Continuity and sequences in metric spaces

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- If X is metrizable, then X is Hausdorff.
- If $(X_i)_{i \in I}$ is a countable family of metrizable spaces, then $\prod_{i \in I} X_i$ is metrizable.

Continuous functions

ϵ - δ definition

Let $f\colon X \to Y$ a function, with X, Y metrizable spaces with metrics d_X, d_Y, respectively. Then f is continuous if and only if for all $x \in X$, $\epsilon > 0$ there is $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \epsilon \text{ if } d_X(x, y) < \delta$$

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Definition (Convergence)

We say that the sequence (x_n) converges to $x \in X$ if for every neighborhood U of x there is a positive integer N such that $x_n \in U$ for all $n \geq N$. We write $x_n \to x$.

Remarks on the definition of convergence

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- If X is Hausdorff, a sequence on X that converges has a unique limit.

Sequence lemma

Theorem (Sequence lemma)

Let X be a topological space, and $A \subseteq X$. If (x_n) is a sequence in A that converges to x, then $x \in \overline{A}$. Conversely, if $x \in \overline{A}$ and X es metrizable, then there is a sequence on A that converges to x

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- Suppose that $x_n \to x$, where $x_n \in A$ for all n. Let U be a neighborhood of x. By definition of convergence, $x_n \in U$ for big enough values of n, and so $x \in \overline{A}$.

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- Now, let X be metrizable with metric d, and let $x \in \overline{A}$. For each n, we choose a point $x_n \in A \cap B(x, \frac{1}{n})$.

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- We claim that $x_n \to x$. Because if U is a neighborhood of x, let $\epsilon > 0$ be such that $B(x, \epsilon) \subseteq U$, and let $N > \frac{1}{\epsilon}$.

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- We claim that $x_n \to x$. Because if U is a neighborhood of x, let $\epsilon > 0$ be such that $B(x, \epsilon) \subseteq U$, and let $N > \frac{1}{\epsilon}$.
- Then, if $n \geq N$, we have that $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, and so $x_n \in U$ si $n \geq N$.

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Remarks to the proof

- The theorem gives a necessary condition for metrizability. We will see examples of $A \subseteq X$, $x \in X$ with $x \in \overline{A}$ such that there is no sequence in A that converges to x. In such cases, X is not metrizable.

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- A countable local base on $x \in X$ is a collection $\{B_n\}_{n\in\mathbb{N}}$ of neighborhoods of x such that for any neighborhood U of x there is n such that $B_n \subseteq U$.

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- A countable local base on $x \in X$ is a collection $\{B_n\}_{n\in\mathbb{N}}$ of neighborhoods of x such that for any neighborhood U of x there is n such that $B_n \subseteq U$.
- The topological space X is first–countable if every $x \in X$ has a countable local base.
- (Exercise) Prove that if X es first–countable, $A \subseteq X$ and $x \in \overline{A}$, there is a sequence (x_n) on A that converges to x.

Continuity and sequences

Theorem (Continuity by sequences)

Let $f\colon X \to Y$ with X metrizable. Then f is continuous if and only if for all convergent sequences $x_n \to x$ on X, the sequence $f(x_n)$ converges to $f(x)$.

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Proof

- Suppose f continuous and let $x_n \to x$. We want to prove $f(x_n) \to f(x)$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x, and so there is N such that $x_n \in f^{-1}(V)$ if $n \geq N$. But then $f(x_n) \in V$ for such values of n.

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- Let $A \subseteq X$, we want to see that $f(\overline{A}) \subseteq \overline{f(A)}$.

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- Let $A \subseteq X$, we want to see that $f(\overline{A}) \subseteq \overline{f(A)}$.
- Let $x \in \overline{A}$. Since X is metrizable, there is a sequence x_n on A such that $x_n \to x$.

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- Let $A \subseteq X$, we want to see that $f(\overline{A}) \subseteq \overline{f(A)}$.
- Let $x \in \overline{A}$. Since X is metrizable, there is a sequence x_n on A such that $x_n \to x$.
- By hypothesis, we have that $f(x_n) \to f(x)$. Since $f(x_n) \in f(A)$, then $f(x) \in \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$.

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Box topology on $\mathbb{R}^{\mathbb{N}}$

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- We will see that the sequence lemma does not hold in $X = \mathbb{R}^{\mathbb{N}}$ with the box topology.
- Let A be the set of all real sequences where the terms are all positive:

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- Let $0 = (0, 0, \ldots) \in X$. Then $0 \in \overline{A}$, since if

$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots$$

is a basic subset that contains 0, then $B \cap A \neq \emptyset$.

Continuation

- However, there is no sequence on A that converges to 0. Suppose that (a_n) is a sequence on A. Let

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  is a neighborhood of $0 \in X$.
- However, B' does not contain points of the sequence (a_n). Hence (a_n) does not converge to 0.

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- Let $A \subseteq \mathbb{R}^J$, where A consists of the (x_α) such that $x_\alpha = 1$ for all but a finite number of $\alpha \in J$, where we have $x_\alpha = 0$. Let 0 denote the "origin" of $\mathbb{R}^J$.

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- We prove that $0 \in \overline{A}$. Let $\prod U_\alpha$ be a basis element such that $0 \in \prod U_\alpha$. Suppose $U_\alpha \neq \mathbb{R}$ for $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$.

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- Then, if (x_α) is defined as $x_\alpha = 0$ for $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$ and $x_\alpha = 1$ for all other values of α, we have that $(x_\alpha) \in A \cap \prod U_\alpha$.

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- We will show then that there is no sequence of points in A that converges to 0. Let (a_n) a sequence in A.

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- Each $a_n \in A$ has only a finite number of values of α such that the αth term of a_n is equal to 0. Let J_n be the set of such values of α.

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- The union of all the J_n is a countable union of finite sets, hence countable. Since J is uncountable, let $\beta \in J$ that is not in any of the J_n. That means that the βth coordinate of all of the a_n is 1.
- Let $U_\beta = (-1, 1)$, and let $U = \pi_\beta^{-1}(U_\beta)$. Then U is a neighborhood of $0 \in \mathbb{R}^J$, and U does not contain any of the a_n. Hence a_n does not converge to 0.

Definition and examples

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- The set $\mathbb{N}$ with the usual order is well-ordered.
- The set $\mathbb{R}$ with the usual order is not well-ordered.

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- *If X is well-ordered and $Y \subseteq X$ has the induced order, then Y is well-ordered.*
- *If X, Y are well-ordered sets, then $X \times Y$ with dictionary order is well-ordered.*

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Corollary

There is a set that is uncountable and well-ordered.

Sections

Definition (Section)

If S is an ordered set and $\alpha \in S$, we denote by S_α the set:

$$S_\alpha = \{x \in S \mid x < \alpha\}.$$

S_α is called the section of S by α.

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- Let X be an uncountable, well-ordered set. Then the set $S = \{1, 2\} \times X$ is also uncountable and well-ordered, and has sections that are uncountable (for example, those of elements of the form $(2, x)$.)

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- The set
$$\{\alpha \in S \mid S_\alpha \text{ is uncountable}\}$$
is nonempty, let Ω its smallest element.

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- The set
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- Then S_Ω is the required set.

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- Since for each $a \in A$, we have that S_a is countable, then $B = \cup_{a \in A} S_a$ is also countable.

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- Since $B \subseteq S_\Omega$, and S_Ω is uncountable, we can choose $x \in S_\Omega - B$.

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- If there was $a \in A$ with $x < a$, then x would be in B, which is a contradiction.

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If $A \subsetneq S_\Omega$ is countable, then A has an upper bound in S_Ω.

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- Since for each $a \in A$, we have that S_a is countable, then $B = \cup_{a \in A} S_a$ is also countable.
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- If there was $a \in A$ with $x < a$, then x would be in B, which is a contradiction.
- Hence x is an upper bound of A. $\square$

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- We have that Ω is a limit point of S_Ω, since any basic element containing Ω has the form $(a, \Omega]$ and must intersect S_Ω. Otherwise we would have $S_\Omega = S_a \cup \{a\}$, but $S_a \cup \{a\}$ is countable.

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- However, there is no sequence in S_Ω that converges to Ω, since if (x_n) is a sequence in S_Ω, there is an upper bound $b \in S_\Omega$ for all the terms in the sequence. Then $(b, \Omega]$ contains no point of the sequence. $\square$

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- However, there is no sequence in S_Ω that converges to Ω, since if (x_n) is a sequence in S_Ω, there is an upper bound $b \in S_\Omega$ for all the terms in the sequence. Then $(b, \Omega]$ contains no point of the sequence. □

We will denote with $\overline{S_\Omega}$ the set $S_\Omega \cup \{\Omega\}$.