CSE 595 Independent Study
Graph Theory

Week 2

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Problem 19 (Bipartite Graphs)

A bipartite graph *G* of order n has partite sets *U* and *W* where $|U| = 10$. Every vertex of *U* has degree 6. In *W*, there are four vertices of degree 2 and three vertices of degree 4. All other vertices of *G* have degree 8. What is *n*?

In a bipartite graph, all nodes in *U* must be connected to nodes in *W*. Therefore, there must be 60 edges from *U* to *W*, by the fact that *U* has 10 nodes each with 6 degrees. Counting the amount of edges given in *W*, it is known that there are $(4 \times 2) + (3 \times 4) = 20$ edges from the four vertices of degree 2, and three vertices of degree 4. The remaining number of vertices is determined by their degree, which is 8. Since $60 - 20 = 40$ edges remain, $\frac{40}{8} = 5$ nodes of degree 8 must exists in *W*.

By definition of a bipartite graph, all nodes of *G* are separated into the sets *U* and *W*. Therefore, we can say

$$n = |W| + |U|$$

$$n = 12 + 10$$

$$\therefore n = 22 \blacksquare$$

Problem 23 (Operations on Graphs)

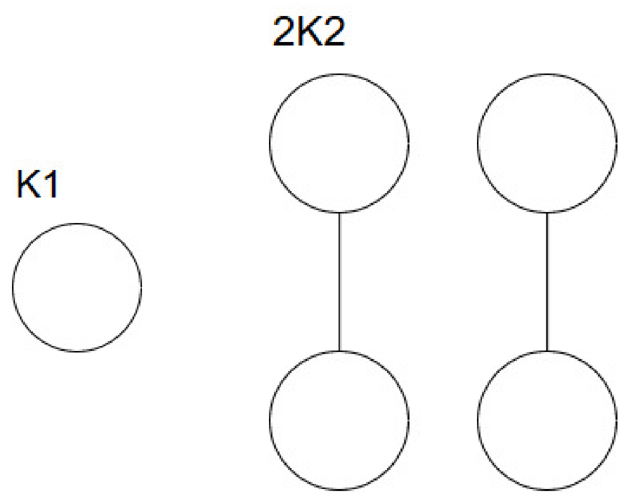
Determine all bipartite graphs G such that $\bar{G}$ is bipartite.

Any graph with a partite set with 3 or more nodes will have a complement with a triangle, and thus in a bipartite setting, both partite sets may not have more than 3 nodes. Therefore, size of *G* will be 4 or less.

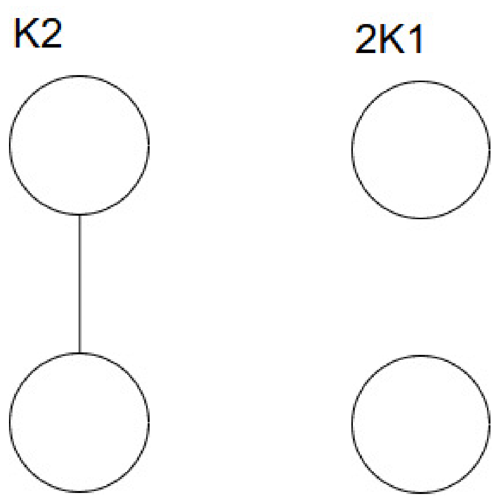
The solution are then the following graphs,

$$K_1, K_2, 2K_1, 2K_2 \blacksquare$$

K_1 and $2K_2$ are self-complementary because they are isomorphic to its complement.



K_2 has a compliment of $2K_1$



Problem 25 (Operations on Graphs)

(a) Show that there are exactly two 4-regular graphs *G* of order 7.
(b) How many 6-regular graphs of order 9 are there?

By Theorem 1.7 in Chartrand [1],

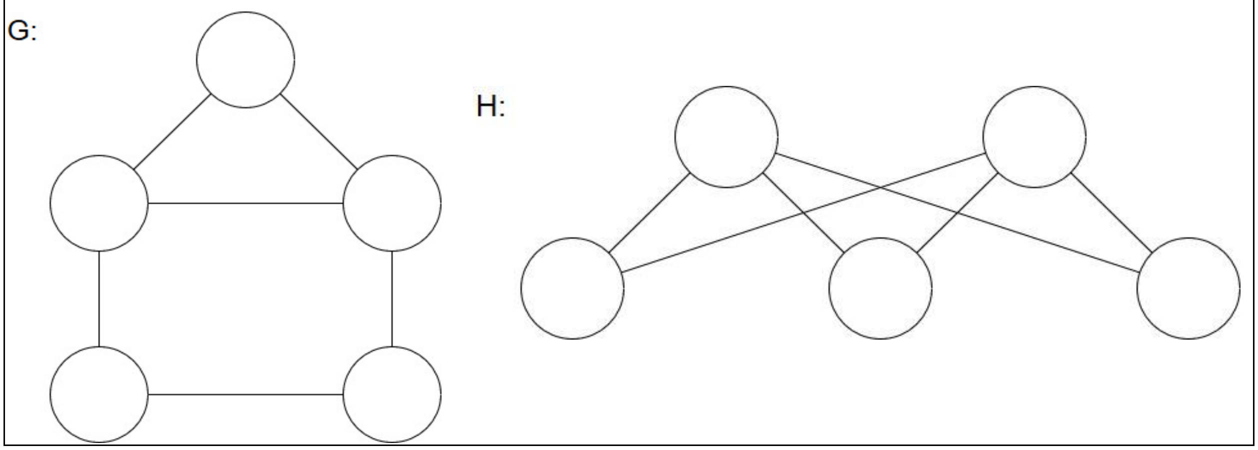
For integers *r* and *n*, there exists an *r*-regular graph of order *n* if and only if $0 \leq r \leq n - 1$ and *r* and *n* are not both odd.

Therefore, we know that solutions exist for both problems.

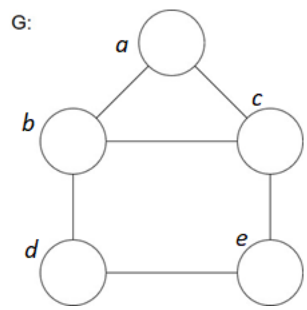
(a) It is easier to look at the compliment of graph G which will be denoted by $\bar{G}$. Thus, by definition [1] the order and size of $\bar{G}$ is determined by G. Since G is of order 7 and each vertex has degree of 6, then size m of G is $\frac{(7)(4)}{2} = 14$, then $\bar{G}$ is of order 7 and size $\binom{7}{2} - 14 = 7$. We can conclude from the order and size that $\bar{G}$ must be a cycle. The only two graphs that may exist are C_7 and $C_4 + C_3$ because of the fact cycles must have order ≥ 3. ∎
(b) Once again, it is better to look at the compliment of *G*. The order of *G* is 9 and each vertex has degree 6, then size m of *G* is =$\frac{(9)(6)}{2} = 27$, then $\bar{G}$ is of order 9 and size $\binom{9}{2} - 27 = 9$. $\bar{G}$ must be constructed from cycles, so the possible cycles are $C_9, C_6 + C_3, C_5 + C_4$, and $3C_3$.∎

Problem 35 (Degree Sequences)

Find a sequence of 2-switches that transforms the graph *G* below into graph *H*.



We may begin by labeling the nodes in *G* in order to transform using 2-switches to transform into *H*.



Thus, we may delete *bc* and *de* from *G* , and add *cd* and *be* In order to create the graph *H.*

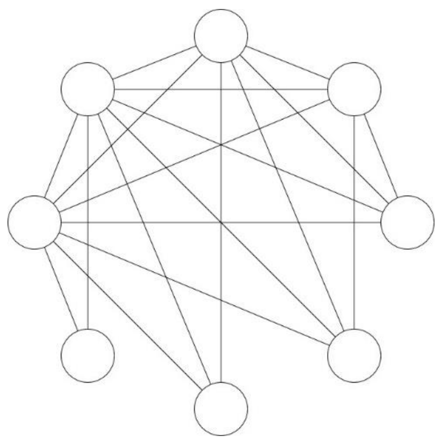
$$G - bc - de + cd + be \blacksquare$$

Problem 37 (Degree Sequences)

Determine whether the following sequences are graphical. If so, construct a graph with appropriate degree sequence.

(a) 4,4,3,2,1
(b) 3,3,2,2,2,2,1,1
(c) 7,7,6,5,4,4,3,2
(d) 7,6,6,5,4,3,2,1
(e) 7,4,3,3,2,2,2,1,1,1

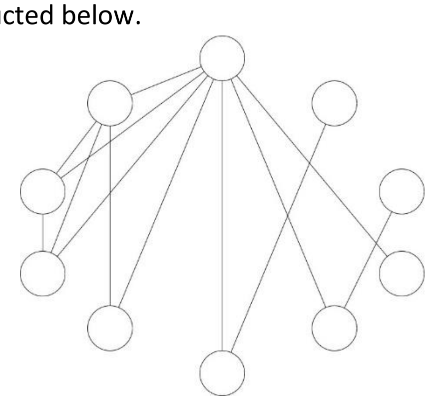
(a) Is not graphical. If graph *G* has 5 vertices, two of which are degree 4, then the other vertices must have degree ≥ 2.
(b) Is graphical and can be constructed using bipartite graphs of $K_{2,3} + K_{1,2}$.
(c) Is graphical and is constructed below.



(d) Is not graphical. Using Theorem 1.12 (Havel-Hakimi Theorem) [1] we may apply the algorithm until either a sequence of 0's is reached, which implies the sequence is graphical, or until a negative number is reached, which implies a nongraphical sequence.

$$s: 7,6,6,5,4,3,2,1$$
$$s_1': 5,5,4,3,2,1,0$$
$$s_2': 4,3,2,1,0,0$$
$$s_3': 2,1,0,-1,0$$

A negative number is reached, and therefore the sequence is nongraphical.
(e) Is graphical and is constructed below.



Works cited

"Introduction." *Graphs & Digraphs*, by Gary Chartrand et al., CRC Press, 2016, pp. 13–24.