Continuous functions

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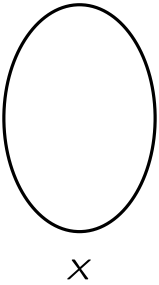

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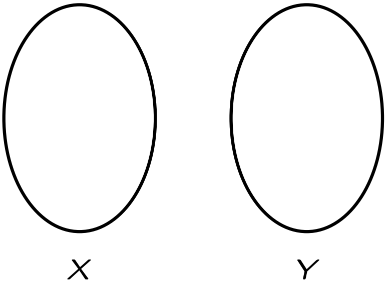

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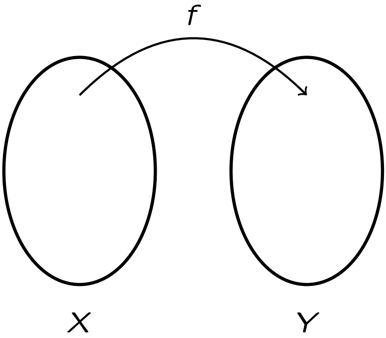

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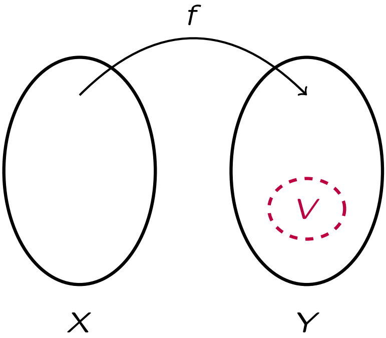

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- Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, $x_0 \in \mathbb{R}$ and $\epsilon > 0$.
- We have that $f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$ is open in $\mathbb{R}$ and contains x_0. Hence there is $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(f(x_0) - \epsilon, f(x_0) + \epsilon)$.

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- Let $\mathbb{R}_l$ be the set of real numbers with the Sorgenfrey topology. Then the identity function $1_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}_l$ is not continuous.
- On the other hand, the identity $1_{\mathbb{R}}\colon \mathbb{R}_l \to \mathbb{R}$ is continuous.

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- *For every closed set C in Y, one has $f^{-1}(C)$ is closed in X.*
- *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

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- *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

When the last condition is satisfied at $x_0 \in X$, we say that f is continuous at x_0.

Exercises

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Exercises

1. Let $f\colon X \to Y$ continuous. If $A \subseteq X$ and x is a limit point of A, is $f(x)$ a limit point of $f(A)$?
2. If the singleton $\{x_0\}$ is open in X, prove that every function $f\colon X \to Y$ is continuous at x_0. Is the converse true?
3. Prove that $f\colon X \to Y$ is continuous if and only if for every $A \subseteq Y$ we have that $f^{-1}(A^\circ) \subseteq (f^{-1}(A))^\circ$.