



# MANE 6960:

## Adjoint for Scientists and Engineers

Lecture 6

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# Adjoint and Over-determined Boundary Value Problems

# Lecture Objective

Today's lecture is a bit of a digression whose purpose is to illustrate an application of the adjoint.

Specifically, we will learn how solutions to the homogeneous adjoint BVP can be used to determine compatibility conditions on the source  $f$  and boundary data  $b$ .

# Motivation

To motivate today's topic, consider the linear (matrix) problem

$$Ax = b, \quad \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

where  $A$  is an  $m \times n$  matrix, with  $m \geq n$ . In other words, we consider potentially **over-determined linear systems**.

— Like the "homogeneous adjoint BVP"

Now, suppose that  $A^T y = 0$  for some  $y \neq 0$ . Then

$$y^T A x = (\cancel{A^T y})^T x = y^T b$$

(A red arrow points from the crossed-out  $A^T y$  to a red  $0$  below it.)

$$\Rightarrow \boxed{y^T b = 0}$$

## Motivation (cont.)

Thus, in order for  $Ax = b$  to have a solution, the right-hand-side vector  $b$  must be orthogonal to **every** nontrivial solution of the homogeneous equation  $A^T y = 0$ .

The solution to the homogeneous adjoint BVP,

$$\begin{aligned} L^* \psi &= 0, & \forall x \in \Omega, \\ B^* \psi &= 0, & \forall x \in \Gamma, \end{aligned} \tag{Adj0}$$

plays a similar role to  $y$  above, as we will now show.

# Compatibility in Over-determined Systems

## Theorem: Compatibility

The primal boundary-value problem

$$\begin{aligned}Lu &= f, & \forall x \in \Omega, \\Bu &= b, & \forall x \in \Gamma,\end{aligned}$$

has a solution only if

$$(\psi, f)_{\Omega} + (C^*\psi, b)_{\Gamma} = 0,$$

for all nontrivial solutions to the homogeneous adjoint BVP (Adj0), where  $C^*$  is the boundary operator from Green's identity.

# Compatibility in Over-determined Systems (cont.)

Proof: From Green's extended identity we have

$$(\Psi, Lu)_{\Omega} - (u, \cancel{L^* \Psi})_{\Omega} = (\cancel{B^* \Psi}, Cu)_{\Gamma} - (C^* \Psi, Bu)_{\Gamma}$$

provided  $\Psi$  satisfies the homogeneous adjoint BVP.

$$\Rightarrow (\Psi, Lu)_{\Omega} = -(C^* \Psi, Bu)_{\Gamma}$$

Substituting  $Lu = f$  and  $Bu = b$  gives the desired result. □

# Compatibility in Over-determined Systems (cont.)

Some remarks are in order regarding this result:

- Although our focus in previous lectures has been adjoints in the context of functionals, this result does not depend on an explicit functional, just Green's identity.

(The functional “hiding” here is the trivial one,  $J(u) = 0$ )

- If the homogeneous adjoint BVP has only the trivial solution,  $\psi = 0$ , then the condition

$$(\psi, f)_{\Omega} + (C^* \psi, b)_{\Gamma} = 0,$$

is satisfied for all  $f$  and  $b$ .



## Example 1: over-constrained point mass

Consider a point mass governed by Newton's second law,

$$m\ddot{x}(t) = f(t), \quad \forall t \in [0, T]$$

and subject to the initial and terminal conditions

$$x(0) = 0, \quad \dot{x}(0) = 0$$

$$x(T) = 0, \quad \dot{x}(T) = 0.$$

What are the compatibility conditions on the external force  $f(t)$ ?

# Example 1: over-constrained point mass (cont.)

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## Example 2: loaded elastic free bar

Consider the following ODE for an elastic bar:

$$I(x) \frac{d^2 u}{dx^2} - \sigma = 0, \quad \forall x \in [0, l]$$
$$\frac{d^2 \sigma}{dx^2} = q(x), \quad \forall x \in [0, l],$$

Are there any conditions on the load  $q(x)$  if the BCs are

$$\sigma(0) = 0, \quad \frac{d\sigma}{dx}(0) = \rho_0,$$
$$\sigma(l) = 0, \quad \frac{d\sigma}{dx}(l) = \rho_l?$$

## Example 2: loaded elastic free bar

## Example 2: loaded elastic free bar (cont.)

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# References