TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Morse Theory: The C^2 Topology

Definition 1.1. Let U be an open subset of $\mathbb{R}^n$. A map $L : U \to \mathbb{R}^m$ is called linear if for any $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$

Lemma 1.2. *Suppose U is an open set in $\mathbb{R}^n$. If $L : U \to \mathbb{R}^m$ is a linear map, then $dL_x = L$ for all $x \in U$.*

Proof.

$$dL_x(h) = \lim_{t\to 0} \frac{L(x+th) - L(x)}{t} = \lim_{t\to 0} \frac{L(x) + tL(h) - L(x)}{t} = \lim_{t\to 0} \frac{tL(h)}{t} = L(h)$$

□

Definition 1.3. We let $Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ denote the set of all linear maps $f : \mathbb{R}^n \to \mathbb{R}$.

Theorem 1.4. *$Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ is a vector space.*

Theorem 1.5. *$Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n$*

Corollary 1.6. *$Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ is a smooth manifold.*

Theorem 1.7. *(Sard's Theorem) Let $f : U \to \mathbb{R}^n$ be a smooth map defined on some open set U of $\mathbb{R}^n$. Let*

$$C = \{x \in U : rank(d_x f) < n\}$$

Then $f(C)$ has Lebesgue measure 0 in $\mathbb{R}^n$.

Lemma 1.8. *If $f : U \to \mathbb{R}$ with U open is a C^2 mapping, then, for almost all linear mappings $L : \mathbb{R}^n \to R$, the function $f + L$ has only nondegenerate critical points.*

Proof. Consider the manifold $U \times Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$. This manifold has a submanifold $M = \{(x, L) | d_x(f + L) = 0\}$. Since $d_x(f + L) = 0$, by Lemma 1.2, we know $d_x f + L = 0$ and thus $L = -d_x f$. Then the correspondence $x \mapsto (x, -df(x))$ is a diffeomorphism of U onto M. Each $(x, L) \in M$ corresponds to a critical point of $f + L$. This critical point is degenerate when the Hessian H is singular. We have the projection $\pi : M \to Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ defined by $\pi(x, L) = L$.

Now, $L = -d_x f$ and so $\pi(x) = -d_x f$. Thus π is critical at $(x, L) \in M$ when $d\pi = -H$ is singular. Thus $f + L$ has a degenerate critical point if and only if L is the image of a critical point of $\pi : M \to Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n$. By Sard's Theorem, the set of these points has measure 0 in $\mathbb{R}^n$. □

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Lemma 1.9. *Let K be a compact subset of an open set U in $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is C^2 and has only non-degenerate critical points in K, then there is a number $\delta > 0$ such that if $g : U \to \mathbb{R}$ is C^2 and at all points of K satisfies*

$$(1)\ \left|\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}\right| < \delta,\ (2)\ \left|\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 g}{\partial x_i \partial x_j}\right| < \delta$$

$i, j = 1, ..., n$, then g likewise has only non-degenerate critical points.

Proof. Let $|df| = \sqrt{(\frac{\partial f}{\partial x_1})^2 + ... + (\frac{\partial f}{\partial x_n})^2}$. Then the condition $|df| + |det(\frac{\partial^2 f}{\partial x_i \partial x_j}) > 0$ holds for all point in K. For δ small enough, we can ensure that any C^2 function satisfying (1) and (2) also satisfies $|df| + |det(\frac{\partial^2 g}{\partial x_i \partial x_j})| > 0$ and thus has no non-degenerate critical points. □

Lemma 1.10. *Suppose $h : U \to U'$ is a diffeomorphism and carries a compact subset $K \subset U$ onto $K' \subset U'$. Given a number $\varepsilon > 0$, there is a number $\delta > 0$ such that if a smooth map $f : U' \to \mathbb{R}$ satisfies*

$$|f| < \delta, |\frac{\partial f}{\partial x_i}| < \delta, |\frac{\partial^2 f}{\partial x_i \partial x_j}| < \delta, i, j = 1, ..., n$$

At all points of $K' \subset U'$, then $f \circ h$ satisfies

$$|f \circ h| < \varepsilon, |\frac{\partial f \circ h}{\partial x_i}| < \varepsilon, |\frac{\partial^2 f \circ h}{\partial x_i \partial x_j}| < \varepsilon, i, j = 1, ..., n$$

at all points of K.

Proof. □

The C^2 topology on a set $C^\infty(M)$ of smooth real valued functions on a compact manifold, M, with boundary may be defined as follows.

Definition 1.11. Let $\{U_\alpha\}$ be a finite coordinate covering with coordinate maps $h_\alpha : U_\alpha \to \mathbb{R}^n$, and let $\{C_\alpha\}$ be a compact refinement of $\{U_\alpha\}$. For every positive constant $\delta > 0$, define a subset $N(\delta)$, of $C^\infty(M)$ consisting of all maps $g : M \to \mathbb{R}$ such that, for all α,

$$|g_\alpha| < \delta, |\frac{\partial g_\alpha}{\partial x_i}| < \delta, |\frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}| < \delta$$

at all points in $h_\alpha(C_\alpha)$ where $g_\alpha = gh_\alpha^{-1}$ and $i, j = 1, ..., n$. If we take the sets $N(\delta)$ as a base of neighborhoods of the zero functions in the additive group $C^\infty(M)$, the resulting topology is called the C^2 topology.

Remark 1.12. The sets of the form $f + N(\delta) = N(f, \delta)$ give a base of neighborhoods for any map $f \in C^\infty(M)$, and $g \in N(F, \delta)$ means that for all α,

$$|f_\alpha - g_\alpha| < \delta, |\frac{\partial f_\alpha}{\partial x_i} - \frac{\partial g_\alpha}{\partial x_i}| < \delta, |\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 g}{\partial x_i \partial x_j}| < \delta$$

at all points of $h_\alpha(C_\alpha)$

Remark 1.13. Lemma C ensures that the topology is independent of coordinate covering.