CSE 595 Independent Study
Graph Theory

Week 1

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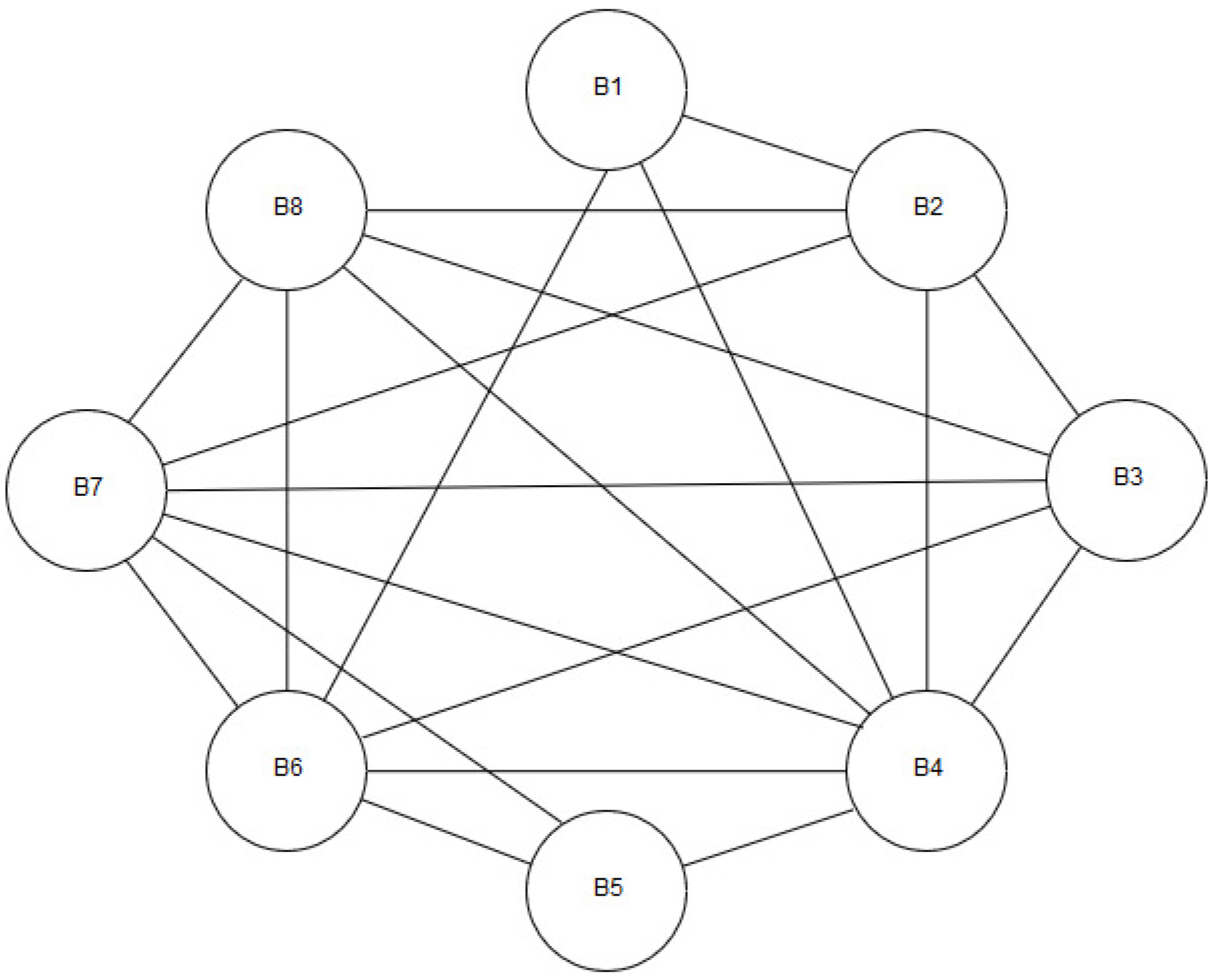
Problem 1 (Introduction)

An electronics company keeps on hand wire segments of a fixed length and of different colors for various purposes. Each wire is either colored blue (b), green (g), purple (p), red (r), silver (s), white (w), or yellow (y). The company has many wire segments of each color. All of the wire segments have been randomly stored in a large barrel and each collection of wires is placed in a box. The boxes are denoted by B_i $(1 \leq i \leq 8)$. The colors of the wire segments in each box are:

$$B_1 = \{b, r\}, B_2 = \{p, r, s, w\}, B_3 = \{p, w, y\}, B_4 = \{g, r, y\}$$

$$B_5 = \{g\}, B_6 = \{b, g, y\}, B_7 = \{g, p, s, w, y\}, B_8 = \{s, w, y\}$$

The following nodes represent boxes with certain color of wires in them and are connected if they have at least one color the same inside of them.



Problem 3 (Degree of a Vertex)

A graph *G* of order 26 and size 58 has 5 vertices of degree 4, 6 vertices of degree 5, and 7 vertices of degree 6. The remaining vertices of *G* all have the same degree. What is this degree?

By Theorem 1.4 in Chartrand [1],

$$\sum_{v \in V(G)} \deg v = 2(m) \therefore 2(m) = 116$$

Where m is the size of *G*.

Finding the remaining vertices that have not been given,

$$n - 5 - 6 - 7 = 8$$

Where n is the order of *G*.

Next find the remaining degrees shared among the 8 vertices,

$$116 - 5(4) - 6(5) - 7(6) = 24$$

Dividing the 24 degrees among the 8 vertices,

$$\frac{24}{8} = 3$$

$$\therefore \deg(v) = 3$$

For the remaining vertices■

Problem 5 (Degree of a Vertex)

The degree of every vertex of a graph *G* is one of three consecutive integers. For each degree *x*, the graph *G* contains exactly *x* vertices of degree *x*. Prove that for every graph *G* contains exactly *x* vertices of degree *x*. Prove that for every graph *G* with this property, two-thirds of the vertices of *G* have odd degree.

Proof

Graph *G* has the following degrees,

$$i + (i+1) + (i+2) = 3(i+1), \qquad i \in \mathbb{Z}^+$$

Odd Case

Substituting i for an odd integer,

$$3[(2n+1)+1], \qquad n \in \mathbb{Z}^+$$
$$= 3[2(n+1)]$$

Which is an even number for any given n which follows Theorem 1.4 [1].

Therefore, there is the following number of vertices with odd degree

$$(2n+1) + [(2n+1)+2] = 2(i+1)$$

Dividing the amount by the total number of vertices

$$\frac{2(i+1)}{3(i+1)} = \frac{2}{3}$$

Even Case

Substituting i for an even integer,

$$3[(2n)+1], \qquad n \in \mathbb{Z}^+$$

This is not possible by Corollary 1.5 [1], which states

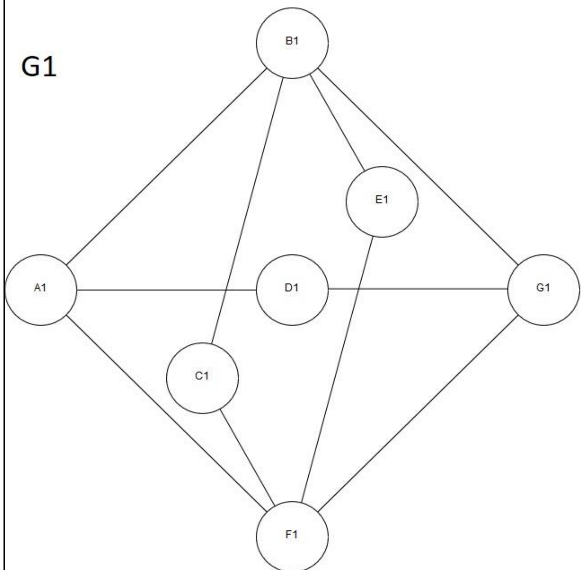
Every graph has an even number of odd vertices

Therefore, there is two-thirds of the vertices of *G* that have odd degree in this case. ■

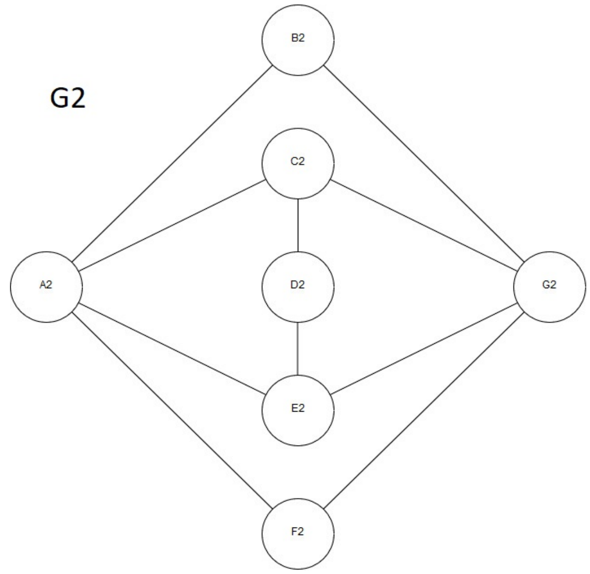
Problem 7 (Isomorphic Graphs)

Consider the pair of graphs G_1, G_2 and H_1, H_2 below

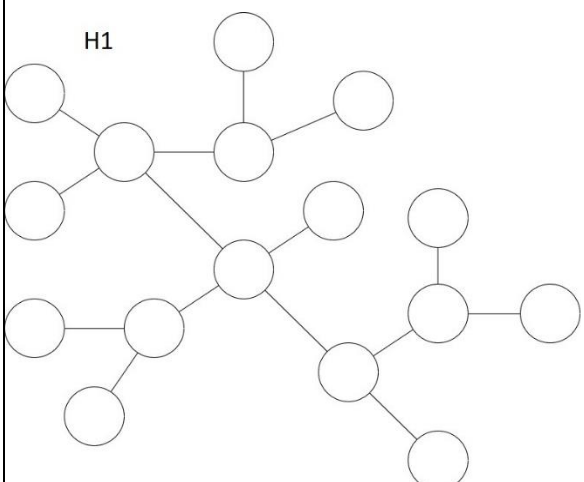
G1



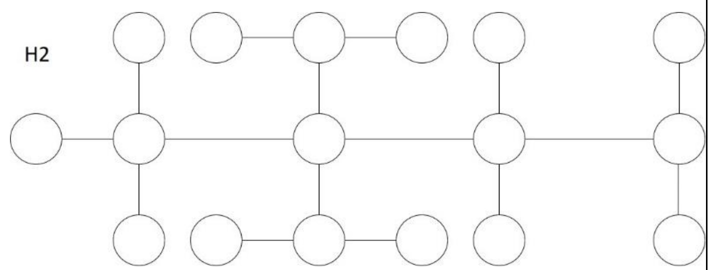
G2



H1



H2



(a) Determine whether $G_1 \cong G_2$. (If they are isomorphic)

(b) Determine whether $H_1 \cong H_2$.

(a) $\phi: V(G_1) \to V(G_2)$ is defined by

$$\phi(A_1) = C_2, \qquad \phi(B_1) = G_2, \qquad \phi(C_1) = B_2$$
$$\phi(D_1) = D_2, \qquad \phi(E_1) = F_2, \qquad \phi(F_1) = A_2, \qquad \phi(A_1) = E_2$$

The function is bijective, and all mutually adjacent vertices in G_1 are also mutually adjacent in G_2. Therefore, $G_1 \cong G_2$. ■

(b) These graphs cannot be isomorphic as they are not of the same order. Therefore, there cannot exist a bijective function such that $\phi: V(H_1) \to V(H_2)$ meaning $H_1 \ncong H_2$. ■

Problem 11 (Regular Graphs)

Show that if G is a nonregular graph of order n and size $\frac{rn}{2}$ for some integer r with $1 \leq r \leq n-2$, then $\Delta(G) - \delta(G) \geq 2$.

By definition, a regular graph has the following property

$$\mathrm{r} = \delta(\mathrm{G}) = \Delta(\mathrm{G})$$

Where $\delta(\mathrm{G})$ refers to the smallest degree of a vertex in G, and $\Delta(\mathrm{G})$ refers to the largest degree of a vertex in G.

Therefore, a nonregular graph must have the following property,

$$\delta(\mathrm{G}) \leq \mathrm{r} \leq \Delta(\mathrm{G})$$

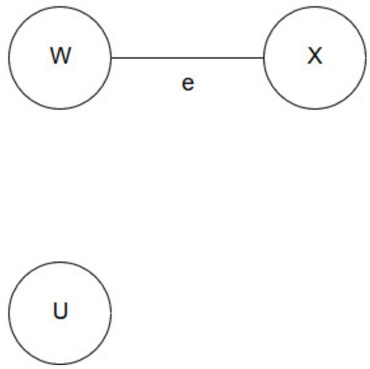
Thus $\Delta(\mathrm{G}) > \delta(G)$, however based on Corollary 1.5 [1] mentioned in problem 5, the difference must be ≥ 2. ■

Problem 13 (Regular Graphs)

Give an example of a nonregular graph G containing an edge e and a vertex u such that $G - e$ and $G - u$ are both regular

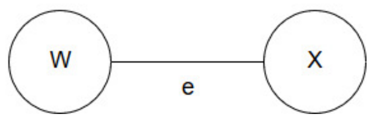
Graph G is nonregular,

G:



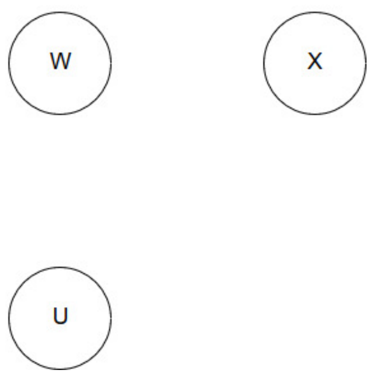
Graph G' deletes node U and creates a 1-regular graph,

G':



Graph G'' deletes edge e and creates a 0-regular graph,

G":



Graph H is deletes both edge e and node U, and creates a 0-regular graph,

H:

Works cited

"Introduction." *Graphs & Digraphs*, by Gary Chartrand et al., CRC Press, 2016, pp. 1–13.