Constructing continuous functions

2016-02-18 9:00 -0500

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- *If $A \subseteq X$ is a subspace, the inclusion $i_A: A \to X$ is continuous.*
- *If $f: X \to Y$ and $g: Y \to Z$ are continuous, then the composition $g \circ f$ is continuous.*

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- *If* $X = \cup U_\alpha$*, where each* U_α *is open and* $f|_{U_\alpha}: U_\alpha \to Y$ *is continuous for each* α*, then* f *is continuous.*

Pasting lemma

Theorem (The pasting lemma)

Let $X = A \cup B$ be a space, with A, B closed in X. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous. Suppose that $f(x) = g(x)$ for every $x \in A \cap B$, and define $h\colon X \to Y$ by:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

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Proof.

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Maps into products

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Let $f\colon A \to X \times Y$ be given by:

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if the functions:

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Embedding

If $f\colon X \to Y$ is injective and the bijection $f\colon X \to f(X)$ (where $f(X) \subseteq Y$ is a subspace) is a homeomorphism, we say that f is an **embedding**.

Exercises

1. If $x_0 \in X$, show that the map $f_{x_0} \colon Y \to X \times Y$ given by $f_{x_0}(y) = (x_0, y)$ is an embedding.

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4. Let $f\colon A \to B$ and $g\colon C \to D$ be continuous functions between topological spaces. Show that the map $f \times g\colon A \times C \to B \times D$ given by $(a, c) \mapsto (f(a), f(c))$ is continuous.