MANE 6760 - FEM for Fluid Dyn. - Lecture 07

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Stabilized FE Options: AD equation

A general stabilized FE form:

$$a(\bar{w},\bar{\phi}) + a_{stab}(\bar{w},\bar{\phi}) = a(\bar{w},\bar{\phi}) + \underbrace{(\hat{\mathcal{L}}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}}_{a_{stab}(\cdot,\cdot)} = (\bar{w}, s)$$

Several options available for $a_{stab}(\cdot,\cdot)$:

- SUPG: $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}^{adv}(\cdot) = -\boldsymbol{a}\cdot\nabla(\cdot)$

$$a_{SUPG}(\bar{w},\bar{\phi}) = (-\mathcal{L}^{adv}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- GLS: $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}(\cdot) = -\left(\boldsymbol{a}\cdot\nabla(\cdot) - \kappa\nabla^2(\cdot)\right)$

$$a_{GLS}(\bar{w},\bar{\phi}) = (-\mathcal{L}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- VMS: $\hat{\mathcal{L}}(\cdot) = \mathcal{L}^*(\cdot) = -\boldsymbol{a}\cdot\nabla(\cdot) - \kappa\nabla^2(\cdot)$

$$a_{VMS}(\bar{w},\bar{\phi}) = (\mathcal{L}^*(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- ... others (residual-free bubbles, etc)

What about stabilization parameter: τ?

Variational Multiscale/VMS: AD equation

Recipe:

- Split/decompose the solution and weight spaces and functions into a coarse scale and a fine scale

$$\mathcal{S} = \bar{\mathcal{S}} \oplus \mathcal{S}' \qquad \mathcal{W} = \bar{\mathcal{W}} \oplus \mathcal{W}'$$
$$\phi = \bar{\phi} + \phi' \qquad w = \bar{w} + w'$$

- Use an *appropriate* projector for this decomposition, for example, $\bar{\phi} = \mathbb{P}(\phi)$ and thus, $\phi' = \mathbb{I}(\phi) - \mathbb{P}(u) = (\mathbb{I} - \mathbb{P})\phi$
- Consider the overall variational form:

$$a(\bar{w} + w', \bar{\phi} + \phi') = (\bar{w} + w', s), \qquad \forall \bar{w} \in \bar{\mathcal{W}}, \quad \forall w' \in \mathcal{W}'$$

Variational Multiscale/VMS: AD equation

Recipe (contd'):

- Split the overall variational problem into two problems: coarse-scale and fine-scale problems (due to the linear independence of w and w')
  Coarse-scale problem:

$$
\begin{aligned}
a(\bar{w}, \bar{\phi}) + a(\bar{w}, \phi') &= (\bar{w}, s), \qquad && \forall \bar{w} \in \bar{\mathcal{W}} \\
a(\bar{w}, \bar{\phi}) + (\mathcal{L}^*(\bar{w}), \phi')_{\hat{\Omega}+\hat{\Gamma}} &= (\bar{w}, s), \qquad && \forall \bar{w} \in \bar{\mathcal{W}}
\end{aligned}
$$

  Fine-scale problem:

$$
\begin{aligned}
a(w', \bar{\phi}) + a(w', \phi') &= (w', s), \qquad && \forall w' \in \mathcal{W}' \\
(w', \mathcal{L}(\bar{\phi}))_{\hat{\Omega}+\hat{\Gamma}} + (w', \mathcal{L}(\phi'))_{\hat{\Omega}+\hat{\Gamma}} &= (w', s), \qquad && \forall w' \in \mathcal{W}'
\end{aligned}
$$

Variational Multiscale/VMS: AD equation

Recipe (contd'):

- Fine-scale solution from the fine-scale problem involves a fine-scale Green's function: M' is a complex integral operator

$$\phi' = M'(s - \mathcal{L}(\bar{\phi})) = -M'R(\bar{\phi})$$

- Model the effect of fine scales on coarse scales using the fine-scale solution from the fine-scale problem:

$$\begin{aligned}(\mathcal{L}^*(\bar{w}), \phi')_{\hat{\Omega}+\hat{\Gamma}} &= (\mathcal{L}^*(\bar{w}), M'(s - \mathcal{L}(\bar{\phi})))_{\hat{\Omega}+\hat{\Gamma}} \\ &= -(\mathcal{L}^*(\bar{w}), M'R(\bar{\phi}))_{\hat{\Omega}+\hat{\Gamma}}\end{aligned}$$

- A practical method uses: i) local effects, and 2) pre-computed stabilization parameter

$$\phi' = \int_{\Omega^e} g'(s - \mathcal{L}(\bar{\phi}))d\Omega^e = -\int_{\Omega^e} g'R(\bar{\phi})d\Omega^e = -\tau R(\bar{\phi})$$

- For linear finite elements (for 1D AD equation):

$$\tau = \frac{1}{|\Omega^e|}\int_{\Omega^e}\int_{\Omega^e} g' d\Omega^e d\Omega^e = \frac{h}{2|a_x|}(\coth(Pe^e) - \frac{1}{Pe^e})$$

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