# Products

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# A base for a topology

**Theorem (Base of the product topology)**

*Let $X$ and $Y$ be topological spaces. Let $\mathcal{B}$ be the collection of all subsets of $X \times Y$ of the form $U \times V$, where $U$ is open in $X$ and $V$ is open in $Y$. Then $\mathcal{B}$ is a base for a topology on $X \times Y$.*

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### Proof

- For any $(x, y) \in X \times Y$, considering that $X \times Y \in \mathcal{B}$, we have that (B1) is satisfied.
- Property $(B2)$ follows from the fact that in general:

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D). \qquad \square$$

# Product topology

**Definition (Product topology)**

The topology $\tau_{\mathcal{B}}$ is called the product topology on the Cartesian product $X \times Y$.

# Product by a base

Theorem (Product topology given by a base)

*Let $X$ be space with base $\mathcal{B}$, and $Y$ be a space with base $\mathcal{C}$. Then*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

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- Let $W$ be an open set in the product topology, and $(x, y) \in W$.
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- By definition of base of the product topology, there are open sets $U$ in $X$ and $V$ in $Y$ with $(x, y) \in U \times V \subseteq W$.
- Again by definition of base, there are $B \in \mathcal{B}$, $C \in \mathcal{C}$ with $x \in B \subseteq U$, $y \in C \subseteq V$. We have then $B \times C \in \mathcal{D}$, and

$$(x, y) \in B \times C \subseteq W. \qquad \square$$

# Examples

Example (Product topology on $\mathbb{R}^2$)

A base for the product of two standard topologies on $\mathbb{R}$ is the collection of all rectangles of the form $(a, b) \times (c, d)$. It follows that the product topology is the same as the metric topology on $\mathbb{R}$.

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**Example (Product of Sierpiński spaces)**

Describe the points and the open sets on the product $X \times X$, when $X$ is a Sierpiński space.

# Projections

Definition (Projections)

The maps $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are called projections of $X \times Y$.

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### Inverse images of projections

If $U$ is open in $X$, then $\pi_1^{-1}(U) = U \times Y$, which is open in $X \times Y$. Similarly, if $V$ is open in $Y$, the set $\pi_2^{-1}(V) = X \times V$, is open in $X \times Y$.

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We have

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$$

# Subbase of the product

**Theorem (Subbase)**

*The collection* $\mathcal{S} = \{\pi_1^{-1}(U)\}_{U \in \tau_X} \cup \{\pi_2^{-1}(V)\}_{V \in \tau_Y}$ *is a subbase for the product topology on* $X \times Y$*.*

# Subspace and products

Theorem (Subspace and product topology)

*Let $A, B$ be subspaces of $X, Y$ respectively. Then the product topology on $A \times B$ is the same as the topology on $A \times B$ as subspace of the product space $X \times Y$.*

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**Proof.**

Exercise □

# Exercises

1. A map $f\colon X \to Y$ between topological space is said to be open if $f(U)$ is open for every $U \subseteq X$ which is an open set. Prove that the projection $\pi_X\colon X \times Y \to X$ is an open set.

# Exercises

1. A map $f\colon X \to Y$ between topological space is said to be open if $f(U)$ is open for every $U \subseteq X$ which is an open set. Prove that the projection $\pi_X\colon X \times Y \to X$ is an open set.
2. Let $I$ denote the closed interval $[0,1] \subseteq \mathbb{R}$ as a subspace of the usual topology on $\mathbb{R}$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the product $I_d \times I$, where $I_d$ denotes discrete topology.

# Links

- Product topology - Wikipedia, the free encyclopedia