4 10

$Ax = \lambda x$ 40/40

$(\lambda - \mu) x = \lambda x - \mu x = Ax - \mu x$

v. good

$= (A - \mu I) x$

$\therefore (A - \mu I) x = (\lambda - \mu) x$

$\Rightarrow \lambda - \mu$ is an ev of $A - \mu I$.

True

$Ax = \lambda x$

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_2 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 = 0$ False

$(2-\lambda)(2-\lambda) = 0$

$\lambda_{1,2} = 2, 2$

$\lambda = 2$ is eigen value of A but $-\lambda = -2$ is not.

$Ax = \lambda x$

all the ev's of A are zero,

$\Rightarrow$ $\det(A) = 0$, $tr(A) = 0$

Let, $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$ with $tr(A) = 0$ and $\det(A) = 0$

$$\lambda_1 + \lambda_2 = 0$$

$$\lambda_1 \lambda_2 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 0$$

$$\begin{vmatrix} 2-\lambda & 4 \\ -1 & -2-\lambda \end{vmatrix} = 0$$

$$= -(2-\lambda)(2+\lambda) + 4 = 0$$

$$-[4-\lambda^2] + 4 = 0$$

$$-4 + \lambda^2 + 4 = 0$$

$$\lambda^2 = 0$$

$\lambda_1 = \lambda_2 = 0$, but $A \neq 0$

$\therefore$ It is false

A is hermitian means $A = A^*$ and its all eigen values are real

Singular values of A are square roots of eigen values of $A^* A$

$A = Q \Lambda Q^* = Q |\Lambda| \, sign(\Lambda) Q^*$ —— (1)

where $|\Lambda|$ is diagonal matrix whose entries are $|\lambda_i|$ and $sign(\Lambda)$ is a diagonal matrix with entries $sign(\lambda_i)$

A is diagonalizable

$\Rightarrow$ $A = X \Lambda X^{-1}$ and its ew's are equal

$A = X \lambda I X^{-1}$ ($\because$ all ew's are equal)

$A = \lambda X X^{-1} = \lambda I$

$\underline{A = \lambda I}$

$\Rightarrow$ A is $\underline{\text{diagonal}}$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \epsilon & \epsilon \\ \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & 0 \end{bmatrix}, \quad \epsilon \le 10^{-3}$$

$A + B$

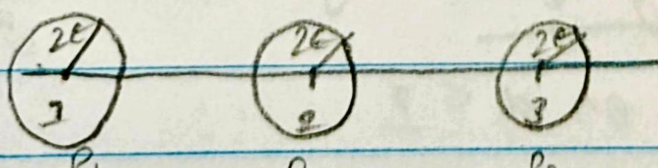
$$= \begin{bmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon \\ \epsilon & \epsilon & 3 \end{bmatrix}$$

eigen values,

$D_1 = |z-1| \le 2\epsilon$

$D_2 = |z-2| \le 2\epsilon$

$D_3 = |z-3| \le 2\epsilon$



D_1 D_2 D_3

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}, \quad d > 0$$

$D^{-1}AD$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/d & 0 \\ 0 & 0 & 1/d \end{bmatrix} \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & \epsilon & \epsilon \\ \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & 0 \end{bmatrix} \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/d & 0 \\ 0 & 0 & 3/d \end{bmatrix} + \begin{bmatrix} 0 & \epsilon & \epsilon \\ \epsilon/d & 0 & \epsilon/d \\ \epsilon/d & \epsilon/d & 0 \end{bmatrix} \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & d & d \\ 1/d & 0 & 1 \\ 1/d & 1 & 0 \end{bmatrix}$$

$D_1(\epsilon) = |z-1| \le 2\epsilon d$

$D_2(\epsilon) = |z-2| \le \frac{\epsilon}{d} + \epsilon = \epsilon\left(1 + \frac{1}{d}\right)$ /

$D_3(\epsilon) = |z-3| \le \epsilon\left(1 + \frac{1}{d}\right)$

What is the best we can do?

The estimate for $\lambda_1 \simeq 1$ can be improved by choosing $d \simeq \frac{\epsilon}{2}$

or $d < 1$.

To prove:-

$$(a)\quad |\tilde{\lambda}_j - \lambda_j| \leq K(V)\|\delta A\|_2$$

Proof:-

using (i) & (iv) conditions of 8.6.1.

$\|\delta A\|_2 \leq \epsilon$ ①, $\|(\tilde{\lambda}_j I - A)^{-1}\|_2 \geq \epsilon^{-1}$

$\Rightarrow \epsilon \|(\tilde{\lambda}_j I - A)^{-1}\|_2 \geq 1$

from ① $\Rightarrow \|\delta A\|_2 \|(\tilde{\lambda}_j I - A)^{-1}\|_2 \leq \epsilon \|(\tilde{\lambda}_j I - A)^{-1}\|_2$

$= \|\delta A\|_2 \|(\tilde{\lambda}_j I - A)^{-1}\|_2 \geq 1$

Now, $A = V\Lambda V^{-1}$

$\Rightarrow \|\delta A\|_2 \|(\tilde{\lambda}_j I - V\Lambda V^{-1})^{-1}\|_2 \geq 1$

$= \|\delta A\|_2 \|(\tilde{\lambda}_j VV^{-1} - V\Lambda V^{-1})^{-1}\|_2 \geq 1$

$= \|\delta A\|_2 \|(V(\tilde{\lambda}_j I - \Lambda)V^{-1})^{-1}\|_2 \geq 1$

$\Rightarrow \|\delta A\|_2 \|V(\tilde{\lambda}_j I - \Lambda)^{-1} V^{-1}\|_2 \geq 1$

$= \|\delta A\|_2 \|V\|_2 \|(\tilde{\lambda}_j I - \Lambda)^{-1}\|_2 \|V^{-1}\|_2 \geq 1$

$= \|\delta A\|_2 \|V^{-1}\|_2 \|V\|_2 \|(\tilde{\lambda}_j I - \Lambda)^{-1}\|_2 \geq 1$

$= \|\delta A\|_2 K(V) \|(\tilde{\lambda}_j I - \Lambda)^{-1}\|_2 \geq 1$

$$\|(\tilde{\lambda}_j I - \Lambda)^{-1}\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|(\tilde{\lambda}_j I - \Lambda)^{-1} x\|_2}{\|x\|_2}$$

$$= \max_{\lambda_j \in \Lambda(A)} \frac{1}{|\tilde{\lambda}_j - \lambda_j|} = \frac{1}{\min_{\lambda_j \in \Lambda(A)} |\tilde{\lambda}_j - \lambda|}$$

$\|\delta A\|_2 K(V) \dfrac{1}{\quad} \geq 1$

In case of normal matrices,

V is unitary matrix,

$\therefore \|V\|_2 = \|V^{-1}\|_2 = 1$

$\Rightarrow K(V) = 1$

$\therefore |\tilde{\lambda}_j - \lambda_j| \le \|\delta A\|_2$ proved

```
%========================= Q(4)=========================================
%it provides [A] matrix
%then it calls function hessenberg to get [W,H] and then, that W is
%used for calling function formQ that provides Q
%finally, it provides error norm
%=======================================================================
clc
clear all
m=5;
for i=1:m
    for j=1:m
        if (i==j)
           A(i,j)=9;
        else
            A(i,j)= 1/(i+j);
        end
    end
end
[W,H]= hessenberg(A);
[Q]= formQ(W);
% A matrix
A1= Q*H*Q';
disp('A1 = QHQ* is')
A1
% Hessenberg matrix
H
% W
W
%Q
Q
%=====================Error=============================================
disp('**************Error**************')
fprintf('||Q*Q-I|| = %8.2e\n',norm(Q'*Q-eye(m),2));
fprintf('||A-QHQ*||=%8.2e \n',norm(A-A1,2));

%=======================================================================
```

```
A1 = QHQ* is

A1 =

    9.0000    0.3333    0.2500    0.2000    0.1667
    0.3333    9.0000    0.2000    0.1667    0.1429
    0.2500    0.2000    9.0000    0.1429    0.1250
    0.2000    0.1667    0.1429    9.0000    0.1111
    0.1667    0.1429    0.1250    0.1111    9.0000


H =

    9.0000   -0.4913         0         0         0
   -0.4913    9.4289    0.1080   -0.0000    0.0000
         0    0.1080    8.8463    0.0400   -0.0000
         0         0    0.0400    8.8507   -0.0208
```

```
        0   -0.0000          0   -0.0208    8.8741

W =

         0         0         0
    0.9161         0         0
    0.2777   -0.8281         0
    0.2222   -0.3721   -0.7992
    0.1851   -0.4194   -0.6010

Q =

    1.0000         0         0         0         0
         0   -0.6785    0.6754   -0.2850   -0.0479
         0   -0.5088   -0.1666    0.7519    0.3847
         0   -0.4071   -0.4524    0.0300   -0.7929
         0   -0.3392   -0.5580   -0.5938    0.4701

**************Error**************
||Q*Q-I|| = 3.52e-16
||A-QHQ*||=6.64e-15
```

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```
%------------------------------------------------------------------------
[m,n]=size(W);
x=eye(m);
 for k=n:-1:1
     x(k:m,:)=x(k:m,:)-2*W(k:m,k)*(W(k:m,k)'*x(k:m,:));
 end
  Q=x;
end
%========================================================================
```

```
function [W,H]= hessenberg(A)
%============================================================================
%this function transforms A to upper hessenberg form
%Input:
%       m by m [A] matrix
%Output:
%       W- matrix of householder vectors (v(k))
%       H- upper hessenberg matrix
%============================================================================
m=size(A,1);
for k=1:m-2
    x=A(k+1:m,k);
    v=sign(x(1))*(norm(x,2)*eye(m-k,1))+x;
    v=v/norm(v,2);
    A(k+1:m,k:m)= A(k+1:m,k:m)-2*v*(v'*A(k+1:m,k:m));
    A(1:m,k+1:m)= A(1:m,k+1:m)-2*(A(1:m,k+1:m)*v)*v';
    W(k+1:m,k)=v;
 end
 H=A ;
end
%============================================================================
```