Executive Summary

The task was to minimize the mass of a wing spar of an unmanned aerial vehicle. The force was uncertain, and the structure could not reach the ultimate strength of carbon fiber, the material. This uncertainty was accounted for by finding the mean and standard deviation of the stress along the spar. The optimal mass was calculated by changing the inner and outer radii of the spar, and the number of elements along the length. The optimal value for the number of elements is 15, giving the spar a mass of 8.57 kg.

Analysis Method

The spar was modelled using the Euler-Bernoulli Beam Theory. There were multiple assumptions made while using this theory. The cross section of the beam had planar symmetry, and the cross section varied smoothly. The internal strain energy accounted only for bending moment deformations. The material was assumed to be elastic and isotropic, and any deformations were negligible.

The displacement of the beam was given by Equation 1 below.

$$f(x,\xi) = \frac{d^2}{dx^2}(EI_{yy}\frac{d^2w}{dx^2}) \qquad \textit{Equation 1}$$

$f(x, \xi)$ was the applied load, E was Young's modulus with a constant value of 70 GPa, I_{yy} was the second moment of area with respect to the y axis, and w was the vertical displacement in the z direction. The loading was modeled by Equation 2 below.

$$f(x,\xi) = f_{nom}(x) + \delta_f(x,\xi) \qquad \textit{Equation 2}$$

The nominal force was shown in Equation 3 below.

$$f_{nom}(x) = \frac{2.5W}{L}\left(1 - \frac{x}{L}\right) \qquad \textit{Equation 3}$$

W was half the operational weight, with a value of 2450 Newton. L was the length of wing spar with a value of 7.5 meters. The uncertainty in the force was given in Equations 4 and 5.

$$\delta_f(x,\xi) = \sum_{n=1}^{4} \xi_n cos(\frac{(2n-1)\pi x}{2L}) \qquad \textit{Equation 4}$$

$$\xi_n \sim N(0, \frac{f_{nom}(0)}{10n}) \qquad \textit{Equation 5}$$

N was a normal distribution where 0 is the mean, μ_n, and $\frac{f_{nom}(0)}{10n}$ was the standard deviation, σ_n. Gauss-Hermite quadrature rules were used, calculating ξ_n with Equation 6 below.

$$\xi_n = \sqrt{2} * \sigma_n * x_i + \mu_n \qquad \textit{Equation 6}$$

x_i was the quadrature locations. I_{yy} was calculated using Equation 7 and 8. Equation 7 was the general form of the second moment of area, and Equation 8 was the specific formula for the second moment of area for a circular annulus.

$$I_{yy} = \iint z^2 dzdy \qquad \textit{Equation 7}$$

$$I_{yy} = \frac{\pi}{4}(rout^4 - rin^4) \qquad \textit{Equation 8}$$

rout was the outer radius and rin was the inner radius. The stress was calculated using Equation 9 below.

$$s(x,\xi) = -z_{max}E\frac{d^2w}{dx^2} \qquad \textit{Equation 9}$$

$s(x,\xi)$ was the stress at each node, and z_{max} was the maximum height of the beam at each node. Due to the uncertainty in the force, there was uncertainty in the stress. To account for this uncertainty, the stress was multiplied by a weight and summed up for each quadrature location. This mean stress equation was shown in Equation 10 below.

$$E(f) = \frac{1}{\sqrt{\pi}}\sum_{i=1}^{m} w_i f(\sqrt{2} * \sigma_n * x_i + \mu_n)$$

Equation 10

E(f) was the expected value of the mean stress, and w_i was the weight at each quadrature point. The function f() outputs the stress after Equation 9. The standard deviation of the stress was calculated by Equation 11 below.

$$\sigma = \sqrt{E(f^2) - E(f)^2}$$

Equation 11

Finally the mass was calculated using the density and volume of the annulus, as shown in Equations 12 and 13.

$$m = \rho V \qquad \textit{Equation 12}$$

$$V = \int_0^L \pi(rout^2 - rin^2)dx \qquad \textit{Equation 13}$$

ρ was the density of carbon fiber with a value 1600kg/m^3 and V was the volume.

Geometry Parameterization

There were two different types of design variables in this task, the inner radius and the outer radius of the annulus. This was organized in a single array of form n inner radius elements, then n outer radius elements, where n was the number of nodes. In order to calculate the thickness, the inner radius was subtracted from the outer radius. The only thing that was changed manually was the number of elements, which is the number of nodes minus one.

Optimization

The objective was to design the spar, the main structural support for a wing, for a new aircraft. The goal was to minimize the mass of the spar. The spar would be a circular annulus. The inner radius must be at least 1 cm, and the outer radius must be at most 5 cm. The thickness must be at least 2.5mm. The aircraft could undergo a maneuver where the total force on the spar was 2.5 times the weight of the entire aircraft, 500kg. However, this force was uncertain, so the force had a mean and standard deviation. Using Gaussian-Hermite quadrature, the mean and standard deviation of the stress along the spar was calculated. The mean stress plus 6 times the stress standard deviation was not able reach the ultimate strength of the carbon fiber, 600 MPa, during this maneuver. The inner radii and the thickness of the annulus were changed by the MATLAB function fmincon to meet these constraints and to minimize mass.

Results

To calculate the optimal mass, the number of design variables were changed for different configurations. The first values calculated were the nominal values, with an inner radii of 4.15 cm and outer radii of 5 cm throughout the entire spar. This gave a nominal mass of 29.32 kg. In order to consider this problem optimized, the spar mass must be 70% lower than this nominal value.

In all the simulations, the radii followed the same trend. A trial with 75 number of elements was attempted, but as it took about an hour per iteration, it was cancelled. Instead, the number of elements was varied from 15 to 25. The calculated mass and the time it took for the code to run is shown below, in Figure 1.

Figure 1

Nelem	10	15	20	25
Mass	8.5334	8.57	8.5833	8.5847
Time(s)	255.233	768.653	2733.366	2776.667

These results were interesting, as increasing the number of elements increased the mass. This was most likely to the space between the nodes being too large for a smaller number of elements. Solving the function at fewer nodes made the difference between function evaluations larger, resulting in a smaller mass. This reveals that accuracy was a tradeoff for computational cost. The optimal solution was where the number of elements was 15. This was because it was more accurate than 10, but waiting over 45 minutes for 20 elements was not worth the extra precision. In Figure 2 below, the optimal inner and outer radii are plotted.

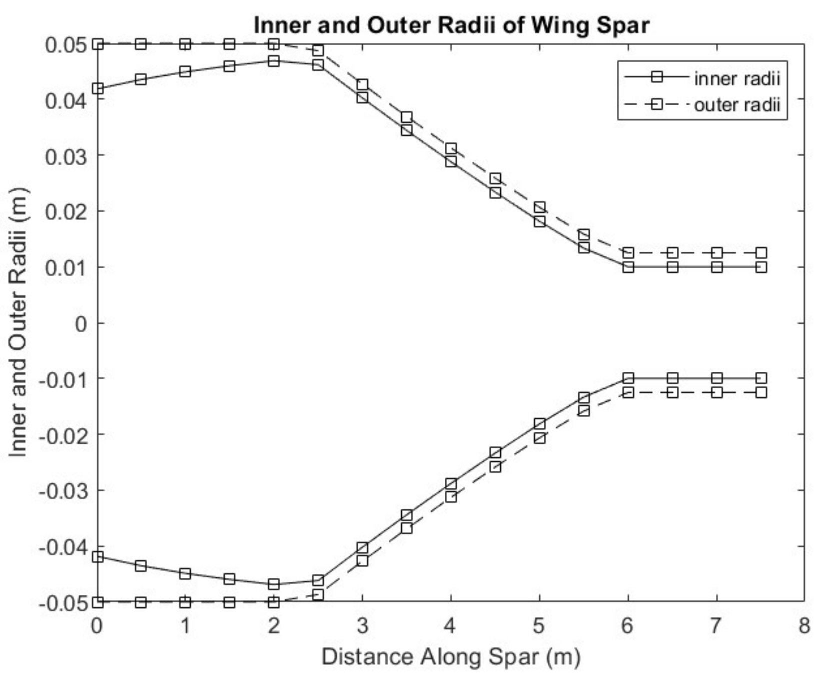


Figure 2

The trend was the same as the inner and outer radii in Project 2, as shown in Figure 3 below.

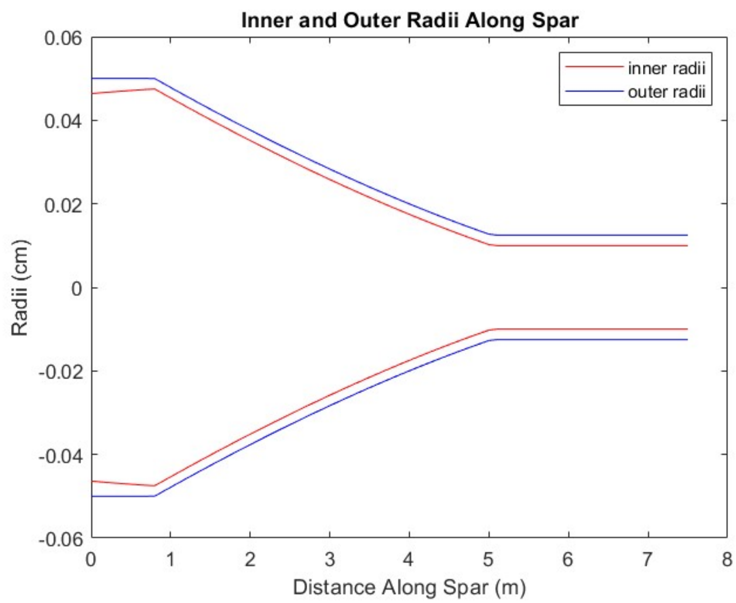


Figure 3

The inner radii of the spar with uncertainty started out at a lower value, and the radii started to decrease around 2 meters into the spar, instead of just below 1 meter. Both of these things were to account for the uncertainty in the stress, as having a larger annulus would decrease some of the stress on the spar. The stress for the spar can be shown in Figure 4 below.

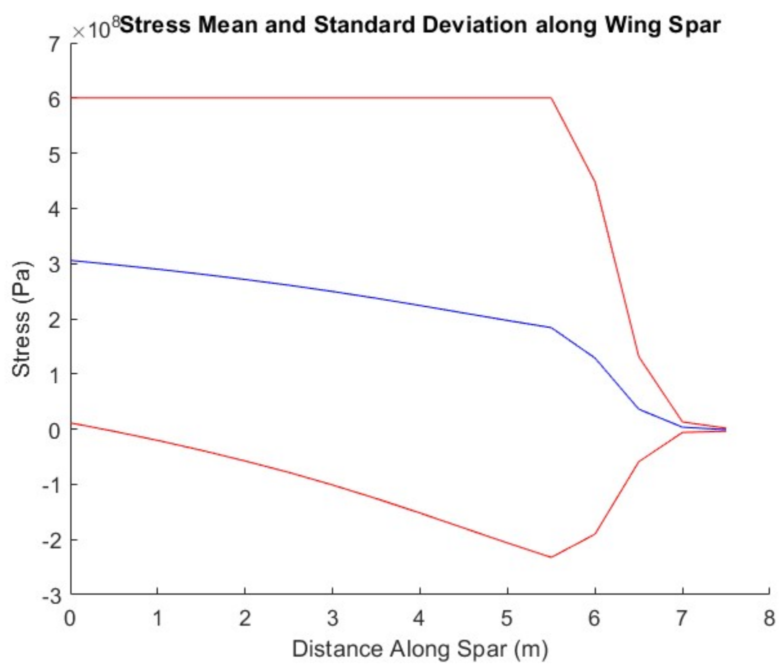


Figure 4

The mean stress decreased throughout the spar. This makes sense, as the stress on the spar should decrease along the distance of the spar, ending at 0 Pa. The standard deviation also drops dramatically to a value of almost 0. However, the standard deviation is the highest in the middle. This is most likely to a large uncertainty in the force at that location. The topmost stress line is similar to the stress line in Project 2. It is not as smooth as in Project 2 due to having only 15 elements, instead of the 75 elements Project 2 had. This can be seen in Figure 5 below.

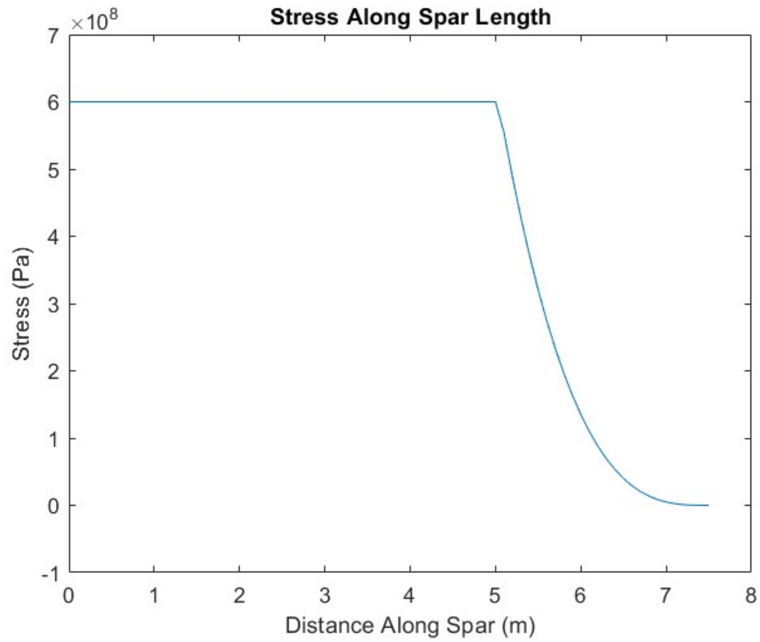
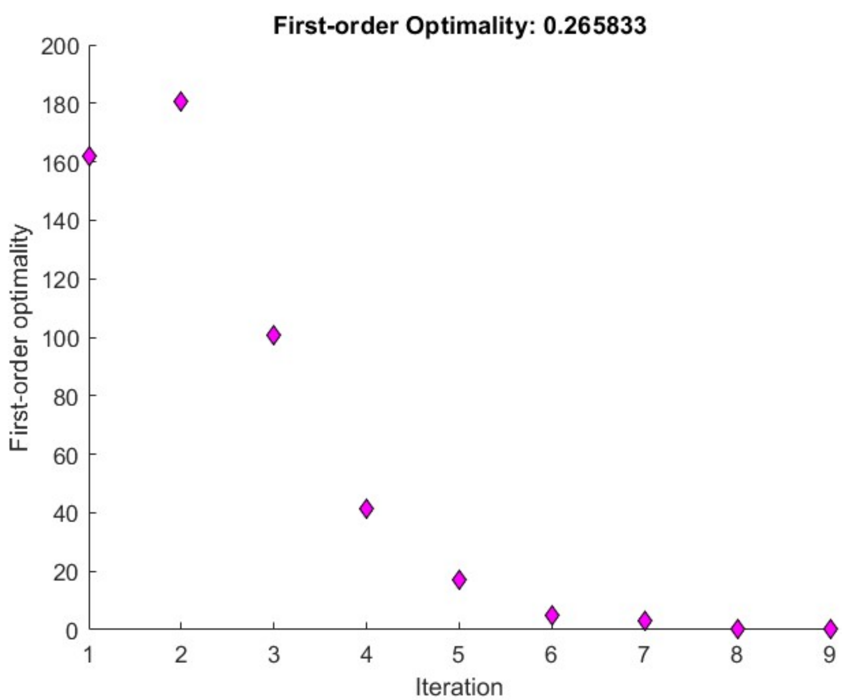


Figure 5

The first order optimality converges to 0.265833, as shown in Figure 6 below. This is close to 0, so this solution is a valid solution.



Conclusion

In conclusion, having the spar being split into 15 elements will allow for an optimized mass of 8.57 kg. The outer radii starts at its upper bound, decreases and flattens out at its lower bound. The inner radii increases a little bit, then decreases to its lower bound. The mean stress decreases throughout the spar to a value of 0 at the tip.

Appendix

A1: mean_force.m

```
function [mean_stress, sd] = mean_force(L, E,force, Nelem,r)
%function to get the mean and standard deviation of stress
%Inputs:
   %L- semi-span in meters
   %E- Young's modulus, Pa
   %force- loading at manueuver
   %Nelem- number of elements
   %r- inner and outer radii

%Outputs:
   %mean stress- array of the mean stress
   %sd- array of the standard deviation of the stress


   x = [0:L/Nelem:L].';
   func = @(pt1, pt2, pt3, pt4) (pt1*cos(pi.*x/(2*L)) + pt2*cos(3*pi.*x/(2*L)) +
pt3*cos(5*pi.*x/(2*L)) + pt4*cos(7*pi.*x/(2*L)));
   %mean and standard deviation of each probabilistic perturbation
   mu = [0; 0; 0; 0];
   sigma = [force(1)/10; force(1)/20; force(1)/30; force(1)/40];
   %using a 4 pt Gauss-Hermite quadrature
   xi = [ -1.65068; -0.524648; .524648; 1.65068];
   wts = [.0813128; .804914; .804914; .0813128]./sqrt(pi);
   %const things
   r_in = r(1:Nelem+1);
   r_out = r(Nelem+2:2*(Nelem+1));
   Iyy = CalcSecondMomentAnnulus(r_in, r_out);
   mean_stress = 0;
   mean_stress2 = 0;
   for i1 = 1:size(xi,1)
      pt1=sqrt(2)*sigma(1)*xi(i1)+mu(1);
      for i2 = 1:size(xi,1)
         pt2 = sqrt(2)*sigma(2)*xi(i2)+mu(2);
         for i3 = 1:size(xi,1)
            pt3 = sqrt(2)*sigma(3)*xi(i3)+mu(3);
            for i4 = 1:size(xi,1)
               pt4 = sqrt(2)*sigma(4)*xi(i4)+mu(4);
```

```
                %take force use to evaluate displacement and stress
                %use weights on stress
                f = force + func(pt1,pt2,pt3,pt4);
                u = CalcBeamDisplacement(L, E, Iyy, f, Nelem);
                stress = CalcBeamStress(L, E, r_out, u, Nelem);
                stress2 = stress.^2;
                mean_stress = mean_stress + wts(i1)*wts(i2)*wts(i3)*wts(i4)*stress;
                mean_stress2 = mean_stress2 + wts(i1)*wts(i2)*wts(i3)*wts(i4)*stress2;
            end
        end
    end
  end
  sd = sqrt(mean_stress2 - mean_stress.^2);
end
```

A2: UncertainConstraint.m

```
function [c, ceq, dcdr, dceqdr] = UncertainConstraint(r, L, E, force, yield, Nelem)
% Computes the nonlinear inequality constraints for the wing-spar problem
% Inputs:
%   r - the DVs; x(1:Nelem+1) inner and x(Nelem+2:2*(Nelem+1) outer radius
%   L - length of the beam
%   E - longitudinal elastic modulus
%   force - force per unit length along the beam axis x
%   yield - the yield stress for the material
%   Nelem - number of finite elements to use
% Outputs:
%   c, ceq - inequality (stress) and equality (empty) constraints
%   dcdr, dceqdr - Jacobians of c and ceq
assert( size(force,1) == (Nelem+1) );
assert( size(r,1) == (2*(Nelem+1)) );

c = CalcInequality(r);
ceq = [];
dcdr = zeros(2*(Nelem+1),Nelem+1);
dceqdr = [];
for k = 1:2*(Nelem+1)
  rc = r;
  rc(k) = rc(k) + complex(0.0, 1e-30);
  dcdr(k,:) = imag(CalcInequality(rc))/1e-30;
end

  function [cineq] = CalcInequality(r)
    %take the mean stress and standard deviation to compute inequality
    [mean_stress, sd] = mean_force(L, E,force, Nelem,r);
    max_stress = mean_stress+6*sd;
    cineq = max_stress./yield - ones(Nelem+1,1);
  end

end
```

A3: opt_spar.m

```
% minimize wing spar weight subject to stress constraints at manuever
clear all;
close all;

% carbon fiber values from http://www.performance-
composites.com/carbonfibre/mechanicalproperties_2.asp
Nelem = 15;
L = 7.5; % semi-span in meters
rho = 1600; % density of standard carbon fiber, kg/m^3
yield = 600e6; % tensile strength of standard carbon fiber, Pa
E = 70e9; % Young's modulus, Pa
W = 0.5*500*9.8; % half of the operational weight, N
force = (2*(2.5*W)/(L^2))*[L:-L/Nelem:0].'; % loading at manueuver

% define function and constraints
fun = @(r) SparWeight(r, L, rho, Nelem);
nonlcon = @(r) UncertainConstraint(r, L, E, force, yield, Nelem);
lb = 0.01*ones(2*(Nelem+1),1);
up = 0.05*ones(2*(Nelem+1),1);
A = zeros(Nelem+1,2*(Nelem+1));
b = -0.0025*ones(Nelem+1,1);
for k = 1:(Nelem+1)
    A(k,k) = 1.0;
    A(k,Nelem+1+k) = -1.0;
end

% define initial guess (the nominal spar)
r0 = zeros(2*(Nelem+1),1);
r0(1:Nelem+1) = 0.0415*ones(Nelem+1,1);
r0(Nelem+2:2*(Nelem+1)) = 0.05*ones(Nelem+1,1);

options = optimset('GradObj','on','GradConstr','on', 'TolCon', 1e-4, ...
    'TolX', 1e-8, 'Display','iter', 'Algorithm', 'active-set'); %, 'DerivativeCheck','on');
[ropt,fval,exitflag,output] = fmincon(fun, r0, A, b, [], [], lb, up, ...
    nonlcon, options);

% plot optimal radii
r_in = ropt(1:Nelem+1);
r_out = ropt(Nelem+2:2*(Nelem+1));
x = [0:L/Nelem:L].';
figure
plot(x, r_in, '-ks');
hold on;
plot(x, r_out, '--ks');
title('Inner and Outer Radii of Wing Spar')
xlabel('Distance Along Spar (m)')
ylabel('Inner and Outer Radii (m)')
hold off

% display weight and stress constraints
[f,~] = fun(ropt);
```

```
[c,~,~,~] = nonlcon(ropt);
[mean_stress, sd] = mean_force(L, E,force, Nelem,ropt);
figure
hold on
plot(x,mean_stress,'b')
plot(x,mean_stress+6*sd, 'r')
plot(x,mean_stress-6*sd, 'r')
xlabel('Distance Along Spar (m)')
ylabel('Stress (Pa)')
title('Mean Stress and  Stress Standard Deviation along Wing Spar')
```