W.D. Henshaw **Math 6800: Problem Set 9** **Due:** Thursday Dec 8, 2022.

NLA = the text-book *Numerical Linear Algebra*, by Trefethen and Bau

1. (10 pts) Let $A \in \mathbb{R}^{m\times m}$ be a real symmetric matrix with eigenvalues λ_i and orthonormal eigenvectors q_i, $i = 1, 2, \ldots, m$.

(a) Prove that for any vector $x \in \mathbb{R}^m$, the Rayeligh quotient $r(x)$, lies in the interval $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\min}$ is the smallest eigenvalue and $\lambda_{\max}$ the largest eigenvalue of A.

(b) Suppose that the eigenvalues of A satisfy,

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots > |\lambda_{m-1}| > |\lambda_m|. \tag{1}$$

Given an initial guess $v^{(0)}$, how fast would the power method converge (in exact arithmetic) if

$$q_1^T v^{(0)} = 0, \qquad q_i^T v^{(0)} \neq 0, \quad i = 2, 3, \ldots, m \ ? \tag{2}$$

Explain your result. What would likely happen using floating point arithmetic with finite precision?

2. (20 pts) Let A be the $m \times m$ tridiagonal matrix with entries

$$\begin{aligned} a_{i,i-1} &= -1, \\ a_{ii} &= 4 + i, \\ a_{i,i+1} &= -1. \end{aligned}$$

(a) Write a Matlab code to use the power method to find the largest eigenvalue (denoted by λ) and corresponding eigenvector (devoted by v). Take $m = 10$ and use an initial guess of

$$v^{(0)} = [1, 1, 1, \ldots, 1]^T. \tag{3}$$

Use the Matlab function `eig` to compute the exact answer for comparison. Perform `maxit=25` iterations, and at each iteration k, print the current estimate $\lambda^{(k)}$, the error in $\lambda^{(k)}$, the 2-norm of the error in $v^{(k)}$ and the ratio of the 2-norm error in $v^{(k)}$ at step k to the previous step $k - 1$. Use the following statement to output the result:

```
fprintf('k=%4d lambda=%18.14f error=%8.2e, v-err=%8.2e ratio=%8.5f\n',k,...
         lambda,abs(lambda-lambda1),vErr,ratio);
```

The convergence rate indicated by `ratio` should approach a certain value. Explain where this value comes from.

(b) Write a Matlab code to use the Rayleigh quotient iteration to find a eigenvalue/eigenvector pair of A. Take $m = 10$ and choose the initial guess

$$v^{(0)} = [1, 1, 1, \ldots, 1]^T, \qquad \lambda^{(0)} = 10.5 \tag{4}$$

Perform `maxit=5` iterations, and at each iteration k print the current estimate $\lambda^{(k)}$, the error in $\lambda^{(k)}$, the 2-norm of the error in the eigenvector $v^{(k)}$ and the ratio of the error in $v^{(k)}$ at step k to *cube of the error* at the previous step $k - 1$ (e.g. `ratio=vErr/(vErrOld^3)`). Use the following statement to output the result:

```
fprintf('k=%4d lambda=%18.14f, error=%8.2e, v-err=%8.2e, ratio=%8.5f\n',k,...
        lambda,abs(lambda-lambda1),vErr,ratio);
```

3. (20 pts) QR program. Let $A \in \mathbb{R}^{m \times m}$ be the real symmetric tridiagonal matrix with 2 on the diagonal and -1 on the subdiagonal and superdiagonal.

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 \end{bmatrix}.$$

Let $\delta(k)$ denote the maximum of the absolute values of the off-diagonal entries of the QR matrix $A^{(k)}$.

Write a program that computes the eigenvalues of A using the UNSHIFTED QR eigenvalue algorithm using the Matlab `qr` function. After each 10 iterations, $k = 10, 20, 30, \ldots$, print k, $\delta(k)$ and the convergence ratio $r(k) = \delta(k)/\delta(k-1)$. For example, you could use a statement such as

```
fprintf('QR: k=%d : delta=%8.2e, ratio=%5.3f\n',k,delta,delta/deltaOld);
```

Continue iterating until $\delta(k) < \text{tol} = 10^{-5}$.

- Show results for $m = 11$. (Hint: you may want to test your code using a smaller value of m).
- After convergence print the eigenvalues found.
- Also print the maximum error in the eigenvalues compared to those obtained from the Matlab routine `eig` (Hint you may want to use the Matlab `sort` function to sort the eigenvalues before comparing).
- What is the numerically observed asymptotic convergence ratio $r(k)$? Is the convergence linear? Can you relate the convergence rate to the eigenvalues of A?

4. (20 pts) Shifted QR program. Repeat question (3) but writing a code using the SHIFTED QR eigenvalue algorithm with deflation (you may write a new code or write one code that can use a shift or not). Choose the shift based on the Wilkinson shift. The eigenvalues should converge from the *bottom* right corner. When the off-diagonal entry next to the bottom corner has magnitude less than $\text{tol} = 10^{-5}$, declare the eigenvalue to be converged and deflate the matrix to be one dimension less by eliminating the last row and last column. Continue the shifted QR with deflation on the remaining smaller matrix.

- Show results for $m = 11$.
- After *each* QR iteration, $k = 1, 2, 3, \ldots$, print k, the shift $\mu(k)$, the current dimension of the matrix $m(k)$, the value of $\delta(k)$ and the convergence ratio $r(k) = \delta(k)/\delta(k-1)$, $k = 1, 2, \ldots$. Continue iterating until all eigenvalues have been found.
- After convergence print the eigenvalues found.

- Also print the maximum error in the eigenvalues compared to those obtained from the Matlab routine `eig`.
- How does the convergence of the shifted QR algorithm compare to the unshifted. Does the convergence rate look linear?