MANE 6760 - FEM for Fluid Dyn. - Lecture 08

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Stabilization Parameter: AD equation

For linear finite elements in 1D (for 1D AD equation):

$$\tau = \tau_{exact} = \frac{1}{|\Omega^e|}\int_{\Omega^e}\int_{\Omega^e} g' d\Omega^e d\Omega^e = \frac{h}{2|a_x|}(\coth(Pe^e) - \frac{1}{Pe^e})$$

- Recall $Pe^e = \frac{|a_x|h}{2\kappa}$
- Consider τ^{adv} and τ^{diff} as follows (where $\frac{\tau^{diff}}{\tau^{adv}} = \frac{1/\tau^{adv}}{1/\tau^{diff}} = Pe^e$):

$$\tau^{adv} = \frac{(h/2)}{|a_x|}$$

$$\tau^{diff} = \frac{(h/2)^2}{\kappa}$$

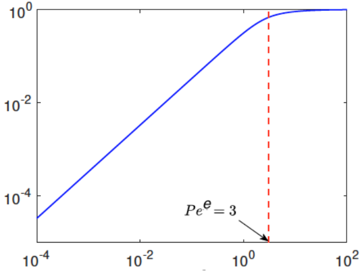
- Advective and diffusive limits of τ are defined as:

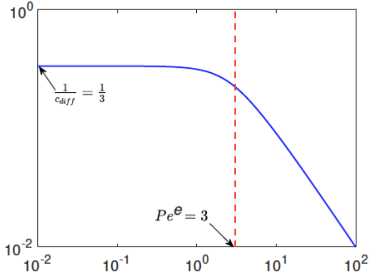
$$\lim_{Pe^e\to\infty} \frac{\tau}{\tau^{adv}} = \lim_{Pe^e\to\infty}(\coth(Pe^e) - \frac{1}{Pe^e}) = 1$$

$$\lim_{Pe^e\to 0} \frac{\tau}{\tau^{diff}} = \lim_{Pe^e\to 0} \frac{1}{Pe^e}(\coth(Pe^e) - \frac{1}{Pe^e}) = \frac{1}{c_{diff}} = \frac{1}{3}$$

Stabilization Parameter: AD equation

- Plot of $\frac{\tau}{\tau^{adv}}$ and $\frac{\tau}{\tau^{diff}}$:





Stabilization Parameter: AD equation

τ approximations

- τ_{exact} involves a transcendental function and is expensive to compute (at each integration point)
- What happens in multiple dimensions, for example, in 2D:
  - Isotropic mesh:
  - Anisotropic mesh:

Stabilization Parameter: AD equation

τ approximations: algebraic versions

- Shakib *et al.* (1991) derived the following:

$$\begin{aligned}\tau_{alg,skb} = \tau_{alg1} : (\tau_{alg,skb})^{-2} &= (\tau^{adv})^{-2} + c_{diff}^2 (\tau^{diff})^{-2} \\ &= \left(\frac{(h/2)}{|a_x|}\right)^{-2} + 9\left(\frac{(h/2)^2}{\kappa}\right)^{-2} \\ &= \left(\frac{2|a_x|}{h}\right)^2 + 9\left(\frac{4\kappa}{h^2}\right)^2\end{aligned}$$

- Codina (1998) derived the following:

$$\begin{aligned}\tau_{alg,cod} = \tau_{alg2} : (\tau_{alg,cod})^{-1} &= (\tau^{adv})^{-1} + (\tau^{diff})^{-1} \\ &= \left(\frac{(h/2)}{|a_x|}\right)^{-1} + \left(\frac{(h/2)^2}{\kappa}\right)^{-1} \\ &= \frac{2|a_x|}{h} + \frac{4\kappa}{h^2}\end{aligned}$$

- ... other options available

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τ approximations: algebraic versions in multiple dimensions

- Define element-level metric tensor: $g_{ij} = \xi_{k,i}\xi_{k,j} = \frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_k}{\partial x_j}$, where in 1D: $g_{11} = (\frac{2}{h})^2$
- In $\tau_{alg,skb}$, use/define advective limit as:

$$(\tau^{adv})^{-2} = a_i g_{ij} a_j$$

- In $\tau_{alg,skb}$, use/define diffusive limit as:

$$(\tau^{diff})^{-2} = g_{ij} g_{ij} \kappa^2$$

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