# GLM With A Quadratic Basis

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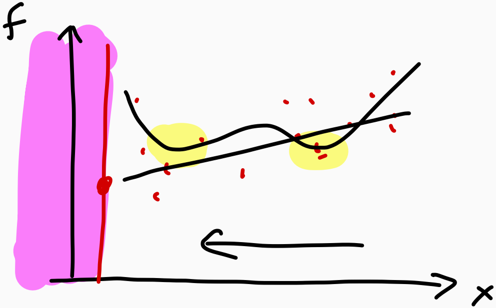
GLMs with linear basis are (perhaps) suitable for approximating the constraints

$$\hat{c}(x, \alpha) \approx c(x).$$

However, a linear surrogate may be less appropriate for the objective



# A Quadratic Basis Is Better Suited For Modeling the Objective

**Definition: GLM with a quadratic basis**

A generalized linear model with a quadratic basis takes the form

$$\hat{f}(x, \alpha) = \alpha_0 + \sum_{k=1}^{n} \alpha_k x_k + \sum_{m=1}^{n} \sum_{k=m}^{n} \alpha_{n-1+m+k} x_m x_k$$

- Note that $\hat{f}$ is linear in $\alpha$
- The parameter vector $\alpha$ has

  $\frac{(n+1)(n+2)}{2}$ elements

# The Vandermonde Matrix Needs To Reflect The Quadratic Basis

Suppose we are interested in a least-squares fit using the quadratic GLM. Then, the residual vector is given by

$$R(\alpha) = V\alpha - y \neq 0.$$

where $y$ is defined as before (for linear regression) but

$$V = \underbrace{\left.\begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & \cdots & x_n^{(1)} & (x_1^{(1)})^2 & \cdots & (x_n^{(1)})^2 \\ 1 & x_1^{(2)} & x_2^{(2)} & \cdots & x_n^{(2)} & (x_1^{(2)})^2 & \cdots & (x_n^{(2)})^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 1 & x_1^{(s)} & x_2^{(s)} & \cdots & x_n^{(s)} & (x_1^{(s)})^2 & \cdots & (x_n^{(s)})^2 \end{bmatrix}\right\}\text{each sample gets a row}}_{\text{each basis function gets a column}}$$

# Least-Squares Parameter Estimation for Polynomial GLMs

**Definition: Least-squares $\alpha$ in polynomial GLM**

The parameters for a GLM using a polynomial basis can be determined by solving the overdetermined system

$$V\alpha = y \qquad R(\alpha) = V\alpha - y$$

where $V$ is the Vandermonde matrix corresponding to the basis functions evaluated at the sample points.

# Example

For the following data points, find the GLM with a quadratic basis using least-squares regression.

$$\begin{bmatrix} x^{(1)} & x^{(2)} & x^{(3)} & x^{(4)} \\ f^{(1)} & f^{(2)} & f^{(3)} & f^{(4)} \end{bmatrix} = \begin{bmatrix} 0.55 & 0.76 & 0.48 & 0.51 \\ 0.45 & 0.99 & 0.99 & 0.64 \end{bmatrix}$$

$$\alpha = (V^T V)^{-1} V^T y$$

$$V = \begin{bmatrix} 1 & x^{(1)} & (x^{(1)})^2 \\ 1 & x^{(2)} & (x^{(2)})^2 \\ 1 & x^{(3)} & (x^{(3)})^2 \\ 1 & x^{(4)} & (x^{(4)})^2 \end{bmatrix}, \quad y = \begin{bmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \\ f^{(4)} \end{bmatrix} \qquad \alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$