TOPICS IN DIFFERENTIAL TOPOLOGY

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1. Morse Theory: Prelims and Motivation

Definition 1.1. Two diffeomorphisms $h_0, h_1 : M \to M'$ are called (smoothly) isotopic if there exists a map $f : M \times I \to M'$ such that

1. f is smooth
2. $f_t(x) = f(x,t)$ is a diffeomorphism for all $t \in I$.
3. $f(x,i) = h_i(x)$ for $i = 0, 1$

Definition 1.2. Two diffeomorphisms $h_0, h_1 : M \to M'$ are called "pseudo"-isotopic if there is a diffeomorphism $g : M \times I \to M' \times I$ such that $g(x,0) = (h_0(x), 0)$ and $g(x,1) = (h_1(x), 1)$.

Lemma 1.3. *Isotopy and pseudo-isotopy are equivalence relations.*

Theorem 1.4. $c_h = c'_h$ *if and only if* h *is pseudo-isotopic to* h'.

Proof. Suppose we have a pseudo-isotopy between h_0 and h_1, $g : M \times I \to M' \times I$. Then we define $h_0^{-1} \times 1 : M' \times I \to M \times I$ by $(h_0^{-1} \times 1)(x,t) = h_0^{-1}(x), t)$. Then $(h^{-1} \times 1) \circ g$ is an equivalence between c_{h_0} and c_{h_1}. □

Let $p \in M$. Then there exists a pair $(U, \phi) \in \mathscr{S}$ such that $x \in U$. $\phi : U \to V \subset \mathbb{R}^n$ is our coordinate mapping. We can identify p with its coordinates $\phi(p) = (p^1, ..., p^n)$ in $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Since $f(x)$ is a vector for every $x \in \mathbb{R}^n$, we can really think of f as a vector of functions $f^1, ..., f^m : \mathbb{R}^n \to \mathbb{R}$ where $f(x) = (f^1(x), ..., f^m(x))$. We can define the derivative $df_p(h) = \lim_{t \to 0} \frac{f(p+th) - f(p)}{t}$ for $h \in \mathbb{R}^n$ this is a linear mapping $df_p : \mathbb{R}^n \to \mathbb{R}^m$. As you know from linear algebra, you can define linear mapping in terms of how they act on the standard basis elements. Here $df_x = [df_p(x^1)...df_p(x^n)]$

$$J|_p = \begin{bmatrix} \frac{\partial f^1}{\partial x^1}(p) & ... & \frac{\partial f^1}{\partial x^n}(p) \\ . & & . \\ . & & . \\ . & & . \\ \frac{\partial f^m(x)}{\partial x^1}(p) & ... & \frac{\partial f^m}{\partial x^m}(p) \end{bmatrix}$$

The matrix J is called the **Jacobian matrix** of f. A point $x \in \mathbb{R}^n$ is called a regular point of f if $J|_x$ has maximal rank. Otherwise, we call x a singular point. Another important matrix is the Hessian, which allows us to generalize the "second derivative test" from freshman calculus.

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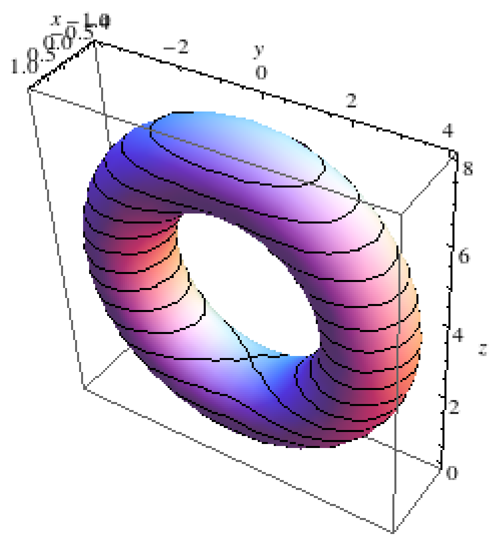
The Hessian matrix of a twice differentiable function $f : \mathbb{R}^n \to R$ is the matrix given by:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^1 x^1} & \cdots & \frac{\partial^2 f}{\partial x^n x^1} \\ . & & . \\ . & & . \\ . & & . \\ \frac{\partial^2 f}{\partial x^1, x^n} & \cdots & \frac{\partial^2 f}{\partial x^n x^n} \end{bmatrix}$$

Remark 1.5. H is a symmetric matrix

A singular point x of f is called **non-degenerate** if $\det H|_x \neq 0$.
Given a cobordism (W, V_0, V_1, h_0, h_1) with $W \subset \mathbb{R}^n$, we can define the height function $f : W \to \mathbb{R}$ where $f(x) = x_n$. Consider $W = torus$ which is a cobordism between $\emptyset$ and $\emptyset$. Then viewing the level curves, we see $f(W)$ has a non-degenerate critical point (in fact, it has four!).



These critical points of the height function tell us something interesting about the topology of the manifold.

Examples 1.6. Let $a, b \in W$ be the two inner critical points of the torus ordered by height. Then if $0 \leq x < f(a)$ we have $W^x = \{p : f(p) < x\}$ is a cobordism between $\emptyset$ and S^1. For $f(a) < x < f(b)$, W^x is a cobordism between $\emptyset$ and $S^1 \sqcup S^1$...etc.

Something about the topology of W^x changes as we pass the critical point. We want to investigate how these functions can help us deconstruct cobordisms. Morse theory will allow us to do that.

Definition 1.7. A morse function on a smooth manifold triad (W, V_0, V_1) is a smooth function $f : W \to [a, b]$ such that:

1. $f^{-1}(a) = V_0, f^{-1}(b) = V_1$
2. All of the critical points of f lie in W°.