Due: Monday April 18, 2022

Problem Set 9

1. (20 pts.) Consider the linear hyperbolic system

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

where A is a constant coefficient matrix. Now introduce the upwind method

$$\mathbf{v}_j^{n+1} = \mathbf{v}_j^n - A_+ \frac{\Delta t}{\Delta x}\left(\mathbf{v}_j^n - \mathbf{v}_{j-1}^n\right) - A_- \frac{\Delta t}{\Delta x}\left(\mathbf{v}_{j+1}^n - \mathbf{v}_j^n\right)$$

where $A_\pm$ are the matrices constructed using the positive/negative eigenvalues and $A = A_+ + A_-$ as discussed in class. For each A given below do the following

   (a) Find the matrices A_+ and A_- in the upwind method above.

   (b) If the PDE is defined for $x \in (-1, 1)$ how many boundary conditions are needed at $x = -1$, and how many at $x = 1$?

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 2 & 3 & -3 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

   (c) Given BCs of the type just derived, determine an exact solution for the second of the two problems. Note that you may find it convenient to use either homogeneous or non-homogeneous BCs, it is your choice.

   (d) Implement the upwind method with BCs. Use your exact solution from (c) to verify first-order convergence.

   (e) (extra credit) Using a method-of-lines approach and RK-4, implement the standard centered second-order discretization for the original linear hyperbolic system. Use your exact solution from (c) to verify second-order convergence.

2. (10 pts.) Consider the scalar conservation equation

$$u_t + [f(u)]_x = 0,$$

for $|x| < \infty$, $t > 0$, and $u(x, 0) = u_0(x)$. For each of the following flux functions $f(u)$

$$f(u) = 2u^4, \qquad\qquad f(u) = e^{2u}$$

   (a) Determine a formula for the propagation speed S of a discontinuity between two states u_L and u_R (L and R for left and right respectively).

   (b) Determine the speed S when $u_L = 2$, and $u_R = 1$.

3. (20 pts.) Consider the conservation equation from #2 above with $f(u) = e^{2u}$.

(a) Find the characteristic form of the equation. What is the characteristic speed?

(b) Assuming $u_0(x) = \frac{1}{2}(u_L + u_R) + \frac{1}{2}(u_R - u_L)\tanh(10x)$, use the characteristics, and the fact that u is constant along characteristics, to sketch qualitative solutions at various times for the following 2 cases;

  i. $u_L = -1$, $u_R = 1$.

  ii. $u_L = 1$, $u_R = -1$.

(c) Write a conservative upwind code and verify your predictions from (b) above.