Products

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A base for a topology

Theorem (Base of the product topology)

Let X and Y be topological spaces. Let $\mathcal{B}$ be the collection of all subsets of $X \times Y$ of the form $U \times V$, where U is open in X and V is open in Y. Then $\mathcal{B}$ is a base for a topology on $X \times Y$.

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- For any $(x, y) \in X \times Y$, considering that $X \times Y \in \mathcal{B}$, we have that (B1) is satisfied.

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Proof

- For any $(x, y) \in X \times Y$, considering that $X \times Y \in \mathcal{B}$, we have that (B1) is satisfied.
- Property $(B2)$ follows from the fact that in general:

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D). \qquad \square$$

Product topology

Definition (Product topology)

The topology $\tau_{\mathcal{B}}$ is called the product topology on the Cartesian product $X \times Y$.

Product by a base

Theorem (Product topology given by a base)

Let X be space with base $\mathcal{B}$, and Y be a space with base $\mathcal{C}$. Then

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

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- Let W be an open set in the product topology, and $(x, y) \in W$.
- By definition of base of the product topology, there are open sets U in X and V in Y with $(x, y) \in U \times V \subseteq W$.
- Again by definition of base, there are $B \in \mathcal{B}$, $C \in \mathcal{C}$ with $x \in B \subseteq U$, $y \in C \subseteq V$. We have then $B \times C \in \mathcal{D}$, and

$$(x, y) \in B \times C \subseteq W. \qquad \square$$

Examples

Example (Product topology on $\mathbb{R}^2$)

A base for the product of two standard topologies on $\mathbb{R}$ is the collection of all rectangles of the form $(a, b) \times (c, d)$. It follows that the product topology is the same as the metric topology on $\mathbb{R}$.

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Example (Product of Sierpiński spaces)

Describe the points and the open sets on the product $X \times X$, when X is a Sierpiński space.

Projections

Definition (Projections)

The maps $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are called projections of $X \times Y$.

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If U is open in X, then $\pi_1^{-1}(U) = U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, the set $\pi_2^{-1}(V) = X \times V$, is open in $X \times Y$.

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We have
$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$$

Subbase of the product

Theorem (Subbase)

The collection $\mathcal{S} = \{\pi_1^{-1}(U)\}_{U \in \tau_X} \cup \{\pi_2^{-1}(V)\}_{V \in \tau_Y}$ *is a subbase for the product topology on* $X \times Y$.

Subspace and products

Theorem (Subspace and product topology)

Let A, B be subspaces of X, Y respectively. Then the product topology on $A \times B$ is the same as the topology on $A \times B$ as subspace of the product space $X \times Y$.

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Proof.

Exercise $\square$

Exercises

1. A map $f\colon X \to Y$ between topological space is said to be open if $f(U)$ is open for every $U \subseteq X$ which is an open set. Prove that the projection $\pi_X\colon X \times Y \to X$ is an open map.

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1. A map $f\colon X \to Y$ between topological space is said to be open if $f(U)$ is open for every $U \subseteq X$ which is an open set. Prove that the projection $\pi_X\colon X \times Y \to X$ is an open map.
2. Let I denote the closed interval $[0, 1] \subseteq \mathbb{R}$ as a subspace of the usual topology on $\mathbb{R}$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the product $I_d \times I$, where I_d denotes discrete topology.

Links

- Product topology - Wikipedia, the free encyclopedia