TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Introduction to Cobordisms

Definition 1.1. A (smooth) k-manifold with boundary M the pair $(M^\circ, \partial M)$ such that $M = M^\circ \sqcup \partial M$, M° is a (smooth) k-manifold and every point $x \in \partial M$ has a neighborhood V and a homeomorphism $f : V \to U \subset \mathbb{R}^k_+$, U open, such that $f(x) = y$ and $y_n = 0$. M° is called the interior and ∂M is called the boundary.

The boundary of a manifold W, can be thought of as the set of points that do not have neighborhoods homeomorphic to an open set in $\mathbb{R}^n$.

Example 1.2. The 2-cylinder $C = S^1 \times [0,1]$ has boundary $\partial C = S^1 \sqcup S^1$

Example 1.3. The 2-disk $D = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ has boundary $\partial D = S^1$

Definition 1.4. A smooth manifold is said to be closed if it is compact and without boundary.

Definition 1.5. $(W; V_0, V_1)$ is a smooth manifold triad if W is a compact smooth n-manifold and $\partial W = V_0 \sqcup V_1$ where V_0, V_1 are two compact smooth $(n-1)$-manifolds without boundary.

If $(W; V_0, V_1)$ and $(W'; V_1', V_2')$ are two smooth manifold triads and $h : V_1 \to V_1'$ is a diffeomorphism, then we can form a third triad $(W \cup_h W'; V_0, V_2')$ where $W \cup_h W'$ is the space given by identifying points of V_1, V_1' under h, according to the following theorem:

Theorem 1.6. *There exists a smoothness structure $\mathscr{S}$ for $W \cup_h W'$ compatable with the given structures. i.e. so the inclusion maps $W \hookrightarrow W \cup_h W' \hookleftarrow W'$ are diffeomorphisms onto their images. $\mathscr{S}$ is unique up to diffeomorphism, leaving $V_0, h(V_1) = V_1'$ and V_2 fixed.*

Proof. We will prove this later. $\square$

Definition 1.7. Given two closed manifolds M_0, M_1, a cobordism from $M_0 \to M_1$ is a 5-tuple $(W; V_0, V_1; h_0, h_1)$ where $(W; V_0, V_1)$ is a smooth manifold triad and $h_i : V_i \to M_i$ is a diffeomorphism for $i = 0, 1$.

Example 1.8.

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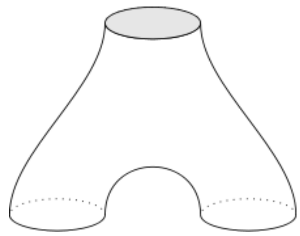

FIGURE 1. A cobordism $S^1 \to S^1 \sqcup S^1$ called a "pair of pants"