Continuous functions

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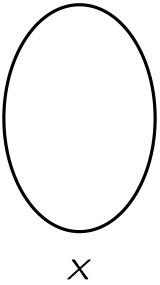

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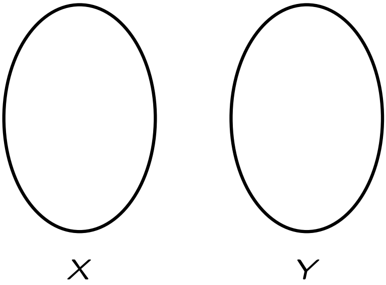

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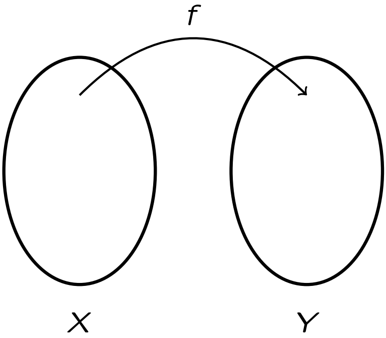

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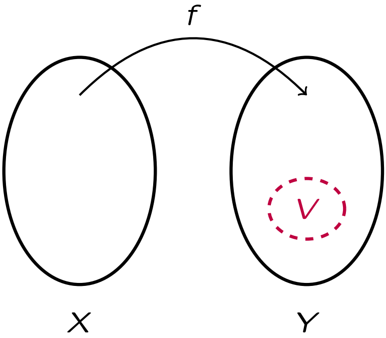

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- A function $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every $x_0 \in \mathbb{R}$ and $\epsilon > 0$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

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- Let $\mathbb{R}_l$ be the set of real numbers with the Sorgenfrey topology. Then the identity function $1_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}_l$ is not continuous.
- On the other hand, the identity $1_{\mathbb{R}}\colon \mathbb{R}_l \to \mathbb{R}$ is continuous.

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Proof.

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Embedding

If $f\colon X \to Y$ is injective and the bijection $f\colon X \to f(X)$ (where $f(X) \subseteq Y$ is a subspace) is a homeomorphism, we say that f is an *embedding*.

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- *If $A \subseteq X$ is a subspace, the inclusion $i_A : A \to X$ is continuous.*
- *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the composition $g \circ f$ is continuous.*

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- *If $f \colon X \to Y$ is continuous and $Z \subseteq Y$ is a subspace such that $f(X) \subseteq Z$, then $f\,| \colon X \to Z$ is continuous. If Z is a space containing Y as a subspace, then $f \colon X \to Z$ is continuous.*

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- *If $X = \cup U_\alpha$, where each U_α is open and $f|_{U_\alpha}: U_\alpha \to Y$ is continuous for each α, then f is continuous.*

Pasting lemma

Theorem (The pasting lemma)

Let $X = A \cup B$ be a space, with A, B closed in X. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous. Suppose that $f(x) = g(x)$ for every $x \in A \cap B$, and define $h\colon X \to Y$ by:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

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Let $f\colon A \to X \times Y$ *be given by:*

$$f(a) = (f_1(a), f_2(a)).$$

Then f *is continuous if and only if the functions:*

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5. Let $f\colon A \to B$ and $g\colon C \to D$ be continuous functions between topological spaces. Show that the map $f \times g\colon A \times C \to B \times D$ given by $(a, c) \mapsto (f(a), f(c))$ is continuous.