Bases

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Theorem (Generated topology)

Let X be a set and $\mathcal{B}$ be a base for a topology on X. Then

$$\tau_{\mathcal{B}} = \{U \subseteq X \mid \forall x \in U \, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}$$

is a topology on X.

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- Assume $x \in U_{\alpha_0}$. Then there is $B \in \mathcal{B}$ such that:

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- Now, let $U_1, U_2 \in \tau_{\mathcal{B}}$, and let $x \in U_1 \cap U_2$. Let $x \in B_i \subseteq U_i$ for $i = 1, 2$.

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- Using (B2), find $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

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- Now, let $U_1, U_2 \in \tau_{\mathcal{B}}$, and let $x \in U_1 \cap U_2$. Let $x \in B_i \subseteq U_i$ for $i = 1, 2$.
- Using (B2), find $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
- Then

  $$x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2,$$

  which proves (T3). $\square$

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Let X be a set and $\mathcal{B}$ a base for a topology. Then $\tau_{\mathcal{B}}$ is equal to the collection of subsets of X that are union of elements of $\mathcal{B}$.

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Proof

- If U is union of elements of $\mathcal{B}$, then, since $\mathcal{B} \subseteq \tau_{\mathcal{B}}$, and $\tau_{\mathcal{B}}$ is closed under arbitrary unions, then $U \in \tau_{\mathcal{B}}$.
- Conversely, if $U \in \tau_{\mathcal{B}}$, let $x \in U$. By definition of $\tau_{\mathcal{B}}$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Hence $U = \cup_{x \in U} B_x$. $\square$

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4. Let $\mathcal{B}$ be a basis for a topology, and let $\mathcal{C} \subseteq P(X)$ be such that $\mathcal{B} \subseteq \mathcal{C}$. Is $\mathcal{C}$ a basis for a topology?
5. Let $\mathcal{B}$ be a basis for a topology on X such that $\tau_{\mathcal{B}}$ is the discrete topology. If $x \in X$, show that $\{x\} \in \mathcal{B}$.

Links

- Base (topology) - Wikipedia, the free encyclopedia