HW5 answers

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① $A, B \in \mathbb{C}^{m\times m} \rightarrow$ nonsingular

$\kappa(A)$ = condition number of A, $\|\cdot\|$ any induced norm.

a) $\|I\| \geq 1$, $\kappa(A) \geq 1 \rightarrow$ Prove.

Proof: $AA^{-1} = I \Rightarrow \|AA^{-1}\| = \|I\| \geq 1$

$\Rightarrow 1 \leq \|AA^{-1}\| \leq \underbrace{\|A\|\|A^{-1}\|}$

$1 \leq \|AA^{-1}\| \leq \kappa(A)$

b) Show $\kappa(AB) \leq \kappa(A)\kappa(B)$ and $\kappa(\alpha A) = \kappa(A)$ for any scalar $\alpha \in \mathbb{C}, \alpha \neq 0$

$\kappa(AB) = \|AB\|\|(AB)^{-1}\| = \|AB\|\|B^{-1}A^{-1}\|$

$\leq \|A\|\|B\|\|B^{-1}\|\|A^{-1}\|$

$\leq \kappa(A)\kappa(B)$ (clubbing terms together.

$\kappa(\alpha A) = \|\alpha A\|\|(\alpha A)^{-1}\|$

$= |\alpha|\|A\|\frac{1}{|\alpha|}\|A^{-1}\| = \kappa(A)$

c) $A \in \mathbb{C}^{m\times m} \rightarrow$ nonsingular with SVD, $A = U\Sigma V^*$, Show that if $Ax = b$, then

$$x = \sum_{i=1}^{m} \frac{u_i^* b}{\sigma_i} v_i$$

$U\Sigma V^* x = b$

$\Sigma V^* x = U^* b$

$V^* x = \Sigma^{-1} U^* b$

$x = V\Sigma^{-1} U^* b$

$= V\Sigma^{-1} \sum_{i=1}^{m} u_i^* b_i$

diagonal matrix

$= V \sum_{i=1}^{m} \frac{u_i^* b_i}{\sigma_i}$

$= \sum_{i=1}^{m} \frac{u_i^* b_i}{\sigma_i} v_i \rightarrow$ Most amplification happens in the direction of smallest σ_i.

② a) $\|A\| < 1$, then

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$$

$I = (I-A)\sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k - A\sum_{k=0}^{\infty} A^k$

$= \sum_{k=0}^{N} A^k - A\sum_{k=0}^{N} A^k$; $N \to \infty$

$= \sum_{k=0}^{N} A^k - \sum_{k=0}^{N} A^{k+1}$, $N \to \infty$

$= \{I + \cancel{A} + \dots + \cancel{A^N}\} - \{\cancel{A} + A^2 + \dots \cancel{A^N} + A^{N+1}\}$

$= I - A^{N+1}$, $N \to \infty$

Now, as $\|A^N\| \leq \|A\|^N \to 0$, with $N \to \infty$

then $\lim_{N\to\infty} (I-A)\sum_{k=0}^{\infty} A^k \to I$

$$\Rightarrow \boxed{\sum_{k=0}^{\infty} A^k = (I-A)^{-1}}$$

b) Show if $\|A\| < 1$, $\|I\| = 1$, then $\|(I-A)^{-1}\| = \frac{1}{1-\|A\|}$

$\|(I-A)^{-1}\| = \|\sum_{k=0}^{\infty} A^k\| \leq \sum_{k=0}^{\infty} \|A\|^k \rightarrow$ triangle inequality.

Geometric progression

$\leq \frac{\|I\|}{\|I\| - \|A\|} \leq \frac{1}{1-\|A\|}$

③ a) $|\varepsilon_i| \leq \varepsilon_{mach}$, $i = 1, 2, \dots, n$ $0 < \varepsilon_{mach} < 1$, then

$(1-\varepsilon_1)(1-\varepsilon_2)\cdots(1-\varepsilon_n) = (1+\varepsilon)^n$ for some ε with $|\varepsilon| \leq \varepsilon_{mach}$

Each $\varepsilon_i = O(\varepsilon_{mach})$. $\Rightarrow$ LHS: $(1 - O(\varepsilon_{mach}))^n$

if all $\varepsilon_i = \varepsilon_1 \rightarrow$ LHS: $(1-\varepsilon_1)^n \simeq 1 - n\varepsilon_1$

all $\varepsilon_i = \varepsilon_n \rightarrow$ LHS $= (1-\varepsilon_n)^n \simeq 1 - n\varepsilon_n$

$\Rightarrow$ some $\varepsilon_i = \varepsilon_1 \pm \varepsilon$ where $|\varepsilon| < \varepsilon_{mach}$.

or $\varepsilon_i = \varepsilon_n \pm \varepsilon$.

if $\varepsilon_1 < \varepsilon_2 \dots < \varepsilon_n$ for all $\varepsilon_i \geq 0$

$\Rightarrow$

$(1-\varepsilon_1)^n \leq (1+\varepsilon)^n \leq (1-\varepsilon_n)^n$

if $\varepsilon_1 < \dots < \varepsilon_n$ & $\varepsilon_i < 0$

$(1-\varepsilon_n)^n \leq (1+\varepsilon)^n \leq (1-\varepsilon_1)^n$

$\therefore$, $(1-\varepsilon_1)(1-\varepsilon_2)\cdots(1-\varepsilon_n) = (1+\varepsilon)^n$

b) $\tilde{f}(x_1, x_2, \dots, x_n) = (\dots(x_1 \oplus x_2) \oplus x_3) \oplus \dots \oplus x_n)$

show

$\tilde{f}(x_1, x_2, \dots, x_n) = x_1(1+\varepsilon_1)^{n-1} + x_2(1+\varepsilon_2)^{n-1} + x_3(1+\varepsilon_3)^{n-2} + \dots + x_n(1+\varepsilon_n)$

for some $|\varepsilon_i| \leq \varepsilon_{mach}$

$\tilde{f}(x_1, x_2, \dots, x_n) = (\dots\{[((x_1 \oplus x_2)(1+\varepsilon_1) + x_3)(1+\varepsilon_2)] + x_4\}(1+\varepsilon_3)\dots)$

$\therefore$, $= x_1(1+\varepsilon_1)(1+\varepsilon_2)\cdots(1+\varepsilon_n) +$

$x_2(1+\varepsilon_1)(1+\varepsilon_2)\cdots(1+\varepsilon_n) +$

$x_3(1+\varepsilon_2)(1+\varepsilon_3)\cdots(1+\varepsilon_n) +$

$x_4(1+\varepsilon_3)(1+\varepsilon_4)\cdots(1+\varepsilon_n) +$

$\vdots$

$x_n(1+\varepsilon_n)$

Since $|\varepsilon_i| \leq \varepsilon_{mach}$ $\varepsilon_i = O(\varepsilon_{mach})$, then replace them by their corresponding terms

$= x_1(1+\varepsilon_1)^n + x_2(1+\varepsilon_2)^n + x_3(1+\varepsilon_3)^{n-1} + \dots + x_n(1+\varepsilon_n)$.

④ a) $x \oplus x = fl(x) \oplus fl(x) = x(1+\varepsilon_1) \oplus x(1+\varepsilon_1)$

$= (x(1+\varepsilon_1) + x(1+\varepsilon_1))(1+\varepsilon_2)$

$= 2x(1+\varepsilon_1)(1+\varepsilon_2)$

$= 2\tilde{x}$ for some $\varepsilon_i \leq \varepsilon_{mach}$]

$\tilde{f}(x) = f(\tilde{x}) \rightarrow$ ①

$\tilde{x} - x = x(\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2)$

$\therefore$, $\frac{\|\tilde{x}-x\|}{\|x\|} = \frac{\|x(\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2)\|}{\|x\|} = \|\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2\| = O(\varepsilon_{mach}) \rightarrow$ ②

from ① and ② $\rightarrow$ this is backwards stable

b) x^2 computed as $x \otimes x$

$fl(x) \otimes fl(x) = x(1+\varepsilon_1) \otimes x(1+\varepsilon_1)$

$= (x(1+\varepsilon_1) \times x(1+\varepsilon_1))(1+\varepsilon_2)$

$= x^2(1+\varepsilon_1)^2(1+\varepsilon_2)$

$\tilde{x} = x(1+\varepsilon_1)\sqrt{1+\varepsilon_2} = $ some $x(1+O(\varepsilon_m))$

$= \tilde{x}^2 = f(\tilde{x}) \Rightarrow$ Backward Stable

c) $x \in \mathbb{C}\setminus\{0\}$, 1 computed as $x \oslash x = \left(\frac{fl(x)}{fl(x)}\right)(1+\varepsilon)$

$f(\tilde{x}) = 1$, but $\tilde{f}(x) = 1+\varepsilon \Rightarrow$ Not Backward stable

d) $x \in \mathbb{C}$, 0 computed as $x \ominus x$

$fl(x) \ominus fl(x) = (fl(x) - fl(x))(1+\varepsilon) = 0$

$f(\tilde{x}) = \tilde{f}(x) = 0 \rightarrow$ Backward stable

e) e computed as $\sum_{k=0}^{\infty} 1/k! = 1 \oplus \frac{1}{1!} \oplus \frac{1}{2!} \oplus \dots$

$= (((1+\varepsilon_1) + \frac{1}{2})(1+\varepsilon_2) + \frac{1}{6})(1+\varepsilon_3) \dots$

error keeps growing $\simeq (1+\varepsilon)^n \simeq 1+n\varepsilon$

as $n\uparrow$, error grows $\Rightarrow$ its <u>not stable</u>!