MANE 6760 - FEM for Fluid Dyn. - Lecture 02

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Strong Form

Strong form of the governing equation

$$R(\phi) = \mathcal{L}(\phi) - s = \frac{\partial \phi}{\partial t} + \nabla \cdot \boldsymbol{F}(\phi) - s = 0, \qquad \phi \in \mathcal{S}_{strong}$$
$$\phi = d_g \quad \text{on} \quad \Gamma_g \in \partial\Omega$$
$$b(\phi) = d_h \quad \text{on} \quad \Gamma_h \in \partial\Omega$$

Boundary conditions are set on Γ_g and Γ_h such that $\Gamma_g \cap \Gamma_h = 0$ and typically $\Gamma_g \cup \Gamma_h = \partial\Omega$

- $\mathcal{S}_{strong}$: solution space
- d_g and d_h: prescribed boundary data
- $b(\cdot)$: boundary operator, e.g., $b(\phi) = \boldsymbol{F}(\phi) \cdot \boldsymbol{n} = h$ or $b(\phi) = (-\kappa \nabla \phi \cdot \boldsymbol{n}) = h$

Weak Form: Infinite-dimensional or Continuous

Weak form of the governing equation (using integration by parts)

$$\int_\Omega wR(\phi)dV = \int_\Omega w\left(\mathcal{L}(\phi) - s\right)dV = (w, \mathcal{L}(\phi) - s) = 0, \phi \in \mathcal{S}, \forall w \in \mathcal{W}$$

$$(w, \mathcal{L}(\phi)) = a(w, \phi) = (w, s), \quad \phi \in \mathcal{S}, \forall w \in \mathcal{W}$$

- $\mathcal{S}$: solution or trial space
- $\mathcal{W}$: weight or test space
- $a(\cdot, \cdot)$: bilinear (or semi-linear) form
- $(\cdot, \cdot)$: L_2 inner product

Galerkin Weak Form: Finite-dimensional or Discretized

Galerkin weak form: find $\tilde{\phi} \in \tilde{\mathcal{S}} \subset \mathcal{S}$ such that

$$a(\tilde{w}, \tilde{\phi}) = (\tilde{w}, s)$$

for all $\tilde{w} \in \tilde{\mathcal{W}} \subset \mathcal{W}$

- $\tilde{(\cdot)}$: denotes a finite-dimensional/discretized quantity based on a discretization (e.g., spectral elements, finite elements, ...)
- $a(\cdot, \cdot)$: bilinear (or semi-linear) form
- $(\cdot, \cdot)$: L_2 inner product

Finite-element (FE) Weak Form

Finite-element based (Galerkin) weak form: find $\phi^{h,p} \in \mathcal{S}^{h,p} \subset \mathcal{S}$ such that

$$a(w^{h,p}, \phi^{h,p}) = (w^{h,p}, s)$$

for all $w^{h,p} \in \mathcal{W}^{h,p} \subset \mathcal{W}$

- $(\cdot)^{h,p}$: denotes a finite-dimensional quantity based on a FE discretization
- h: element size
- p: basis order
- $a(\cdot,\cdot)$: bilinear (or semi-linear) form
- $(\cdot,\cdot)$: L_2 inner product

FE Form: AD equation

FE form for AD equation: find $\phi^{h,p} \in \mathcal{S}^{h,p} \subset \mathcal{S}$ such that

$$a(w^{h,p}, \phi^{h,p}) = (w^{h,p}, s)$$

$$\int_\Omega w^{h,p} \frac{\partial \phi^{h,p}}{\partial t} dV + \int_\Omega \nabla w^{h,p} \cdot \boldsymbol{F}(\phi^{h,p}) dV - \int_{\Gamma_h} w^{h,p} \underbrace{d_h}_{\boldsymbol{F}\cdot\boldsymbol{n}} = (w^{h,p}, s)$$

$$\int_\Omega w^{h,p} \frac{\partial \phi^{h,p}}{\partial t} dV + \int_\Omega \nabla w^{h,p} \cdot (\boldsymbol{a}\phi^{h,p} - \kappa \nabla \phi^{h,p}) dV - \int_{\Gamma_h} w^{h,p} \underbrace{d_h}_{\boldsymbol{F}\cdot\boldsymbol{n}} = (w^{h,p}, s)$$

for all $w^{h,p} \in \mathcal{W}^{h,p} \subset \mathcal{W}$

- $\mathcal{S}^{h,p}$: $\{\phi^{h,p} | \phi^{h,p} \in H^1, \phi^{h,p} = d_g \text{ on } \Gamma_g\}$
- $\mathcal{W}^{h,p}$: $\{w^{h,p} | w^{h,p} \in H^1, w^{h,p} = 0 \text{ on } \Gamma_g\}$

FE Form: AD equation

For brevity we set FE form for AD equation: find $\phi^h = \bar{\phi} \in \mathcal{S}^h = \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\int_\Omega w^h \frac{\partial \phi^h}{\partial t} dV + \int_\Omega \nabla w^h \cdot (\boldsymbol{a}\phi^h - \kappa \nabla \phi^h) dV - \int_{\Gamma_h} w^h \underbrace{d_h}_{\boldsymbol{F}\cdot\boldsymbol{n}} = (w^h, s)$$

$$\int_\Omega \bar{w} \frac{\partial \bar{\phi}}{\partial t} dV + \int_\Omega \nabla \bar{w} \cdot (\boldsymbol{a}\bar{\phi} - \kappa \nabla \bar{\phi}) dV - \int_{\Gamma_h} \bar{w} \underbrace{d_h}_{\boldsymbol{F}\cdot\boldsymbol{n}} = (\bar{w}, s)$$

for all $w^h = \bar{w} \in \mathcal{W}^h = \bar{\mathcal{W}} \subset \mathcal{W}$

FE Form: AD equation

A number of simplifications:

- Steady
- 1D domain: $x \in [0, L]$
- No source term: $s{=}0$
- Only Dirichlet/essential boundary conditions and no Neumann/flux boundary condition: $\Gamma_h = \emptyset$, i.e., $\Gamma_g = \partial\Omega = \{x = 0, x = L\}$

Find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\boxed{}$$

for all $w^h = \bar{w} \in \mathcal{W}^h = \bar{\mathcal{W}} \subset \mathcal{W}$

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