MATH-6840
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Due: Monday February 7, 2022

Problem Set 3

NPDE is the textbook *Numerical Partial Differential Equations.* Submissions are due in the LMS, and must be typeset (e.g. LaTeX).

1. (10 pts.) Consider the smooth function $u(x)$ to be known at integer grid points $x_j = j\Delta x$ and use the notation $u_j = u(x_j)$

   (a) Is it possible to approximate $u_x(0)$ with error $\mathcal{O}(\Delta x^3)$ for general $u(x)$ using solution values u_j, for $j = -1, 0, 1$?

   (b) Under what restrictions on $u(x)$ can one approximate $u_x(0)$ with error $\mathcal{O}(\Delta x^3)$ using solution values u_j, for $j = -1, 0, 1$?

   (c) Using the solution values u_j, for $j = -2, -1, 0, 1, 2$, derive as accurate an approximation to $u_x(0)$ as possible. What is the order of accuracy?

   (d) Using the solution values u_j, for $j = -2, -1, 0, 1, 2$, derive as accurate an approximation to $u_{xxx}(0)$ as possible. What is the order of accuracy?

2. (15 pts.) Again consider the smooth function $u(x)$ to be known at integer grid points $x_j = j\Delta x$ and continue to use the notation $u_j = u(x_j)$.

   (a) Derive an infinite expansion for the exact value of $u_{xx}(0)$ using the discrete operators $D_{\pm}$ and D_0 and assuming u_j is know at all relevant locations.

   (b) Using the representation in (a) above, derive a nonlinear equation whose solution gives the coefficients in the expansion in (a).

   (c) Using Taylor series, solve for the coefficients in your expansion and derive a 10th order accurate approximation to $u_{xx}(0)$. Present the discrete approximation. Note you are permitted to use symbolic software such as Maple or Mathematica.

3. (10 pts.) *Adopted from NPDE exercise 1.5.12:*

   (a) Write a code to approximately solve

$$
\begin{aligned}
u_t &= \nu u_{xx}, && x \in (0,1), && t > 0 \\
u(x,0) &= \sin(2\pi x), && x \in (0,1) && \\
u(0,t) &= 0, && && t \geq 0 \\
u(1,t) &= 0, && && t \geq 0.
\end{aligned}
$$

   Use the grid $x_j = j\Delta x$, with $j = -1, 0, 1, \ldots, N, N+1$, and $\Delta x = 1/N$ (as described in the text), and apply the fourth-order centered spatial discretization with forward Euler time integration for $j = 1, 2, \ldots, N-1$, (BCs specified below) i.e.

$$
D_{+t} v_j^n = \nu D_{+x} D_{-x} \left(I - \frac{\Delta x^2}{12} D_{+x} D_{-x} \right) v_j^n .
$$

(b) Set $\nu = 1/6$, $\Delta t = 0.02$, and $N = 10$. Define ghost values using

$$\begin{aligned} v_{-1}^n &= 2v_0^n - v_1^n \\ v_0^n &= 0 \\ v_N^n &= 0 \\ v_{N+1}^n &= 2v_N^n - v_{N-1}^n, \end{aligned}$$

and compute approximate solutions at $t = 0.06$, $t = 0.1$, and $t = 0.9$.

(c) Again set $\nu = 1/6$, $\Delta t = 0.02$, and $N = 10$. Now define ghost values using

$$\begin{aligned} v_{-1}^n &= 0 \\ v_{N+1}^n &= 0, \end{aligned}$$

and compute approximate solutions at $t = 0.06$, $t = 0.1$, and $t = 0.9$.

(d) Discuss your results in comparison to each other and to those from PS2 #1.

4. (15 pts.) Consider the heat equation

$$u_t - u_{xx} = f(x,t), \qquad 0 < x < 1, \qquad t > 0$$

with initial conditions $u(x, t = 0) = u_0(x)$ and boundary conditions of the form

$$\begin{aligned} u(x = 0, t) &= \gamma_L(t) \\ u_x(x = 1, t) &= \gamma_R(t). \end{aligned}$$

(a) Determine $f(x,t)$, $u_0(x)$, $\gamma_L(t)$, and $\gamma_R(t)$ so that the exact solution to the problem is $u(x,t) = 2\cos(x)\cos(t)$.

(b) Write a code to solve this problem using the scheme

$$D_{+t}v_j^n = \nu D_{+x}D_{-x}v_j^n + f_j^n$$

on the grid defined by $x_j = j\Delta x$, $j = 0, 1, \ldots, N$, $\Delta x = 1/N$, with the parameter $r = \Delta t/\Delta x^2$. You can include ghost points as you need them, but you must ensure that your boundary conditions are at least second-order accurate.

(c) Perform a grid refinement study using $N = 20, 40, 80, 160, 320, 640$ by computing the maximum errors in the approximation at $t = 1.1$. Discuss the observed order of accuracy of the method.