W.D. Henshaw **Math 6800: Problem Set 8** **Due:** Thurs. Dec 1, 2022.

NLA = the text-book *Numerical Linear Algebra*, by Trefethen and Bau

1. NLA 24.1 *For each of the following statements, prove that ...*

2. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \epsilon & \epsilon \\ \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & 0 \end{bmatrix},$$

with ϵ a small positive perturbation, with $\epsilon \le 10^{-3}$.

(a) Estimate the locations of the eigenvalues of $A + B$ by using Gershgorin's theorem.

(b) Improve the estimate for $\lambda_1 \approx 1$ by judicious choice of diagonal similiarity transformation of the form

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix},$$

for some $d > 0$.

3. (NLA 26.3) One of the best known results of eigenvalue perturbation theory is the *Bauer-Fike theorem. Suppose $A \in \mathbb{C}^{m\times m}$ is diagonalizable with $A = V\Lambda V^{-1}$, and let $\delta A \in \mathbb{C}^{m\times m}$ be arbitrary. The every eigenvalue of $A + \delta A$ lies in at least one of the m circular disks in the complex plane of radius $\kappa(V)\|\delta A\|_2$ centred at the eigenvalues of A, where κ is the 2-norm condition number.*

(a) Prove the Bauer-Fike thereom by using the equivalence of conditions (i) and (iv) in Exercise 26.1.

(b) Suppose that A is normal. Show that for each eigenvalue $\tilde{\lambda}_j$ of $A + \delta A$, there is an eigenvalue λ_j of A such that

$$|\tilde{\lambda}_j - \lambda_j| \le \|\delta A\|_2. \tag{1}$$

4. Write a Matlab code `[W,H] = hessenberg(A)` to transform an $m \times m$ matrix A to upper Hessenberg form, H, by similarity transformations using Householder reflectors,

$$A = QHQ^*.$$

Here Q is represented implicitly in terms of the Householder vectors v_k stored in W. Also write a Matlab function `[Q] = formQh(W)` that takes W and generates the matrix Q.

Test your routine on the $m \times m$ matrix $A = [a_{ij}]$ with entries

$$\begin{aligned} a_{ij} &= 9, && \text{for } i = j, \\ a_{ij} &= \frac{1}{(i+j)} && \text{for } i \ne j \end{aligned}$$

and $m = 5$. Check that your routines are correct by confirming that H is upper Hessenberg, Q is unitary and $A = QHQ^*$.

Output A, H, W, Q, $\|Q^*Q - I\|_2$, and $\|A - QHQ^*\|_2$.