W.D. Henshaw **Math 6800: Problem Set 5** **Due:** Thursday Oct. 20, 2022.

NLA = the text-book *Numerical Linear Algebra*, by Trefethen and Bau

1. Let $A, B \in \mathbb{C}^{m\times m}$ be nonsingular and let $\kappa(A)$ be the condition number of A. Let $\|\cdot\|$ denote any induced matrix norm.

(a) Show that for any induced norm where $\|I\| \ge 1$, then $\kappa(A) \ge 1$.
(b) Show that $\kappa(AB) \le \kappa(A)\kappa(B)$ and $\kappa(\alpha A) = \kappa(A)$ for any scalar $\alpha \in \mathbb{C}$, $\alpha \neq 0$.
(c) Let $A \in \mathbb{C}^{m\times m}$ be nonsingular with SVD $A = U\Sigma V^*$. Show that if $Ax = b$ then

$$x = \sum_{i=1}^{m} \frac{u_i^* b}{\sigma_i} v_i,$$

where U has columns u_i and V has columns v_i. In which direction will perturbations in b be amplified the most?

2. Let $A \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|$ denote any induced matrix norm.
(a) Show that if $\|A\| < 1$ then

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

This shows that when $\|A\| < 1$, then $I - A$ is nonsingular. (Hint, consider $I - A$ times the partial sums $S_N = \sum_{k=0}^{N} A^k$ and let $N \to \infty$).

(b) Show that if $\|A\| < 1$, and $\|I\| = 1$ then

$$\|(I - A)^{-1}\| \le \frac{1}{1 - \|A\|}.$$

3. (a) Show that if $|\epsilon_i| \le \epsilon_{\text{machine}}$, $i = 1, 2, \ldots, n$ where $0 < \epsilon_{\text{machine}} < 1$ then

$$(1 - \epsilon_1)(1 - \epsilon_2)\cdots(1 - \epsilon_n) = (1 + \epsilon)^n$$

for some ϵ with $|\epsilon| \le \epsilon_{\text{machine}}$. (Hint: Consider the maximum and minimum values that the left and right-hand sides can take).

(b) Let $\tilde{f}(x_1, x_2, \ldots, x_n)$ denote the algorithm to sum n floating point numbers $x_i \in \mathbb{F}$, evaluated, in floating point arithmetic from left to right, that is

$$\tilde{f}(x_1, x_2, \ldots, x_n) = (\ldots((x_1 \oplus x_2) \oplus x_3) \oplus \ldots \oplus x_n).$$

Show that

$$\tilde{f}(x_1, x_2, \ldots, x_n) = x_1(1 + \epsilon_1)^{n-1} + x_2(1 + \epsilon_2)^{n-1} + +x_3(1 + \epsilon_3)^{n-2} + \ldots + x_n(1 + \epsilon_n),$$

for some ϵ_i with $|\epsilon_i| \le \epsilon_{\text{machine}}$.

4. NLA 15.1 : *Each of the following problems describes an algorithm implemented on a computer satisfying axioms (13.5) and (13.7). For each ...*