TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Topology Review

Since Differential Topology assumes familiarity with topology, we will give a short review

Notation 1.1. We denote Euclidean Space by

$$\mathbb{R}^n = \{(x_1, ..., x_n) : x_i \in \mathbb{R}\, i = 1...n\}$$

Where $\mathbb{R}$ denotes all real numbers. We denote the Euclidean half space by:

$$\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n \geq 0, i = 1...n\}$$

Definition 1.2. A topological space is a set X (called the underlying set) with a collection of subsets τ (called open sets) such that:

1. $X, \emptyset \in \tau$
2. Any union of open sets is open.
3. Finite intersections of open sets are open.

τ is also called a topology on X. We usually denote topological spaces by their undelying set. We say any member of τ is open in its underlying set.

Example 1.3. $\mathbb{R}$ with $\tau = \{\text{open intervals in } \mathbb{R}\}$. This is the "usual" or "standard" topology on $\mathbb{R}$.

Example 1.4. $\mathbb{R}^n$ with $\tau = \bigcup_{\varepsilon \in \mathbb{R}_+} \bigcup_{x \in \mathbb{R}^n} B_\epsilon(x)$ where $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\}$ = open ball of radius ε and center x.

Example 1.5. Given X, we let $\tau = \{\text{Power set of } X\}$. This is called the "discrete topology".

Example 1.6. Given X, we let $\tau = \{\emptyset, X\}$. This is called the "trivial topology"

Construction 1.7. A subset Y of a topological space X with topology τ can be given a topology called the subspace topology τ'. Here we say $V \in \tau'$ if there exists $U \in \tau$ such that $U \cap Y = V$

Definition 1.8. A neighborhood of a point $x \in X$ is a set V containing an open set U such that $x \in U \subset V$.

Definition 1.9. A family of sets $\mathscr{B}$ is called a basis for a topology on a space X if the following properties hold:

1. $\forall x \in \mathscr{B}$, there is a $B \in \mathscr{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

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Definition 1.10. If $\mathscr{B}$ is a basis for a topology on X, the topology τ generated by $\mathscr{B}$ is the collection of open sets defined as follows: a subset U of X is said to be open if $\forall x \in U$ there is a $B \in \mathscr{B}$ such that $x \in B \subset U$.

Remark 1.11. Bases are not unique but they do generate a unique topology.

Definition 1.12. An open cover of a topological space X is a collection of subsets $\{U_\alpha\}_{\alpha \in I}$ such that $\bigcup_{\alpha \in I} U_\alpha = X$.

Definition 1.13. A topological space X is called compact if for every open cover $\{U_\alpha\}_{\alpha \in I}$, we can find a finite subcover $\{U_i\}_{i=1}^n$.

Definition 1.14. Let X, Y be topological spaces. A function $f : X \to Y$ is continuous if $f^{-1}(U)$ is open in X for any U open in Y

Remark 1.15. This notion of continuity coincides with the definition from freshman calculus under the standard topology on $\mathbb{R}$.

Definition 1.16. A continuous function $f : X \to Y$ is called a homeomorphism if $\exists g : Y \to X$ continuous such that $f \circ g = id_Y$ and $g \circ f = id_X$.

Definition 1.17. A topological space X is called Hausdorff or T_2 if $\forall x, y \in X$ with $x \neq y$, there exists neighborhoods V_x and V_y, of x and y respectively, such that $V_x \cap V_y = \emptyset$. In other words, two distinct points have disjoint neighborhoods.

We have a nice category **Top** in which to work in. Its objects are topological spaces. Its morphisms are continuous functions. In this category, homeomorphism corresponds to our notion of isomorphism.

We will have to get used to this definition now that we have the language to express it:

Definition 1.18. A topological n-manifold (or just manifold) is a Hausdorff topological space X such that every $x \in X$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 1.19. A manifold M is called compact if M is compact as a topological space.