Topological spaces

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Outline

X a set

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$\mathbb{R}$ the set of real numbers.

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1. If τ is a topology on X, the pair (X, τ) is called a *topological space*.
2. The elements of τ are called *open sets*.

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- Let X be any set. Then $\tau_i = \{\emptyset, X\}$ is a topology, called the *indiscrete topology*.
- Let $X = \{1, 2\}$. Then $\tau = \{\emptyset, X, \{1\}\}$ is a topology, and (X, τ) is called the *Sierpiński space*.

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- Let
  - (X, d) be any metric space. For $x \in X$, let $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$.
  - $\tau = \{U \subseteq X \mid \forall x \in U\, \exists \epsilon > 0, \text{such that } B_\epsilon(x) \subseteq U\}$.
  - Then τ is called the *metric topology* on X.

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Example (Cofinite topology)

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- Let $U_\alpha \in \tau$ for all $\alpha \in I$. We need to show that either $\cup_\alpha U_\alpha$ is empty or $X - \cup_\alpha U_\alpha$ is finite.

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- Let $U_1, U_2 \in \tau$. We need to show that either $U_1 \cap U_2$ is empty or $X - U_1 \cap U_2$ is finite.

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Exercises

1. Show that if X is any set, there is a metric on X such that the metric topology is the same as the discrete topology.

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2. Show that in general there is no metric on X such that the metric topology is the same as the indiscrete topology.
3. Show that if τ_α is a topology on X for each $\alpha \in I$, then $\tau = \cap \tau_\alpha$ is a topology on X.

Links

- Topological space - Wikipedia, the free encyclopedia

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- How to get intuition in topology concerning the definitions? - Mathematics Stack Exchange

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- Why is a topology made up of 'open' sets? - MathOverflow
- Why does topology rarely come up outside of topology? - Mathematics Stack Exchange