MANE 6760 - FEM for Fluid Dyn. - Lecture 15

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Stabilized FE Options: Non-linear "AD"/Burgers Eqn

A general stabilized FE form:

$$a(\bar{w},\bar{u}) + a_{stab}(\bar{w},\bar{u}) = a(\bar{w},\bar{u}) + \underbrace{(\hat{\mathcal{L}}(\bar{w}), -\tau R(\bar{u}))_{\hat{\Omega}}}_{a_{stab}(\cdot,\cdot)} = (\bar{w}, s)$$

Several options available for $a_{stab}(\cdot,\cdot)$:

- SUPG: $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}^{adv}(\cdot) = -\boldsymbol{u}\cdot\nabla(\cdot)$

$$a_{stab}(\bar{w},\bar{u}) = a_{SUPG}(\bar{w},\bar{u}) = (-\mathcal{L}^{adv}(\bar{w}), -\tau R(\bar{u}))_{\hat{\Omega}}$$

- GLS: $\hat{\mathcal{L}}(\cdot) = -\mathcal{L}(\cdot) = -\left(\boldsymbol{u}\cdot\nabla(\cdot) - \nu\nabla^2(\cdot)\right)$

$$a_{stab}(\bar{w},\bar{u}) = a_{GLS}(\bar{w},\bar{u}) = (-\mathcal{L}(\bar{w}), -\tau R(\bar{u}))_{\hat{\Omega}}$$

- VMS: $\hat{\mathcal{L}}(\cdot) = \mathcal{L}^*(\cdot) = -\boldsymbol{u}\cdot\nabla(\cdot) - \nu\nabla^2(\cdot))$

$$a_{stab}(\bar{w},\bar{u}) = a_{VMS}(\bar{w},\bar{u}) = (\mathcal{L}^*(\bar{w}), -\tau R(\bar{u}))_{\hat{\Omega}}$$

- ... others (residual-free bubbles, etc)

What about stabilization parameter: τ?

Stabilization Parameter: Non-linear “AD”/Burgers Eqn

τ approximation in 1D: algebraic version by Shakib *et al.* (1991):

$$\tau_{alg,skb} = \tau_{alg1} : (\tau_{alg,skb})^{-2} = \left(\frac{(h/2)}{|u|}\right)^{-2} + 9\left(\frac{(h/2)^2}{\nu}\right)^{-2}$$

$$= \left(\frac{2|u|}{h}\right)^{2} + 9\left(\frac{4\nu}{h^2}\right)^{2}$$

$$\tau_{alg,skb} = \tau_{alg1} = \frac{1}{\sqrt{\left(\frac{2|u|}{h}\right)^{2} + 9\left(\frac{4\nu}{h^2}\right)^{2}}}$$

τ approximation in multiple dimensions:

$$(\tau_{alg,skb})^{-2} = (\tau_{alg1})^{-2} = u_i g_{ij} u_j + c_{diff}^2 g_{ij} g_{ij} \nu^2$$

$$\tau_{alg,skb} = \tau_{alg1} = \frac{1}{\sqrt{u_i g_{ij} u_j + c_{diff}^2 g_{ij} g_{ij} \nu^2}}$$

Simplified: 1D Non-linear "AD"/Burgers Eqn

A number of simplifications:

- Steady
- 1D domain: $x \in [0, L]$
- Only Dirichlet/essential boundary conditions

Strong form:

$$
\begin{gathered}
R(u) = \mathcal{L}(u) - s = u\frac{du}{dx} - \nu\frac{d^2u}{dx^2} - s = 0, \qquad u \in \mathcal{S}_{strong} \\
x \in [0, L] \\
u(x = 0) = u_0 \quad \text{on} \quad x = 0 \\
u(x = L) = u_L \quad \text{on} \quad x = L
\end{gathered}
$$

Method of Manufactured Sol.: (Simplified) 1D Burgers Eqn

Method of manufactured solution: assume an exact solution (a form/expression) and determine BCs and source term, and use these BCs and source term (in FE code to compute an approximate FE solution).

For example (γ is a "free" parameter):

$$
\begin{aligned}
u(x) &= 1 + \frac{x}{L} - \frac{e^{-\gamma(L-x)} - e^{-\gamma L}}{1 - e^{-\gamma L}} \\
u(x=0) &= u_0 = 1 \\
u(x=L) &= u_L = 1 \\
s(x) &= \dots
\end{aligned}
$$

Stabilized FE Form: (Simplified) 1D Burgers Eqn

Stabilized FE forms: find $\bar{u} \in \bar{\mathcal{S}} \subset \mathcal{S}$ such that

$$\left\{ \int_0^L \left(\underbrace{-\bar{w}_{,x}\bar{u}\bar{u}}_{1} + \underbrace{\bar{w}_{,x}\nu\bar{u}_{,x}}_{2} + \underbrace{\ldots\ldots}_{3} \underbrace{-\bar{w}s}_{4} + \underbrace{\ldots\ldots}_{5} \right) dx = 0 \right.$$

for all $\bar{w} \in \bar{\mathcal{W}} \subset \mathcal{W}$

$$\bar{w}_{,x} \underbrace{\bar{u}\,\tilde{\tau}\,\bar{u}}_{\nu_{num}} \bar{u}_{,x}$$

$$-\bar{w}_{,x}\bar{u}\tau s$$

$$\hat{w}_A G_A = 0 \quad \text{and} \quad G_A = 0 \ \forall A$$

Non-linear Iterations: (Simplified) 1D Burgers Eqn

Non-linear iterations: require non-linear weak residual and tangent matrix at every iteration

$$G_A = \int_0^L \left(\underbrace{-N_{A,x}\bar{u}\bar{u}}_{1} + \underbrace{N_{A,x}\nu\bar{u}_{,x}}_{2} + \underbrace{\ldots\ldots}_{3} \underbrace{-N_A s}_{4} + \underbrace{\ldots\ldots}_{5} \right) dx$$

$$\frac{\partial G_A}{\partial \hat{u}_B} = \underbrace{\ldots\ldots}_{1}$$

$$+ \underbrace{\ldots\ldots}_{2} \quad N_{A,x}\,\nu\,\frac{\partial}{\partial \hat{u}_B}\left(N_{C,x}\hat{u}_C\right) = N_{A,x}\,\nu\,\overbrace{N_{C,x}\,\delta_{CB}}^{N_{B,x}}$$

$$+ \underbrace{\ldots\ldots}_{3}$$

$$+ \underbrace{\ldots\ldots}_{4} = 0$$

$$+ \underbrace{\ldots\ldots}_{5}$$

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