# Topological spaces

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# Outline

Notation

Topological spaces

Exercises

Links

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$\mathbb{R}$ the set of real numbers.

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1. If $\tau$ is a topology on $X$, the pair $(X, \tau)$ is called a *topological space*.
2. The elements of $\tau$ are called *open sets*.

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- Let $X = \{1, 2\}$. Then $\tau = \{\emptyset, X, \{1\}\}$ is a topology, and $(X, \tau)$ is called the *Sierpiński space*.

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  - $\tau = \{U \subseteq X \mid \forall x \in U \, \exists \epsilon > 0, \text{such that } B_\epsilon(x) \subseteq U\}$.
  - Then $\tau$ is called the *metric topology* on $X$.

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- Let $U_1, U_2 \in \tau$. We need to show that either $U_1 \cap U_2$ is empty or $X - (U_1 \cap U_2)$ is finite.

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2. Show that in general there is no metric on $X$ such that the metric topology is the same as the indiscrete topology.
3. Show that if $\tau_\alpha$ is a topology on $X$ for each $\alpha \in I$, then $\tau = \cap \tau_\alpha$ is a topology on $X$.

# Links

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- Why does topology rarely come up outside of topology? - Mathematics Stack Exchange