Q1 - Q5

Thursday, November 10, 2022 2:01 AM

① $A \in \mathbb{C}^{m\times n}$, $\operatorname{rank}(A) = n$
$b \in \mathbb{C}^{m}$

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix};$$

$$I_{m\times m} \times r_{m\times 1} + A_{m\times n} x_{n\times 1} = b_{m\times 1}$$

$$A^*_{n\times m} r_{m\times 1} = 0$$

$$r_{m\times 1} = b_{m\times 1} - A_{m\times n} x_{n\times 1}$$

$$A^*_{n\times m}\left(b_{m\times 1} - A_{m\times n} x_{n\times 1}\right) = 0$$

$$A^*_{n\times m} A_{m\times n} x_{n\times 1} = A^*_{n\times m} b_{m\times 1}$$

② $A \in \mathbb{C}^{m\times m} \rightarrow$ non singular,
Prove it has an LU Factorization iff $\forall k \in [1, m]$, the upper left $k\times k$ block is nonsingular.

Proof: for a 1×1 matrix, we need, $a_{11} \neq 0$ which is the leading block matrix and we dont want it to be singular.
Suppose

$$A \in \mathbb{C}^{m\times m} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1m} \\ \vdots & \ddots & & \vdots & & \vdots \\ a_{k1} & \cdots & \cdots & a_{kk} & & \vdots \\ \vdots & & & & & \\ a_{m1} & \cdots & \cdots & \cdots & \cdots & a_{mm} \end{bmatrix}$$

suppose $a_{11} = 0$, then any row operations arent valid to make column entries a_{21} to $a_{m1} = 0$. So the $\underline{\det(a_{11}) \neq 0}$.

$$L_1 = \begin{bmatrix} 1 & \cdots & & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{a_{m1}}{a_{11}} & & & 1 \end{bmatrix}; \quad A^{(2)} = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1m} \\ 0 & -\frac{a_{21}}{a_{11}}\cdot a_{12} + a_{22} & \cdots & -\frac{a_{21}}{a_{11}} a_{1m} + a_{2m} \\ \vdots & & & \vdots \\ 0 & -\frac{a_{m1}}{a_{11}} a_{12} + a_{m2} & \cdots & -\frac{a_{m1}}{a_{11}} a_{1m} + a_{mm} \end{bmatrix} = L_1 A.$$

Now, for second case, we need $-\frac{a_{21}}{a_{11}} a_{12} + a_{22} \neq 0$.

or in other words, $-a_{21} a_{11} + a_{11} a_{22} \neq 0$. (if $a_{11} \neq 0$), this is the determinant of the submatrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

So, determinant of sub matrix $\overbrace{\begin{bmatrix} a_{11} & a_{12} \\ 0 & -\frac{a_{21}}{a_{11}}\cdot a_{12} + a_{22} \end{bmatrix}}^{\text{submatrix } A^{(2)}_{1:2,1:2}} =$

determinant of $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Suppose we go till column k, then we know that determinant of $A^{(k+1)} = \det(A_{1:k,1:k})$; in which case, if the entry, $\underline{A^{(k+1)}_{kk} = \frac{1}{\beta}\det(A_{1:k,1:k})}$, $(\beta \in \mathbb{C} \neq 0)$. should not be 0 for us to proceed further.

Suppose LU decomposition isn't unique, then for and $M \neq I$, $M \in \mathbb{C}^{m\times m}$, $LU = LDD^{-1}U$
where $(LD)(D^{-1}U) = L'U'$, $L' \rightarrow$ lower Δ^{lr}, $U' \rightarrow$ upper Δ^{lr}.

LD for any non-singular D will not be strictly lower Δ^{lr} unless D is a diagonal matrix, but if its diagonal, we cant have L's diagonal to be 1 when diagonal entries of D arent $= 1$. which means, there exists no $D \neq I_n$ that can guarantee this and hence, <u>LU decomposition is unique.</u>

③ Bandwith $= 2p+1$

$p = 1 \Rightarrow BW = 3$

$a_{ij} = 0$ for $|i-j| > 1$

$\Rightarrow$ for $A \in \mathbb{C}^{3\times 3}$, $\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$

for $A \in \mathbb{C}^{4\times 4}$, $\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$

$p = 2 \Rightarrow BW = 5 \quad |i-j| > 2$

$A \in \mathbb{C}^{5\times 5}$, $\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & a_{53} & a_{54} & a_{55} \end{bmatrix}$

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & \alpha_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} = A^{(2)}.$$

$$L_2 A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{a_{32}}{\alpha_{22}} & 1 \end{bmatrix} A^{(2)} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & \alpha_{22} & a_{23} \\ 0 & 0 & \alpha_{33} \end{bmatrix} = A^{(3)}$$

$$L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & \frac{a_{32}}{\alpha_{22}} & 1 \end{bmatrix} = U$$

L, U will have <u>same sparsity pattern of</u> A which means, we can figure out the rows on which the row reduction need not be performed and avoid excess FLOPS.

④ a) if A is non-singular, such a factorization always exists.

Proof: We need the $\det(A_{1:k,1:k}) \neq 0$; so, even if that seems to be the case, this can be avoided through pivoting. We can permute those corresponding rows and we can make sure $AQ = LU$ is formed.

ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, LU with pivoting possible.

b) if A is singular; $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (ex)

In this case, we dont have an LU Factorization at all even with pivoting. Hence, we cant always prove its existence

⑤ a) one way is to augment the matrix as following;

$AA = [A \mid I]$ where $I \in \mathbb{C}^{m\times m}$, $A \in \mathbb{C}^{m\times m}$.

Now performing and reducing AA to its row echelon form unit the reduced matrix becomes <u>$AA^* = [I \mid B]$</u> will generate <u>$A^{-1} = B$</u>.

The number of divisions $\left(\frac{m.(m+1)}{2}\right)$ dont change, but the number of multiplications and subtractions are on the rows of AA, which has <u>2m</u> columns.

So,

$$\left.\begin{aligned} \otimes &\rightarrow \left((2m)^3 + 3(2m)^2 - 5(2m)\right)/6 \\ \ominus &\rightarrow \left((2m)^3 + 3(2m)^2 - 5(2m)\right)/6 \end{aligned}\right\} + \approx \frac{8m^3}{3}$$

b) For all $|i-j| > p$, we can avoid divisions, multiplications and subtractions on those rows (i). $k < i \leq (m)$.

$\Rightarrow$ we can subtract off this part from each $\otimes$, $\ominus$

giving

$$2 \times \frac{(2m)^3 + 3(2m)^2 - 5(2m)}{6} - \left(\text{operations determined by } p.\right)$$

$$\left(\quad '' \quad\right) - \sum_{i=1}^{(m-f(p))} 2(k-i)^2$$

$\rightarrow$ should reduce it to order $\sim 2m^3$

c) i) We can augment A to make it $AA = [A \mid B]$ and perform LU until it becomes $AA^* = [U \mid Y]$

Number of colums have become <u>$m+n$</u>

So,

$$\otimes \rightarrow \frac{(m+n)^3 + 3(m+n)^2 - 5(m+n)}{6}$$

$$\ominus \rightarrow \frac{(m+n)^3 + 3(m+n)^2 - 5(m+n)}{6}$$

$$\hookrightarrow \frac{2(m+n)^3}{6} \approx \frac{(m+n)^3}{3}$$

ii) Each entry of $A^{-1}b_j$ will have $2m$ flops. (m, $\otimes$ and m $\oplus$). There are m entries as well adding $2m$, m times $\approx 2m^2$

Then we need to do this 'n' times for this system of equations $\Rightarrow$ FLOPS $\sim 2m^2 n$

To find A^{-1}, we'll have $\sim \frac{8m^3}{3}$ flops

$\therefore$, total operation count $\sim \underline{\underline{\frac{8m^3}{3} + 2m^2 n}}$