W.D. Henshaw

Math 6800: Solutions for Problem Set 1

1. NLA exercise 1.1 *Let B be... Solution:*
Let A_k denote the operation in step k, so that

$$A_5A_3A_2BA_1A_4A_6A_7$$

and

$$\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & .5 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} B \begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Written as ABC:

$$\begin{bmatrix}1 & -1 & .5 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & .5 & 0\\ 0 & -1 & 0 & 1\end{bmatrix} B \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

2. NLA exercise 2.2 *The Pythagorean...*
Solution:

(a) Using $x_1^*x_2 = 0$ then

$$\|x_1 + x_2\|^2 = (x_1 + x_2)^*(x_1 + x_2) = x_1^*x_1 + x_1^*x_2 + x_2^*x_1 + x_2^*x_2 = \|x_1\|^2 + \|x_2\|^2$$

(b) We prove by induction on n, we have shown the case $n = 2$. In general

$$\Big\|\sum_{i=1}^{n} x_i\Big\|^2 = \|x_1 + z\|^2, \qquad z = \sum_{i=1}^{n-1} x_i$$

where $x_1^*z = 0$ and thus from (a)

$$\|x_1 + z\|^2 = \|x_1\|^2 + \|z\|^2$$

and by induction

$$\Big\|\sum_{i=1}^{n} x_i\Big\|^2 = \|x_1 + z\|^2 = \|x_1\|^2 + \sum_{i=1}^{n-1} \|x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

proving the result.

3. NLA exercise 2.3 Let A ...
Solution:

(a) Given $A = A^*$. Let $x \neq 0$ by an eigenvector with eigenvalue λ then

$$Ax = \lambda x \to x^*A^* = \bar{\lambda}x^* \to x^*A = \bar{\lambda}x^*$$

thus

$$x^*Ax = \lambda x^*x = \bar{\lambda}x^*x$$

Thus $\lambda = \bar{\lambda}$, which implies $\lambda \in \mathbb{R}$.

(b) Suppose x and y are eigenvectors corresp. to distinct eigenvalues,

$$Ax = \lambda x, \qquad Ay = \mu y,$$

then

$$\begin{aligned} &Ax = \lambda x \to y^*Ax = \lambda y^*x, \\ &Ay = \mu y \to y^*A^* = \mu y^* \to y^*Ax = \mu y^*x \end{aligned}$$

and thus $y^*Ax = \lambda y^*x = \mu y^*x$ and thus

$$(\lambda - \mu)y^*x = 0$$

and since $\lambda \neq \mu$ then $y^*x = 0$.

4. NLA exercise 2.5 Let $S \in \mathbb{C}^{m\times m}$ be skew-hermitian...
Solution:
(a) Show that the eigenvalues of $S = -S^*$ are pure imaginary.

Solution 1. (using 2.3) Note that $A = iS$ is Hermitian, since

$$A^* = \bar{i}S^* = iS = A. \tag{1}$$

Since A has real eigenvalues, S has pure imaginary eigenvalues.

Solution 2.
Let λ be an eigenvalue and $x \neq \mathbf{0}$ be an eigenvector of S. Then

$$Sx = \lambda x, \tag{2}$$
$$\Longrightarrow x^*Sx = \lambda x^*x, \tag{3}$$

and taking the complex conjugate of this last expression and using $S^* = -S$ implies

$$-x^*Sx = \bar{\lambda}x^*x, \tag{4}$$

There for

$$(\lambda - \bar{\lambda})x^*x = 0 \tag{5}$$

which implies $\bar{\lambda} = -\lambda$, and λ is pure imaginary.

(b) Show that $B = I - S$ is nonsingular. The eigenvalues of B are $1 - \lambda(S)$ and thus cannot be zero since the eigenvalues of S are pure imaginary. Therefore B is nonsingular.

(c) Show that $Q = (I - S)^{-1}(I + S)$ is unitary. Note that

$$Q^* = \Big((I - S)^{-1}(I + S)\Big)^* \tag{6}$$
$$= (I + S)^*(I - S)^{-*} \tag{7}$$
$$= (I + S)^*[(I - S)^*]^{-1} \tag{8}$$
$$= (I - S)(I + S)^{-1} \tag{9}$$

Whence

$$Q^*Q = \Big((I-S)^{-1}(I+S)\Big)^*(I-S)^{-1}(I+S), \tag{10}$$
$$= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S), \tag{11}$$
$$= (I-S)(I-S)^{-1}(I+S)^{-1}(I+S), \tag{12}$$
$$= I \tag{13}$$

where we have used the fact that $(I+S)^{-1}$ and $(I-S)^{-1}$ commute. This can be seen from

$$(I-S)^{-1}(I+S)^{-1} = \Big[(I+S)(I-S)\Big]^{-1} = \Big[I-S^2\Big]^{-1}, \tag{14}$$
$$(I+S)^{-1}(I-S)^{-1} = \Big[(I-S)(I+S)\Big]^{-1} = \Big[I-S^2\Big]^{-1} \tag{15}$$

5. NLA exercise 2.6 If u and v are ...
Solution:

$$A = I + uv^*, \qquad u, v \in \mathbb{C}^m,$$

Suppose $A^{-1} = I + \alpha uv^*$ for some $\alpha \in \mathbb{C}$, then

$$\begin{aligned} AA^{-1} &= (I+uv^*)(I+\alpha uv^*) \\ &= I + (\alpha + 1 + \alpha v^*u)uv^* \end{aligned}$$

and thus $AA^{-1} = I$ when

$$\alpha = \frac{-1}{1+v^*u}.$$

The inverse in this form exists provided $v^*u \neq -1$. Conversely if $v^*u = -1$ then A is singular since $Au = 0$.

Now if $x \in null(A)$ then $Ax = 0$ and

$$(I+uv^*)x = 0 \ \to x = -(v^*x)u$$

and thus x is parallel to u, say $x = \beta u$. Substituting $x = \beta u$ into $x = -(v^*x)u$ gives $\beta u = -\beta(v^*u)u$ which has a non trivial solution only if $v^*u = -1$ in which case

$$null(A) = span(u), \ \ \text{provided } v^*u = -1.$$

6. NLA exercise 3.1 Prove that if W...
Solution: Let W be a nonsingular matrix and let $\|\cdot\|$ be any vector norm. Define

$$\|x\|_W \equiv \|Wx\|$$

To show that $\|x\|_W$ is a vector norm we must show it satisfies the 3 properties.
Property (i) Here we use W is non-singular which implies $Wx = 0$ iff $x = 0$.

$$\begin{aligned} &\|x\|_W = \|Wx\| \geq 0, \\ &\|x\|_W = 0 \to \|Wx\| = 0 \to Wx = 0 \to x = 0 \end{aligned}$$

Property (ii)

$$\|x+y\|_W = \|W(x+y)\| = \|Wx+Wy\| \leq \|Wx\| + \|Wy\| = \|x\|_W + \|y\|_W$$

Property (iii)

$$\|\alpha x\|_W = \|\alpha W x\| = |\alpha| \|Wx\| = |\alpha| \|x\|_W.$$

7. NLA exercise 3.3 *Vector and matrix p-norms...*
Solution:

(a)

$$\|x\|_\infty = \max_{1 \leq i \leq m} |x_i| \leq \left(\sum_{j=1}^{m} |x_j|^2 \right)^{1/2} = \|x\|_2$$

and this bound is achieved with $x = e_k$.

(b) Let $|x_k| = \max_i |x_i|$. Then

$$\|x\|_2^2 = \sum_{j=1}^{m} |x_j|^2 \leq \sum_{j=1}^{m} |x_k|^2 = m|x_k|^2 \rightarrow \|x\|_2 \leq \sqrt{m}\|x\|_\infty$$

This bound is achieved with $x = [1,1,1,\ldots,1]^T$.

(c) Let $A \in \mathbb{C}^{m \times n}$ and $x \in \mathbb{C}^n$. By definition

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

From (a) and (b), for any $x \in \mathbb{C}^n$

$$\|Ax\|_\infty \leq \|Ax\|_2, \qquad \|x\|_2 \leq \sqrt{n}\|x\|_\infty.$$

Thus for any $x \neq 0$,

$$\frac{\|Ax\|_\infty}{\sqrt{n}\|x\|_\infty} \leq \frac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_2$$

Therefore for any $x \neq 0$

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \sqrt{n}\|A\|_2$$

Taking the maximum of the left-hand-side gives

$$\|A\|_\infty \leq \sqrt{n}\|A\|_2.$$

This bound is achieved, e.g., for a matrix A with all one's in the first row and zeros elsewhere,

$$A = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \end{bmatrix}$$

Then $\|A\|_\infty = n$ (max row-sum). Letting $w = [1, 1, 1, \ldots, 1]^T$ be the first row of A then (using the Cauchy-Schwartz inequality)

$$\|Ax\|_2 = |w^T x| \leq \|w\|_2 \|x\|_2 = \sqrt{n}\|x\|_2$$

with equality when $x = w$. Thus $\|A\|_2 = \sqrt{n}$. Therefore with this A, we have achieved equality $\|A\|_\infty = \sqrt{n}\|A\|_2$.

(d) From (a) and (b), for any $x \in \mathbb{C}^n$

$$\|Ax\|_2 \leq \sqrt{m}\|Ax\|_\infty, \qquad \|x\|_\infty \leq \|x\|_2.$$

Thus for any $x \neq 0$,

$$\frac{\|Ax\|_2}{\|x\|_2} \leq \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty} \leq \sqrt{m}\|A\|_\infty$$

Therefore for any $x \neq 0$

$$\frac{\|Ax\|_2}{\|x\|_2} \leq \sqrt{m}\|A\|_\infty$$

Taking the maximum of the left-hand-side gives

$$\|A\|_2 \leq \sqrt{m}\|A\|_\infty.$$

This bound is achieved, e.g., for the matrix A with entries one in the first column,

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & \cdots \end{bmatrix},$$

for then $\|A\|_\infty = 1$ (max row-sum) and

$$\|Ax\|_2 = \|[x_1, x_1, \ldots x_1]^T\|_2 = \sqrt{m}|x_1| \leq \sqrt{m}\|x\|_2$$

with equality when $x = [1, 0, 0, \ldots]$. Therefore $\|A\|_2 = \sqrt{m}$ and for this matrix we have $\|A\|_2 = \sqrt{m}\|A\|_\infty$.

8. Let $A \in \mathbb{C}^{m\times n}$ with columns a_i, and $B \in \mathbb{C}^{p\times n}$ with columns b_i

$$A = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \ldots & a_n \end{array}\right], \qquad B = \left[\begin{array}{c|c|c|c} b_1 & b_2 & \ldots & b_n \end{array}\right],$$

Show that

$$AB^* = a_1 b_1^* + a_2 b_2^* + \ldots + a_n b_n^*$$

Solution:

Solution 1. The result can be proved by *brute force* using the definitions of the entries for matrix-matrix products. The ij entry of the matrix AB^* is

$$(AB^*)_{ij} = \sum_{k=1}^{n} a_{ik}\bar{b}_{jk},$$

The ij entry of the rank-one matrix $a_k b_k^*$ is

$$(a_k b_k^*)_{ij} = a_{ik}\bar{b}_{jk},$$

and thus

$$a_1 b_1^* + a_2 b_2^* + \ldots + a_n b_n^* = \sum_{k=1}^{n} (a_k b_k^*)_{ij} = \sum_{k=1}^{n} a_{ik}\bar{b}_{jk}$$

which equals $(AB^*)_{ij}$.

Solution 2. Secondly we can use block matrix multiplication,

$$AB^* = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \end{array}\right] \left[\begin{array}{c} b_1^* \\ \hline b_2^* \\ \hline \vdots \\ \hline b_n^* \end{array}\right] = a_1 b_1^* + a_2 b_2^* + \ldots + a_n b_n^*$$

Solution 3. An alternative proof is to use $I_{n\times n} = e_1 e_1^* + e_2 e_2^* + \ldots e_n e_n^*$ and thus

$$AB^* = A\ I\ B^* = A(e_1 e_1^* + e_2 e_2^* + \ldots e_n e_n^*)B^* = \sum_{k=1}^{n} (Ae_k)(e_k^* B^*) = \sum_{k=1}^{n} (Ae_k)(Be_k)^* = \sum_{k=1}^{n} a_k b_k^*$$