$ \begin{array}{lll} \text{Step} & \text{Algorithm: } A := \text{LU_UNB_VARI}(A) \\ \text{1a} & \left\{A = \widehat{A} \right. \\ & A \to \left(\frac{A_{TL}}{A_{BL}} \middle  A_{TR} \right), L \to \left(\frac{L_{TL}}{L_{BL}} \middle  L_{BR} \right), U \to \left(\frac{U_{TL}}{U_{BL}} \middle  U_{TR} \right) \\ & \text{where } A_{TL} \text{ is } 0 \times 0, L_{TL} \text{ is } 0 \times 0, U_{TL} \text{ is } 0 \times 0 \\ & 2 & \left\{\left(\frac{A_{TL}}{A_{RL}} \middle  A_{RR} \right) = \left(\frac{L \setminus U_{TL}}{\widehat{A}_{BL}} \middle  \widehat{A}_{RR} \right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \\ & 3 & \text{while } m(A_{TL}) < m(A) \\ & 2,3 & \left\{\left(\frac{A_{TL}}{A_{BL}} \middle  A_{RR} \right) \to \left(\frac{A_{00}}{A_{00}} \middle  a_{01} \middle  A_{02} \right) \\ & \left(\frac{A_{TL}}{A_{BL}} \middle  A_{RR} \right) \to \left(\frac{A_{00}}{A_{00}} \middle  a_{01} \middle  A_{02} \right) \\ & \left(\frac{A_{TL}}{A_{BL}} \middle  A_{RR} \right) \to \left(\frac{A_{00}}{A_{00}} \middle  a_{01} \middle  A_{02} \right) \\ & \left(\frac{A_{TL}}{A_{BL}} \middle  A_{RR} \right) \to \left(\frac{A_{00}}{A_{00}} \middle  a_{01} \middle  A_{02} \right) \\ & \left(\frac{A_{00}}{A_{01}} \middle  a_{11} \middle  a_{12} \middle  a_{21} \right) = \left(\frac{L \setminus U_{00}}{A_{00}} \middle  a_{01} \middle  A_{02} \right) \\ & \left(\frac{L_{TL}}{A_{BL}} \middle  L_{RR} \right) \to \cdots, \left(\frac{U_{TL}}{U_{TR}} \middle  U_{TR} \right) \to \cdots \\ & \left(\frac{U_{TL}}{U_{RL}} \middle  U_{RR} \right) \to \cdots \\ & \left(\frac{U_{TL}}{A_{RL}} \middle  U_{RR} \right) \to \cdots \\ & \left(\frac{U_{TL}}{U_{RL}} \middle  U_{R$	C.	
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	_	
$ \begin{array}{c} \text{where } A_{TL} \text{ is } 0 \times 0, \ L_{TL} \text{ is } 0 \times 0, \ U_{TL} \text{ is } 0 \times 0 \\ \hline 2 & \left\{ \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \\ \hline 3 & \text{while } m(A_{TL}) < m(A) \text{ do} \\ \hline 2.3 & \left\{ \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \wedge m(A_{TL}) < m(A) \\ \hline 5a & \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix} & \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \rightarrow \cdots, \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \rightarrow \cdots \\ \hline & & \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L \setminus U_{00} & \widehat{a}_{01} & \widehat{A}_{02} \\ \widehat{a}_{10}^T & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{A}_{02} \\ \widehat{a}_{10}^T & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{02} \\ \widehat{a}_{10}^T & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{02} \\ \widehat{a}_{10}^T & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{02} \\ \widehat{a}_{10}^T & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{02} \\ \widehat{a}_{10}^T & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{02} \\ \widehat{A}_{10} & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{10} \\ \widehat{A}_{10} & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{10} \\ \widehat{A}_{10} & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{10} \\ \widehat{A}_{10} & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{01} & \widehat{a}_{10} \\ \widehat{A}_{10} & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} & \begin{pmatrix} L_{D0} & \widehat{a}_{10} & \widehat{a}_{10} \\ \widehat{A}_{$	la la	A = A
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	4	
$ \begin{array}{c} 2.3 & \left\{ \begin{array}{c} \left( \begin{array}{c} A_{TL} \mid A_{TR} \\ A_{BL} \mid A_{BR} \end{array} \right) = \left( \begin{array}{c} L \backslash U_{TL} \mid \widehat{A}_{TR} \\ \widehat{A}_{BL} \mid \widehat{A}_{BR} \end{array} \right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \wedge m(A_{TL}) < m(A) \\ \end{array} \right. \\ \left. \begin{array}{c} \left( \begin{array}{c} A_{TL} \mid A_{TR} \\ A_{BL} \mid A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c} A_{00} \mid a_{01} \mid A_{02} \\ a_{10}^{T} \mid \alpha_{11} \mid a_{12}^{T2} \\ A_{20} \mid a_{21} \mid A_{22} \end{array} \right), \left( \begin{array}{c} L_{LL} \mid L_{TR} \\ L_{BL} \mid L_{BR} \end{array} \right) \rightarrow \cdots, \left( \begin{array}{c} U_{TL} \mid U_{TR} \\ U_{BL} \mid U_{BR} \end{array} \right) \rightarrow \cdots \\ \text{where } \alpha_{11} \text{ is } 1 \times 1, \lambda_{11} \text{ is } 1 \times 1, v_{11} \text{ is } 1 \times 1 \end{array} \right. \\ \left. \begin{array}{c} \left( \begin{array}{c} A_{00} \mid a_{01} \mid A_{02} \\ a_{10} \mid \alpha_{11} \mid a_{12}^{T2} \\ A_{20} \mid a_{21} \mid A_{22} \end{array} \right) = \left( \begin{array}{c} L \backslash U_{00} \mid \widehat{a}_{01} \\ \widehat{\alpha}_{10} \mid \widehat{\alpha}_{11} \mid \widehat{\alpha}_{12}^{T2} \\ \widehat{A}_{20} \mid \widehat{a}_{21} \mid \widehat{A}_{22} \end{array} \right) \\ \left( \begin{array}{c} a_{01} := u_{01} = L_{00}^{-1} a_{01} \\ A_{20} \mid a_{21} \mid A_{22} \end{array} \right) = \left( \begin{array}{c} L \backslash U_{00} \mid \widehat{a}_{01} \\ \widehat{\alpha}_{20} \mid \widehat{a}_{21} \mid \widehat{A}_{22} \end{array} \right) \\ \left( \begin{array}{c} L_{00} \mid I_{01} \mid \widehat{a}_{12} \\ \widehat{A}_{20} \mid \widehat{a}_{21} \mid \widehat{A}_{22} \end{array} \right) \\ \left( \begin{array}{c} A_{00} \mid a_{01} \mid A_{02} \\ \alpha_{10} := l_{10}^{T} = a_{11} - l_{10}^{T} u_{01} \end{array} \right) \\ \left( \begin{array}{c} U_{00} \mid i \mid \widehat{a}_{02} \\ \widehat{a}_{11} \mid \widehat{a}_{12} \end{aligned} \right) \\ \left( \begin{array}{c} A_{00} \mid a_{01} \mid A_{02} \\ \alpha_{11} := v_{11} = \alpha_{11} - l_{10}^{T} u_{01} \end{array} \right) \\ \left( \begin{array}{c} U_{00} \mid i \mid \widehat{a}_{02} \\ \widehat{A}_{20} \mid \widehat{a}_{21} \mid \widehat{A}_{22} \end{array} \right) \\ \left( \begin{array}{c} A_{00} \mid a_{01} \mid A_{02} \\ \widehat{a}_{10} \mid \widehat{a}_{11} \mid \widehat{a}_{12}^{T} \\ \widehat{A}_{20} \mid \widehat{a}_{21} \mid \widehat{A}_{22} \end{array} \right) \\ \left( \begin{array}{c} L_{00} U_{00} = \widehat{a}_{00} \\ I_{10} U_{00} = \widehat{a}_{10} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{01} \\ I_{10} U_{01} \mid \widehat{a}_{01} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{01} \\ I_{10} U_{01} = \widehat{a}_{01} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{02} \\ I_{10} U_{01} \mid \widehat{a}_{02} \\ \widehat{A}_{20} \mid \widehat{a}_{21} \mid \widehat{A}_{22} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{02} \\ I_{10} U_{01} \mid \widehat{a}_{01} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{01} \\ I_{10} U_{01} \mid \widehat{a}_{02} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{01} \\ I_{10} U_{01} \mid \widehat{a}_{01} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{01} \\ I_{10} U_{01} \mid \widehat{a}_{01} \end{array} \right) \\ \left( \begin{array}{c} I_{10} U_{01} \mid \widehat{a}_{01} \\ I_{$	2	$\left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL}$
$ \begin{array}{c} 5a & \left( \begin{array}{c} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & a_{11} & a_{12}^T \\ a_{20} & a_{21} & a_{22} \end{array} \right), \left( \begin{array}{c} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{array} \right) \rightarrow \cdots, \left( \begin{array}{c} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{array} \right) \rightarrow \cdots \\ & \text{where } \alpha_{11} \text{ is } 1 \times 1, \lambda_{11} \text{ is } 1 \times 1, \nu_{11} \text{ is } 1 \times 1, \nu_{11} \text{ is } 1 \times 1 \\ \hline \\ 6 & \left\{ \begin{array}{c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array} \right) = \left( \begin{array}{c} L \setminus U_{00} & \widehat{a}_{01} & \widehat{A}_{02} \\ \widehat{a}_{10}^T & \widehat{a}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{array} \right) \wedge L_{00} U_{00} = \widehat{A}_{00} \\ \hline \\ 8 & a_{10}^T := U_{10}^T = a_{10}^T U_{00}^{-1} \\ \alpha_{11} := U_{11}^T = a_{11} - U_{10}^T u_{01} \end{array} \right) \qquad (U_{00} \text{ is stored in the strictly lower triangular part of } A_{00}) \\ \hline \\ 7 & \left\{ \begin{array}{c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array} \right\} = \left( \begin{array}{c} L \setminus U_{00} & u_{01} & \widehat{A}_{02} \\ U_{10}^T & v_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{array} \right) \wedge \left( \begin{array}{c} L_{00} U_{00} = \widehat{A}_{00} \\ U_{10}^T U_{00} = \widehat{a}_{10}^T \end{array} \right) \\ \hline \\ 7 & \left\{ \begin{array}{c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array} \right) = \left( \begin{array}{c} L \setminus U_{00} & u_{01} & \widehat{A}_{02} \\ U_{10}^T & v_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{array} \right) \wedge \left( \begin{array}{c} L_{00} U_{00} = \widehat{A}_{00} \\ U_{10}^T U_{00} = \widehat{a}_{10}^T \end{array} \right) \\ \hline \\ 7 & \left\{ \begin{array}{c} A_{11} & A_{12} \\ A_{20} & a_{21} & A_{22} \end{array} \right\} \leftarrow \left( \begin{array}{c} A_{00} & a_{01} \\ A_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{array} \right) \wedge \left( \begin{array}{c} L_{00} U_{00} = \widehat{A}_{00} \\ U_{10}^T U_{00} = \widehat{a}_{10}^T \end{array} \right) \\ \hline \\ 7 & \left\{ \begin{array}{c} A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{array} \right\} \leftarrow \left( \begin{array}{c} A_{11} & A_{12} \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{array} \right) \wedge \left( \begin{array}{c} L_{00} U_{00} = \widehat{A}_{00} \\ U_{10}^T U_{00} = \widehat{a}_{10}^T \end{array} \right) \\ \hline \\ 7 & \left\{ \begin{array}{c} A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{array} \right\} \leftarrow \left( \begin{array}{c} A_{11} & A_{12} \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{array} \right) \wedge \left( \begin{array}{c} L_{11} & L_{12} \\ U_{10} U_{00} = \widehat{a}_{10}^T \end{array} \right) \\ \hline \\ 7 & \left\{ \begin{array}{c} A_{11} & A_{12} \\ A_{11} & A_$	3	while $m(A_{TL}) < m(A)$ do
$\begin{array}{c} \text{where } \alpha_{11} \text{ is } 1 \times 1, \lambda_{11} \text{ is } 1 \times 1, \lambda_{11} \text{ is } 1 \times 1 \\ \\ \left\{ \begin{array}{c} A_{00}  a_{01}  A_{02} \\ a_{10}^T  \alpha_{11}  a_{12}^T \\ A_{20}  a_{21}  A_{22} \end{array} \right) = \begin{pmatrix} L \setminus U_{00}  \widehat{a}_{01}  \widehat{A}_{02} \\ \widehat{a}_{10}^T  \widehat{\alpha}_{11}  \widehat{a}_{12}^T \\ \widehat{A}_{20}  \widehat{a}_{21}  \widehat{A}_{22} \end{pmatrix} \wedge L_{00}U_{00} = \widehat{A}_{00} \\ \\ \left\{ \begin{array}{c} a_{01} := u_{01} =  L_{00}^{-1}a_{01} \\ A_{10} := l_{10}^T =  a_{10}^TU_{00}^{-1} \\ a_{10}^T := l_{10}^T =  a_{10}^TU_{00}^{-1} \\ A_{10} := l_{10}^T =  a_{10}^TU_{00}^{-1} \\ A_{10} := l_{10}^T =  a_{11}^T - l_{10}^Tu_{01} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  a_{01}  A_{02} \\ a_{10}^T  \alpha_{11}  a_{12}^T \\ A_{20}  a_{21}  A_{22} \\ \end{array} \right\} = \begin{pmatrix} L \setminus U_{00}  u_{01}  \widehat{A}_{02} \\ l_{10}^T  v_{11}  \widehat{a}_{12}^T \\ \widehat{A}_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  a_{01}  A_{02} \\ A_{20}  a_{21}  A_{22} \\ \end{array} \right\} = \begin{pmatrix} L \setminus U_{00}  u_{01}  \widehat{A}_{02} \\ l_{10}^T  v_{11}  \widehat{a}_{12}^T \\ \widehat{A}_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  a_{01}  A_{02} \\ A_{20}  a_{21}  A_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  a_{01}  A_{02} \\ A_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  a_{01}  A_{02} \\ A_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  A_{01}  A_{02} \\ A_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  A_{01}  A_{02} \\ A_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  A_{01}  A_{02} \\ A_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{00}  A_{01}  A_{02} \\ A_{20}  \widehat{a}_{21}  A_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{01}  A_{01}  A_{02} \\ A_{20}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{01}  A_{02} \\ A_{01}  A_{01}  A_{02} \\ A_{02}  \widehat{a}_{21}  \widehat{A}_{22} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{01}  A_{02} \\ A_{01}  A_{01}  A_{02} \\ A_{02}  A_{01}  A_{02} \\ A_{02}  A_{01}  A_{02} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{01}  A_{01} \\ A_{01}  A_{01}  A_{02} \\ A_{02}  A_{01}  A_{02} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{01}  A_{01}  A_{02} \\ A_{01}  A_{01}  A_{02} \\ A_{02}  A_{01}  A_{02} \\ \end{array} \right. \\ \left\{ \begin{array}{c} A_{01}  A_{01}  A_{02} \\ A_{01}  A_{01}  A_{02} \\ A_{02}  A_{01}  A_{02} \\ \end{array} \right. \\ $	2,3	$\left\{ \begin{array}{c c} \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge m(A_{TL}) < m(A) \end{array} \right\}$
$ \begin{cases} \begin{cases} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{cases} = \begin{pmatrix} L \setminus U_{00} & \widehat{a}_{01} & \widehat{A}_{02} \\ \widehat{a}_{10}^T & \widehat{\alpha}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} \wedge L_{00}U_{00} = \widehat{A}_{00} \\ \begin{cases} a_{01} := u_{01} = & L_{00}^{-1}a_{01} \\ a_{10}^T := l_{10}^T = & a_{10}^T U_{00}^{-1} \end{cases} & (L_{00} \text{ is stored in the strictly lower triangular part of } A_{00}) \\ \begin{cases} a_{10}^T := l_{10}^T = & a_{10}^T U_{00}^{-1} \\ a_{11} := v_{11} = & \alpha_{11} - l_{10}^T u_{01} \end{cases} & (U_{00} \text{ is stored in the upper triangular part of } A_{00}) \\ \end{cases} \\ $	5a	where $\alpha_{11}$ is $1 \times 1$ , $\lambda_{11}$ is $1 \times 1$ , $v_{11}$ is $1 \times 1$
$ \begin{array}{c} a_{10}^{T} := l_{10}^{T} = a_{10}^{T} U_{00}^{-1} & (U_{00} \text{ is stored in the upper triangular part of } A_{00}) \\ \alpha_{11} := v_{11} = \alpha_{11} - l_{10}^{T} u_{01} \\ 7  \left\{ \begin{array}{c} A_{00} \ a_{01} \ A_{02} \\ a_{10}^{T} \ \alpha_{11} \ a_{12}^{T} \\ A_{20} \ a_{21} \ A_{22} \end{array} \right\} = \begin{pmatrix} L \setminus U_{00} \ u_{01} \ \hat{A}_{02} \\ l_{10}^{T} \ v_{11} \ \hat{\alpha}_{12}^{T} \\ \hat{A}_{20} \ \hat{a}_{21} \ \hat{A}_{22} \end{array} \right) \wedge \begin{pmatrix} L_{00} U_{00} = \hat{A}_{00} \ L_{00} u_{01} = \hat{a}_{01} \\ l_{10}^{T} U_{01} = \hat{a}_{01} \end{array} $ $ \begin{array}{c} D_{10}^{T} U_{00} = \hat{a}_{10}^{T} \\ D_{10}^{T} U_{00} = \hat{a}_{10}^{T} \end{array}  \begin{bmatrix} L_{00} u_{01} = \hat{a}_{01} \\ l_{10}^{T} U_{01} = \hat{a}_{01} \end{array} \right] $ $ \begin{array}{c} D_{10}^{T} U_{00} = \hat{a}_{10}^{T} \\ D_{10}^{T} U_{01} + v_{11} = \hat{\alpha}_{11} \end{array} $ $ \begin{array}{c} D_{10}^{T} U_{01} = \hat{a}_{01} \end{array}  \begin{bmatrix} L_{00} u_{01} + \hat{a}_{01} \\ L_{10}^{T} U_{00} = \hat{a}_{10}^{T} \end{array}  \begin{bmatrix} L_{10}^{T} U_{01} + v_{11} = \hat{\alpha}_{11} \end{array}  \begin{bmatrix} L_{10}^{T} U_{01} + v_{11} + \hat{\alpha}_{11} \end{array}  \begin{bmatrix} $	6	$ \left\{ \begin{array}{c ccc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array} \right\} = \left( \begin{array}{c ccc} L \setminus U_{00} & \widehat{a}_{01} & \widehat{A}_{02} \\ \widehat{a}_{10}^T & \widehat{\alpha}_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{array} \right) \wedge L_{00}U_{00} = \widehat{A}_{00} $
5b $\left(\begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c c} L_{TL} & L_{TR} \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \cdots, \left(\begin{array}{c c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \leftarrow \cdots$ $2  \left\{\begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c c} L \setminus U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL}$ endwhile $2,3  \left\{\left(\begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c c} L \setminus U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array}\right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \wedge \neg (m(A_{TL}) < m(A))$	8	$a_{10}^T := l_{10}^T = a_{10}^T U_{00}^{-1}$ ( $U_{00}$ is stored in the upper triangular part of $A_{00}$ )
$2  \left\{ \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right\} = \left( \begin{array}{c c} L \setminus U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL}$ endwhile $2,3  \left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & \widehat{A}_{BR} \end{array} \right) = \left( \begin{array}{c c} L \setminus U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \wedge \neg (m(A_{TL}) < m(A))$	7	$ \left\{  \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L \setminus U_{00} & u_{01} & \widehat{A}_{02} \\ l_{10}^T & v_{11} & \widehat{a}_{12}^T \\ \widehat{A}_{20} & \widehat{a}_{21} & \widehat{A}_{22} \end{pmatrix} \wedge \begin{matrix} L_{00}U_{00} = \widehat{A}_{00} & L_{00}u_{01} = \widehat{a}_{01} \\ l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + v_{11} = \widehat{\alpha}_{11} \end{pmatrix} $
endwhile $2,3  \left\{ \left( \frac{A_{TL} \mid A_{TR}}{A_{BL} \mid A_{BR}} \right) = \left( \frac{L \setminus U_{TL} \mid \widehat{A}_{TR}}{\widehat{A}_{BL} \mid \widehat{A}_{BR}} \right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \wedge \neg (m(A_{TL}) < m(A)) \right\}$	5b	$ \left(\begin{array}{c c} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c c} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{array}\right) \leftarrow \cdots, \left(\begin{array}{c c} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{array}\right) \leftarrow \cdots $
$2,3  \left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge \neg (m(A_{TL}) < m(A))$	2	$\left\{ \begin{array}{c c} \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL}$
		endwhile
	2,3	$\left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \setminus U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \wedge \neg (m(A_{TL}) < m(A)) \right\}$
	1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

Step	Algorithm: $A := LU_{UNB\_VAR1}(A)$	
1a	{	
4	where	
2		
3	while do	
2,3		
5a	where	
6		
8	$egin{aligned} a_{01} &:= u_{01} = & L_{00}^{-1} a_{01} \ a_{10}^T &:= l_{10}^T = & a_{10}^T U_{00}^{-1} \ lpha_{11} &:= v_{11} = & lpha_{11} - l_{10}^T u_{01} \end{aligned}$	
7	$\begin{bmatrix} L_{00}u_{01} = \widehat{a}_{01} \\ l_{10}^T U_{00} = \widehat{a}_{10}^T & l_{10}^T u_{01} + v_{11} = \widehat{\alpha}_{11} \end{bmatrix}$	
5b		
2		
	endwhile	
2,3	$\bigg\{ \qquad \qquad \land \neg ( \qquad )$	
1b	{	,

Step	Algorithm: $A := LU_{UNB\_VAR1}(A)$
1a	$\{A = \widehat{A}\}$
4	where
2	
3	while do
2,3	
5a	
	where
6	$egin{array}{c c} a_{01} & \widehat{a}_{01} \ \hline a_{10}^T & \widehat{a}_{11} \ \hline \end{array} egin{array}{c} \widehat{a}_{01} & \hline \widehat{a}_{11} \ \hline \end{array}$
8	$egin{aligned} a_{01} &:= u_{01} = & L_{00}^{-1} a_{01} \ a_{10}^T &:= l_{10}^T = & a_{10}^T U_{00}^{-1} \ lpha_{11} &:= v_{11} = & lpha_{11} - l_{10}^T u_{01} \end{aligned}$
7	$\begin{bmatrix} L_{00}u_{01} = \widehat{a}_{01} \\ l_{10}^T U_{00} = \widehat{a}_{10}^T \end{bmatrix} \begin{bmatrix} l_{10}^T u_{01} + v_{11} = \widehat{\alpha}_{11} \end{bmatrix}$
5b	
2	
	endwhile
2,3	$\bigg\{ \qquad \qquad \land \neg ( \qquad )$
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

Step	Algorithm: $A := LU_{UNB\_VAR1}(A)$
1a	$\{A = \widehat{A}\}$
4	where
2	$\left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \right\}$
3	while do
2,3	$\left\{ \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge \right\}$
5a	where
6	$\left\{\begin{array}{c c}a_{01}\\\hline a_{10}^T&\alpha_{11}\\\hline \end{array}\right. \begin{array}{c c}\widehat{a}_{01}\\\hline \widehat{a}_{10}^T&\widehat{\alpha}_{11}\\\hline \end{array}$
8	$a_{01} := u_{01} = L_{00}^{-1} a_{01}$ $a_{10}^{T} := l_{10}^{T} = a_{10}^{T} U_{00}^{-1}$ $\alpha_{11} := v_{11} = \alpha_{11} - l_{10}^{T} u_{01}$
7	$\begin{bmatrix} L_{00}u_{01} = \widehat{a}_{01} \\ i_{10}^T U_{00} = \widehat{a}_{10}^T & i_{10}^T u_{01} + v_{11} = \widehat{\alpha}_{11} \end{bmatrix}$
5b	
2	$\left\{ \begin{array}{c c} \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \end{array} \right\}$
	endwhile
2,3	$ \left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge \neg ( )  $
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

Step	Algorithm: $A := LU_{UNB\_VAR1}(A)$
1a	$\{A = \widehat{A}\}$
4	where
2	$ \left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL}  $
3	while $m(A_{TL}) < m(A)$ do
2,3	$\left\{ \begin{array}{c c} \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge m(A_{TL}) < m(A) \end{array} \right\}$
5a	where
6	$\left\{egin{array}{c c} a_{01} & \widehat{a}_{01} \ \hline a_{10}^T & \widehat{lpha}_{10} & \widehat{lpha}_{11} \end{array} ight.$
8	$\begin{array}{cccc} a_{01} := u_{01} = & L_{00}^{-1} a_{01} \\ \\ a_{10}^T := l_{10}^T = & a_{10}^T U_{00}^{-1} \\ \\ \alpha_{11} := v_{11} = & \alpha_{11} - l_{10}^T u_{01} \end{array}$
7	$\begin{bmatrix} L_{00}u_{01} = \widehat{a}_{01} \\ l_{10}^T U_{00} = \widehat{a}_{10}^T \end{bmatrix} \begin{array}{c} L_{00}u_{01} = \widehat{a}_{01} \\ l_{10}^T u_{01} + v_{11} = \widehat{\alpha}_{11} \end{bmatrix}$
5b	
2	$\left\{ \begin{array}{c c} \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \end{array} \right\}$
	endwhile
2,3	$\left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge \neg (m(A_{TL}) < m(A)) \right\}$
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

Step	Algorithm: $A := LU_{UNB_VAR1}(A)$
1a	$\{A = \widehat{A} $
4	$A \to \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}, L \to \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix}, U \to \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix}$ where $A_{TL}$ is $0 \times 0$ , $L_{TL}$ is $0 \times 0$ , $U_{TL}$ is $0 \times 0$
2	where $A_{TL}$ is $0 \times 0$ , $L_{TL}$ is $0 \times 0$ , $U_{TL}$ is $0 \times 0$ $ \left\{ \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \right\} $
3	while $m(A_{TL}) < m(A)$ do
2,3	$\left\{ \begin{array}{c c} \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge m(A_{TL}) < m(A) \end{array} \right\}$
5a	where
6	$\left\{\begin{array}{c c}a_{01}\\\hline a_{10}^T&\alpha_{11}\\\hline \end{array}\right.  \left.\begin{array}{c c}\widehat{a}_{01}\\\hline \widehat{a}_{10}^T&\widehat{\alpha}_{11}\\\hline \end{array}\right.$
8	$a_{01} := u_{01} = L_{00}^{-1} a_{01}$ $a_{10}^T := l_{10}^T = a_{10}^T U_{00}^{-1}$ $\alpha_{11} := v_{11} = \alpha_{11} - l_{10}^T u_{01}$
7	$\begin{bmatrix} L_{00}u_{01} = \widehat{a}_{01} \\ l_{10}^T U_{00} = \widehat{a}_{10}^T \end{bmatrix} \begin{bmatrix} l_{10}^T u_{01} + v_{11} = \widehat{\alpha}_{11} \end{bmatrix}$
5b	
2	$\left\{ \begin{array}{c c} \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \end{array} \right\}$
	endwhile
2,3	$\left\{ \left( \begin{array}{c c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) = \left( \begin{array}{c c} L \backslash U_{TL} & \widehat{A}_{TR} \\ \hline \widehat{A}_{BL} & \widehat{A}_{BR} \end{array} \right) \wedge L_{TL} U_{TL} = \widehat{A}_{TL} \wedge \neg (m(A_{TL}) < m(A)) \right\}$
1b	$\left\{ A = L \backslash U \wedge LU = \widehat{A} \right\}$

Step Algorithm: 
$$A := LU_LUNB_LVAR1(A)$$

1a  $\{A = \widehat{A}\}$ 
 $A \rightarrow \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}$ ,  $L \rightarrow \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix}$ ,  $U \rightarrow \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix}$ 

where  $A_{TL}$  is  $0 \times 0$ ,  $L_{TL}$  is  $0 \times 0$ ,  $U_{TL}$  is  $0 \times 0$ 

2  $\{\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \land L_{TL}U_{TL} = \widehat{A}_{TL} \end{pmatrix}$ 

3 while  $m(A_{TL}) < m(A)$  do

2,3  $\{\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \land L_{TL}U_{TL} = \widehat{A}_{TL} \land m(A_{TL}) < m(A)$ 

5a  $\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ A_{20} & a_{21} & A_{22} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} , \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \rightarrow \cdots, \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \rightarrow \cdots$ 

8  $\begin{pmatrix} a_{TL} & A_{TR} \\ a_{H} & a_{H} \end{pmatrix} \rightarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ A_{20} & a_{21} & A_{22} \\ A_{20} & a_{21} & A_{22} \end{pmatrix} , \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \rightarrow \cdots, \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \rightarrow \cdots$ 

7  $\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \leftarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & a_{22}^T \end{pmatrix} , \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \leftarrow \cdots, \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \leftarrow \cdots$ 

2  $\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & a_{22}^T \end{pmatrix} , \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \leftarrow \cdots, \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \leftarrow \cdots$ 

2  $\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \land L_{TL}U_{TL} = \widehat{A}_{TL} \land \neg (m(A_{TL}) < m(A))$ 

1b  $\{A = L \setminus U \wedge LU = \widehat{A}\}$ 

Step Algorithm: 
$$A := LU \ UNB \ VAR1(A)$$

1a  $A = \widehat{A}$ 
 $A \to \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}, L \to \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix}, U \to \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix}$ 

where  $A_{TL}$  is  $0 \times 0$ ,  $L_{TL}$  is  $0 \times 0$ ,  $U_{TL}$  is  $0 \times 0$ 

2  $\left\{\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \end{pmatrix}$ 

3 while  $m(A_{TL}) < m(A)$  do

2.3  $\left\{\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \wedge L_{TL}U_{TL} = \widehat{A}_{TL} \wedge m(A_{TL}) < m(A) \end{pmatrix}$ 

5a  $\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ A_{20} & a_{01} & A_{22} \\ A_{20} & a_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \rightarrow \cdots, \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \rightarrow \cdots$ 

where  $a_{11}$  is  $1 \times 1$ ,  $\lambda_{11}$  is  $1 \times 1$ ,

Step Algorithm: 
$$A := LU_LNB_LVAR1(A)$$

1a  $\{A = \widehat{A}\}$ 

A  $\rightarrow \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}$ ,  $L \rightarrow \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix}$ ,  $U \rightarrow \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix}$ 

where  $A_{TL}$  is  $0 \times 0$ ,  $L_{TL}$  is  $0 \times 0$ ,  $U_{TL}$  is  $0 \times 0$ 

2  $\{\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} = \begin{pmatrix} L \setminus U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \land L_{TL}U_{TL} = \widehat{A}_{TL} \\ \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{CU} & A_{CU} & \widehat{A}_{CU} \\ A_{BL} & A_{BR} & \widehat{A}_{BL} & \widehat{A}_{BR} \end{pmatrix} \land L_{TL}U_{TL} = \widehat{A}_{TL} \land m(A_{TL}) < m(A)$ 

5a  $\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{CU} & A_{CU} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} & \widehat{A}_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{TL} & \widehat{A}_{TR} \\ \widehat{A}_{BL} & \widehat{A}_{BR} & \widehat{A}_{BR} \end{pmatrix} \land L_{TL}U_{TL} = \widehat{A}_{TL} \land m(A_{TL}) < m(A)$ 

5a  $\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{CU} & A_{CU} & A_{CU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \rightarrow \cdots , \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \rightarrow \cdots$ 

where  $\alpha_{11}$  is  $1 \times 1$ ,  $\lambda_{11}$  is  $1 \times 1$ ,  $\nu_{11}$  is  $1 \times 1$ 

6  $\begin{pmatrix} A_{CU} & A_{CU} & \widehat{A}_{CU} & \widehat{A}_{CU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{CU} & \widehat{A}_{CU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{CU} & \widehat{A}_{CU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{20} & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{UU} & \widehat{A}_{UU} & \widehat{A}_{UU} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{UU} & \widehat{A}_{UU} & \widehat{A}_{UU} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{UU} & \widehat{A}_{UU} & \widehat{A}_{UU} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{U} & \widehat{A}_{UU} & \widehat{A}_{UU} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{U} & \widehat{A}_{UU} & \widehat{A}_{UU} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{UU} & \widehat{A}_{UU} & \widehat{A}_{UU} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{UU} & \widehat{A}_{UU} & \widehat{A}_{UU} \end{pmatrix} \rightarrow \begin{pmatrix} L_{U}U_{UU} & \widehat{A}_{UU} \\ \widehat{A}_{UU} & \widehat$ 

Algorithm: $A := LU_{UNB\_VAR1}(A)$
$A \to \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}, L \to \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix}, U \to \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix}$ where $A_{TL}$ is $0 \times 0$ , $L_{TL}$ is $0 \times 0$ , $U_{TL}$ is $0 \times 0$
while $m(A_{TL}) < m(A)$ do
$ \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \to \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} L_{TL} & L_{TR} \\ L_{BL} & L_{BR} \end{pmatrix} \to \cdots, \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix} \to \cdots $ where $\alpha_{11}$ is $1 \times 1$ , $\lambda_{11}$ is $1 \times 1$ , $v_{11}$ is $1 \times 1$
$a_{01} := u_{01} = L_{00}^{-1}a_{01}$ ( $L_{00}$ is stored in the strictly lower triangular part of $A_{00}$ ) $a_{10}^T := l_{10}^T = a_{10}^T U_{00}^{-1}$ ( $U_{00}$ is stored in the upper triangular part of $A_{00}$ ) $\alpha_{11} := v_{11} = \alpha_{11} - l_{10}^T u_{01}$
$ \left(\begin{array}{c c} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c c} L_{TL} & L_{TR} \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \cdots, \left(\begin{array}{c c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right) \leftarrow \cdots $
endwhile

## Algorithm: $A := LU_{UNB\_VAR1}(A)$

$$A o \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \,,\, L o \left( \begin{array}{c|c} L_{TL} & L_{TR} \\ \hline L_{BL} & L_{BR} \end{array} \right) \,,\, U o \left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right)$$

where  $A_{TL}$  is  $0 \times 0$ ,  $L_{TL}$  is  $0 \times 0$ ,  $U_{TL}$  is  $0 \times 0$ 

while  $m(A_{TL}) < m(A)$  do

$$\left(\begin{array}{c|c|c}
A_{TL} & A_{TR} \\
\hline
A_{BL} & A_{BR}
\end{array}\right) \to \left(\begin{array}{c|c|c}
A_{00} & a_{01} & A_{02} \\
\hline
a_{10}^T & \alpha_{11} & a_{12}^T \\
A_{20} & a_{21} & A_{22}
\end{array}\right), \left(\begin{array}{c|c|c}
L_{TL} & L_{TR} \\
\hline
L_{BL} & L_{BR}
\end{array}\right) \to \cdots, \left(\begin{array}{c|c|c}
U_{TL} & U_{TR} \\
\hline
U_{BL} & U_{BR}
\end{array}\right) \to \cdots$$

where  $\alpha_{11}$  is  $1 \times 1$ ,  $\lambda_{11}$  is  $1 \times 1$ ,  $v_{11}$  is  $1 \times 1$ 

$$a_{01} := u_{01} = L_{00}^{-1} a_{01}$$

 $a_{01} := u_{01} = L_{00}^{-1} a_{01}$  ( $L_{00}$  is stored in the strictly lower triangular part of  $A_{00}$ )

$$a_{10}^T := l_{10}^T = a_{10}^T U_{00}^{-1}$$

 $(U_{00} \text{ is stored in the upper triangular part of } A_{00})$ 

$$\alpha_{11} := v_{11} = \alpha_{11} - l_{10}^T u_{01}$$

$$\left(\begin{array}{c|c|c}
A_{TL} & A_{TR} \\
\hline
A_{BL} & A_{BR}
\end{array}\right) \leftarrow \left(\begin{array}{c|c|c}
A_{00} & a_{01} & A_{02} \\
\hline
a_{10}^T & \alpha_{11} & a_{12}^T \\
\hline
A_{20} & a_{21} & A_{22}
\end{array}\right), \left(\begin{array}{c|c|c}
L_{TL} & L_{TR} \\
\hline
L_{BL} & L_{BR}
\end{array}\right) \leftarrow \cdots, \left(\begin{array}{c|c|c}
U_{TL} & U_{TR} \\
\hline
U_{BL} & U_{BR}
\end{array}\right) \leftarrow \cdots$$

endwhile