**Due: Monday January 24, 2022**

# Problem Set 1

1. (15 pts.) Consider the equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = u_y - 2u.$$

For each of the following cases, determine if the equation is elliptic, hyperbolic, or parabolic and determine any real characteristics. Also, determine a coordinate transformation that brings the equation to a canonical form (e.g. Poisson's equation for elliptic, the wave equation for hyperbolic, and the heat equation for parabolic).

   (a) $A = 1, \quad B = 1, \quad C = -3$
   (b) $A = 1, \quad B = 1, \quad C = 2$
   (c) $A = 1, \quad B = 1, \quad C = 1$

2. (15 pts.) Consider the PDE $u_t = \nu u_{xx}$ with $\nu > 0$, $x \in \mathbb{R}$, and $t > 0$.

   (a) Determine the dispersion relation for the PDE.
   (b) Determine the exact solution subject to the initial conditions $u(x, t = 0) = \cos(kx)$, and create a surface plot of the solution with $\nu = 1$, $k = 2$.
   (c) Determine the exact solution subject to the initial conditions $u(x, t = 0) = H(x)$ where $H(x)$ is the Heaviside function, and create a surface plot of the solution with $\nu = 1$.
   Hint: Recall that the exact solution for the heat equation with initial condition $f(x)$ can be written as
$$u(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4\nu t}\, d\xi.$$

3. (15 pts.) Consider the PDE $u_t = au_x + \nu u_{xx}$ with $a \in \mathbb{R}$, $\nu > 0$, $x \in \mathbb{R}$, and $t > 0$.

   (a) Determine the dispersion relation for the PDE.
   (b) Determine the exact solution subject to the initial conditions $u(x, t = 0) = \cos(kx)$, and create a surface plot of the solution with $a = 1$, $\nu = 1$, $k = 2$.
   (c) Determine the exact solution to the IVP subject to the initial conditions $u(x, t = 0) = H(x)$, and create a surface plot of the solution with $a = 1$ and $\nu = 1$.

4. (15 pts.) Consider the PDE $u_{tt} = c^2 u_{xx} - 2au_{tx}$ with $c \in \mathbb{R}$, $a \in \mathbb{R}$, $x \in \mathbb{R}$, and $t > 0$.

   (a) Determine the dispersion relation for the PDE.
   (b) Determine the exact solution subject to the initial conditions $u(x, t = 0) = \cos(kx)$ and $u_t(x, t = 0) = k(a + \sqrt{c^2 + a^2}) \sin(kx)$ where $k \in \mathbb{R}$. Create a surface plot of the solution with $c = 1$, $a = 1$, $k = 2$.
   (c) Determine the exact solution subject to the initial conditions $u(x, t = 0) = H(x)$ and $u_t(x, t = 0) = 1$. Create a surface plot of the solution with $c = 1$, $a = 1$. (Hint: Think along the line of the d'Alembert solution here.)