TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Morse Theory

Definition 1.1. Let $f : U \to \mathbb{R}^n$ be a smooth function. A point $p \in U$ is called a critical point if $rank d_x f < n$. A critical point is degenerate if the hessian $H_x(f)$ is singular.

Lemma 1.2. *(Morse's Lemma): If p is a non-degenerate critical point of a smooth function f, then in some coordinate system about p,*

$$f(x_1, ..., x_n) = c - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2$$

λ is called the index of the critical point p.

Corollary 1.3. *Non-degenerate critical points are isolated.*

Proof. $\nabla f = \langle -2x_1, -2x_2, ..., -2x_\lambda, 2x_{\lambda+1}, ..., 2x_n \rangle$ □

Definition 1.4. The Morse number μ of (W, V_0, V_1) is the minimum over all Morse functions f of the number of critical points of f.

The goal for the next few weeks will be to prove the following theorem:

Theorem 1.5. *Every smooth manifold triad has a morse function.*

We need a few lemmas before we are able to prove this.

Definition 1.6. A smooth partition of unity is a collection of smooth functions $\{\phi_i\}$ such that

$$\sum_i \phi_i = 1$$

Definition 1.7. The support of a function f denoted *supp* f is the closure of the set $\{x : f(x) \neq 0\}$.

Definition 1.8. A partition of unity $\{\phi_i\}$ is said to be subordinate to a cover $\{U_i\}$ if *supp* $\phi_i \subset U_i$.

Lemma 1.9. *Every open cover of a smooth manifold has a partition of unity subordinate to it.*

Lemma 1.10. *There exists a smooth function $f : W \to [0, 1]$ with $f^{-1}(0) = V_0$ and $f^{-1}(1) = V_1$.*

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Proof. Let $U_1, ..., U_k$ be a cover of W by coordinate neighborhoods. On each U_i we define a map
$$f_i : U_i \to [0,1]$$
as follows. If $U_i \cap V_0 \neq 0$ then $f_i(x) = x_n$. If $U_i \cap V_1 \neq 0$ then we let $f_i(x) = 1 - x_n$. Otherwise we let $f_i = 1/2$. Choose a partition of unity subordinate to our open cover and define $f : W \to [0,1]$ by
$$f(p) = \phi_1(p) f_1(p) + ... + \phi_k(p) f_k(p)$$
Then we see f is a smooth map with the first two desired properties. □

Lemma 1.11. *f as defined above has no critical points in a neighborhood of the boundary of W.*

Proof. Suppose $q \in V_0$. Then for some i, we have $\phi_i(q) > 0$ and $q \in U_i$ Let $h_i(p) = (x^1(p), ..., x^n(p))$. Then
$$\frac{\partial f}{\partial x^n} = \sum_{j=1}^{k} f_j \frac{\partial \phi_j}{\partial x^n} + (\phi_1 \frac{\partial f_1}{\partial x^n} + ... + \phi_i \frac{\partial f_i}{\partial x^n} + ...)$$
Since $q \in V_0$, $f_j(q) = x_n = 0$. If $q \in V_1$, since
$$\sum_{j=1}^{k} \frac{\partial \phi_j}{\partial x^n} = \frac{\partial}{\partial x^n} (\sum_{j=1}^{k} \phi_j) = 0$$
the first summand vanishes. $\frac{\partial f_i}{\partial x^n}(q) = 1$ if $q \in V_0$ and -1 if $q \in V_1$. Also $\frac{\partial f_j}{\partial x^n}(q)$ is either 0 or the same sign as $\frac{\partial f_i}{\partial x^n}(q), j = 1, ...k$. Therefore $\frac{\partial f}{\partial x^n}(q) \neq 0$ and thus $df \neq 0$ on ∂W. Thus $df \neq 0$ in a neighborhood of ∂W. □