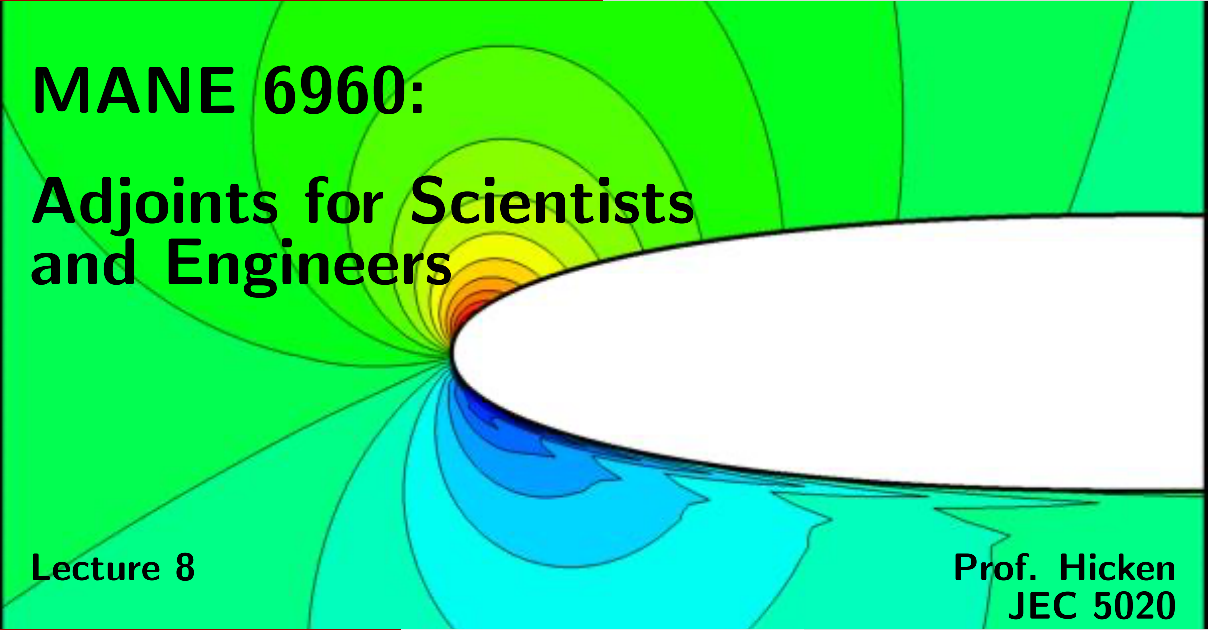


MANE 6960:

Adjoints for Scientists and Engineers

Lecture 8

Prof. Hicken
JEC 5020

Case Study: Adjoint BVP for the 2D Euler Equations

Introduction

Last class we derived the adjoint BVP for nonlinear problems. The analysis was quite general, but it was also abstract.

This lecture we will try to make the derivation more concrete by deriving the adjoint for the Euler equations of gas dynamics.

Euler Equations

The steady two-dimensional Euler equations are a system of nonlinear partial differential equations given by

$$N(q) = \frac{\partial F_x(q)}{\partial x} + \frac{\partial F_y(q)}{\partial y} = 0, \qquad \forall x \in \Omega \subset \mathbb{R}^2,$$

where q, F_x, and F_y are defined below.

The Euler equations must be augmented with appropriate boundary conditions for well-posedness; BCs will be introduced later when we discuss the adjoint boundary operators.

Euler Equations (cont.)

The (primal) state is

$$q = \begin{bmatrix} \rho & \rho u & \rho v & e \end{bmatrix}^T,$$

where

- ρ is the density;
- ρu and ρv are the x and y components of the momentum per unit volume, respectively;
- and e is the total energy per unit volume.

Note: We will use q rather than u for the state to avoid confusion with the x component of the velocity field.

Euler Equations (cont.)

The Euler fluxes in the x and y directions are

$$F_x(q) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho v u \\ (e+p)u \end{pmatrix}, \qquad \text{and} \qquad F_y(q) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)v \end{pmatrix},$$

respectively.

- The pressure is defined using the ideal-gas equation of state:

$$\begin{aligned} p &= (\gamma - 1)\left(e - \frac{\rho}{2}(u^2 + v^2)\right) \\ &= (\gamma - 1)\left(q_4 - \frac{1}{2q_1}(q_2^2 + q_3^2)\right). \end{aligned}$$

Adjoint Differential Operator

The first step in deriving the adjoint differential operator for the Euler equations is to evaluate the Fréchet derivative of $N(q)$.

$$
\begin{aligned}
N'[q]w &= D_w N(q) \\
&= \left[\frac{\partial}{\partial \varepsilon} \left(\frac{\partial}{\partial x} F_x(q + \varepsilon w) + \frac{\partial}{\partial y} F_y(q + \varepsilon w) \right) \right]_{\varepsilon = 0} \\
&= \left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial \varepsilon} F_x(q + \varepsilon w) \right) + \frac{\partial}{\partial y} \left(\frac{\partial}{\partial \varepsilon} F_y(q + \varepsilon w) \right) \right]_{\varepsilon = 0} \\
&= \frac{\partial}{\partial x} (A_x w) + \frac{\partial}{\partial y} (A_y w)
\end{aligned}
$$

Adjoint Differential Operator (cont.)

$A_x = \frac{\partial F_x}{\partial q}, \quad A_y = \frac{\partial F_y}{\partial q}$

where A_x and A_y are the Euler-flux Jacobians:

$$A_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 + \phi^2 & u - (\gamma - 2)u & -(\gamma-1)v & (\gamma - 1) \\ -vu & v & u & 0 \\ \frac{\gamma u e}{\rho} & \frac{\gamma e}{\rho} - \phi^2 - (\gamma-1)u^2 & -(\gamma-1)uv & \gamma u \end{bmatrix}$$

$$A_y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -vu & v & u & 0 \\ -v^2 + \phi^2 & -(\gamma-1)u & v - (\gamma-2)v & (\gamma-1) \\ \frac{\gamma v e}{\rho} & -(\gamma-1)uv & \frac{\gamma e}{\rho} - \phi^2 - (\gamma-1)v^2 & \gamma v \end{bmatrix},$$

where $\phi^2 \equiv \frac{1}{2}(\gamma - 1)(u^2 + v^2)$.

Adjoint Differential Operator (cont.)

Next, we take the inner product of $N'[q]w$ with ψ on the domain Ω, and then use integration by parts to obtain Green's identity.

$$(\psi, N'[q]w)_\Omega = \int_\Omega \psi^T N'[q]w \, d\Omega$$

$$= \int_\Omega \psi^T \left[\frac{\partial}{\partial x} \overbrace{(A_x w)}^{x\text{-flux}} + \frac{\partial}{\partial y} \overbrace{(A_y w)}^{y\text{-flux}} \right] d\Omega$$

$z \in \mathbb{R}$

$z = z^T$

$(y^T A x = x^T A^T y)$

$$= \int_\Omega \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \left(\psi^T A_x w, \ \psi^T A_y w \right) d\Omega$$

$$- \int_\Omega \left[\frac{\partial \psi^T}{\partial x} A_x w + \frac{\partial \psi^T}{\partial y} A_y w \right] d\Omega$$

Adjoint Differential Operator (cont.)

$$\int_\Omega \psi^T N'[q] w \, d\Omega = \int_\Gamma \psi^T A_n w \, d\Gamma + \int_\Omega w^T \underbrace{\left[-A_x^T \frac{\partial \psi}{\partial x} - A_y^T \frac{\partial \psi}{\partial y} \right]}_{N'[q]^* \psi} d\Omega$$

where

$$A_n \overset{\text{def}}{\equiv} n_x A_x + n_y A_y = \frac{\partial}{\partial q}\left(n_x F_x + n_y F_y \right)$$

is the Jacobian of the normal flux.

Adjoint Differential Operator (cont.)

The adjoint differential operator for the two-dimensional Euler equations is

$$N'[q]^*\psi = -A_x^T \frac{\partial \psi}{\partial x} - A_y^T \frac{\partial \psi}{\partial y},$$

where A_x and A_y are the flux Jacobians.

- The negative signs indicate a reversal of flow, which will have implications for the adjoint boundary conditions.

Boundary Operators

Comparing the generic Green's identity from last lecture against the identity obtained for the Euler equations, we conclude that

$$\begin{aligned}\int_\Gamma \psi^T A_n w \, d\Gamma &= \int_\Gamma \left(B'[q]^* \psi\right)^T C'[q] w \, d\Gamma - \int_\Gamma \left(C'[q]^* \psi\right)^T B'[q] w \, d\Gamma \\ &= \int_\Gamma \psi^T \Big\{ \underbrace{\left(B'[q]^*\right)^T C'[q] - \left(C'[q]^*\right)^T B'[q]}_{A_n} \Big\} w \, d\Gamma\end{aligned}$$

This equality holds if and only if

$$A_n = \left(B'[q]^*\right)^T C'[q] - \left(C'[q]^*\right)^T B'[q]$$

Boundary Operators (cont.)

Alternatively, we can express this condition as [GP97]

$$A_n = (T^*)^T T$$

where

$$T^* \equiv \begin{bmatrix} -C'[q]^* \\ B'[q]^* \end{bmatrix}, \qquad \text{and} \qquad T \equiv \begin{bmatrix} B'[q] \\ C'[q] \end{bmatrix}.$$

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}^{\text{example for inlet}}$$

- T and T^* are just matrices, despite all the notation flying around.
- The above decomposition of A_n is useful beyond the Euler equations; see [GP97]. For second- and higher-order PDEs, the variables w and ψ must be augmented to include normal derivatives.

Digression into Characteristic Variables

We will need to make use of the so-called characteristic variables to determine the adjoint boundary operators for the Euler equations.

To this end, consider the normal-flux Jacobian and its eigendecomposition:

$$\begin{aligned} A_n \equiv n_x A_x + n_y A_y &= \frac{\partial}{\partial q}\left[n_x F_x(q) + n_y F_y(q)\right], \\ &= R \Lambda_n R^{-1}, \end{aligned}$$

where R and R^{-1} are the right and left eigenvectors of A_n, and Λ_n is the diagonal matrix of eigenvalues.

Digression into Characteristic Variables (cont.)

For completeness [Pul86],

$$R^{-1} = \begin{bmatrix} (1-\phi^2/a^2) & (\gamma-1)u/a^2 & (\gamma-1)v/a^2 & -(\gamma-1)/a^2 \\ -u_t/\rho & n_y/\rho & -n_x/\rho & 0 \\ \beta(\phi^2-au_n) & \beta[n_x a-(\gamma-1)u] & \beta[n_y a-(\gamma-1)v] & \beta(\gamma-1) \\ \beta(\phi^2+au_n) & -\beta[n_x a+(\gamma-1)u] & -\beta[n_y a+(\gamma-1)v] & \beta(\gamma-1) \end{bmatrix},$$

and $$\Lambda_n = \begin{bmatrix} u_n & & & \\ & u_n & & \\ & & u_n + a & \\ & & & u_n - a \end{bmatrix},$$

where $u_n = un_x + vn_x$ is the normal velocity, $u_t = n_y u - n_x v$ is the tangential velocity, $a = \sqrt{\gamma p/\rho}$ is the speed of sound, $\beta = 1/(\sqrt{2}\rho a)$, and $\phi^2 \equiv \frac{1}{2}(\gamma-1)(u^2+v^2)$.

Digression into Characteristic Variables (cont.)

Definition: Characteristic Variables (Euler)

Let A_n denote the Jacobian of the normal flux, with eigendecomposition $A_n = R\Lambda_n R^{-1}$. Then the characteristic primal and dual variables in the direction (n_x, n_y) are

$$w_c \equiv R^{-1} w,$$

and $$\psi_c \equiv R^T \psi,$$

respectively.

Digression into Characteristic Variables (cont.)

The characteristic variables simplify the boundary term as follows:

$$\begin{aligned} \psi^T A_n w &= \psi_c^T \Lambda w_c \\ &= \psi_c^T (T_c^*)^T T_c w_c. \end{aligned}$$

In other words

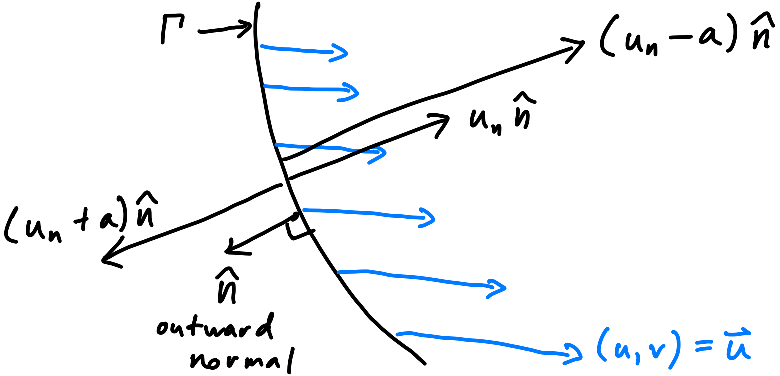
$$(T_c^*)^T T_c = \begin{bmatrix} u_n & & & \\ & u_n & & \\ & & u_n + a & \\ & & & u_n - a \end{bmatrix}.$$

- We can easily determine appropriate boundary and functional operators for the characteristic variables, and then convert back if necessary.

Subsonic Inlet Boundary

$$\boxed{u_n \equiv \vec{u}\cdot\hat{n} \le 0}$$

Consider a far-field, subsonic inlet boundary, where $u_n - c < u_n < 0 < u_n + c$. Consequently, there are three incoming characteristics for which we need to supply boundary conditions: $w_{c,1}$, $w_{c,2}$ and $w_{c,4}$.



Subsonic Inlet Boundary (cont.)

Therefore, the appropriate boundary operator at an inlet is

$$B'_c[q] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{array}{ll} & \underline{\lambda \text{ values}} \\ \leftarrow & u_n \\ \leftarrow & u_n \\ \leftarrow & u_n - c \end{array}$$

and we can use $w_{c,3}$ for the functional operator, since it remains unspecified in $B'_c[q]$:

$$C'_c[q] = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

Subsonic Inlet Boundary (cont.)

recall

$$(T_c^*)^T T_c = \Lambda_n$$

Thus, the compound operator is

$$T_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which is clearly nonsingular. Consequently, we can invert T_c to find T_c^*:

$$T_c^* = (T_c^{-1})^T \Lambda_n^T = \begin{bmatrix} u_n & 0 & 0 & 0 \\ 0 & u_n & 0 & 0 \\ 0 & 0 & 0 & u_n - a \\ 0 & 0 & u_n + a & 0 \end{bmatrix}.$$

Subsonic Inlet Boundary (cont.)

Recall: matrix dimensions of $C_c'[q]^*$ and $B_c'[q]^*$ must agree with $B_c'[q]$ and $C_c'[q]$, respectively

Recalling that $T_c^* = \begin{bmatrix} -C_c'[q]^* \\ B_c'[q]^* \end{bmatrix}$, we find that

$$-C_c'[q]^* = \begin{bmatrix} u_n & 0 & 0 & 0 \\ 0 & u_n & 0 & 0 \\ 0 & 0 & 0 & u_n - a \end{bmatrix}$$

and

$$B_c'[q]^* = \begin{bmatrix} 0 & 0 & u_n + a & 0 \end{bmatrix}$$

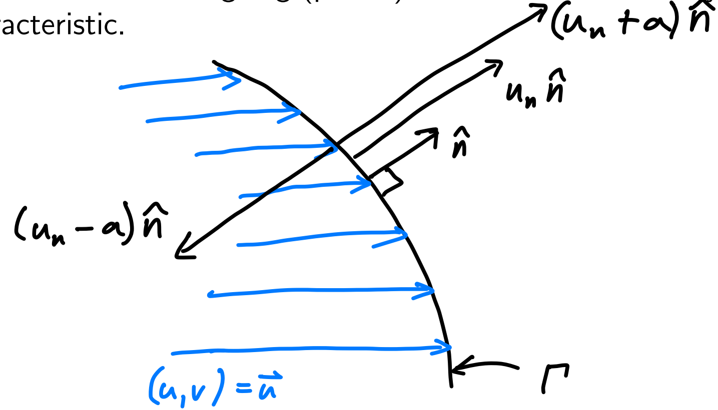
- The adjoint boundary operator is applied to only one characteristic variable, namely $\psi_{c,3}$, which is an in-going characteristic for the adjoint problem (recall the flow reversal in $N'[q]^*$).

Subsonic Outlet Boundary

$$u_n = (\vec{u} \cdot \hat{n}) \geq 0$$

The analysis at a subsonic outlet boundary is similar:

- In this case there are three out-going (primal) characteristics and only one in-coming characteristic.



Ω

Subsonic Outlet Boundary (cont.)

The primal boundary-condition and functional operators become

λ values

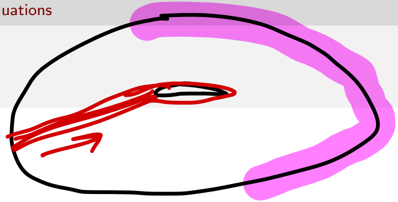
$$B_c'[q] = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

$\leftarrow$ $u_n - a \leq 0$

and

$$C_c'[q] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Subsonic Outlet Boundary (cont.)



The adjoint boundary and functional operators are

$$B_c'[q]^* = \begin{bmatrix} u_n & 0 & 0 & 0 \\ 0 & u_n & 0 & 0 \\ 0 & 0 & u_n + a & 0 \end{bmatrix}$$

and

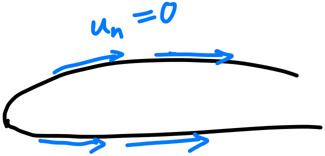
$$-C_c'[q]^* = \begin{bmatrix} 0 & 0 & 0 & u_n - a \end{bmatrix}$$

- Here, the primal problem has three out-going characteristics, so the adjoint boundary operator is applied to three variables.

Solid Wall Boundary

At a solid wall we impose a no-penetration condition, $u_n = 0$. This complicates the approach we used for the far-field boundaries, since now two of the four eigenvalues are zero:

$$\Lambda_n = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & a & \\ & & & -a \end{bmatrix},$$



so A_n is singular. Nevertheless, we still have

$$(T_c^*)^T T_c = \Lambda_n.$$

Solid Wall Boundary (cont.)

- To find T_c^*, we will consider only the non-zero eigenvalues and use the corresponding pseudo-inverse of T_c

Solid Wall Boundary (cont.)

Let's begin by differentiating the boundary operator, $B(q) = u_n = n_x u + n_y v$, and relating this to the characteristic operator.

$$B'[q]w = D_w\left(n_x(q_2/q_1) + n_y(q_3/q_1)\right) \in \mathbb{R}^4$$

$$= \left[\frac{-n_x q_2 - n_y q_3}{q_1^2}, \frac{n_x}{q_1}, \frac{n_y}{q_1}, 0\right] w$$

$$= \frac{1}{\rho}[\underbrace{-n_x u - n_y v}_{0\text{, since } u_n = 0}, n_x, n_y, 0] w$$

One can show

$$\underbrace{\frac{1}{\sqrt{2}}[0, 0, 1, -1] w_c}_{B'_c[q]w_c} = \frac{1}{\rho}[0, n_x, n_y, 0] w$$

Solid Wall Boundary (cont.)

The functional boundary operator at a solid wall is typically the pressure, which can be used to compute the lift and drag, e.g.

pressure

$$D(q) = \int_{\Gamma} (n_x \cos(\alpha) + n_y \sin(\alpha)) p \, d\Gamma,$$

where α is the angle of attack in radians.

Thus, $C(q) = p(q)$. As with the boundary-condition operator, we need to differentiate $C(q)$ and relate it to the characteristic operator.

$$C'[q]w = \left[\, \phi^2, \; -(\gamma-1)u, \; -(\gamma-1)v, \; (\gamma-1) \,\right] w$$

$$= \frac{\rho a}{\sqrt{2}} \left[\, 0, \; 0, \; 1, \; 1 \,\right] w_c = C_c'[q] w_c$$

Solid Wall Boundary (cont.)

Summarizing, we have the following at a solid wall:

$$(T_c^*)^T T_c = \Lambda_n$$

$$\Rightarrow \qquad \begin{bmatrix} -C_c'[q]^* \\ B_c'[q]^* \end{bmatrix}^T \begin{bmatrix} B_c'[q] \\ C_c'[q] \end{bmatrix} = \Lambda_n$$

$B_c'[q]$

$$\Rightarrow \qquad \begin{bmatrix} -C_c'[q]^* \\ B_c'[q]^* \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{\rho a}{\sqrt{2}} & \frac{\rho a}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & a & \\ & & & -a \end{bmatrix}.$$

$C_c'[q]$

Solid Wall Boundary (cont.)

Equating the lower-right block matrices on the left- and right-hand sides, we conclude that

$$\begin{bmatrix} -C_c'[q]^* \\ B_c'[q]^* \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{\sqrt{2}} & \frac{1}{\sqrt{2}\rho} \\ 0 & 0 & \frac{a}{\sqrt{2}} & -\frac{1}{\sqrt{2}\rho} \end{bmatrix}.$$

Thus, the characteristic adjoint boundary operators at a solid wall are

$$C_c'[q]^* = \begin{bmatrix} 0 & 0 & -\frac{a}{\sqrt{2}} & -\frac{a}{\sqrt{2}} \end{bmatrix},$$
$$B_c'[q]^* = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}\rho} & -\frac{1}{\sqrt{2}\rho} \end{bmatrix}.$$

Solid Wall Boundary (cont.)

The adjoint boundary conditions at a solid wall in an Euler flow are given by

$$B_c'[q]^* \psi_c = \frac{1}{\sqrt{2}\rho} \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix} \psi_c$$

or, equivalently,

$$\begin{aligned} B'[q]^* \psi &= \begin{bmatrix} 0 & n_x & n_y & 0 \end{bmatrix} \psi \\ &= n_x \psi_2 + n_y \psi_3 \end{aligned}$$

- Thus, the solid-wall adjoint boundary-condition operator is just a scaled version of the primal boundary operator.

Exercise

If the functional is drag, what is the adjoint boundary value? That is, determine $c'[C(q)]$ in

$$B'[q]^*\psi = n_x\psi_2 + n_y\psi_3 = c'[C(q)].$$

$$c'[C(q)] = (n_x \cos(\alpha) + n_y \sin(\alpha))$$

c depends linearly on $p = C(q)$

References

[GP97] M. B. Giles and N. A. Pierce, *Adjoint equations in CFD: duality, boundary conditions, and solution behaviour*, 13th AIAA Computational Fluid Dynamics Conference (Snowmass Village, CO), no. AIAA–97–1850, June 1997.

[Pul86] T. H. Pulliam, *Efficient solution methods for the Navier-Stokes equations*, Tech. report, Lecture Notes for the von Kármán Inst. for Fluid Dynamics Lecture Series: Numerical Techniques for Viscous Flow Computation in Turbomachinery Bladings, Rhode-Saint-Genèse, Belgium, January 1986.