MANE 6760 - FEM for Fluid Dyn. - Lecture 06

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Regular FE Form: AD equation

Regular/standard FE (Galerkin) form: find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$a(\bar{w}, \bar{\phi}) = (\bar{w}, s)$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

Leads to numerically spurious oscillations for an advection-dominated case with high (global) Peclet number, $Pe = Pe^G = |\boldsymbol{a}|L/\kappa$, particular when cell/element Peclet number, $Pe^e = |\boldsymbol{a}|h/(2\kappa)$ is above $\mathcal{O}(1)$ (note for uniform mesh $Pe^e = Pe^G/(2N_e)$)

Stabilized FE Form: AD equation

Stabilized/generalized FE (Galerkin) form: find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$a(\bar{w}, \bar{\phi}) + a_{stab}(\bar{w}, \bar{\phi}) = (\bar{w}, s)$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

Several options available for $a_{stab}(\cdot, \cdot)$:

- Streamline-upwind Petrov Galerkin/SUPG
- Galerkin least squares/GLS
- Variational multiscale/VMS
- ... others (residual-free bubbles, etc)

Stabilized FE Options: AD equation

A general stabilized FE form:

- Consider a general operator $\hat{\mathcal{L}}(\cdot)$ acting on weight function (becomes specific when a particular choice is made for $\hat{\mathcal{L}}(\cdot)$)

$$a(\bar{w},\bar{\phi}) + a_{stab}(\bar{w},\bar{\phi}) = (\bar{w},s)$$

$$a(\bar{w},\bar{\phi}) + \underbrace{(\hat{\mathcal{L}}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}}_{a_{stab}(\cdot,\cdot)} = (\bar{w},s)$$

  where $\hat{\Omega}$ is the element interiors, τ is the stabilization parameter and recall $R(\cdot)$ is the strong-form residual leading to a consistent method in that when $\bar{u} = u$ (i.e., exact solution) then $a_{stab}(\cdot,\cdot) = 0$
- Why element interiors? (i.e., $\hat{\Omega}$)

Stabilized FE Options: AD equation

Streamline-upwind Petrov-Galerkin/SUPG form:

- $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}^{adv}(\cdot) = -\boldsymbol{a} \cdot \nabla(\cdot)$

$$a_{SUPG}(\bar{w}, \bar{\phi}) = (-\mathcal{L}^{adv}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- Why is it referred to as streamline-upwind?

- Why is it referred to as Petrov-Galerkin?

Stabilized FE Options: AD equation

Galerkin least squares/GLS form:

- $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}(\cdot) = -\left(\mathcal{L}^{adv}(\cdot) + \mathcal{L}^{diff}(\cdot)\right) = -\left(\boldsymbol{a} \cdot \nabla(\cdot) - \kappa \nabla^2(\cdot)\right)$

$$a_{GLS}(\bar{w}, \bar{\phi}) = (-\mathcal{L}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- Why is it referred to as Galerkin least squares?

Stabilized FE Options: AD equation

Variational multiscale/VMS form:

- $\hat{\mathcal{L}}(\cdot) = \mathcal{L}^{*}(\cdot) = -\mathcal{L}^{adv}(\cdot) + \mathcal{L}^{diff}(\cdot) = -\boldsymbol{a} \cdot \nabla(\cdot) - \kappa \nabla^2(\cdot)$ (adjoint operator)

$$a_{VMS}(\bar{w}, \bar{\phi}) = (\mathcal{L}^{*}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- Why is it referred to as variational multiscale?

- Note that all stabilized forms are equivalent when using linear (simplicial) finite elements for steady, linear, scalar advection-diffusion equation

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