MATH-6800
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Problem Set 2

1. NLA exercise 4.1 *Determine SVDs of ...*

   (a)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}; \ A^* = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$
$$AA^* = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Since, its a square diagonal matrix, its eigen-values are given by $\lambda = 9, 4$. This means the singular values are calculated as the square root of the eigen-values $\implies \sigma = 3, 2$. Proceeding to find the $null\,(\lambda I - AA^*)$ will help us find the left singular vectors of A, given by $U = [u_1|u_2]$.

$$null\,(\lambda I - AA^*) = null\left(\begin{bmatrix} \lambda - 9 & 0 \\ 0 & \lambda - 4 \end{bmatrix}\right)$$

If $\lambda = 9$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 5u_{21} = 0, \quad \text{therefore, I set, } u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If $\lambda = 4$,

$$\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -5u_{12} = 0, \quad \text{therefore, I set, } u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since $Av_i = \sigma_i u_i$, we can use the left singular vectors to figure out what the right singular vector are,

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 3u_1 \implies v_{11} = 1 \text{ and } v_{21} \text{ can be anything, therefore, I'll set it to be } v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 2u_2 \implies v_{22} = -1 \text{ and } v_{12} \text{ can be anything, therefore, I'll set it to be } v_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Hence, the final SVD of this matrix is written as $A = U\Sigma V^*$, where,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and, } V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   (b)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; \ A^* = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$
$$AA^* = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

Similar to the first part, since its a square diagonal matrix, the eigen values are given by $\lambda = 4, 9$. This means the singular values are calculated as $\sigma = 2, 3$. In an ordered way, its $\sigma_1 = 3, \sigma_2 = 2$, since $\sigma_1 \geq \sigma_2 \geq 0$. Proceeding to find the $null\,(\lambda I - AA^*)$ will help in finding the left singular vectors $U = [u_1|u_2]$.

$$null\,(\lambda I - AA^*) = null\left(\begin{bmatrix} \lambda - 4 & 0 \\ 0 & \lambda - 9 \end{bmatrix}\right)$$

If, $\lambda = 9$,

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 5u_{11} = 0, \text{ therefore, I set, } u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If, $\lambda = 4$,

$$\begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -5u_{22} = 0, \text{ therefore, I set, } u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since, $Av_i = \sigma_i u_i$, we use the left singular vectors to find the right singular vectors.

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 3u_1 \implies v_{11} = 0, v_{21} = 1. \text{ Hence, } v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 2u_2 \implies v_{12} = 1, v_{22} = 0. \text{ Hence, } v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, $A = U\Sigma V^*$, where,

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and, } V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} ; A^* = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

The eigen-values are $\lambda = 0, 4$ and the singular values in order are $2, 0$. The $null\,(\lambda I - A^*A)$ gives us the right singular vectors in this case.

$$null\,(\lambda I - A^*A) = null\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda - 4 \end{bmatrix}\right)$$

If $\lambda = 4$,

$$\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_{11} = 0, \text{ therefore I choose, } v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If $\lambda = 0$,

$$\begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_{22} = 0, \text{ therefore I choose, } v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using, $Av_i = \sigma_i u_i$, we can try to find all u_i's. Since, we only have 1 non-zero singular value, we can populate the other two left singular vectors to be orthogonal to u_1 and to each other. We can find u_1, as $u_1 = \left(\frac{1}{\sigma_1}\right) Av_1$. $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Two other left singular vectors that are orthogonal to u_1 are

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The SVD of $A = U\Sigma V^*$, is

$$U = I_{3\times 3}, \ \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \ ; A^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$
$$AA^* = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

The eigen-values are $\lambda = 2, 0$ and hence the singular values are $\sigma = \sqrt{2}, 0$.

$$null\left(\lambda I - AA^*\right) = null\left(\begin{bmatrix} \lambda - 2 & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

If $\lambda = 2$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies u_{21} = 0, \text{ therefore, I set, } u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

If $\lambda = 0$,

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies u_{12} = 0, \text{ therefore, I set, } u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Using the fact that $Av_i = \sigma_i u_i$,

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \sqrt{2} u_1 \implies v_{11} + v_{21} = \sqrt{2}. \text{ Hence, I choose, } v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Since, we no-longer have any non-zero singular values, we can choose a vector orthogonal to v_1 and one that makes V a unitary matrix $\implies v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ fits the requirements. Therefore, the SVD of $A = U\Sigma V^*$ where,

$$U = I_{2\times 2}, \ \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, \text{ and, } V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(e)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \; ; \; A^* = A$$

$$AA^* = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

We can find the eigen-values of this by finding the roots of the characteristic polynomial formed by the $\det(\lambda I - AA^*) = 0$.

$$\det\left(\begin{bmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{bmatrix}\right) = \lambda\left(\lambda - 4\right) = 0$$

The eigen-values are $\lambda = 4, 0$ and hence the singular values are $\sigma = 2, 0$. Now, since $AV = U\Sigma$,

$$\begin{bmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \sigma_1 u_1 & \sigma_2 u_2 \\ | & | \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} v_{11} + v_{21} \\ v_{11} + v_{21} \end{bmatrix} & \begin{bmatrix} v_{12} + v_{22} \\ v_{12} + v_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sigma_1 \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

Since, $v_{12} + v_{22} = 0$, I choose them to be negative of each other and hence $v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Now, a vector orthogonal to this makes, $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. This makes $u_{11} = u_{21} = 1/\sqrt{2}$. Therefore a vector orthogonal to that will be $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. Therefore the final SVD of $A = U\Sigma V^*$, where,

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \; \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and, } V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

All the cases and its solutions have been verified and listed in Listing 1 and the answers are in the PDF attached below.

```
clc;
clear all;
%% Problem 1
U = eye(2);
S = [3 0; 0 2];
V = [1 0; 0 -1];
disp("part(a) A = ");
disp(U*S*V');

U = [0 1; 1 0];
S = [3 0; 0 2];
V = [0 1; 1 0];
disp("part(b) A = ");
disp(U*S*V');

U = eye(3);
S = [2 0; 0 0; 0 0];
```

```
V = [0 1; 1 0];
disp("part(c) A = ");
disp(U*S*V');

U = eye(2);
S = [sqrt(2) 0; 0 0];
V = 1/(sqrt(2))*[1 1; 1 -1];
disp("part(d) A = ");
disp(U*S*V');

U = (1/sqrt(2))*[1 1; 1 -1];
S = [2 0; 0 0];
V = (1/sqrt(2))*[1 -1; 1 1];
disp("part(e) A = ");
disp(U*S*V');
```

Listing 1: Q1-verification script

```
part(a) A =
     3     0
     0    -2

part(b) A =
     2     0
     0     3

part(c) A =
     0     2
     0     0
     0     0

part(d) A =
     1     1
     0     0

part(e) A =
    1.0000    1.0000
    1.0000    1.0000

>>
```

2. NLA exercise 4.4 *Two matrices ...*
Let $A, B \in \mathbb{C}^{m\times m}$ be unitarily equivalent. Then, $A = QBQ^*$ where $Q \in \mathbb{C}^{m\times m}$ and unitary. The matrix B will have it own SVD where $B = U_B\Sigma_B V_B^*$. The matrix A also has its own SVD which makes it, $A = U_A\Sigma_A V_A^*$.

$$\begin{aligned} A &= QBQ^* \\ &= (QU_B)\,\Sigma_B\,(V_B^* Q^*) \\ &= (QU_B)\,\Sigma_B\,(QV_B)^* \end{aligned}$$

Lets check if QU_B and QV_B are unitary. $(QU_B)^*(QU_B) = U_B^*Q^*QU_B = I$. Therefore, it is unitary. Hence,

$$(QU_B)\,\Sigma_B\,(QV_B)^* = U_A\Sigma_A V_A^*$$

Since, the singular values are uniquely determined we cant have two Σ's and therefore, if A and B are unitarily equivalent, they have the same singular values. Now, lets look at the alternate case. Let, A, and B have the same singular values, Σ. From B's SVD we can say that $\Sigma = U_B^* B V_B$.

$$\begin{aligned} AV_A &= U_A\Sigma \\ AV_A &= U_A U_B^* B V_B \\ A &= U_A U_B^* B V_B V_A^* \end{aligned}$$

We know that $Q = U_A U_B^*$ is unitary, but it is not necessarily equal to $V_A V_B^*$. Therefore, we cant say the inverse is true. Hence, if matrices A, and B are unitarily equivalent, then they have the same Σ, but its not true to say that if they have the same Σ they are unitarily equivalent.

3. NLA exercise 5.1 *In example 3.1 ...*

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

From the SVD property we know that the $||A||_2 = \sigma_1$ which is the maximum singular value of matrix A. Lets proceed to find its singular values.

$$AA^* = \begin{bmatrix} 5 & 4 \\ 4 & 4 \end{bmatrix}$$

The $\det(\lambda I - AA^*) =$

$$\det\left(\begin{bmatrix} \lambda - 5 & -4 \\ -4 & \lambda - 4 \end{bmatrix}\right) = \lambda^2 - 9\lambda + 4 = 0$$

$\lambda = \frac{9\pm\sqrt{67}}{2}$. $\sigma = \left(\frac{9\pm\sqrt{67}}{2}\right)^{1/2} = 2.9313267294014, 0.63821909565899$.
Therefore the $\sigma_{max}(A) = \sigma_1 = 2.9313267294014 \approx 2.9208$ and $\sigma_{min}(A) = \sigma_2 = 0.63821909565899$.

4. NLA exercise 5.3 *Consider the matrix ...*

(a)

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \; ; \; AA^* = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$$

$$\det\left(\lambda I - AA^*\right) = 0 \; ; \; \det\left(\begin{bmatrix} \lambda - 125 & -75 \\ -75 & \lambda - 125 \end{bmatrix}\right) = 0$$

This gives us $\lambda^2 + (200)(50) - 250\lambda = 0 \implies \lambda = 200, 50$. Hence, $\sigma = 10\sqrt{2}, 5\sqrt{2}$. The null $\left(\begin{bmatrix} \lambda - 125 & -75 \\ -75 & \lambda - 125 \end{bmatrix}\right)$ gives us the left singular vectors.
If $\lambda = 200$,

$$\begin{bmatrix} 75 & -75 \\ -75 & 75 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies u_{11} = u_{21}, \text{ and I choose them to be } u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

If $\lambda = 50$,

$$\begin{bmatrix} -75 & -75 \\ -75 & -75 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies u_{11} = -u_{21}$$

This leaves us with 2 choices, where we can have u_2 to either be $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ or $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.
We know $Av_i = \sigma_i u_i$, so

$$\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \implies v_1 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \text{ or } \begin{bmatrix} -5 \\ 5 \end{bmatrix} \implies v_2 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \text{ or } v_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

Since, we are required to take the option with the least negative signs, the SVD of $A = U\Sigma V^*$ where,

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \; \Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}, \text{ and, } V = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

(b) For this part, I have attached the code in Listing 2.

```
clc;
clear all;
%%
U = (1/sqrt(2))*[1 1; 1 -1];
S = sqrt(2)*[10 0; 0 5];
V = (1/5)*[-3 4; 4 3];

A = U*S*V';
US = U*S;

% plotting a unit circle
t = (0:pi/50:2*pi);
x = cos(t);
y = sin(t);

X = [0, 0];
Y = X;
```

```

figure(1)
plot(x,y,'r','LineWidth',2);
hold on;
grid on;
axis equal;
xline(0);
yline(0);
quiver(X, Y, V(1,:), V(2,:),0,'b');
text(V(1,1)-0.25,V(2,1)+0.02, ...
    "$("+num2str(V(1,1))+","+num2str(V(2,1))+")$",'Interpreter','latex');
text(V(1,2)+0.02,V(2,2)+0.02, ...
    "$("+num2str(V(1,2))+","+num2str(V(2,2))+")$",'Interpreter','latex');
text(-0.5,0.75,"$v_1$",'Interpreter','latex');
text(0.5,0.5,"$v_2$",'Interpreter','latex');
title('Pre-Image of unit circle');


% plotting the ellipse
e = A*[x;y];

figure(2)
plot(e(1,:),e(2,:),'r','LineWidth',2);
grid on;
hold on;
axis equal;
xline(0);
yline(0);
quiver(X,Y,US(1,:),US(2,:),0,'b');
text(US(1,1)+0.2,US(2,1)+0.2, ...
    "$("+num2str(US(1,1))+","+num2str(US(2,1))+")$",'Interpreter','latex'
    );
text(US(1,2)+0.2,US(2,2)-0.2, ...
    "$("+num2str(US(1,2))+","+num2str(US(2,2))+")$",'Interpreter','latex'
    );
text(7.2,9,"$\sigma_1 u_1$",'Interpreter','latex');
text(2.3,-2,"$\sigma_2 u_2$",'Interpreter','latex');
title('Image of unit circle after mapping by A');
```

Listing 2: Code to plot Pre-Image and Image of a unit circle under operation by A

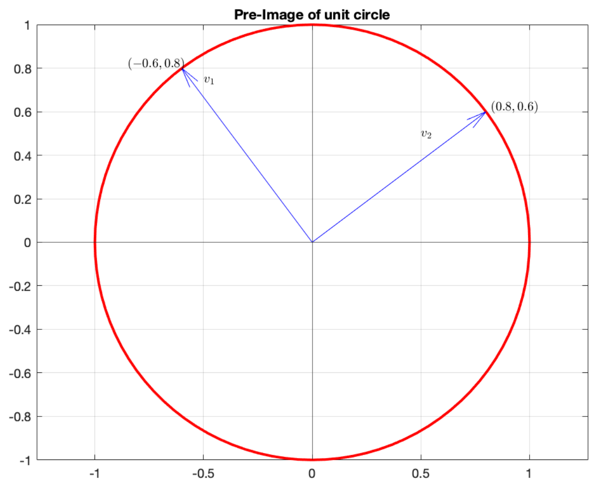
(c) The following matrix norms are:

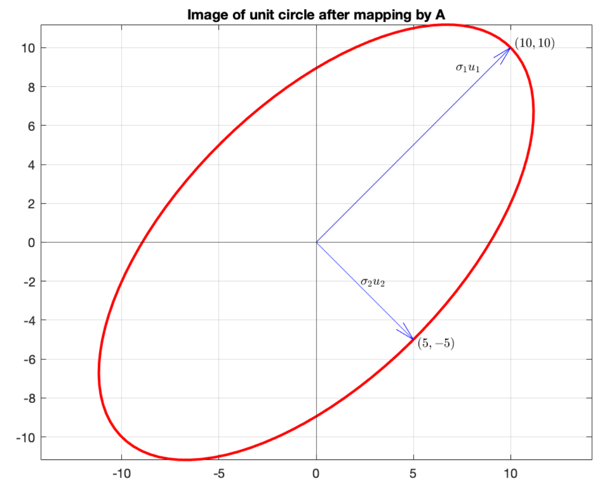
$$||A||_1 = \max_{1\leq i\leq n} ||a_i||_1 = \max(12, 16) = 16$$

$$||A||_2 = \sigma_2 = 10\sqrt{2}$$

$$||A||_\infty = \max_{1\leq i\leq n} ||a_i^*||_\infty = \max(13, 16) = 16$$

$$||A||_F = \sqrt{\mathrm{Tr}\,(AA*)} = \sqrt{250} = 5\sqrt{10}$$





(d)

$$\begin{aligned}
A &= U\Sigma V^* \\
A^{-1} &= (U\Sigma V^*)^{-1} \\
&= (V^*)^{-1}\Sigma^{-1}U^{-1}, \; U, \text{ and } V \text{ are Unitary matrices} \\
&= \left(V^{-1}\right)^*\Sigma^{-1}U^* \\
&= V\Sigma^{-1}U^*
\end{aligned}$$

Since, Σ is a diagonal matrix, its inverse is $\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & 1/\sigma_2 \end{bmatrix}$. Computing this we get,

$$A^{-1} = \begin{bmatrix} 5/100 & -11/100 \\ 1/10 & -2/100 \end{bmatrix}$$

(e) The eigen-values of A can be found out by finding its characteristic polynomial $=$ $\det(\lambda I - A) = 0$

$$\begin{aligned}
\det(\lambda I - A) &= 0 \\
\det\left(\begin{bmatrix} \lambda + 2 & -11 \\ 10 & \lambda - 5 \end{bmatrix}\right) &= 0 \\
\lambda^2 - 3\lambda + 100 &= 0
\end{aligned}$$

Solving for λ, we get $\lambda = \frac{3 \pm i\sqrt{391}}{2}$. These are the eigen-values of A.

(f) $\det(A) = (-2 \times 5) - (11 \times -10) = 100$.
$\lambda_1\lambda_2 = \frac{3+i\sqrt{391}}{2} \times \frac{3-i\sqrt{391}}{2} = \frac{400}{4} = 100$. Hence, both are the same.

(g) The area of the ellipsoid that A maps the unit circle is given by Area $= \pi ab$, where a is the length of the semi-major axis and b is the length of the semi-minor axis. We known that as the singular values σ_1, and σ_2. Therefore, Area $= \pi\sigma_1\sigma_2 = 100\pi \;\; (unit)^2$.

Figure 1: Original image

5. Use the SVD to show that if $A \in \mathbb{C}^{m \times n}$ has rank n, then

$$\left|\left|A\left(A^*A\right)^{-1}A^*\right|\right|_2 = 1$$

Any $A = U\Sigma V^*$ where $U \in \mathbb{C}^{m\times m}$, and $V \in \mathbb{C}^{n\times n}$ are unitary matrices. Let us look at $A\left(A^*A\right)^{-1}A^*$

$$\begin{aligned}
A\left(A^*A\right)^{-1}A^* &= U\Sigma V^*\left(V\left(\Sigma^*\Sigma\right)V^*\right)^{-1}V\Sigma^*U^* \\
K &= U\Sigma V^*\left(V^*\right)^{-1}\left(\Sigma^*\Sigma\right)^{-1}V^{-1}V\Sigma^*U^* \\
&= U\Sigma\left(\Sigma^*\Sigma\right)^{-1}\Sigma^*U^* \\
U^*KU &= \Sigma\left(\Sigma^*\Sigma\right)^{-1}\Sigma^* \\
U^*KU\Sigma &= \Sigma \\
U^*KU\Sigma\Sigma^*\left(\Sigma\Sigma^*\right)^{-1} &= \Sigma\Sigma^*\left(\Sigma\Sigma^*\right)^{-1} \\
U^*KU &= I \\
UU^*KUU^* &= UU^* = I \\
K &= I \\
A\left(A^*A\right)^{-1}A^* &= I \\
\left|\left|A\left(A^*A\right)^{-1}A^*\right|\right|_2 &= ||I||_2 = 1
\end{aligned}$$

6. The original image I used for this is Fig 1. Then, after performing SVD with 4 different choices for the number of Singular Values to use, the compressed images are attached in Fig 2, and Fig 3. The Singular values used in each of these cases are plotted in Fig 4. The code I used to generate these compressed images and their corresponding SVD plots are in Listing 3.

```
clc;
clear all;
%%
[Im] = imread("chelsea-giroud.jpeg"); % reads image
Imd  = im2double(Im);

```

Figure 2: Rank 10 and Rank 50 approximations

Figure 3: Rank 150 and Rank 200 approximations

```
max_rank_approx = [10 50 150 200];

[m,n,l] = size(Imd);
U = zeros([m,m,3]);
S = zeros([m,n,3]);
V = zeros([n,n,3]);

for i=1:3
[U(:,:,i),S(:,:,i),V(:,:,i)] = svd(Imd(:,:,i));
end

for i=1:length(max_rank_approx)
    ImC = zeros(size(Imd));
    for j = 1:3
        for k = 1:max_rank_approx(i)
            ImC(:,:,j) = ImC(:,:,j) + S(k,k,j)*U(:,k,j)*V(:,k,j)';
        end
    end
    figure
    imshow(ImC);
end

figure(1)
title("Rank "+num2str(max_rank_approx(1))+" Approximation",'FontSize',12);
print('Rank-10-approx','-dpng');

figure(2)
title("Rank "+num2str(max_rank_approx(2))+" Approximation",'FontSize',12);
print('Rank-50-approx','-dpng');

figure(3)
title("Rank "+num2str(max_rank_approx(3))+" Approximation",'FontSize',12);
print('Rank-150-approx','-dpng');

figure(4)
title("Rank "+num2str(max_rank_approx(4))+" Approximation",'FontSize',12);
print('Rank-200-approx','-dpng');

for i=1:length(max_rank_approx)
    v = zeros(max_rank_approx(i),3);
    for k=1:3
        for j=1:max_rank_approx(i)
            v(j,k) = S(j,j,k);
        end
    end
    figure
    plot(v(:,1),'r');
    hold on;
    grid on;
    plot(v(:,2),'g');
    plot(v(:,3),'b');
    legend('Red','Green','Blue');
    xlabel('$k$','Interpreter','latex');
    ylabel('$\sigma_i$','Interpreter','latex');
    title('Singular values (vs) k', 'FontSize',12);
    print("Sigma_v_k_"+num2str(i),'-dpng');
```

```
end
```

Listing 3: Code to perform SVD compression of Image

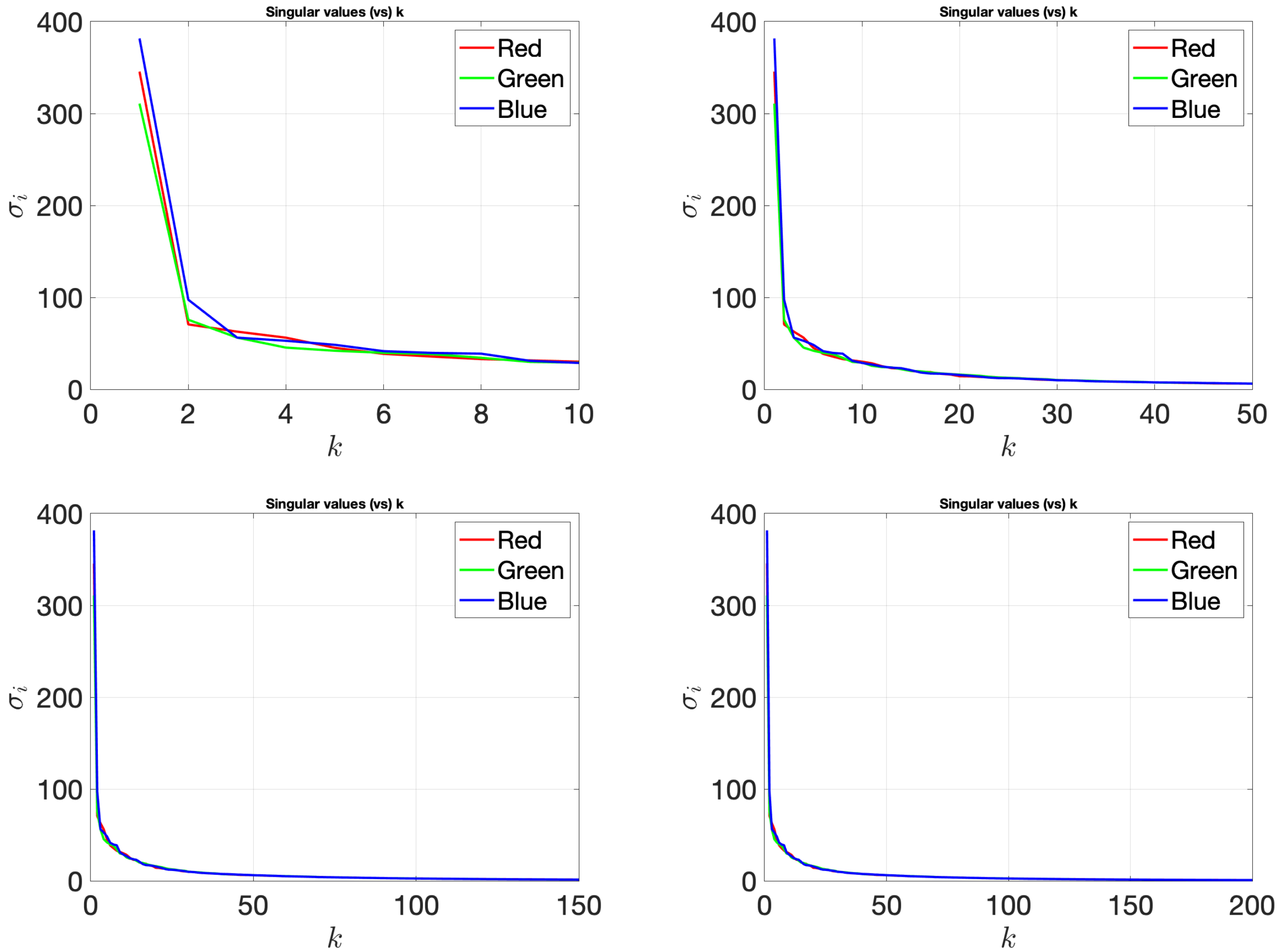


Figure 4: Singular values used in cases $k = 10, 50, 150, 200$