Arbitrary products

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Products

Definition (Arbitrary Cartesian product as set)

Let X_α be a topological space for each $\alpha \in I$. The *Cartesian product* of the sets X_α is denoted as:

$$\prod_{\alpha \in I} X_\alpha$$

and consists of all maps $f: I \to \cup_\alpha X_\alpha$ such that $f(\alpha) \in X_\alpha$. If $f(\alpha) = x_\alpha$, we denote f as $(x_\alpha)_\alpha$.

Definition (Projections)

For each $\alpha_0 \in I$, there is a *projection map*

$$p_{\alpha_0} \colon \prod_{\alpha \in I} X_\alpha \to X_{\alpha_0},$$

given by $p_{\alpha_0}((x_\alpha)_\alpha) = x_{\alpha_0}$.

Box topology

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Definition (Box topology)

The collection of all subsets of X of the form

$$\prod_{\alpha \in I} U_\alpha,$$

where U_α is open in X_α for all $\alpha \in I$, is a basis for a topology on X, called the *box topology*.

Product topology

Definition (Product topology)

The collection

$$\mathcal{S} = \cup_{\alpha\in I}\{p_\alpha^{-1}(U_\alpha) \mid U_\alpha \text{ open in } X_\alpha\}$$

is a subbasis for a topology on X, called the *product topology*.

Remark

The product topology has as basis the subsets of $X = \prod X_\alpha$ of the form

$$\prod_{\alpha \in I} U_\alpha,$$

where U_α is open in X_α for all $\alpha \in I$, and $U_\alpha = X_\alpha$ for all but finitely many values of $\alpha \in I$

Continuous functions into the product

Theorem

Let $f\colon A \to \prod_{\alpha\in I} X_\alpha$ be given by:

$$f(a) = (f_\alpha(a))_{\alpha\in I},$$

where $f_\alpha\colon A \to X_\alpha$ for each α. Suppose that $\prod X_\alpha$ has the product topology. Then f is continuous if and only if each f_α is continuous.

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Remark

The last theorem is not true if we use the box topology on the Cartesian product

Exercises

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    $$\overline{\prod A_\alpha} = \prod \overline{A_\alpha}.$$
    - Which of the last two are true if we use the box topology?
2. We say that the sequence x_n of points in X converges to $x \in X$ if for any neighborhood U of x there is N such that $n \geq N$ implies $x_n \in U$. Show that a sequence x_n in $\prod_{\alpha \in I} X_\alpha$ converges to (x_α) if and only if $\pi_\alpha(x_n)$ converges to x_α for each $\alpha \in I$.