MANE 6760 - FEM for Fluid Dyn. - Lecture 10

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ADR Equation: Linear and Scalar

ADR equation:

$$\phi_{,t} + \nabla \cdot (\boldsymbol{a}\phi - \kappa \nabla \phi) + c\phi = s$$
$$\phi_{,t} + (a_i \phi - \kappa \phi_{,i})_{,i} + c\phi = s$$
$$\phi_{,t} + (a_i \phi - \kappa \phi_{,i})_{,i} - r\phi = s$$

Note that Einstein summation notation is used for repeated indices. Also, $c < 0$ or $r > 0$ implies production, and $c > 0$ or $r < 0$ implies destruction.

Peclet and Damköhler numbers characterize the solution, where $Da = |c|L/|a_i|$ (and recall $Pe^G = |a_i|L/\kappa$). Similarly, cell Damköhler number is: $Da^e = |c|h/|a_i|$ (and recall $Pe^e = |a_i|h/(2\kappa)$). Note that (global or cell) Damköhler number is reported along with sign of reactive term: production or destruction.

FE Form: (Simplified) ADR Equation

A number of simplifications:

- Steady
- 1D domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\int_0^L \left(-\bar{w}_{,x}(a_x\bar{\phi} - \kappa\bar{\phi}_{,x}) + \bar{w}c\bar{phi}\right) dx = \int_0^L \bar{w}sdx$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

System of Equations: ADR

ADR system of equations
(where $\phi = [\phi_1, \phi_2, \ldots, \phi_M]^T$ for M solution variables):

Vector of matrices

$$\phi_{,t} + \nabla \cdot (\underline{\mathcal{A}}\phi - \mathcal{K}\nabla\phi) + \mathcal{C}\phi = \boldsymbol{s}$$

$$\phi_{,t} + (\mathcal{A}_i\phi - \underline{\mathcal{K}}\phi_{,i})_{,i} + \mathcal{C}\phi = \boldsymbol{s}$$

huge matrix

$$\phi_{,t} + (\mathcal{A}_i\phi - \mathcal{K}\phi_{,i})_{,i} - \mathcal{R}\phi = \boldsymbol{s}$$

$$\phi_{l,t} + ((\mathcal{A}_{lm})_i\phi_m - \mathcal{K}_{lm}\phi_{m,i})_{,i} + \mathcal{C}_{lm}\phi_m = s_l$$

$$\phi_{l,t} + ((\mathcal{A}_{lm})_i\phi_m - \mathcal{K}_{lm}\phi_{m,i})_{,i} - \mathcal{R}_{lm}\phi_m = s_l$$

l is in [1,M]; m is in [1 M]]

$$[\phi]_{,t} + \nabla\cdot\left([A]_i[\phi] - [K][\phi]_{,i}\right) + [C][\phi] = [s]$$

A - vector of matrices; i: 1… to n_sd
each of A_i is MxM size

K, C are of size MxM each
K - isotropic diffusion case

Stabilization Parameter: ADR

A practical way to design stabilization parameter is to consider each differential equation independently (not best suited from a theoretical viewpoint):

$$(\tau_{alg,skb})_I = (\tau_{alg1})_I = \frac{1}{\sqrt{(\mathcal{A}_{\underline{II}})_i g_{ij} (\mathcal{A}_{\underline{II}})_j + c_{diff}^2 g_{ij} g_{ij} \mathcal{K}_{\underline{II}} + \mathcal{C}_{\underline{II}}^2}}$$

A theoretical was to design stabilization parameter is to apply an eigenanalysis and possibly diagonalize the differential system of equations involving diagonal matrices $\tilde{\mathcal{A}}_i$, $\tilde{\mathcal{K}}$, and $\tilde{\mathcal{C}}$ (not best suited from a practical viewpoint):

$$(\tilde{\tau}_{alg,skb})_I = (\tilde{\tau}_{alg1})_I = \frac{1}{\sqrt{(\tilde{\mathcal{A}}_{\underline{I}})_i g_{ij} (\tilde{\mathcal{A}}_{\underline{I}})_j + c_{diff}^2 g_{ij} g_{ij} \tilde{\mathcal{K}}_I + \tilde{\mathcal{C}}_I^2}}$$

FE Setup and Procedure: Multiple Dimensions

Jacobian matrix (related to mapping between $\boldsymbol{x}$ and $\boldsymbol{\xi}$):

Local/element level operations (matrices and vectors):

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