Quotient spaces

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Saturated subsets

Definition (Fiber)

Let $p\colon X \to Y$ be a *surjective* map between sets and $y \in Y$. The inverse image set

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Definition (Saturated set)

Let $p\colon X \to Y$ be a *surjective* map between sets. We say that $A \subseteq X$ is saturated (with respect to p) if $A \cap p^{-1}(y) \neq \emptyset$ implies $p^{-1}(y) \subseteq A$

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Remark

One must check that:

$$\tau = \{U \subseteq A \mid p^{-1}(U) \text{ is open in } X\}$$

is a topology on A.

Partitions

Definition (Partition)

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Quotient space

Let X be a topological space, $\tilde{X}$ be a partition of X, and $p\colon X \to \tilde{X}$ the natural surjection. If we give $\tilde{X}$ the quotient topology induced by p, the space $\tilde{X}$ is called a quotient space of X

Quotients and subspaces

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Theorem (Quotients and subspaces)

Let $p: X \to Y$ be a quotient map and $A \subseteq X$ a subspace, that is saturated with respect to p, and let $q: A \to p(A)$ the restriction of p. Then:

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- *if p is open or closed, then q is a quotient map.*

Proof

- We verify the following:

$$
\begin{aligned}
q^{-1}(V) &= p^{-1}(V) && \text{if } V \subseteq p(A), \\
p(U \cap A) &= p(U) \cap p(A) && \text{if } U \subseteq X.
\end{aligned}
$$

Quotients and compositions

Theorem

Composition of quotient maps is a quotient map.

Fundamental theorem

Theorem (Fundamental theorem)

Let $p\colon X \to Y$ be a quotient map. Let Z be a space, and $g\colon X \to Z$ a continuous map that is constant on each fiber of p, that is $p(x) = p(x')$ implies $g(x) = g(x')$. Then g induces a continuous map $\overline{g}\colon Y \to Z$ such that $\overline{g} \circ p = g$. The function $\overline{g}$ is unique with the property that $\overline{g} \circ p = g$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, dashed, "\overline{g}"'] & \end{tikzcd}$$

Fundamental theorem

Theorem (Fundamental theorem, for equivalence relations)

Let X be a space, and $\sim$ an equivalence relation on X. Let $g\colon X \to Z$ be a continuous map such that $g(x) = g(x')$ whenever $x \sim x'$. Then, if we denote by $p\colon X \to X/{\sim}$ the quotient map, the map g induces a unique continuous map $\overline{g}\colon X/{\sim} \to Z$ such that $\overline{g} \circ p = g$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \\ X/{\sim} \arrow[ur, dashed, "\overline{g}"'] & \end{tikzcd}$$