MANE 6760 - FEM for Fluid Dyn. - Lecture 20

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Stabilized FE Form: (Simplified) 1D TAD Eqn

Stabilized/SUPG FE *semi-discrete* form (with $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}^{adv}(\cdot)$):
find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\int_0^L \big(\underbrace{\bar{w}\bar{\phi}_{,t}}_{1} + \underbrace{\bar{w}_{,x} a_x \tau \bar{\phi}_{,t}}_{2} + \underbrace{-\bar{w}_{,x} a_x \bar{\phi}}_{3} + \underbrace{\bar{w}_{,x} \kappa \bar{\phi}_{,x}}_{4} + \underbrace{\bar{w}_{,x} \overbrace{\kappa_{num}}^{\kappa_{num}=a_x\tau a_x} \bar{\phi}_{,x}}_{5}$$

$$+ \underbrace{(-\bar{w}s)}_{6} + \underbrace{(-\bar{w}_{,x} a_x \tau s)}_{7}\big) dx = 0$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

Time Integration/Marching: (Simplified) 1D TAD Eqn

Consider N_t time intervals in time integration/marching: $\{t_0 = t_{min}, t_1, \ldots, t_{N_t} = t_{max}\}$, and a time step from t_n to t_{n+1}, where $n = 0$ corresponds to the start time of $t_{n=0} = t_{min}$ for which solution is known as the initial condition: $\phi(x, t = t_{n=0}) = \phi_{IC}(x)$, and $n = N_t$ corresponds to the end time of $t_{n=N_t} = t_{max}$ at which computation ends. Denote:

$\dot{\hat{\boldsymbol{\Phi}}}^{(n)} = \hat{\boldsymbol{\Phi}}^{(n)}_{,t} = [\hat{\phi}_{1,t}(t_n), \ldots, \hat{\phi}_{Ns,t}(t_n)]^t$, and

$\hat{\boldsymbol{\Phi}}^{(n)} = [\hat{\phi}_1(t_n), \ldots, \hat{\phi}_{Ns}(t_n)]^t$ (similarly, $\dot{\hat{\boldsymbol{\Phi}}}^{(n+1)}$ and $\hat{\boldsymbol{\Phi}}^{(n+1)}$).

Explicit methods:

$$\boldsymbol{M}\dot{\hat{\boldsymbol{\Phi}}}^{(n)} + \boldsymbol{A}\hat{\boldsymbol{\Phi}}^{(n)} - \boldsymbol{b}^{(n)} = 0$$

$$\underline{\underline{K}}\,\underline{\hat{\phi}}^{(n+1)} = \underline{d} \qquad \underline{\underline{K}}\,\underline{x} = \underline{d}$$

Implicit methods:

$$\boldsymbol{M}\dot{\hat{\boldsymbol{\Phi}}}^{(n+1)} + \boldsymbol{A}\hat{\boldsymbol{\Phi}}^{(n+1)} - \boldsymbol{b}^{(n+1)} = 0$$

$$\underline{\underline{K}}\,\underline{\hat{\phi}}^{(n+1)} = \underline{d}$$

Time Integration/Marching: (Simplified) 1D TAD Eqn

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Explicit methods: forward Euler with $\dot{\hat{\mathbf{\Phi}}}^{(n)} = \frac{\hat{\mathbf{\Phi}}^{(n+1)} - \hat{\mathbf{\Phi}}^{(n)}}{t_{n+1} - t_n}$

$$\boldsymbol{M}\hat{\mathbf{\Phi}}^{(n+1)} = \boldsymbol{M}\hat{\mathbf{\Phi}}^{(n)} + (t_{n+1} - t_n)\left(\boldsymbol{b}^{(n)} - \boldsymbol{A}\hat{\mathbf{\Phi}}^{(n)}\right)$$

$\underline{d}$

$\underline{\underline{K}}$

Implicit methods: backward Euler $\dot{\hat{\mathbf{\Phi}}}^{(n+1)} = \frac{\hat{\mathbf{\Phi}}^{(n+1)} - \hat{\mathbf{\Phi}}^{(n)}}{t_{n+1} - t_n}$

$\underline{d}$

$$\boldsymbol{M}\hat{\mathbf{\Phi}}^{(n+1)} + (t_{n+1} - t_n)\boldsymbol{A}\hat{\mathbf{\Phi}}^{(n+1)} = \boldsymbol{M}\hat{\mathbf{\Phi}}^{(n)} + (t_{n+1} - t_n)\boldsymbol{b}^{(n+1)}$$

$\underline{\underline{K}}$

Simplified: 1D Non-linear (NL) TAD Eqn

A number of simplifications:

- 1D (spatial) domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Strong form:

$$R(\phi) = \mathcal{L}(\phi) - s = \frac{\partial \phi}{\partial t} + a_x \frac{\partial \phi}{\partial x} - \frac{\partial}{\partial x}\left(\kappa(\phi)\frac{\partial \phi}{\partial x}\right) - s = 0, \quad \phi \in \mathcal{S}_{strong}$$

$$\begin{aligned} x &\in [0, L] \\ t &\in [t_{min}, t_{max}] \\ \phi(x, t = t_{min}) &= \phi_{IC}(x)\, \forall x \\ \phi(x = 0, t) &= \phi_0(t) \quad \text{on} \quad x = 0\, \forall t \\ \phi(x = L, t) &= \phi_L(t) \quad \text{on} \quad x = L\, \forall t \end{aligned}$$

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