MANE 6760 - FEM for Fluid Dyn. - Lecture 11

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F22: 7th Oct 2022

Stabilized FE Options: ADR equation

A general stabilized FE form:

$$a(\bar{w}, \bar{\phi}) + a_{stab}(\bar{w}, \bar{\phi}) = a(\bar{w}, \bar{\phi}) + \underbrace{(\hat{\mathcal{L}}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}}_{a_{stab}(\cdot,\cdot)} = (\bar{w}, s)$$

Several options available for $a_{stab}(\cdot, \cdot)$:

- SUPG: $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}^{adv}(\cdot) = -\boldsymbol{a} \cdot \nabla(\cdot)$

$$a_{stab}(\bar{w}, \bar{\phi}) = a_{SUPG}(\bar{w}, \bar{\phi}) = (-\mathcal{L}^{adv}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- GLS: $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}(\cdot) = -\left(\boldsymbol{a} \cdot \nabla(\cdot) - \kappa \nabla^2(\cdot) + c(\cdot)\right)$

$$a_{stab}(\bar{w}, \bar{\phi}) = a_{GLS}(\bar{w}, \bar{\phi}) = (-\mathcal{L}(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- VMS: $\hat{\mathcal{L}}(\cdot) = \mathcal{L}^*(\cdot) = -\boldsymbol{a} \cdot \nabla(\cdot) - \kappa \nabla^2(\cdot) + c(\cdot)$

$$a_{stab}(\bar{w}, \bar{\phi}) = a_{VMS}(\bar{w}, \bar{\phi}) = (\mathcal{L}^*(\bar{w}), -\tau R(\bar{\phi}))_{\hat{\Omega}}$$

- ... others (residual-free bubbles, etc)

What about stabilization parameter: τ?

Stabilization Parameter: ADR equation

τ approximation in 1D: algebraic version by Shakib *et al.* (1991):

$$\tau_{alg,skb} = \tau_{alg1} : (\tau_{alg,skb})^{-2} = \left(\frac{(h/2)}{|a_x|}\right)^{-2} + 9\left(\frac{(h/2)^2}{\kappa}\right)^{-2} + (\frac{1}{c})^{-2}$$

$$= \left(\frac{2|a_x|}{h}\right)^2 + 9\left(\frac{4\kappa}{h^2}\right)^2 + c^2$$

$$\tau_{alg,skb} = \tau_{alg1} = \frac{1}{\sqrt{\left(\frac{2|a_x|}{h}\right)^2 + 9\left(\frac{4\kappa}{h^2}\right)^2 + c^2}}$$

τ approximation in multiple dimensions:

$$(\tau_{alg,skb})^{-2} = (\tau_{alg1})^{-2} = a_i g_{ij} a_j + c_{diff}^2 g_{ij} g_{ij} \kappa^2 + c^2$$

$$\tau_{alg,skb} = \tau_{alg1} = \frac{1}{\sqrt{a_i g_{ij} a_j + c_{diff}^2 g_{ij} g_{ij} \kappa^2 + c^2}}$$

Simplified: 1D ADR equation

A number of simplifications:

- Steady
- 1D domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Strong form:

$$R(\phi) = \mathcal{L}(\phi) - s = a_x \frac{d\phi}{dx} - \kappa \frac{d^2\phi}{dx^2} + c\phi - s = 0 \qquad \phi \in \mathcal{S}_{strong}$$

$$x \in [0, L]$$

$$\phi(x = 0) = \phi_0 \quad \text{on} \quad x = 0$$

$$\phi(x = L) = \phi_L \quad \text{on} \quad x = L$$

Exact Solution: 1D ADR equation

General form of the solution (with a non-zero c):

$$\phi^{exact} = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \frac{s}{c}$$

λ_1 and λ_2 are given as:

$$\lambda_{\{1,2\}} = \frac{a_x \mp \sqrt{a_x^2 + 4\kappa c}}{2\kappa}$$

c_1 and c_2 are solved for to satisfy the boundary conditions. For example, when $\phi(0) = \phi_0$ and $\phi(L) = \phi_L$, then:

$$c_1 = \frac{(\phi_0 - \frac{s}{c})e^{\lambda_2 L} - (\phi_L - \frac{s}{c})}{e^{\lambda_2 L} - e^{\lambda_1 L}} \qquad c_2 = \frac{(\phi_0 - \frac{s}{c})e^{\lambda_1 L} - (\phi_L - \frac{s}{c})}{e^{\lambda_1 L} - e^{\lambda_2 L}}$$

Sign of $a_x^2 + 4\kappa c$ determines the solution regime:

- $a_x^2 + 4\kappa c \geq 0$: exponential solution
- $a_x^2 + 4\kappa c < 0$: propagating solution (exponentially modulated)

Stabilized FE Forms: 1D ADR equation

A number of simplifications:

- Steady
- 1D domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Stabilized FE forms: find $\bar{\phi} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\int_0^L \left(-\bar{w}_{,x}(a_x\bar{\phi} - \kappa\bar{\phi}_{,x}) + \bar{w}c\bar{\phi}\right) dx$$

$$\ldots \qquad \ldots = \int_0^L \bar{w} s dx$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

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