MATH-6840
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Due: Monday March 21, 2022

Problem Set 6

1. (25 pts.) Consider the advection-diffusion equation

$$\begin{aligned} u_t + au_x - \nu u_{xx} &= f(x,t), \qquad x \in (0,1), \qquad 0 < t \le T_f \\ u(x,0) &= u_0(x), \qquad x \in (0,1) \\ u(0,t) &= \alpha(t), \\ u_x(1,t) &= \beta(t), \qquad t \ge 0 \qquad t \ge 0. \end{aligned}$$

   (a) Determine $f(x,t)$, $u_0(x)$, $\alpha(t)$, and $\beta(t)$ so that the exact solution to the problem is $u(x,t) = 2\cos(3x)\cos(t)$.

   (b) Now using the computational grid defined by $x_j = j\Delta x$, $-1, 0, \ldots, N+1$, with $\Delta x = 1/N$ (note there is a ghost cell at left and right), define a discrete treatment of the boundary conditions that is at least second-order accurate.

   (c) Write a code to solve this problem using the Crank-Nicolson scheme

$$D_{+t} v_j^n = (-aD_{0x} + \nu D_{+x} D_{-x}) \frac{v_j^{n+1} + v_j^n}{2} + \frac{f_j^{n+1} + f_j^n}{2}.$$

   for all interior j (exact values may depend on your discrete BCs), along with the BCs you defined in part (b) above.

   (d) Perform a grid refinement study with $a = 1$, $\nu = 1$, and $\Delta t = \Delta x$. Present results for the maximum error in the approximation at $t = 1$. Discuss the observed order-of-accuracy.

   (e) In your computations, you should have observed stability for $\Delta t = \Delta x$. Perform a stability analysis for the Cauchy problem (i.e. the infinite domain problem with no BCs) to partially explain this.

2. (25 pts.) Consider the initial-boundary value problem

$$u_t = \nu(u_{xx} + u_{yy}), \quad 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0$$

   with initial condition $u(x, y, t = 0) = u_0(x, y)$, and boundary conditions

$$\begin{aligned} u(0,y,t) &= u(\pi,y,t) = 0 \\ u_y(x,0,t) &= u_y(x,\pi,t) = 0. \end{aligned}$$

   (a) Define a computational grid and second-order accurate discrete BCs. You can use ghost cells or not, as you see fit.

   (b) Write a code to solve this problem using the ADI scheme of Peaceman and Rachford.

   (c) Setting $u_0(x,y) = \sin(x)(\cos(y) - 3\cos(2y))$, compare the numerical and exact solutions at $t = 1$.

   (d) Perform a grid refinement study to verify second-order convergence in both space and time.

   (e) Find a numerical solution using 40 grid lines in both physical dimensions for the case when

$$u_0(x,y) = \begin{cases} 1 & \text{if} \quad (x - \frac{\pi}{2})^2 + (y - \frac{\pi}{2})^2 < \frac{1}{2} \\ 0 & \text{else.} \end{cases}$$

   Plot your results at $t = 0$, $t = .1$, and $t = .5$