

Continuous functions

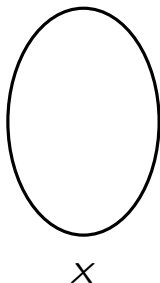
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Definition (Continuous functions)

Let X and Y be topological spaces. The function $f: X \rightarrow Y$ is **continuous** if for any open set V in Y , we have that $f^{-1}(V)$ is open in X .

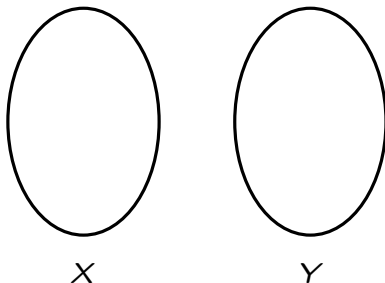
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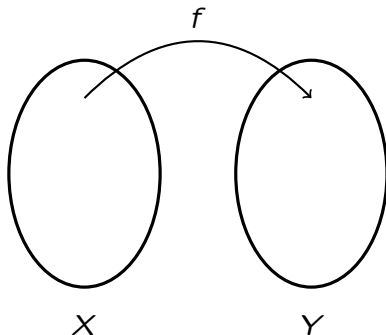
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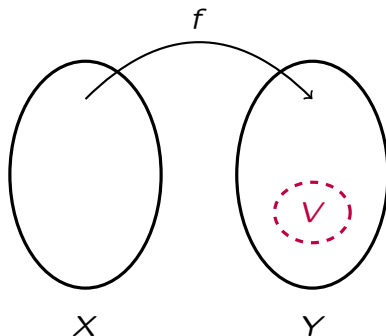
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- A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if for every $x_0 \in \mathbb{R}$ and $\epsilon > 0$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

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- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous, $x_0 \in \mathbb{R}$ and $\epsilon > 0$.
 - We have that $f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$ is open in \mathbb{R} and contains x_0 . Hence there is $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(f(x_0) - \epsilon, f(x_0) + \epsilon)$.

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- Let \mathbb{R}_l be the set of real numbers with the Sorgenfrey topology. Then the identity function $1_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}_l$ is not continuous.
- On the other hand, the identity $1_{\mathbb{R}}: \mathbb{R}_l \rightarrow \mathbb{R}$ is continuous.

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- *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

When the last condition is satisfied at $x_0 \in X$, we say that f is **continuous at x_0** .

Exercises

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1. Let $f: X \rightarrow Y$ continuous. If $A \subseteq X$ and x is a limit point of A , is $f(x)$ a limit point of $f(A)$?
2. If the singleton $\{x_0\}$ is open in X , prove that every function $f: X \rightarrow Y$ is continuous at x_0 . Is the converse true?
3. Prove that $f: X \rightarrow Y$ is continuous if and only if for every $A \subseteq X$ we have that $f^{-1}(A^\circ) \subseteq (f^{-1}(A))^\circ$.