Topological spaces

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2. The elements of τ are called open sets.

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- Let $X = \{1, 2\}$. Then $\tau = \{\emptyset, X, \{1\}\}$ is a topology, and (X, τ) is called the Sierpiński space.

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- Let $U_1, U_2 \in \tau$. We need to show that either $U_1 \cap U_2$ is empty or $X - (U_1 \cap U_2)$ is finite.

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9. Let $X = \{1, 2, 3\}$. Enumerate all topologies on X.

Links

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- Why does topology rarely come up outside of topology? - Mathematics Stack Exchange