MANE 6760 - FEM for Fluid Dyn. - Lecture 09

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Stabilization Parameter: AD equation

τ approximations: algebraic versions

- Shakib *et al.* (1991) derived the following:

$$\begin{aligned}\tau_{alg,skb} = \tau_{alg1} : (\tau_{alg,skb})^{-2} &= (\tau^{adv})^{-2} + c_{diff}^2(\tau^{diff})^{-2} \\ &= \left(\frac{(h/2)}{|a_x|}\right)^{-2} + 9\left(\frac{(h/2)^2}{\kappa}\right)^{-2} \\ &= \left(\frac{2|a_x|}{h}\right)^2 + 9\left(\frac{4\kappa}{h^2}\right)^2\end{aligned}$$

- Codina (1998) derived the following:

$$\begin{aligned}\tau_{alg,cod} = \tau_{alg2} : (\tau_{alg,cod})^{-1} &= (\tau^{adv})^{-1} + (\tau^{diff})^{-1} \\ &= \left(\frac{(h/2)}{|a_x|}\right)^{-1} + \left(\frac{(h/2)^2}{\kappa}\right)^{-1} \\ &= \frac{2|a_x|}{h} + \frac{4\kappa}{h^2}\end{aligned}$$

- ... other options available

Stabilization Parameter: AD equation

τ approximations: algebraic versions in multiple dimensions

- Define element-level metric tensor: $g_{ij} = \xi_{k,i}\xi_{k,j} = \frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_k}{\partial x_j}$, where in 1D: $g_{11} = (\frac{2}{h})^2$
- In $\tau_{alg,skb}$, use/define advective limit as:

$$(\tau^{adv})^{-2} = a_i g_{ij} a_j$$

- In $\tau_{alg,skb}$, use/define diffusive limit as:

$$(\tau^{diff})^{-2} = g_{ij} g_{ij} \kappa^2$$

- $\tau_{alg,skb}$ in multiple dimensions becomes:

$$(\tau^{alb,skb})^{-2} = a_i g_{ij} a_j + c_{diff}^2 g_{ij} g_{ij} \kappa^2$$

$$\tau^{alb,skb} = \frac{1}{\sqrt{a_i g_{ij} a_j + c_{diff}^2 g_{ij} g_{ij} \kappa^2}}$$

Stabilization Parameter: AD equation

Note that when a_i or κ are a function of spatial coordinate x_i (i.e., not constants), then τ is a non-trivial function of spatial coordinate x_i. However, this aspect is typically ignored during numerical integration in finite element calculations. τ or its approximation (e.g., algebraic τ) is still evaluated at integration points (and its value can change within an element from one integration point to another).

Advection-diffusion-reaction (ADR) Equation: Linear and Scalar

ADR equation:

$$\phi_{,t} + \nabla \cdot (\boldsymbol{a}\phi - \kappa \nabla \phi) + c\phi = s$$
$$\phi_{,t} + (a_i\phi - \kappa \phi_{,i})_{,i} + c\phi = s$$
$$\phi_{,t} + (a_i\phi - \kappa \phi_{,i})_{,i} - r\phi = s$$

Note that Einstein summation notation is used for repeated indices. $c < 0$ or $r > 0$ implies production, and $c > 0$ or $r < 0$ implies destruction. For now, we assume:

$$|a_i|^2 + 4c\kappa \geq 0$$
$$|a_i|^2 - 4r\kappa \geq 0$$

Peclet and Damköhler numbers characterize the solution, where $Da = |c|L/|a_i|$ (and recall $Pe^G = |a_i|L/\kappa$). Similarly, cell Damköhler number is: $Da^e = |c|h/|a_i|$ (and recall $Pe^e = |a_i|h/(2\kappa)$).

FE Form: (Simplified) ADR Equation

A number of simplifications:

- Steady
- 1D domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

Stabilization Parameter: ADR equation

τ approximation: algebraic version due to Shakib *et al.* (1991):

$$\tau_{alg,skb} = \tau_{alg1} : (\tau_{alg,skb})^{-2} = \left(\frac{(h/2)}{|a_x|}\right)^{-2} + 9\left(\frac{(h/2)^2}{\kappa}\right)^{-2} (\frac{1}{c})^{-2}$$

$$= \left(\frac{2|a_x|}{h}\right)^2 + 9\left(\frac{4\kappa}{h^2}\right)^2 + c^2$$

$$(\tau_{alg,skb}) = \frac{1}{\sqrt{\left(\frac{2|a_x|}{h}\right)^2 + 9\left(\frac{4\kappa}{h^2}\right)^2 + c^2}}$$

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