Bases and subbases

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A topology smaller than other

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Lemma (Condition of smaller topology)

Let $\mathcal{B}, \mathcal{B}'$ be bases for topologies on X. Then the following are equivalent:

- *$\tau_{\mathcal{B}}$ is smaller than $\tau_{\mathcal{B}'}$.*
- *For each $x \in X$ and $B \in \mathcal{B}$ with $x \in B$, there is $B' \in \mathcal{B}'$ with $x \in B' \subseteq B$.*

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- Now assume the second condition, and let $U \in \tau_{\mathcal{B}}$. We have to prove $U \in \tau_{\mathcal{B}'}$.

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- Now assume the second condition, and let $U \in \tau_{\mathcal{B}}$. We have to prove $U \in \tau_{\mathcal{B}'}$.
- Let $x \in U$. Since $U \in \tau_{\mathcal{B}}$, there is $B \in \mathcal{B}$ with $x \in B \subseteq U$.

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- Now assume the second condition, and let $U \in \tau_{\mathcal{B}}$. We have to prove $U \in \tau_{\mathcal{B}'}$.
- Let $x \in U$. Since $U \in \tau_{\mathcal{B}}$, there is $B \in \mathcal{B}$ with $x \in B \subseteq U$.
- By the assumed condition, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

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- Since $B \in \mathcal{B} \subseteq \tau_{\mathcal{B}} \subseteq \tau_{\mathcal{B}'}$, by definition of the generated topology $\tau_{\mathcal{B}'}$, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.
- Now assume the second condition, and let $U \in \tau_{\mathcal{B}}$. We have to prove $U \in \tau_{\mathcal{B}'}$.
- Let $x \in U$. Since $U \in \tau_{\mathcal{B}}$, there is $B \in \mathcal{B}$ with $x \in B \subseteq U$.
- By the assumed condition, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.
- Hence $x \in B' \subseteq U$, which proves $U \in \tau_{\mathcal{B}'}$. $\square$

From a topology to a base

Lemma

Let (X, τ) be a topological space. Let $\mathcal{C} \subseteq \tau$ be such that for any $U \in \tau$ and $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is base for a topology on X, and $\tau_{\mathcal{C}} = \tau$.

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Proof

- The collection $\mathcal{C}$ satisfies (B1) by hypothesis.
- Let $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$. Since $C_1 \cap C_2$ is open, by hypothesis there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. So $\mathcal{C}$ satisfies (B2) and is base for a topology.

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- Now, to prove $\tau_{\mathcal{C}} = \tau$, let $U \in \tau_{\mathcal{C}}$. Since U is union of elements of $\mathcal{C}$, and these are in τ, then $U \in \tau$.

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Proof

- The collection $\mathcal{C}$ satisfies (B1) by hypothesis.
- Let $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$. Since $C_1 \cap C_2$ is open, by hypothesis there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. So $\mathcal{C}$ satisfies (B2) and is base for a topology.
- Now, to prove $\tau_{\mathcal{C}} = \tau$, let $U \in \tau_{\mathcal{C}}$. Since U is union of elements of $\mathcal{C}$, and these are in τ, then $U \in \tau$.
- Finally, if $U \in \tau$, then for any $x \in U$, by our hypothesis, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Hence $U \in \tau_{\mathcal{C}}$. $\square$

Why bases are useful

- We will now give several examples of topological spaces defined by bases. As a matter of fact, most of the topologies we meet in the course are given that way.

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- Many theorems on certain topologies are easier to prove using properties of the base that generate them.

Standard topology on $\mathbb{R}$

Definition (The standard topology)

If $X = \mathbb{R}$, the collection of all open intervals $(a, b) \subseteq \mathbb{R}$ with $a < b$, for $a, b \in \mathbb{R}$ is a base for a topology, called standard topology.

Sorgenfrey line

Definition (The Sorgenfrey line)

The collection of all intervals of the form

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

for $a, b \in \mathbb{R}$, is also a base for a topology on $\mathbb{R}$. The topological space obtained is called the Sorgenfrey line.

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Lemma (Sorgenfrey line is bigger)

The Sorgenfrey topology is strictly bigger than the standard topology.

Subbases

Definition (Subbase)

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Theorem (Topology generated by a subbase)

The collection of all finite intersections of elements of a subbase $\mathcal{S}$ is a base for a topology on X.

Proof

- Let $\mathcal{B}$ be the collection of all finite intersections of elements of $\mathcal{S}$. Observe that $\mathcal{S} \subseteq \mathcal{B}$

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- Given $x \in X$, by our hypothesis, there is $S \in \mathcal{S}$ such that $x \in S$. Since $S \in \mathcal{B}$, we have (B1).

Proof

- Let $\mathcal{B}$ be the collection of all finite intersections of elements of $\mathcal{S}$. Observe that $\mathcal{S} \subseteq \mathcal{B}$
- Given $x \in X$, by our hypothesis, there is $S \in \mathcal{S}$ such that $x \in S$. Since $S \in \mathcal{B}$, we have (B1).
- Let $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$. Suppose that there are $S_1, \ldots, S_n, S'_1, \ldots, S'_m \in \mathcal{S}$ such that:

$$B_1 = S_1 \cap \cdots \cap S_n, \quad B_2 = S'_1 \cap \cdots \cap S'_m.$$

Proof

- Let $\mathcal{B}$ be the collection of all finite intersections of elements of $\mathcal{S}$. Observe that $\mathcal{S} \subseteq \mathcal{B}$
- Given $x \in X$, by our hypothesis, there is $S \in \mathcal{S}$ such that $x \in S$. Since $S \in \mathcal{B}$, we have (B1).
- Let $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$. Suppose that there are $S_1, \dots, S_n, S_1', \dots, S_m' \in \mathcal{S}$ such that:

$$B_1 = S_1 \cap \cdots \cap S_n, \quad B_2 = S_1' \cap \cdots \cap S_m'.$$

- Then, taking

$$B_3 = S_1 \cap \cdots \cap S_n \cap S_1' \cap \cdots \cap S_m',$$

we obtain $B_3 \in \mathcal{B}$, and $x \in B_3 \subseteq B_1 \cap B_2$. Hence (B2) is satisfied. $\square$

Exercises

1. Let $X = \{1, 2, 3, 4, 5, 6\}$, and $\mathcal{S} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{4, 5, 6\}\}$. Enumerate the open sets of $\tau_{\mathcal{S}}$.

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2. Show that if $\mathcal{S}$ is a subbase for a topology on X, then $\tau_{\mathcal{S}}$ is equal to the intersection of the topologies on X that contain $\mathcal{S}$.
3. Let $\mathcal{S} = \{[a, b] \subseteq \mathbb{R} \mid a < b, a, b \in \mathbb{Q}\}$. Prove that $\mathcal{S}$ is a subbase for a topology. Prove also that if $\mathcal{B} = \mathcal{S} \cup \{\{a\} \mid a \in \mathbb{Q}\}$, then $\mathcal{B}$ is base of a topology, and $\tau_{\mathcal{S}} = \tau_{\mathcal{B}}$.

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3. Let $\mathcal{S} = \{[a, b] \subseteq \mathbb{R} \mid a < b, a, b \in \mathbb{Q}\}$. Prove that $\mathcal{S}$ is a subbase for a topology. Prove also that if $\mathcal{B} = \mathcal{S} \cup \{\{a\} \mid a \in \mathbb{Q}\}$, then $\mathcal{B}$ is base of a topology, and $\tau_{\mathcal{S}} = \tau_{\mathcal{B}}$.
4. Given a natural number n, denote by $[n] = \{kn \in \mathbb{N} \mid k \in \mathbb{N}\}$. Prove that $\mathcal{B} = \{[n] \mid n \in \mathbb{N}\}$ is a base for a topology on $\mathbb{N}$. Show also that $\mathcal{S} = \{[p^r] \mid p \text{ is prime}, r \in \mathbb{N}\}$ is a subbase for a topology on $\mathbb{N}$, and $\tau_{\mathcal{S}} = \tau_{\mathcal{B}}$.

Links

- Lower limit topology - Wikipedia, the free encyclopedia

Links

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- Subbase - Wikipedia, the free encyclopedia