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- The set $[0, 1) \subseteq \mathbb{R}$ is not closed (nor open).
- In a discrete topology, any set is closed.
- In a cofinite topology on X, exactly X and their finite subsets are closed.

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Exercise. □

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Remark

We can prove that if a collection of subsets of X satisfies the previous conditions, then their complements form a topology.

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Let Y be a subspace of X. Then $A \subseteq Y$ is closed in Y if and only if there is $C \subseteq X$ closed in X such that $A = C \cap Y$.

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Theorem (Closed in closed)

Let Y be a subspace of X. If Y is closed in X and $A \subseteq Y$ is closed in Y, then A is closed in X.

Links

- Closed set - Wikipedia, the free encyclopedia

Exercises

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- Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.