W.D. Henshaw **Math 6800: Problem Set 4** **Due:** Thurs. Oct. 6, 2022.

NLA = the text-book *Numerical Linear Algebra*, by Trefethen and Bau

1. Consider the Householder reflector,

$$F = I - 2uu^*, \quad u^*u = 1.$$

Determine the eigenvalues and eigenvectors, determinant, and singular values of F.

2. General Householder reflector. Let $x, y \in \mathbb{C}^m$, with $m > 1$. Show explicitly (using algebra) that if $\|x\|_2 = \|y\|_2$ then there is Householder reflector $F = I - 2uu^*$, $\|u\|_2 = 1$, such that $Fx = \alpha y$ where $\alpha = \pm \text{sign}(y^* x)$. Note: for $z \in \mathbb{C}$, with $z = re^{i\theta}$, $r, \theta \in \mathbb{R}$ and $r \geq 0$ then $\text{sign}(z) \equiv e^{i\theta}$. (Hint: if $x \neq \alpha y$ consider $v = x - \alpha y$, $u = v/\|v\|_2$.)

3. Write a Matlab function `[W,R] = house(A)` that computes an implicit representation of a full or reduced QR factorization for $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ using Householder reflections. The output variables are a lower triangular matrix $W \in \mathbb{C}^{m\times n}$ whose columns are the Householder vectors v_k, and an upper triangular matrix $R \in \mathbb{C}^{n\times n}$.

Also write a Matlab function `Q = formQ(W)` that takes the matrix W from `house` and generates the full matrix $Q \in \mathbb{C}^{m\times m}$.

(a) Test your programs on the Vandermonde matrix from problem set 3 with $m = 5$. Compare Q and R from the Matlab function `[Q,R]=qr(A)` to the output from `house` and `formQ`. Print A along with Q and R for each case along with $\|A - QR\|_2$, and $\|Q^*Q - I\|_2$ for each factorization.

(b) (NLA 10.3) Let Z be the matrix

$$Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{bmatrix}.$$

Compute three reduced QR factorizations of Z in Matlab: by the Gram-Schmidt routine `mgs`, by the householder routines `house` and `formQ` and by Matlab's builtin command `qr`. Compare the three results and comment on the differences.

4. Take $m = 50$, $n = 12$. Create a vector $t \in \mathbb{R}^m$ of equally spaced points for $t \in [0, 1]$. Create the $m \times n$ Vandermonde matrix using the points t to build the matrix associated with solving a least squares fit of a polynomial of degree $n - 1$ to the function $\cos(4t)$ so that the right-hand-side is $b = \cos(4t)$.

Solve the least squares problem in 7 ways:

(a) by forming and solving the normal equations.

(b) using a QR factorization and Classical Gram-Schmidt, `clgs`.

(c) using a QR factorization and modified Gram-Schmidt, `mgs`.

(d) using a QR factorization and the Householder triangularization function `house` (from ex. 3.).

(e) using a QR factorization and Matlab's `qr`.

(f) using Matlab's backslash function: $x = A\backslash b$.

(g) using the SVD, using Matlab's `svd` function.

For each of the cases, output the solution to 16 digits formatted in 7 columns (4 columns followed by 3 columns) using the following Matlab code:

```
fprintf('       Normal                  CLGS                  MGS                    HOUSE\n');
for i=1:n
  fprintf(' %22.15e %22.15e %22.15e %22.15e\n',xa(i),xb(i),xc(i),xd(i))
end;

fprintf('       Matlab QR         Matlab backslash           SVD\n');
for i=1:n
  fprintf(' %22.15e %22.15e %22.15e\n',xe(i),xf(i),xg(i))
end;
```

The results are assumed to be stored in the arrays `xa, xb, xc, xd, xe, xf, xg`. In each case highlight the digits (e.g. underline in red, or with a hilighter marker) that appear to be wrong (due to rounding errors). Comment on the differences you observe. Do the normal equations exhibit numerical instability (large errors due to roundoff) ? You do not have to explain the cause of the differences.