The subspace topology

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Definition

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$$\tau_Y = \{Y \cap U \mid U \in \tau\}$$

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- Let $V_\alpha \in \tau_Y$ for $\alpha \in I$. For each $\alpha \in I$, there is $U_\alpha \in \tau$ such that $V_\alpha = Y \cap U_\alpha$.

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- Let $V_\alpha \in \tau_Y$ for $\alpha \in I$. For each $\alpha \in I$, there is $U_\alpha \in \tau$ such that $V_\alpha = Y \cap U_\alpha$.
- Then:

$$Y \cap (\cup U_\alpha) = \cup (Y \cap U_\alpha) = \cup V_\alpha,$$

and we obtain (T2). The proof of (T3) is an exercise.

Open sets relative to Y

If Y is a subspace of X, and $U \subseteq Y$, it could be that U is open in Y but not open in X.

Base of the subspace topology

Lemma

Let $\mathcal{B}$ be a base for the topology of X. Then

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}\}$$

is a base for the subspace topology on Y.

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Lemma

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is a base for the subspace topology on Y.

Proof

Let U be open in X and $y \in Y \cap U$. Let $B \in \mathcal{B}$ such that $y \in B \subseteq U$. Then $Y \cap B \in \mathcal{B}_Y$, and $y \in Y \cap B \subseteq Y \cap U$. $\square$

Example

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- A base for the subspace topology on Y is the collection of all sets of the form $(a, b) \cap Y$.
- They all are of one of the following forms: (a, b), $[0, b)$, $(a, 1]$, Y, $\emptyset$.
- We obtain that these elements generate the order topology on $[0, 1]$.

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- Consider now $Y = [0,1) \cup \{2\} \subseteq X = \mathbb{R}$.
- In the subspace topology, the set $\{2\}$ is open.
- But in the order topology on Y, considering that 2 is $\max Y$, any open set that contains 2 has to contain other points.

Order topology in intervals

Lemma (Order topology in intervals)

If X is an ordered set that has the order topology, and $Y \subseteq X$ is an interval, then the order topology and the subspace topology on Y are the same.

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Exercise □

Exercises

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Exercises

1. Show that if we consider in $\mathbb{R}$ the usual topology, then the topology on $\mathbb{N}$ as a subspace of $\mathbb{R}$ is discrete.
2. Let $Y \subseteq X$ be open. Show that $A \subseteq Y$ is open in Y if and only if it is open in X.
3. Show that if X is a topological space, and $A \subseteq Y \subseteq X$, then the topology of A as a subspace of Y is the same as a subspace of X (Y is a subspace of X).

Links

- Subspace topology - Wikipedia, the free encyclopedia