MANE 6760 - FEM for Fluid Dyn. - Lecture 24

Prof. Onkar Sahni, RPI

F23: 2nd Dec 2022

Transient Non-Linear System of Equations: Navier-Stokes

Conservative variables:
$\boldsymbol{\mathcal{U}} = [\rho, \rho u_1, \ldots, \rho e_{tot}]^T$

$$\tilde{\mathcal{L}}(\boldsymbol{\mathcal{U}}) = \frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \nabla \cdot \left(\tilde{\boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{U}})\boldsymbol{\mathcal{U}} - \tilde{\boldsymbol{\mathcal{K}}}(\boldsymbol{\mathcal{U}})\nabla\boldsymbol{\mathcal{U}} \right) = \boldsymbol{S}$$

Primitive variables:
$\boldsymbol{Y} = [p, u_1, \ldots, T]^T$ (pressure-primitive variables) or
$\boldsymbol{Y} = [\rho, u_1, \ldots, T]^T$ (density-primitive variables)

$$\mathcal{L}(\boldsymbol{Y}) = \boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{Y}}{\partial t} + \nabla \cdot (\boldsymbol{\mathcal{A}}(\boldsymbol{Y})\boldsymbol{Y} - \boldsymbol{\mathcal{K}}(\boldsymbol{Y})\nabla \boldsymbol{Y}) = \boldsymbol{S}$$

Entropy variables:
$\boldsymbol{V} = [\vartheta_1, \vartheta_2, \ldots, \vartheta_{n_{sd}+2}]^T$

$$\hat{\mathcal{L}}(\boldsymbol{V}) = \hat{\boldsymbol{\mathcal{A}}}_0 \frac{\partial \boldsymbol{V}}{\partial t} + \nabla \cdot \left(\hat{\boldsymbol{\mathcal{A}}}(\boldsymbol{V})\boldsymbol{V} - \hat{\boldsymbol{\mathcal{K}}}(\boldsymbol{V})\nabla \boldsymbol{V} \right) = \boldsymbol{S}$$

Compressible Navier-Stokes: SUPG FE Form for Entropy Variables

$\int \underline{\overline{w}} \cdot \underline{\underline{\hat{A}}}_0 \underline{\overline{V}}_{,t}$

SUPG (stabilized) finite element form for $\boldsymbol{V}$:

$$B^{SUPG}_{stab}(\bar{\boldsymbol{W}}, \bar{\boldsymbol{V}}) = \sum_e \int_{\Omega_e} \hat{\mathcal{L}}^T_{stab}(\bar{\boldsymbol{W}}) \cdot \hat{\boldsymbol{\tau}}(\hat{\mathcal{L}}(\bar{\boldsymbol{V}}) - \boldsymbol{S}) d\Omega_e$$

$$\hat{\mathcal{L}}^T_{stab}(\cdot) = \hat{\boldsymbol{\mathcal{A}}}_i^{T\,(sym)}(\bar{\boldsymbol{V}})(\cdot)_{,i} = \hat{\boldsymbol{\mathcal{A}}}_i^{(sym)}(\bar{\boldsymbol{V}})(\cdot)_{,i}$$

$= a_i (\cdot)_{,i}$

5×5   5×5   5×5   5×5

$$\hat{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}}_{\boldsymbol{V}1} = \hat{\boldsymbol{\mathcal{A}}}_0^{-1} \left((\frac{2}{\Delta t})^2 \boldsymbol{I} + \hat{\boldsymbol{\mathcal{A}}}_i g_{ij} \hat{\boldsymbol{\mathcal{A}}}_j + c^2_{diff}\, g_{hk} g_{ij} \hat{\boldsymbol{\mathcal{K}}}_{hi} \hat{\boldsymbol{\mathcal{K}}}_{jk} \right)^{-\frac{1}{2}}$$

$$\hat{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}}_{\boldsymbol{V}2} = \hat{\boldsymbol{\mathcal{A}}}_0^{-1} (\boldsymbol{B}_i \boldsymbol{B}_i)^{-\frac{1}{2}}, \qquad \boldsymbol{B}i = \xi_{i,j} \hat{\boldsymbol{\mathcal{A}}}_j(\bar{\boldsymbol{V}})$$

Compressible Navier-Stokes: SUPG FE Form for Conservative Variables

SUPG (stabilized) finite element form for $\boldsymbol{\mathcal{U}}$:

$$B_{stab}(\bar{\boldsymbol{W}}, \bar{\boldsymbol{\mathcal{U}}}) = \sum_{e} \int_{\Omega_e} \tilde{\mathcal{L}}^T_{stab}(\bar{\boldsymbol{W}}) \cdot \tilde{\boldsymbol{\tau}}(\tilde{\mathcal{L}}(\bar{\boldsymbol{\mathcal{U}}}) - \boldsymbol{S}) d\Omega_e$$

$$\tilde{\mathcal{L}}^T_{stab}(\cdot) = \tilde{\boldsymbol{\mathcal{A}}}_i^T(\boldsymbol{\mathcal{U}})(\cdot)_{,i}$$

$$\tilde{\boldsymbol{\tau}} = \tilde{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{U}}1} = \left((\frac{2}{\Delta t})^2 \boldsymbol{I} + \tilde{\boldsymbol{\mathcal{A}}}_i g_{ij} \tilde{\boldsymbol{\mathcal{A}}}_j + c_{diff}\, g_{hk} g_{ij} \tilde{\boldsymbol{\mathcal{K}}}_{hi} \tilde{\boldsymbol{\mathcal{K}}}_{jk} \right)^{-\frac{1}{2}}$$

Compressible Navier-Stokes: SUPG FE Form for Primitive Variables

$\int \bar{\underline{w}} \cdot \underline{\underline{A}}_0 \bar{\underline{Y}}_{,t}$

AD

$\int \bar{w}_{,x} \, \bar{\Phi}_{,x}$

SUPG (stabilized) finite element form for $\boldsymbol{Y}$:

$$B_{stab}(\bar{\boldsymbol{W}}, \bar{\boldsymbol{Y}}) = \sum_e \int_{\Omega_e} \mathcal{L}^T_{stab}(\bar{\boldsymbol{W}}) \cdot \boldsymbol{\tau}(\mathcal{L}\underbrace{(\bar{\boldsymbol{Y}}) - \boldsymbol{S})}_{A_i(\bar{Y})_{,i}} d\Omega_e$$

$k_{num} = a_x \tau a_x$

$$\mathcal{L}^T_{stab}(\cdot) = \boldsymbol{\mathcal{A}}_i^T(\boldsymbol{Y})(\cdot)_{,i}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\boldsymbol{Y}1} = \boldsymbol{Y}_{,\boldsymbol{\mathcal{U}}} \tilde{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{U}}1} = \boldsymbol{\mathcal{A}}_0^{-1} \tilde{\boldsymbol{\tau}}_{\boldsymbol{\mathcal{U}}1}$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\boldsymbol{Y}2} = \boldsymbol{Y}_{,\boldsymbol{V}} \hat{\boldsymbol{\tau}}_{\boldsymbol{V}1}$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\boldsymbol{Y}3} = \boldsymbol{Y}_{,\boldsymbol{V}} \hat{\boldsymbol{\tau}}_{\boldsymbol{V}2}$$

Compressible Navier-Stokes: Discontinuity Capturing (DC)

Additional dissipation mechanisms are needed when shock waves form in compressible flows, and thus, "discontinuity capturing" (DC) operator/term is used.

DC term for ϕ:

$$B_{DC}(\bar{w}, \bar{\phi}) = \sum_e \int_{\Omega_e} \bar{w}_{,i} \kappa_{DC} \bar{\phi}_{,i} d\Omega_e$$

DC term for $\boldsymbol{\mathcal{U}}$:

$$B_{DC}(\bar{\boldsymbol{W}}, \bar{\boldsymbol{\mathcal{U}}}) = \sum_e \int_{\Omega_e} \bar{\boldsymbol{W}}_{,i} \cdot \tilde{\boldsymbol{\kappa}}_{DC} \bar{\boldsymbol{\mathcal{U}}}_{,i} d\Omega_e$$

DC term for $\boldsymbol{Y}$:

$$\begin{aligned} B_{DC}(\bar{\boldsymbol{W}}, \bar{\boldsymbol{Y}}) &= \sum_e \int_{\Omega_e} \bar{\boldsymbol{W}}_{,i} \cdot \underbrace{\tilde{\boldsymbol{\kappa}}_{DC} \boldsymbol{\mathcal{A}}_0}_{\boldsymbol{\kappa}_{DC}} \bar{\boldsymbol{Y}}_{,i} d\Omega_e \\ &= \sum_e \int_{\Omega_e} \bar{\boldsymbol{W}}_{,i} \cdot \boldsymbol{\kappa}_{DC} \bar{\boldsymbol{Y}}_{,i} d\Omega_e \end{aligned}$$

Intentionally Left Blank