MATH-6840
J. W. Banks
Due: Tuesday February 22, 2022

Problem Set 5

1. (15 pts.) Consider the IBVP

$$
\begin{aligned}
u_t &= \nu u_{xx}, & x &\in (0,1), & 0 < t \le T_f \\
u(x,0) &= f(x), & x &\in (0,1) \\
u_x(0,t) &= \alpha(t), & u(1,t) &= \beta(t), & t \ge 0.
\end{aligned}
$$

Using the grid $x_j = j\Delta x$, $j = -1, 0, \dots, N+1$, $\Delta x = 1/N$, apply the following discretization

$$
\begin{aligned}
D_{+t} v_j^n &= \nu D_{+x} D_{-x}\left(\theta v_j^{n+1} + (1-\theta) v_j^n\right), && \text{for } j = 0, 1, \dots, N,\ n = 1, 2, \dots \\
v_j^0 &= f(x_j) && \text{for } j = 0, 1, \dots, N \\
D_{0x} v_0^n &= \alpha(t_n) && \text{for } n = 0, 1, \dots \\
\nu D_{+x} D_{-x} v_N^n &= \beta^{'}(t_n) && \text{for } n = 0, 1, \dots,
\end{aligned}
$$

where $\theta \in [0,1]$ is a parameter (note $\theta = 1$ corresponds to backward Euler, $\theta = \frac{1}{2}$ corresponds to the trapezoidal rule, and $\theta = 0$ corresponds to forward Euler.

   (a) Determine the order-of-accuracy (consistency) of the scheme including both the interior discretization and the boundary conditions.

   (b) Using normal mode stability theory, determine the stability of the scheme taking account of the boundary conditions. Please use the notation that $r = \frac{\nu\Delta t}{\Delta x^2}$. Hint: for stability you need only consider the error equation so that the boundary conditions can be taken as homogeneous.

   (c) Based on the above, how do you expect the scheme converge with respect to grid parameters? Why?

2. (20 pts.) Here you will take steps to implement the discretization described in #1.

   (a) Carefully write down the $N+3$ linear equations that must be solved at each time step. Present this linear system.

   (b) Now implement the scheme in code using the solution $u_{ex} = e^{-\nu k^2 t}\sin(kx)$, from which you must determine $f(x)$, $\alpha(t)$, and $\beta(t)$. Note that much of the infrastructure can be adopted from the solution to PS #3 problem #4.

   (c) Taking $\nu = 1$, $k = 2$ and $T_f = .4$, perform a convergence study with $N = 20, 40, 80, 160$ using $\theta = 1$ and $r = \frac{\nu\Delta t}{\Delta x} \approx 0.9$ (as usual, the time step may be slightly modified so the simulation actually attains the final time). Present plots of the solution and plots of the error at the final time for each grid resolution. Also present a log-log plot of the maximum error vs. the grid size, as well as a reference line indicating the expected convergence rate.

   (d) Taking $\nu = 1$, $k = 2$ and $T_f = .4$, perform a convergence study with $N = 20, 40, 80, 160$ using $\theta = \frac{1}{2}$ and $r = \frac{\nu\Delta t}{\Delta x} \approx 0.9$ (as usual, the time step may be slightly modified so the simulation actually attains the final time). Present plots of the solution and plots of the error at the final time for each grid resolution. Also present a log-log plot of the maximum error vs. the grid size, as well as a reference line indicating the expected convergence rate.

3. (10 pts.) In HW #2 problem #2, we investigated the leapfrog scheme with a centered spatial discretization for the heat equation and experienced some difficulty in computing solutions.

   (a) Determine the amplification factor of the discrete operator, and make surface plots of the amplitude of the amplification factor as a function of discrete wave number and the parameter $r = \frac{\nu \Delta t}{\Delta x^2}$. Note there are two roots and you should produce one plot for each root.

   (b) Determine if the scheme is stable for any choice of the parameter $r > 0$ (hint: you may find it useful to use the plots from (a) as a guide). How do these results help to explain the behavior that we experienced in HW #2 problem #2.