MANE 6760 - FEM for Fluid Dyn. - Lecture 21

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Simplified: 1D Non-linear (NL) TAD Eqn

A number of simplifications:

- 1D (spatial) domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Strong form:

$$
\begin{aligned}
R(\phi) = \mathcal{L}(\phi) - s = \frac{\partial \phi}{\partial t} + a_x \frac{\partial \phi}{\partial x} - \frac{\partial}{\partial x}\left(\kappa(\phi)\frac{\partial \phi}{\partial x}\right) - s = 0, \quad \phi \in \mathcal{S}_{strong} \\
x \in [0, L] \\
t \in [t_{min}, t_{max}] \\
\phi(x, t = t_{min}) = \phi_{IC}(x) \, \forall x \\
\phi(x = 0, t) = \phi_0(t) \quad \text{on} \quad x = 0 \, \forall t \\
\phi(x = L, t) = \phi_L(t) \quad \text{on} \quad x = L \, \forall t
\end{aligned}
$$

Non-linear Transient Equations: (Semi-discrete) FE for NL TAD

$$\bar{\phi} = \sum_A N_A(x)\,\hat{\phi}_A(t)$$

Derive a finite-element based non-linear weak residual:

$$\hat{w}_A G_A = 0, \qquad \text{where } G_A = G_A^{galk} \text{ or } G_A = G_A^{stab}$$

$$\bar{w} = \sum_A N_A(x)\,\hat{w}_A$$

Derive a non-linear system of (ordinary differential) equations:

$$G_A(\dot{\hat{\boldsymbol{\Phi}}}, \hat{\boldsymbol{\Phi}}) = G_A^L(\dot{\hat{\boldsymbol{\Phi}}}, \hat{\boldsymbol{\Phi}}) + G_A^{NL}(\dot{\hat{\boldsymbol{\Phi}}}, \hat{\boldsymbol{\Phi}}) = 0, \qquad \forall A$$

semi-discrete

Linear

$$G_A(\dot{\hat{\boldsymbol{\Phi}}}, \hat{\boldsymbol{\Phi}}) = G_A^T(\dot{\hat{\boldsymbol{\Phi}}}, \hat{\boldsymbol{\Phi}}) + G_A^S(\hat{\boldsymbol{\Phi}}) = M_{AB}\dot{\hat{\phi}}_B + G_A^S(\hat{\boldsymbol{\Phi}}) = 0, \qquad \forall A$$

Use a time integration scheme/ODE solver, e.g., backward Euler:

fully-discrete

$$\mathcal{G}_A(\dot{\hat{\boldsymbol{\Phi}}}^{(n+1)}, \hat{\boldsymbol{\Phi}}^{(n+1)}, \hat{\boldsymbol{\Phi}}^{(n)}) = M_{AB}\frac{\hat{\phi}_B^{(n+1)} - \hat{\phi}_B^{(n)}}{t_{n+1} - t_n} + G_A^S(\hat{\boldsymbol{\Phi}}^{(\overset{n+1}{\cancel{n}})}) = 0, \qquad \forall A$$

Use a non-linear solver (e.g., Newton Raphson) for each time step:

$$\frac{\partial \mathcal{G}_A}{\partial \hat{\phi}_B^{(n+1)}} \Delta\hat{\phi}_B^{(n+1)} + \mathcal{G}_A = 0, \qquad \forall A$$

$$\rho c_p(T)\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) = 0$$

Stabilized FE Form: (Simplified) 1D NL TAD Eqn

Stabilized/SUPG FE *semi-discrete* form (with $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}^{adv}(\cdot)$):
find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\int_0^L \Big(\underbrace{\bar{w}\bar{\phi}_{,t}}_{1} + \underbrace{\bar{w}_{,x} a_x \tau \bar{\phi}_{,t}}_{2} + \underbrace{(-\bar{w}_{,x} a_x \bar{\phi})}_{3} + \underbrace{\bar{w}_{,x} \kappa(\bar{\phi}) \bar{\phi}_{,x}}_{4} + \underbrace{\bar{w}_{,x} \overbrace{\kappa_{num}}^{\kappa_{num}=a_x \tau a_x} \bar{\phi}_{,x}}_{5} + \underbrace{(-\bar{w}s)}_{6} + \underbrace{(-\bar{w}_{,x} a_x \tau s)}_{7} \Big) dx = 0$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

Non-linear Iterations: (Simplified) 1D NL TAD Eqn

Non-linear iterations: require *fully discrete* non-linear weak residual and tangent matrix at every iteration within each time step

semi discrete

$$G_A = \int_0^L \Big(\underbrace{N_A\bar{\phi}_{,t}}_{1} + \underbrace{N_{A,x}a_x\tau\bar{\phi}_{,t}}_{2} + \underbrace{(-N_{A,x}a_x\bar{\phi})}_{3} + \underbrace{N_{A,x}(\kappa(\bar{\phi}) + \kappa_{num})\bar{\phi}_{,x}}_{4+5} + \underbrace{(-N_A s)}_{6} + \underbrace{(-N_{A,x}a_x\tau s)}_{7}\Big)dx$$

fully discrete

$$\mathcal{G}_A = \int_0^L (\ldots\ldots)dx$$

$$\frac{\partial \mathcal{G}_A}{\partial \hat{\phi}_B} = \underbrace{\ldots\ldots}_{1} + \underbrace{\ldots\ldots}_{2} + \ldots\ldots$$

$$M_{AB} = ?$$

$$\stackrel{?}{=} \int_0^L \big(N_A N_B + \ldots\big)\,dx$$

1 $\boxed{N_A N_B \dot{\hat{\phi}}_B}$

2 $\boxed{N_{A,x} a_x \tau N_B \dot{\hat{\phi}}_B}$

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