MANE 6961:

Adjoints for Scientists and Engineers

Lecture 6

Prof. Hicken
JEC 2036

Adjoints and Over-determined Boundary Value Problems

Lecture Objective

Today's lecture is a bit of a digression whose purpose is to illustrate an application of the adjoint.

Specifically, we will learn how solutions to the homogeneous adjoint BVP can be used to determine compatibility conditions on the source f and boundary data b.

Motivation

To motivate today's topic, consider the linear (matrix) problem

$$Ax = b,$$

where A is an $m \times n$ matrix, with $m \geq n$. In other words, we consider potentially over-determined linear systems.

Now, suppose that $A^T y = 0$ for some $y \neq 0$. Then

Motivation (cont.)

Thus, in order for $Ax = b$ to have a solution, the right-hand-side vector b must be orthogonal to every nontrivial solution of the homogeneous equation $A^T y = 0$.

The solution to the homogeneous adjoint BVP,

$$\begin{aligned} L^*\psi &= 0, & \forall x \in \Omega, \\ B^*\psi &= 0, & \forall x \in \Gamma, \end{aligned}$$

(Adj0)

plays a similar role to y above, as we will now show.

Compatibility in Over-determined Systems

Theorem: Compatibility

The primal boundary-value problem

$$Lu = f, \qquad \forall x \in \Omega,$$
$$Bu = b, \qquad \forall x \in \Gamma,$$

has a solution only if

$$(\psi, f)_\Omega + (C^*\psi, b)_\Gamma = 0,$$

for all nontrivial solutions to the homogeneous adjoint BVP (Adj0), where C^* is the boundary operator from Green's identity.

Compatibility in Over-determined Systems (cont.)

Proof:

Compatibility in Over-determined Systems (cont.)

Some remarks are in order regarding this result:

- Although our focus in previous lectures has been adjoints in the context of functionals, this result does not depend on an explicit functional, just Green's identity.

  (The functional "hiding" here is the trivial one, $J(u) = 0$)

- If the homogeneous adjoint BVP has only the trivial solution, $\psi = 0$, then the condition

$$(\psi, f)_\Omega + (C^*\psi, b)_\Gamma = 0,$$

  is satisfied for all f and b.

Example 1: over-constrained point mass

Consider a point mass governed by Newton's second law,

$$m\ddot{x}(t) = f(t), \qquad \forall t \in [0, T]$$

and subject to the initial and terminal conditions

$$x(0) = 0, \qquad \dot{x}(0) = 0$$
$$x(T) = 0, \qquad \dot{x}(T) = 0.$$

What are the compatibility conditions on the external force $f(t)$?

Example 1: over-constrained point mass (cont.)

Example 1: over-constrained point mass (cont.)

Example 1: over-constrained point mass (cont.)

Example 2: loaded elastic free bar

Consider the following ODE for an elastic bar:

$$I(x)\frac{d^2u}{dx^2} - \sigma = 0, \qquad \forall x \in [0, l]$$

$$\frac{d^2\sigma}{dx^2} = q(x), \qquad \forall x \in [0, l],$$

Are there any conditions on the load $q(x)$ if the BCs are

$$\sigma(0) = 0, \qquad \frac{d\sigma}{dx}(0) = \rho_0,$$

$$\sigma(l) = 0, \qquad \frac{d\sigma}{dx}(l) = \rho_l?$$

Example 2: loaded elastic free bar

Example 2: loaded elastic free bar (cont.)

Example 2: loaded elastic free bar (cont.)

Example 2: loaded elastic free bar (cont.)

References