CS722/822 Machine Learning

Homework #1

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Homework #1 Part 1

Given: $f(u,v) = 10u^2v^3 + 3v^2 + 4 \cdot u$

$g(u,v,w) = x \cdot \log(u) + yuvw^2 + 10x^2$

$$h(u,v) = \sum_{i=1}^{m} \frac{1}{2}(x^{(i)}u + y^{(i)}v)^2$$

Find:

1.) $\boxed{\frac{\delta}{\delta u} f(u,v) = 20u \cdot v^3 + 4}$

2.) $\boxed{\frac{\delta}{\delta v} f(u,v) = 30u^2v^2 + 6v}$

3.) $\frac{\delta}{\delta u} g(u,v,w) = \ldots$

$\frac{d}{dx} \ln(x) = 1/x$;

charge of base

$$\log_b(x) = \frac{\log_e(x)}{\log_e(b)} = \frac{\ln(x)}{\ln(b)}$$

$$\frac{d}{dx}\log_{10}(x) = \frac{d}{dx}\left\{\frac{\ln(x)}{\ln(10)}\right\} = \frac{1}{x \cdot \ln(10)}$$

3.) $\boxed{\frac{\delta}{\delta u} g(u,v,w) = \frac{x}{u \cdot \ln(10)} + yvw^2}$

4.) $\boxed{\frac{\delta}{\delta v} g(u,v,w) = yuw^2}$

5.) $\boxed{\frac{\delta}{\delta w} g(u,v,w) = 2 \cdot uvw}$

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$$h(u,v) = \sum_{i=1}^{m} \frac{1}{2}\left(x^{(i)}u + y^{(i)}v\right)^2$$

6.) $\frac{\delta}{\delta u} h(u,v) = ?$

let $K = \left(x^{(i)}u + y^{(i)}v\right)^2 = x^{(i)2}u^2 + 2 \cdot x^{(i)} u y^{(i)} v + y^{(i)2} v^2$

$$\frac{\delta}{\delta u} K(u,v) = 2 \cdot x^{(i)2} u + 2 \cdot x^{(i)} y^{(i)} v$$

$$\frac{\delta}{\delta u} h(u,v) = \sum_{i=1}^{m} \left\{ x^{(i)2} u + x^{(i)} y^{(i)} v \right\}$$

$$\boxed{\frac{\delta}{\delta u} h(u,v) = u \cdot \sum_{i=1}^{m} (x^{(i)})^2 + v \cdot \sum_{i=1}^{m} x^{(i)} y^{(i)}}$$

7.) $\frac{\delta}{\delta v} h(u,v) = ?$

$$\frac{\delta}{\delta v} K(u,v) = 2 \cdot x^{(i)} u y^{(i)} + 2 y^{(i)2} v$$

$$\boxed{\frac{\delta}{\delta v} h(u,v) = v \sum_{i=1}^{m} (y^{(i)})^2 + u \sum_{i=1}^{m} x^{(i)} y^{(i)}}$$

Homework 1 Part 2

1.) Given: $s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, s_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

a) $2 \cdot s_1 + 3 \cdot s_2 + 4 \cdot s_3 = V$

$$2 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = V = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$$

$$\underset{3 \times 3}{S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \; ; \; \underset{3 \times 1}{X = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}$$

$$SX = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 \\ 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 0 \\ 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} = V$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

jth collumn a_j

ith row a_i^T

$s_1^T = [1 \; 0 \; 0], \quad s_2^T = [1 \; 1 \; 0], \quad s_3^T = [1 \; 1 \; 1]$

$$s_1^T \odot X = [1 \; 0 \; 0] \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 1 \cdot 2 + 0 \cdot 3 + 0 \cdot 4 = 2$$

$$s_2^T \odot X = [1 \; 1 \; 0] \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 1 \cdot 2 + 1 \cdot 3 + 0 \cdot 4 = 5$$

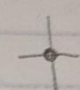
$$s_3^T \odot X = [1 \; 1 \; 1] \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 = 9$$

b.) Given: $M = [2S_1, 3S_2, 4S_3]$

Find: $M = SA$

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 4 \end{bmatrix} ; \; S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} ; \; A^{3\times3} = ?$$

intersect



$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

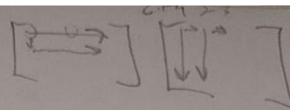
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-a_{11} & a_{22} & a_{23} \\ a_{31}-a_{11} & a_{32} & a_{33} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-a_{11} & a_{22} & a_{23} \\ a_{31}-a_{11} & a_{32}-a_{22} & a_{33}-a_{23} \end{bmatrix}$$

$$\downarrow$$

$$I * I^{-1} = I$$

$$M = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{A} = \begin{bmatrix} 1\cdot2 & 0 & 0 \\ 1\cdot2 & 1\cdot3 & 0 \\ 1\cdot2 & 1\cdot3 & 1\cdot4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 4 \end{bmatrix}$$



2) Verify matrix multiplication is distributive

$$A(B+C) = AB + AC$$

Product Definition

$$A \in \mathbb{R}^{m \times n}, \quad B \in^{n \times p}, \quad C = AB \in \mathbb{R}^{m \times p}$$

$$C_{ij} = \sum_{k=1} A_{ik} B_{kj} \; ;$$

Sum

$$D = A + B; \quad D_{ij} = A_{ij} + B_{ij}$$

$$(A(B+C))_{ij} = \sum_{k=1} A_{ik} (B+C)_{kj} = \sum_{k=1} A_{ik} (B_{kj} + C_{kj})$$

$$= \sum_{k=1} A_{ik} B_{kj} + \sum_{k=1} A_{ik} B_{kj}$$

$$= (AB)_{ij} + (AC_{ij})$$

$$= (AB + AC)_{ij}$$

$$\therefore$$

$$A(B+C) = AB + AC$$

Homework #1 Part 2

3.) Given: A is symmetric & positive-semi-definite
Show: All eigenvalues of A are non-negative

Definitions:

Symmetric - a square matrix that is equal to its transpose

$$A = A^T \; ; \; a_{ij} = a_{ji}$$

Positive Semi-Definite -

a symetric $n \times n$ real matrix M if
$\vec{x}^T M \vec{x} \geq$ for all $X \in R^n$

$$\left(\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & \ddots & & \vdots \\ \vdots & & & \\ m_{n1} & \cdots & & m_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right)_{ij} \geq 0$$

Non formal Eigen Value - factor by which an eigen vector is stretched or squished durring transformation or axis of rotation

$$\underbrace{A\vec{v}}_{\text{eigen vector}} = \underbrace{\lambda \vec{v}}_{\text{eigenvalue}} = (\lambda I)\vec{v} \; ; \; (A - \lambda I)\vec{v} = \vec{0}$$

$$\det(A - \lambda I) = 0$$

let $C = \vec{X}^T M \vec{X} \; ; \; C_{ij} =$

Homework #1 Part 2

4.) Find the inverse of the follwing matrices

$A=\begin{bmatrix}0&3\\4&0\end{bmatrix}$; $B=\begin{bmatrix}2&0\\4&2\end{bmatrix}$; $C=\begin{bmatrix}3&4\\5&7\end{bmatrix}$

$M \in R^2 \quad M^{-1} = \frac{adj(M)}{det(M)}$

a.) $det(A) = 0-(4\cdot3) = -12$

$adj(A) = \begin{bmatrix}0&-3\\-4&0\end{bmatrix}$; $\boxed{A^{-1} = \frac{-1}{12}\begin{bmatrix}0&-3\\-4&0\end{bmatrix} = \begin{bmatrix}0&\frac{1}{4}\\\frac{1}{3}&0\end{bmatrix}}$

b.) $det(B) = 2\cdot2-0 = 4$; $adj(B) = \begin{bmatrix}2&0\\-4&2\end{bmatrix}$

$\boxed{B^{-1} = \frac{1}{4}\begin{bmatrix}2&0\\-4&2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&0\\-1&\frac{1}{2}\end{bmatrix}}$

c.) $det(C) = 3\cdot7-5\cdot4 = 21-20 = 1$

$adj(C) = \begin{bmatrix}7&-4\\-5&3\end{bmatrix}$

$\boxed{C^{-1} = \begin{bmatrix}7&-4\\-5&3\end{bmatrix}}$

1.) Homework # 1, Part 3

Initially,

$P(A = W) = \frac{2}{3} \rightarrow P(X)$; $P(A = b) = \frac{1}{3}$

$P(B = W) = \frac{1}{6}$; $P(B = b) = \frac{5}{6}$

Consider Both Possible Outcomes

a) The ball drawn from A was white (B'_w)

$P(B^*_w = W) = \frac{2}{7} \rightarrow P(Y|X)$

$P(A_w|W) = 0.5$

b.) The ball drawn from B was black

$P(B^*_b = W) = \frac{1}{7} = 0.16\overline{66}$

$P(A_b|W) = 0$

At this point the overall probability of drawing a white ball is

$$P(B^* = W) = \frac{N_b + P(A=W)}{N_B + 1} = \frac{1 + 2/3}{7} = \frac{5}{21} \rightarrow P(Y)$$

Bayes' Theorum: $P(X|Y) = P(Y|X) \cdot P(X) / P(Y)$

$$\boxed{P(A=W|B'=W) = \frac{P(B^*=W|A=W) \cdot P(A=W)}{P(B^*=W)} = \frac{\left(\frac{2}{7}\right) \cdot \left(\frac{2}{3}\right)}{(5/21)} = \frac{(4/21)}{(5/21)} = \frac{4}{5}}$$

Homework #1 Part 3

2.) $P(w) = 0.9$, $P(d|\bar{w}) = 0.8$, $P(d|w) = 0.15$

$P(d) = P(w) \cdot P(d|w) + (1 - P(w)) \cdot P(d|\bar{w})$

$P(d) = (0.9) \cdot (0.15) + (0.1)(0.8) = 0.215$

a.) $\boxed{P(\bar{d}) = 1 - P(d) = 0.785}$

b.) $\boxed{P(\bar{w}|d) = \frac{P(d|\bar{w}) \cdot P(\bar{w})}{P(d)} = \frac{(0.8) \cdot (0.1)}{0.215} = 0.372}$

3.) Given: $P(X=0) = 1 - P(X=1)$, $E[X] = 3Var(X) = \mu$

Find: $P(X=0) = ?$

Note: X is binary (0 or 1)

Expected value $E[X]$ of a random variable X is the theoretical mean of the random variable
(Not based on sample data, based on distribution)

$\mu = E[X] = \sum_{x} x \cdot P(x)$

values probability

Variance of X: $E[(X-\mu)^2] = \sum_{x} (X-\mu)^2 \cdot P(x)$

$E[(X-\mu)^2] = E(X^2) - (E[X])^2 = E(X^2) - \mu^2$

$\sum_{x} x^2 P(x)$

Cont'd →

$\mu = 3\cdot E[(X-\mu)^2] = 3\cdot\{E(X^2)-\mu^2\}$

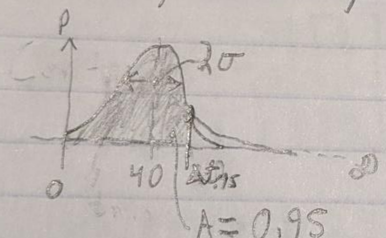
$\mu + 3\mu^2 = 3\cdot E(X^2) = 3\cdot\sum_{x} x^2\cdot P(x)$

because $P(X=0) = (1-P(X=0))$

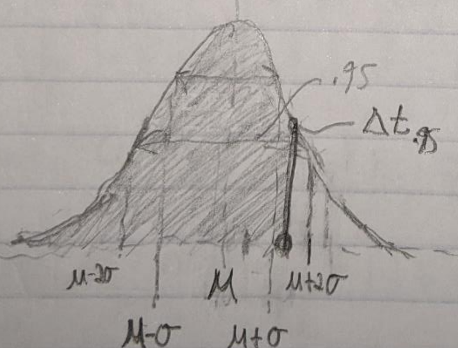
$\mu + 3\cdot\mu^2 = 3\cdot P(X=1) = 3\cdot(1-P(X=0))$

$\boxed{P(X=0) = 1-\mu^2-\mu/3}$

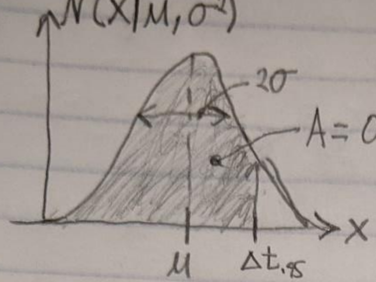
4.) $\mu = 40$, $\sigma^2 = 7$, $t_s = 1{:}00$ p.m.; Certainty $= 0.95$



$\Delta t_{.95} = t_s - t_i = ?$



4. $\mu = 40$, $\sigma^2 = 7$; $t_s = 1{:}00$ p.m., Certainty $= 0.95$



$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\,dx = 1$$

$$\int_{-\infty}^{\Delta t_{.95}} \mathcal{N}(x|\mu,\sigma^2)\,dx = 0.95$$

$$\int_{\Delta t_{.95}}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\,dx = 0.05$$

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \cdot \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

let $C = 2\pi\sigma^2$

$$\int_{-\infty}^{\Delta t_{.95}} \mathcal{N}(x|\mu,\sigma^2)\,dx = \frac{1}{\sqrt{C}} \cdot \int_{-\infty}^{\Delta t_{.95}} e^{-c\cdot(x-\mu)^2}$$

let $K = x-\mu$

$$\frac{1}{\sqrt{C}} \int_{-\infty}^{\Delta t_{.95}+\mu} e^{-cK^2} = ?$$