MATH-6840
Vignesh Ramakrishnan
RIN: 662028006
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Problem Set 4

NPDE is the textbook *Numerical Partial Differential Equations*. Submissions are due in the LMS, and must be typeset (e.g. LaTeX).

1. (0 pts.) Rework PS 3, number 4 using the backward Euler time integrator

$$D_{+t}v_j^n = \nu D_{+x}D_{-x}v_j^{n+1} + f_j^{n+1}$$

Notice now that you can take a large time step (e.g. $\nu\Delta t/\Delta x$ fixed), but in doing so the observed temporal accuracy is only $O(\Delta t)$. For reference refer to Section 2.6 in the text.

The following discretization scheme is expanded and written as

$$v_j^{n+1} = v_j^n + r\left(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}\right) + \Delta t f_j^{n+1}$$

This can be re-written in the form,

$$-rv_{j-1}^{n+1} + (1+2r)\,v_j^{n+1} - rv_{j+1}^{n+1} = v_j^n + \Delta t f_j^{n+1}$$

Now, to deal with the boundary conditions, I chose to use Implicit compatibility condition for the Dirchlet BC specified on the left boundary and an Implicit Neumann for the BC specified on the right boundary.

$$v_{-1}^{n+1} - 2v_0^{n+1} + v_1^{n+1} = \Delta x^2\frac{\gamma_L^{'}(t_n)}{\nu}$$
$$v_{N+1}^{n+1} - v_{N-1}^{n+1} = 2\Delta x\gamma_R(t_n)$$

Listing 1: Heat Equation - Order 4 (Q.3b)

```
function [err_norm,x,uhat,u_ex,A] = HeatEqn_ImplicitForcing(N,r,xlim1,xlim2,...
                                                           tlim1,tlim2)
% $Author: Vignesh Ramakrishnan$
% $RIN: 662028006$
% u_t - \nu u_{xx} = f(x,t)
% s.t u(x,0) = u_0(x)
% u(0,t)     = \Gamma_L(t)
% u_x(1,t)   = \Gamma_R(t)
% u_{ex} = 2\cos(x)\cos(t)
% This function is a method to prove that the Difference methods work and
% will be a good approximate to the exact solution. The task is to find
% functions f(x,t), u_0(x), Gamma_L(t) and Gamma_R(t), plug it in
% and solve using the scheme: D+t v^n_j = \nu D+xD-xv^{n+1}_j + f^{n+1}_j
% Inputs: N          - Number of elements
%         r          - CFL number
```

```
%           xlim1    - left end of the spatial boundary
%           xlim2    - right end of the spatial boundary
%           tlim1    - start time of simulation
%           tmin2    - end time of simulation
% Output: err_norm - L2 norm of the error between the exact solution and
%                    numerical solution

nu   = 1;                                   % Co-efficient of heat conduction

u0   = @(x) 2*cos(x);                       % Initial condition
gL   = @(t) 2*cos(t);                       % Drichlet BC on Left Boundary
gR   = @(t) -2*sin(1)*cos(t);               % Neumann BC on Right Boundary
u0t  = @(t) -2*sin(t);                      % u_t @ x=0
f    = @(x,t) 2*cos(x)*(cos(t)-sin(t));     % Forcing function f

dx   = (xlim2-xlim1)/N;                     % dx -spatial discretization

dt     = r*dx^2;                            % r = CFL number

ng   = 1   ;                                % number of ghost points at BC
NP   = N+1+2*ng;                            % Total number of spatial points
ja   = ng+1;                                % xlim1's index number
jb   = NP-ng;                               % xlim2's index number

x    = (xlim1:dx:xlim2);                    % Spatial locations  x
t    = (tlim1:dt:tlim2);                    % Temporal locations  t

u_prev   = zeros(NP,1);                     % Solution at previous tstep
u_curr   = zeros(NP,1);                     % Solution at current tstep

% set initial conditions for the spatial grid
u_prev(ja:jb)      = u0(x);
u_prev(ja)         = gL(tlim1);

% Set Compatability boundary condition
u_prev(ng)         = (u0t(tlim1) - f(xlim1,tlim1))*dx^2 ...
                           + 2*u_prev(ja) - u_prev(ja+1);
% Set Neumann boundary condition
u_prev(NP)         = u_prev(jb-1) + 2*gR(tlim1)*dx;

% create Matrix A
A = zeros(NP);

for j = ng:NP
    if (j==ng)
        A(j,ng)    = 1;
        A(j,ja)    = -2;
        A(j,ja+1)  = 1;
```

```
    elseif (j==NP)
        A(j,jb-1) = -1;
        A(j,NP)   = 1;
    else
        A(j,j-1)  = -r;
        A(j,j)    = 1+2*r;
        A(j,j+1)  = -r;
    end
end

RHS = zeros(NP,1);

% find implicit solution each time step
for i=2:length(t)
    for j=ng:NP
        if j==ng
            RHS(j) = dx^2*u0t(t(i))/nu;
        elseif j==NP
            RHS(j) = 2*dx*gR(t(i));
        else
            RHS(j) = u_prev(j) + f(x(j-1),t(i))*dt;
        end
    end
    u_curr = A\RHS;
end

u_ex = 2*cos(x)*cos(tlim2);
uhat = u_curr(ja:jb);
err      = u_ex - u_prev(ja:jb);
err_norm = norm(err);

end
```

In Fig 1, $r = 0.6$ which should be unstable in the previous discretization scheme, but it produces results in this Implicit scheme but the error varies linearly with time step.

2. (10 pts.) Prove that the Lax-Friedrichs scheme

$$v_j^{n+1} = \frac{1}{2}(v_{j+1}^n + v_{j-1}^n) - \frac{R}{2}(v_{j+1}^n - v_{j-1}^n)$$

is convergent in the max-norm to the solution of the PDE $u_t + au_x = 0$ for $|R| \leq 1$ with $R = a\Delta t/\Delta x$.

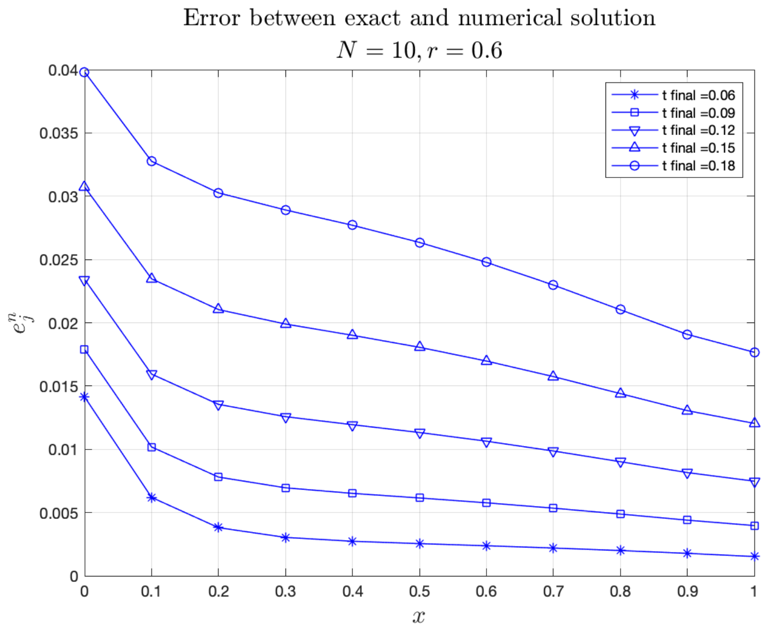

Figure 1: Error between exact and numerical solution

$$v_j^{n+1} = \left(\frac{1-R}{2}\right) v_{j+1}^n + \left(\frac{1+R}{2}\right) v_{j-1}^n$$

Let, $e_j^n = v_j^n + u_j^n$

(Assume u is smooth and $u_j^n$is a discrete value of u in the spatial grid.)

$$e_j^{n+1} = \left(\frac{1-R}{2}\right) e_{j+1}^n + \left(\frac{1+R}{2}\right) e_{j-1}^n - \left\{u_j^{n+1} - \left(\frac{1-R}{2}\right) u_{j+1}^n - \left(\frac{1+R}{2}\right) u_{j-1}^n\right\}$$

We can compute the truncation error as follows,

$$u_j^{n+1} = \left[u + \Delta t u_t + \frac{\Delta t^2}{2!} u_{tt} + \frac{\Delta t^3}{3!} u_{ttt} + \mathcal{O}(\Delta t^4)\right]_j^n$$

$$u_{j+1}^n = \left[u + \Delta x u_x + \frac{\Delta x^2}{2!} u_{xx} + \frac{\Delta x^3}{3!} u_{xxx} + \mathcal{O}(\Delta x^4)\right]_j^n$$

$$u_{j-1}^n = \left[u - \Delta x u_x + \frac{\Delta x^2}{2!} u_{xx} - \frac{\Delta x^3}{3!} u_{xxx} + \mathcal{O}(\Delta x^4)\right]_n^j$$

$$
\begin{aligned}
u_j^{n+1} - \left(\frac{1-R}{2}\right) u_{j+1}^n - \left(\frac{1+R}{2}\right) u_{j-1}^n = \\
&[\Delta t u_t + \frac{\Delta t^2}{2!} u_{tt} + \mathcal{O}(\Delta t^3) + \\
&R\Delta x u_x + R\frac{\Delta x^3}{3!} u_{xxx} + \dots \dots \\
&- \frac{\Delta x^2}{2!} u_{xx} - \frac{\Delta x^4}{4!} u_{xxxx} + \mathcal{O}(\Delta x^6)]_j^n
\end{aligned}
$$

Combining the terms in blue, we get,

$$(\Delta t \ u_t + R\Delta x \ u_x) = \Delta t \left(u_t + a u_x\right) = 0$$

Now, combining this we get the truncation error to be,

$$u_j^{n+1} - \left(\frac{1-R}{2}\right) u_{j+1}^n - \left(\frac{1+R}{2}\right) u_{j-1}^n = \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta t \Delta x^2) + \mathcal{O}(\Delta x^2) \approx A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right)$$

$$\tau_j^n = A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right)$$

The error equation becomes,

$$
\begin{aligned}
e_j^{n+1} &= \left(\frac{1-R}{2}\right) e_{j+1}^n + \left(\frac{1+R}{2}\right) e_{j-1}^n + \tau_j^n \\
e_j^{n+1} &= \left(\frac{1-R}{2}\right) e_{j+1}^n + \left(\frac{1+R}{2}\right) e_{j-1}^n + A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right)
\end{aligned}
$$

Take the max norm of the above equation (absolute value) and set $E^n = \max\limits_j e_j^n$

$$
\begin{aligned}
\text{If, } k = 1 - R = 1 + (-R) \ \text{ and, } 0 \le (|R| = m^2) \le 1 \\
0 \le \left|\frac{1-R}{2}\right| \le \frac{1}{2}\left(|1| + |-R|\right) \le 1 \\
\text{Similarly, } 0 \le \left|\frac{1+R}{2}\right| \le \frac{1}{2}\left(|1| + |R|\right) \le 1
\end{aligned}
$$

$$\left|e_j^{n+1}\right| \le \left(\frac{1-R}{2}\right)\left|e_{j+1}^n\right| + \left(\frac{1+R}{2}\right)\left|e_{j-1}^n\right| + A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right)$$

$$
\begin{aligned}
E^{n+1} &\le E^n + A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right) \\
&\le E^{n-1} + 2A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right) \\
&\le E^{n-2} + 3A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right) \\
&\ . \\
&\ . \\
&\le E^0 + (n+1)\, A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right) \\
&\le \ (n+1)\, A\left(\Delta t^2 + \Delta t \Delta x^2 + \Delta x^2\right), \ \text{as } E^0 = 0 \\
&\le (n+1)\, A\left(\frac{R^2}{a^2}\Delta x^2 + \frac{R}{a}\Delta x^3 + \Delta x^2\right) \to 0, \ \text{as, } \Delta x \to 0
\end{aligned}
$$

Hence, it is convergent.

3. (10 pts.) Prove that the scheme

$$D_{+t}v_j^n = \nu D_{+x}D_{-x}v_j^n + aD_{0x}v_j^n$$

is convergent in the max-norm to the solution of the PDE $u_t = \nu u_{xx} + au_x$, under certain constraints on Δx and Δt. What are these constraints? Use the notation $r = \nu\Delta t/\Delta x^2$ and $\sigma = a\Delta t/\Delta x$.

$$\begin{aligned}
\frac{v_j^{n+1} - v_j^n}{\Delta t} &= \frac{\nu}{\Delta x^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) + \frac{a}{2\Delta x}\left(v_{j+1}^n - v_{j-1}^n\right) \\
v_{j+1}^n &= v_j^n + r\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) + \frac{\sigma}{2}\left(v_{j+1}^n - v_{j-1}^n\right) \\
v_{j+1}^n &= \left(r - \frac{\sigma}{2}\right)v_{j-1}^n + (1-2r)\,v_j^n + \left(r + \frac{\sigma}{2}\right)v_{j+1}^n \\
v_{j+1}^n &= (1-2r)\,v_j^n + r\left[v_{j+1}^n + v_{j-1}^n\right] + \frac{\sigma}{2}\left(v_{j+1}^n - v_{j-1}^n\right)
\end{aligned}$$

Performing a DFT,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi}v_j^{n+1} &= \hat{V}^{n+1} \\
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi}v_j^{n} &= \hat{V}^{n} \\
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-i(j)\xi}v_{j+1}^{n} &= \frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-i(m-1)\xi}v_m^n = e^{i\xi}\ \hat{V}^n \\
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi}v_{j-1}^{n} &= e^{-i\xi}\hat{V}^{n}
\end{aligned}$$

Substituting it into the equation,

$$\begin{aligned}
\hat{V}^{n+1} &= (1-2r)\,\hat{V}^n + r\left[e^{i\xi} + e^{-i\xi}\right]\hat{V}^n + \frac{\sigma}{2}\left[e^{i\xi} - e^{-i\xi}\right]\hat{V}^n \\
\hat{V}^{n+1} &= (1 - 2r + 2r\cos\xi + i\sigma\sin\xi)\,\hat{V}^n \\
\hat{V}^{n+1} &= (1 - 2r\,(1-\cos\xi) + i\sigma\sin\xi)\,\hat{V}^n \\
|a(\xi)|^2 &= (1 - 2r\,(1-\cos\xi))^2 + (\sigma\sin\xi)^2 \le 1 \\
\implies |a(\xi)|^2 &= 1 + 4r^2\,(1-\cos\xi)^2 - 4r\,(1-\cos\xi) + \sigma^2\left(1-\cos^2\xi\right) \le 1
\end{aligned}$$

Let $z = \cos\xi$

$$4r^2\,(1-z) - 4r + \sigma^2\,(1+z) \le 0$$

If $z = -1$, $r \le \frac{1}{2}$.
If $z = 1$, $\sigma^2 \le 2r \le 1$.
Therefore, the 2 constraints are,

$$\Delta x^2 - a^2\Delta t^2 \ge 0 \tag{1}$$

$$\Delta x^2 - 2\nu\Delta t \ge 0 \tag{2}$$

4. (15 pts.) Determine the formal order of accuracy of the following difference equations (i.e. investigate the truncation error). Throughout, use the definitions $r = \nu\Delta t/\Delta x^2$ and $\sigma = a\Delta t/\Delta x$.

(a) For the PDE $u_t = \nu u_{xx} - a u_x$ consider the discretization

$$D_{+t}v_j^n = \nu D_{+x}D_{-x}v_j^n - aD_{0x}v_j^n.$$

Let $e_j^n = v_j^n - u_j^n$,

$$D_{+t}\left(e_j^n + u_j^n\right) = \nu D_{+x}D_{-x}\left(e_j^n + u_j^n\right) - aD_{0x}\left(e_j^n + u_j^n\right)$$
$$D_{+t}e_j^n = \nu D_{+x}D_{-x}e_j^n - aD_{0x}e_j^n - \left(D_{+t}u_j^n - \nu D_{+x}D_{-x}u_j^n + aD_{0x}u_j^n\right)$$

$$D_{+t}u_j^n = \left[u_t + \frac{\Delta t}{2}u_{tt} + \mathcal{O}(\Delta t^2)\right]_j^n = \left[u_t + \mathcal{O}(\Delta t)\right]_j^n$$
$$D_{+x}D_{-x}u_j^n = \left[u_{xx} + \frac{\Delta x^2}{12}u_{xxxx} + \mathcal{O}(\Delta x^4)\right]_j^n = \left[u_{xx} + \mathcal{O}(\Delta x^2)\right]_j^n$$
$$D_{0x}u_j^n = \left[u_x + \frac{\Delta x^2}{3!}u_{xxx} + \mathcal{O}(\Delta x^4)\right]_j^n = \left[u_x + \mathcal{O}(\Delta x^2)\right]_j^n$$

$$D_{+t}e_j^n = \nu D_{+x}D_{-x}e_j^n - aD_{0x}e_j^n - \left(\left[u_t - \nu u_{xx} + a u_x + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)\right]_j^n\right)$$

The terms in blue sum up to 0 since it is the actual PDE.
Therefore, truncation error is,

$$\tau_j^n = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$$

(b) For the PDE $u_t = \nu u_{xx} - a u_x$ consider the discretization

$$D_{+t}v_j^n = \nu D_{+x}D_{-x}v_j^{n+1} - aD_{0x}v_j^{n+1}.$$

Similar to last problem,

$$D_{+t}e_j^n = \nu D_{+x}D_{-x}e_j^{n+1} - aD_{0x}e_j^{n+1} - \left(D_{+t}u_j^n - \nu D_{+x}D_{-x}u_j^{n+1} + aD_{0x}u_j^{n+1}\right)$$

$$D_{+t}u_j^n = \left[u_t + \frac{\Delta t}{2}u_{tt} + \mathcal{O}(\Delta t^2)\right]_j^n = \left[u_t + \mathcal{O}(\Delta t)\right]_j^n$$
$$\begin{aligned}D_{+x}D_{-x}u_j^{n+1} &= \left[u_{xx} + \frac{\Delta x^2}{12}u_{xxxx} + \mathcal{O}(\Delta x^4)\right]_j^{n+1} \\ &= \left[u_{xx}\right]_j^{n+1} + \frac{\Delta x^2}{12}\left[u_{xxxx}\right]_j^{n+1} + \left[\mathcal{O}(\Delta x^4)\right]_j^{n+1} \\ &= \left[u_{xx} + \Delta t\ u_{txx} + \mathcal{O}(\Delta t^2)\right]_j^n + \frac{\Delta x^2}{12}\left[u_{xxxx} + \Delta t\ u_{txxxx} + \mathcal{O}(\Delta t^2)\right]_j^n + +\mathcal{O}(\Delta x^4)\end{aligned}$$
$$\begin{aligned}D_{0x}u_j^{n+1} &= \left[u_x + \frac{\Delta x^2}{3!}u_{xxx} + \mathcal{O}(\Delta x^4)\right]_j^{n+1} \\ &= \left[u_x + \Delta t\ u_{tx} + \mathcal{O}(\Delta t^2)\right]_j^n + \frac{\Delta x^2}{3!}\left[u_{xxx} + u_{txxx} + \mathcal{O}(\Delta t^2))\right]_j^n + \mathcal{O}(\Delta x^4)\end{aligned}$$

$$D_{+t}u_j^n - \nu D_{+x}D_{-x}u_j^{n+1} + aD_{0x}u_j^{n+1} = \left[u_t - \nu u_{xx} + au_x + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)\right]_j^n$$
$$\tau_j^n = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$$

(c) For the PDE $u_t = -au_x$ consider the discretization

$$D_{+t}v_j^n = -aD_{-x}v_j^{n+1}.$$

Similar to the previous problem,

$$D_{+t}e_j^n = -\,aD_{-x}e_j^{n+1} - \left(D_{+t}u_j^n + aD_{-x}u_j^{n+1}\right)$$

$$\begin{aligned}
D_{+t}u_j^n &= \left[u_t + \frac{\Delta t}{2}u_{tt} + \mathcal{O}(\Delta t^2)\right]_j^n = \left[u_t + \mathcal{O}(\Delta t)\right]_j^n \\
D_{-x}u_j^{n+1} &= \left[u_x - \frac{\Delta x^2}{2!}u_{xx} + \frac{\Delta x^3}{3!}u_{xxx} + \mathcal{O}(\Delta x^4)\right]_j^{n+1} \\
&= \left[u_x + \Delta t u_{tx} + \mathcal{O}(\Delta t^2)\right]_j^n - \frac{\Delta x^2}{2!}\left[u_{xx} + \Delta t u_{txx} + \mathcal{O}(\Delta t^2)\right]_j^n + \mathcal{O}(\Delta x^4)
\end{aligned}$$

$$\tau_j^n = \left[u_t + au_x + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)\right]_j^n = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$$

5. (15 pts.)Investigate the stability of the schemes from number 4 above, and discuss any limitations on parameters that you find are required to guarantee stability. Again use the definitions $r = \nu\Delta t/\Delta x^2$ and $\sigma = a\Delta t/\Delta x$. Hint: using the DFT is probably simplest.

I'm going to assume that the coefficients ν, a are positive in all three cases considered. So if, $\Delta t, \Delta x$ are set to be positive numbers, the parameters r, σ are positive constants.

(a) For the PDE $u_t = \nu u_{xx} - au_x$, consider the discretization

$$D_{+t}v_j^n = \nu D_{+x}D_{-x}v_j^n - aD_{0x}v_j^n$$

$$\begin{aligned}
\frac{v_j^{n+1} - v_j^n}{\Delta t} &= \frac{\nu}{\Delta x^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) - \frac{a}{2\Delta x}\left(v_{j+1}^n - v_{j-1}^n\right) \\
v_{j+1}^n &= v_j^n + r\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) - \frac{\sigma}{2}\left(v_{j+1}^n - v_{j-1}^n\right) \\
v_{j+1}^n &= \left(r + \frac{\sigma}{2}\right)v_{j-1}^n + (1 - 2r)\,v_j^n + \left(r - \frac{\sigma}{2}\right)v_{j+1}^n \\
v_{j+1}^n &= (1 - 2r)\,v_j^n + r\left[v_{j+1}^n + v_{j-1}^n\right] - \frac{\sigma}{2}\left(v_{j+1}^n - v_{j-1}^n\right)
\end{aligned}$$

Performing a DFT,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} v_j^{n+1} =& \hat{V}^{n+1} \\
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} v_j^{n} =& \hat{V}^{n} \\
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-i(j)\xi} v_{j+1}^{n} =& \frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-i(m-1)\xi} v_m^{n} = e^{i\xi}\ \hat{V}^{n} \\
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} v_{j-1}^{n} =& e^{-i\xi}\hat{V}^{n}
\end{aligned}$$

Substituting it into the equation,

$$\begin{aligned}
\hat{V}^{n+1} =& \ (1-2r)\hat{V}^n + r\left[e^{i\xi}+e^{-i\xi}\right]\hat{V}^n - \frac{\sigma}{2}\left[e^{i\xi}-e^{-i\xi}\right]\hat{V}^n \\
\hat{V}^{n+1} =& \ (1-2r+2r\cos\xi - i\sigma\sin\xi)\hat{V}^n \\
\hat{V}^{n+1} =& \ (1-2r(1-\cos\xi) - i\sigma\sin\xi)\hat{V}^n \\
|a(\xi)|^2 =& (1-2r(1-\cos\xi))^2 + (\sigma\sin\xi)^2 \le 1 \\
\implies |a(\xi)|^2 =& 1+4r^2(1-\cos\xi)^2 - 4r(1-\cos\xi) + \sigma^2\left(1-\cos^2\xi\right) \le 1
\end{aligned}$$

Let $z=\cos\xi$

$$4r^2(1-z) - 4r + \sigma^2(1+z) \le 0$$

If $z=-1$, $r\le\frac{1}{2}$.
If $z=1$, $\sigma^2\le 2r\le 1$.
If these conditions are met, then the scheme is stable.

(b) For the PDE $u_t=\nu u_{xx}-au_x$, consider the discretization

$$D_{+t}v_j^n = \nu D_{+x}D_{-x}v_j^{n+1} - aD_{0x}v_j^{n+1}$$

$$v_j^{n+1}-v_j^n = \left(r-\frac{\sigma}{2}\right)v_{j+1}^{n+1} - 2rv_j^{n+1} + \left(r+\frac{\sigma}{2}\right)v_{j-1}^{n+1}$$

Taking a DFT,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} v_{j+1}^{n+1} =& \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} e^{-i(m-1)\xi} v_m^{n+1} \equiv e^{i\xi}\hat{V}^{n+1} \\
\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{-ij\xi} v_{j-1}^{n+1} =& \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} e^{-i(m+1)\xi} v_m^{n+1} \equiv e^{-i\xi}\hat{V}^{n+1} \\
\cos\xi =& \frac{e^{i\xi}+e^{-i\xi}}{2} \\
\sin\xi =& \frac{e^{i\xi}-e^{-i\xi}}{2i}
\end{aligned}$$

$$\begin{aligned}
\hat{V}^{n+1} - \hat{V}^n &= \left(r - \frac{\sigma}{2}\right) e^{i\xi} \hat{V}^{n+1} - 2r\hat{V}^{n+1} + \left(r + \frac{\sigma}{2}\right) e^{-i\xi} \hat{V}^{n+1} \\
\hat{V}^{n+1} &= \frac{1}{[1 + 2r(1 - \cos\xi) + i\sigma \sin\xi]} \hat{V}^n
\end{aligned}$$

$$\begin{aligned}
\implies a(\xi) &= \frac{[1 + 2r(1 - \cos\xi) - i\sigma \sin\xi]}{(1 + 2r(1 - \cos\xi))^2 + \sigma^2 \sin^2\xi} \\
|a(\xi)|^2 &= \frac{1}{(1 + 2r(1 - \cos\xi))^2 + \sigma^2 \sin^2\xi}
\end{aligned}$$

If $|a(\xi)| \leq 1$, then this is a stable scheme.

$$\begin{aligned}
&\text{If, } \cos\xi = -1, \\
&\quad 8r^2 + 4r \geq 0 \\
&\quad 4r(2r + 1) \geq 0
\end{aligned}$$

This is always true since we assumed $r \geq 0$

$$\begin{aligned}
&\text{If, } \cos\xi = 1, \\
&\quad 4r + 2\sigma^2 \geq 0
\end{aligned}$$

This is also true since $r \geq 0$ and $\sigma \geq 0$. Hence this is a stable scheme as $|a(\xi)| \leq 1$ always.

(c) For the PDE $u_t = -au_x$ consider the discretization

$$D_{+t} v_j^n = -aD_{-x} v_j^{n+1}.$$

$$\begin{aligned}
v_j^{n+1} - v_j^n &= -\sigma v_j^{n+1} + \sigma v_{j-1}^{n+1} \\
\hat{V}^{n+1} &= \frac{1}{1 + \sigma\left(1 - e^{-i\xi}\right)} \hat{V}^n \\
e^{-i\xi} &= \cos\xi - i\sin\xi \\
\implies |a(\xi)|^2 &= \frac{1}{(1 + \sigma(1 - \cos\xi))^2 + \sigma^2(1 - \cos^2\xi)}
\end{aligned}$$

The denominator is positive and always greater than 1, this stable is always stable since $|a(\xi)| \leq 1$