MATH-6800
Vignesh Ramakrishnan
RIN: 662028006
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# Problem Set 1

1. NLA exercise 1.1 Let $B$ be a $4 \times 4$ matrix ...

- double column 1: $B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$
- halve row 3: $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0.5&0\\0&0&0&1\end{bmatrix}B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$
- add row 3 to row 1: $\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0.5&0\\0&0&0&1\end{bmatrix}B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$
- interchange columns 1 and 4: $\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0.5&0\\0&0&0&1\end{bmatrix}B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}$
- subtract row 2 from each of the other rows:
$$\begin{bmatrix}1&-1&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0.5&0\\0&0&0&1\end{bmatrix}B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}$$
- replace column 4 by column 3:
$$\begin{bmatrix}1&-1&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0.5&0\\0&0&0&1\end{bmatrix}B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}$$
- delete column 1:
$$\begin{bmatrix}1&-1&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0.5&0\\0&0&0&1\end{bmatrix}B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

When written as a product of three matrices $= ABC$, where,

$$A = \begin{bmatrix}1&-1&0.5&0\\0&1&0&0\\0&-1&0.5&0\\0&-1&0&1\end{bmatrix}, \text{and } C = \begin{bmatrix}0&0&0\\1&0&0\\0&1&1\\0&0&0\end{bmatrix}$$

```
%% Q1
clc
clear all;

L1 = [1 0 0 0; 0 1 0 0; 0 0 0.5 0; 0 0 0 1];
L2 = [1 0 1 0; 0 1 0 0; 0 0 1 0; 0 0 0 1];
L3 = [1 -1 0 0; 0 1 0 0; 0 -1 1 0; 0 -1 0 1];
A  = L3*L2*L1;
disp("A = ");
disp(A);

R1 = [2 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1];
R2 = [0 0 0 1; 0 1 0 0; 0 0 1 0; 1 0 0 0];
R3 = [1 0 0 0; 0 1 0 0; 0 0 1 1; 0 0 0 0];
R4 = [0 0 0; 1 0 0; 0 1 0; 0 0 1];
C  = R1*R2*R3*R4;
disp("C = ");
disp(C);

B = eye(4);
disp("B = ");
disp(B);
disp("double column 1");
disp(B*R1);
disp("halve row 3");
disp(L1*B*R1);
disp("add row 3 to row 1");
disp(L2*L1*B*R1);
disp("interchange columns 1 and 4");
disp(L2*L1*B*R1*R2);
disp("subtract row 2 from each of the other rows");
disp(L3*L2*L1*B*R1*R2);
disp("replace column 4 by column 3");
disp(L3*L2*L1*B*R1*R2*R3);
disp("delete column 1");
disp(L3*L2*L1*B*R1*R2*R3*R4);
disp("ABC = ");
disp(A*B*C);
%%
```

Listing 1: script to find matrices $A, C$ and to verify it works

```
A =
    1.0000   -1.0000    0.5000         0
         0    1.0000         0         0
         0   -1.0000    0.5000         0
         0   -1.0000         0    1.0000

C =
     0     0     0
     1     0     0
     0     1     1
     0     0     0

B =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1

double column 1
     2     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1

halve row 3
    2.0000         0         0         0
         0    1.0000         0         0
         0         0    0.5000         0
         0         0         0    1.0000

add row 3 to row 1
    2.0000         0    0.5000         0
         0    1.0000         0         0
         0         0    0.5000         0
         0         0         0    1.0000

interchange columns 1 and 4
         0         0    0.5000    2.0000
         0    1.0000         0         0
         0         0    0.5000         0
    1.0000         0         0         0

subtract row 2 from each of the other rows
         0   -1.0000    0.5000    2.0000
         0    1.0000         0         0
         0   -1.0000    0.5000         0
    1.0000   -1.0000         0         0

replace column 4 by column 3
         0   -1.0000    0.5000    0.5000
         0    1.0000         0         0
         0   -1.0000    0.5000    0.5000
    1.0000   -1.0000         0         0

delete column 1
   -1.0000    0.5000    0.5000
    1.0000         0         0
   -1.0000    0.5000    0.5000
```

```
   -1.0000         0         0

ABC =
   -1.0000    0.5000    0.5000
    1.0000         0         0
   -1.0000    0.5000    0.5000
   -1.0000         0         0

>>
```

2. NLA exercise 2.2 *The Pythagorean theorem asserts that for a set ...*

(a)

$$
\begin{aligned}
||x_1 + x_2||^2 &= ||x_1||^2 + ||x_2||^2 + 2\cancelto{0}{\langle x_1, x_2\rangle} \\
||x_1 + x_2||^2 &= ||x_1||^2 + ||x_2||^2
\end{aligned}
$$

Since $x_1, x_2$ are orthogonal to each other, the inner product is 0.

(b) Assuming this form holds true for some $k < n$,

$$\implies \left|\left|\sum_{i=1}^{k} x_i\right|\right|^2 = \sum_{i=1}^{k} ||x_i||^2$$

Now, using result from part(a),

$$
\begin{aligned}
\left|\left|\left(\sum_{i=1}^{k} x_i\right) + x_{k+1}\right|\right|^2 &= \left|\left|\sum_{i=1}^{k} x_i\right|\right|^2 + ||x_{k+1}||^2 + 2\left\langle \sum_{i=1}^{k} x_i, x_{k+1}\right\rangle \\
\left|\left|\left(\sum_{i=1}^{k} x_i\right) + x_{k+1}\right|\right|^2 &= \left|\left|\sum_{i=1}^{k} x_i\right|\right|^2 + ||x_{k+1}||^2 + 2\left(\sum_{i=1}^{k} \cancelto{0}{\langle x_i, x_{k+1}\rangle}\right) \\
\left|\left|\left(\sum_{i=1}^{k} x_i\right) + x_{k+1}\right|\right|^2 &= \sum_{i=1}^{k+1} ||x_i||^2
\end{aligned}
$$

Now, if $k = n - 1$, the above equation becomes,

$$\left|\left|\sum_{i=1}^{n} x_i\right|\right|^2 = \sum_{i=1}^{n} ||x_i||^2$$

Hence Proved through induction.

3. NLA exercise 2.3 Let $A \in \mathbb{C}^{m\times m}$ be hermitian. An eigenvector ...
$A \in \mathbb{C}^{m\times m}$ is hermitian, $x \in \mathbb{C}^m$ is a non-zero eigen-vector of $A$ and,

$$Ax = \lambda x$$

where $\lambda \in \mathbb{C}$ is an eigenvalue.

(a) Prove that all eigenvalues are real.

$$\langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda\, ||x||^2$$

$$
\begin{aligned}
(x^* Ax)^* &= (Ax)^*\, x = x^* A^* x \\
&= (\lambda x)^*\, x = x^* Ax \quad \text{, since } A \text{ is hermitian} \\
&= \bar{\lambda}\, ||x||^2 = \lambda\, ||x||^2 \quad \text{, from above}
\end{aligned}
$$

Since $x$ is a non-zero eigen-vector, $\bar{\lambda} = \lambda$, therefore all eigen-values $\in \mathbb{R}$.

(b)

$$Ax = \lambda_1 x, \qquad\qquad Ay = \lambda_2 y$$

Prove that $x, y$ are orthogonal to each other if all the eigen-values are distinct.

$$\begin{aligned}
\langle y, Ax\rangle =& y^* Ax = y^* \lambda_1 x \\
=& y^* A^* x = \lambda_1 y^* x \quad ,A \text{ is hermitian} \\
=& (Ay)^* x = \lambda_1 y^* x \\
=& \lambda_2 y^* x = \lambda_1 y^* x \quad ,\lambda\text{'s are real numbers} \\
\implies \ y^* x (\lambda_2 - \lambda_1) = 0 &
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle x, Ay\rangle =& x^* Ay = x^* \lambda_2 y \\
=& x^* A^* y = \lambda_2 x^* y \\
=& (Ax)^* y = \lambda_2 x^* y \\
=& \lambda_1 x^* y = \lambda_2 x^* y
\end{aligned}$$

This means, $y^* x = 0, x^* y = 0$ as $\lambda$'s are distinct and real. Hence the vectors $x, y$ are orthogonal to each other.

4. NLA exercise 2.5 Let $S \in \mathbb{C}^{m \times m}$ be skew-hermitian ...

$$S^* = -S$$

(a) If $S$ is skew-hermitian, $iS$ is a hermitian matrix because,

$$(iS)^* = -iS^* = iS$$

So, if $iS$ is a hermitian matrix, from the previous question, we know that all its eigenvalues are real. In which case,

$$\begin{aligned}
(iS)\, x =& \lambda x \quad , \text{ where } \lambda \in \mathbb{R} \\
-i\,(iS)\, x =& -i\lambda x \\
Sx = (i\lambda)\, x =& \beta x \quad , \text{ where } \beta \in \mathbb{C}
\end{aligned}$$

Therefore, $S$ has purely imaginary eigenvalues because $\lambda \in \mathbb{R}$, then $i\lambda$ is purely an imaginary number.

(b) Prove that, $(I - S)$ is non-singular.
A property of a non-singular matrix is that its inverse exists and that

$$(I - S)\, x = 0 \implies x = 0$$

Suppose there is an $x \neq 0$ that satisfies $(I - S)x = 0$, then

$$\begin{aligned}
x - Sx &= 0 \\
x &= Sx \\
x^* x &= x^* Sx
\end{aligned}$$

If $x \neq 0$ is an eigen-vector, then the following relationship would become

$$\begin{aligned} x^*x - x^*Sx =& 0 \\ x^*x + x^*S^*x =& x^*x + (Sx)^* x = 0 \\ ||x||_2^2 + \lambda ||x||_2^2 =& 0 \\ (1+\lambda) ||x||_2^2 =& 0 \end{aligned}$$

Since, the eigen-vector is a non-zero vector, $||x||_2^2 \neq 0$. Then $1 + \lambda = 0$. This means its eigen-values are $= -1$ which contradicts the result in part(a) where we proved that it has purely imaginary eigen-values. Hence, the only possibility that $(I - S)x = 0$ is when $x = 0$ vector.

(c) Show that the matrix $Q = (I - S)^{-1}(I + S)$ is unitary.
Let us start with finding out what is $Q^*Q$:

$$\begin{aligned} Q^*Q =& \left((I-S)^{-1}(I+S)\right)^* (I-S)^{-1}(I+S) \\ &= (I+S)^* \left((I-S)^*\right)^{-1} (I-S)^{-1}(I+S) \\ &= (I+S^*)(I-S^*)^{-1}(I-S)^{-1}(I+S) \\ &= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S) \\ &= (I-S)\left((I-S)(I+S)\right)^{-1}(I+S), \text{ multiplying by } (I-S)(I-S)^{-1} = I \\ &= (I-S)\left(I-S^2\right)^{-1}\left(I-S^2\right)(I-S)^{-1} \\ &= I \end{aligned}$$

Since, $Q^*Q = I$, this matrix is unitary.

5. NLA exercise 2.6 If $u$ and $v$ are $m$-vectors, the matrix $I + uv^*$ is known ...
If $A$ is a singular matrix then there should be some vector $x \neq 0$ that is in the $null(A)$. So, $(I + uv^*)x = 0$. This means, $x + uv^*x = 0$, where $v^*x = \alpha \in \mathbb{C} \neq 0$. Therefore, some $x = -\alpha u$, which is a linear combination of $u$, falls into the null-space of $A$. $null(A) = -\alpha u$. Now, substituting this back to $(I+uv^*)x = (u+uv^*u)(-\alpha) = 0$ which means that if $v^*u = -1$, then matrix $A$ will be singular.
If $A$ is a non-singular matrix, then its inverse exists, $A \in \mathbb{C}^{m\times m}$ and $AA^{-1} = I$. Let,

$$A^{-1} = \begin{bmatrix} | & | & \dots & | \\ x_1 & x_2 & \dots & x_m \\ | & | & \dots & | \end{bmatrix}$$

where, each of its columns are vectors $x_i$. Then,

$$\begin{aligned} AA^{-1} =& (I+uv^*)\begin{bmatrix} | & | & \dots & | \\ x_1 & x_2 & \dots & x_m \\ | & | & \dots & | \end{bmatrix} = I \\ =& \begin{bmatrix} | & | & \dots & | \\ (I+uv^*)\,x_1 & (I+uv^*)\,x_2 & \dots & (I+uv^*)\,x_m \\ | & | & \dots & | \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ e_1 & e_2 & \dots & e_m \\ | & | & \dots & | \end{bmatrix} \end{aligned}$$

This means, $(I + uv^*)\, x_i = e_i, \implies x_i + uv^* x_i = e_i$. Here, $v^* x_i = \beta_i \in \mathbb{C}$. Therefore, $x_i = e_i - \beta_i u$, where $\beta_i \in \mathbb{C}$ and substituting this back into $A^{-1}$, we get

$$
\begin{aligned}
A^{-1} &= \begin{bmatrix} | & | & \dots & | \\ e_1 - \beta_1 u & e_2 - \beta_2 u & \dots & e_m - \beta_m u \\ | & | & \dots & | \end{bmatrix} \\
&= I - \begin{bmatrix} | & | & \dots & | \\ \beta_1 u & \beta_2 u & \dots & \beta_m u \\ | & | & \dots & | \end{bmatrix} \\
&= I - u\beta^*, \text{ also proven in question 8}
\end{aligned}
$$

where $\beta = [\beta_1, \beta_2, \dots, \beta_m]^T$. Now, using this form in $AA^{-1} = I$, we get,

$$
\begin{aligned}
(I + uv^*)\,(I - u\beta^*) &= I \\
I - u\beta^* + uv^* - uv^* u\beta^* &= I, \; v * u = \gamma \in \mathbb{C}
\end{aligned}
$$

$$
-\begin{bmatrix} | & | & \dots & | \\ \beta_1 u & \beta_2 u & \dots & \beta_m u \\ | & | & \dots & | \end{bmatrix} + \begin{bmatrix} | & | & \dots & | \\ v_1 u & v_2 u & \dots & v_m u \\ | & | & \dots & | \end{bmatrix} - \gamma \begin{bmatrix} | & | & \dots & | \\ \beta_1 u & \beta_2 u & \dots & \beta_m u \\ | & | & \dots & | \end{bmatrix} = 0
$$

Some general form here looks like, $-u\beta_i + uv_i - \gamma u \beta_i = 0$ and solving for $\beta_i$ gives us, $\beta_i = \frac{v_i}{1+\gamma} = \alpha v_i, \alpha \in \mathbb{C}$. Therefore, putting this back to $A^{-1}$ tells that

$$
A^{-1} = I + \alpha uv^*
$$

6. NLA exercise 3.1 Prove that if $W$ is an arbitrary nonsingular matrix, ...
   To prove this, we need to make sure the norm satisfies all the 3 properties of a vector norm.

$$
||x||_W = ||Wx|| \geq 0
$$

   This is because all norms are positive quantities and its equal to 0 only when $x = 0$ because of the non-singularity of $W$.

$$
\begin{aligned}
||x + y||_W &= ||W\,(x + y)|| \leq ||Wx|| + ||Wy|| \\
&\leq ||x||_W + ||y||_W
\end{aligned}
$$

   This proves the second property of the vector norm. Finally,

$$
||\alpha x||_W = ||W\alpha x|| = |\alpha|\, ||Wx|| = |\alpha|\, ||x||_W
$$

   Satisfies the third property as well and hence, this is a vector-norm.

7. NLA exercise 3.3 *Vector and matrix p-norms are related by various inequalities, ...*

(a) Let, $||x||_\infty = \max_{1\le i\le m}|x_i| = |x_k|$, where $1 < k < m$.

$$||x||_2 = \left(\sum_{i=1}^{m}|x_i|^2\right)^{1/2}$$
$$||x||_2 = \left(\left(\sum_{i=1}^{k-1}|x_i|^2\right) + |x_k|^2 + \left(\sum_{j=k+1}^{m}|x_j|^2\right)\right)^{1/2}$$
$$||x_2|| = \Big(\text{positive quantity } 1 + ||x||_\infty^2 + \text{positive quantity } 2\Big)^{1/2} \ge ||x||_\infty$$

One example when both these norms are equal is if $x = [1, 0]$ where both norms are $= 1$.

(b) Both $||x||_\infty$ and $||x||_2$ are $\ge 0$.

$$||x||_\infty^2 = \max_{1\le i\le m}|x_i|^2$$
$$||x||_2^2 = \sum_{i=1}^{m}|x_i|^2 \le m \max_{1\le i\le m}|x_i|^2$$
$$\le m\,||x||_\infty^2$$
$$\implies ||x||_2 \le \sqrt{m}\,||x||_\infty$$

(c) First computing $||Ax||_\infty$ and $||Ax||_2$

$$||Ax||_\infty = \left|\left|\sum_{i=1}^{n}(\underline{a}_i x_i)\right|\right|_\infty \le \sum_{i=1}^{n}||\underline{a}_i x_i||_\infty = \sum_{i=1}^{n}|x_i|\,||\underline{a}_i||_\infty \le \sum_{i=1}^{n}|x_i|\,||\underline{a}_i||_2 = ||Ax||_2$$
$$||Ax||_\infty \le ||Ax||_2$$

Now from part(b) we know that,

$$||x||_\infty \ge \frac{1}{\sqrt{n}}\,||x||_2$$
$$\sqrt{n}\frac{1}{||x||_2} \ge \frac{1}{||x||_\infty}$$
$$\sqrt{n}\frac{||Ax||_\infty}{||x||_2} \ge \frac{||Ax||_\infty}{||x||_\infty}$$

This means

$$\frac{||Ax||_\infty}{||x||_\infty} \le \sqrt{n}\frac{||Ax||_\infty}{||x||_2} \le \sqrt{n}\frac{||Ax||_2}{||x||_2} \quad \text{, from before}$$

Therefore,
$$||A||_\infty \le \sqrt{n}\,||A||_2$$

(d)

$$||Ax||_2 = \left|\left|\sum_{i=1}^{n}(\underline{a}_i x_i)\right|\right|_2 \le \sum_{i=1}^{n}||\underline{a}_i x_i||_2 = \sum_{i=1}^{n}(|x_i|\,||\underline{a}_i||_2)$$

From part(2), $||\underline{a}_i||_2 \leq \sqrt{m}\,||\underline{a}_i||_\infty$ So,

$$||Ax||_2 \leq \sqrt{m}\,||Ax||_\infty$$

Dividing by $||x||_\infty$ on both sides gives

$$\frac{||Ax||_2}{||x||_\infty} \leq \sqrt{m}\,||A||_\infty$$

Since, $||x||_\infty \leq ||x||_2$, from part(1)

$$\frac{||Ax||_2}{||x||_2} \leq \frac{||Ax||_2}{||x||_\infty} \leq \sqrt{m}\,||A||_\infty$$

$$||A||_2 \leq \sqrt{m}\,||A||_\infty$$

8. Let $A \in \mathbb{C}^{m\times n}$ with columns $a_i$ and $B \in \mathbb{C}^{p\times n}$ with columns $b_i$

$$A = [a_1|a_2|\ldots|a_n]\,, B = [b_1|b_2|\ldots|b_n]$$

, Show that

$$AB^* = a_1b_1^* + a_2b_2^* + \ldots + a_nb_n^*$$

, in two ways

(a) first using the component-wise definition for the elements of the product of two matrices: Let $C = AB^*$

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}^*$$

Then the $C$ matrix can be written as:

$$C = \begin{bmatrix} \sum_{k=1}^n a_{1k}b_{k1}^* & \sum_{k=1}^n a_{1k}b_{k2}^* & \ldots & \sum_{k=1}^n a_{1k}b_{kp}^* \\ \sum_{k=1}^n a_{2k}b_{k1}^* & \sum_{k=1}^n a_{2k}b_{k2}^* & \ldots & \sum_{k=1}^n a_{2k}b_{kp}^* \\ \vdots & \vdots & \ddots & \ldots \\ \sum_{k=1}^n a_{mk}b_{k1}^* & \sum_{k=1}^n a_{mk}b_{k2}^* & \ldots & \sum_{k=1}^n a_{mk}b_{kp}^* \end{bmatrix}$$
$$= \sum_{k=1}^{n} \begin{bmatrix} a_{1k}b_{k1}^* & a_{1k}b_{k2}^* & \ldots & a_{1k}b_{kp}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{mk}b^*k1 & a_{mk}b_{k2}^* & \ldots & a_{mk}b_{kp}^* \end{bmatrix}$$
$$= \sum_{k=1}^{n} \underline{a}_k\underline{b}_k^*$$
$$= \underline{a}_1\underline{b}_1^* + \underline{a}_2\underline{b}_2^* + \ldots + \underline{a}_n\underline{b}_n^*$$

(b) Using block-matrix approach: Any block-matrix multiplication can be represented as:

$$\left[\begin{array}{ccc|c} a_{1,1} & \dots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & \dots & a_{m,n-1} & a_{m,n} \end{array}\right]_{m\times n} \left[\begin{array}{ccc} — & \underline{b}_1^* & — \\ & \vdots & \\ — & \underline{b}_{n-1}^* & — \\ \hline — & \underline{b}_n^* & — \end{array}\right]_{n\times p} =$$

$$\left[ \left[\begin{array}{ccc|c} a_{1,1} & \dots & a_{1,n-2} & a_{1,n-1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & \dots & a_{m,n-2} & a_{m,n-1} \end{array}\right]_{m\times (n-1)} \left[\begin{array}{ccc} — & \underline{b}_1^* & — \\ & \vdots & \\ — & \underline{b}_{n-2}^* & — \\ \hline — & \underline{b}_{n-1}^* & — \end{array}\right]_{(n-1)\times p} + \underline{a}_n \underline{b}_n^* \right]_{m\times p}$$

$\vdots$

using the same block-matrix multiplication approach, you can generalize it as

$$\left[ \left[\begin{array}{ccc|c} a_{1,1} & \dots & a_{1,n-k-1} & a_{1,n-k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & \dots & a_{m,n-k-1} & a_{m,n-k} \end{array}\right]_{m\times (n-k)} \left[\begin{array}{ccc} — & \underline{b}_1^* & — \\ & \vdots & \\ — & \underline{b}_{n-k-1}^* & — \\ \hline — & \underline{b}_{n-k}^* & — \end{array}\right]_{(n-k)\times p} + \underline{a}_{k+1} \underline{b}_{k+1}^* + \dots + \underline{a}_n \underline{b}_n^* \right]_{m\times p}$$

$\vdots$

and it can be further generalized to

$= [\underline{a}_1 \underline{b}_1^* + \underline{a}_2 \underline{b}_2^* + \dots + \underline{a}_n \underline{b}_n^*]_{m\times p}$