TOPICS IN DIFFERENTIAL TOPOLOGY

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1. The Category of Smooth Manifolds

Definition 1.1. Let $V \subset \mathbb{R}^n$. A map $f : V \to \mathbb{R}^n$ is called smooth or differentiable of class C^∞ if f can be extended to a map $g : U \to \mathbb{R}^n$ with $U \supset V$ is open in $\mathbb{R}^n$ and all partial derivatives of g exists and are continuous.

Definition 1.2. A smooth function $f : V \to U$ with $V, U \subset \mathbb{R}^n$ is a diffeomorphism if $\exists g : U \to V$ smooth with $f \circ g = g \circ f = id$. We say U and V are diffeomorphic.

Definition 1.3. A smooth n-manifold M is a topological n-manifold with a countable basis together with a smoothness structure $\mathscr{S}$ on M. $\mathscr{S}$ is a collection of pairs (U_i, ϕ_i) satisfying:

1. Each $(U_i, \phi_i) \in \mathscr{S}$ (called charts) consists of an open set $U \subset M$ (called the coordinate patch) and a homeomorphism $\phi_i : U \to V$ (called the coordinate map) which maps U onto some open subset of $\mathbb{R}^n$ or $\mathbb{R}^n_+$.
2. $\bigcup U_i = M$
3. if $(U_i, \phi_i), (U_j, \phi_j) \in \mathscr{S}$ for $i \neq j$, then $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \mathbb{R}^n$ or $\mathbb{R}^n_+$ is smooth.
4. $\mathscr{S}$ is minimal with respect to 3. i.e. if $(U, \phi) \notin \mathscr{S}$ is adjoined to $\mathscr{S}$, then property 3. fails.

Construction 1.4. A function $f : M \to N$ from an m-manifold M to an n-manifold N is called smooth if for every $x \in M$ there is a chart (U, ϕ) in M containing x and a chart (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$ is smooth.

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