Connectedness

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Separations

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- $U \cup V = X$,
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Definition (Connected space)

A topological space is connected if it has no separation.

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This is because if U, V form a separation, then both are open and closed. Conversely, if $A \subseteq X$ is not the empty set and not X, and is open and closed, then $A, X - A$ form a separation.

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Let X be a space and Y a subspace of X. Then a separation of Y is a pair of disjoint nonempty sets A, B with union Y, and none containing a limit point of the other.

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- Let A, B be a separation of Y. Then both A, B are open and closed in Y. We have that the closure of A in Y is $\overline{A} \cap Y$, and since A is closed, we have $A = \overline{A} \cap Y$. Since A, B are disjoint, we have $\overline{A} \cap B = \emptyset$.

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- Let A, B be a separation of Y. Then both A, B are open and closed in Y. We have that the closure of A in Y is $\overline{A} \cap Y$, and since A is closed, we have $A = \overline{A} \cap Y$. Since A, B are disjoint, we have $\overline{A} \cap B = \emptyset$.
- Now, suppose $A, B \subseteq Y$ are disjoint, nonempty, and none containing a limit point of the other. Then $\overline{A} \cap B = \emptyset$, hence $\overline{A} \cap Y = A$. It follows that A is closed in Y, hence B is open in Y.

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- If X is a space with only one point, then it is connected.
- The subspace $Y = [-1, 0) \cup (0, 1]$ of $\mathbb{R}$ is not connected.
- $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is not connected.

Properties of connected spaces

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If A, B form a separation of X, and $Y \subseteq X$ is connected, then either $Y \subseteq A$ or $Y \subseteq B$.

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We have that $Y \cap A, Y \cap B$ are both open in Y, they are disjoint and its union is Y. Since Y is connected they cannot be both nonempty, and if $Y \cap A = \emptyset$, then $Y \subseteq B$.

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Lemma

The union of connected subspaces with a point in common is connected.

Let $X = \cup_{\alpha \in I} A_\alpha$ with A_α connected, and $x_0 \in \cap A_\alpha$. Let U, V be a separation of X. Suppose $x_0 \in U$. By the previous lemma, we must have $A_\alpha \subseteq U$ for all $\alpha \in I$. But then $V = \emptyset$, which is a contradiction.

More theorems

Theorem

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Proof

Suppose that C, D is a separation of B. By a previous lemma, we may assume without loss that $A \subseteq C$. But then $\overline{A} \subseteq \overline{C}$, and, since $\overline{C}$ and D are disjoint and $B \subseteq \overline{A}$, we must have $B \cap D = \emptyset$, which is a contradiction. $\square$

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Proof

Let $Z = f(X)$, and consider the surjective map $f\colon X \to Z$, which is also continuous. Let A, B be a separation of Z. Then $f^{-1}(A), f^{-1}(B)$ would form a separation of X, which is impossible. Therefore, there is no separation of Z, hence $f(X)$ is connected. $\square$

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- Let $(a, b) \in X \times Y$. Then $X \times \{b\}$ and $\{x\} \times Y$ are connected, for all $y \in Y$.
- Hence $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected, since it is the union of two connected spaces with a point in common.

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- Hence $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected, since it is the union of two connected spaces with a point in common.
- Then, $X \times Y = \cup_{x \in X} T_x$ is the union of connected spaces with the point (a, b) in common.

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- Hence $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected, since it is the union of two connected spaces with a point in common.
- Then, $X \times Y = \cup_{x \in X} T_x$ is the union of connected spaces with the point (a, b) in common.
- By induction, we obtain that the product of any finite number of connected spaces is connected.

- Now, let X_α be a connected space, for $\alpha \in I$. We want to prove that $\prod_\alpha X_\alpha$ is connected. Choose $b = (b_\alpha) \in X$.

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- Given $\{\alpha_1, \ldots, \alpha_n\} \subseteq I$, we define $X(\alpha_1, \ldots, \alpha_n)$ as the subspace of X with points such that $x_\alpha = b_\alpha$ for $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$.

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- Let Y be the subspace of X that is the union of all $X(\alpha_1, \ldots, \alpha_n)$, for $\{\alpha_1, \ldots, \alpha_n\} \subseteq I$ finite. Then Y is connected, as is the union of connected spaces with the point b in common.

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- We finally prove that $\overline{Y} = X$. Let $x = (x_\alpha) \in X$, and $U = \prod U_\alpha$ be a basis element of the product topology. We have that $U = X_\alpha$ for all $\alpha \in I - \{\alpha_1, \ldots, \alpha_n\}$.

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- Let $y = (y_\alpha)$ defined as $y_\alpha = x_\alpha$ for $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$, and $y_\alpha = b_\alpha$ otherwise. Then $y \in X(\alpha_1, \ldots, \alpha_n)$, so $Y \cap U \neq \emptyset$. $\square$

Linear continuum

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- If $x < y$, there is z such that $x < z < y$.

Theorem

If L is a linear continuum, then L is connected in the order topology. (Hence, every interval and every ray in L is connected.)

Proof

- We say that a subset $Y \subseteq L$ is convex if $a, b \in Y$ with $a < b$ implies $[a, b] \subseteq Y$. We will prove that any convex subset Y of L is connected.

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- Suppose that $Y = A \cup B$, with A, B open in Y, disjoint and nonempty. Let $a \in A$, $b \in B$, and assume that $a < b$.
- Since Y is convex, we have $[a, b] \subseteq Y$. Then $[a, b]$ is the union of the disjoint nonempty sets A_0, B_0 given by:

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- Note that A_0, B_0 are open in $[a, b]$, and thus form a separation of $[a, b]$.

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- Note that A_0, B_0 are open in $[a, b]$, and thus form a separation of $[a, b]$.
- The interval $[a, b]$ is a subspace of L. Therefore the topology on $[a, b]$ is the order topology.
- Let $c = \sup A_0$. Then $c \in [a, b]$.

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- Hence d is an upper bound of A_0, which contradicts the choice of c.

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- Then $c \in A_0$. Given that $c \neq b$, we have that either $c = a$ or $c \in (a, b)$.

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- Then $c \in A_0$. Given that $c \neq b$, we have that either $c = a$ or $c \in (a, b)$.
- In any case, we have that there is d such that $[c, d) \subseteq A_0$.
- By Property 2 of linear continuum, there is z such that $c < z < d$. But this contradicts that c is upper bound of A_0. $\square$

Connected subsets of $\mathbb{R}$

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Theorem

If X and Y are linear continuum, and Y is bounded above and below, then $X \times Y$ is a linear continuum with dictionary order.

Intermediate value theorem

Theorem

Let $f\colon X \to Y$ be a continuous map, where X is connected and Y is ordered and has the order topology. Suppose that $a, b \in X$ and $y \in Y$ are such that $f(a) < y < f(b)$. Then there is $c \in X$ such that $f(c) = y$.

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- A path on a space X from $x \in X$ to $y \in X$ is a continuous function $f\colon [a,b] \to X$ from some interval $[a,b] \subseteq \mathbb{R}$, such that $f(a) = x$, $f(b) = y$.

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Examples

The converse is not true, see $[0,1] \times [0,1]$ with dictionary order and the *deleted comb space*.