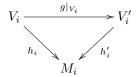
TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Cobordism Constructions

Definition 1.1. Two cobordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V'_0, V'_1; h'_0, h'_1)$ from M_0 to M_1 are equivalent if there exists a diffeomorphism $g: W \to W'$ carrying $V_i \to V'_i$ for i = 0, 1. We have the following commutative diagram:



Now we have a category: (objects: Smooth manifolds, arrows: equivalence clsses of cobordisms).

This means that cobordisms follow these properties:

- 1. Given cobordism equivalence classes c from $M_0 \to M_1$ and c' from $M_1 \to M_2$, there is a well defined cobordism class cc' from $M_0 \to M_2$. This composition operation is associative. This is in accordance with a theorem we mentioned in the last lecture.
- 2. For every closed manifold M there is an identity cobordism class i_M which is the equivalence class of $(M \times I; M \times 0, M \times 1; p_0, p_1), p_i(x, i) = x, x \in M, i = 0, 1$. That is, if c is a cobordism class from $M_1 \to M_2$, then $i_{M_1}c = c = ci_{M_2}$. We will show this in detail later.

Notice that is is possible that $cc' = i_M$ but c is not i_M .

$$c = S^1 \rightarrow disconnected \ shape \rightarrow S^1$$

here c has a right inverse that is not a left inverse. Note that manifolds in cobordisms are not assumed to be connected.

Consider the cobordism classes from M to itself. M fixed. These form a monoid H_M , i.e. a set with an associative composition with an identity. The invertible cobordisms in H_M form a group G_M . We can construct some elements of G_M by taking M = M' below.

Construction 1.2. Given diffeomorphism $h: M \to M'$, define c_h as the class of $(M \times I; M \times 0, M \times 1, j, h_1)$ where j(x, 0) = x and $h_1(x, 1) = h(x), x \in M$.

Theorem 1.3. $c_h c_{h'} = c_{h'} c_h$ for any two diffeomorphisms $h: M \to M'$ and $h': M' \to M''$.

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Proof. We consider $W = M \times [0,1] \cup_h M' \times [0,1]$ as in $c_h c_{h'}$. We let

$$j_h: M \times [0,1] \to M, \ j_{h'}: M' \times [0,1] \to M'$$

be defined by $j_h(x,t)=x$ and $j_{h'}(x,t)=x$. Let $W'=M\times [0,1]$ as in the definition of $c_{h'h}$. Then we can define an equivalence of cobordisms $g:W'\to W$ defined by

$$g(x,t) = \begin{cases} j_h(x,2t) & 0 \le t \le 1/2 \\ j'_h(x,2t-1) & 1/2 \le t \le 1 \end{cases}$$