MATH-6840
J. W. Banks
Due: Monday April 25, 2022

Problem Set 10

1. (20 pts. extra credit) Consider the inviscid Burgers' equation $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ with initial conditions

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

   (a) Show that the following is a weak solution for $\alpha < 1$

$$u(x,t) = \begin{cases} -1 & \text{if } x < -(1-\alpha)t/2 \\ \alpha & \text{if } -(1-\alpha)t/2 \leq x < 0 \\ -\alpha & \text{if } 0 \leq x < (1-\alpha)t/2 \\ 1 & \text{if } x \geq (1-\alpha)t/2. \end{cases}$$

   (b) Plot (or sketch) the solution from part (a) for $\alpha = .5$, and plot (or sketch) the characteristics in the $x-t$ plane. Is this solution an entropy satisfying solution (give a reason as to why or why not)?

   (c) Construct the entropy satisfying weak solution to this problem.

   (d) Create a conservative upwind code to run this case and present computed solution at $t = 0.5$. Compare the numerical solution to the exact solution from part (c) above.

2. (20 pts. extra credit) Now let us revisit the annular section problem from PS7, and consider the steady state of a heat conduction problem in an annular section

$$0 = \left[\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}\right], \quad 1 < r < 2, \quad 0 < \theta < \frac{\pi}{2},$$

with boundary conditions

$$\begin{aligned} u_r(1,\theta) &= 0, & u_r(2,\theta) &= 0 \\ u(r,0) &= 0 & u(r,\tfrac{\pi}{2}) &= (r-1)^2(r-2)^2. \end{aligned}$$

   (a) Starting from your code from PS7 (or the one from SS7), write a second-order accurate code to solve this problem using centered differencing. Hint, as before you may find it useful to use manufactured solutions to verify the accuracy of your code.

   (b) Using 40 grid lines in both the radial and angular coordinate directions, compute a numerical solutions to this problem, and create a surface plots of the solution. In addition, create a single line plot with two curves showing the solution along the inner radius ($r = 1$), and the outer radius ($r = 2$), as a function of θ.

   (c) Finally, compare the steady state solution you compute here to the solutions of the time-dependent problem from PS7. In particular show the approach of the time-dependent solutions to the steady solution along the inner radius $r = 1$.