MANE 6760 - FEM for Fluid Dyn. - Lecture 01

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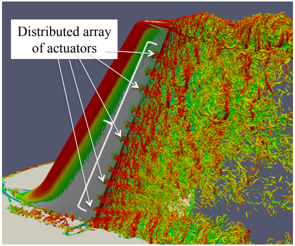
Complex Problems of Interest

Many applications of interest involve fluid flows with complex dynamics (e.g., flow around aircrafts, buildings, wind turbines, etc.)



Scientific Approaches

Major pillars of scientific inquiry

1. Theoretical/analytical methods
2. Physical experimentation/lab or field testing
3. Computation/computer simulation (e.g., computational fluid dynamics/CFD)
4. Data science

Why a Computational Approach or CFD?

- Insights
- Prediction/forecast
- Design and optimization
- Reduce physical testing (it cannot be replaced fully)
- Data for data-driven models

Landscape of Any Computation

- Observe physical phenomena: observations
- Form a mathematical description/model, or models: modeling
- Perform computation on computer: simulation
- Process simulation data to obtain quantities of interest: post-processing
- Improve models and computation: overall accuracy/raliability
- Apply drivers: physical insights, design, optimization, uncertainty quantification, train data-driven models, ...

Basic Steps of Any Computation

- Generate a CAD model/geometry that represents the computational domain
- Discretize the computational domain into a grid/mesh
- Apply boundary conditions
- Set appropriate model(s) as well as numerical procedures and parameters
- Perform the computation/simulation – may require significant computational resources
- Analyze the results (qualitatively and quantitatively)
- Repeat this process (based on drivers such as accuracy, design, optimization, uncertainty quantification, etc.)

Common Numerical Methods/Techniques

- Finite difference methods
- Finite volume methods
- Finite element methods
- Spectral methods
- Particle methods
- $\cdots$

Advection-diffusion (AD) Equation: Scalar and Linear

$$\int_{\Omega}\left(\frac{\partial\phi}{\partial t}+\nabla\cdot\boldsymbol{F}(\phi)\right)dV=\int_{\Omega}\frac{\partial\phi}{\partial t}dV+\int_{\partial\Omega}\boldsymbol{F}(\phi)\cdot\boldsymbol{n}dS=\int_{\Omega}s dV$$

$$\frac{\partial\phi}{\partial t}+\nabla\cdot\boldsymbol{F}(\phi)=s$$

$$\boldsymbol{F}(\phi)=\boldsymbol{F}^{adv}+\boldsymbol{F}^{diff}=\underbrace{\boldsymbol{a}\phi}_{\boldsymbol{F}^{adv}}+\underbrace{(-\kappa\nabla\phi)}_{\boldsymbol{F}^{diff}}$$

Note:

$\rightarrow$ ϕ: (scalar) solution variable, i.e., $\phi(x,y,z,t)$ or $\phi(x_1,x_2,x_3,t)$ in 3D
$\rightarrow$ bold font: non-scalar quantity, e.g., a vector or a 2nd-order tensor
$\rightarrow$ $\boldsymbol{F}(\phi)$: (vector) flux function, i.e., $\boldsymbol{F}=F_x\hat{i}+F_y\hat{j}+F_z\hat{k}$ or (F_1,F_2,F_3)
$\rightarrow$ Ω, $\partial\Omega$ and $\boldsymbol{n}$: domain, boundary and unit outward normal vector
$\rightarrow$ s: (volumetric) source term
$\rightarrow$ $\nabla\cdot(\_)$: div. operator, $\nabla\cdot\boldsymbol{F}=\partial F_x/\partial x+\partial F_y/\partial y+\partial F_z/\partial z=F_{i,i}$
$\rightarrow$ $\nabla(\_)$: grad. operator, $\nabla\phi=(\partial\phi/\partial x,\partial\phi/\partial y,\partial\phi/\partial z)=\phi_{,i}$
$\rightarrow$ $\boldsymbol{a}$: advection/convection/flow velocity vector (i.e., units of L/T)
$\rightarrow$ κ: diffusivity (i.e., units of L^2/T)

Advection-diffusion (AD) Equation: Scalar and Linear

Can be derived using the control-volume approach: rate of change of a quantity in a (fixed) control volume is equal to the net flux of quantity through boundaries and any (volumetric) source term

$$\frac{d}{dt}\int_{\Omega}\phi dV = -\int_{\partial\Omega}\boldsymbol{F}(\phi)\cdot\boldsymbol{n}dS + \int_{\partial\Omega} sdV$$

$$\int_{\Omega}\frac{\partial\phi}{\partial t}dV + \int_{\partial\Omega}\boldsymbol{F}(\phi)\cdot\boldsymbol{n}dS = \int_{\Omega} sdV$$

$$\int_{\Omega}\left(\frac{\partial\phi}{\partial t} + \nabla\cdot\boldsymbol{F}(\phi)\right)dV = \int_{\Omega} sdV$$

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\boldsymbol{F}(\phi) = s$$

Some Examples of AD Equation

Consider $\phi = \delta Q/\delta V$, i.e., $\delta Q = \phi \delta V$, $dQ = \phi dV$ or $Q = \int_\Omega \phi dV$, where Q is a conserved quantity (e.g., mass, momentum, energy)

- Mass transport: $\phi = \rho$ is density

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\boldsymbol{a}\rho) = 0$$

- Species transport: $\phi = c$ is concentration, D is diffusivity of species and s is source term

$$\frac{\partial c}{\partial t} + \nabla \cdot (\boldsymbol{a}c - D\nabla c) = s$$

- Heat/thermal transport (assuming constant density ρ and heat capacity c_p): $\phi = T$ is temperature and α is thermal diffusivity

$$\frac{\partial T}{\partial t} + \nabla \cdot (\boldsymbol{a}T - \alpha \nabla T) = 0$$

Strong and Weak Forms

- Strong form of the governing equations

$$R(\phi) = \mathcal{L}(\phi) - s = \frac{\partial \phi}{\partial t} + \nabla \cdot \boldsymbol{F}(\phi) - s = 0, \qquad \phi \in \mathcal{S}_{strong}$$

- Weak form of the governing equations

$$\int_\Omega w R(\phi) dV = \int_\Omega w \left(\mathcal{L}(\phi) - s\right) dV = 0, \qquad \phi \in \mathcal{S} \qquad \forall w \in \mathcal{W}$$

Galerkin Weak Form

Galerkin weak form: find $\tilde{\phi} \in \tilde{\mathcal{S}} \subset \mathcal{S}$ such that

$$a(\tilde{w}, \tilde{\phi}) = (\tilde{w}, s)$$

for all $\tilde{w} \in \tilde{\mathcal{W}} \subset \mathcal{W}$

- $a(\cdot, \cdot)$: bilinear form
- $(\cdot, \cdot)$: L_2 inner product

Finite-element Weak Form

Finite-element based (Galerkin) weak form: find $\phi^{h,p} \in \mathcal{S}^{h,p} \subset \mathcal{S}$ such that

$$a(w^{h,p}, \phi^{h,p}) = (w^{h,p}, s)$$

for all $w^{h,p} \in \mathcal{W}^{h,p} \subset \mathcal{W}$

- h: element size
- p: basis order
- $a(\cdot, \cdot)$: bilinear form
- $(\cdot, \cdot)$: L_2 inner product

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