Compact spaces

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Definition (Compact space)

A topological space X is called compact if every open cover of X has a finite subcollection that is also an open cover of X.

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- The subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ is compact.
- Any finite topological space is compact.
- The subspace $(0, 1] \subseteq \mathbb{R}$ is not compact.

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Lemma

Let $Y \subseteq X$ be a subspace. Then Y is compact if an only if every covering of Y by open sets (in X) contains a finite subcollection covering Y.

Proof

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- Then $\{U_\alpha \cap Y\}$ is a cover of Y by open sets in Y. By compactness of Y, we have that there is a finite subset $\{U_1, U_2, \ldots, U_r\} \subseteq \mathcal{O}$ such that

$$Y = (U_1 \cap Y) \cup \cdots \cup (U_r \cap Y).$$

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- Now, let $\mathcal{O}'$ be a cover of Y by open sets in Y. For each $U'_\alpha \in \mathcal{O}'$, let U_α open in X such that $U'_\alpha = U_\alpha \cap Y$. Then $\{U_\alpha\}$ covers Y.

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- Now, if $\{U_{\alpha_i}\}_{i=1}^r$ covers Y, we have that $\{U'_{\alpha_i}\}_{i=1}^r \subseteq \mathcal{O}'$ covers Y as well.

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- Then $\mathcal{O}' = \mathcal{O} \cup \{X - Y\}$ is an open cover of X. Since X is compact, a finite subcollection $\mathcal{F} \subseteq \mathcal{O}'$ covers X.

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- Then $\mathcal{F} - \{X - Y\}$ is a subcollection of $\mathcal{O}$ and is a finite subcover of Y. $\square$

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- Let $Y \subseteq X$ with Y compact and X Hausdorff. We will prove that $X - Y$ is open.
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- Let $x_0 \in X - Y$. For each $y \in Y$, let U_y, V_y be disjoint neighborhoods of x_0, y respectively.
- Then $\{V_y\}_{y \in Y}$ is an open cover of Y. Choose $y_1, y_2, \ldots, y_r \in Y$ such that $Y \subseteq V_{y_1} \cup \cdots \cup V_{y_r}$.

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- It follows that $U_{y_1} \cap \cdots \cap U_{y_r}$ is a neigborhood of x_0 that is disjoint from $V_{y_1} \cup \cdots \cup V_{y_r}$ and so is contained in $X - Y$. □

A corollary

This statement follows from the proof of the previous theorem:

Corollary

If $Y \subseteq X$ is compact, where X is Hausdorff, and $x \notin Y$, there are disjoint open sets U, V such that $x \in U$ and $Y \subseteq V$.

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- Let $\{V_\alpha\}$ be an open cover of $f(X)$.
- Then $\{f^{-1}(V_\alpha)\}$ is an open cover of X. Since X is compact, there are $\alpha_1, \ldots, \alpha_n$ such that $X \subseteq f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})$.

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- It follows that $f(X) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. $\square$

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- Since f is continuous and biyective, in order to show that f is a homeomorphism it is enough to show it is closed.
- Let $C \subseteq X$ be closed. Since X is compact, we have that C is compact.
- Since f is continuous, we have that $f(C)$ is compact.
- Finally, since Y is Hausdorff, we have that $f(C)$ is closed. $\square$

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- For each $y \in Y$, since $(x_0, y) \in N$, which is open in $X \times Y$, there is a basis element $U_y \times V_y \subseteq N$ containing (x_0, y). The collection $\{U_y \times V_y\}$ covers the compact set $\{x_0\} \times Y$, and so we can cover it with finitely many such basis elements: $U_1 \times V_1, \ldots, U_n \times V_n$.

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- Let $W = U_1 \cap \cdots \cap U_n$. Then W is a neighborhood of x_0. We prove the finite subcover also cover the *tube* $W \times Y$.
- Let $(x, y) \in W \times Y$. Then $(x_0, y) \in U_j \times V_j$ for some j. Since $x \in W$ implies $x \in U_j$, it follows that $(x, y) \in U_j \times V_j$, and so $W \times Y \subseteq N$.

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- By the *tube lemma*, there is a neighborhood W_x of x such that $W \times Y \subseteq N$. Then the collection $\{W_x\}$ covers X. Since X is compact, there is a finite subcollection $W_1, \ldots, W_m$ covering X.

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- By the *tube lemma*, there is a neighborhood W_x of x such that $W \times Y \subseteq N$. Then the collection $\{W_x\}$ covers X. Since X is compact, there is a finite subcollection $W_1, \ldots, W_m$ covering X.
- Finally, since the finite collection of tubes $W_1 \times Y, \ldots, W_m \times Y$ covers $X \times Y$, and each tube can be covered by a finite subcollection of $\mathcal{O}$, we are done. $\square$

Property of finite intersection

Definition (Finite intersection property)

Let $\mathcal{C}$ be a collection of subsets of X. We say that $\mathcal{C}$ has the finite intersection property if for every finite subcolection $\{F_1, \ldots, F_r\} \subseteq \mathcal{C}$ we have that $F_1 \cap \cdots \cap F_r \neq \emptyset$.

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  - $\mathcal{O}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.
  - $\mathcal{O}$ is a cover of X if and only if $\cap\mathcal{C} = \emptyset$.
  - The finite subcolection $\{U_1, \ldots, U_n\} \subseteq \mathcal{O}$ covers X if and only if the intersection of the corresponding $X - U_i$

Closed intervals on ordered sets

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- Let $a, b \in X$, $a < b$, and let $\mathcal{O}$ be a covering of $[a, b]$ by sets open in $[a, b]$ with respect to the subspace topology (which is the same as the order topology on $[a, b]$). We wish to prove there is a finite subcover of $\mathcal{O}$.

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- We prove first that if $x \in [a, b]$, $x \neq b$, then there is a point $y \in [a, b]$, $y > x$, such that the inverval $[x, y]$ can be covered by at most two elements of $\mathcal{O}$.

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- If x has an immediate successor $y \in X$, then $y \in [a, b]$ and $[x, y] = \{x, y\}$

Corollary

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Theorem

A subset $A \subseteq \mathbb{R}^n$ is compact if and only if is closed and bounded in the euclidian metric d, or in the metric ρ.
($\rho(x, y) = \max |x_i - y_i|$)

Maximum value theorem

Theorem (Maximum value theorem)

Let $f\colon X \to Y$ be continuous, where Y is a totally ordered set with the order topology. If X is compact, there are points $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

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