A Benchmark for Evaluating Approximated Algorithms for Fair Wireless Channel Allocation

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***Abstract*—We present a benchmark for the performance evaluation of heuristic and meta-heuristic approaches to fair distribution of indivisible goods. The specific problem reflected by the benchmark data sets is Wireless Channel Allocation (WCA), and the approach to fair distribution is to choose from feasible allocations by the maximum set of a fairness relation between their corresponding allocation performances. The effort for exhaustive search for such maximum sets is rapidly increasing and even problems with 10 users may already be beyond today's computing capabilities. Here we present the results for up to 7 users and also discuss some general aspects of using fairness relations in the prescribed manner, also indicating the efficiency of the approach in terms of establishing rather small maximum sets with much overlap of maximal elements among different fairness relations.**

***Keywords*-maxmin fairness; parabolic fairness; proportional fairness; wireless channel allocation; meta-heuristics; heuristics; benchmark**

I. Introduction

An increasing number of resource sharing problems in telecommunication appears to be of discrete nature, often referring to the distribution of indivisible goods among a number of users. We can easily demonstrate how a number of recent "hot" problems promotes this aspect, like cognitive radio, cooperative relaying, or queue utilisation in a network of processors. Among these problems, one can consider the Wireless Channel Allocation [1] as a basic abstraction of combinatorial and optimality aspects in these domains. It is a special case of fair distribution of indivisible goods. The "goods" here are transmission channels, e.g. in a wireless communication among a number of mobile agents (users), and a scheduler has to assign at most one channel per user for each time slot. The physical and infrastructure conditions of channel utilisation of a specific mobile user at a specific timeslot are put into so-called *channel coefficients*. Usually, they are reals from $[0, 1]$ and used as factors to model transmission properties like channel fading, etc. Also, the pairs timeslot-channel can be more simply stated as *cells* with a corresponding matrix of channel coefficients per cell and user. Then for each *allocation* of users to cells there is a performance per user (the sum of channel coefficients of cells allocated to this user) comprising a performance vector. This is the *performance* of an allocation, obviously a multi-criterion entity.

Distribution of indivisible goods has been studied in economics for decades, and among the approaches that are under discussion is social choice theory. Recently, Suzumura brought the attention to the concept of rationalisation of social choice [2]. It is a purely set theoretic approach, basing on the central concept of relations as a subset of the direct product of sets to represent preference. Concomitant with the specification of a particular relation, we have its extreme set representing elements of the relation domain to which no other element is in relation or which are in relation to all other. In case that for a social choice there is a relation such that social choice and extreme set coincide, the social choice is considered to be rationalisable. Assuming a preference for fair allocations among a number of available states, the same approach can be applied in resource allocation problems, as long as we can specify corresponding fairness relations that allow for a pairwise comparison of allocation states.

Recently, a number of such potential candidates for fairness relations has been presented. In addition to the classical relations, such as maxmin fairness [3], leximin fairness and proportional fairness [4], we can find newer approaches including alpha-fairness [5], parabolic fairness [6], majority fairness [7], ordered proportional fairness [8], maxmin multi-fairness [9]. We have to note that several of these proposals were not made under a strict relation-theoretic framework, but under the notion of stability, where a state is considered stable if it establishes a logical property against all other possible states. However, this corresponds exactly with the specification of this state as belonging to the best set of a relation that compares two states by that logical property. This is an extension of the common concept of maximizing some functional that represents the goodness or quality of states.

The approach has a number of advantages, mostly resulting from revealing generic aspects of the relational design. But there are also some disadvantages. One is that the search domain for feasible allocations is squaring. It means that in order to find best or maximal elements of a relation (then to be used as chosen fair allocations) we need to do a pairwise comparison of states. This is not making the problem intractable, as there is no exponential growth in complexity, but it can hurt nevertheless. For example, to perform an exhaustive search for maximal elements in a set of 1000 states requires 10^6 pairwise comparisons. Having one Mio. states, these are already 10^{12} comparisons, rendering such approaches infeasible as well (we may call this "jigsaw puzzle complexity"). On the other hand, we are not aware of exact algorithms with linear complexity in discrete domains that can be used to directly find (or better: algorithmically construct) maximal or best states.

Therefore, it is a good opportunity to study heuristic and meta-heuristic approaches in order to approximate extreme states of relations. A number of potential algorithms have been presented in the domain of evolutionary multi-objective optimisation employing the Pareto dominance relation (also known as vector inequality) or variants. Such algorithms can be easily adjusted to handle fairness relations as well.

But having a number of potential algorithms, judging their performance can become a conundrum: for knowing the quality of their approximation to extreme states, we need the exact solutions. But for finding these exact solutions, we would need good algorithms. A possible approach has already been presented in [10] where the performance of relational optimisation by a fixed algorithm was justified by means of comparing with "same effort" random search. However, the method appears to be unable to differentiate among algorithms for larger problem scales (in this case beyond the level of 20 users) since the random search becomes unable to provide a sufficient number of good quality solutions.

Here, we want to assist the development and design of such algorithms by providing a benchmark of specific problems beyond "toy world" scale along with the exact maximum sets for various fairness relations. For finding the maximum sets, concurrent computing has been employed. The paper will recall the definition of the used fairness relations (Section II), describe the scope of the benchmark (Setion III), present the structure of the benchmark and report about its design and implementation (Section IV), and provide some statistics and base performance values (Section V) before concluding.

II. Fairness relations

We recall a number of definitions for fairness relations that appear in the benchmark. We refer to the literature to find more information about their motivation and usage. In all cases, we consider vector relations, esp. from vectors with positive components. Then, a binary vector relation R has the *domain* $A \subseteq R_n^+$ and is given as subset of $A \times A$ (note that several definitions can be extended to R_n). First we recall the basic vector inequality of Pareto dominance. All definitions assume states x and y to be from the domain A and have components x_i, y_i resp. with $i = 1, \ldots, n$. We also use the notation $x_{(i)}$ to indicate the i-th smallest component of a general vector x. In all these definitions, we use the "$\geq$" symbol to represent the aspect of reflexivity. In general, each of these relations comes in pairs, with a corresponding "$>$" relation as its asymmetric part (i.e. among the set of all pairs $x, y \in A$ all pairs such that $(x, y) \in R$ but not $(y, x) \in R$).

Definition 1. *Pareto dominance: $x \geq_p y$ if for all i $x_i \geq y_i$.*

Now we consider maxmin fairness.

Definition 2. *Maxmin fairness: $x \geq_{mmf} y$ if for all i with $x_i < y_i$ there exists a j such that (1) $x_j \leq x_i$ and (2) $x_j > y_j$.*

There are a number of ways to simplify the definition. Especially from the implementation point of view above the definition is rather inconvenient.

A possible way is to consider the set of all least components of x where x and y differ, and to check whether at least one of the corresponding y components is smaller. This also helps to explain the name "maxmin." The maxmin fairness was explicitly introduced in [3].

Related to maxmin fairness (in some ways of consideration even appearing to be the same concept) is the leximin relation, lexicographic minimum, lexicographic maxmin fairness etc. It is a sorted version of maxmin fairness.

Definition 3. *Leximin relation: $x \geq_{lm} y$ if either for all i $x_{(i)} = y_{(i)}$ or there is an index i such that $x_{(k)} = y_{(k)}$ for $0 < k < i$ and $x_{(i)} > y_{(i)}$.*

Parabolic fairness refers to a concept of expanding maxmin fairness to different domains. We need to specify an ordered weighted aggregation operator. Given a set of weights w_i then the ordered weighted averaging (OWA) is defined as $\mathrm{OWA}_w(x) = \sum_i w_i x_{(i)}$. In parabolic fairness, we focus on comparison by OWA with decreasing weight vectors.

Definition 4. *Parabolic fairness: given a set of strictly decreasing weights w_i then $x \geq_{pb} y$ iff $\mathrm{OOWA}_w(x) \geq \mathrm{OOWA}_w(y)$.*

To indicate the decrease of weights (i.e. the smaller component is multiplied with the larger weight) we can also write ordered-ordered weighted averaging (OOWA) but the choice for this name is rather random. There are three specific ways for providing the weights: from lowest to largest increasing exponentially, by Fibonacci series, and linear. The corresponding definitions:

1) **expOOWA** realises parabolic fairness by choosing the weights such that for $n \geq i > 1$ $w_i > \sum_{0<j<i} w_j$. A suitable choice is $w_i = 2^{n-i}$.
2) **linOOWA** realises parabolic fairness by any decreasing set of weights, with $w_i = n - i + 1$ being a suitable choice.
3) **FibOOWA** relaxes the rapid decay of weights in expOOWA by the condition $w_2 > w_1$ and $w_i > w_{i-1} + w_{i-2}$ for $i > 2$. A suitable choice is $w_i = F_{n-i+3} - 1$ where F_i is the i-th Fibonacci number.

We will write $x \geq_{eo} y, x \geq_{lo} y$ and $x \geq_{fo} y$ correspondingly.

We continue with the "family" of proportional fairness relations, initiated by Kelly's seminal paper [4].

Definition 5. *Proportional fairness: $x \geq_{pf} y$ if and only if*

$$\sum_{i=1}^{n} \frac{y_i - x_i}{x_i} \leq 0 \tag{1}$$

Definition 6. *Alpha fairness: given an integer $\alpha \geq 1$ $x \geq_{af} y$ if and only if*

$$\sum_{i=1}^{n} \frac{y_i - x_i}{x_i^{\alpha}} \leq 0 \tag{2}$$

Note that for $\alpha = 1$ this is the same as proportional fairness, and in some restricted sense for $\alpha \rightarrow \infty$ alpha fairness approximates maxmin fairness [5]. But with regard to an adjustment between proportional fairness and maxmin fairness, ordered proportional fairness was introduced in [9].

Definition 7. *Ordered Proportional Fairness: $x \geq_{opf} y$ if and only if*

$$\sum_{i=1}^{n} \frac{y_{(i)} - x_{(i)}}{x_{(i)}} \leq 0 \tag{3}$$

At last, another way of "balancing" proportional and maxmin fairness is introduced here:

Definition 8. *Self-weighted proportional fairness: $x \geq_{swpf} y$ if and only if*

$$\sum_{i=1}^{n} \left[\sum_{j \neq i} x_i \right] \times \frac{y_i - x_i}{x_i} \leq 0 \tag{4}$$

where each term in the indicator expression for proportional fairness is weighted by the average of all other components.

Each relation can be decomposed into an asymmetric and a symmetric part.

Definition 9. *Given a relation R over a domain A. The* asymmetric part *$P(R)$ (or $P_A(R)$) is the set of all ordered pairs (x, y) with $x, y \in A$ where $(x, y) \in R$ but not $(y, x) \in R$. The* symmetric part *$I(R)$ (or $I_A(R)$) is the set of all ordered pairs (x, y) with $x, y \in A$ where $(x, y) \in R$ and $(y, x) \in R$.*

Obviously $R = P(R) \cup I(R)$ and $P(R) \cap I(R) = \emptyset$. Generally, for any comparing relation, if the relation is to be understood in the sense of "at least as" then the asymmetric part is a corresponding "more than" relation (for example, for the $\geq$-relation among real numbers, the asymmetric part is the corresponding $>$-relation). This concept is needed to properly define the concept of a maximum set.

Definition 10. *For a relation R over domain A an element $x \in A$ is maximal if and only if there is no $y \in A$ with $(y, x) \in P(R)$. The set of all maximal elements is called the maximum set M_R (or $M_R(A)$ if A is not understood from context).*

Then, generally (relational) optimisation can be interpreted as the task of finding the maximum set of a given relation R. If R is given by a real-valued n-variate function $f : R^n \rightarrow R$ it coincides with "standard" function optimisation (actually maximization). But the relevant aspect about its definition is that it applies to *each relation*, including all listed fairness relations, and thus each definition of a relation specifies a corresponding optimisation task.

III. Scope of the Benchmark

The goal is to provide a number of test instances for evaluating and comparing the performanc of model-free optimisation approaches. The goal of optimisation here is approximating the maximum sets for some fairness relation within a given feasible space. We selected the WCA problem (see next section for the exact specification) as it represents the generic aspect of many resource distribution problems appearing in networked communication these days. Since the approach does abstract from other modeling aspects, design and utilization of meta-heuristic algorithms seems to be a promising approach in this direction. However, these algorithms are facing a scaling problem, as the size of feasible spaces grows exponentially, and exact algorithms are not known so far. For keeping problem scales in the scope of present day computing capabilities, we have to focus on a time instant situation (a "snapshot") and ignore time varying aspects. Note that there is ongoing research into a better understanding of time varying issues with regard to fairness-based control, e.g. as represented by the concept of balanced fairness [11].

Said this, there are limitations on the benchmark, basically also to avoid combinatorial explosion of the number of possible choices. Future versions of the benchmark may address some of these issues, if there is a particular insterest. It means that in general the benchmark cannot cover aspects that would need additional models for the corresponding processes, including: changing number of users by some probability distribution, point processes for the creation of traffic or bandwidth demands, prizing or prioritization models to solicit user demands. With regard to the last item, we also want to refer to Kelly's aproach to proportional fairness [4], where a case was considered where users pay for resource access. Then, a separation is possible: if the system maintains proportional fairness, then for each utility function there is a price model maximizing total utility for all users on a lowest cost level. We may consider this separation of combined aspects and modalities of such a problem as a general aspect.

IV. Specification and Implementation

Definition 11 (Wireless Channel Allocation (WCA) Problem)**.** *Given a set of n users U and m cells C and an $n \times m$ matrix CC of* channel coefficients*, i.e. reals from $[0, 1]$. A* channel allocation *is a mapping $A : C \rightarrow U$ where to each cell c_i with $i = 1, \ldots, m$ exactly one user u_j with $j = 1, \ldots, n$ is allocated. The notation is $u_j = A(c_i)$. An allocation is* feasible *if at least one cell is allocated to each user. The* performance *of user u_j in allocation A is $p_j = \sum_{i, A(c_i) = u_j} CC_{ji}$. The task of wireless channel allocation (WCA) is to find a feasible allocation a that "maximizes" the performances for all users.*

If "maximizing" is seen as maximizing the total sum of performances for all users, then the solution would be simply to select for each cell one of the users with largest channel coefficient. However, this might not be a feasible allocation, i.e. there can be users to which no cell will be allocated this way. What we are considering here is to maximize performances by selecting solutions from the maximum set of a relation.

Structure of the benchmark and implementation

A $n \times m$ matrix CC of *channel coefficients* defines a particular setting (number of channels, number of users and channels coefficients), setting called *run* in the following. For each run, a maximum set is obtained for each considered fairness relation, which defines a benchmark for that run.

The whole benchmark is provided as a set of plain text files, each file for a choice of number of users and number of cells. Each file contains the results for a fixed number of *runs*, 30 runs for smaller problems, 10 for larger problems. Each run is specified by a set of channel coefficients, followed by the maximum sets for the 10 relations that were listed in the foregoing section: alpha fairness for $\alpha = 2$ and $\alpha = 3$ (note that larger values for α make problems with numerical precision), maxmin fairness, proportional fairness, ordered proportional fairness, self-weighted proportional fairness, parabolic fairness using expOOWA, FibOOWA and linOOWA, and finally the leximin relation. For each relation, all pairs of performance vectors and allocations of maximal elements are listed.

Only feasible allocations, where each users receives at least one cell, were considered. Thus, the number of feasible allocations for n user and m cells is $n!S_2(m,n)$ where $S_2(m,n)$ is the Stirling number of second kind – this counts the number of surjective mappings from the set of cells to the set of users. Nevertheless, the evaluation had to scan all n^m possible allocations, since no simple constructive procedure for surjective allocations is known.

It is also planned to provide a complete list of heuristic results for these problems. So far, results for random sampling are provided, to serve as a base performance measure for future comparisons. The benchmark is publicly available via a thematic webpage[1].

The benchmark was generated using a program written in the Go programing language (using the recently released version Go 1.0). The Go programming language was selected because of (1) its flexibility to create concurrent programs – which also allows easy parallelization –, (2) because it is a compiled language – which allows a fast execution, being in our experience about 2 times the running time of the same program written in C–, and (3) because of its semantics, libraries and tools allow fast prototyping, development and debugging. The total running time of the whole benchmark was close to 30 hours. For each run of the experiments, the running time for all of the 10 presented fairness relations takes a few seconds for small problems (e.g. 3 users and 5 channels) and several minutes for larger problems (e.g 5 users and 7 channels). These times were obtained on a Intel(R) Core(TM) i7-2600 CPU @ 3.40GHz computer with 8 cores and 16GB RAM.

In the Appendix, an example of a dataset is provided. It shows the first run for the problem of 5 users and 6 cells.

V. Characterization of the Benchmark

We present a number of features and statistic evaluations of the data set, also for the purpose to gain insight into the behaviour of fairness relations for increasing problem scales. For space reasons, we will only consider a selected number of aspects for the larger scale problems.

Table I shows basic performance statistics of the maximum sets for all 10 fairness relations considered here. In the first column, the performance range is per single user performance, and it can be hold against the case of an expected performance of a random assignment. For example, in the case of 6 cells and 6 users, one user might in average receive one channel, thus the expected performance equals the expected channel coefficient: 0.5 in our unbiased model. But we can see that the performances in fair allocations are generally much larger, like 0.8 median in worst case. We can also see that this worst case relates to the lower difference between number of channels and number of users. The lower the difference, the lower the gain that users can expect in average.

Next column of Table I compares the total performance of a maximum fair allocation with the maximum possible total performance (which appears when we assign to each cell the user with largest corresponding channel coefficient). We can see that choosing a fair allocation instead of a maximizing one usually will not incur a substantial loss in total performance. In all cases, the loss is about 10% and seems to be rather constant.

The last column also indicates a convenient aspect of fair allocations: there are always allocations selected by multiple fairness relations (by being maximal with regard to more than one relation). The sets shown in each cell indicate the number of fair allocation that are maximal for one relation (first element), two relations (second element) etc. up to the last component indicating allocations appearing for all 10 relations. The numbers were counted over all runs and relations. A related evaluation (not shown here, as it can be derived from the set values) is that the average number of repeated occurrences of maximal allocations is about 3. The good point is that the application of several fairness relations provides overlapping maximum sets, and thus allows to select allocations that appear for a larger number of fairness relations. Also here, the effect seems stronger when number of users and cells become similar.

Table II gives some information about sizes of maximum sets. The relations not listed here have all exactly one maximum element per definition. As a general observation, the maximum sets remain tractable, i.e. there is no notable “explosion” of their size, if compared to e.g. Pareto dominance with a confirmed exponential increase. Thus, they allow for efficient selection. Among the six relations, maxmin fairness and ordered proportional fair-

[1] Link is http://www.ndrc.kyutech.ac.jp/wcabenchmark/. Mirror on http://rodrigo.verschae.org/wca/

users, cells	allocations	performance range	maximum performance ratio	repeating of maximal elements
4, 6	1560	0.375 - 0.85 - **0.972** - 1.315 - 2.453	0.652 - 0.877 - **0.923** - 0.965 - 1.0	(94 43 45 83 19 22 8 7 5 6)
4, 7	8400	0.454 - 0.968 - **1.29** - 1.598 - 2.786	0.678 - 0.889 - **0.944** - 0.977 - 1.0	(46 21 14 22 8 4 2 3 1 2)
5, 6	1800	0.185 - 0.77 - **0.89** - 0.977 - 1.945	0.726 - 0.88 - **0.916** - 0.95 - 1.0	(106 36 37 87 32 10 6 4 7 8)
5, 7	16800	0.292 - 0.824 - **0.932** - 1.267 - 1.904	0.697 - 0.873 - **0.923** - 0.95 - 1.0	(40 48 27 50 17 6 1 0 4 1)
6, 6	720	0.078 - 0.616 - **0.8** - 0.919 - 0.999	0.626 - 0.861 - **0.91** - 0.942 - 1.0	(73 35 17 24 14 6 4 5 2 16)
6, 7	15120	0.302 - 0.759 - **0.91** - 0.979 - 1.961	0.757 - 0.884 - **0.92** - 0.951 - 0.998	(24 29 16 32 10 4 6 1 0 1)
7, 7	5040	0.026 - 0.717 - **0.81** - 0.912 - 0.998	0.63 - 0.878 - **0.916** - 0.942 - 0.964	(48 8 4 0 6 2 0 2 2 6)

Table I
SOME STATISTICS FOR THE LARGER SCALE PROBLEMS. PERFORMANCE RANGE FOR SINGLE USERS AND RATIO TO MAXIMUM POSSIBLE TOTAL PERFORMANCE ARE GIVEN AS RANGES: MINIMUM - 25%-QUANTILE - MEDIAN - 75%-QUANTILE - MAXIMUM. THE COUNT OF REPETITIONS OF MAXIMAL ELEMENTS IS GIVEN AS ARRAY, FIRST ELEMENT THE NUMBER OF MAXIMAL ALLOCATIONS (AMONG ALL RUNS AND ALL 10 RELATIONS) APPEARING ONE TIME, SECOND ELEMENT APPEARING MAXIMAL FOR TWO RELATIONS ETC. FOR THE SAME WCA PROBLEM.

users, cells	$>_{af2}$	$>_{af3}$	$>_{mmf}$	$>_{pf}$	$>_{opf}$	$>_{swpf}$
4, 6	7.2 ± 2.0	7.3 ± 2.0	4.0 ± 1.2	6.0 ± 1.6	1.5 ± 0.8	6.2 ± 1.5
4, 7	6.9 ± 1.4	7.8 ± 1.5	3.4 ± 1.1	5.1 ± 1.6	1.4 ± 0.9	6.3 ± 1.6
5, 6	6.5 ± 2.4	7.0 ± 2.8	4.5 ± 2.3	5.9 ± 2.1	1.6 ± 0.76	6.2 ± 2.1
5, 7	14.0 ± 2.8	15.2 ± 3.6	3.6 ± 2.2	9.7 ± 4.3	1.7 ± 1.1	10.9 ± 4.4
6, 6	3.5 ± 3.6	4.5 ± 5.3	4.0 ± 3.5	2.0 ± 1.9	1.1 ± 0.3	2.3 ± 2.2
6, 7	7.7 ± 3.6	8.1 ± 3.1	3.5 ± 2.1	7.1 ± 2.7	1.4 ± 0.5	7.4 ± 3.4
7, 7	2.2 ± 1.9	3.0 ± 2.9	7.0 ± 7.0	1.8 ± 1.4	1.2 ± 0.4	2.0 ± 1.6

Table II
AVERAGE SIZES AND STANDARD DEVIATION OF MAXIMUM SETS FOR THE LARGER SCALE PROBLEMS.

ness appear to have rather constant sizes (except the case of 7 users and 7 cells for maxmin fairness) while for other relations, the maximum sets tend to become larger when the number of users and cells differ more strongly. But this can be also related to the increase of the number of feasible solutions, see second column in Table I.

Table III shows the quality of random search with 1000 samples for selected fairness relations and larger problem scales. The range values indicate the distribution of smallest and largest shortest pairwise Euclidean distances between all elements of the maximum sets of 1000 samples and elements of the known correct maximum sets. The purpose here is to give a base measure for future evaluation of other algorithms (more "intelligent" than random search), but they also demonstrate the general hardness of finding good approximations to maximum sets. We have to note that such investigations might include the need to also find good performance measures for search algorithms, as we have only limited evidence that the Euclidean distance is a good choice to prescribe similarity of approximated and real maximum sets.

The values listed in the table give some indication that smaller maximum sets can be found more easily, despite of being more sparse elements in feasible space, while other searches are subject to a scaling effect and the performance decays about linearly with problem scale.

VI. CONCLUSIONS

Benchmarking in general may always serve two purposes. The obvious purpose is to provide a means for justification of solution quality. But the other purpose is also the stimulation and increase of awareness for the underlying technical challenge. Both purposes are served by the presented WCA benchmark. In addition to allow for a quantification of future algorithm performances, it also should help to become more aware of the relational approach to fairness as a "social choice" from a number of available states by using fairness relations to represent the underlying preferences. Then, the datasets in the benchmark also demonstrate the efficiency of the approach, following from the appearance of rather small maximum sets along with much overlap between the maximum sets for various relations. Even for problems with feasible spaces of size around 10,000 (thus 10^8 comparisons are needed to find all maximum elements) the sizes of maximum sets usually stay below 10, and in average each maximal element appears for about 3 of 10 relations. A good base for specifying suitable algorithms to find the maximum sets has been already established by numerous evolutionary multi-objective optimisation algorithms, employing Pareto dominance relation for internal justifications. We express a strong expectation that these algorithms can be applied to the generalising relational optimisation problem by suitable modifications as well and that the presented benchmark can help for their development.

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users, cells	range minimum distances	range Hausdorff distances
	Maxmin Fairness	
4, 7	0.0 - 0.147384 - **0.226919** - 0.341545 - 0.731937	0.176003 - 0.479681 - **0.635258** - 0.718378 - 1.18324
5, 7	0.0 - 0.236108 - **0.355949** - 0.450506 - 0.788894	0.280708 - 0.542024 - **0.709582** - 0.833534 - 1.20676
6, 7	0.0744312 - 0.353999 - **0.460859** - 0.608199 - 0.984624	0.321145 - 0.798776 - **0.905503** - 1.07927 - 1.4934
	Proportional Fairness	
4, 7	0.0 - 0.0 - **0.277916** - 0.341606 - 0.804337	0.0 - 0.668604 - **0.846364** - 1.01774 - 1.25849
5, 7	0.0 - 0.0 - **0.233341** - 0.408969 - 0.670712	0.204863 - 0.977704 - **1.07597** - 1.26048 - 1.65592
6, 7	0.0 - 0.389891 - **0.447506** - 0.57173 - 0.769509	0.0 - 1.4578 - **1.60417** - 1.75633 - 2.59928
	Ordered Proportional Fairness	
4, 7	0.0 - 0.359477 - **0.672203** - 0.875952 - 1.38887	0.0 - 0.479213 - **0.862431** - 1.11058 - 1.72604
5, 7	0.0 - 0.37806 - **0.715634** - 1.02487 - 1.62217	0.0 - 0.654368 - **0.957394** - 1.15764 - 1.91955
6, 7	0.218002 - 0.646561 - **0.823563** - 0.983349 - 1.38393	0.218002 - 0.799959 - **0.984393** - 1.20118 - 1.8094
	Parabolic Fairness (expOOWA)	
4, 7	0.0602992 - 0.359477 - **0.668123** - 1.01656 - 1.38887	
5, 7	0.0 - 0.408036 - **0.647562** - 0.80851 - 1.24902	
6, 7	0.221084 - 0.498017 - **0.609827** - 0.752659 - 1.13065	

Table III

PERFORMANCE OF RANDOM SEARCH WITH 1000 SAMPLES FOR SELECTED FAIRNESS RELATIONS AND LARGER PROBLEM SCALES. THE RANGE VALUES INDICATE THE DISTRIBUTION OF SMALLEST AND LARGEST SHORTEST EUCLIDIAN DISTANCES BETWEEN MAXIMUM SETS OF 1000 SAMPLES AND KNOWN CORRECT MAXIMUM SETS. THE SHOWN VALUES HAVE THE SAME MEANING AS IN TABLE I. FOR EXPOOWA, MAXIMUM AND MINIMUM DISTANCE COINCIDE SINCE THERE IS ALWAYS ONLY ONE MAXIMAL ELEMENT. ALL RESULTS ARE BASED ON 10 REPETITIONS FOR EACH OF THE 10 RUNS IN THE BENCHMARK.

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APPENDIX

Example dataset

We present the entry for run 1 for 5 users and 6 cells. The used channel coefficients were:

User	Channel Coefficients
0	(0.736 0.330 0.480 0.028 0.229 0.496)
1	(0.306 0.412 0.385 0.396 0.571 0.950)
2	(0.702 0.076 0.818 0.857 0.993 0.587)
3	(0.180 0.688 0.814 0.780 0.893 0.720)
4	(0.117 0.398 0.135 0.597 0.120 0.924)

In this example, the allocation for maximum total performance is (0 3 2 2 2 1) (for each cell the user with maximal channel coefficient is selected) with performances (0.736 0.924 0.688 2.668 0.0). In this case, user 4 would not receive any allocation.

The maximum set for **alpha fairness** with $\alpha = 2$:

Performance	Allocation
(0.736 0.950 1.811 0.688 0.597)	(0 3 2 4 2 1)
(1.066 0.950 0.993 0.814 0.597)	(0 0 3 4 2 1)
(0.736 1.362 0.993 0.814 0.597)	(0 1 3 4 2 1)
(0.736 0.950 0.993 1.502 0.597)	(0 3 3 4 2 1)
(0.736 0.950 0.993 0.814 0.995)	(0 4 3 4 2 1)
(0.736 0.571 1.675 0.688 0.924)	(0 3 2 2 1 4)
(0.736 0.571 0.857 1.502 0.924)	(0 3 3 2 1 4)

The maximum set for **alpha fairness** with $\alpha = 3$:

Performance	Allocation
(0.736 0.950 1.811 0.688 0.597)	(0 3 2 4 2 1)
(1.066 0.950 0.993 0.814 0.597)	(0 0 3 4 2 1)
(0.736 1.362 0.993 0.814 0.597)	(0 1 3 4 2 1)
(0.736 0.950 0.993 1.502 0.597)	(0 3 3 4 2 1)
(0.736 0.950 0.993 0.814 0.995)	(0 4 3 4 2 1)
(1.066 0.571 0.857 0.814 0.924)	(0 0 3 2 1 4)
(0.736 0.571 0.857 1.502 0.924)	(0 3 3 2 1 4)

The maximum set for **maxmin fairness**:

Performance	Allocation
(1.066 0.950 0.993 0.814 0.597)	(0 0 3 4 2 1)
(0.736 0.950 0.993 1.502 0.597)	(0 3 3 4 2 1)
(0.736 0.950 0.993 0.814 0.995)	(0 4 3 4 2 1)
(1.066 0.950 0.818 0.893 0.597)	(0 0 2 4 3 1)
(0.736 0.950 0.818 0.893 0.995)	(0 4 2 4 3 1)
(1.066 0.571 0.857 0.814 0.924)	(0 0 3 2 1 4)
(0.736 0.983 0.857 0.814 0.924)	(0 1 3 2 1 4)

The maximum set for **proportional fairness**:

Performance	Allocation
(0.736 0.950 1.811 0.688 0.597)	(0 3 2 4 2 1)
(0.736 0.950 0.993 1.502 0.597)	(0 3 3 4 2 1)
(0.736 0.950 0.993 0.814 0.995)	(0 4 3 4 2 1)
(0.736 0.571 1.675 0.688 0.924)	(0 3 2 2 1 4)
(0.736 0.571 0.857 1.502 0.924)	(0 3 3 2 1 4)
(0.736 0.412 1.850 0.814 0.924)	(0 1 3 2 2 4)
(0.736 0.412 1.675 0.893 0.924)	(0 1 2 2 3 4)

The maximum set for **ordered proportional fairness**:

Performance	Allocation
(0.736 0.950 0.993 1.502 0.597)	(0 3 3 4 2 1)
(0.736 0.950 0.993 0.814 0.995)	(0 4 3 4 2 1)

The maximum set for **self-weighted proportional fairness**:

Performance	Allocation
(0.736 0.950 1.811 0.688 0.597)	(0 3 2 4 2 1)
(1.066 0.950 0.993 0.814 0.597)	(0 0 3 4 2 1)
(0.736 1.362 0.993 0.814 0.597)	(0 1 3 4 2 1)
(0.736 0.950 0.993 1.502 0.597)	(0 3 3 4 2 1)
(0.736 0.950 0.993 0.814 0.995)	(0 4 3 4 2 1)
(0.736 0.571 1.675 0.688 0.924)	(0 3 2 2 1 4)
(0.736 0.571 0.857 1.502 0.924)	(0 3 3 2 1 4)
(0.736 0.412 1.675 0.893 0.924)	(0 1 2 2 3 4)

The last four relations produce maximum sets with exactly one element:

- **expOOWA**: performance (0.736 0.950 0.993 0.814 0.995) for allocation (0 4 3 4 2 1)
- **FibOOWA**: performance (0.736 0.950 0.993 0.814 0.995) for allocation (0 4 3 4 2 1)
- **linOOWA**: performance (0.736 0.950 0.993 0.814 0.995) for allocation (0 4 3 4 2 1)
- **leximin**: performance (0.736 0.950 0.818 0.893 0.995) for allocation (0 4 2 4 3 1)