Closure and interior

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Lemma

D is closed if and only if $D = \overline{D}$.

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Theorem (Properties of closure)

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Theorem (Characterization of closure)

The first three properties in the last theorem characterize $\overline{D}$, *in the sense that if* R *is a closed set that contains* D, *and that is contained in any closed set that contains* D, *then we must have* $R = \overline{D}$.

Closure and subspaces

Note

If Y is a subspace of X and $D \subseteq Y$, the closure of D in Y and the closure of D in X might be different. The notation $\overline{D}$ will always denote the closure in X.

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Theorem (Closure and subspaces)

Let Y be a subspace of X and $D \subseteq Y$. Then the closure of D in Y equals $\overline{D} \cap Y$.

Proof

We prove that $\overline{D} \cap Y$ satisfies the properties that characterize the closure of D in Y.

Other characterizations of closure

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Theorem (Points in closure)

Let X be a space, $A \subseteq X$ and $x \in X$. Then

- *$x \in \overline{A}$ if and only if every neighborhood of x intersects A.*
- *If the topology of X is given by the basis $\mathcal{B}$, then $x \in \overline{A}$ if and only if every $B \in \mathcal{B}$ that contains x intersects A.*

Limit points

Definition (Limit point)

Let X be a space, $A \subseteq X$ and $x \in X$. We say that x is a limit point of A if every neighborhood of x intersects A in a point different from x. We denote the set of all limit points of A by A'.

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- Let $x \in \overline{A}$. If $x \in A$, then x is in the set on the right side. If $x \notin A$, let U be a neighborhood of x. Then U intersects A in a point different from x.

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- Let $x \in \overline{A}$. If $x \in A$, then x is in the set on the right side. If $x \notin A$, let U be a neighborhood of x. Then U intersects A in a point different from x.
- From the definition it follows immediately that $A' \subseteq \overline{A}$, and also $A \subseteq \overline{A}$ always. $\square$

Remark

It follows that A is closed if and only if $A' \subseteq A$.

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- *If $U \subseteq X$ is open and $U \subseteq D$, then $U \subseteq D^\circ$.*
- *U is open if and only if $U = U^\circ$.*

Links

- Closure (topology) - Wikipedia, the free encyclopedia

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- Let X be a totally ordered set with the order topology. Show that $\overline{(a, b)} \subseteq [a, b]$. Is equality always true?
- Show that A' is closed for any $A \subseteq X$.