MANE-6760
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MANE 6760 (FEM for Fluid Dyn.) Fall 2022: HW5

1. (5 points) For the compressible Navier-Stokes equations in pressure-primitive variables and with Nobel-Able equations of state: $\rho = \rho(p, T) = \frac{p}{RT+bp}$, where R and b are some constants. Determine: $(\mathcal{A}_0)_{l=1,m=1}$ in terms of p and T (recall that $\mathcal{A}_0 = \mathcal{U}_{,\mathcal{Y}}$).

   For this form of the Navier-Stokes equations,

$$\begin{aligned}\mathcal{A}_0 &= \mathcal{U}_{,\mathcal{Y}} \\ &= \begin{bmatrix} \frac{\partial \mathcal{U}_1}{\partial \mathcal{Y}_1} & \cdots & \frac{\partial \mathcal{U}_1}{\partial \mathcal{Y}_{n_{sd}+2}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{U}_{n_{sd}+2}}{\partial \mathcal{Y}_1} & \cdots & \frac{\partial \mathcal{U}_{n_{sd}+2}}{\partial \mathcal{Y}_{n_{sd}+2}} \end{bmatrix}\end{aligned}$$

   Here, $(\mathcal{A}_0)_{11} = \frac{\partial \mathcal{U}_1}{\partial \mathcal{Y}_1}$ where, $\mathcal{U}_1 = \rho$ and $\mathcal{Y}_1 = p$. It is also given that, $\rho = \frac{p}{RT+bp}$ where R, b are some constants. Therefore,

$$\begin{aligned}(\mathcal{A}_0)_{11} &= \frac{\partial}{\partial p}\left(\frac{p}{RT+bp}\right) \\ &= \frac{\partial p}{\partial p}\left(\frac{1}{RT+bp}\right) + p\frac{\partial}{\partial p}\left(\frac{1}{RT+bp}\right) \\ &= \left(\frac{1}{RT+bp}\right) + p\frac{\partial}{\partial\left(RT+bp\right)}\left(\frac{1}{RT+bp}\right)\frac{\partial(RT+bp)}{\partial p} \\ &= \left(\frac{1}{RT+bp}\right) - \frac{bp}{(RT+bp)^2}\end{aligned}$$

2. (10 points) For 1D, steady compressible Navier-Stokes equations with no source terms, consider the stabilized FE form for linear finite elements to be:

$$B_{stab}\left(\bar{\mathbf{W}}, \bar{\mathbf{Y}}\right) = \sum_e \int_{\Omega_e} \mathcal{A}_1^T \bar{\mathbf{W}}_{,1} \cdot \tau \mathcal{A}_1 \bar{\mathbf{Y}}_{,1}\, d\Omega_e = \sum_e \int_{\Omega_e} \bar{\mathbf{W}}_{,1} \cdot \mathcal{K}_{num} \bar{\mathbf{Y}}_{,1}\, d\Omega_e$$

   where $\mathcal{A}_1, \tau$ and $\mathcal{K}_{num}$ are $(n_{sd}+2)\times(n_{sd}+2) = 3\times 3$ matrices. Expand out $(\mathcal{K}_{num})_{lm}$ in terms of entries of the $\mathcal{A}_1$ and τ matrices, i.e., in terms of $(\mathcal{A}_1)_{11}, (\mathcal{A}_1)_{12}, \ldots, \tau_{11}, \tau_{12}, \ldots$, leading to a form such as: $(\mathcal{K}_{num})_{lm} = (\mathcal{A}_1)_{??}\,\tau_{??}\,(\mathcal{A}_1)_{??} + (\mathcal{A}_1)_{??}\,\tau_{??}\,(\mathcal{A}_1)_{??} + \ldots + (\mathcal{A}_1)_{??}\,\tau_{??}\,(\mathcal{A}_1)_{??}$. Specifically, expand out the following

   (a) $(\mathcal{K}_{num})_{l=2,m=3}$

   It is important to note that the term $\mathcal{A}_1^T \bar{\mathbf{W}}_{,1} \cdot \tau \mathcal{A}_1 \bar{\mathbf{Y}}_{,1}$ can be written as, $\left(\mathcal{A}_1^T \bar{\mathbf{W}}_{,1}\right)^T \tau \mathcal{A}_1 \bar{\mathbf{Y}}_{,1}$ using Linear Algebra.

$$\left(\mathcal{A}_1^T \bar{\mathbf{W}}_{,1}\right)^T \tau \mathcal{A}_1 \bar{\mathbf{Y}}_{,1} = \bar{\mathbf{W}}_{,1}^T \mathcal{A}_1 \tau \mathcal{A}_1 \bar{\mathbf{Y}}_{,1}$$

This means $\mathcal{K}_{num} = \mathcal{A}_1 \tau \mathcal{A}_1$. After working out this Matrix multiplication, we get

$$\begin{aligned}(\mathcal{K}_{num})_{23} = &(\mathcal{A}_1)_{13}\left((\mathcal{A}_1)_{21}\,\tau_{11} + (\mathcal{A}_1)_{22}\,\tau_{21} + (\mathcal{A}_1)_{23}\,\tau_{31}\right) + \\ &(\mathcal{A}_1)_{23}\left((\mathcal{A}_1)_{21}\,\tau_{12} + (\mathcal{A}_1)_{22}\,\tau_{22} + (\mathcal{A}_1)_{23}\,\tau_{32}\right) + \\ &(\mathcal{A}_1)_{33}\left((\mathcal{A}_1)_{21}\,\tau_{13} + (\mathcal{A}_1)_{22}\,\tau_{23} + (\mathcal{A}_1)_{23}\,\tau_{33}\right)\end{aligned}$$

To perform this Matrix multiplication, I wrote a script in Matlab and it is attached in Listing 1.

```
clc
clear

%%
s1 = 'a';
s2 = 't';

for i=1:3
    for j=1:3

        temp1 = strcat(s1,num2str(i));
        temp1 = strcat(temp1,num2str(j));

        temp2 = strcat(s2,num2str(i));
        temp2 = strcat(temp2,num2str(j));

        A(i,j)= str2sym(temp1);
        T(i,j)= str2sym(temp2);
    end
end

Knum = A*T*A;
disp(Knum(2,3));
```

Listing 1: $\mathcal{K}_{num}$ symbolic calculation