MANE 6760 - FEM for Fluid Dyn. - Lecture 17

Prof. Onkar Sahni, RPI

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Simplified: 1D Non-linear (NL) AD Eqn

A number of simplifications:

- Steady
- 1D domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Strong form:

$$R(\phi) = \mathcal{L}(\phi) - s = a_x \frac{d\phi}{dx} - \frac{d}{dx}\left(\kappa(\phi)\frac{d\phi}{dx}\right) - s = 0 \qquad \phi \in \mathcal{S}_{strong}$$

$$x \in [0, L]$$

$$\phi(x = 0) = \phi_0 \quad \text{on} \quad x = 0$$

$$\phi(x = L) = \phi_L \quad \text{on} \quad x = L$$

Method of Manufactured Sol.: (Simplified) 1D NL AD Eqn

Method of manufactured solution: assume an exact solution (a form/expression) and determine BCs and source term, and use these BCs and source term (in FE code to compute an approximate FE solution).

For example (γ is a "free" parameter):

$$\phi(x) = 1 + \frac{x}{L} - \frac{e^{-\gamma(L-x)} - e^{-\gamma L}}{1 - e^{-\gamma L}}$$

$$\phi(x = 0) = \phi_0 = 1$$

$$\phi(x = L) = \phi_L = 1$$

$$s(x) = \dots$$

Stabilized FE Form: (Simplified) 1D NL AD Eqn

Stabilized FE forms: find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\int_0^L \left(\underbrace{-\bar{w}_{,x} a_x \bar{\phi}_{,x}}_{1} + \underbrace{\bar{w}_{,x} \kappa(\bar{\phi}) \bar{\phi}_{,x}}_{2} + \underbrace{\ldots\ldots}_{3} + \underbrace{(-\bar{w} s)}_{4} + \underbrace{\ldots\ldots}_{5} \right) dx = 0$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

Non-linear Iterations: (Simplified) 1D NL AD Eqn

Non-linear iterations: require non-linear weak residual and tangent matrix at every iteration

$$G_A = \int_0^L \left(\underbrace{-N_{A,x} a_x \bar{\phi}_{,x}}_{1} + \underbrace{N_{A,x} \kappa(\bar{\phi}) \bar{\phi}_{,x}}_{2} + \underbrace{\ldots\ldots}_{3} + \underbrace{(-N_A s)}_{4} + \underbrace{\ldots\ldots}_{5} \right) dx$$

$$\begin{aligned} \frac{\partial G_A}{\partial \hat{u}_B} = & \underbrace{\ldots\ldots}_{1} \\ & + \underbrace{\ldots\ldots}_{2} \\ & + \underbrace{\ldots\ldots}_{3} \simeq N_{A,x} k_{num} N_{B,x} \\ & + \underbrace{\ldots\ldots}_{4} = 0 \\ & + \underbrace{\ldots\ldots}_{5} \simeq 0 \end{aligned}$$

Examples: (Simplified) 1D NL AD Eqn

$s = 1$

- $\kappa(\phi) = \kappa_0$
- $\kappa(\phi) = \kappa_0 \phi$
- $\kappa(\phi) = \kappa_0 \phi^2$
- ...

Simplified: 1D Unsteady/Transient AD (TAD) Eqn

A number of simplifications:

- 1D (spatial) domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Strong form:

$$
\begin{aligned}
R(\phi) = \mathcal{L}(\phi) - s = \frac{\partial \phi}{\partial t} + a_x \frac{\partial \phi}{\partial x} - \kappa \frac{\partial^2 \phi}{\partial x^2} - s &= 0 \qquad \phi \in \mathcal{S}_{strong} \\
x &\in [0, L] \\
t &\in [t_0, t_1] \\
\phi(x, t = t_0) &= \phi_{IC}(x) \, \forall x \\
\phi(x = 0, t) &= \phi_0(t) \quad \text{on} \quad x = 0 \, \forall t \\
\phi(x = L, t) &= \phi_L(t) \quad \text{on} \quad x = L \, \forall t
\end{aligned}
$$

FE Setup and Procedure

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