

MANE 6760 - FEM for Fluid Dyn. - Lecture 10

Prof. Onkar Sahni, RPI

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ADR Equation: Linear and Scalar

ADR equation:

$$\phi_{,t} + \nabla \cdot (\mathbf{a}\phi - \kappa \nabla \phi) + c\phi = s$$

$$\phi_{,t} + (\mathbf{a}_i \phi - \kappa \phi_{,i})_{,i} + c\phi = s$$

$$\phi_{,t} + (\mathbf{a}_i \phi - \kappa \phi_{,i})_{,i} - r\phi = s$$

Note that Einstein summation notation is used for repeated indices. Also, $c < 0$ or $r > 0$ implies production, and $c > 0$ or $r < 0$ implies destruction.

Peclet and Damköhler numbers characterize the solution, where $Da = |c|L/|a_i|$ (and recall $Pe^G = |a_i|L/\kappa$). Similarly, cell Damköhler number is: $Da^e = |c|h/|a_i|$ (and recall $Pe^e = |a_i|h/(2\kappa)$). Note that (global or cell) Damköhler number is reported along with sign of reactive term: production or destruction.

FE Form: (Simplified) ADR Equation

A number of simplifications:

- ▶ Steady
- ▶ 1D domain: $x \in [0, L]$
- ▶ Only Dirichlet/essential boundary conditions

Find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\int_0^L (-\bar{w}_{,x}(a_x \bar{\phi} - \kappa \bar{\phi}_{,x}) + \bar{w} c p \bar{h} i) dx = \int_0^L \bar{w} s dx$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

System of Equations: ADR

ADR system of equations

(where $\phi = [\phi_1, \phi_2, \dots, \phi_M]^T$ for M solution variables):

$$\phi_{,t} + \nabla \cdot (\underbrace{\mathcal{A}}_{\text{vector of matrices}} \phi - \mathcal{K} \nabla \phi) + \mathcal{C} \phi = \mathbf{s}$$

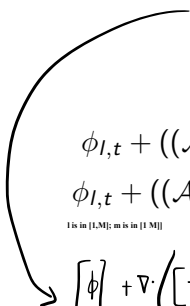
$$\phi_{,t} + (\mathcal{A}_i \phi - \mathcal{K} \phi_{,i})_{,i} + \mathcal{C} \phi = \mathbf{s}$$

$$\phi_{,t} + (\mathcal{A}_i \phi - \mathcal{K} \phi_{,i})_{,i} - \mathcal{R} \phi = \mathbf{s}$$

$$\phi_{l,t} + ((\mathcal{A}_{lm})_i \phi_m - \mathcal{K}_{lm} \phi_{m,i})_{,i} + \mathcal{C}_{lm} \phi_m = s_l$$

$$\phi_{l,t} + ((\mathcal{A}_{lm})_i \phi_m - \mathcal{K}_{lm} \phi_{m,i})_{,i} - \mathcal{R}_{lm} \phi_m = s_l$$

l is in $[1,M]$; m is in $[1,M]$



$$\left[\phi \right]_t + \nabla \cdot \left(\left[\mathcal{A} \right]_i \left[\phi \right] - \left[\mathcal{K} \right] \left[\phi \right]_{,i} \right) + \left[\mathcal{C} \right] \left[\phi \right] = \left[\mathbf{s} \right]$$

\mathcal{A} - vector of matrices; $i: 1 \dots$ to n_{sd}
each of \mathcal{A}_i is $M \times M$ size

\mathcal{K}, \mathcal{C} are of size $M \times M$ each
 \mathcal{K} - isotropic diffusion case

Stabilization Parameter: ADR

A practical way to design stabilization parameter is to consider each differential equation independently (not best suited from a theoretical viewpoint):

$$(\tau_{alg,skb})_I = (\tau_{alg1})_I = \frac{1}{\sqrt{(\mathcal{A}_{II})_i g_{ij} (\mathcal{A}_{II})_j + c_{diff}^2 g_{ij} g_{ij} \mathcal{K}_{II} + \mathcal{C}_{II}^2}}$$

A theoretical way to design stabilization parameter is to apply an eigenanalysis and possibly diagonalize the differential system of equations involving diagonal matrices $\tilde{\mathcal{A}}_i$, $\tilde{\mathcal{K}}$, and $\tilde{\mathcal{C}}$ (not best suited from a practical viewpoint):

$$(\tilde{\tau}_{alg,skb})_I = (\tilde{\tau}_{alg1})_I = \frac{1}{\sqrt{(\tilde{\mathcal{A}}_I)_i g_{ij} (\tilde{\mathcal{A}}_I)_j + c_{diff}^2 g_{ij} g_{ij} \tilde{\mathcal{K}}_I + \tilde{\mathcal{C}}_I^2}}$$

FE Setup and Procedure: Multiple Dimensions

Jacobian matrix (related to mapping between \mathbf{x} and $\boldsymbol{\xi}$):

Local/element level operations (matrices and vectors):

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