CSE 595 Independent Study
Graph Theory

Week 3

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Chapter 1 Problem 53 (Multigraphs)

Give an example of an irregular multigraph (if such a multigraph exists) having degree sequence

(a) 5,4,3,2,1
(b) 6,5,4,3,2,1
(c) 7,6,5,4,3,2,1

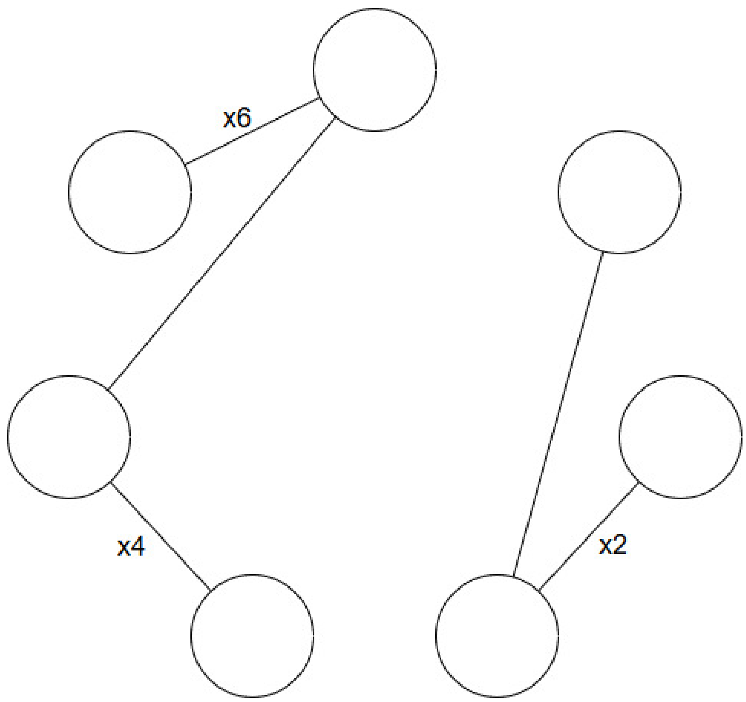
(a) This sequence is not capable of a multigraph because of Corollary 1.5 [1] which states

Every graph has an even number of odd vertices.

It is, however, capable of a pseudograph, where a loop (an edge which is connected to the same node) exists.

(b) This sequence also contradicts the Corollary 1.5 from part a, and therefore a multigraph does not exist.

(c) The following does exist as a multigraph, shown below.



Chapter 2 Problem 3 (Connected Graphs)

Let G_1, G_2 and G_3 be three graphs of order *n* and size *m* having adjacency matrices A_1, A_2, and A_3 respectively.

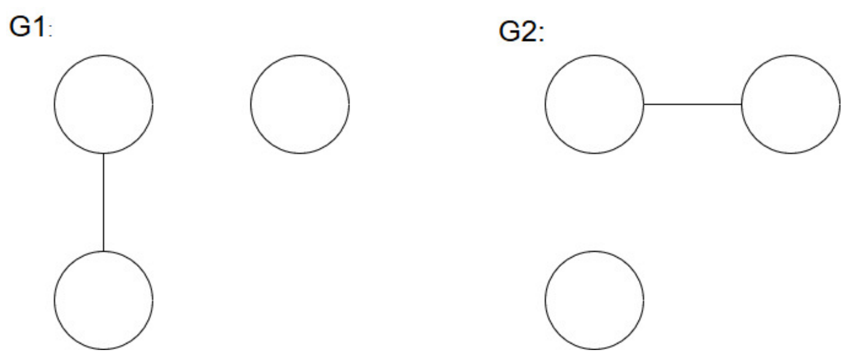
(a) Prove or disprove: If $A_1 = A_2$, then $G_1 \cong G_2$
(b) Prove or disprove: If $A_1 \neq A_2$, then $G_1 \ncong G_2$

(a) Let u, v be vertices in graph G_1, G_2 respectively. A matrix is equal iff they have the same dimensionality and the corresponding elements are the same. Therefore, if $A_1 = A_2$ then these graphs must be isomorphic with each vertex $v_{i,j} \in G_1$ having the same exact adjacent vertices as its counterpart $u_{i,j} \in G_2$. Hence, if $A_1 = A_2$, then $G_1 \cong G_2$ is true. ■

(b) Assume that this statement is true. Therefore, there should not exists two graphs G_1, G_2 such that $A_1 \neq A_2$ and $G_1 \cong G_2$. Examining the following two adjacency matrices A_1, A_2

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

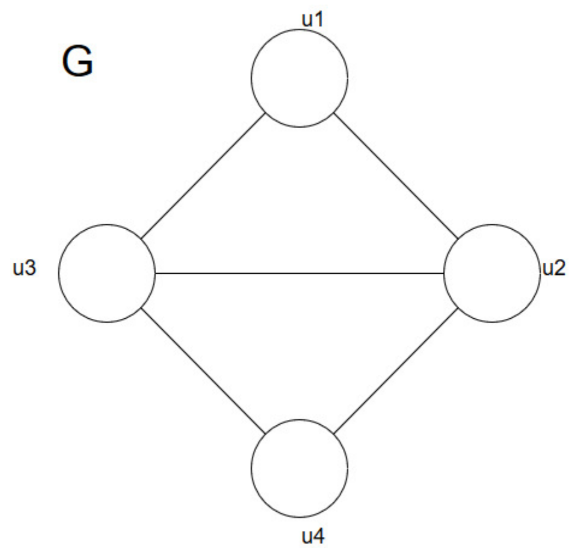
Obviously, these matrices are not equal, however the graphs are isomorphic, as seen below.



Thus, the statement if $A_1 \neq A_2$, then $G_1 \ncong G_2$ is not true. ■

Chapter 2 Problem 5 (Connected Graphs)

Determine the adjacency matrix of the graph G below. Then determine A^2 and A^3 without multiplying matrices.



The adjacency matrix of graph *G* is,

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The adjacency matrix of graph *G* with walk length 2 is,

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

The adjacency matrix of graph *G* with walk length 3 is,

$$A^3 = \begin{bmatrix} 2 & 4 & 4 & 2 \\ 4 & 4 & 3 & 4 \\ 4 & 3 & 4 & 4 \\ 2 & 4 & 4 & 2 \end{bmatrix}$$

Chapter 2 Problem 7 (Connected Graphs)

Determine the graph G with adjacency matrix A for which

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \ and\ A^3 = \begin{bmatrix} 2 & 2 & 3 & 1 & 1 \\ 2 & 2 & 3 & 1 & 1 \\ 3 & 3 & 2 & 4 & 0 \\ 1 & 1 & 4 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix}$$

Upon trying to solve this problem and using the solutions and hints portion in Chartrand [1], it was determined that the matrix A^3 is not actually the power of matrix A. The solution, according the text $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_5\}$. This graph has the corresponding adjacency matrix,

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Multiplying $A \times A$ gives the correct A^2 matrix. Multiplying $A^2 \times A$ does not get the A^3 matrix given in the problem. Instead, the matrix A^3 is

$$A^3 = \begin{bmatrix} 2 & 3 & 4 & 1 & 1 \\ 3 & 2 & 4 & 1 & 1 \\ 4 & 4 & 2 & 4 & 0 \\ 1 & 1 & 4 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix}$$

I have verified that the A^3 would not be correct, as performing $A^{-2} \times A^3 \neq A$ and in fact if this operation is performed, non-integer numbers are achieved.

Problem 25 (Distance in Graphs)

Let *u*, *v*, and *w* be three vertices in a connected graph *G*. Prove that $d(u,v) + d(u,w) + d(v,w) \geq 2d(u,w)$.

First reduce the problem algebraically

$$d(u,v) + d(u,w) + d(v,w) \geq 2d(u,w)$$

$$d(u,v) + d(v,w) \geq d(u,w)$$

Thus, based on the triangle inequality in Chartrand [1] which states

$$d(u,v) + d(v,w) \geq d(u,w) \textit{ for all } u, v, w \in V(G)$$

However, for the sake of further examination of the problem,

If we assume the case $u = v$

$$d(u,v) = 0 \to d(u,v) + d(v,w) = d(u,w)$$

$$\therefore d(u,w) = d(u,w)$$

All other cases if it obvious to see that if the path $u - v \in u - w$ geodesic then,

$$d(u,v) + d(v,w) = d(u,w)$$

Otherwise,

$$d(u,v) > 0 \to d(u,v) + d(v,w) > d(u,w) \blacksquare$$

Problem 31 (Distance in Graphs)

Let *a* and *b* be positive integers with $a \leq b \leq 2a$. Show that there exists a connected graph *G* with $rad(G) = a$ and $diam(G) = b$.

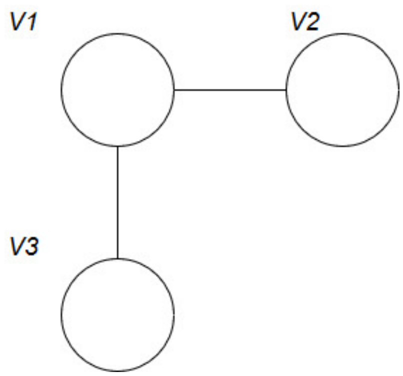
By definition of the radius and diameter of a graph, the following portion of the inequality,

$$a \leq b$$

Is obviously true, so the other inequality to show is,

$$b \leq 2a$$

Consider the following graph,



$$e(v_1) = 1, \qquad e(v_2) = e(v_3) = 2$$

Therefore,

$$Diam(G) = 2, \qquad Rad(G) = 1$$

Thus, a graph *G* with the property exists $a \leq b \leq 2a$. ■

Works cited

“Connected Graphs and Digraphs.” *Graphs & Digraphs*, by Gary Chartrand et al., CRC Press, 2016, pp. 25–55.