = V_0, f^{-1}(1) = V_1$
2. f has no critical points in a neighborhood of the boundary

We make an attempt to eliminate the degenerate critical points in $W - \partial W$ while preserving the above properties. Let U be an open neighborhood of ∂W where f has no critical points. Since W is a normal space, we can find an open neighborhood V of ∂W such that $\bar{V} \subset U$.

Let U_i be a finite coordinate covering such that each set U_i lies in U or in $W - \bar{V}$. Let $\{C_i\}$ be a compact refinement of $\{U_i\}$ and let C_0 be the union of all C_i that lie in U. As before, we can use Lemma B to show that there is some neighborhood of f such that no function can have degenerate critical points in C_0.

Also note that $f \neq 0 \neq 1$ on $W - V$. Thus, there is some neighborhood N' of f such that every function $g \in N'$ satisfies

$$0 < g < 1, \ on \ W - V$$

Let $N_0 = N \cap N'$. Thus our remaining coordinate neighborhoods in $W - V$ are $U_1, ..., U_k$. Then repeat the argument used in the previous theorem. □

We can further extend the results to find Morse functions that lie on different levels

Lemma 1.10. *Let* $f : W \to [0,1]$ *be a Morse function for the triad* $(W; V_0, V_1)$ *with critical points* $P_1, ..., P_k$. *Then* f *can be approximated by a Morse function* g *with the same critical points such that* $g(p_i) \neq g(p_j)$ *for* $i \neq j$.

Lemma 1.11. *Let* $f : (W; V_0, V_1) \to ([0,1], 0, 1)$ *be a Morse function and suppose that* $0 < c < 1$ *where* c *is not a critical value of* f. *Then* $f^{-1}[0,c]$ *and* $f^{-1}[c,1]$ *are smooth manifolds with boundary.*

Corollary 1.12. *Any cobordism can be expressed as a composition of cobordisms with Morse number 1.*