MATH-6840
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Due: Thursday April 7, 2022

Problem Set 8

1. (25 pts.) Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad t > 0$$

with initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ (boundary conditions will be added later).

   (a) Derive a sixth-order accurate (in space and time) discretization of this equation using centered spatial differences and the 3-level modified equation time stepper discussed in class. Hint: Useful discretizations may be found on the 2nd page of this document.

   (b) Using Fourier mode analysis, derive an expression for the amplification factors. Create a surface plot of the magnitude of each of the two roots for $\sigma = c\Delta t/\Delta x \in [-1.1, 1.1]$ and for the discrete wave number $\xi \in [-\pi, \pi]$.

   (c) Now restrict consideration to the finite domain $x \in [-1, 1]$ with boundary conditions $u_x(-1,t) = \alpha(t)$, $u(1,t) = \beta(t)$. Using the computational grid defined by $x_j = -1 + j\Delta x$, $0 \le j \le N$, $\Delta x = 2/N$, introduce ghost cells as needed and define appropriate compatibility boundary conditions suitable for 6th order accuracy.

   (d) Write a code implementing the sixth-order method. Perform a convergence study using the exact solution $u(x,t) = \sin(5(x - ct)) + \cos(2(x + ct))$ with $c = .9$.

2. (25 pts.) Consider the wave propagation problem in an annular section

$$u_{tt} = c^2 \left[\frac{1}{r}(ru_r)_r + \frac{1}{r^2} u_{\theta\theta} \right], \quad \frac{1}{2} < r < 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad t > 0$$

with initial condition $u(r,\theta,0) = f(r,\theta)$, $u_t(r,\theta,0) = g(r,\theta)$, and boundary conditions

$$u\left(\frac{1}{2}, \theta, t\right) = 0, \qquad u_r\,(1,\theta,t) = 0$$
$$u\left(r, -\frac{\pi}{2}, t\right) = 0 \qquad u_\theta\left(r, \frac{\pi}{2}, t\right) = 0.$$

   (a) Write a second-order accurate code to solve this problem using centered differencing and the 3-level modified equation time stepper discussed in class. That is to say you must to treat this as a variable coefficient operator rather than performing a chain rule (e.g. you must discretize $(ru_r)_r$ as it sits and **not** convert it to $u_r + ru_{rr}$). Note this code will have a maximal stable time step and your code will need to be constructed to satisfy this constraint.

   (b) Verify the accuracy of your code using a manufactured solution. Here you should use $N_\theta = 3N_r$ so that in physical space the grids are approximately square. Note that you will likely need to consider non-homogeneous boundary conditions since your exact solution may not satisfy the given BCs.

(c) Using $c = 1$, $Nr = 160$ and $N_\theta = 480$, compute numerical solutions to this problem using $f(r,\theta) = \exp(-100((r-0.75)^2 + (\theta)^2))$, $g(r,\theta) = 0$ at $t = 0, .5, 1.5, 2.5$. Create surface plots of the solution for each time. In addition, create a single line plot with four curves showing the solution along the outer radius ($r = 1$), as a function of θ for all four times.

The following discrete approximations may be useful for problem (1)

$$
\begin{aligned}
u_x(x_j) &= \frac{u_{j+3} - 9u_{j+2} + 45u_{j+1} - 45u_{j-1} + 9u_{j-2} - u_{j-3}}{60\Delta x} + O(\Delta x^6) \\
u_{xx}(x_j) &= \frac{2u_{j+3} - 27u_{j+2} + 270u_{j+1} - 490u_j + 270u_{j-1} - 27u_{j-2} + 2u_{j-3}}{180\Delta x^2} + O(\Delta x^6) \\
u_{xxx}(x_j) &= \frac{-u_{j+3} + 8u_{j+2} - 13u_{j+1} + 13u_{j-1} - 8u_{j-2} + u_{j-3}}{8\Delta x^3} + O(\Delta x^4) \\
u_{xxxx}(x_j) &= \frac{-u_{j+3} + 12u_{j+2} - 39u_{j+1} + 56u_j - 39u_{j-1} + 12u_{j-2} - u_{j-3}}{6\Delta x^4} + O(\Delta x^4) \\
u_{xxxxx}(x_j) &= \frac{u_{j+3} - 4u_{j+2} + 5u_{j+1} - 5u_{j-1} + 4u_{j-2} - u_{j-3}}{2\Delta x^5} + O(\Delta x^2) \\
u_{xxxxxx}(x_j) &= \frac{u_{j+3} - 6u_{j+2} + 15u_{j+1} - 20u_j + 15u_{j-1} - 6u_{j-2} + u_{j-3}}{\Delta x^6} + O(\Delta x^2)
\end{aligned}
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