A Relational Approach to Collaborative Fairness

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***Abstract*—In the relational approach to fairness, fairness is considered as a social choice that coincides with the maximum set of a fairness relation. Here we consider the application of this approach to achieve general fairness in collaborative systems. The approach is based on posing additional conditions on the fairness relation based on representation of collaboration among agents by a social graph, and various social types of agents. The relation can be used for the formal analysis of various collaboration scenarios by justifying tractable sizes of maximum sets of corresponding collaborative fairness relations. As an example result, the introduction of cliques of larger size appears to be in favor of achievability of collaborate fairness.**

***Keywords*-collaborative systems; fairness; maxmin fairness; proportional fairness**

I. INTRODUCTION

These days we can witness a growing impact of the networking paradigm in various fields of economic, cultural and political activities, including social networks, hardwired network infrastructure, power grids, distribution of trade goods, road traffic, regional accessibility, large-scale institution schedules etc. This has also stated new challenges and demands on the general problem of sharing and distributing goods. It is a situation where the global maximization approach demonstrates more and more weak points, especially with regard to the parallel fulfillment of a particular conflicting suite of additional demands like proportionality, envy-freeness, equity and efficiency, commonly put together under the term *fairness*.

Commonly, fairness is seen as an issue of a set of agents in a joint state (for example, a state reflecting the shares of a good allocated to the agents), justifying their present situation by mutually referring to each other's state. In this approach, the notion of stability appears where some states appear in a fixed relation to all other feasible states, and is often accompanied by the provision of two functions: an *utility function*, expressing the value of a share for each agent separately, and an *aggregation function*, fusing the values of utilities (or raw states) into a single real value that can be employed for direct numerical comparisons. Typical examples here are Jain's Fairness Indicator, or the Gini Index. However, these approaches unanimously refer to maximization by having all states equal, and measure the deviation from such a state on more or less incomparable scales.

The interplay of proportionality and envy-freeness can be seen as an extension of this rather simple approach, not referring to aggregation and not requiring equality. A proportional fair sharing is a state where all n agents, according to their own utilities, receive at least $1/n$ of the total value of a good [1]. An envy-free state is seen as a state where no agent receives more than the other, always seen in his or her own micro-universe of pre-assigned utility values. It often comes out that proportionality is under-specified while envy-freeness is over-specified, i.e. that it is rather easy to provide proportional fair sharings, but the number of choices can be too high, while it might be impossible to yield envy-freeness. In combination with the additional consideration of efficiency (where one state can only be improved if declining at least one other state), equity (where shares of same value are transferable between agents) and majority (maximization of the ordered aggregation of states) [2] one gets both: a challenging mixture of demands on new optimization approaches, and a mixture of demands that easily can become more and more conflicting and only be applicable to a restricted set of sharing scenarios.

In this context, we want to promote a *relational approach to fairness*, where we essentially compare among different state vectors and not among components of a same state vector, and consider fairness as a *social choice* among all feasible state vectors. A generic approach to social choice was introduced by Suzumura [3]. For this *rationalization approach to social choice*, social preference is expressed in terms of a set-theoretic relation R among states. Given a set X of states, a relation R is a subset of $X \times X$. Furthermore, one assumes that a social choice is formalized as a *social preference function* that assigns to each subset A of X a (non-empty) subset of A. Then, given a relation R we can consider extreme elements of X: greatest elements that are in relation to all other elements, or maximal elements to which no other element of X is in relation. Then, for short, a social choice is *rationalizable* if there is a relation R such that the social choices coincide with extreme elements of R itself. A model of a society then is expressed in terms of so-called axioms, and it is explored how these axioms entail specific properties of the social choice relation.

We are considering the application of this approach to model fair states as rationalizable social choices, while using a *fairness relation* for rationalization of the choices. In comparison to the social choice approach, there are only two differences: (1) we are not going to use specific axioms, but provide the specific relations itself in order to model the economic context, (2) we are focusing on maximal elements instead of greatest elements. Item (2) refers to an understanding of a fairness relation between two states $x \geq_R y$ as follows: agents in state y would "envy" agents in state x (by pure state comparison, or the unfair manner of transition from x to y). Therefore, the maximum set can be characterized as a set of envy-free states.

The most prominent fairness relations are Pareto dominance, maxmin fairness [4] and proportional fairness [5]. Assuming two states $x, y \in X$ they are formally given as:

Definition 1. *Pareto dominance: $x \geq_p y$ if for all i $x_i \geq y_i$.*

Definition 2. *Maxmin fairness: $x \geq_{mmf} y$ if for all i with $x_i < y_i$ there exists a j such that (1) $x_j \leq x_i$ and (2) $x_j > y_j$.*

Definition 3. *Proportional fairness: $x \geq_{pf} y$ if and only if*

$$\sum_{i=1}^{n} \frac{y_i - x_i}{x_i} \leq 0 \tag{1}$$

Here we use a $\geq$-notation for the relation, in order to express a meaning of "at least as good" - the corresponding "better"-relation, using $>$-notation is per suggestion of Suzumura [3] expressed as *asymmetric part* of such a relation:

$$P(R) = \{(x, y) | (x, y) \in R \wedge (y, x) \notin R\} \tag{2}$$

We cannot discuss all details of the relational approach here and wanted to just cover the basics. For more details, see the provided references as well as [6]. We only want to mention two more properties of relations in general and their implication for fairness, since we will have to refer to them in this paper.
A relation R is said to be *cycle-free* if there is no sequence x_i with $i = 1, .., k$ and $k \geq 2$ of states such that $x_1 \geq_R x_2, x_2 \geq_R x_3, \ldots, x_{k-1} \geq_R x_k$ and $x_k >_R x_1$. Note that $>_R$ stands for the asymmetric part of a $\geq_R$-relation. The advantage of a cycle-free relation is that each finite set has a non-empty maximum set. For each finite set, it also allows to introduce a ranking of elements: rank 1 are all elements of the maximum set, rank 2 are the elements of the maximum set after removal of the rank 1 elements from X, then removing rank 1 and rank 2 elements from X and selecting the maximum set gives rank 3 elements etc.
The adual of a relation is defined as

$$(x, y) \in R^a \leftrightarrow (x, y) \in R \vee (y, x) \notin R \tag{3}$$

and for a given $\geq_R$-relation expressing "at least as good" will express "at least as efficient" since, in extension to R it also refers to non-membership. A typical example is Pareto-efficiency, seen as a state where an agent can only improve its state if another agent's state becomes declined — this is actually the adual relation to Pareto dominance.

The relational approach to fairness reveals opportunities for elaborated modeling of fairness situations. As an example, the extension to multi-fairness was presented in [7]. The reason for this are the fairly generic ingredients of the approach: it only needs to specify the sets X and R, where the latter one is a means for comparing two states, however provided.

Groups of agents also undergo social relations and this can have an influence on judgments about fairness of a state or between states. Networking issues have been regarded to collaboration for various aspects. Many works focus on the issue of community detection and evaluation. Clique percolation is presented as a means for the analysis of network dynamics under collaboration in the seminal paper [8] with a result that stability of large cliques is related to their dynamic change, while small cliques have to remain unchanged. Follow-up studies identified communities from evaluation of log files [9] or general evaluation of graphs [10]. The focus on procedural collaboration beyond identification of collaboration can be found in other works like collaborative tagging [11] with regard to the modeling of folksonomy, and collaborative filtering, e.g. for recommendation systems or spam filtering [12]. For networking control, collaboration has been considered especially with regard to trust in communication [13]. An example for collaborative radio spectrum access is studied in [14] and for collaborative file access in P2P networking in [15]. In [16] a packet forwarding fairness protocol is proposed that identifies a malicious (i.e. considered as acting unfair) node by communicating node *reputation* among neighboring nodes in a wireless mesh network.

However, the question about a collaborative fairness, taking social relations among agents into account, did not find much attention so far. One example is the proposal of a time-stamp based approach to achieve data consistency as well as fair resource sharing and jitter compensation in a collaborative virtual environment [17]. But there, fairness is seen as the unbiased treatment of waiting request in a queue, based on imposing expected delays to each such request. It means there is no strict formal concept of fairness itself. In [18] we can find a study demonstrating the evolution of unfairness in complex cooperation networks by highly connected collaborators contributing little while extracting high payoffs. Also here, we do not see any formal specification of fairness. One reason for such shortages can be seen in the weak point of the utility and single-state based approaches. The only choice to represent dependable justifications among agents then would be the provision of epicycles over epicycles of conditional utility functions, which seems both: hard to specify, and hard to employ. The relational approach to fairness does not need such a specification, and can be based on the manner of judging between two states (considering structural information of a social network) alone. For short: it does not need numbers for comparison but procedures.

The main contribution of this paper is the provision of such a judgment procedure, taking a fairness relation and a social network of relationships into account. This gives raise to a formal definition of collaborative fairness as a refinement of a fairness relation, where additional constraints are derived from fairness among peers. Then, we can study and analyze various features of the social network, the ambition of agents etc. The definition of collaborative fairness based on a social graph, an allocation of social types of agents, and a base fairness relation will be provided in Section II. Then we demonstrate computational means for comparing various forms of collaborations with regard to achievability of fairness. The main indicator will be the estimated size of maximum sets, judging a scenario where the maximum sets are becoming large as intractable with regard to collaborative fairness. More details on this approach, and a number of examples will be provided in Section III. The paper concludes with an Outlook section, as we feel that the presented approach offers a lot of flexibility and potential extensions to study more refined models of collaboration.

II. Formal Approach to Collaborative Fairness

We consider a task of resource sharing of dividable or individable goods G among a group A of agents. The utility of an allocation of goods to agents is represented as a state vectors x. The dimension of a state vector is the same as the number of agents $n = |A|$ and all state vectors are restricted to a feasible domain S as a subset of R_n^+ (we exclude allocations where some agents receive no share of any available good). For example, the goods can be channels of a base station in a wireless network schedule, traffic rates in a wired network with link capacity constraints, relays in a cooperative networking architecture, or queues with limited buffer capacity in a network of processors. The states then can represent traffic rates.

So far, this is the "standard" way of expressing quality of resource sharing. In addition, we have a directed social graph G_s representing a relation between agents. The relation aR_sb between two agents a and b can be, for example, understood as "a cares for b" in a sense related to the utility of goods that are allocated to b. At the same time, and only then there is an edge from node a to node b in the social graph G_s.

The question now is about a fair allocation of goods, taking the social graph into account. Generally, when speaking about fairness, we have to utilize viewpoints of one agent on behalf of other agents - agent a judges the advantage of a situation not only based on his or her own advantages, but also on accompanying advantages or disadvantages sensed by other agents. With regard to the social graph, we will relate this perspective to the agent's "peers," i.e. the set of all agents to which he or she is in social relation (and there is a directed link in the social graph). Then we consider an agent to be "altruistic" in the sense that an agent will give up an attempted improvement, if there is a concomitant disadvantage for his or her peers. Stability of a state then is achieved whenever an improvement for an altruistic agent results in a decline for his or her peers.

We may also assume that our small society of agents can become "infected" by a few "envy" agents. Here, a decline of an envy agent is only possible if there is also a decline for his or her peers.

With regard to the social graph, in addition to random graphs, we will also consider the appearance of *cliques* in the social graphs, with the following property: Each agent of a clique is connected to all other agents of that clique, and only to them.

Now we want to introduce a relation that, given a set of states S, a social graph G_s, a *typing* T of all agents as either altruistic or envy[1] and a "base relation" R will represent the *collaborative fairness* between states x and y from S. We also need a few more formal notations: by G_i we indicate the set of all peers of agent i, and by $R|I$ with index set $I \subseteq \{1, \ldots, n\}$ we indicate the relation R reduced to sub-vectors with components indexed by I. For example, having 7 agents, $I = \{1, 3, 4\}$ and if taking the proportional fairness relation as R:

$$x \geq_{R|I} y \leftrightarrow \frac{y_1 - x_1}{x_1} + \frac{y_3 - x_3}{x_3} + \frac{y_4 - x_4}{x_4} \leq 0 \quad (4)$$

Finally, I_a indicates the index set of all altruistic agents, and I_e of all envy agents (thus, $I_a \cap I_e = \emptyset$ and $I_a \cup I_e = \{1, \ldots, n\}$), and we write $T = (I_a, I_e)$.

Using this notation, we may now state the definition for collaborative fairness.

Definition 4 (Collaborative Fairness)**.** *For two states x and y it is said that x is collaborative fair against y (alternatively (R, G_s, T)-collaborative fair), $x \geq_{cf} y$ (or $x \geq_{cf(R,G_s,T)} y$) if and only if the following three conditions are met:*

$$\begin{aligned} &(1)\ x \geq_R y \\ &(2)\ \forall i \in I_a : (y_i > x_i) \rightarrow x \geq_{R|G_i} y \\ &(3)\ \forall i \in I_e : (x_i > y_i) \rightarrow x \geq_{R|G_i} y \end{aligned} \quad (5)$$

Surely, this definition needs a few explanatory words. While loosing a little bit formal rigor, one may read "$x >_R y$" likewise as "compared to x, y appears unfair" or "x declines towards y" or "y envies x" and the corresponding $\geq_R$-relation with an additional "at most" modifier. In this sense, condition (1) says that the collaborative fairness is a refinement of a *base relation* R. Condition (2) considers all altruistic agents and would read as: whenever there is an improvement for an altruistic agent, there is a decline for the agent's peers; or: the peers would envy x; or: the improvement would appear as unfair to the agent's peers. By maximizing the collaborative fairness, we are also seeking a state where each improvement for an altruistic agent appears unfair to his or her peers.

Then, condition (3) expresses a related concern of envy agents (as an anti-thesis to collaboration, effectively disturbing the collaboration): whenever there is a decline for an envy agent, there must be a decline for the agent's peers as well.

Alternatively we can rewrite the conditions (2) and (3) in Def. 4 by taking the logical equivalence $A \rightarrow B \equiv$

[1]The case that an agent is both, which is logically possible, will not be considered here in order to keep the focus on the grouping aspects.

$\neg A \vee B \equiv \neg(A \wedge \neg B)$ into account

$$\begin{aligned}
&(1)\ x \geq_R y \\
&(2)\ \neg\exists i \in I_a : (y_i > x_i) \wedge x \ngeq_{R|G_i} y \qquad (6)\\
&(3)\ \neg\exists i \in I_e : (x_i > y_i) \wedge x \ngeq_{R|G_i} y
\end{aligned}$$

By putting the definition this way, we also ensure a number of properties.

1) $x \geq_{cf(R,G_s,T)} y$ implies $x \geq_R y$, which follows directly from condition (1) in Def. 4.
2) Whenever x Pareto dominates y and Pareto dominance implies R then also $x \geq_{cf} y$. Thus, it fulfills (1), but also condition (2) since there is no $y_i > x_i$. Condition (3) then is fulfilled for all envy agents since Pareto dominance implies Pareto dominance for each reduction to a subset of indices.
3) Whenever R is cycle-free the corresponding collaborative fairness relation is cycle-free as well. Otherwise, since collaborative fairness implies R, from a cycle of the $\geq_{cf}$-relation a cycle of R would follow. Thus, it also ensures the existence of maximal elements and ranks for any finite sets of states.
4) If there are no altruistic and envy agents at all, collaborative fairness equals R. The same holds if the social graph is empty, i.e. the agents have no peers.

We conclude the discussion of the definition by the provision of an example.

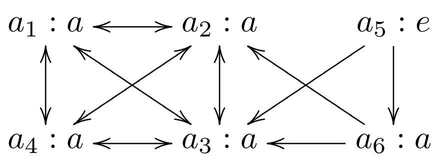

Figure 1. Social graph of six agents a_1 to a_6. Agents a_1 to a_4 form a clique of altruistic agents (notation $a_i : a$) and there is an additional altruistic agent e_6 and an envy agents a_5 (notation $a_i : e$).

Consider as an example proportional fairness for states of 6 agents, where agents 1 to 4 are altruistic and belong to one clique, agents 5 is envy with peers a_3, a_6 and agent a_6 is altruistic with peers a_2, a_3. Figure 1 shows the corresponding social graph. This means $T = (\{1, 2, 3, 4, 6\}, \{5\})$. Then, if we compare two states x and y and we assume a situation, as an example, where $x_1 < y_1, x_4 < y_4, x_6 < y_6$ and for all other i $x_i > y_i$, the relation $x \geq_{cf} y$ is tested by validating $x \geq_{pf} y$ and

$$\begin{aligned}
&\frac{y_2 - x_2}{x_2} + \frac{y_3 - x_3}{x_3} + \frac{y_4 - x_4}{x_4} \leq 0 \\
\wedge\ &\frac{y_1 - x_1}{x_1} + \frac{y_2 - x_2}{x_2} + \frac{y_3 - x_3}{x_3} \leq 0 \\
\wedge\ &\frac{y_3 - x_3}{x_3} + \frac{y_6 - x_6}{x_6} \leq 0 \\
\wedge\ &\frac{y_2 - x_2}{x_2} + \frac{y_3 - x_3}{x_3} \leq 0
\end{aligned}$$

Here, the first line is a consequence of $x_1 < y_1$, i.e. the first altruistic agents notes an improvement in y and checks if there is a disadvantage for his or her peers (agents 2, 3 and 4). The second line is a consequence of $x_4 < y_4$ in a similar manner. The third line is following from $x_5 > y_5$. Agent 5 notes a disadvantage for him or her in y and confirms that there is an disadvantage in y for his or her peers as well. The fourth line follows like the first and second line from $x_6 < y_6$ and the fact that a_2, a_3 are the peers of agent 6. If all of these tests are positive, state y is not considered as alternative to x.

III. Relational Analysis of Collaboration

The proposed approach to collaborative fairness allows for an analysis of a rich set of configurations with regard to different structures of social graphs and distribution of social types of agents. Now we consider an indicator value and the way to obtain it, in order to also have a computable means for comparison. This can be achieved by studying the expected size of maximum sets of collaborative fairness relations.

Table I gives an impression for the bounds of maximum sets for base relations. We remind on the fact that always Pareto dominance implies collaborative fairness, while collaborative fairness implies its base relation. Thus, the maximum set of a collaborative fairness for a (finite) set of states is a subset of the Pareto set, and the maximum set of the base relation is a subset of the maximum set of the collaborative fairness relation. This follows directly from the specification of maximum sets. Then, the sizes of maximum sets for collaborative fairness relations are also bound between the sizes for the base relation and the sizes for Pareto dominance.

For obtaining the values shown in Table I, 1000 states as elements of $(0, 1]^n$ (dimension n is the number of agents) were randomly sampled and the size of the maximum sets for the corresponding relation was evaluated. The experiment was repeated 30 times. The cell values in the table (as well as in all following tables) show the distribution of the maximum set sizes:

$$av\ (min - q_{25} - med - q_{75} - max)$$

where av is the average value, min, max are the minimum and maximum sizes appearing in the 30 repetitions, med the median value, and q_{25}, q_{75} are the 25%- and 75% quantiles resp.

We can see from Table I the well-known fact that Pareto dominance becomes highly inefficient with increasing dimensions. Even for the rather small number of 12 agents, nearly all states are maximal. This is related to the exponential decay of the relation occurrence (see [19] for related discussion and proofs). On the other hand, maxmin fairness (with about linear decay) and proportional fairness appear with rather small maximum sets.

Now we continue the same analysis for various configurations of collaborative fairness. To get a general picture, except cliques we do not consider any other specific structure of social graphs here. Even then, we can consider a large number of cases and need some restrictions. Especially two cases are of interest, dubbed as

Dimension	Pareto dominance	Proportional Fairness	Maxmin Fairness
8	541.3 (453 - 522 - 540 - 569 - 622)	4.5 (1 - 2 - 3 - 6 - 10)	5.6 (2 - 5 - 6 - 6 - 8)
12	914.7 (880 - 903 - 916 - 927 - 948)	13.8 (1 - 5 - 12 - 16 - 41)	7.2 (5 - 6 - 7 - 8 - 10)
20	957.5 (940 - 947 - 950 - 955 - 1000)	43.5 (6 - 21 - 38 - 60 - 110)	11.1 (7 - 10 - 11 - 12 - 15)

Table I
ESTIMATED SIZES OF MAXIMUM SETS FOR THE BASE RELATIONS.

Agents/Size of Clique	Envy = 0	Envy = 1	Envy = 5
Proportional Fairness			
8/0	13.33 (3 - 8 - 11 - 18 - 38)	19.20 (3 - 12 - 15 - 23 - 56)	35.67 (11 - 25 - 34 - 45 - 83)
8/2	11.87 (2 - 8 - 12 - 16 - 26)	30.73 (10 - 17 - 26 - 39 - 97)	58.40 (19 - 47 - 54 - 65 - 142)
8/4	11.77 (2 - 8 - 11 - 16 - 24)	16.83 (7 - 11 - 16 - 20 - 38)	40.47 (13 - 34 - 39 - 52 - 76)
8/8	4.13 (1 - 2 - 4 - 6 - 12)	4.83 (1 - 3 - 4 - 7 - 10)	14.7 (2 - 10 - 15 - 20 - 31)
12/0	48.20 (12 - 33 - 44 - 62 - 91)	55.13 (28 - 44 - 50 - 66 - 93)	90.57 (47 - 69 - 94 - 106 - 138)
12/2	46.9 (22 - 37 - 44 - 53 - 114)	85.4 (44 - 68 - 82 - 109 - 142)	108.5 (44 - 73 - 107 - 124 - 288)
12/4	44.57 (20 - 34 - 40 - 52 - 128)	54.0 (32 - 41 - 50 - 65 - 87)	100.4 (29 - 76 - 100 - 121 - 162)
12/12	12.10 (2 - 6 - 12 - 17 - 25)	12.43 (2 - 7 - 12 - 17 - 29)	26.27 (4 - 18 - 22 - 34 - 53)
20/0	189.9 (49 - 155 - 198 - 228 - 291)	203.5 (92 - 155 - 206 - 235 - 342)	253.6 (104 - 169 - 246 - 320 - 508)
20/2	209.9 (53 - 190 - 210 - 256 - 304)	270.3 (118 - 205 - 285 - 330 - 386)	285.1 (106 - 222 - 287 - 344 - 424)
20/10	154.9 (29 - 126 - 150 - 201 - 232)	176.0 (73 - 123 - 179 - 226 - 321)	208.8 (71 - 161 - 206 - 262 - 313)
20/20	37.53 (6 - 17 - 28 - 60 - 86)	45.23 (6 - 21 - 43 - 65 - 92)	70.8 (17 - 50 - 78 - 96 - 134)
Maxmin Fairness			
8/0	25.00 (12 - 18 - 22 - 29 - 52)	40.67 (13 - 22 - 28 - 37 - 365)	79.43 (38 - 55 - 62 - 76 - 529)
8/2	17.87 (9 - 15 - 18 - 22 - 25)	51.87 (26 - 42 - 50 - 55 - 109)	117.9 (50 - 77 - 89 - 108 - 416)
8/4	23.83 (13 - 19 - 22 - 27 - 76)	41.93 (21 - 34 - 40 - 45 - 93)	78.47 (37 - 69 - 78 - 91 - 115)
8/8	5.17 (2 - 4 - 5 - 6 - 8)	8.10 (5 - 7 - 8 - 9 - 12)	25.10 (18 - 22 - 25 - 29 - 33)
12/0	62.23 (34 - 53 - 56 - 71 - 118)	72.93 (45 - 64 - 72 - 80 - 123)	122.8 (52 - 103 - 124 - 142 - 196)
12/2	64.43 (25 - 52 - 64 - 77 - 98)	123.7 (47 - 95 - 121 - 146 - 232)	161.4 (95 - 132 - 162 - 186 - 231)
12/4	76.57 (38 - 63 - 79 - 85 - 146)	109.9 (69 - 96 - 106 - 125 - 161)	166.6 (97 - 148 - 164 - 190 - 243)
12/12	7.33 (5 - 6 - 8 - 8 - 11)	9.60 (6 - 8 - 10 - 11 - 13)	23.70 (15 - 20 - 24 - 28 - 34)
20/0	294.1 (184 - 237 - 294 - 331 - 481)	306.0 (222 - 262 - 312 - 329 - 406)	386.3 (179 - 336 - 389 - 441 - 540)
20/2	303.5 (225 - 284 - 301 - 323 - 394)	380.9 (290 - 348 - 393 - 411 - 489)	421.2 (225 - 350 - 426 - 477 - 611)
20/10	182.6 (124 - 150 - 184 - 201 - 253)	197.4 (123 - 177 - 196 - 211 - 316)	262.4 (151 - 228 - 266 - 288 - 350)
20/20	10.83 (8 - 10 - 11 - 12 - 14)	12.50 (6 - 11 - 13 - 14 - 16)	23.70 (15 - 21 - 23 - 26 - 32)

Table II
ESTIMATION OF MAXIMUM SET SIZES OF 1000 RANDOM STATES, 30 REPETITIONS: MIXED SOCIETY RESULTS.

mixed society and *sorted society*. In general, mixed society refers to a set of agents were a clique of size m coexists with a random social graph of the agents outside of the clique. Then, sorted society refers to the subdivision of the whole set of agents into cliques of same size m. The setup for a mixed society of n agents, clique size m and e envy agents in detail:

- The first m agents comprise a clique of altruistic agents.
- The remaining $n - m$ agents select a random subset of peers from all available agents (except themself) and their social type is set to *altruistic* as well.
- e different agents are randomly selected, and their social type is changed to *envy*. The selection does not differentiate between agents belonging to the clique or not.

The set up of a sorted society is as follows:

- The agents are divided into n/m groups of same size (we use values for m that divide n).
- Within each group, each agent is connected to each other agent, thus comprising a clique, and the social type is set to *altruistic*.
- The selection of envy agents is the same as for the mixed society case.

Then, we restrict the values of n, m, e to study a few special cases. The number n of agents was varied between moderate sizes 8, 12 and 20. The clique size varied between small, median and full. In addition, for the mixed society case we also consider the case $m = 0$ which means no cliques appear, and the social graph is a random graph. The number of envies was kept rather small, we considered the cases $e = 1$ and $e = 5$ only. We used proportional fairness and maxmin fairness as base relations.

To obtain estimates for maximum set sizes we followed the same procedure as for the base relation. For each configuration, 1000 random states with components from $(0, 1]$ were sampled, the maximum set was computed (based on 1 Mio. pairwise comparisons of states by the corresponding collaborative fairness relation) and its size was stored. Then, average and quantiles were computed for 30 repetitions of the procedure. The results are shown in Table II for the mixed society and Table III for the sorted society.

From these results, a number of observations can be made. In the following, we will consider a case with a smaller estimate of the maximum set size as more *effective* in the sense that it allows for a more tight specification of a fair state as maximal state of the corresponding

Agents/Size of Clique	Envy = 0	Envy = 1	Envy = 5
Proportional Fairness			
8/2	13.57 (7 - 11 - 14 - 16 - 26)	34.07 (18 - 27 - 34 - 39 - 59)	133.03 (68 - 94 - 140 - 169 - 192)
8/4	11.57 (5 - 9 - 12 - 13 - 20)	19.23 (10 - 15 - 18 - 21 - 39)	54.40 (30 - 44 - 51 - 67 - 80)
8/8	4.00 (1 - 2 - 3 - 5 - 11)	6.10 (1 - 4 - 6 - 8 - 15)	13.33 (2 - 9 - 12 - 16 - 35)
12/2	51.47 (25 - 47 - 52 - 59 - 75)	88.77 (60 - 70 - 94 - 107 - 116)	302.0 (202 - 274 - 292 - 329 - 414)
12/4	55.77 (35 - 43 - 56 - 66 - 89)	77.80 (38 - 68 - 72 - 92 - 114)	178.7 (128 - 161 - 175 - 193 - 263)
12/12	10.43 (2 - 5 - 10 - 15 - 24)	18.73 (7 - 14 - 17 - 25 - 37)	28.50 (6 - 17 - 29 - 36 - 56)
20/2	259.0 (168 - 232 - 264 - 280 - 354)	343.9 (240 - 309 - 350 - 384 - 437)	606.5 (491 - 567 - 592 - 653 - 707)
20/10	113.1 (38 - 85 - 126 - 141 - 191)	139.2 (47 - 111 - 137 - 176 - 206)	204.3 (98 - 178 - 210 - 237 - 293)
20/20	34.70 (4 - 13 - 40 - 49 - 75)	45.86 (9 - 25 - 46 - 67 - 87)	60.76 (8 - 29 - 58 - 87 - 125)
Maxmin Fairness			
8/2	16.37 (12 - 14 - 16 - 18 - 23)	47.6 (35 - 41 - 46 - 52 - 64)	173.8 (115 - 137 - 176 - 200 - 244)
8/4	21.50 (15 - 19 - 21 - 24 - 30)	40.87 (27 - 35 - 42 - 47 - 53)	111.1 (78 - 104 - 114 - 121 - 129)
8/8	5.00 (3 - 4 - 5 - 6 - 8)	8.27 (5 - 7 - 8 - 10 - 14)	25.73 (13 - 22 - 26 - 29 - 38)
12/2	52.93 (42 - 48 - 54 - 58 - 64)	109.8 (76 - 99 - 110 - 120 - 132)	330.8 (227 - 272 - 332 - 358 - 466)
12/4	98.07 (70 - 90 - 100 - 107 - 124)	141.6 (105 - 128 - 144 - 152 - 167)	314.8 (270 - 289 - 316 - 340 - 366)
12/12	6.80 (4 - 6 - 7 - 8 - 9)	9.60 (6 - 8 - 10 - 11 - 13)	26.07 (13 - 21 - 25 - 31 - 41)
20/2	252.0 (210 - 241 - 248 - 264 - 297)	350.4 (312 - 337 - 350 - 360 - 411)	649.0 (515 - 606 - 644 - 697 - 755)
20/10	124.5 (102 - 113 - 125 - 132 - 143)	143.0 (118 - 131 - 140 - 155 - 175)	219.6 (171 - 208 - 222 - 233 - 270)
20/20	11.00 (7 - 10 - 11 - 12 - 14)	12.80 (10 - 11 - 13 - 14 - 17)	23.37 (10 - 20 - 23 - 27 - 35)

Table III
ESTIMATION OF MAXIMUM SET SIZES OF 1000 RANDOM STATES, 30 REPETITIONS: SORTED SOCIETY RESULTS.

collaborative fairness relation. Then, a case where the maximum set sizes grow rapidly can be considered as *intractable*, at least with regard to achieving fairness efficiently.

- Even for the same number of agents, the maximum set sizes vary strongly with the different configurations. We can find values close to the lower bound given by the base relation as well as values up to 50% of the sample size. We can confirm a strong influence of the social graph and the distribution of social types. We can also confirm the suitability of the maximum set size as indicator for achievability of fairness.
- Generally, the maximum sets become larger for increasing number of agents. We can observe the same tendency as for the base relations, which means that fairness among an increasing number of agents becomes more and more intractable (with the smallest increase for maxmin fairness).
- There are no strong differences in size relations between the mixed society case, where a single clique co-exists with a random social graph, and the sorted society case, where all agents are grouped into cliques of same size. This indicates that the effectiveness of fairness is mostly driven by the share of agents organized into cliques.
- The maximum sets tend to become smaller for increasing size of cliques, while reaching the lower bound if there is only a single clique. It can be understood as that fracturing a society into a larger number of cliques makes it harder to achieve collaborative fairness, while large and few cliques diminish the influence of collaboration on fairness aspects.
- The maximum set sizes for maxmin fairness exceed the sizes for proportional fairness: they tend to become larger where proportional fairness as base relation produces larger sets, and smaller in the opposite case. It can be seen as a reflection of the "all-or-none" aspect of maxmin fairness, which basically judges from the value of the agent with lowest state. Between the extreme cases, proportional fairness seems to be the better mediator.
- The introduction of even a few envy agents has a strong influence on achievable fairness. In cases where the maximum sets are already large, they become rapidly larger. We can find cases where a single envy can double the size of maximum sets, esp. for small clique sizes.
- The growth of envy agent influence with increasing number of envies seems larger for small cliques in the sorted society. We assume, with regard to fairness, that envies do not only disturb the clique to which they belong, but also all other cliques, and that smaller cliques are more affected by envies than larger cliques.

The observations can be summarized into a sort of "recipe" for the design of collaborative structures.

Size of cliques: The size of cliques has strong influence on effectiveness of specifying a fair state in collaboration. Small cliques can make fairness intractable, while large cliques render the collaboration aspect void. A smaller number of larger cliques seems to be a good trade-off between both extremes.

Number of cliques: The influence seems to be less strong than the size of cliques. This can be understood by the *inbound* character of a clique and the justification of fairness independent of any other clique.

Envy agents: It has to be noted that fairness does not necessarily refer to effectiveness in the sense that all agents receive allocations that are as high as possible, but to the aspect of a balanced allocation. Therefore, the impact of envy agents is not the one of a "selfish" agent who would only focus on large shares and not care for peers. The envy agent will accept a decline only if all his peers face a decline as well. This attitude can disrupt fair collaboration more effectively than plain selfish behavior (which simply stands out of any collaboration).

IV. Outlook

In the sections before, we have provided a specification of a collaborative fairness relation, and the use of this relation to analyze various structures of collaboration. In some sense, this might not seem to be sufficient for practical applications, and we also need to provide a perspective how to tackle related issues in more detail. This will be done in the following subsection.

A. Further analysis of collaborative structures

Feasible states: So far, no assumptions were made about the collaborative task itself. In fact, the only reference was given to random state vectors. This was to learn about general feasibility of the approach. In a specific context, like eLearning, eCommerce, eGovernment, the feasible state vectors will be domain-specific, and the estimates of maximum set sizes can become strongly influenced. Same holds for the general change to discrete and bounded domains. However, the formal approach will be the same, and based on the specification of R alone.

Calibration: We have already demonstrated how different social graphs influence achievability of collaborative fairness. Based on this, some social graphs appear to be more attractive than others. However, a critique might be that a social structure is not a matter of design, but a matter of grown relations among individuals, often based on their personal history. This is true, but the organization of collaboration will always provide some flexibility. Confining a set of agents at some location will surely increase the establishment of social relations among these agents. Splitting students in a class into groups by a teacher will enforce the same. Thus, relational analysis might help to identify the goal situations of a promoted social relationship.

Another aspect appears with regard to the social graph. Beyond cliques and purely random graphs, social graphs can provide a lot of structural properties. These properties can refer to the existence of substructures like dominated sets, or global measures like diameter, connectivity, assortativity etc. Relational analysis will allow for characterization of social graphs in the sense of another calibration as well.

Refined agent models: In Def. 4 the set of conditions is not fixed, one could add other roles as social types as well. Without elaborating much on this point, we just mention possible approaches:

- A *teacher* is characterized as taking care for fairness among his or her peers, without dependency on self-improvement. Thus, the further condition could be like $\forall i \in I_t : x \geq_{R|G_i} y$ and relational analysis could help to answer the question if its better to allocate one teacher per clique, or one teacher for all cliques.
- A *leader*, which does not consider the relation but its adual, in order to take the more broad perspective of efficiency than just "betterness" into account.
- A *lurker* that represents a combination of the altruistic and the envy social type: its state can only be increased if the peers are not declining, and only decline if peer's states are declining as well.

These are just examples, presented here without validation. However, the perspective of extending Def. 4 might have become clearer.

Detection: Another relevant task with regard to collaboration is the detection and identification of counter-collaborative attitudes. We have seen how strongly a single envy agent can influence the maximum set size indicator. But this was not just a qualitative claim, it was also quantitative, i.e. the relation analysis also allows to numerically express the influence of such attitudes, and in return, a given real-world situation can be fitted to such model cases.

B. Limitations

We will conclude the paper by noting that the presented approach also has some limitations. Some of them appeared already throughout the exposition of this paper.

- On first glance, the number of agents appears to be rather limited. While real-world social networks can have millions of members, here using the approach for more than 20 agents appears already infeasible. However, taking the aspect of ranking into account (see corr. comment about cycle-free relations) one can apply the same framework, but focus on finding states of a rank that is as low as possible (low means that the number of the rank is small). The effect of increasing problem scale then is that it is just getting more and more unlikely to find elements of rank 1, but nevertheless a good rank can be possible. However, the maximum set size criterion would have to be modified as well. Independently, in a specific case maximum sets might be easier to find then for the general case of pure random sampling.
- Increasing the number of different social types gives additional restrictions on the relation occurrence, thus also expanding maximum sets by making the relation sparser. Therefore, care should be taken about not "overloading" the formal specification. Alternatively, the conditions can be handled in other logical constructs than just AND-ing.
- For the computation of maximum sets, only in few cases exact algorithms with linear complexity are known (for example, Bottleneck Flow Control to achieve maxmin fairness [4][20]). Exhaustive search requires the comparison between all possible pairs of states, thus the computational effort increases with the square of the number of states. This is commonly considered as still tractable, but in specific cases it can easily become an obstacle. However, in [21] it was shown that nevertheless, No-Free-Lunch does not hold for that kind of search and we can consider

some algorithms to be more efficient in searching maximum sets, while probably being still not known. Last but not least, meta-heuristic approaches [22] are a general family of algorithms to approximate maximum sets to any level of desired accuracy.

- Maximum sets will usually include more than one element, so it seems to be a disadvantage to not having a uniquely specified solution at the end. Usually, reference is taken to a *decision maker* to select from such sets. In case of a fairness relation, the good point is that a decision maker can still decide on global optimality, since this criterion has not been used in the approach so far. This is in contrary to e.g. selecting from Pareto fronts in multi-criterion decision-making, where the selecting criterion cannot be used anymore.

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