TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Cobordism Constructions

Definition 1.1. Two cobordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_0', V_1'; h_0', h_1')$ from M_0 to M_1 are equivalent if there exists a diffeomorphism $g : W \to W'$ carrying $V_i \to V_i'$ for $i = 0, 1$. We have the following commutative diagram:

$$\begin{tikzcd} V_i \arrow[rr, "g|_{V_i}"] \arrow[dr, "h_i"'] & & V_i' \arrow[dl, "h_i'"] \\ & M_i & \end{tikzcd}$$

Now we have a category: (objects: Smooth manifolds, arrows: equivalence clsses of cobordisms).

This means that cobordisms follow these properties:

1. Given cobordism equivalence classes c from $M_0 \to M_1$ and c' from $M_1 \to M_2$, there is a well defined cobordism class cc' from $M_0 \to M_2$. This composition operation is associative. This is in accordance with a theorem we mentioned in the last lecture.
2. For every closed manifold M there is an identity cobordism class i_M which is the equivalence class of $(M \times I; M \times 0, M \times 1; p_0, p_1), p_i(x, i) = x, x \in M, i = 0, 1$. That is, if c is a cobordism class from $M_1 \to M_2$, then $i_{M_1} c = c = c i_{M_2}$. We will show this in detail later.

Notice that is is possible that $cc' = i_M$ but c is not i_M.

$$c = S^1 \to \textit{disconnected shape} \to S^1$$

here c has a right inverse that is not a left inverse. Note that manifolds in cobordisms are not assumed to be connected.

Consider the cobordism classes from M to itself. M fixed. These form a monoid H_M, *i.e.* a set with an associative composition with an identity. The invertible cobordisms in H_M form a group G_M. We can construct some elements of G_M by taking $M = M'$ below.

Construction 1.2. Given diffeomorphism $h : M \to M'$, define c_h as the class of $(M \times I; M \times 0, M \times 1, j, h_1)$ where $j(x, 0) = x$ and $h_1(x, 1) = h(x), x \in M$.

Theorem 1.3. *$c_h c_{h'} = c_{h'} c_h$ for any two diffeomorphisms $h : M \to M'$ and $h' : M' \to M''$.*

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Proof. We consider $W = M \times [0,1] \cup_h M' \times [0,1]$ as in $c_h c_{h'}$. We let

$$j_h : M \times [0,1] \to M,\ j_{h'} : M' \times [0,1] \to M'$$

be defined by $j_h(x,t) = x$ and $j_{h'}(x,t) = x$. Let $W' = M \times [0,1]$ as in the definition of $c_{h'h}$. Then we can define an equivalence of cobordisms $g : W' \to W$ defined by

$$g(x,t) = \begin{cases} j_h(x,2t) & 0 \leq t \leq 1/2 \\ j_h'(x,2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

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