The order topology

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Definition (Totally ordered set)

In this case, the pair $(X, <)$ is called a totally ordered set. If $x < y$ or $x = y$, we write $x \leq y$. An example of a total order is given by $X = \mathbb{R}$ with usual order.

Intervals

Definition (Intervals)

If X is a totally ordered set, and $a, b \in X$, we will use the following notation for intervals:

$$
\begin{aligned}
[a, b] &= \{x \in X \mid a \le x \le b\} \\
(a, b) &= \{x \in X \mid a < x < b\} \\
(a, b] &= \{x \in X \mid a < x \le b\} \\
[a, b) &= \{x \in X \mid a \le x < b\} \\
(a, \infty) &= \{x \in X \mid x > a\} \\
(-\infty, b) &= \{x \in X \mid x < b\}
\end{aligned}
$$

The order topology

Definition (Order topology)

Let X be a totally ordered set with more than one element. Let $\mathcal{S}$ be the collection:

$$\mathcal{S} = \{(-\infty, a) \mid a \in X\} \cup \{(b, \infty) \mid b \in X\}.$$

Then $\mathcal{S}$ is subbase for a topology on X, called order topology.

Base for the order topology

Lemma (Base for the order topology)

Let X be a totally ordered set with more than one element. Let $\mathcal{B}$ be the collection of all intervals of the form (a, b), together with those of the form $[m, b)$, in case X has a minimum m, and those of the form $(a, M]$, in case X has a maximum M. Then $\mathcal{B}$ is a base for the order topology on X.

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Proof

The set of finite intersections of the intervals in the subbasis can be obtained as union of elements of the intervals described. $\square$

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Lemma (The dictionary order is a total order)

The order on $X \times Y$ just described is a total order, called the dictionary order.

Proof that the dictionary order is a total order

- Let $(x, y) \neq (x', y')$ with $(x, y), (x', y') \in X \times Y$. If $x \neq x'$, then one of the pairs is less than the other. Otherwise $x = x'$, and since the pairs are distinct, we must have $y \neq y'$, and so (O1) follows.

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- If we had $(x, y) < (x, y)$, then it would follow that either $x < x$ or $x = x, y < y$. Since none of these is possible, we conclude that (O2) is true.

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- If we had $(x, y) < (x, y)$, then it would follow that either $x < x$ or $x = x, y < y$. Since none of these is possible, we conclude that (O2) is true.
- The proof of (O3) is left as an exercise.

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- The order topology on $\mathbb{R} \times \mathbb{R}$ with the dictionary order is different from the usual metric topology on $\mathbb{R}^2$.
- The order topology on $\{0, 1\} \times \mathbb{N}$ has almost all singletons as open sets.

Exercises

1. Let $(X, <)$ be a totally ordered set, and let $Y \subseteq X$. If, for $y, y' \in Y$ we define $y <' y'$ whenever $y < y'$ in X, show that $(Y, <')$ is a totally ordered set.

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2. If X is a totally ordered set, and $Y \subseteq X$, show that Y can have at most one smallest element.

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2. If X is a totally ordered set, and $Y \subseteq X$, show that Y can have at most one smallest element.
3. If X is a totally ordered set, and we have that $x < y$ and there are no $z \in X$ such that $x < z < y$, we say that y is an immediate succesor of x. Show that any $x \in X$ has at most one immediate succesor.

Exercises

1. Let $(X, <)$ be a totally ordered set, and let $Y \subseteq X$. If, for $y, y' \in Y$ we define $y <' y'$ whenever $y < y'$ in X, show that $(Y, <')$ is a totally ordered set.
2. If X is a totally ordered set, and $Y \subseteq X$, show that Y can have at most one smallest element.
3. If X is a totally ordered set, and we have that $x < y$ and there are no $z \in X$ such that $x < z < y$, we say that y is an immediate succesor of x. Show that any $x \in X$ has at most one immediate succesor.
4. Let $(X, <)$ be a totally ordered set with no minimum element, and let $\mathcal{C} = \{(a, \infty) \mid a \in X\}$. Show that $\mathcal{C}$ is a basis for a topology on X. Is $\tau_{\mathcal{C}}$ in general, the same as the order topology?

Links

- Total order - Wikipedia, the free encyclopedia

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