Due: Monday March 28, 2022

Problem Set 7

1. (25 pts.) Consider the variable coefficient diffusion equation

$$u_t = (Du_x)_x, \qquad 0 < x < 1, \qquad t > 0$$

where $D > 0$ is a function of x, with initial conditions $u(x,0) = u_0(x)$ and subject to boundary conditions $u_x(0,t) = 0$, $u(1,t) = 0$.

   (a) Show that this problem is well-posed by proving the energy estimate

$$\frac{d}{dt}||u||^2 \le 0,$$

   where $||u||^2 = (u,u) = \int_0^1 u^2\,dx$.

   (b) Propose a computational grid, a semi-discretization of the PDE in space, and a treatment of the boundary conditions for which you can prove a discrete energy estimate of the form $\frac{d}{dt}||u||_h^2 \le 0$ for an appropriately defined discrete norm $||\cdot||_h$. Prove your discrete energy estimate.

   (c) Prove a fully discrete energy estimate for Crank-Nicolson integration of the scheme from part (b) above.

   (d) Write a code to implement the scheme in part (d). Demonstrate second-order convergence using a manufactured solution.

2. (25 pts.) Consider a heat conduction problem in an annular section

$$u_t = \nu\left[\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}\right], \quad 1 < r < 2, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0$$

with initial condition $u(r,\theta,0) = u_0(r,\theta)$, and boundary conditions

$$\begin{aligned} u_r(1,\theta,t) &= \alpha_1(\theta,t), \qquad & u_r(2,\theta,t) &= \alpha_2(\theta,t) \\ u(r,0,t) &= \beta_1(r,t) \qquad & u(r,\tfrac{\pi}{2},t) &= \beta_2(r,t). \end{aligned}$$

   (a) Write a second-order accurate code to solve this problem using centered differencing and Crank-Nicolson time integration.

   (b) Verify the accuracy of your code using manufactured solutions.

   (c) Now set

$$\begin{aligned} u_0(r,\theta) &= 0 \\ \alpha_1(\theta,t) &= 0 \\ \alpha_2(\theta,t) &= 0 \\ \beta_1(r,t) &= 0 \\ \beta_2(r,t) &= (r-1)^2(r-2)^2. \end{aligned}$$

   Using $\nu = 1$ and 40 grid lines in both the radial and angular coordinate directions, compute numerical solutions to this problem at $t = 0, .1, .5, 1.5$. Create surface plots of the solution for each time.

   (d) Create a single line plot with four curves showing the solutions from part (c) along the inner radius ($r = 1$), as a function of θ for all four times $t = 0, .1, .5, 1.5$.