



# MANE 6961:

## Adjoint for Scientists and Engineers

Lecture 4

Prof. Hicken  
JEC 2036

# The Continuous and Discrete Approaches to the Adjoint

# Lecture Objective

Our objective in this lecture is to summarize the two approaches of deriving the (discretized) adjoint equations. We will also start to examine how these approaches differ.

Over the last two classes, we have learned how to derive the following adjoint boundary-value problem given a primal boundary-value problem and a functional.

$$\begin{aligned} L^*\psi &= g, & \forall x \in \Omega, \\ B^*\psi &= c, & \forall x \in \Gamma, \end{aligned} \tag{Adj}$$

# Continuous-Adjoint Approach

As with the primal BVP, we cannot solve the adjoint BVP analytically, at least in general. Therefore, we need to discretize the adjoint PDE and BCs.

- you can apply your “favorite” discretization to discretize the adjoint BVP
- however, we shall see next class that some methods are better than others

We will denote the discretized adjoint BVP as

$$(L^*)_h \psi_h = g_h$$

## Continuous-Adjoint Approach (cont.)

This sequence — derive the adjoint BVP then discretize it — defines the continuous-adjoint approach.

### **Definition: Continuous-Adjoint Approach**

The continuous-adjoint approach consists of deriving the adjoint BVP first, and then discretizing the adjoint BVP.

- The continuous-adjoint method is also known as the “differentiate-then-discretize” approach or the “optimize-then-discretize” approach.

# Discrete Adjoint Approach

The adjoint derivation in the first lecture followed the so-called discrete-adjoint approach. Let's review it again here, following the method we used to derive the continuous adjoint.

Suppose we are given a primal BVP:

$$\begin{aligned} Lu &= f, & \forall x \in \Omega, \\ Bu &= b, & \forall x \in \Gamma. \end{aligned}$$

After discretization, we get the linear system

$$L_h u_h = f_h.$$

## Discrete Adjoint Approach (cont.)

We have to discretize the functional as well to derive the discrete adjoint. Recall that the general functional had the form

$$J(u) = (g, u)_{\Omega} + (c, Cu)_{\Gamma}$$

The discrete version of this is

$$J_h(u_h) = (g_h, u_h)_h = g_h^T u_h$$

- Like  $f_h$  in the primal discretization,  $g_h$  combines both volume and boundary terms, i.e.  $g$  and  $c$ .
- Depending on how  $(,)_h$  is defined,  $g_h$  will also hold quadrature weights.

# Discrete Adjoint Approach (cont.)

$$\begin{aligned}
 (\psi_h, L_h u_h)_h &= \psi_h^T L_h u_h \\
 &= u_h^T L_h^T \psi_h \\
 &= (u_h, L_h^T \psi_h)_h
 \end{aligned}$$

As in the continuous BVP, we also have duality in the discrete case:

$$\begin{aligned}
 J_h(u_h) &= (g_h, u_h)_h \\
 &= (g_h, u_h)_h - (\psi_h, L_h u_h - f_h)_h \\
 &= (g_h, u_h)_h - (\psi_h, L_h u_h)_h + (\psi_h, f_h)_h \\
 &= (f_h, \psi_h)_h - (u_h, L_h^T \psi_h - g_h)_h
 \end{aligned}$$

Setting  $L_h^T \psi_h = g_h$ , we get that

$$\begin{aligned}
 J_h(u_h) &= (g_h, u_h)_h \\
 &= (f_h, \psi_h)_h \\
 &\equiv \mathcal{J}_h^*(\psi_h)
 \end{aligned}$$



## Discrete Adjoint Approach (cont.)

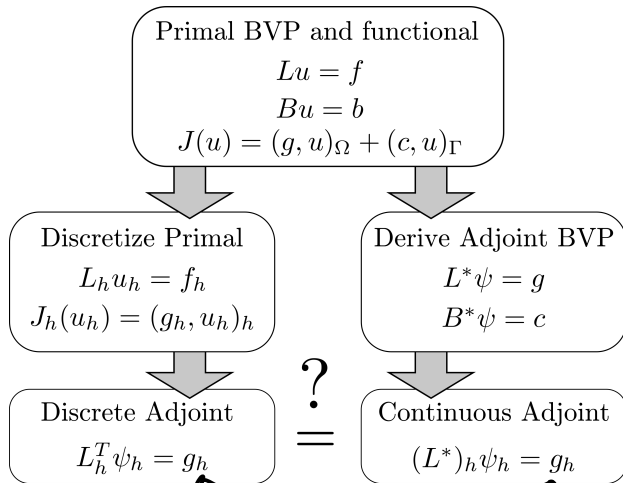
This sequence — discretize the primal BVP and functional then derive the adjoint — defines the discrete adjoint approach.

### **Definition: Discrete-Adjoint Approach**

The discrete-adjoint approach refers to discretizing the primal BVP and functional first, and then deriving the adjoint.

- The discrete-adjoint method is also known as the “discretize-then-differentiate” approach or the “discretize-then-optimize” approach.

# Summary: Paths to the Adjoint



*these not the same  
in general*

## Summary: Paths to the Adjoint (cont.)

It is important to recognize that the two paths may produce different solutions. In other words

$$(L^*)_h \neq L_h^T$$

in general.

- The right-hand sides of the adjoint equation,  $g_h$ , will also differ in general.

More on this topic next class.

Consider the Poisson's equation in one dimension:

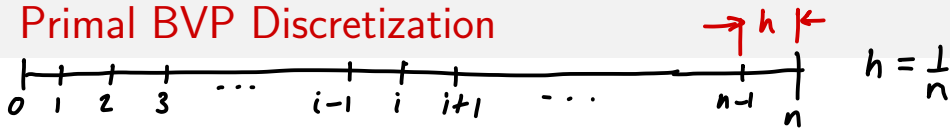
$$Lu = \frac{d}{dx} \left( \nu \frac{du}{dx} \right) = f, \quad \forall x \in [0, 1]$$

$$Bu = \begin{cases} u, & x = 0 \\ \frac{du}{dx}, & x = 1 \end{cases} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where  $\nu(x) > 0$  is a spatially varying diffusion coefficient.

- 1 Derive the discrete-adjoint equation; use a second-order accurate finite-difference method for the primal BVP
- 2 Compare the resulting adjoint discretization with the continuous BVP

# Example: Primal BVP Discretization



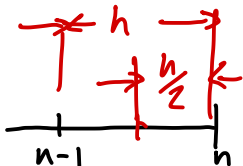
$$\left[ \frac{d}{dx} \left( \nu \frac{du}{dx} \right) - f \right]_i \approx \frac{1}{h} \left[ \nu_{i+\frac{1}{2}} \left( \frac{u_{i+1} - u_i}{h} \right) - \nu_{i-\frac{1}{2}} \left( \frac{u_i - u_{i-1}}{h} \right) \right] - f_i$$

$$= \underbrace{\left( \frac{\nu_{i-\frac{1}{2}}}{h^2} \right)}_{[L_h]_{i,i-1}} u_{i-1} - \underbrace{\left( \frac{\nu_{i-\frac{1}{2}} + \nu_{i+\frac{1}{2}}}{h^2} \right)}_{[L_h]_{i,i}} u_i + \underbrace{\left( \frac{\nu_{i+\frac{1}{2}}}{h^2} \right)}_{[L_h]_{i,i+1}} u_{i+1} - f_i$$

$$(Bu)_{i=0} = u_{i=0} = u_0 = 0 \Rightarrow [L_h]_{0,0} = 1$$

$$(Bu)_{i=n} =$$

$$\frac{2}{h} \left[ \nu_n 0 - \nu_{n-\frac{1}{2}} \left( \frac{u_n - u_{n-1}}{h} \right) \right] - f_n = 0$$



# Example: Primal BVP Discretization (cont.)

$$\underbrace{\begin{bmatrix} 1 & & & & \\ \frac{\nu_{\frac{1}{2}}}{h^2} & -\frac{(\nu_{\frac{1}{2}} + \nu_{\frac{3}{2}})}{h^2} & \frac{\nu_{\frac{3}{2}}}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\nu_{n-\frac{3}{2}}}{h^2} & -\frac{(\nu_{n-\frac{3}{2}} + \nu_{n-\frac{1}{2}})}{h^2} & \frac{\nu_{n-\frac{1}{2}}}{h^2} \\ & & \frac{2\nu_{n-\frac{1}{2}}}{h^2} & -\frac{2\nu_{n-\frac{1}{2}}}{h^2} & \end{bmatrix}}_{L_h} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}}_{u_h} = \underbrace{\begin{bmatrix} 0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}}_{f_h}$$

## Example: Functional Discretization

Recall that the general (compatible) functional in this case is

$$J(u) = \int_{\Omega} g u \, d\Omega - c_0 \left( \nu \frac{du}{dx} \right)_{x=0} - c_1 (\nu u)_{x=1}$$

$\approx \frac{du}{dx} \Big|_{x=0}$

This can be discretized as

$$J_h(u_h) = \sum_{i=0}^n g_i u_i w_i - c_0 \nu_0 \left( \frac{u_1 - u_0}{h} \right) - c_1 \nu_n u_n$$

where

$$w_i = \begin{cases} h, & i = 1, 2, \dots, n-1 \\ h/2, & i = 0, n \end{cases}$$

i.e. trapezoidal method.

## Example: Functional Discretization (cont.)

We need to get the discrete functional in the form

$$J_h(u_h) = (g_h, u_h)_h = g_h^T u_h$$

$$(g_h)_i = g_i h \quad , \quad i = 2, 3, \dots, n-1$$

$$(g_h)_0 = g_0 \frac{h}{2} + \frac{C_0 V_0}{h} \quad ,$$

$$(g_h)_1 = g_1 h - \frac{C_0 V_0}{h}$$

$$(g_h)_n = g_n \frac{h}{2} - C_1 V_n$$



# Example: Discrete Adjoint

The bilinear identity gives us

$$\begin{aligned}
 (\psi_h, L_h u_h)_h &= \sum_{i=0}^n (\psi_h)_i [L_h u_h]_i \\
 &= \sum_{i=0}^n (\psi_h)_i \sum_{j=0}^n [L_h]_{ij} (u_h)_j \\
 &= \sum_{j=0}^n (u_h)_j \sum_{i=0}^n [L_h]_{ij} (\psi_h)_i \\
 &= (u_h, L_h^T \psi_h)_h
 \end{aligned}$$

# Example: Discrete Adjoint (cont.)

$$\underbrace{\begin{bmatrix} 1 & \frac{\nu_{\frac{1}{2}}}{h^2} & & & \\ 0 & -\frac{(\nu_{\frac{1}{2}} + \nu_{\frac{3}{2}})}{h^2} & \frac{\nu_{\frac{3}{2}}}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\nu_{n-\frac{3}{2}}}{h^2} & -\frac{(\nu_{n-\frac{3}{2}} + \nu_{n-\frac{1}{2}})}{h^2} & \frac{2\nu_{n-\frac{1}{2}}}{h^2} \\ & & & \frac{\nu_{n-\frac{1}{2}}}{h^2} & -\frac{2\nu_{n-\frac{1}{2}}}{h^2} \end{bmatrix}}_{L_h^T} \underbrace{\begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{n-1} \\ \psi_n \end{bmatrix}}_{\Psi_n} = \underbrace{\begin{bmatrix} g_0 \frac{h}{2} + \frac{c_0 \nu_0}{h} \\ g_1 h - \frac{c_0 \nu_0}{h} \\ g_2 h \\ \vdots \\ g_{n-1} h \\ g_n \frac{h}{2} - c_1 \nu_n \end{bmatrix}}_{g_h}$$

## Example: Continuous Adjoint

Recall that the adjoint BVP was self-adjoint; therefore, the natural discretization for the adjoint BVP is the same one used for the primal BVP:

$$(L^*)_h = L_h$$

And the right-hand side would be given by

$$g_h^T = \left[ c_0 \quad g_1 \quad g_2 \quad \dots \quad g_{n-1} \quad g_n - \frac{2\nu_n c_1}{h} \right]$$

Comparing  $(L^*)_h$  with  $L_h^T$ , it is clear that they differ. The same is true for the  $g_h$ .

- Let's analyze the discrete adjoint to see what it approximates.

## Example: Adjoint, Interior Scheme

$$i = 2, 3, \dots, n-1$$

$$(L_h^T \Psi_h)_i = \left( \frac{v_{i-\frac{1}{2}}}{h^2} \right) \Psi_{i-1} - \left( \frac{v_{i-\frac{1}{2}} + v_{i+\frac{1}{2}}}{h^2} \right) \Psi_i + \left( \frac{v_{i+\frac{1}{2}}}{h^2} \right) \Psi_{i+1} = g_i h$$

Scaling off on R.H.S. :  $g_h \rightarrow 0$  as  $h \rightarrow 0$

Correct by defining  $\bar{\Psi}_i = \begin{cases} \Psi_i / h & , i = 1, 2, \dots, n-1 \\ 2\Psi_i / h & , i = 0, n \end{cases}$

$$\Rightarrow \left( \frac{v_{i-\frac{1}{2}}}{h^2} \right) \bar{\Psi}_{i-1} - \left( \frac{v_{i-\frac{1}{2}} + v_{i+\frac{1}{2}}}{h^2} \right) \bar{\Psi}_i + \left( \frac{v_{i+\frac{1}{2}}}{h^2} \right) \bar{\Psi}_{i+1} = g_i$$

# Example: Adjoint, Interior Scheme (cont.)

# Example: Adjoint, Right Boundary

# Example: Adjoint, Left Boundary

## Example: Adjoint, Left Boundary (cont.)



# References