Strings, Sets, and Binomial Coefficients

Much of combinatorial mathematics can be reduced to the study of strings, as they form the basis of all written human communications. Also, strings are the way humans communicate with computers, as well as the way one computer communicates with another. As we shall see, sets and binomial coefficients are topics that fall under the string umbrella. So it makes sense to begin our in-depth study of combinatorics with strings.

Strings: A First Look

Let n be a positive integer. Throughout this text, we will use the shorthand notation $[n]$ to denote the n-element set $\{1, 2, \ldots, n\}$. Now let X be a set. Then a function $s\colon [n] \to X$ is also called an *X-string of length n*. In discussions of X-strings, it is customary to refer to the elements of X as *characters*, while the element $s(i)$ is the i^{th} character of s. Whenever practical, we prefer to denote a string s by writing $s =$ "$x_1x_2x_3 \ldots x_n$", rather than the more cumbersome notation $s(1) = x_1$, $s(2) = x_2$, ..., $s(n) = x_n$. There are several alternatives for the notation and terminology associated with strings. First, the characters in a string s are frequently written using subscripts as $s_1, s_2, \ldots, s_n$, so the i^{th}-term of s can be denoted s_i rather than $s(i)$. Strings are also called *sequences*, especially when X is a set of numbers and the function s is defined by an algebraic rule. For example, the sequence of odd integers is defined by $s_i = 2i - 1$.

Alternatively, strings are called *words*, the set X is called the *alphabet* and the elements of X are called *letters*. For example, *aababbccabcbb* is a 13-letter word on the 3-letter alphabet $\{a, b, c\}$.

In many computing languages, strings are called *arrays*. Also, when the character $s(i)$ is constrained to belong to a subset $X_i \subseteq X$, a string can be considered as an element of the cartesian product $X_1 \times X_2 \times \cdots \times X_n$, which is normally viewed as n-tuples of the form $(x_1, x_2, \ldots, x_n)$ such that $x_i \in X_i$ for all $i \in [n]$.

Example: In the state of Georgia, license plates consist of four digits followed by a space followed by three capital letters. The first digit cannot be a 0. How many license plates are possible?

Let X consist of the digits $\{0, 1, 2, \ldots, 9\}$, let Y be the singleton set whose only element is a space, and let Z denote the set of capital letters. A valid license plate is just a string from

$$(X - \{0\}) \times X \times X \times X \times Y \times Z \times Z \times Z$$

so the number of different license plates is $9 \times 10^3 \times 1 \times 26^3 = 158\,184\,000$, since the size of a product of sets is the product of the sets' sizes. We can get a feel for why this is the case by focusing just on the digit part of the string here. We can think about the digits portion as being four blanks that need to be filled. The first blank has 9 options (the digits 1 through 9). If we focus on just the digit strings beginning with 1, one perspective is that they range from 1000 to 1999, so there are 1000 of them. However, we could also think about there being 10 options for the second spot, 10 options for the third spot, and 10 options for the fourth. Multiplying $10 \times 10 \times 10$ gives 1000. Since our analysis of

filling the remaining digit blanks didn't depend on our choice of a 1 for the first position, we see that each of the 9 choices of initial digit gives $1\,000$ strings, for a total of $9\,000 = 9 \times 10^3$.
In the case that $X = \{0, 1\}$, an X-string is called a 0–1 string (also a *binary string* or *bit string.*). When $X = \{0, 1, 2\}$, an X-string is also called a *ternary* string.
Example: A machine instruction in a 32-bit operating system is just a bit string of length 32. Thus, there are 2 options for each of 32 positions to fill, making the number of such strings $2^{32} = 4\,294\,967\,296$. In general, the number of bit strings of length n is 2^n.

Example: Suppose that a website allows its users to pick their own usernames for accounts, but imposes some restrictions. The first chracter must be an upper-case letter in the English alphabet. The second through sixth characters can be letters (both upper-case and lower-case allowed) in the English alphabet or decimal digits (0–9). The seventh position must be '@' or '.'. The eighth through twelfth positions allow lower-case English letters, '*', '%', and '#'. The thirteenth position must be a digit. How many users can the website accept registrations from?
We can visualize the options by thinking of the 13 positions in the string as blanks that need to be filled in and putting the options for

that blank above. Below, we've used U to denote the set of upper-case letters, L for the set of lower-case letters, and D for the set of digits.

U	U
26	62

Figure 1: String Template

							#	#	#	#	#	
	D	D	D	D	D		%	%	%	%	%	
	L	L	L	L	L	.	*	*	*	*	*	
U	U	U	U	U	U	@	L	L	L	L	L	D
26	62	62	62	62	62	2	29	29	29	29	29	10

Below each position in the string, we've written the number of options for that position. (For example, there are 62 options for the second position, since there are 52 letters once both cases are accounted for and 10 digits. We then multiply these possibilities together, since each choice is independent of the others. Therefore, we have

$$26 \times 62^5 \times 2 \times 29^5 \times 10 = 9\,771\,287\,250\,890\,863\,360$$

total possible usernames.

Permutations

In the previous section, we considered strings in which repetition of symbols is allowed. For instance, “01110000” is a perfectly good bit string of length eight. However, in many applied settings where a string is an appropriate model, a symbol may be used in at most one position.

Example: Imagine placing the 26 letters of the English alphabet in a bag and drawing them out one at a time (without returning a letter once it's been drawn) to form a six-character string. We know there are 26^6 strings of length six that can be formed from the English alphabet. However, if we restrict the manner of string formation, not all strings are possible. The string “yellow” has six characters, but it uses the letter “l” twice and thus cannot be formed by drawing letters from a bag. However, “jacket” can be formed in this manner. Starting from a full bag, we note there are 26 choices for the first letter. Once it has been removed, there are 25 letters remaining in the bag. After drawing the second letter, there are 24 letters remaining. Continuing, we note that immediately before the sixth letter is drawn from the bag, there are 21 letters in the bag. Thus, we can form $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21$ six-character strings of English letters by drawing letters from a bag, a little more than half the total number of six-character strings on this alphabet.

To generalize the preceding example, we now introduce permutations. To do so, let X be a finite set and let n be a positive integer. An X-string $s = x_1 x_2 \dots x_n$ is called a *permutation* if all n characters used in s are distinct. Clearly, the existence of an X-permutation of length n requires that $|X| \geq n$.

When n is a positive integer, we define $n!$ (read "*n factorial*") by

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

By convention, we set $0! = 1$. As an example, $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$. Now for integers m, n with $m \geq n \geq 0$ define $P(m, n)$ by

$$P(m, n) = \frac{m!}{(m-n)!} = m(m-1)\cdots(m-n+1).$$

For example, $P(9, 3) = 9 \cdot 8 \cdot 7 = 504$ and $P(8, 4) = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$. Also, a computer algebra system will quickly report that

$$P(68, 23) = 20732231223375515741894286164203929600000.$$

Proposition 1

If X is an m-element set and n is a positive integer with $m \geq n$, then the number of X-strings of length n that are permutations is $P(m, n)$.

Proof.

The proposition is true since when constructing a permutation $s = x_1x_2, \ldots x_n$ from an m-element set, we see that there are m choices for x_1. After fixing x_1, we have that for x_2, there are $m-1$ choices, as we can use any element of $X - \{x_1\}$. For x_3, there are $m-2$ choices, since we can use any element in $X - \{x_1, x_2\}$. For x_n, there are $m-n+1$ choices, because we can use any element of X except $x_1, x_2, \ldots x_{n-1}$. Noting that

$$P(m,n) = \frac{m!}{(m-n)!} = m(m-1)(m-2)\ldots(m-n+1),$$

our proof is complete. $\square$

Note that the answer we arrived at in Example 1 is simply $P(26, 20)$ as we would expect in light of Proposition 1.

Example: It's time to elect a slate of four class officers (President, Vice President, Secretary and Treasurer) from the pool of 80 students enrolled in Applied Combinatorics. If any interested student could be elected to any position (Alice contends this is a big "if" since Bob is running), how many different slates of officers can be elected?

To count possible officer slates, work from a set X containing the names of the 80 interested students (yes, even poor Bob). A permutation of length four chosen from X is then a slate of officers by considering the first name in the permutation as the President, the second as the Vice President, the third as the Secretary, and the fourth as the Treasurer. Thus, the number of officer slates is $P(80, 4) = 37957920$.

Example: Let's return to the license plate question of Example 1. Suppose that Georgia required that the three letters be distinct from each other. Then, instead of having $26^3 = 17\,576$ ways to fill the last three positions on the license plate, we'd have $P(26, 3) = 26 \times 25 \times 24 = 15\,600$ options, giving a total of $140\,400\,000$ license plates.

As another example, suppose that repetition of letters were allowed but the three digits in positions two through four must all be distinct from each other (but could repeat the first digit, which must still be nonzero). Then there are still 9 options for the first position and 26^3 options for the letters, but the three remaining digits can be completed in $P(10, 3)$ ways. The total number of license plates would then be $9 \times P(10, 3) \times 26^3$. If we want to prohibit repetition of the digit in the first position as well, we need a bit more thought. We first have 9 choices for that initial digit. Then, when filling in the next three positions with digits, we need a permutation of length 3 chosen from the remaining 9 digits. Thus, there are $9 \times P(9,3)$ ways to complete the digits portion, giving a total of $9 \times P(9,3) \times 26^3$ license plates.

Combinations

To motivate the topic of this section, we consider another variant on the officer election problem from Example 1. Suppose that instead of electing students to specific offices, the class is to elect an executive council of four students from the pool of 80 students. Each position on the executive council is equal, so there would be no difference between Alice winning the “first” seat on the executive council and her winning the “fourth” seat. In other words, we just want to pick four of the 80 students without any regard to order. We’ll return to this question after introducing our next concept.

Let X be a finite set and let k be an integer with $0 \leq k \leq |X|$. Then a k-element subset of X is also called a *combination* of size k. When $|X| = n$, the number of k-element subsets of X is denoted $\binom{n}{k}$. Numbers of the form $\binom{n}{k}$ are called *binomial coefficients*, and many combinatorists read $\binom{n}{k}$ as "*n choose k*." When we need an in-line version, the preferred notation is $C(n, k)$. Also, the quantity $C(n, k)$ is referred to as the number of combinations of n things, taken k at a time.
Bob notes that with this notation, the number of ways a four-member executive council can be elected from the 80 interested students is $C(80, 4)$. However, he's puzzled about how

to compute the value of $C(80,4)$. Alice points out that it must be less than $P(80,4)$, since each executive council could be turned into 4! different slates of officers. Carlos agrees and says that Alice has really hit upon the key idea in finding a formula to compute $C(n,k)$ in general.

Proposition 2

If n and k are integers with $0 \leq k \leq n$, then

$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Proof.

Let X be an n-element set. The quantity $P(n,k)$ counts the number of X-permutations of length k. Each of the $C(n,k)$ k-element subsets of X can be turned into $k!$ permutations, and this accounts for each permutation exactly once. Therefore, $k!C(n,k) = P(n,k)$ and dividing by $k!$ gives the formula for the number of k-element subsets. $\square$

Using Proposition 2, we can now determine that $C(80,4) = 1581580$ is the number of ways a four-member executive council could be elected from the 80 interested students.

Our argument above illustrates a common combinatorial counting strategy. We counted one thing and determined that the objects we wanted to count were *overcounted* the same number of times each, so we divided by that number ($k!$ in this case).

The following result is tantamount to saying that choosing elements to belong to a set (the executive council election winners) is the same as choosing those elements which are to be denied membership (the election losers).

Proposition 3

For all integers n and k with $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Example: A Southern restaurant lists 21 items in the "vegetable" category of its menu. (Like any good Southern restaurant, macaroni and cheese is *one* of the vegetable options.) They sell a vegetable plate which gives the customer four different vegetables from the menu. Since there is no importance to the order the vegetables are placed on the plate, there are $C(21, 4) = 5985$ different ways for a customer to order a vegetable plate at the restaurant.

Our next example introduces an important correspondence between sets and bit strings that we will repeatedly exploit in this text.

Example: Let n be a positive integer and let X be an n-element set. Then there is a natural one-to-one correspondence between subsets of X and bit strings of length n. To be precise, let $X = \{x_1, x_2, \ldots, x_n\}$. Then a subset $A \subseteq X$ corresponds to the string s where $s(i) = 1$ if and only if $i \in A$. For example, if $X = \{a, b, c, d, e, f, g, h\}$, then the subset $\{b, c, g\}$ corresponds to the bit string 01100010. There are $C(8, 3) = 56$ bit strings of length eight with precisely three 1's. Thinking about this correspondence, what is the total number of subsets of an n-element set?

Combinatorial Proofs

Combinatorial arguments are among the most beautiful in all of mathematics. Oftentimes, statements that can be proved by other, more complicated methods (usually involving large amounts of tedious algebraic manipulations) have very short proofs once you can make a connection to counting. In this section, we introduce a new way of thinking about combinatorial problems with several examples. Our goal is to help you develop a “gut feeling” for combinatorial problems.

Example: Let n be a positive integer. Explain why

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Consider an $(n+1) \times (n+1)$ array of dots as depicted in 2. There are $(n+1)^2$ dots altogether, with exactly $n+1$ on the main diagonal. The off-diagonal entries split naturally into two equal size parts, those above and those below the diagonal.

Furthermore, each of those two parts has $S(n) = 1 + 2 + 3 + \cdots + n$ dots. It follows that

$$S(n) = \frac{(n+1)^2 - (n+1)}{2}$$

and this is obvious! Now a little algebra on the right hand side of this expression produces the formula given earlier.

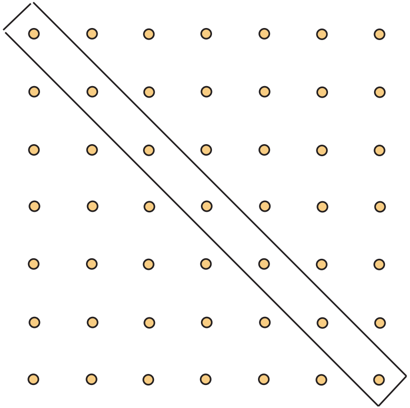

Figure 2: The sum of the first n integers

Example: Let n be a positive integer. Explain why

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2.$$

The left hand side is just the sum of the first n odd integers. But as suggested in 3, this is clearly equal to n^2.

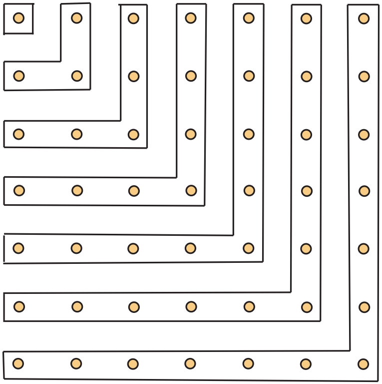

Figure 3: The sum of the first n odd integers

Example: Let n be a positive integer. Explain why

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Both sides count the number of bit strings of length n, with the left side first grouping them according to the number of 0's.

Example: Let n and k be integers with $0 \leq k < n$. Then

$$\binom{n}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k}.$$

To prove this formula, we simply observe that both sides count the number of bit strings of length n that contain $k+1$ 1's with the right hand side first partitioning them according to the last occurence of a 1. (For example, if the last 1 occurs in position $k+5$, then the remaining k 1's must appear in the preceding $k+4$ positions, giving $C(k+4, k)$ strings of this type.) Note that when $k=1$ (so $k+1=2$), we have the same formula as developed earlier for the sum of the first n positive integers.

Example: Explain the identity

$$3^n = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^n.$$

Both sides count the number of $\{0, 1, 2\}$-strings of length n, the right hand side first partitioning them according to positions in the string which are not 2. (For instance, if 6 of the positions are not 2, we must first choose those 6 positions in $C(n, 6)$ ways and then there are 2^6 ways to fill in those six positions by choosing either a 0 or a 1 for each position.)

Example: For each non-negative integer n,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Both sides count the number of bit strings of length $2n$ with half the bits being 0's, with the right side first partitioning them according to the number of 1's occurring in the first n positions of the string. Note that we are also using the trivial identity $\binom{n}{k} = \binom{n}{n-k}$.

The Ubiquitous Nature of Binomial Coefficients

In this section, we present several combinatorial problems that can be solved by appeal to binomial coefficients, even though at first glance, they do not appear to have anything to do with sets.

Example: The office assistant is distributing supplies. In how many ways can he distribute 18 identical folders among four office employees: Audrey, Bart, Cecilia and Darren, with the additional restriction that each will receive at least one folder?

Imagine the folders placed in a row. Then there are 17 gaps between them. Of these gaps, choose three and place a divider in each. Then this choice divides the folders into four non-empty sets.

The first goes to Audrey, the second to Bart, etc. Thus the answer is $C(17, 3)$. In Figure 4, we illustrate this scheme with Audrey receiving 6 folders, Bart getting 1, Cecilia 4 and Darren 7.

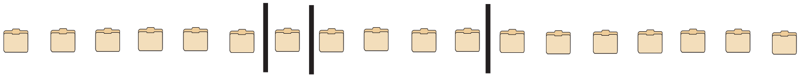

Figure 4: Distributing Identical Objects into Distinct Cells

Example: Suppose we redo the preceding problem but drop the restriction that each of the four employees gets at least one folder. Now how many ways can the distribution be made?
The solution involves a "trick" of sorts. First, we convert the problem to one that we already know how to solve. This is accomplished by *artificially* inflating everyone's allocation by one. In other words, if Bart will get 7 folders, we say that he will get 8. Also, artificially inflate the number of folders by 4, one for each of the four persons. So now imagine a row of $22 = 18 + 4$ folders. Again, choose 3 gaps. This determines a non-zero allocation for

each person. The actual allocation is one less—and may be zero. So the answer is $C(21,3)$.

Example: Again we have the same problem as before, but now we want to count the number of distributions where only Audrey and Cecilia are guaranteed to get a folder. Bart and Darren are allowed to get zero folders. Now the trick is to artificially inflate Bart and Darren's allocation, but leave the numbers for Audrey and Cecilia as is. So the answer is $C(19,3)$.

Example: Here is a reformulation of the preceding discussion expressed in terms of integer solutions of inequalities.

We count the number of integer solutions to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 538$$

subject to various sets of restrictions on the values of $x_1, x_2, \ldots, x_6$. Some of these restrictions will require that the inequality actually be an equation.

The number of integer solutions is:

1. $C(537, 5)$, when all $x_i > 0$ and equality holds.
2. $C(543, 5)$, when all $x_i \geq 0$ and equality holds.
3. $C(291, 3)$, when $x_1, x_2, x_4, x_6 > 0$, $x_3 = 52$, $x_5 = 194$, and equality holds.

4. $C(537, 6)$, when all $x_i > 0$ and the inequality is strict. *Hint:* Imagine a new variable x_7 which is the balance. Note that x_7 must be positive.
5. $C(543, 6)$, when all $x_i \geq 0$ and the inequality is strict. *Hint:* Add a new variable x_7 as above. Now it is the only one which is required to be positive.
6. $C(544, 6)$, when all $x_i \geq 0$.

A classical enumeration problem (with connections to several problems) involves counting lattice paths. A *lattice path* in the plane is a sequence of ordered pairs of integers:

$$(m_1, n_1), (m_2, n_2), (m_3, n_3), \ldots, (m_t, n_t)$$

so that for all $i = 1, 2, \ldots, t-1$, either

1. $m_{i+1} = m_i + 1$ and $n_{i+1} = n_i$, or
2. $m_{i+1} = m_i$ and $n_{i+1} = n_i + 1$.

In Figure 5, we show a lattice path from $(0, 0)$ to $(13, 8)$.

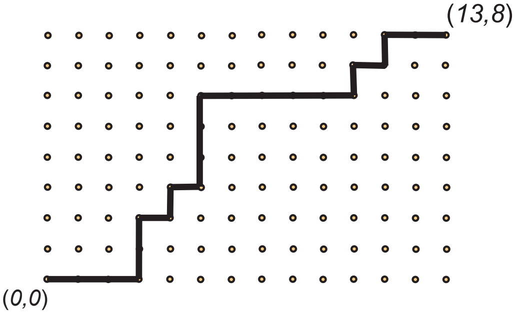

Figure 5: A Lattice Path

Example: The number of lattice paths from (m, n) to (p, q) is $C((p-m)+(q-n), p-m)$.
To see why this formula is valid, note that a lattice path is just an X-string with $X = \{H, V\}$, where H stands for *horizontal* and V stands for *vertical*. In this case, there are exactly $(p-m)+(q-n)$ moves, of which $p-m$ are horizontal.
Example:
Let n be a non-negative integer. Then the number of lattice paths from $(0,0)$ to (n, n) which never go above the diagonal line $y = x$ is the Catalan number

$$C(n) = \frac{1}{n+1}\binom{2n}{n}.$$

To see that this formula holds, consider the family $\mathcal{P}$ of all lattice paths from $(0, 0)$ to (n, n). A lattice path from $(0, 0)$ to (n, n) is just a $\{H, V\}$-string of length $2n$ with exactly n H's. So $|\mathcal{P}| = \binom{2n}{n}$. We classify the paths in $\mathcal{P}$ as *good* if they never go over the diagonal; otherwise, they are *bad*. A string $s \in \mathcal{P}$ is good if the number of V's in an initial segment of s never exceeds the number of H's. For example, the string "*HHVHVVHHHVHVVV*" is a good lattice path from $(0, 0)$ to $(7, 7)$, while the path "*HVHVHHVVVHVHHV*" is bad. In the second case, note that after 9 moves, we have 5 V's and 4 H's.

Let $\mathcal{G}$ and $\mathcal{B}$ denote the family of all good and bad paths, respectively. Of course, our goal is to determine $|\mathcal{G}|$.

Consider a path $s \in \mathcal{B}$. Then there is a least integer i so that s has more V's than H's in the first i positions. By the minimality of i, it is easy to see that i must be odd (otherwise, we can back up a step), and if we set $i = 2j + 1$, then in the first $2j + 1$ positions of s, there are exactly j H's and $j + 1$ V's. The remaining $2n - 2j - 1$ positions (the "tail of s") have $n - j$ H's and $n - j - 1$ V's. We now transform s to a new string s' by replacing the H's in the tail of s by V's and the V's in the tail of s by H's and leaving the initial $2j + 1$ positions unchanged. For example, see Figure 6, where the path s is shown solid and s' agrees with s until it crosses the line $y = x$ and then is the dashed path. Then s' is a string of length $2n$ having $(n-j)+(j+1) = n+1$ V's and $(n-j-1)+j = n-1$ H's, so s' is a lattice path from $(0,0)$ to $(n-1, n+1)$. Note that there are $\binom{2n}{n-1}$ such lattice paths.

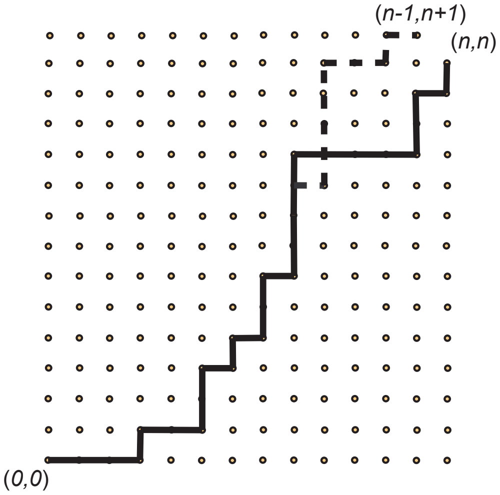
(n-1,n+1)
(n,n)
(0,0)

Figure 6: Transforming a Lattice Path

We can also observe that the transformation we've described is in fact a bijection between $\mathcal{B}$ and $\mathcal{P}'$, the set of lattice paths from $(0,0)$ to $(n-1,n+1)$. To see that this is true, note that every path s' in $\mathcal{P}'$ must cross the line $y=x$, so there is a first time it crosses it, say in position i. Again, i must be odd, so $i=2j+1$ and there are j H's and $j+1$ V's in the first i positions of s'. Therefore the tail of s' contains $n+1-(j+1)=n-j$ V's and $(n-1)-j$ H's, so interchanging H's and V's in the tail of s' creates a new string s that has n H's and n V's and thus represents a lattice path from $(0,0)$ to (n,n), but it's still a bad lattice path, as we did not adjust the first part of the path, which results in crossing the line $y = x$ in position i. Therefore, $|\mathcal{B}| = |\mathcal{P}'|$ and thus

$$C(n) = |\mathcal{G}| = |\mathcal{P}| - |\mathcal{B}| = |\mathcal{P}| - |\mathcal{P}'| = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n},$$

after a bit of algebra.

It is worth observing that in the preceding example, we made use of two common enumerative techniques: giving a bijection between two classes of objects, one of which is "easier" to count than the other, and counting the objects we do *not* wish to enumerate and deducting their number from the total.

The Binomial Theorem

Here is a truly basic result from combinatorics kindergarten.

Theorem 4 (Binomial Theorem)

Let x and y be real numbers with x, y and $x + y$ non-zero. Then for every non-negative integer n,

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Proof.

View $(x+y)^n$ as a product

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)(x+y)\dots(x+y)(x+y)}_{n \text{ factors}}.$$

Each term of the expansion of the product results from choosing either x or y from one of these factors. If x is chosen $n-i$ times and y is chosen i times, then the resulting product is $x^{n-i}y^i$. Clearly, the number of such terms is $C(n,i)$, i.e., out of the n factors, we choose the element y from i of them, while we take x in the remaining $n-i$. $\square$

Example: There are times when we are interested not in the full expansion of a power of a binomial, but just the coefficient on one of the terms. The Binomial Theorem gives that the coefficient of x^5y^8 in $(2x-3y)^{13}$ is $\binom{13}{5}2^5(-3)^8$.

Multinomial Coefficients

Let X be a set of n elements. Suppose that we have two colors of paint, say red and blue, and we are going to choose a subset of k elements to be painted red with the rest painted blue. Then the number of different ways this can be done is just the binomial coefficient $\binom{n}{k}$. Now suppose that we have three different colors, say red, blue, and green. We will choose k_1 to be colored red, k_2 to be colored blue, with the remaining $k_3 = n - (k_1 + k_2)$ colored green. We may compute the number of ways to do this by first choosing k_1 of the n elements to paint red, then from the remaining

$n - k_1$ choosing k_2 to paint blue, and then painting the remaining k_3 green. It is easy to see that the number of ways to do this is

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!(n-(k_1+k_2))!} = \frac{n!}{k_1!k_2!k_3!}$$

Numbers of this form are called *multinomial coefficients*; they are an obvious generalization of the binomial coefficients. The general notation is:

$$\binom{n}{k_1, k_2, k_3, \ldots, k_r} = \frac{n!}{k_1!k_2!k_3!\ldots k_r!}.$$

For example,

$$\binom{8}{3,2,1,2} = \frac{8!}{3!2!1!2!} = \frac{40320}{6 \cdot 2 \cdot 1 \cdot 2} = 1680.$$

Note that there is some “overkill” in this notation, since the value of k_r is determined by n and the values for k_1, $k_2, \ldots, k_{r-1}$. For example, with the ordinary binomial coeffients, we just write $\binom{8}{3}$ and not $\binom{8}{3,5}$.

Example: How many different rearrangements of the string:

MITCHELTKELLERANDWILLIAMTTROTTERAREREGENIUSES!!

are possible if all letters and characters must be used?
To answer this question, we note that there are a total of 45 characters distributed as follows: 3 A's, 1 C, 1 D, 7 E's, 1 G, 1 H, 4 I's, 1 K, 5 L's, 2 M's, 2 N's, 1 O, 4 R's, 2 S's, 6 T's, 1 U, 1 W, and 2 !'s. So the number of rearrangements is

$$\frac{45!}{3!1!1!7!1!1!4!1!5!2!2!1!4!2!6!1!1!2!}.$$

Just as with binomial coefficients and the Binomial Theorem, the multinomial coefficients arise in the expansion of powers of a multinomial:

Theorem 5 (Multinomial Theorem)

Let $x_1, x_2, \dots, x_r$ be nonzero real numbers with $\sum_{i=1}^{r} x_i \neq 0$. Then for every $n \in \mathbb{N}_0$,

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \dots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$$

Example: What is the coefficient of $x^{99}y^{60}z^{14}$ in $(2x^3 + y - z^2)^{100}$? What about $x^{99}y^{61}z^{13}$?

By the Multinomial Theorem, the expansion of $(2x^3 + y - z^2)^{100}$ has terms of the form

$$\binom{100}{k_1, k_2, k_3}(2x^3)^{k_1} y^{k_2} (-z^2)^{k_3} = \binom{100}{k_1, k_2, k_3} 2^{k_1} x^{3k_1} y^{k_2} (-1)^{k_3} z^{2k_3}.$$

The $x^{99}y^{60}z^{14}$ arises when $k_1 = 33$, $k_2 = 60$, and $k_3 = 7$, so it must have coefficient

$$-\binom{100}{33, 60, 7} 2^{33}.$$

For $x^{99}y^{61}z^{13}$, the exponent on z is odd, which cannot arise in the expansion of $(2x^3 + y - z^2)^{100}$, so the coefficient is 0.

Discussion
Over coffee, Xing said that he had been experimenting with the Maple software discussed in the introductory chapter. He understood that Maple was treating a big integer as a string. Xing enthusiastically reported that he had asked Maple to find the sum $a + b$ of two large integers a and b, each having more than 800 digits. Maple found the answer about as fast as he could hit the enter key on his netbook. "That's not so impressive" Alice interjected. "A human, even Bob, could do this in a couple of minutes using pencil and paper." "Thanks for your kind remarks,"

replied Bob, with the rest of the group noting that that Alice was being pretty harsh on Bob and not for any good reason.
Dave took up Bob's case with the remark that "Very few humans, not even you Alice, would want to tackle finding the product of a and b by hand." Xing jumped back in with "That's the point. Even a tiny netbook can find the product very, very quickly. In fact, I tried it out with two integers, each having more than one thousand digits. It found the product in about one second." Ever the skeptic, Zori said "You mean you carefully typed in two integers of that size?" Xing quickly replied "Of course not. I just copied and pasted

the data from one source to another." Yolanda said "What a neat trick that is. Really cuts down the chance of an error."
Dave said "What about factoring? Can your netbook with its fancy software for strings factor big integers?" Xing said that he would try some sample problems and report back. Carlos said "Factoring an integer with several hundred digits is likely to be very challenging, not only for a netbook, but also for a super computer. For example, suppose the given integer was either a prime or the product of two large primes. Detecting which of these two statements holds could be very difficult."

Undeterred, Dave continued "What about exponentiation? Can your software calculate a^b when a and b are large integers?" Xing said "That shouldn't be a problem. After all, a^b is just multiplying a times itself a total of b times, and if you can do multiplication quickly, that's just a loop." Yolanda said that the way Xing was describing things, he was actually talking about a program with nested loops so it might take a long time for such a program to halt. Carlos was quiet but he thought there might be ways to speed up such computations.

By this time, Alice reinserted herself into the conversation "Hey guys. While you were talking, I was looking for big integer topics on the web and found this problem. Is 838200020310007224300 a Catalan number? How would you answer this? Do you have to use special software?"

Zori was not happy. She gloomily envisioned a future job hunt in which she was compelled to use big integer arithmetic as a job skill. Arrgghh.

Exercises

1. The Hawaiian alphabet consists of 12 letters. How many six-character strings can be made using the Hawaiian alphabet?
2. How many $2n$-digit positive integers can be formed if the digits in odd positions (counting the rightmost digit as position 1) must be odd and the digits in even positions must be even and positive?

3. Matt is designing a website authentication system. He knows passwords are most secure if they contain letters, numbers, and symbols. However, he doesn't quite understand that this additional security is defeated if he specifies in which positions each character type appears. He decides that valid passwords for his system will begin with three letters (uppercase and lowercase both allowed), followed by two digits, followed by one of 10 symbols, followed by two uppercase letters, followed by a digit, followed by one of 10 symbols. How many different passwords are there for his website system? How does this compare to the total number of strings of length 10 made

from the alphabet of all uppercase and lowercase English letters, decimal digits, and 10 symbols?

4. How many ternary strings of length $2n$ are there in which the zeroes appear only in odd-numbered positions?
5. Suppose we are making license plates of the form $l_1 l_2 l_3 - d_1 d_2 d_3$ where l_1, l_2, l_3 are capital letters in the English alphabet and d_1, d_2, d_3 are decimal digits (i.e., in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) subject to the restriction that at least one digit is nonzero and at least one letter is K. How many license plates can we make?

6. Mrs. Steffen's third grade class has 30 students in it. The students are divided into three groups (numbered 1, 2, and 3), each having 10 students.

   6.1 The students in group 1 earned 10 extra minutes of recess by winning a class competition. Before going out for their extra recess time, they form a single file line. In how many ways can they line up?

   6.2 When all 30 students come in from recess together, they again form a single file line. However, this time the students are arranged so that the first student is from group 1, the second from group 2, the third from group 3, and from there on, the students continue to alternate by group in this order. In how many ways can they line up to come in from recess?

7. How many strings of the form $l_1 l_2 d_1 d_2 d_3 l_3 l_4 d_4 l_5 l_6$ are there where
   - for $1 \leq i \leq 6$, l_i is an uppercase letter in the English alphabet;
   - for $1 \leq i \leq 4$, d_i is a decimal digit;
   - l_2 is not a vowel (i.e., $l_2 \notin \{A, E, I, O, U\}$); and
   - the digits d_1, d_2, and d_3 are distinct (i.e., $d_1 \neq d_2 \neq d_3 \neq d_1$).
8. In this exercise, we consider strings made from uppercase letters in the English alphabet and decimal digits. How many strings of length 10 can be constructed in each of the following scenarios?
   8.1 The first and last characters of the string are letters.
   8.2 The first character is a vowel, the second character is a consonant, and the last character is a digit.

8.3 Vowels (not necessarily distinct) appear in the third, sixth, and eighth positions and no other positions.
8.4 Vowels (not necessarily distinct) appear in exactly two positions.
8.5 Precisely four characters in the string are digits and no digit appears more than one time.

9. A database uses 20-character strings as record identifiers. The valid characters in these strings are upper-case letters in the English alphabet and decimal digits. (Recall there are 26 letters in the English alphabet and 10 decimal digits.) How many valid record identifiers are possible if a valid record identifier must meet *all* of the following criteria:
   - Letter(s) from the set $\{A, E, I, O, U\}$ occur in *exactly* three positions of the string.

- The last three characters in the string are *distinct* decimal digits that do not appear elsewhere in the string.
- The remaining characters of the string may be filled with any of the remaining letters or decimal digits.

10. Let X be the set of the 26 lowercase English letters and 10 decimal digits. How many X-strings of length 15 satisfy *all* of the following properties (at the same time)?
    - The first and last symbols of the string are distinct digits (which may appear elsewhere in the string).
    - Precisely four of the symbols in the string are the letter 't'.
    - Precisely three characters in the string are elements of the set $V = \{a, e, i, o, u\}$ and these characters are all distinct.

11. A donut shop sells 12 types of donuts. A manager wants to buy six donuts, one each for himself and his five employees.

11.1 Suppose that he does this by selecting a specific type of donut for each person. (He can select the same type of donut for more than one person.) In how many ways can he do this?

11.2 How many ways could he select the donuts if he wants to ensure that he chooses a different type of donut for each person?

11.3 Suppose instead that he wishes to select one donut of each of six *different* types and place them in the breakroom. In how many ways can he do this? (The order of the donuts in the box is irrelevant.)

12. The sport of korfball is played by teams of eight players. Each team has four men and four women on it. Halliday High School has seven men and 11 women interested in playing korfball. In how many ways can they form a korfball team from their 18 interested students?
13. Twenty students compete in a programming competition in which the top four students are recognized with trophies for first, second, third, and fourth places.
    13.1 How many different outcomes are there for the top four places?

13.2 At the last minute, the judges decide that they will award honorable mention certificates to four individuals who did not receive trophies. In how many ways can the honorable mention recipients be selected (after the top four places have been determined)? How many total outcomes (trophies plus certificates) are there then?

14. An ice cream shop has a special on banana splits, and Xing is taking advantage of it. He's astounded at all the options he has in constructing his banana split:

- He must choose three different flavors of ice cream to place in the asymmetric bowl the banana split is served in. The shop has 20 flavors of ice cream available.

- Each scoop of ice cream must be topped by a sauce, chosen from six different options. Xing is free to put the same type of sauce on more than one scoop of ice cream.
- There are 10 sprinkled toppings available, and he must choose three of them to have sprinkled over the entire banana split.

14.1 How many different ways are there for Xing to construct a banana split at this ice cream shop?

14.2 Suppose that instead of requiring that Xing choose exactly three sprinkled toppings, he is allowed to choose between zero and three sprinkled toppings. In this scenario, how many different ways are there for him to construct a banana split?

15. Suppose that a teacher wishes to distribute 25 identical pencils to Ahmed, Barbara, Casper, and Dieter such that Ahmed and Dieter receive at least one pencil each, Casper receives no more than five pencils, and Barbara receives at least four pencils. In how many ways can such a distribution be made?
16. How many integer-valued solutions are there to each of the following equations and inequalities?
    - 16.1 $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i > 0$
    - 16.2 $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$
    - 16.3 $x_1 + x_2 + x_3 + x_4 + x_5 \leq 63$, all $x_i \geq 0$
    - 16.4 $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$, $x_2 \geq 10$
    - 16.5 $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$, $x_2 \leq 9$

17. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 132$$

provided that $x_1 > 0$, and $x_2, x_3, x_4 \geq 0$? What if we add the restriction that $x_4 < 17$?

18. How many integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 782$$

provided that $x_1, x_2 > 0$, $x_3 \geq 0$, and $x_4, x_5 \geq 10$?

19. A teacher has 450 identical pieces of candy. He wants to distribute them to his class of 65 students, although he is willing to take some leftover candy home. (He does not insist on taking any candy home, however.) The student who won a contest in the last class is to receive at least 10 pieces of candy as a reward. Of the remaining students, 34 of them insist on receiving at least one piece of candy, while the remaining 30 students are willing to receive no candy.

19.1 In how many ways can he distribute the candy?

19.2 In how many ways can he distribute the candy if, in addition to the conditions above, one of his students is diabetic and can receive at most 7 pieces of candy? (This student is one of the 34 who insist on receiving at least one piece of candy.)

20. Give a combinatorial argument to prove the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Hint: Think of choosing a team with a captain.

21. Let m and w be positive integers. Give a combinatorial argument to prove that for integers $k \geq 0$,

$$\sum_{j=0}^{k}\binom{m}{j}\binom{w}{k-j} = \binom{m+w}{k}.$$

22. How many lattice paths are there from $(0, 0)$ to $(10, 12)$?
23. How many lattice paths are there from $(3, 5)$ to $(10, 12)$?
24. How many lattice paths are there from $(0, 0)$ to $(10, 12)$ that pass through $(3, 5)$?
25. How many lattice paths from $(0, 0)$ to $(17, 12)$ are there that pass through $(7, 6)$ and $(12, 9)$?
26. How many lattice paths from $(0, 0)$ to $(14, 73)$ are there that do *not* pass through $(6, 37)$?

27. A small-town bank robber is driving his getaway car from the bank he just robbed to his hideout. The bank is at the intersection of 1^{st} Street and 1^{st} Avenue. He needs to return to his hideout at the intersection of 7^{th} Street and 5^{th} Avenue. However, one of his lookouts has reported that the town's one police officer is parked at the intersection of 4^{th} Street and 4^{th} Avenue. Assuming that the bank robber does not want to get arrested and drives only on streets and avenues, in how many ways can he safely return to his hideout? (Streets and avenues are uniformly spaced and numbered consecutively in this small town.)

28. The setting for this problem is the fictional town of Mascotville, which is laid out as a grid. Mascots are allowed to travel only on the streets, and not “as the yellow jacket flies.” Buzz, the Georgia Tech mascot, wants to go visit his friend Thundar, the North Dakota State University mascot, who lives 6 blocks east and 7 blocks north of Buzz’s hive. However, Uga VIII has recently moved into the doghouse 2 blocks east and 3 blocks north of Buzz’s hive and already has a restraining order against Buzz. There’s also a pair of tigers (mother and cub) from Clemson who live 1 block east and 2 blocks north of Uga VIII, and they’re known for setting traps for Buzz. Buzz wants

to travel from his hive to Thundar's pen every day without encountering Uga VIII or The Tiger and The Tiger Cub. However, he wants to avoid the boredom caused by using a route he's used in the past. What is the largest number of consecutive days on which Buzz can make the trip to visit Thundar without reusing a route (you may assume the routes taken by Buzz only go east and north)?

29. Determine the coefficient on $x^{15}y^{120}z^{25}$ in $(2x + 3y^2 + z)^{100}$.
30. Determine the coefficient on $x^{12}y^{24}$ in $(x^3 + 2xy^2 + y + 3)^{18}$. (Be careful, as x and y now appear in multiple terms!)

31. For each word below, determine the number of rearrangements of the word in which all letters must be used.

31.1 OVERNUMEROUSNESSES

31.2 OPHTHALMOOTORHINOLARYNGOLOGY

31.3 HONORIFICABILITUDINITATIBUS (the longest word in the English language consisting strictly of alternating consonants and vowels[1])

32. How many ways are there to paint a set of 27 elements such that 7 are painted white, 6 are painted old gold, 2 are painted blue, 7 are painted yellow, 5 are painted green, and 0 of are painted red?

33. There are many useful sets that are enumerated by the Catalan numbers. (Volume two of R.P. Stanley's *Enumerative Combinatorics* contains a famous (or perhaps infamous) exercise in 66 parts asking readers to find bijections that will show that the number of various combinatorial structures is $C(n)$, and his `http://www-math.mit.edu/~rstan/ec/catadd.pdf`web page boasts an additional list of at least 100 parts.) Give bijective arguments to show that each class of objects below is enumerated by $C(n)$. (All three were selected from the list in Stanley's book.)

33.1 The number of ways to fully-parenthesize a product of $n+1$ factors as if the "multiplication" operation in question were not necessarily associative. For example, there is one way to parenthesize a product of two factors $(a_1 a_2)$, there are two ways to parenthesize a product of three factors $((a_1(a_2a_3))$ and $((a_1a_2)a_3))$, and there are five ways to parenthesize a product of four factors:

$$(a_1(a_2(a_3a_4))), (a_1((a_2a_3)a_4)), ((a_1a_2)(a_3a_4)), ((a_1(a_2a_3))a_4), (((a_1a_2)a_3)a_4).$$

33.2 Sequences of n 1's and n -1's in which the sum of the first i terms is nonnegative for all i.

33.3 Sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$. For example, for $n = 3$, the sequences are

$$111 \qquad 112 \qquad 113 \qquad 122 \qquad 123.$$

Hint: Think about drawing lattice paths on paper with grid lines and (basically) the number of boxes below a lattice path in a particular column.

[1] http://www.rinkworks.com/words/oddities.shtml