MANE 6760 - FEM for Fluid Dyn. - Lecture 25

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Compressible Navier-Stokes: Discontinuity Capturing (DC)

Additional dissipation mechanisms are needed when shock waves form in compressible flows, and thus, "discontinuity capturing" (DC) operator/term is used.

DC term for ϕ:

$$B_{DC}(\bar{w},\bar{\phi}) = \sum_e \int_{\Omega_e} \bar{w}_{,i} \kappa_{DC} \bar{\phi}_{,i} d\Omega_e$$

DC term for $\boldsymbol{\mathcal{U}}$:

$$B_{DC}(\bar{\boldsymbol{W}},\bar{\boldsymbol{\mathcal{U}}}) = \sum_e \int_{\Omega_e} \bar{\boldsymbol{W}}_{,i} \cdot \tilde{\mathcal{K}}_{DC} \bar{\boldsymbol{\mathcal{U}}}_{,i} d\Omega_e$$

$= A_0 \bar{Y}_{,i}$

DC term for $\boldsymbol{Y}$:

$$B_{DC}(\bar{\boldsymbol{W}},\bar{\boldsymbol{Y}}) = \sum_e \int_{\Omega_e} \bar{\boldsymbol{W}}_{,i} \cdot \underbrace{\tilde{\mathcal{K}}_{DC} \boldsymbol{\mathcal{A}}_0}_{\mathcal{K}_{DC}} \bar{\boldsymbol{Y}}_{,i} d\Omega_e$$

$$= \sum_e \int_{\Omega_e} \bar{\boldsymbol{W}}_{,i} \cdot \mathcal{K}_{DC} \bar{\boldsymbol{Y}}_{,i} d\Omega_e$$

Compressible Navier-Stokes: Discontinuity Capturing (DC)

DC "viscosity" $\tilde{\boldsymbol{\mathcal{K}}}_{DC}$:

3D 5×5

$$\tilde{\boldsymbol{\mathcal{K}}}_{DC} = (\tilde{\boldsymbol{\mathcal{K}}}_{DC})_{\text{diag1}} = \text{diag}(\tilde{\mathcal{K}}_c, \tilde{\mathcal{K}}_m, \ldots, \tilde{\mathcal{K}}_e)$$

Diagonal entries are given in terms of strong-form residual of the continuity, momentum, and energy equations (i.e., $R_c(\bar{\boldsymbol{\mathcal{U}}})$, $\boldsymbol{R}_m(\bar{\boldsymbol{\mathcal{U}}})$ and $R_e(\bar{\boldsymbol{\mathcal{U}}})$):

first diagonal term $$\tilde{\mathcal{K}}_c = C_c h \frac{|R_c(\bar{\boldsymbol{\mathcal{U}}})|}{|\nabla \bar{\mathcal{U}}_1|}$$

middle diagonal term(s) $$\tilde{\mathcal{K}}_m = C_m h \frac{|\boldsymbol{R}_m(\bar{\boldsymbol{\mathcal{U}}})|}{|\nabla \bar{\boldsymbol{\mathcal{U}}}_{2:n_{sd}+1}|}$$

last diagonal term $$\tilde{\mathcal{K}}_e = C_e h \frac{|R_e(\bar{\boldsymbol{\mathcal{U}}})|}{|\nabla \bar{\mathcal{U}}_{n_{sd}+2}|}$$

3D

$$\begin{bmatrix} \tilde{K}_c & & & & \\ & \tilde{K}_m & & & \\ & & \tilde{K}_m & & \\ & & & \tilde{K}_m & \\ & & & & \tilde{K}_e \end{bmatrix}$$

C_c, C_m and C_e – $\mathcal{O}(1)$

Transient Non-Linear Incompressible Navier-Stokes

Incompressible Navier-Stokes equations (vector and index forms, with momentum equation scaled by density):

force per unit mass

$$R_c(\boldsymbol{u}) = \nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{R}_m(\boldsymbol{u}, p) = \frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \nabla \cdot \boldsymbol{\tau}^{visc,sym} - \boldsymbol{f} = \boldsymbol{0}$$

Newtonian viscous fluid $\boxed{2\nu \underline{\underline{S}}}$

$$R_c(u_j) = u_{j,j} = 0$$

$$(R_m(u_j, p))_i = u_{i,t} + (u_i u_j)_{,j} + p_{,i} - \tau^{visc,sym}_{ij,j} - f_i = 0$$

Incompressible Navier-Stokes: VMS FE Form

VMS (stabilized) finite element form for u_i and p:

$$B(\{\bar{w}_i, \bar{q}\}, \{\bar{u}_j, \bar{p}\}) + B_{stab}(\{\bar{w}_i, \bar{q}\}, \{\bar{u}_j, \bar{p}\}) = (\bar{w}_i, f_i)$$

only Dirichlet BCs

time-derivative for unsteady case

continuity eqn

$$B(\{\bar{w}_i, \bar{q}\}, \{\bar{u}_i, \bar{p}\}) = \int_\Omega (\bar{w}_{i,j}(-\bar{u}_i\bar{u}_j - \bar{p}\delta_{ij} + \bar{\tau}_{ij}^{visc,sym}) - \bar{q}_{,j}\bar{u}_j) \, d\Omega$$

$$(\bar{w}_i, f_i) = \int_\Omega \bar{w}_i f_i \, d\Omega$$

addition from VMS

SUPG

$$B_{stab}(\{\bar{w}_i, \bar{q}\}, \{u'_j, p'\}; \{\bar{u}_j, \bar{p}\}) = \sum_e \int_{\Omega_e} (\bar{w}_{i,j}(-u'_i\bar{u}_j - \bar{u}_i u'_j - u'_i u'_j - p'\delta_{ij}) - \bar{q}_{,j} u'_j) \, d\Omega_e$$

LSIC

pressure-stabilizing/Petrov Galerkin PSPG

Incompressible Navier-Stokes: VMS FE Form

Fine-scale model for u'_i and p':

$$\begin{Bmatrix} u'_1 \\ \vdots \\ p' \end{Bmatrix} = -\boldsymbol{\tau} \begin{Bmatrix} (R_m(\bar{u}_j, \bar{p}))_1 \\ \vdots \\ R_c(\bar{u}_j) \end{Bmatrix}$$

Stabilization parameters τ_m and τ_c:

?

$$\boldsymbol{\tau} = \mathrm{diag}(\tau_m, \ldots, \tau_c)$$

$$\tau_m = \frac{1}{\sqrt{(\frac{2}{\Delta t})^2 + \bar{u}_i g_{ij} \bar{u}_j + c_{diff}^2 g_{ij} g_{ij} \nu^2}}$$

$$\tau_c = \frac{1}{\tau_m tr(g_{ij})}$$

$$B_{stab}(\{\bar{w}_i, \bar{q}\}, \{\bar{u}_j, \bar{p}\}) = \sum_e \int_{\Omega_e} \left(\underbrace{\ldots}_{1} + \underbrace{\ldots}_{2} + \underbrace{\ldots}_{3} + \underbrace{\ldots}_{4} + \underbrace{\ldots}_{5} \right) d\Omega_e$$

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