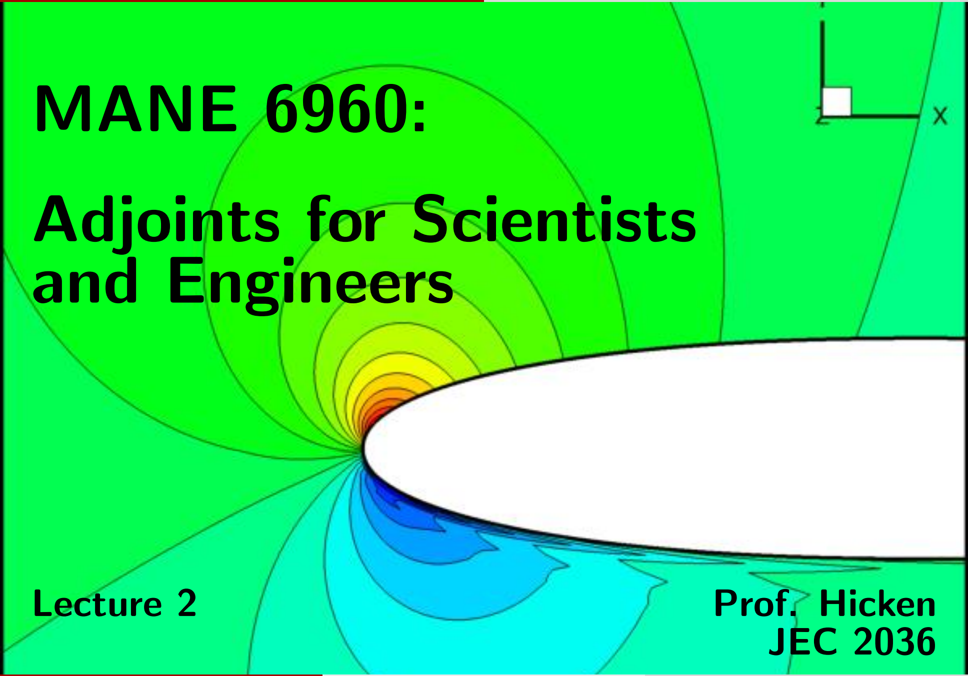

MANE 6960:

Adjoints for Scientists and Engineers

Lecture 2

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Green's Identity and Extended Identity

Lecture objective

Our objective in this lecture is to learn how to derive the adjoint operator for linear partial differential equations (PDEs).

We will use $L : \mathcal{V} \to V$ to denote the differential operator that appears in the (primal) PDE, where $\mathcal{V}$ is an appropriate function space.

Examples of L:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \overbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}^{L} u$$

$$A_x \frac{\partial u}{\partial x} + A_y \frac{\partial u}{\partial y} = \underbrace{\left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y}\right)}_{L} u$$

Review of discrete adjoint

Recall the generic discrete adjoint equation introduced last class:

$$L_h^T \psi_h = -g_h$$

where $L_h \equiv \underbrace{\partial R_h / \partial u_h}_{\text{matrix}}$ and $g_h = \underbrace{(\partial J_h / \partial u_h)}_{\text{vector}}^T$.

Notation: moving forward, I will use a subscript h whenever I am referring to a finite-dimensional object (e.g. vector, matrix).

- Our objective is to determine the analog of L_h^T for L.
- We will call this the adjoint operator and denote it by L^*.

Bilinear identity

Idea: To find L^* we will generalize the bilinear identity:

$$\psi_h^T L_h u_h - u_h^T L_h^T \psi_h = 0.$$

$(x^T A y) = y^T A^T x$

In order to generalize the bilinear identity, it is helpful (I think) to make the implicit inner product above explicit.

- For example, let $(u_h, v_h)_h \equiv u_h^T v_h$.

Then the bilinear identity becomes

$$(\psi_h, L_h u_h)_h - (u_h, L_h^T \psi_h)_h = 0.$$

Bilinear identity (cont.)

Let's make some connections between the discrete and continuous case.

discrete	**continuous**
u_h, ψ_h: vectors	u, ψ: functions
L_h, L_h^T: matrices	L, L^*: operators
$(u_h, v_h)_h$: dot product	$(u, v)_\Omega$: integral product

Integral Inner Product

Definition: Integral Inner Product

The integral inner product between two scalar, real-valued functions u and v, defined on the domain Ω, is denoted $(u, v)_\Omega$ and is defined by

$$(u, v)_\Omega \equiv \int_\Omega uv \, d\Omega.$$

- If u and v are vector-valued functions, the integrand is simply replaced with $u^T v$.
- If u and v are complex-valued functions, the integrand is replaced with $u^* v$, where u^* denotes the complex conjugate of u.

Green's identity

We now have the pieces necessary to define the analog of the bilinear identity.

Definition: Green's Identity [Lan61]

For any linear differential operator L we can uniquely define the adjoint operator L^* such that

$$(\psi, Lu)_{\Omega} - (u, L^*\psi)_\Omega = \int_\Omega \left(\psi Lu - \overbrace{uL^*\psi}^{(L^*\psi)^T u}\right) \, d\Omega = 0,$$

for any pair of sufficiently differentiable functions u and ψ that satisfy the proper boundary conditions.

Green's identity (cont.)

What does "proper boundary conditions" mean?

- For u, the "proper boundary conditions" will be give by the original PDE.
- For ψ, the "proper boundary conditions" will be discussed next class.

Extended Green's identity

Since the boundary conditions distract from our current focus on the adjoint differential operators, we will drop these requirements on u and ψ for now.

Thus, any pair of sufficiently differentiable functions u and ψ will satisfy the extended Green's Identity

$$
\begin{aligned}
(\psi, Lu)_{\Omega} - (u, L^{*}\psi)_{\Omega} &= \int_{\Omega} (\psi L u - u L^{*}\psi)\, d\Omega \\
&= \text{boundary terms},
\end{aligned}
$$

where "boundary terms" refers to integrals over the boundary $\partial\Omega$.

The 1D case

To see how the (extended) Green's identity is used to derive L^*, let's consider the one-dimensional case, where L is an ordinary differential operator.

Lemma

For a given ordinary differential operator L and sufficiently differentiable functions u and ψ, there exists L^* such that

$$\psi(x) L u(x) - u(x) L^* \psi(x) = \frac{d}{dx} F(\psi, u),$$

where $F(\psi, u)$ is a bilinear function of u, ψ, and their derivatives.

The 1D case (cont.)

Note that if the above lemma is true, then

$$(\psi, Lu)_\Omega - (u, L^*\psi)_\Omega = \int_\Omega \frac{d}{dx} F(\psi, u)\, dx = F(\psi, u)\big|_{\text{boundary}},$$

by the fundamental theorem of calculus.

Proof of the lemma:

The operator L is generally of the form

$$Lu = \sum_{k=0}^{r} p_k(x) \frac{d^k u}{dx^k}$$

It is sufficient to consider the generic term $p_k(x) \dfrac{d^k u}{dx^k}$.

The 1D case (cont.)

Now, you can verify that, for $f(x)$ and $g(x)$ in $C^k[\Omega]$,

$$f(x)\frac{d^k g(x)}{dx^k} - (-1)^k g(x)\frac{d^k f(x)}{dx^k}$$

$$= \frac{d}{dx}\left[f(x)\frac{d^{k-1}g(x)}{dx^{k-1}} - \frac{df(x)}{dx}\frac{d^{k-2}g(x)}{dx^{k-2}} + \cdots \right.$$

$$\left. + (-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}g(x) \right]$$

The 1D case (cont.)

Identify $g(x)$ with $u(x)$ and $f(x)$ with $\psi(x)\, p_k(x)$

Then,

$$\psi(x)\, p_k(x) \frac{d^k u(x)}{dx} - (-1)^k u(x) \frac{d^k}{dx^k}\left[\psi(x)\, p_k(x)\right] = \frac{d}{dx} F(\psi, u)$$

as required $\square$

Therefore, the adjoint operator for

$$Lu = p_k(x) \frac{d^k u(x)}{dx^k}$$

is

$$L^* \psi = (-1)^k \frac{d^k}{dx^k}\left[p_k(x)\, \psi(x)\right]$$

The 1D case (cont.)

In summary, the adjoint for the linear, ordinary differential operator

$$Lu = \sum_{k=0}^{r} p_k(x) \frac{d^k}{dx^k} u(x)$$

is given by

$$L^* \psi = \sum_{k=0}^{r} (-1)^k \frac{d^k}{dx^k} \left[p_k(x) \psi(x) \right].$$

- coefficients change position, e.g. outside to inside
- odd derivatives get a negative sign

Exercise

Determine the adjoint operator for

$$Lu = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u.$$

Using the above result,

$$L^*\psi = \frac{d^2}{dx^2}\left[a(x)\psi(x)\right] - \frac{d}{dx}\left[b(x)\psi(x)\right] + c(x)\psi(x)$$

$$= \underbrace{a(x)}_{\text{"}a^*(x)\text{"}}\frac{d^2\psi(x)}{dx} + \underbrace{\left[2\frac{da}{dx} - b\right]}_{\text{"}b^*(x)\text{"}}\frac{d\psi}{dx}$$

$$+ \underbrace{\left[\frac{d^2a}{dx^2} - \frac{db}{dx} + c\right]}_{\text{"}c^*(x)\text{"}}\psi$$

Exercise

Determine the adjoint operator for

$$Lu = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{du_1}{dx} \\ \frac{du_2}{dx} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$$Lu = A(x) \frac{d}{dx} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B(x) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $A(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B(x) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

$$\underbrace{\psi^T A \frac{d}{dx} u}_{\text{“}\rho_n(x)\frac{du}{dx}\text{”}} - \overbrace{(-1)^1 \frac{d}{dx}\left[\psi^T A\right]}^{\text{“}(-1)\, d/dx[\rho_n(x)\psi]\text{”}} u = \frac{d}{dx} F(\psi, u)$$

Exercise (cont.)

$$\left\{(-1)\frac{d}{dx}\left[\psi^T A\right]u\right\}^T = \left\{u^T(-1)\frac{d}{dx}\left[A^T\psi\right]\right\}$$

$$L^*\psi = -\frac{d}{dx}\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^T\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix}\right\} + \begin{bmatrix}0 & -1\\0 & 0\end{bmatrix}^T\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix}$$

$$= -\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\frac{d}{dx}\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} + \begin{bmatrix}0 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix}$$

so,

$$L^*\psi_1 = -\frac{d\psi_1}{dx}$$

$$L^*\psi_2 = -\frac{d\psi_2}{dx} - \psi_1$$

compare with primal:

$$L u_1 = \frac{du_1}{dx} - u_2$$

$$L u_2 = \frac{du_2}{dx}$$

Exercise

Determine the adjoint operator for

$$Lu = A(x,y)\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}u(x,y).$$

eg. $i=2,\ j=3$ $\quad \dfrac{\partial^5}{\partial x^2 \partial y^3}$

Idea: attack one spatial dimension at a time.

$$\underbrace{\psi^T A}_{\text{"}\psi\,p_h\text{"}} \frac{\partial^i}{\partial x^i}\underbrace{\left[\frac{\partial^j u}{\partial y^j}\right]}_{\text{"}u\text{"}} = (-1)^i \underbrace{\frac{\partial^i}{\partial x^i}\left[\psi^T A\right]}_{\text{"}\psi\,p_h\text{"}} \underbrace{\frac{\partial^j u}{\partial y^j}}_{\text{"}\frac{d^h u}{dx^h}\text{"}} + \frac{\partial F_x}{\partial x}(\psi, u)$$

$$= (-1)^i(-1)^j \frac{\partial^j}{\partial y^j}\frac{\partial^i}{\partial x^i}\left[\psi^T A\right]u + \frac{\partial F_x}{\partial x} + \partial F_y/\partial y$$

Exercise (cont.)

and transpose R.H.S.

Integrate to get Green's extended identity

$$\int_\Omega \psi^T A \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j} u \, d\Omega = \int_\Omega (-1)^i (-1)^j u^T \frac{\partial^j}{\partial y^j}\frac{\partial^i}{\partial x^i}\left[A^T \psi\right] d\Omega$$

$$+ \underbrace{\int_\Omega \left\{ \frac{\partial F_x(\psi, u)}{\partial x} + \frac{\partial F_y}{\partial y}(\psi, u) \right\} d\Omega}_{\text{Boundary terms}}$$

$$\therefore \quad L^* \psi = (-1)^{i+j} \frac{\partial^j}{\partial y^j}\frac{\partial^i}{\partial x^i}\left[A^T \psi\right]$$

References

[Lan61] Cornelius Lanczos, *Linear Differential Operators*, D. Van Nostrand Company, Limited, London, England, 1961.