MANE 6760 - FEM for Fluid Dyn. - Lecture 19

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Simplified: 1D Unsteady/Transient AD (TAD) Eqn

A number of simplifications:

- 1D (spatial) domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Strong form:

$$R(\phi) = \mathcal{L}(\phi) - s = \boxed{\frac{\partial \phi}{\partial t} + a_x \frac{\partial \phi}{\partial x}} - \kappa \frac{\partial^2 \phi}{\partial x^2} - s = 0, \qquad \phi \in \mathcal{S}_{strong}$$

$$x \in [0, L]$$

$$t \in [t_0, t_1]$$

$$\phi(x, t = t_0) = \phi_{IC}(x)\,\forall x$$

$$\phi(x = 0, t) = \phi_0(t) \quad \text{on} \quad x = 0\,\forall t$$

$$\phi(x = L, t) = \phi_L(t) \quad \text{on} \quad x = L\,\forall t$$

$$\underline{\hat{a}} \cdot \tilde{\nabla} \phi = \begin{bmatrix} 1 \\ a_x \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \phi}{\partial x} \end{bmatrix}$$

FE Setup and Procedure

Space-time methods:

$$\bar{\phi}(x,t) = \sum_A \hat{\phi}_A N_A(x,t)$$

$$\int_{\Omega_T}\int_{\Omega_x} \bar{w}(x,t) R(\bar{\phi})\, dx\, dt$$

Semi-discrete methods:

$$\bar{\phi}(x,t) = \sum_A \hat{\phi}_A(t) N_A(x)$$

$$\int_{\Omega_x} \bar{w}(x) \underline{\qquad}\, dx$$

Time marching

Stabilization Parameter: TAD equation

τ approximation in 1D: algebraic version by Shakib *et al.* (1991):

$$\tau_{alg,skb} = \tau_{alg1} : (\tau_{alg,skb})^{-2} = \left(\frac{\Delta t}{2}\right)^{-2} + \left(\frac{(h/2)}{|a_x|}\right)^{-2} + 9\left(\frac{(h/2)^2}{\kappa}\right)^{-2}$$

$$= \left(\frac{\Delta t}{2}\right)^{-2} + \left(\frac{2|a_x|}{h}\right)^{2} + 9\left(\frac{4\kappa}{h^2}\right)^{2}$$

$$\hat{a}_i \, \hat{g}_{ij} \, \hat{a}_j$$

$$\begin{bmatrix} 1 & a_i \end{bmatrix} \begin{bmatrix} \left(\frac{2}{\Delta t}\right)^2 & 0 \\ 0 & g_{ij} \end{bmatrix} \begin{bmatrix} 1 \\ a_j \end{bmatrix}$$

$$\tau_{alg,skb} = \tau_{alg1} = \frac{1}{\sqrt{(\frac{2}{\Delta t})^2 + \left(\frac{2|a_x|}{h}\right)^2 + 9\left(\frac{4\kappa}{h^2}\right)^2}}$$

τ approximation in multiple dimensions:

$$(\tau_{alg,skb})^{-2} = (\tau_{alg1})^{-2} = \left(\frac{\Delta t}{2}\right)^{-2} + a_i g_{ij} a_j + c_{diff}^2 g_{ij} g_{ij} \kappa^2$$

$$\tau_{alg,skb} = \tau_{alg1} = \sqrt{\frac{1}{(\frac{2}{\Delta t})^2 + a_i g_{ij} a_j + c_{diff}^2 g_{ij} g_{ij} \kappa^2}}$$

Stabilized FE Form: (Simplified) 1D TAD Eqn

Stabilized/SUPG FE forms (with $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}^{adv}(\cdot)$): find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$K_{mn} = (a_x \tau a_x)$$

$$\int_0^L (\underbrace{\bar{w}\bar{\phi}_{,t}}_{1} + \underbrace{\ldots\ldots}_{2} + \underbrace{-\bar{w}_{,x} a_x \bar{\phi}}_{3} + \underbrace{\bar{w}_{,x} \kappa \bar{\phi}_{,x}}_{4} + \underbrace{\ldots\ldots}_{5}$$

$$\bar{w}_{,x} K_{mn} \bar{\phi}_{,x}$$

$$+ \underbrace{(-\bar{w} s)}_{6} + \underbrace{\ldots\ldots}_{7}) dx = 0$$

$$\hat{\mathcal{L}} \bar{w} = -a_x \bar{w}_{,x}$$

$$(-\bar{w}_{,x} a_x \tau s)$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

$$u' = -\tau R(\bar{\phi})$$

$$(+\bar{w}_{,x} a_x \tau \bar{\phi}_{,t})$$

$$\hat{w}_A M^{stab}_{AB} \hat{\phi}_{B,t} + \hat{w}_A A_{AB} \hat{\phi}_B = \hat{w}_A b_A$$

Time Integration/Marching: (Simplified) 1D TAD Eqn

Explicit methods: 'n' is current step and 'n+1' being computed/next step

$$M_{AB}\hat{\phi}_{B,t} + A_{AB}\hat{\phi}_B(t_n) = b_A$$

finite difference in time (ODE)

assume 's' is constant in time

Implicit methods:

$$M_{AB}\hat{\phi}_{B,t} + A_{AB}\hat{\phi}_B(t_{n+1}) = b_A$$

finite difference in time (ODE)

assume 's' is constant in time

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