Topological spaces

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$\mathbb{R}$ the set of real numbers.

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1. If τ is a topology on X, the pair (X, τ) is called a *topological space*.
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- Let $X = \{1, 2\}$. Then $\tau = \{\emptyset, X, \{1\}\}$ is a topology, and (X, τ) is called the *Sierpiński space*.

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  - Then τ is called the *metric topology* on X.

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- Let $U_1, U_2 \in \tau$. We need to show that either $U_1 \cap U_2$ is empty or $X - (U_1 \cap U_2)$ is finite.

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9. Let $X = \{1, 2, 3\}$. Enumerate all topologies on X.

Links

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