MANE 6760 - FEM for Fluid Dyn. - Lecture 22

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Transient Equations: Fully discrete form

CFL numbers and conditions/restrictions on time step size (due to stability; note that a condition/restriction on time step size due to accuracy is taken into account separately)

Depends on the scheme (time integration scheme)

- Advective regime

$\Delta t \leq (\_ \; \_ \; \_)$

$$\gamma_{CFL,adv} = \frac{|a_x|\Delta t}{\Delta x}$$

$$\gamma_{CFL,adv} \leq \gamma_{adv,limit}$$

Upper limit/bound (max allowable value for CFL number)

Given: 'ax' (or 'kappa') and 'dx', then how large can one go for 'dt'?

- Diffusive regime

$\Delta t \leq (\_ \; \_ \; \_)$

$$\gamma_{CFL,diff} = \frac{\kappa\Delta t}{\Delta^2 x}$$

$$\gamma_{CFL,diff} \leq \gamma_{diff,limit}$$

Upper limit/bound (max allowable value for CFL number)

Depends on the scheme (time integration scheme)

Transient Non-Linear System of Equations: AD (Conservative Variables)

Consider the vector of conserved solution variables (e.g., species energy): $\boldsymbol{\psi} = [\psi_1, \psi_2, \ldots, \psi_M]^T$

Physicists
$$\frac{\partial \boldsymbol{\psi}}{\partial t} + \underbrace{\nabla \cdot \boldsymbol{F}}_{\nabla\cdot(\boldsymbol{F}^{adv}+\boldsymbol{F}^{diff})} = \boldsymbol{s}$$
$\mathcal{L}(\cdot) = S$

$$\boxed{\frac{\partial \boldsymbol{\psi}}{\partial t} + \nabla \cdot \left(\tilde{\boldsymbol{\mathcal{A}}} \boldsymbol{\psi} - \tilde{\boldsymbol{\mathcal{K}}} \nabla \boldsymbol{\psi} \right) = \boldsymbol{s}}$$
quasi-linear

$$\psi_{l,t} + \left((\tilde{\mathcal{A}}_{lm})_i \psi_m - (\tilde{\mathcal{K}}_{lm})_{ij} \phi_{m,j} \right)_{,i} = s_l$$

$$\mathcal{L}(\cdot) = \overbrace{\frac{\partial (\cdot)}{\partial t}}^{\mathcal{L}^{t}(\cdot)} + \overbrace{\nabla \cdot \left(\underbrace{\tilde{A}(\cdot)}_{\mathcal{L}^{adv}(\cdot)} - \underbrace{\tilde{K} \nabla (\cdot)}_{\mathcal{L}^{diff}(\cdot)} \right)}^{\mathcal{L}^{s}(\cdot)} = S$$

Transient Non-Linear System of Equations: AD (Primitive Variables)

Consider the vector of primitive solution variables (e.g., species temperature): $\boldsymbol{\phi} = [\phi_1, \phi_2, \ldots, \phi_M]^T$ such that $\boldsymbol{\mathcal{A}}_0 = \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{\phi}}$

Engineer

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{\phi}}{\partial t} = \frac{\partial \psi}{\partial t}$$

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{\phi}}{\partial t} + \underbrace{\nabla \cdot \boldsymbol{F}}_{\nabla \cdot (\boldsymbol{F}^{adv} + \boldsymbol{F}^{diff})} = \boldsymbol{s}$$

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{\phi}}{\partial t} + \nabla \cdot (\boldsymbol{\mathcal{A}} \boldsymbol{\phi} - \boldsymbol{\mathcal{K}} \nabla \boldsymbol{\phi}) = \boldsymbol{s}$$

$$(\mathcal{A}_0)_{lm} \phi_{m,t} + \left((\mathcal{A}_{lm})_i \phi_m - (\mathcal{K}_{lm})_{ij} \phi_{m,j} \right)_{,i} = s_l$$

Transient Non-Linear System of Equations: AD (Entropy Variables)

Consider the vector of entropy solution variables (e.g., species "entropy"): $\boldsymbol{\vartheta} = [\vartheta_1, \vartheta_2, \ldots, \vartheta_M]^T$ such that $\hat{\boldsymbol{\mathcal{A}}}_0 = \frac{\partial \psi}{\partial \boldsymbol{\vartheta}}$

Mathematician

$$\hat{\boldsymbol{\mathcal{A}}}_0 \frac{\partial \boldsymbol{\vartheta}}{\partial t} + \underbrace{\nabla \cdot \boldsymbol{F}}_{\nabla \cdot (\boldsymbol{F}^{adv} + \boldsymbol{F}^{diff})} = \boldsymbol{s}$$

$$\hat{\boldsymbol{\mathcal{A}}}_0 \frac{\partial \boldsymbol{\vartheta}}{\partial t} + \nabla \cdot \left(\hat{\boldsymbol{\mathcal{A}}} \boldsymbol{\vartheta} - \hat{\boldsymbol{\mathcal{K}}} \nabla \boldsymbol{\vartheta} \right) = \boldsymbol{s}$$

$$(\hat{\boldsymbol{\mathcal{A}}}_0)_{lm} \vartheta_{m,t} + \left((\hat{\mathcal{A}}_{lm})_i \vartheta_m - (\hat{\mathcal{K}}_{lm})_{ij} \vartheta_{m,j} \right)_{,i} = s_l$$

Transient Non-Linear System of Equations: Navier-Stokes

Conservative variables:
$\boldsymbol{\mathcal{U}} = [\rho, \rho u_1, \ldots, \rho e_{tot}]^T$

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \nabla \cdot \left(\tilde{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{U}} - \tilde{\boldsymbol{\mathcal{K}}} \nabla \boldsymbol{\mathcal{U}} \right) = \boldsymbol{S}$$

Primitive variables:
$\boldsymbol{Y} = [p, u_1, \ldots, T]$ (pressure) or
$[\rho, u_1, \ldots, T]^T$ (density)

$$\boldsymbol{\mathcal{A}}_0 \frac{\partial \boldsymbol{Y}}{\partial t} + \nabla \cdot (\boldsymbol{\mathcal{A}} \boldsymbol{Y} - \boldsymbol{\mathcal{K}} \nabla \boldsymbol{Y}) = \boldsymbol{S}$$

Entropy variables:
$\boldsymbol{V} = [\vartheta_1, \vartheta_2, \ldots, \vartheta_{n_{sd}+2}]^T$

$$\hat{\boldsymbol{\mathcal{A}}}_0 \frac{\partial \boldsymbol{V}}{\partial t} + \nabla \cdot \left(\hat{\boldsymbol{\mathcal{A}}} \boldsymbol{V} - \hat{\boldsymbol{\mathcal{K}}} \nabla \boldsymbol{V} \right) = \boldsymbol{S}$$

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