1.1

Graph is defined by G = (V,E) where V is vertices, and E is edges.

V(G) is the vertex set denoted by {u,v}, and E(G) is the edge set denoted by {uv}

If uv is an edge, then u and v are **adjacent vertices**

Neighborhoods of a vertex are all the other vertices connected by an edge to that vertex

Adjacent edges are considered edges connected by a vertex.

Order of a graph is the number of vertices

Size of a graph is the number of edges

Graph of order 1 is called **Trivial**, and therefore **Nontrivial** is a graph that has two or more vertices

Graph of size 0 is called an **empty** graph, any other graph is considered **nonempty.**

Complete graph is every all distinct vertices are adjacent

1.2

Degree of a vertex is the number of vertices adjacent to the vertex

Vertex of degree 0 is an **isolated vertex**

Vertex of degree 1 is an **end-vertex** or a **leaf**

Maximum degree of G is the largest degree of any vertex in G denoted by Δ (uppercase delta)

Minimum degree of G is the smallest degree of any vertex in G denoted by δ (lowercase delta)

Theorem 1.4 (First Theorem of Graph Theory) – let m be the size of a graph, then the summation of all vertex degrees is equal to **2m**

Average degree or order n and size m is **2m/n**

A vertex in graph G is **even or odd** depending on if its degree is even or odd

Corollary 1.5 - A graph can have an even or odd number of **even** vertices, but **must** have an even number of **odd** vertices.

1.3

Isomorphic graphs mean has the same structure

Function ϕ is called **isomorphism** from graph G to H (or $\phi: V(G) \rightarrow V(H)$)

If G and H are isomorphic, we write $G \cong H$

Theorem 1.6 – If two graphs G and H are isomorphic, then they have the same order and the same size, and the degrees of the vertices of G are the same as the degrees of the vertices of H.

H is a **subgraph** of G if V(H) is in V(G) and E(H) is in E(G), notation is H in G

G is a **supergraph** of H if H is a subgraph of G.

If V(H)=V(G) then H is a **spanning subgraph** of G.

If H is a subgraph of G and is not isomorphic to G, then H is a **proper subgraph** of G.

If a graph is an **induced subgraph** then there is a nonempty set S of V(G) that creates H=G[S]

If a graph is **edge induced** is there is a nonempty subset X of E(G) such that H=E[X]

1.4

A graph G is **regular** if the vertices of G have the same degree and is **regular of degree** r if this degree is r.

Also called r-**regular**.

Theorem 1.7 – For integers r and n, there exists an r-regular graph of order n if and only if 0 <= r <= n-1 and r and n are not both odd.

A Petersen Graph is a 3-regular graph also called **cubic**

1.5

A **bipartite graph** can be partitioned into two sets, U and W such that very edge in G joins a vertex of U with a vertex off W.

Since [U,W] is the set of edges connecting partite sets then E(G) = [U,W].

Theorem 1.8 The size of every bipartite graph of order n is at most $\left\lfloor \frac{n^2}{4} \right\rfloor$

Theorem 1.9 Every graph of order $n \geq 3$ and size $m > \left\lfloor \frac{n^2}{4} \right\rfloor$ contains a triangle

1.6

Complement of graph *G* is *G'*, where vertex set V(G) have adjacent vertices, G' does not have those vertices, and vice versa.

A graph *G* is **self-complementary** if *G* is isomorphic to *G'*.

Self-complementary graph *G* of order *n* has size $m = \frac{\binom{n}{2}}{2} = \frac{n(n-1)}{4}$

Union of a graph *G = G1 + G2*. The union of G+G of two disjoint copies of G is denoted by 2G.

Join G=G1vG2 of G1 and G2 has vertex set of V(G)=V(G1)UV(G2) and edge set E(G)=E(G1)UE(G2)U{all vertex of each graph & connect them}.

1.7

Degree sequence of a graph *G* of order *n* if the vertices of *G* can be labeled v1,v2,…,vn so that deg vi=di for 1<= I <= n.

Graphical sequence if s is finite nonnegative integer set

2-switch is deleting two edges, and adding two different edges into a graph, which will contain the same degree sequence.

Theorem 1.10 – Let $s: d_1, d_2, \dots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and let $\mathcal{g}_s$ be the set of all graphs *F* with degree sequence *s* such that $V(F) = \{v_1, v_2, \dots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$. Then every graph $H \in \mathcal{g}_s$ can be transformed into a graph $G \in \mathcal{g}_s$ by a sequence of 2-switches such that $N_G(v_1) = \{v_2, v_3, \dots, v_{\Delta+1}\}$

2.1

Walk *W* in *G* is a sequence of vertices in *G,* beginning with *u* and ending at *v*. Nonconsecutive vertices need not be distinct.

Length of a walk *W* is the number of edges encountered in *W*.

Open Walk is a walk whose initial and terminal vertices are distinct.

Closed walk is a walk whose initial and terminal vertices are NOT distinct.

Trivial walk is a walk of a single vertex (edges are 0)

Trail in *G* no edge is repeated

Path in *G* no vertex is repeated (every nontrivial path is necessarily an open walk)

Theorem 2.1 – Let *u* and *v* be two vertices of a graph *G*. For every *u-v* walk *W* in *G*, there exists a *u-v* path *P* such that every edge of *P* belongs to *W*.

Adjacency Matrix of *G* is the *n x n* zero-one matrix

Theorem 2.2 – Let *G* be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and adjacency matrix *A*. For each positive integer k, the number of different $v_i - v_k$ walk of length *k* in *G* is the (i, j) – entry in the matrix A^k.

Circuit is a closed walk in a graph *G* in which no edge is repeated.

Cycle is a circuit where the vertices are distinct.

Cycle *C* is called a **k-Cycle**

Triangle is a 3-cycle

Even cycle is if length is even, **odd cycle** is if the length is odd.

Girth of *G* is the length of the smallest cycle denoted g(G).

Circumference of *G* is the length of the longest cycle denoted by c(G).

If two vertices contain a path, then they are **connected.**

A graph *G* is **connected** if every two vertices are connected. If a graph is not connected it is **disconnected.**

A connected subgraph *H* of a graph *G* is a **component** of *G* if *H* is not a proper subgraph of any connected subgraph of *G*. Number of components in G is denoted k(G).

Theorem 2.3 – If *G* is a nontrivial graph of order n such that $\deg u + \deg v \geq n - 1$ for every two nonadjacent vertices *u* and *v* of *G*, then *G* is connected.

Corollary 2.4 – If *G* is a graph of order *n* with $\delta(G) \geq \frac{n-1}{2}$ then *G* is connected.

Theorem 2.5 – If *G* is a graph of order $n \geq 2$ and size $m \geq \binom{n-1}{2} + 1$, then *G* is connected.

2.2

Distance $d_G(u, v)$ from a vertex *u* to a vertex *v* in a connected graph *G* is the smallest length of a *u* – *v* path in *G*.

If a *u* – *v* path of length d(u,v) it is called *u* – *v* **Geodesic**.

Symmetric property – $d(u, v) = d(v, u)$ for all $u, v \in V(G)$

Triangle inequality - $d(u, v) + d(v, w) \geq d(u, w)$ for all $u, v, w \in V(G)$

If *d* satisfies 4 properties on pg 45, then *d* is a **metric** on V(G) and (V(G),*d*) is a **metric space**.

Theorem 2.6 – A nontrivial graph *G* is a bipartite graph if and only if *G* contains no odd cycles.

Eccentricity e(v) of a vertex *v* in a connected graph *G* is the distance between *v* and a vertex farthest from *v* in *G*.

Theorem 2.7 – If *u* and *v* are adjacent vertices in a connected graph *G*, then |e(u)-e(v)|<= 1

Diameter diam(G) of a connected graph *G* is the largest eccentricity among the vertices of *G*, while the **radius** rad(G) is the smallest eccentricity among the vertices of *G*.

A **central vertex** is a vertex *v* with e(v) = rad(G).

A vertex e(v)=diam(G) is called a **peripheral vertex** of *G*.

Two vertices *u* and *v* of *G* with d(u,v) = diam(G) are **antipodal vertices** of *G*.

Theorem 2.8 – For every nontrivial connected graph *G*, $rad(G) \leq diam(G) \leq 2rad(G)$

Subgraph induced by the central vertices of a connected graph *G* is the **center** of *G* denoted by Cen(*G*).

If every vertex of *G* is a central vertex, then Cen(*G*) = *G* and *G* is **self-centered**.

Periphery of *G* is subgraph induced by the peripheral vertices of a connected graph *G*, denoted Per(G).

Theorem 2.9 – Every graph is the center of some graph.

Theorem 2.11 – A nontrivial graph *G* is the periphery of some graph if and only if every vertex of *G* has eccentricity 1 or no vertex of *G* has eccentricity 1.

3.1

Cut-vertex is when a vertex v in a connected graph G if G-v is disconnected

Theorem 3.1 – Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

Theorem 3.2 – A vertex v in a graph G is a cut-vertex of G if and only if there are two vertices u and w distinct from v such that v lies on every u – w path in G

A nontrivial connected graph containing no cut-vertices is a **Nonseparable Graph**.

Theorem 3.3 – Let G be a graph of order 3 or more. Then G is Nonseparable if and only if every two vertices of G lie on a common cycle of G.

For two distinct vertices u and v in a graph G two u-v paths are **internally disjoint** if they have only u and v in common.

Corollary 3.4 – A connected graph G of order 3 or more is Nonseparable if and only if for every tow distint vertices u and v in G there are two internally disjoint u – v paths

Corollary 3.5 – Let u and w be two distinct vertices in a Nonseparable graph G. If H is obtained from G by adding a new vertex v and joining v tto u and w, then H Is Nonseparable.

Corollary 3.6 – Iff U and W are disjoint sets of vertices in a Nonseparable graph G of order 4 or more with |U| = |W| = 2, then G contains two disjoint paths connecting the vertices of U and the vertices of W.

A **block** of G is a maximal Nonseparable subgraph of G.

A block of G containing exactly one cut-vertex of G is called an **end-block.**

Theorem 3.7 – Every connected graph containing cut-vertices has at least two end-blocks.

Theorem 3.8 – Let G be a connected graph with at least one cut-vertex. Then G contains a cut-vertex v with the property that, with at most one exception, all blocks of G containing v are end-blocks.

Theorem 3.9 – The center of every connected graph G lies in a single block of G.

A cut-vertex v in a graph G has **branches** that are the blocks connected to v

3.2

An edge e=uv in a connected graph G whose removal results in a disconnected graph is a **bridge**.

Theorem 3.10 – An edge in a graph G is a bridge of G if and only if e lies on no cycle in G.

An **acyclic graph** has no cycles.

A **tree** is a connected acyclic graph.

A **central vertex** of $K_{1,t}$ (K is a star graph) is the vertex of degree T (degree n-1)

A tree containing exactly two vertices that are not leaves (which must be adjacent) is called a **double star.**

A **caterpillar** is a tree T of order 3 or more, the removal of whose leaves produces a path (which is called the **spine** of T).

Theorem 3.11 – A graph G is a tree if and only if every two vertices of G are connected by a unique path.

Corollary 3.12 – Every nontrivial tree contains at least two leaves

Theorem 3.13 – If T is a tree of order n and size m, then m=n-1

Theorem 3.14 – Let T be a tree of order n >= 3 having maximum degree Δ and containing n_i vertices of degree i ($1 \leq i \leq \Delta$). Then the number n_1 of leaves of T is given by.

Theorem 3.15 – A sequence s: $d_1, d_2, .., d_n$ of n>=2 positive integers is the degree sequence of a tree of order n if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

Corollary 3.16 – The size of a forest of order n having k components is n – k

Theorem 3.17 – Let G be a graph of order n and size m. If G has no cycles and m=n-1, then G is a tree.

Theorem 3.18 – Let G be a graph of order n and size m. If G is connected and m = n – 1, then G is a tree.

Theorem 3.19 – Lett G be a graph of order n and size m. If G satisfies any two of the following three properties then G is a tree: 1. G is connected, 2. G has no cycles, 3. m=n-1

Theorem 3.20 – Let T be a tree of order k. If G is a graph for which $\delta(G) \geq k - 1$ then G contains a subgraph that is isomorphic to T.

Two labelings of the same graph from the same set of labels are considered **distinct labelings** if they produce different edge sets.

Theorem 3.23 – For each positive integer n, there are n^(n-2) distinct labeled trees of order n having the same vertex set.

A **spanning tree** of a graph G is a spanning subgraph of G that is a tree.

m-n+1 is the **cycle rank** of G, where m is the size and n is the order. (trees have cycle rank 0).

Graphs connected with one cycle are **unicyclic graphs** and n=m, m-n+1=1

Theorem 3.24 – If G is a connected graph of order n, then rad(G) <= n/2