

# Quotient spaces

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# Saturated subsets

## Definition (Fiber)

Let  $p: X \rightarrow Y$  be a *surjective* map between sets and  $y \in Y$ . The inverse image set

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## Definition (Saturated set)

Let  $p: X \rightarrow Y$  be a *surjective* map between sets. We say that  $A \subseteq X$  is **saturated (with respect to  $p$ )** if  $A \cap p^{-1}(y) \neq \emptyset$  implies  $p^{-1}(y) \subseteq A$



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# Quotient topology

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Let  $X$  be a space,  $A$  a set, and  $p: X \rightarrow A$  a *surjective* map. Then there is exactly one topology on  $A$  that makes  $p$  a quotient map, such topology is called **quotient topology**.

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## Remark

One must check that:

$$\tau = \{U \subseteq A \mid p^{-1}(U) \text{ is open in } X\}$$

is a topology on  $A$ .

# Partitions

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## Quotient space

Let  $X$  be a topological space,  $\tilde{X}$  be a partition of  $X$ , and  $p: X \rightarrow \tilde{X}$  the natural surjection. If we give  $\tilde{X}$  the quotient topology induced by  $p$ , the space  $\tilde{X}$  is called a **quotient space** of  $X$

# Quotients and subspaces

If  $p: X \rightarrow Y$  is a quotient map and  $A \subseteq X$  is a subspace, it does not necessarily follow that the restriction of  $p$ :  $A \rightarrow p(A)$  is a quotient map. However we have:

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## Theorem (Quotients and subspaces)

*Let  $p: X \rightarrow Y$  be a quotient map and  $A \subseteq X$  a subspace, that is saturated with respect to  $p$ , and let  $q: A \rightarrow p(A)$  the restriction of  $p$ . Then:*

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# Proof

- We verify the following:

$$\begin{array}{ll} q^{-1}(V) = p^{-1}(V) & \text{if } V \subseteq p(A), \\ p(U \cap A) = p(U) \cap p(A) & \text{if } U \subseteq X. \end{array}$$

# Quotients and compositions

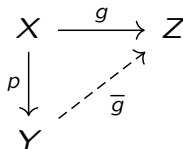
## Theorem

*Composition of quotient maps is a quotient map.*

# Fundamental theorem

## Theorem (Fundamental theorem)

Let  $p: X \rightarrow Y$  be a quotient map. Let  $Z$  be a space, and  $g: X \rightarrow Z$  a continuous map that is constant on each fiber of  $p$ , that is  $p(x) = p(x')$  implies  $g(x) = g(x')$ . Then  $g$  induces a continuous map  $\bar{g}: Y \rightarrow Z$  such that  $\bar{g} \circ p = g$ . The function  $\bar{g}$  is unique with the property that  $\bar{g} \circ p = g$





# Fundamental theorem

Theorem (Fundamental theorem, for equivalence relations)

Let  $X$  be a space, and  $\sim$  an equivalence relation on  $X$ . Let  $g: X \rightarrow Z$  be a continuous map such that  $g(x) = g(x')$  whenever  $x \sim x'$ . Then, if we denote by  $p: X \rightarrow X/\sim$  the quotient map, the map  $g$  induces a unique continuous map  $\bar{g}: X/\sim \rightarrow Z$  such that  $\bar{g} \circ p = g$

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ p \downarrow & \nearrow \bar{g} & \\ X/\sim & & \end{array}$$