CSE 595 Independent Study
Graph Theory

Week 4

California State University - San Bernardino

Richard Vargas

Supervisor – Dr Owen Murphy

Chapter 3 Problem 1 (Nonseparable Graphs)

Let *G* be a nontrivial connected graph and let $u \in V(G)$. If v is a vertex that is farthest from u in *G*, then v is not a cut-vertex of *G*.

Looking at Theorem 3.2 in Chartrand [1] which states,

> *A vertex v in a graph G is a cut-vertex of G if and only if there are two vertices u and w distinct from v such that v lies on every $u -\ w$ path in G.*

If a vertex w were to lie further past v, then the eccentricity $e(u) \neq v$.

Furthermore, if we were to delete v from G and there was another vertex w with the same distance from u then the shortest path $u - w$ does not include v. ■

Chapter 3 Problem 7 (Nonseparable Graphs)

(a) Show that no graph has a cut-vertex of degree 1

(b) Show that if *G* is a graph with $\delta(G) \geq 2$ containing a cut-vertex of degree 2, then *G* has at least three cut-vertices

(c) Show, for every integer $k \geq 2$, that there is a graph containing a cut-vertex of degree k.

(a) If a graph has a vertex of degree 1, then this implies the vertex is a leaf. If this vertex is a leaf, then the graph $G - v$ has the same number of connected graphs as G■.

(b) If $\delta(G) \geq 2$, then order of $G \geq 3$. If this is true, then Theorem 3.3 [1] can be applied which states the following,

> Let G be a graph of order 3 of more. Then G is nonseparable if and only if every two vertices of G lie on a common cycle of G.

If we follow that there is a cut-vertex of degree 2, then this implies that there are two subgraphs of G connected by the cut-vertex. If the cut-vertex is removed, then we have the two subgraphs. If we remove either adjacent vertex of the cut-vertex, then the cut-vertex becomes a leaf for which ever graph the adjacent vertex belongs to that was not removed■.

(c) If we assume that for a graph G of order n, that the cut-vertex has a degree of $n - 1$, and all other vertices are of degree 1 which are connected to the cut-vertex. This allows for a graph with degree of $k = n - 1$ with $k \geq 2$, which will always create k subgraphs of order 1 ■.

Chapter 3 Problem 17 (Nonseparable Graphs)

Lets *G* be a nontrivial connected graph.

(a) Prove that no cut-vertex of *G* is a peripheral vertex of *G.*

(b) Prove or disprove: Every peripheral vertex of *G* belongs to an end-block of *G*.

(a) Chapter 3 Problem 1 can be looked in this exercise. A peripheral vertex in a graph G has the largest eccentricity, meaning it has that largest distance from the vertex to another vertex in G. Thus, if this is a peripheral vertex, then it follows the earlier above problem and a cut-vertex of G cannot be a peripheral vertex of G ■.

(b) This is a false statement. Proving by counterexample, we can examine a graph G with order 1. Although the one vertex in G is a peripheral vertex, it is not a part of an end-block of *G*. In order to be an end-block, the subgraph of G must contain one cut-vertex. A graph of order 1 does not contain any cut-vertex■.

Chapter 3 Problem 19 (Introduction to Trees)

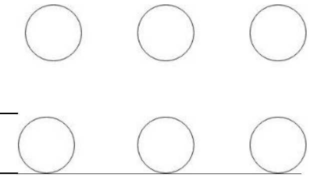
Let *G* be a connected graph of order 3 or more. Prove that if $e = uv$ is a bridge of *G*, then at least one of u and v is a cut-vertex of *G*.

Following Theorem 3.10 [1] which states,

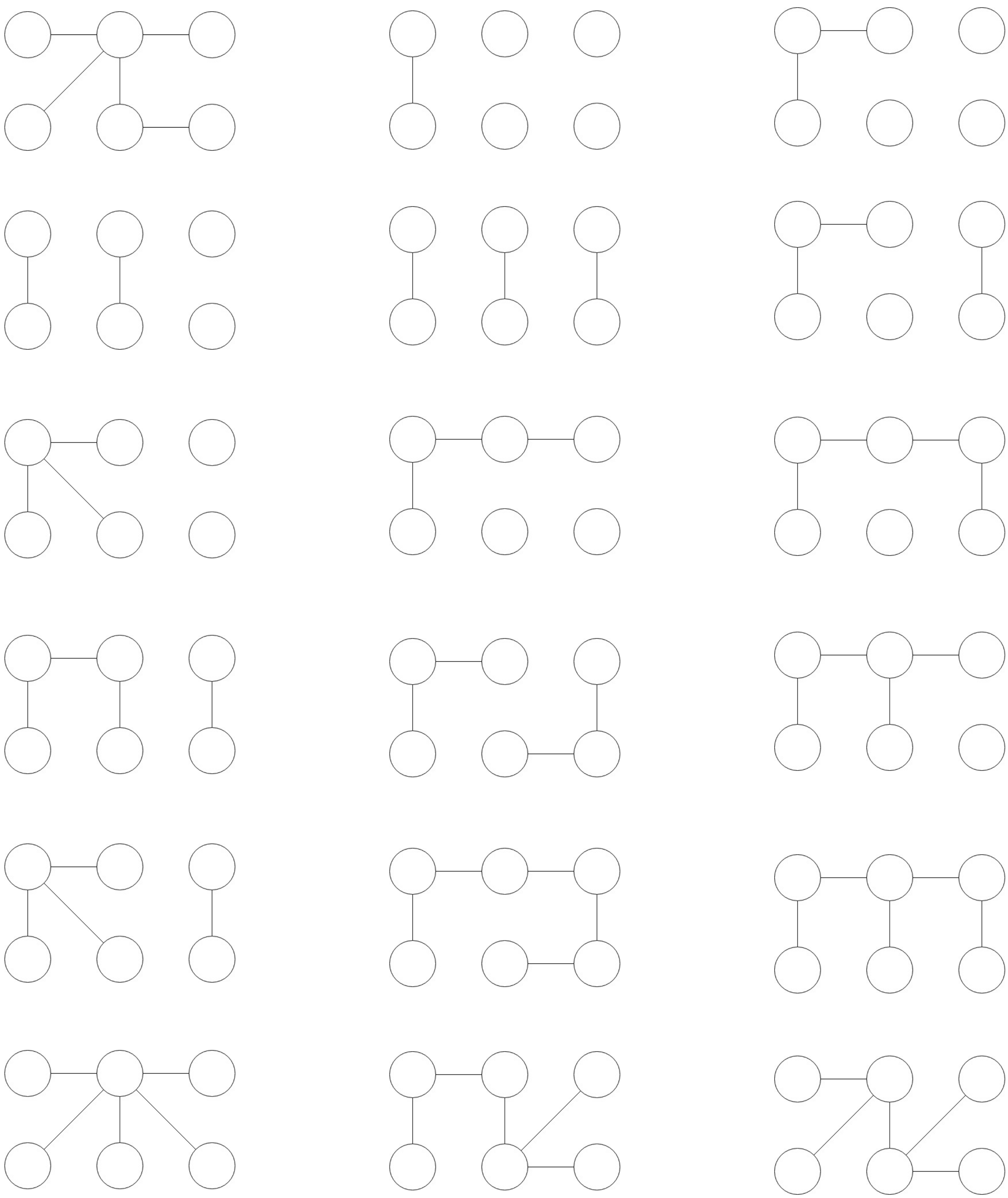
An edge e in a graph G is a bridge of G if and only if e lies on no cycle in G

Therefore, we know that e does not lie on a cycle in G. Since the order is of 3 or more, $\deg{(u)} \geq 2$ or $\deg(v) \geq 2$. At least one of these vertices must therefore have one or more distinct vertices, and therefore is a cut-vertex of G■.

Chapter 3 Problem 23 (Introduction to Trees)



Draw all forests of order 6.



Chapter 3 Problem 29 (Introduction to Trees)

A tree is called **central** if its center is K_1 and **bicentral** is its center is at K_2. Show that every tree is either central or bicentral.

Looking at Theorem 3.9 [1] which states,

The center of every connected graph G lies in a single block of G.

Since the center of a graph G is a vertex $v \in G$ where $\min e(v_i)\,, v_i \in G$. There can only be a maximum of two vertices where $\min e(v_i)\,, v_i \in G$ is true in the case of a tree.

Two cases may be looked at then, a tree of order 1, and order 2

Order 1 – If the tree is of order 1, it is just a single vertex. Therefore, the graph K_1 could be a subgraph of a countably infinite number of supergraphs where cut-vertices have been removed.

Order 2 – If the tree is order 2, there are 2 vertices and one edge connecting them. Similarly to the case above, the K_2 graph may also be a subgraph of a countably infinite number of supergraphs where cut-vertices have been removed.

All other cases of trees will be separable graphs, who have a center block of either order 1 or order 2 ∎.

Works cited

"Trees." *Graphs & Digraphs*, by Gary Chartrand et al., CRC Press, 2016, pp. 57–69.