W.D. Henshaw **Math 6800: Solutions for Problem Set 6**

1. NLA 19.1 *Given $A \in \mathbb{C}^{m\times n}$ of rank n and $b \in \mathbb{C}$, ...*

Solution:

Written out the equations are

$$
\begin{aligned}
r + Ax &= b, \\
A^* r &= 0.
\end{aligned}
$$

Multiplying the first equation by A^* and using the second equation gives

$$
A^*Ax = A^*b
$$

These are the normal equations and thus x is a solution to the least squares problem and $r = b - Ax$ is the residual by definition. Since $A \in \mathbb{C}^{m\times n}$ has full rank, A^*A is nonsingular and there is a unique solution $x = (A^*A)^{-1}A^*b$. Given x, r is uniquely determined from $r = b - Ax$. Thus $[r, x]^T$ is the unique solution, x is the least squares solution and r is the residual.

2. NLA 20.1 *Let $A \in \mathbb{C}^{m\times m}$ be nonsingular. Show that A has an LU factorization if and only if for each k with $1 \le k \le m$, the upper left $k \times k$ block $A_{1:k,1:k}$ is nonsingular. Prove that this LU factorization is unique.*

Solution:

$\Longrightarrow$ Suppose that A has an LU factorization, $A = LU$, where L is unit lower triangular and U is upper triangular. Since A is nonsingular,

$$
\det(A) = \det(L)\det(U) = \det(U) = \prod_{i=1}^{m} u_{ii} \neq 0,
$$

and thus $u_{ii} \neq 0$, $i = 1, 2, \ldots, m$. Let $A_{11} = A_{1:k,1:k}$ and consider the blocked form of $A = LU$,

$$
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}
$$

where L_{11}, L_{22} are unit lower triangular and U_{11}, U_{22} are upper triangular. Then $A_{11} = L_{11}U_{11}$ and $\det(A_{11}) = \det(L_{11})\det(U_{11}) = \det(U_{11}) = \prod_{i=1}^{k} u_{ii} \neq 0$. Thus $A_{1:k,1:k}$ is nonsingular for $k = 1, 2, \ldots, m$.

$\Longleftarrow$ Suppose that each $A_{1:k,1:k}$ is nonsingular. At each stage in the Gaussian elimination, the pivot u_{kk} (e.g. $u_{11} = a_{11}$) will be nonzero since $\det(A_{1:k,1:k}) = \det(U_{1:k,1:k}) = \prod_{i=1}^{k} u_{ii} \neq 0$. Thus we can carry out the steps of Gaussian elimination to completion to form $A = LU$. This shows that A has an LU factorization.

Alternatively we can prove that A has an LU factorization by induction on the dimension of A. For $m = 1$, $A = [a_{11}] = LU$ where $L = I_{1\times 1}$ and $U = [u_{11}] = [a_{11}]$ and where $u_{11} \neq 0$ since $A_{1:1,1:1}$ is nonsingular and thus U is nonsingular. Now consider an $m \times m$ matrix A . By the induction hypothesis $A_{1:m-1,1:m-1}$ has an LU factorization $A_{1:m-1,1:m-1} = L_{1:m-1,1:m-1}U_{1:m-1,1:m-1}$ where $U_{1:m-1,1:m-1}$ is nonsingular. Let us see if we can find an LU factorization for A of the form

$$A = \begin{bmatrix} A_{11} & a_2 \\ a_1^* & a_{m,m} \end{bmatrix} = \begin{bmatrix} L_{1:m-1,1:m-1} & 0 \\ b^* & 1 \end{bmatrix} \begin{bmatrix} U_{1:m-1,1:m-1} & c \\ 0 & u_{m,m} \end{bmatrix}$$

Multiplying these out implies that we must satisfy

$$\begin{aligned}
&A_{1:m-1,1:m-1} = L_{1:m-1,1:m-1}U_{1:m-1,1:m-1}, \\
&L_{1:m-1,1:m-1}c = a_2, \\
&b^*U_{1:m-1,1:m-1} = a_1^*, \\
&a_{m,m} = b^*c + u_{m,m}.
\end{aligned}$$

The first equation is true by the induction hypotheses. We can solve for b and c since $L_{1:m-1,1:m-1}$ and $U_{1:m-1,1:m-1}$ are nonsingular,

$$\begin{aligned}
&c = L_{1:m-1,1:m-1}^{-1}a_2, \\
&b^* = a_1^*U_{1:m-1,1:m-1}^{-1},
\end{aligned}$$

and then $u_{m,m}$ is given by,

$$u_{m,m} = b^*c - a_{m,m}.$$

Since A is nonsingular, $u_{m,m} \neq 0$ and thus $U_{1:m,1:m}$ is nonsingular. Thus A has an LU factorization with U nonsingular. This completes the proof.

To show that this LU factorization is unique, suppose that there are two LU factorizations

$$A = L_1U_1 = L_2U_2$$

where L_p, $p = 1, 2$ are both lower triangular with unit diagonal and U_p are upper triangular. Multiplying on the left by L_1^{-1} and by U_2^{-1} on the right implies

$$U_1U_2^{-1} = L_1^{-1}L_2$$

Since L_1^{-1} is lower triangular with unit diagonal, and the product of two lower triangular matrices with unit diagonals is another lower triangular matrix with unit diagonal, it follows that $L_1^{-1}L_2$ is lower triangular with unit diagonal. On the other hand $U_1U_2^{-1}$ is upper triangular since U_2^{-1} is upper triangular. The only way an upper triangular matrix $U_1U_2^{-1}$ can equal a lower triangular matrix $L_1^{-1}L_2$ is if they are both diagonal. Since $L_1^{-1}L_2$ has unit diagonal it follows that

$$U_1U_2^{-1} = I_{m\times m}, \qquad L_1^{-1}L_2 = I_{m\times m}.$$

Multiplying $U_1U_2^{-1} = I_{m\times m}$ by U_2 on the right gives $U_1 = U_2$. Multiplying $L_1^{-1}L_2 = I_{m\times m}$ by L_1 on the left gives $L_1 = L_2$. Therefore the LU decomposition is unique.

3. NLA 20.2 *Suppose $A \in \mathbb{C}^{m\times m}$ satisfies the condition of Exercise 20.1 and is banded with bandwidth $2p+1$, i.e. $a_{ij} = 0$ for $|i-j| > p$. What can you say about the sparsity patterns of the factors L and U of A?*

Solution:

A is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,p+1} & 0 & \dots & & & \\ a_{21} & a_{22} & \dots & & a_{2,p+2} & 0 & \dots & & \\ \vdots & & & & & & & & \\ a_{p+1,1} & a_{p+1,2} & \dots & & & a_{p+1,2p+1} & 0 & \dots & \\ 0 & a_{p+2,2} & a_{p+2,3} & \dots & & & a_{p+2,2p+2} & 0 & \dots \\ 0 & 0 & \ddots & \dots & \dots & & \ddots & \ddots & \\ 0 & 0 & 0 & a_{m,m-p} & a_{m,m-p+1} & \dots & a_{m,m} & & \end{bmatrix}$$

Factor L_1 in the Gaussian elimination will be of the form

$$L_1 = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & & & \\ -l_{21} & 0 & \dots & 0 & 0 & 0 & \dots & & \\ \vdots & & & & & & & & \\ -l_{p+1,1} & 0 & \dots & & & 0 & 0 & \dots & \\ 0 & 0 & 0 & \dots & & & 0 & 0 & \dots \\ 0 & 0 & \ddots & \dots & \dots & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & & \end{bmatrix}$$

and L_1A with have zeros in the first column below a_{11}. Note that the bandwidth of L_1A will not increase since we are adding multiples of the first row to rows $2, \dots p+1$. Also note that the inverse of L_1 is just equal to L with the entries l_{ij} changing sign. At stage 2, L_2 will be the identity plus non-zero values $-l_{3,2}, \dots, -l_{p+2,2}$. The bandwidth of L_2L_1A will not increase. Continuing in this way we see that $L = L_{m-1}^{-1} \cdots L_2^{-1} L_1$ with be a lower triangular matrix with at most p non-zero sub-diagonals.

$$L = \begin{bmatrix} 1 & 0 & \dots & & & & & & \\ l_{21} & 1 & 0 & \dots & & & & & \\ l_{31} & l_{32} & 1 & 0 & \dots & & & & \\ \vdots & \vdots & \ddots & \ddots & & & & & \\ l_{p+1,1} & l_{p+1,2} & \dots & l_{p+1,p} & 1 & 0\dots & & & \\ 0 & l_{p+2,2} & \dots & & l_{p+2,p+1} & 1 & 0 & \dots & \\ 0 & 0 & \ddots & \dots & \dots & & \ddots & 0 & \\ 0 & 0 & 0 & l_{m,m-p} & l_{m,m-p+1} & \dots & l_{m,m-1} & 1 & \end{bmatrix}$$

Note that when multiplying by L_1 followed by L_2, etc., to L_k, no new non-zero entries will appear in $U_{1:k,1:k}$ above the p-th super diagonal. Thus U will be upper triangular with at most p non-zero super-diagonals.

$$U = \begin{bmatrix} u_{11} & \dots & u_{1,p+1} & & \\ & u_{22} & \dots & u_{2,p+2} & \\ & & \ddots & \ddots & \\ & & & u_{m-1,m-1} & u_{m-1,m} \\ & & & & u_{m,m} \end{bmatrix}$$

4. NLA 21.3 *Consider Gaussian elimination carried out with pivoting by columns instead of rows, leading to a factorization $AQ = LU$, where Q is a permutation matrix.*
(a) Show that if A is nonsingular, such a factorization exists.
(b) Show that if A is singular, such a factorization does not always exist.

(a) At the start of stage i in Gaussian elimination, $i = 1, 2, \dots, m-1$, we have constructed

$$A^{(i)} \equiv L_{i-1} \cdots L_2 L_1 A Q_1 Q_2 \cdots Q_{i-1}.$$

We now look at elements in row i, $a_{ij}^{(i)}$ $j = i, i+1, \dots m$ for a pivot. If A is nonsingular then these elements cannot all be zero or else det$(A^{(i)})$ would be zero (since the entire row of $A^{(i)}$ would be zero). But det$(A^{(i)}) = 0$ would imply det$(A) = 0$ which is a contradiction since A is nonsingular. Thus we can always find a pivot and proceed to the next stage.

(b) The matrix A given by

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

is singular and does not have a factorization $AQ = LU$ since there is no non-zero pivot in the first row.

5. *NLA 21.4* Gaussian elimination can be used to compute the inverse A^{-1} of a nonsingular matrix...

(a) To compute A^{-1} we can

1. *form the $PA = LU$ factorization at an asymptotic cost of $(2/3)m^3$ flops.*
2. *Let z_j denote the jth row of A^{-1}. Then to find z_j we solve*

   $$Az_j = e_j$$

   using the factorization found in step 1. Each solve costs $2m^2$ flops.

The total cost is then $(2/3)m^3 + 2m^3 = (8/3)m^3$ flops.

(b) Consider solving (assume $P = I$, the argument is easy to adjust for general P)

$$LUz_j = e_j$$

We first perform the forward solves

$$Ly_j = e_j$$

which take the form

$$\begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \vdots & & & & \\ \vdots & & & & \\ \vdots & & & & \\ l_{m1} & l_{m2} & l+m3 & \dots & l_{mm} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{j-1} \\ y_j \\ y_{j+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

Thus we see that $y_1 = y_2 = \ldots = y_{j-1} = 0$ and there is no need to compute these. The remaining lower triangular matrix is of size $m - j$ and the asymptotic cost to solve is $(m-j)^2$ flops. Thus the total cost of ALL the forward solves is

$$\sum_{j=1}^{m}(m-j)^2 = \sum_{j=1}^{m-1} j^2 \sim \frac{1}{3}m^3.$$

The total cost of the all the back-substitutions is $m \times m^2$. The total cost is then $(2/3)m^3 + (1/3)m^3 + m^3 = 2m^3$ flops.

(c) To solve the n equations, $Ax_j = b_j$, $j = 1, 2, \ldots, n$ we can

1. *Factor $PA = LU$, then perform n solves for the asymptotic cost of*

$$C_1 \sim (2/3)m^3 + 2nm^2 \text{ flops.}$$

2. *Form A^{-1} and then multiply $A^{-1}b_j$, for the asymptotic cost of*

$$C_2 \sim 2m^3 + 2nm^2 \text{ flops.}$$