The metric topology

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Definition of metric

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Metric space

If d is a metric on X, we say that the pair (X, d) is a *metric space*.

Metric topology

Definition (Ball)

If (X, d) is a metric space and $\epsilon > 0$, we call

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\},$$

the ϵ-*ball* centered on x.

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Theorem

If (X, d) is a metric space, the collection

$$\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$$

is a basis for a topology on X

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- Usual metric on $\mathbb{R}^n$.

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Remark

One can prove that $U \subseteq X$ is open in τ_d if and only if for all $x \in U$ there is $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$ (that is, the ball can be chosen with x as center)

Metrizable spaces

Definition (Metrizable space)

We say that the topological space (X, τ) is *metrizable* if there is a metric d on X such that $\tau_d = \tau$.

Bounded metrics

Definition (Bounded set)

Let (X, d) be a metric space. We say that $A \subseteq X$ is *bounded* if there is M such that $d(x, y) \leq M$ for all $x, y \in A$.

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Theorem (Bounded metric)

Let (X, d) be a metric space. If we define $\overline{d}$ as:

$$\overline{d}(x, y) = \min\{d(x, y), 1\},$$

then $\overline{d}$ is a metric on X, and $\tau_d = \tau_{\overline{d}}$.

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Theorem

The product topology on $\mathbb{R}^n$ is the same as τ_d and the same as τ_ρ.

Uniform metric

Definition (Uniform metric)

If I is a set, and $\overline{d}(x, y) = \min\{|x - y|, 1\}$ for $x, y \in \mathbb{R}$, we can define the metric $\overline{\rho}$ on $\mathbb{R}^I = \prod_{i \in I} \mathbb{R}$ by:

$$\overline{\rho}(x, y) = \sup\{\overline{d}(x_i, y_i) \mid i \in I\}.$$

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Exercise

Show that if I is infinite, the uniform topology is different from both the product and the box topology on $\mathbb{R}^I$

Metric in $\mathbb{R}^{\mathbb{N}}$

Definition (The function D)

For $x, y \in \mathbb{R}^{\mathbb{N}}$, we define:

$$D(x, y) = \sup\{\frac{\overline{d}(x_i, y_i)}{i} \mid i \in \mathbb{N}\}$$

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Lemma (D is metric)

D defines a metric on $\mathbb{R}^{\mathbb{N}}$.

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- We know that D is a metric. Let τ the product topology on $X = \mathbb{R}^{\mathbb{N}}$. We prove that $\tau_D \subseteq \tau$.

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- Let $U \in \tau_D$, $x \in U$, we will show that there is $V \in \tau$ with $x \in V \subseteq U$.

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- Let $\epsilon > 0$ be such that $B_D(x, \epsilon) \subseteq U$. Let N such that $\frac{1}{N} < \epsilon$.

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- Let $\epsilon > 0$ be such that $B_D(x, \epsilon) \subseteq U$. Let N such that $\frac{1}{N} < \epsilon$.
- Let

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

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$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

- We have $x \in V$ and $V \in \tau$, we will show $V \subseteq U$.

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- First, note that if $i \geq N, y \in \mathbb{R}^{\mathbb{N}}$, then $\overline{d}(x_i, y_i) \leq 1$ and $\frac{1}{i} \leq \frac{1}{N}$, hence $\frac{\overline{d}(x_i, y_i)}{i} \leq \frac{1}{N}$.

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- Now, let $y \in V$. We have that $|x_i - y_i| < \epsilon$ for $i = 1, 2, \ldots, N$, and so for such values of i we have:

$$\frac{\overline{d}(x_i, y_i)}{i} \leq \overline{d}(x_i, y_i) \leq d(x_i, y_i) = |x_i - y_i| < \epsilon,$$

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- Finally:

$$\begin{aligned} D(x, y) &= \sup\{\frac{\overline{d}(x_i, y_i)}{i} \mid i \in \mathbb{N}\} \\ &\leq \max\{\frac{\overline{d}(x_1, y_1)}{1}, \ldots \frac{\overline{d}(x_N, y_N)}{N}, \frac{1}{N}\} < \epsilon \end{aligned}$$

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- Hence: $y \in B_D(x, \epsilon) \subseteq U$.

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- Let U an element of the base of τ and $x \in U$. We will see that there is $V \in \tau_D$ such that $x \in V \subseteq U$.
- Let $U = \prod U_i$, with U_i open in $\mathbb{R}$. Suppose that $U_i = \mathbb{R}$ if $i \notin \{\alpha_1, \ldots, \alpha_n\}$. Let ϵ_i such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$ for $i \in \{\alpha_1, \ldots, \alpha_n\}$. Without loss we can assume $\epsilon_i \leq 1$.

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- Let $\epsilon = \min\{\frac{\epsilon_i}{i} \mid i = \alpha_1, \ldots, \alpha_n\}$. We claim that $B_D(x, \epsilon) \subseteq U$. Let $y \in B_D(x, \epsilon)$.

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- For $i \in \{\alpha_1, \ldots, \alpha_n\}$ we have that:

$$\frac{\overline{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon \leq \frac{\epsilon_i}{i}$$

and so: $\overline{d}(x_i, y_i) < \epsilon_i \leq 1$.

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  and so: $\overline{d}(x_i, y_i) < \epsilon_i \leq 1$.
- It follows that if $i \in \{\alpha_1, \ldots, \alpha_n\}$, then $d(x_i, y_i) = \overline{d}(x_i, y_i) < \epsilon_i$. Hence $y \in \prod U_i$. $\square$

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   - Show that if X has a topology τ such that $d\colon X \times X \to \mathbb{R}$ is continuous, then $\tau_d \subseteq \tau$.
3. Let $A \subseteq \mathbb{R}^{\mathbb{N}}$ to consist on all sequences that are eventually zero. What is $\overline{A}$? (where $\mathbb{R}^{\mathbb{N}}$ has the uniform topology).