CSE 595 Independent Study Graph Theory

Week 5

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Chapter 3 Problem 45 (Spanning Trees)

Determine the labeled tree having Prufer code (4, 5, 7, 2, 1, 1, 6, 6, 7).

Following Algorithm 3.22 in Chartrand [1],

Allow $S_1 = \{1 \dots n\}$ be the vertex set of Tree T, where n is $2 \; +$ the length of the Prufer sequence, s_1, given above. The edge set T will update until the set $S_1 = \{\emptyset\}$.

$$S_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}, \qquad s_1 = \{4,5,7,2,1,1,6,6,7\}, \qquad T = \{\emptyset\}$$

$$S_1 = \{v_1, v_2, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}, \qquad s_1 = \{5,7,2,1,1,6,6,7\}, \qquad T = \{v_3v_4\}$$

$$S_1 = \{v_1, v_2, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}, \qquad s_1 = \{7,2,1,1,6,6,7\}, \qquad T = \{v_3v_4, v_4v_5\}$$

$$S_1 = \{v_1, v_2, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}, \qquad s_1 = \{2,1,1,6,6,7\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7\}$$

$$S_1 = \{v_1, v_2, v_6, v_7, v_9, v_{10}, v_{11}\}, \qquad s_1 = \{1,1,6,6,7\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7, v_8v_2\}$$

$$S_1 = \{v_1, v_6, v_7, v_9, v_{10}, v_{11}\}, \qquad s_1 = \{1,6,6,7\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7, v_8v_2, v_2v_1\}$$

$$S_1 = \{v_1, v_6, v_7, v_{10}, v_{11}\}, \qquad s_1 = \{6,6,7\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7, v_8v_2, v_2v_1, v_9v_1\}$$

$$S_1 = \{v_6, v_7, v_{10}, v_{11}\}, \qquad s_1 = \{6,7\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7, v_8v_2, v_2v_1, v_9v_1, v_1v_6\}$$

$$S_1 = \{v_6, v_7, v_{11}\}, \qquad s_1 = \{7\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7, v_8v_2, v_2v_1, v_9v_1, v_1v_6, v_{10}v_6\}$$

$$S_1 = \{v_7, v_{11}\}, \qquad s_1 = \{\emptyset\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7, v_8v_2, v_2v_1, v_9v_1, v_1v_6, v_{10}v_6, v_6v_7\}$$

$$S_1 = \{\emptyset\}, \qquad s_1 = \{\emptyset\}, \qquad T = \{v_3v_4, v_4v_5, v_5v_7, v_8v_2, v_2v_1, v_9v_1, v_1v_6, v_{10}v_6, v_6v_7, v_7v_{11}\} \blacksquare$$

Chapter 3 Problem 49 (Spanning Trees)

Show that Theorem 3.24 is sharp by giving an example of a graph G of order n with $rad(G) = \frac{n}{2}$ for every integer $n \geq 2$.

We may examine the graph

$$G = P_2$$

Therefore,

$$rad(G) = 1$$

Since order $n = 2$ in P_2, then we have proven for an all paths of P_n based on the properties of paths,

$$rad(P_n) = \frac{n}{2} \blacksquare$$

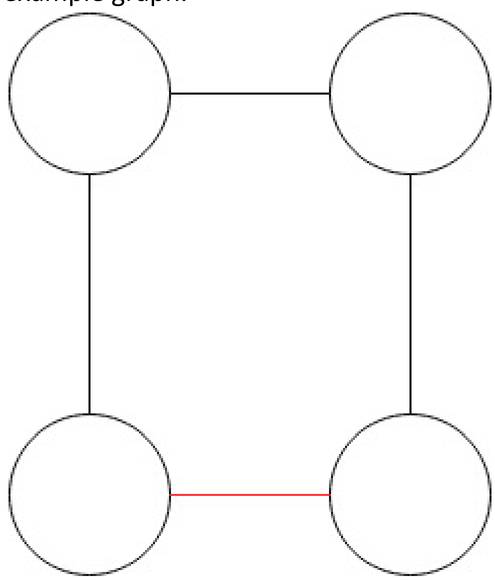
Chapter 3 Problem 59 (Spanning Trees)

Prove that an edge e of a connected graph is a bridge if and only if e belongs to every spanning tree of G

By providing a counter example we can prove the above. Suppose that edge is bridge e, designated as the red line, but does not belong to every spanning tree of graph G. Chartrand [1] defines a bridge as,

an edge $e = uv$ in a connected graph G whose removal results in a disconnected graph

We may consider the following example graph.



The graph $G - e$ is a spanning tree that exists without e, thus by contradiction it is shown that e must exists in every spanning tree of a graph G in order to be considered a bridge.∎

Chapter 3 Problem 63 (Spanning Trees)

Show that there is only one positive integer k such that no graph contains exactly k spanning trees.

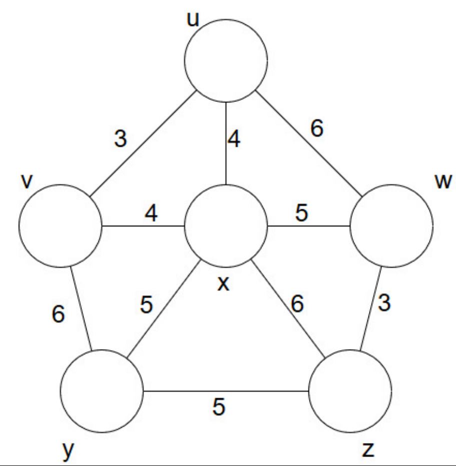
When $k = 2$ there is no graph that contains exactly two spanning trees, because there is only the graph T_2 that is a spanning tree.

For the case of $k = 1$ it is obvious that G_1 only has one spanning tree, so this remains true.

For $k \geq 3$, looking at C_k ,that has k nodes and k edges, spanning trees may be produce by removing one of the edges, always producing k spanning trees. ■

Chapter 3 Problem 67 (The Minimum Spanning Tree Problem)

Apply both Kruskal's and Prim's algorithms to find a minimum spanning tree in the weighted graph below. In each case, show how this tree is constructed.



First following Algorithm 3.27 (Kruskal's Algorithm) in Chartrand [1], an edge set s will be built consisting of edges with the lowest values, so long as no cycles are induced.

Starting with the lowest edge weight, 3, the edges uv and wz, so adding on edge at a time,

$$s = \{uv\}$$

$$s = \{uv, wz\}$$

$$s = \{uv, wz, vx\}$$

$$s = \{uv, wz, vx, wx\}$$

$$s = \{uv, wz, vx, wx, yx\}$$

Thus, a minimum spanning tree is induced by Kruskal's Algorithm.

Next following Algorithm 3.29 (Prim's Algorithm) in Chartrand [2], an edge set s will be built, starting with any node, and adding the lowest weight adjacent edge and adding its adjacent node to the graph.

Starting with node u, the smallest edge weight is 3 which connects to node v,

$$s = \{uv\}$$

Next the smallest edge weight is 4, connecting to node x either form node u or node v,

$$s = \{uv, ux\}$$

$$s = \{uv, ux, xy\}$$

$$s = \{uv, ux, xy, yz\}$$

$$s = \{uv, ux, xy, yz, zw\}$$

Chapter 3 Problem 71 (The Minimum Spanning Tree Problem)

Let G be a connected weighted graph whose edges have distinct weights. Show that G has exactly one minimum spanning tree.

If the Kruskal's Algorithm, it is obvious that the graph G will have a distinct minimum spanning tree since

$$weight(e_1) < weight(e_i) < weight(e_k)$$

When adding any edge using this algorithm, there will always be a distinct weight, and therefore no alternative graphs will be induced.

Following Prim's Algorithm, a distinct graph will also be induced. Given and node to start, selecting the minimum edge weight for each edge adjacent to the induced graph and adding it will cause for the same graph, if the graph is connected.■

Works cited

"Trees." *Graphs & Digraphs*, by Gary Chartrand et al., CRC Press, 2016, pp. 57–94.