Components and local connectedness

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Definition

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- *Each connected nonempty subset of X intersects exactly one of them.*
- *They are closed, and if there are finitely many components, they are also open.*

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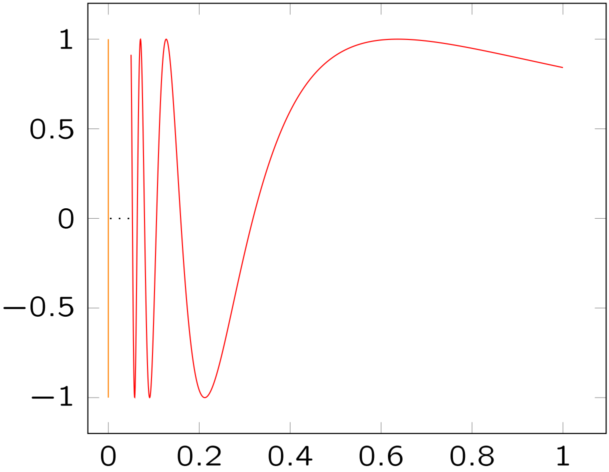
- *Each nonempty path-connected subset of X intersects exactly one of them.*

The topologist's sine curve



$$X = \{(x, sin\frac{1}{x}) \mid x > 0\} \cup \{(0, x) \mid x \in [-1, 1]\}$$

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Examples

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- Any interval in $\mathbb{R}$ is connected and locally connected.
- The subspace $[0,1] \cup [2,3] \subseteq \mathbb{R}$ is locally connected and not connected.
- The topologist's sine curve is connected but not locally connected.
- $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is neither connected nor locally connected.

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- Suppose that X is locally connected, and let C be a component of the open set $U \subseteq X$.
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- Since V is connected, it must be contained in C.
- Now suppose the components of open sets in X are open.
- Let $x \in X$ and a neighborhood U of x. If C is the component of U that contains x, then C is a connected neighborhood of x. $\square$

The proof of the following theorem is similar and left as an exercise.

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- Let C be a component of X, let $x \in C$, and let P be the path component of X containg x. Since P is connected, $P \subseteq C$.

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- *If X is locally path connected, the components and the path components are the same.*

Proof

- Let C be a component of X, let $x \in C$, and let P be the path component of X containg x. Since P is connected, $P \subseteq C$.
- Suppose now that X is locally path connected, and we wish to prove $P = C$. Assume that $P \subsetneq C$.

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- Since X is locally path connected, each path component of X is open in X. Hence P and S are open, disjoint and nonempty sets with union C. This contradicts that C is connected. $\square$