MANE 6760 - FEM for Fluid Dyn. - Lecture 23

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Transient Non-Linear System of Equations: Navier-Stokes

Conservative variables:
$\boldsymbol{\mathcal{U}} = [\rho, \rho u_1, \ldots, \rho e_{tot}]^T$

$$\tilde{\mathcal{L}}(\boldsymbol{\mathcal{U}}) = \frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \nabla \cdot \left(\tilde{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{U}} - \tilde{\boldsymbol{\mathcal{K}}} \nabla \boldsymbol{\mathcal{U}} \right) = \boldsymbol{S}$$

Primitive variables:
$\boldsymbol{Y} = [p, u_1, \ldots, T]^T$ (pressure-primitive variables) or
$\boldsymbol{Y} = [\rho, u_1, \ldots, T]^T$ (density-primitive variables)

$$\mathcal{L}(\boldsymbol{Y}) = \boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{Y}}{\partial t} + \nabla \cdot (\boldsymbol{\mathcal{A}} \boldsymbol{Y} - \boldsymbol{\mathcal{K}} \nabla \boldsymbol{Y}) = \boldsymbol{S}$$

Entropy variables:
$\boldsymbol{V} = [\vartheta_1, \vartheta_2, \ldots, \vartheta_{n_{sd}+2}]^T$

$$\hat{\mathcal{L}}(\boldsymbol{V}) = \hat{\boldsymbol{\mathcal{A}}}_0 \frac{\partial \boldsymbol{V}}{\partial t} + \nabla \cdot \left(\hat{\boldsymbol{\mathcal{A}}} \boldsymbol{V} - \hat{\boldsymbol{\mathcal{K}}} \nabla \boldsymbol{V} \right) = \boldsymbol{S}$$

Compressible Navier-Stokes: Conservative Variables

Conservative variables: $\boldsymbol{\mathcal{U}}$

$e = f(\mathcal{U}) = ?$

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \nabla \cdot (\boldsymbol{F}) = \boldsymbol{S}$$

$F_{i,i} = \boxed{A_i} \mathcal{U}_{,i}$

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \nabla \cdot \left(\tilde{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{U}} - \tilde{\boldsymbol{\mathcal{K}}} \nabla \boldsymbol{\mathcal{U}} \right) = \boldsymbol{S}$$

$F_{i,\mathcal{U}}$

$$\mathcal{U}_{l,t} + \left((\tilde{\mathcal{A}}_{lm})_i \mathcal{U}_m - (\tilde{\mathcal{K}}_{lm})_{ij} \mathcal{U}_{m,j} \right)_{,i} = S_l$$

$\mathcal{U}_1 =$

$\mathcal{U}_2 =$

F_i^c $\quad F_i^p$ $\quad F_i^d$

$$\boldsymbol{\mathcal{U}} = \left\{ \begin{array}{c} \rho \\ \rho u_1 \\ \vdots \\ \rho(e + \boldsymbol{u} \cdot \boldsymbol{u}/2) \end{array} \right\} \qquad \boldsymbol{F}_i = \rho u_i \left\{ \begin{array}{c} 1 \\ u_1 \\ \vdots \\ e \end{array} \right\} + p \left\{ \begin{array}{c} 0 \\ \delta_{1i} \\ \vdots \\ u_i \end{array} \right\} - \left\{ \begin{array}{c} 0 \\ \sigma^d_{1i} \\ \vdots \\ \sigma^d_{ij} u_j - q_i \end{array} \right\}$$

Supplemented with the thermodynamic relations (equation of state): $p = p(\rho, e)$ and so on

+ constitutive laws (for visc. stresses and heat conduction)

Compressible Navier-Stokes: Stab. FE Form for Conservative Variables

$\int \bar{w}_\ell \bar{U}_{\ell,t}$

$\int \underline{\bar{w}} \cdot \underline{\bar{U}}_{,t} + \quad + \int \underline{\bar{w}}_{,i} \cdot \underline{\underline{K}}_{ij} \underline{\bar{U}}_{,j}$

Stabilized finite element form:

$(\bar{W}, \mathcal{L}(\bar{U}))$

$$B(\bar{\boldsymbol{W}},\bar{\boldsymbol{\mathcal{U}}}) + B_{stab}(\bar{\boldsymbol{W}},\bar{\boldsymbol{\mathcal{U}}}) = (\bar{\boldsymbol{W}},\boldsymbol{S})$$

$$B_{stab}(\bar{\boldsymbol{W}},\bar{\boldsymbol{\mathcal{U}}}) = \sum_e \int_{\Omega_e} \tilde{\mathcal{L}}^T_{stab}(\bar{\boldsymbol{W}}) \cdot \tilde{\tau}(\tilde{\mathcal{L}}(\bar{\boldsymbol{\mathcal{U}}}) - \boldsymbol{S}) d\Omega_e$$

$$\tilde{\mathcal{L}}^T_{stab}(\cdot) = (\cdot)_{,t} + \tilde{\boldsymbol{\mathcal{A}}}_i^T (\cdot)_{,i} - (\tilde{\boldsymbol{\mathcal{K}}}_{ij}^T (\cdot)_{,j})_{,i}$$

Compressible Navier-Stokes: Primitive Variables

Primitive variables: $\boldsymbol{Y}$

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{Y}}{\partial t} + \nabla \cdot (\boldsymbol{F}) = \boldsymbol{S}$$

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{Y}}{\partial t} + \nabla \cdot (\boldsymbol{\mathcal{A}} \boldsymbol{Y} - \boldsymbol{\mathcal{K}} \nabla \boldsymbol{Y}) = \boldsymbol{S}$$

$$(\mathcal{A}_0)_{lm} Y_{m,t} + \left((\mathcal{A}_{lm})_i Y_m - (\mathcal{K}_{lm})_{ij} Y_{m,j}\right)_{,i} = S_l$$

$$\boldsymbol{Y} = \begin{Bmatrix} p \\ u_1 \\ \vdots \\ T \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \rho \\ u_1 \\ \vdots \\ T \end{Bmatrix}$$

Supplemented with the thermodynamic relations (equation of state): $\rho = \rho(p, T)$ or $p = p(\rho, T)$, and so on

Compressible Navier-Stokes: Primitive Variables

$\boldsymbol{A}_0 = \boldsymbol{\mathcal{U}}_{,\boldsymbol{Y}}$ matrix

$\frac{\partial U_1}{\partial Y_2}$

pressure primitive

density primitive

$$\boldsymbol{A}_0 = \begin{bmatrix} \left(\frac{\partial \rho}{\partial p}\right)_T & 0 & \dots \\ u_1 \left(\frac{\partial \rho}{\partial p}\right)_T & \rho & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & \dots \\ u_1 & \rho & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$\frac{\partial U_2}{\partial Y_2}$

$\boldsymbol{A}_1 = \boldsymbol{F}^{adv}_{1,\boldsymbol{Y}}$ matrix

$$\boldsymbol{A}_1 = \begin{bmatrix} u_1 \left(\frac{\partial \rho}{\partial p}\right)_T & \rho & \dots \\ u_1^2 \left(\frac{\partial \rho}{\partial p}\right)_T + 1 & 2\rho u_1 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u_1 & \rho & \dots \\ u_1^2 + \frac{1}{\left(\frac{\partial \rho}{\partial p}\right)_T} & 2\rho u_1 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Similarly, other $\boldsymbol{A}_i$ as well as $\mathcal{K}_{ij}$ matrices are defined

Compressible Navier-Stokes: Stab. FE Form for Primitive Variables

Stabilized finite element form:

$$B(\bar{\boldsymbol{W}}, \bar{\boldsymbol{Y}}) + B_{stab}(\bar{\boldsymbol{W}}, \bar{\boldsymbol{Y}}) = (\bar{\boldsymbol{W}}, \boldsymbol{S})$$

$$B_{stab}(\bar{\boldsymbol{W}}, \bar{\boldsymbol{Y}}) = \sum_e \int_{\Omega_e} \mathcal{L}_{stab}^T(\bar{\boldsymbol{W}}) \cdot \boldsymbol{\tau}(\mathcal{L}(\bar{\boldsymbol{Y}}) - \boldsymbol{S}) d\Omega_e$$

$$\mathcal{L}_{stab}^T(\cdot) = (\cdot)_{,t} + \boldsymbol{\mathcal{A}}_i^T(\cdot)_{,i} - (\boldsymbol{\mathcal{K}}_{ij}^T(\cdot)_{,j})_{,i}$$

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