W.D. Henshaw **Math 6800: Solutions for Problem Set 2**

1. NLA exercise 4.1. *Determine SVDs of the following matrices by hand calculation*

$$(a)\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix},\quad (b)\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix},\quad (c)\begin{bmatrix}0 & 2\\ 0 & 0\\ 0 & 0\end{bmatrix},\quad (d)\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix},\quad (e)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

Solution:

(a)

$$A = \begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}$$

$\sigma_1 = \|A\|_2 = 3$ *(2-norm of a diagonal matrix), with* $Ae_1 = 3e_1$ *(*$Av_1 = \sigma_1 u_1$*) gives after the first step in the proof of Theorem 4.1,*

$$U_1^* A V_1 = \begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix} = \begin{bmatrix}3 & 0\\ 0 & B\end{bmatrix},\qquad U_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad V_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

Now $\|B\|_2 = 2$ *and* $Bv_2 = 2u_2$, $v_2 = [-1]$, $u_2 = [1]$. *Thus*

$$A = \begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = U\Sigma V^*$$

(b) $\sigma_1 = \|A\|_2 = 3$, $\sigma_2 = 2$ *and we can determine:*

$$A = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = U\Sigma V^*$$

(c)

$$A = \begin{bmatrix}0 & 2\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

$\|A\|_2 = \max \|Ax\|/\|x\| = 2|x_2|/\sqrt{|x_1|^2 + |x_2|^2} \le 2$ *with equality* $x = [0,1]^*$ *Thus* $\sigma_1 = 2$, $Av_1 = 2u_1$ *where* $v_1 = [0,1]^*$ *and* $u_1 = [1,0,0]^*$.

$$A = \begin{bmatrix}0 & 2\\ 0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = U\Sigma V^*$$

(d)

$$A = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

$\|A\|_2 = \max \|Ax\|/\|x\| = |x_1 + x_2|/\sqrt{|x_1|^2 + |x_2|^2} \leq \sqrt{2}$ *with equality* $x = [1, 1]^*$ *Thus* $\sigma_1 = \sqrt{2}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = U\Sigma V^*$$

(e)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\|A\|_2 = \max \|Ax\|/\|x\| = \sqrt{|x_1 + x_2|^2 + |x_1 + x_2|^2}/\sqrt{|x_1|^2 + |x_2|^2} \leq 2$ *with equality* $x = [1, 1]^*$ *Thus* $\sigma_1 = 2$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = U\Sigma V^*$$

2. *NLA exercise 4.4* Two matrices $A, B \in \mathbb{C}^{m \times m}$ are unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m \times m}$. Is is true or false that A and B are unitarily equivalent if and only if they have the same singular values? Solution:

$\rightarrow$ *Suppose* A *and* B *are unitarily equivalent,* $A = QBQ^*$. *Let* $B = U\Sigma V^*$ *be the the SVD of* B, *then*

$$A = QU\Sigma V^* Q^* = (QU)\Sigma(QV)^*$$

which is an SVD for A *since* QU *and* QV *are unitary. Thus unitarily equivalent implies the same singular values.*

$\leftarrow$ *Suppose* A *and* B *have the same singular values. If* $A = QBQ^*$ *then* A *and* B *have the same eigenvalues. But*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

have different eigenvalues but the same singular values, 2 *and* 1. *Thus is is NOT true that matrices with the same singular values are unitarily equivalent.*

3. *NLA exercise 5.1* In example 3.1 we considered the matrix (3.7)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix},$$

and asserted, among other things, that its 2-norm is approximately 2.9208. Using the SVD, work out (on paper) the exact values of $\sigma_{min}(A)$ and $\sigma_{max}(A)$ for this matrix.
Solution:

The singular values are the square roots of the eigenvalues of A^*A *and* AA^*

$$A^*A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

with eigenvalues $\lambda = (9 \pm \sqrt{65})/2$ *Thus*

$$\sigma_{\max}(A) = \sqrt{(9+\sqrt{65})/2} \approx 2.9208,$$
$$\sigma_{\min}(A) = \sqrt{(9-\sqrt{65})/2} \approx .68474$$

4. *NLA exercise 5.3* Consider the matrix ...

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$$

Solution:

(a) Let $A = U\Sigma V^*$ *denote the SVD of A. Then*

$$A^*A = V\Sigma^*\Sigma V^*,$$
$$AA^* = U\Sigma\Sigma^* U^*$$

The singular values are the eigenvalues of A^*A *or* AA^*. *The right singular vectors* v_i *are the eigenvectors of* A^*A *while left singular vectors are the eigenvectors of* AA^*. *Now*

$$B = A^*A = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}, \qquad C = AA^* = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$$

with eigenvalues satisfying,

$$\lambda^2 - 250\lambda + 1000 = 0,$$
$$\lambda_1 = 200,\ \lambda_2 = 50.$$

and thus $\sigma_1 = \sqrt{200}$, $\sigma_2 = \sqrt{50}$. *The eigenvectors of B are*

$$v_1 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix},$$

We compute u_1 *and* u_2 *from* $Av_i = \sigma_i u_i$,

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},$$

and thus

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{200} & 0 \\ 0 & \sqrt{50} \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} = U\Sigma V^T$$

(b) The singular values and vectors are given above. See Figure 1 for the drawings.

(c)

$$\|A\|_1 = 16,$$
$$\|A\|_2 = \sigma_1 = \sqrt{200} \approx 14.1421,$$
$$\|A\|_\infty = 15,$$
$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{250} = 5\sqrt{10}.$$

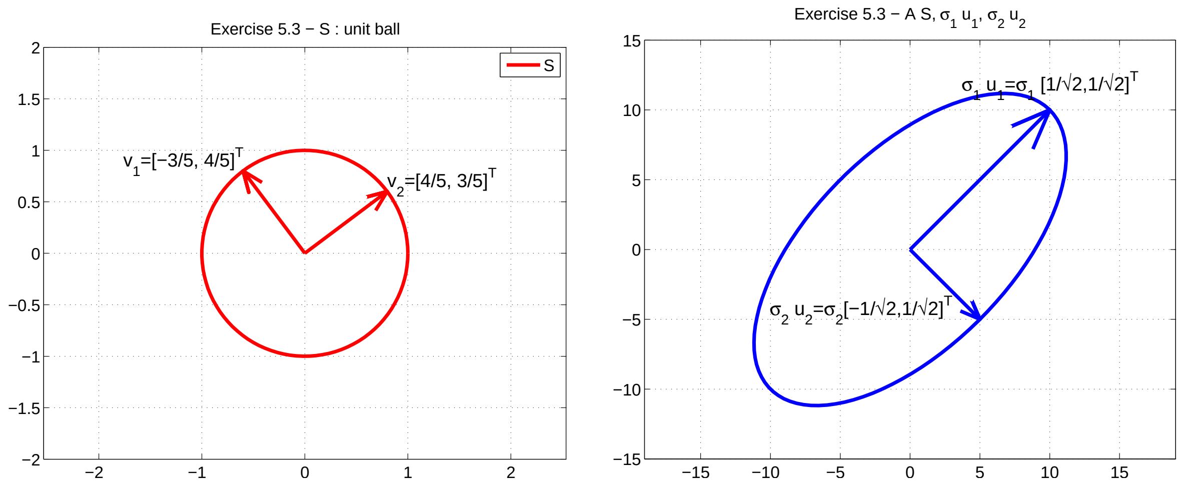

Figure 1: Unit ball S (left) and image under A.

(d)

$$A^{-1} = V\Sigma^{-1}U^* = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{200}} & 0 \\ 0 & \frac{1}{\sqrt{50}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \frac{1}{100} \begin{bmatrix} 5 & -11 \\ 10 & -2 \end{bmatrix}$$

(e) The eigenvalues of A satisfy

$$\det(A - \lambda I) = 0 \to (-2-\lambda)(5-\lambda) + 110 = 0 \to \lambda^2 - 3\lambda + 100 = 0,$$
$$\lambda = \frac{3}{2} \pm i\frac{\sqrt{391}}{2}$$

(f)

$$\begin{aligned} &det(A) = 100, \\ &\lambda_1\lambda_2 = 100, \\ &\sigma_1\sigma_2 = \sqrt{200}\sqrt{50} = \sqrt{10^4} = 100 \end{aligned}$$

(g) The area of the ellipse is $A = \pi\sigma_1\sigma_2 = 100\pi$.

5. *Use the SVD to show that if $A \in \mathbb{C}^{m\times n}$ has rank n then*

$$\|A(A^*A)^{-1}A^*\|_2 = 1.$$

Solution:

Let $A = U\Sigma V^*$ be an SVD for A. Then

$$\begin{aligned} A(A^*A)^{-1}A^* &= U\Sigma V^*(V\Sigma^*\Sigma V^*)^{-1}V\Sigma^*U^* \\ &= U\Sigma V^*(V(\Sigma^*\Sigma)^{-1}V^*)V\Sigma^*U^*, \\ &= U\Sigma(\Sigma^*\Sigma)^{-1}\Sigma^*U^* \end{aligned}$$

Thus

$$\|A(A^*A)^{-1}A^*\|_2 = \|\Sigma(\Sigma^*\Sigma)^{-1}\Sigma^*\|_2$$

Now if $\Sigma \in \mathbb{C}^{m\times n}$, $m \geq n$, *then* Σ *is of the form*

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & \\ 0 & \sigma_2 & 0 & \dots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \sigma_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \end{bmatrix},$$

where $\sigma_i > 0$ *since* A *is full rank. Whence,*

$$\Sigma^*\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots \\ 0 & \sigma_2^2 & 0 & \dots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \sigma_n^2 \end{bmatrix},$$

$$(\Sigma^*\Sigma)^{-1} = \begin{bmatrix} \sigma_1^{-2} & 0 & 0 & \dots \\ 0 & \sigma_2^{-2} & 0 & \dots \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \sigma_n^{-2} \end{bmatrix},$$

$$\Sigma(\Sigma^*\Sigma)^{-1}\Sigma^* = \begin{bmatrix} I_{n\times n} & 0 \\ 0 & 0 \end{bmatrix}_{m\times m}$$

Thus $\|A(A^*A)^{-1}A^*\|_2 = 1$ *from the 2-norm for a diagonal matrix.*

6. *Compress an image using the SVD. Results are shown in figure 2.*

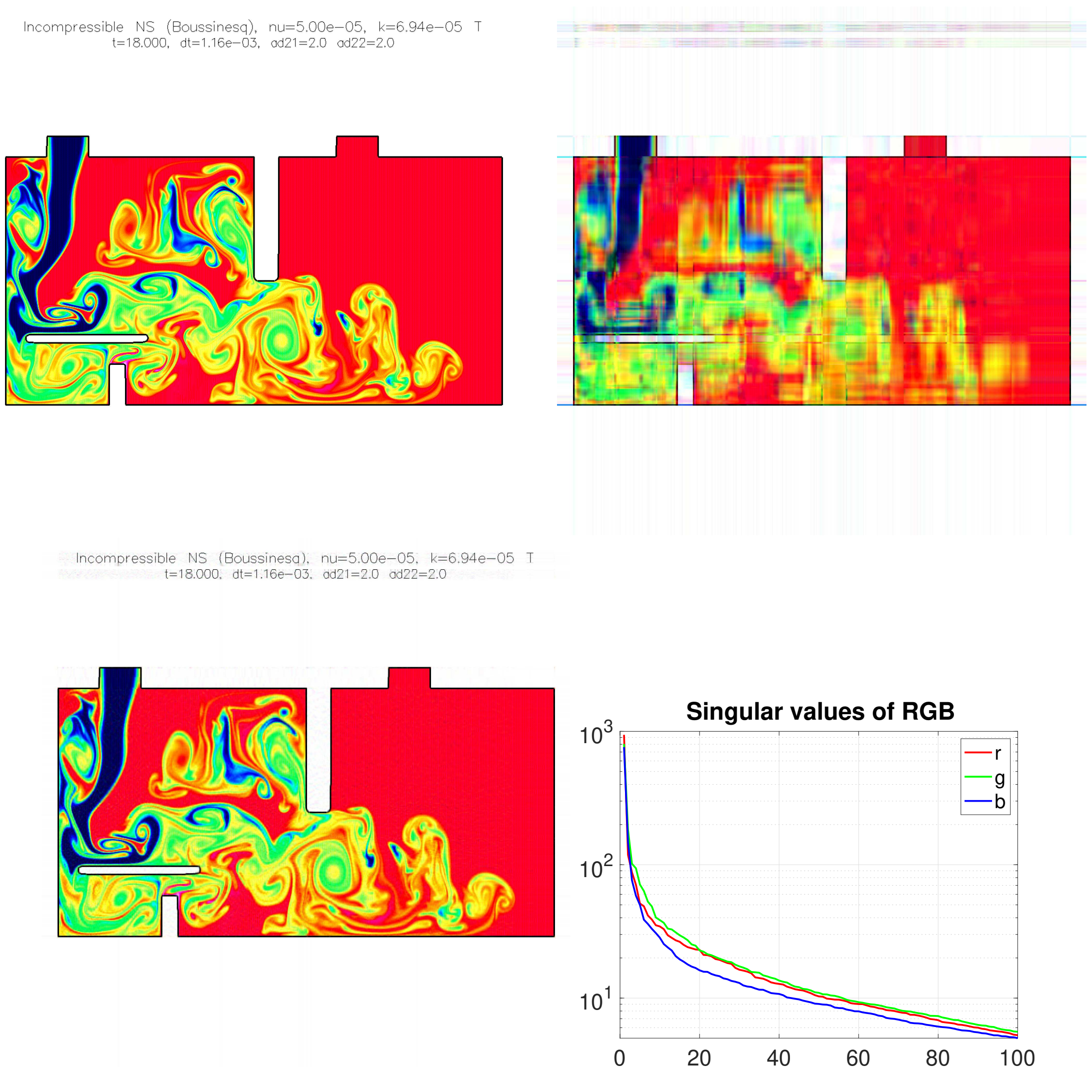

Figure 2: Compressing an image using the SVD. Top left: original image. Top right: keeping 10 singular values. Bottom left: keeping 100 singular values. Bottom right: Log plot of singular values.