MANE-6760
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Due: 11pm November 22, 2022

MANE 6760 (FEM for Fluid Dyn.) Fall 2022: HW4

1. (20 points) Consider the formulation provided in the course for the stabilized/SUPG finite element (FE) method for the transient, 1D, scalar AD equation. Use the template code provided to implement the backward Euler scheme. Try three different values of $N_t = 10, 50$ and 250. Keep all the other settings the same (e.g., a_x, κ, N_e, etc.). Provide the updated Python code and the three solution plots (one for each value of N_t).

   The main part of this problem is identifying the Mass matrix (M) (and its contribution that comes from Stabilization) and the A matrix. This is achieved by writing the code in Listing 1 and the solutions for the suggested time discretizations are provided in Fig 1.

```
    for idx_a in range(nes): # loop index in [0,nes-1]
        be[idx_a] = 0.0
        de[idx_a] = de[idx_a] +  \
                        0.0   # ... to be implemented ...
    for idx_b in range(nes): # loop index in [0,nes-1]
        Me[idx_a,idx_b]=Me[idx_a,idx_b]+shp[idx_a,q]*shp[idx_b,q]*wdetj \
                        +shpdgbl[idx_a,q]*(ax*tau)*shp[idx_b,q]*wdetj
        Ae[idx_a,idx_b]=Ae[idx_a,idx_b]-shpdgbl[idx_a,q]*ax*shp[idx_b,q]*wdetj \
                        + shpdgbl[idx_a,q]*(kappa+kappa_num)*shpdgbl[idx_b,q]*wdetj # ... to be implemented ...
        Ke[idx_a,idx_b] = Me[idx_a,idx_b] + dt*Ae[idx_a,idx_b]
        de[idx_a] = de[idx_a] + Me[idx_a,idx_b]*phi_sfem[ien[e,idx_b]]

```

Listing 1: code to Numerically assemble element matrices for TAD equation

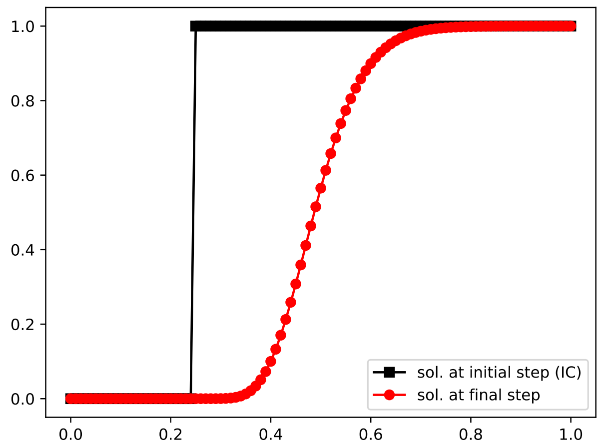
2. (10 points) For the transient, 1D, non-linear Burgers equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = s$, consider the stabilized/SUPG finite element (FE) method leading to the following semi-discrete non-linear weak residual:

$$G_A = \int_0^L \left(\ldots + N_{A,x}\bar{u}\tau\bar{u}_{,t} + \ldots + \ldots\right)\, dx$$

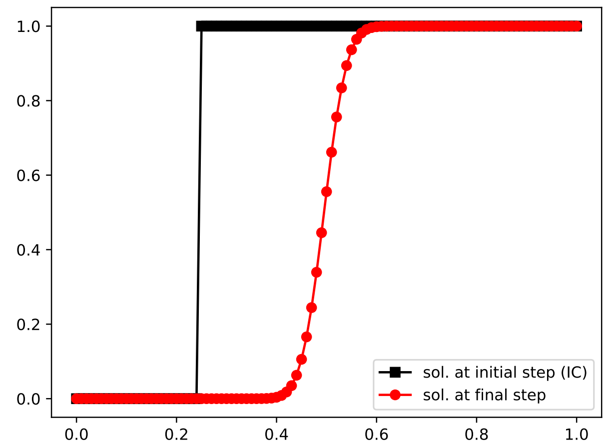
   Find the contribution of the (only) term shown above to the tangent/LHS matrix $\frac{\partial G_A}{\partial \hat{\phi}_B^{n+1}}$ for the fully discrete form based on the following two time integration schemes:

   Let $\bar{u} = \sum N_i\hat{\phi}_i(t)$ and there is a need to compute,

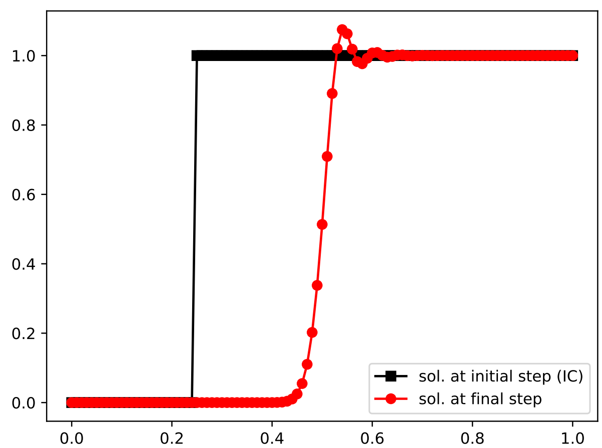
$$\begin{aligned}
\frac{\partial \mathcal{G}_A}{\partial \hat{\phi}_B^{n+1}} &= \ldots + \frac{\partial\left(N_{A,x}\bar{u}\tau\bar{u}_{,t}\right)}{\partial \hat{\phi}_B^{n+1}} + \ldots + \ldots \\
&= \ldots + N_{A,x}\frac{\partial \bar{u}}{\partial \hat{\phi}_B^{n+1}}\tau\bar{u}_{,t} + N_{A,x}\bar{u}\tau\frac{\partial \sum N_i\,\dot{\hat{\phi}}_i}{\partial \hat{\phi}_B^{n+1}}\ldots + \ldots \\
&= \ldots + N_{A,x}\frac{\partial \sum N_i\hat{\phi}_i}{\partial \hat{\phi}_B^{n+1}}\tau\bar{u}_{,t} + \frac{1}{\Delta t}N_{A,x}\bar{u}\tau\frac{\partial \sum N_i\left(\hat{\phi}_i^{n+1} - \hat{\phi}_i^{n}\right)}{\partial \hat{\phi}_B^{n+1}}\ldots + \ldots
\end{aligned}$$



a. $N_t = 10$



b. $N_t = 50$



c. $N_t = 250$

Figure 1: Solutions for different N_t

(a) Forward Euler

$$= \frac{1}{\Delta t} N_{A,x} \bar{u}^n \tau N_B$$

(b) Backward Euler

$$= N_{A,x} N_B \tau \bar{u}_{,t} + \frac{1}{\Delta t} N_{A,x} \bar{u}^{n+1} \tau N_B$$