Backpropagation

Overview

In this exercise we will use backpropagation to train a multi-layer perceptron (with a single hidden layer). We will experiment with different patterns and see how quickly or slowly the weights converge. We will see the impact and interplay of different parameters such as learning rate, number of iterations, and number of data points.

In [1]:
```
#Setup
import numpy as np
import matplotlib.pyplot as plt
```

In this exercise, we will prepare code to create a multi-layer perceptron with a single hidden layer (with 4 nodes) and train it via back-propagation. We will take the following steps:

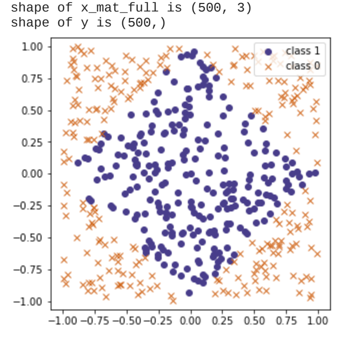
1. Initialize the weights to random values between -1 and 1
2. Perform the feed-forward computation
3. Compute the loss function
4. Calculate the gradients for all the weights via back-propagation
5. Update the weight matrices (using a learning_rate parameter)
6. Execute steps 2-5 for a fixed number of iterations
7. Plot the accuracies and log loss and observe how they change over time

Once the code is running, we can address the following questions:

- Which patterns was the neural network able to learn quickly and which took longer?
- What learning rates and numbers of iterations worked well?

In [2]:
```
## This code below generates two x values and a y value according to different patterns
## It also creates a "bias" term (a vector of 1s)
## The goal is then to learn the mapping from x to y using a neural network via back-propagation

num_obs = 500
x_mat_1 = np.random.uniform(-1,1,size = (num_obs,2))
x_mat_bias = np.ones((num_obs,1))
x_mat_full = np.concatenate( (x_mat_1,x_mat_bias), axis=1)

# # Diamond Pattern
y = ((np.abs(x_mat_full[:,0]) + np.abs(x_mat_full[:,1]))<1).astype(int)

print('shape of x_mat_full is {}'.format(x_mat_full.shape))
print('shape of y is {}'.format(y.shape))

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x_mat_full[y==1, 0],x_mat_full[y==1, 1], 'ro', label='class 1', color='darkslateblue')
ax.plot(x_mat_full[y==0, 0],x_mat_full[y==0, 1], 'bx', label='class 0', color='chocolate')
# ax.grid(True)
ax.legend(loc='best')
ax.axis('equal');
```

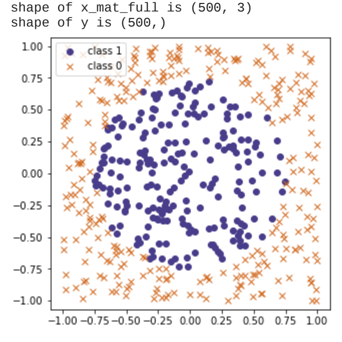
```
shape of x_mat_full is (500, 3)
shape of y is (500,)
```



checking other with other patterns

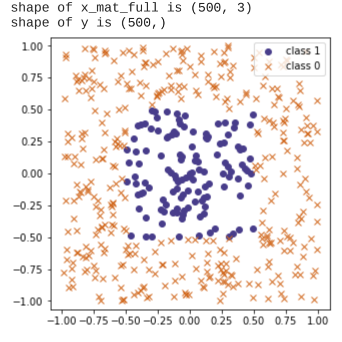
In [3]:
```
num_obs = 500
x_mat_1 = np.random.uniform(-1,1,size = (num_obs,2))
x_mat_bias = np.ones((num_obs,1))
x_mat_full = np.concatenate( (x_mat_1,x_mat_bias), axis=1)

 # Circle pattern
y = (np.sqrt(x_mat_full[:,0]**2 + x_mat_full[:,1]**2)<.75).astype(int)

print('shape of x_mat_full is {}'.format(x_mat_full.shape))
print('shape of y is {}'.format(y.shape))

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x_mat_full[y==1, 0],x_mat_full[y==1, 1], 'ro', label='class 1', color='darkslateblue')
ax.plot(x_mat_full[y==0, 0],x_mat_full[y==0, 1], 'bx', label='class 0', color='chocolate')
# ax.grid(True)
ax.legend(loc='best')
ax.axis('equal');
```

```
shape of x_mat_full is (500, 3)
shape of y is (500,)
```



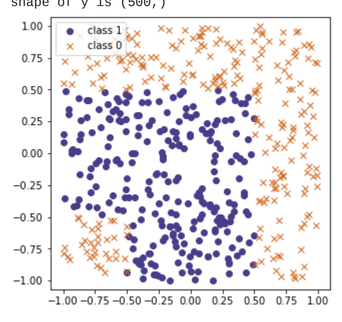
In [7]:
```
num_obs = 500
x_mat_1 = np.random.uniform(-1,1,size = (num_obs,2))
x_mat_bias = np.ones((num_obs,1))
x_mat_full = np.concatenate( (x_mat_1,x_mat_bias), axis=1)

# # Centered square
y = ((np.maximum(np.abs(x_mat_full[:,0]), np.abs(x_mat_full[:,1])))<.5).astype(int)

print('shape of x_mat_full is {}'.format(x_mat_full.shape))
print('shape of y is {}'.format(y.shape))

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x_mat_full[y==1, 0],x_mat_full[y==1, 1], 'ro', label='class 1', color='darkslateblue')
ax.plot(x_mat_full[y==0, 0],x_mat_full[y==0, 1], 'bx', label='class 0', color='chocolate')
# ax.grid(True)
ax.legend(loc='best')
ax.axis('equal');
```

```
shape of x_mat_full is (500, 3)
shape of y is (500,)
```



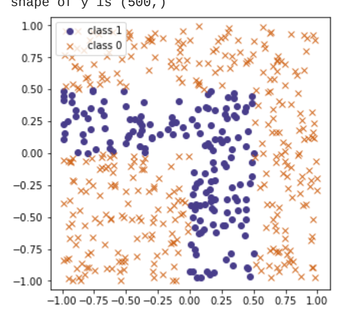
In [8]:
```
num_obs = 500
x_mat_1 = np.random.uniform(-1,1,size = (num_obs,2))
x_mat_bias = np.ones((num_obs,1))
x_mat_full = np.concatenate( (x_mat_1,x_mat_bias), axis=1)

# # Thick Right Angle pattern
y = (((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))<.5) & ((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))>-.5)).astype(int)

print('shape of x_mat_full is {}'.format(x_mat_full.shape))
print('shape of y is {}'.format(y.shape))

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x_mat_full[y==1, 0],x_mat_full[y==1, 1], 'ro', label='class 1', color='darkslateblue')
ax.plot(x_mat_full[y==0, 0],x_mat_full[y==0, 1], 'bx', label='class 0', color='chocolate')
# ax.grid(True)
ax.legend(loc='best')
ax.axis('equal');
```

```
shape of x_mat_full is (500, 3)
shape of y is (500,)
```



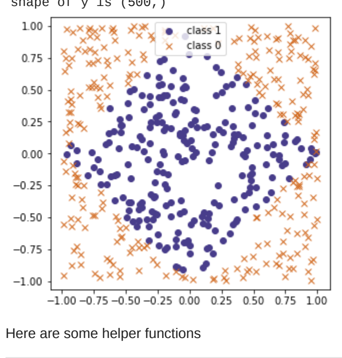
In [9]:
```
num_obs = 500
x_mat_1 = np.random.uniform(-1,1,size = (num_obs,2))
x_mat_bias = np.ones((num_obs,1))
x_mat_full = np.concatenate( (x_mat_1,x_mat_bias), axis=1)

# # Thin right angle pattern
y = (((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))<.5) & ((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))>0)).astype(int)


print('shape of x_mat_full is {}'.format(x_mat_full.shape))
print('shape of y is {}'.format(y.shape))

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x_mat_full[y==1, 0],x_mat_full[y==1, 1], 'ro', label='class 1', color='darkslateblue')
ax.plot(x_mat_full[y==0, 0],x_mat_full[y==0, 1], 'bx', label='class 0', color='chocolate')
# ax.grid(True)
ax.legend(loc='best')
ax.axis('equal');
```

```
shape of x_mat_full is (500, 3)
shape of y is (500,)
```



In [10]:
```
## This code below generates two x values and a y value according to different patterns
## It also creates a "bias" term (a vector of 1s)
## The goal is then to learn the mapping from x to y using a neural network via back-propagation

num_obs = 500
x_mat_1 = np.random.uniform(-1,1,size = (num_obs,2))
x_mat_bias = np.ones((num_obs,1))
x_mat_full = np.concatenate( (x_mat_1,x_mat_bias), axis=1)

# PICK ONE PATTERN BELOW and comment out the rest.

# # Circle pattern
# y = (np.sqrt(x_mat_full[:,0]**2 + x_mat_full[:,1]**2)<.75).astype(int)

# # Diamond Pattern
y = ((np.abs(x_mat_full[:,0]) + np.abs(x_mat_full[:,1]))<1).astype(int)

# # Centered square
# y = ((np.maximum(np.abs(x_mat_full[:,0]), np.abs(x_mat_full[:,1])))<.5).astype(int)

# # Thick Right Angle pattern
# y = (((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))<.5) & ((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))>-.5)).astype(int)

# # Thin right angle pattern
# y = (((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))<.5) & ((np.maximum((x_mat_full[:,0]), (x_mat_full[:,1])))>0)).astype(int)


print('shape of x_mat_full is {}'.format(x_mat_full.shape))
print('shape of y is {}'.format(y.shape))

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x_mat_full[y==1, 0],x_mat_full[y==1, 1], 'ro', label='class 1', color='darkslateblue')
ax.plot(x_mat_full[y==0, 0],x_mat_full[y==0, 1], 'bx', label='class 0', color='chocolate')
# ax.grid(True)
ax.legend(loc='best')
ax.axis('equal');
```

```
shape of x_mat_full is (500, 3)
shape of y is (500,)
```

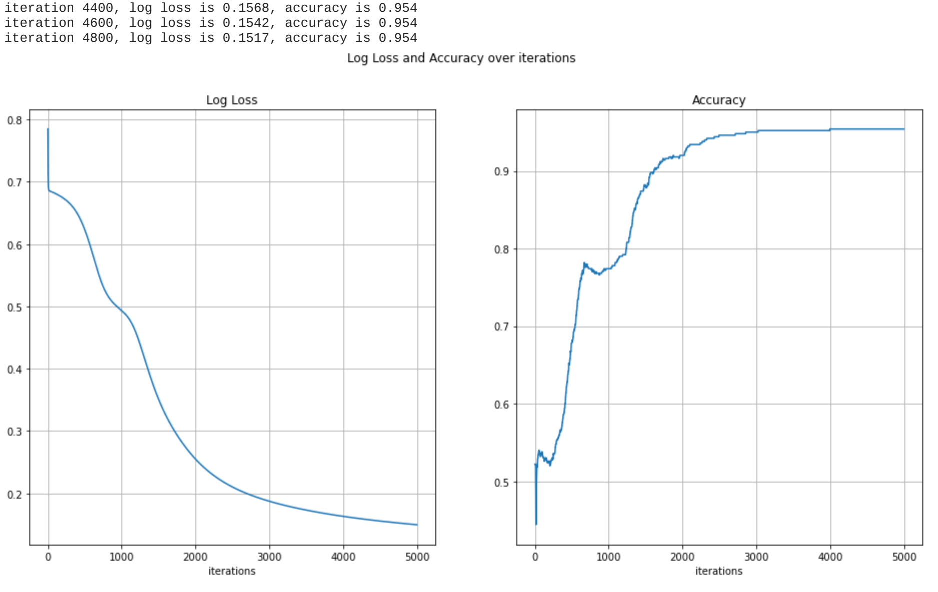


Here are some helper functions

In [22]:
```
def sigmoid(x):
    """
    Sigmoid function
    """
    return 1.0 / (1.0 + np.exp(-x))


def loss_fn(y_true, y_pred, eps=1e-16):
    """
    Loss function we would like to optimize (minimize)
    We are using Logarithmic Loss
    http://scikit-learn.org/stable/modules/model_evaluation.html#log-loss
    """
    y_pred = np.maximum(y_pred,eps)
    y_pred = np.minimum(y_pred,(1-eps))
    return -(np.sum(y_true * np.log(y_pred)) + np.sum((1-y_true)*np.log(1-y_pred)))/len(y_true)


def forward_pass(W1, W2):
    """
    Does a forward computation of the neural network
    Takes the input `x_mat` (global variable) and produces the output `y_pred`
    Also produces the gradient of the log loss function
    """
    global x_mat
    global y
    global num_
    # First, compute the new predictions `y_pred`
    z_2 = np.dot(x_mat, W_1)
    a_2 = sigmoid(z_2)
    z_3 = np.dot(a_2, W_2)
    y_pred = sigmoid(z_3).reshape((len(x_mat),))
    # Now compute the gradient
    J_z_3_grad = -y + y_pred
    J_W_2_grad = np.dot(J_z_3_grad, a_2)
    a_2_z_2_grad = sigmoid(z_2)*(1-sigmoid(z_2))
    J_W_1_grad = (np.dot((J_z_3_grad).reshape(-1,1), W_2.reshape(-1,1).T)*a_2_z_2_grad).T.dot(x_mat).T
    gradient = (J_W_1_grad, J_W_2_grad)

    # return
    return y_pred, gradient


def plot_loss_accuracy(loss_vals, accuracies):
    fig = plt.figure(figsize=(16, 8))
    fig.suptitle('Log Loss and Accuracy over iterations')

    ax = fig.add_subplot(1, 2, 1)
    ax.plot(loss_vals)
    ax.grid(True)
    ax.set(xlabel='iterations', title='Log Loss')

    ax = fig.add_subplot(1, 2, 2)
    ax.plot(accuracies)
    ax.grid(True)
    ax.set(xlabel='iterations', title='Accuracy');
```

In [26]:
```
#### Initialize the network parameters

np.random.seed(1241)

W_1 = np.random.uniform(-1,1,size=(3,4)) # (3,4) 3 dimensions as x1,x2 and bias term with 4 nodes
W_2 = np.random.uniform(-1,1,size=(4))
num_iter = 5000
learning_rate = .001
x_mat = x_mat_full


loss_vals, accuracies = [], []
for i in range(num_iter):
    ### Do a forward computation, and get the gradient
    y_pred, (J_W_1_grad, J_W_2_grad) = forward_pass(W_1, W_2)

    ## Update the weight matrices
    W_1 = W_1 - learning_rate*J_W_1_grad
    W_2 = W_2 - learning_rate*J_W_2_grad

    ### Compute the loss and accuracy
    curr_loss = loss_fn(y,y_pred)
    loss_vals.append(curr_loss)
    acc = np.sum((y_pred>=.5) == y)/num_obs
    accuracies.append(acc)

    ## Print the loss and accuracy for every 200th iteration
    if((i%200) == 0):
        print('iteration {}, log loss is {:.4f}, accuracy is {}'.format(
            i, curr_loss, acc
        ))
plot_loss_accuracy(loss_vals, accuracies)
```

```
iteration 0, log loss is 0.7848, accuracy is 0.522
iteration 200, log loss is 0.6746, accuracy is 0.524
iteration 400, log loss is 0.6489, accuracy is 0.59
iteration 600, log loss is 0.5871, accuracy is 0.748
iteration 800, log loss is 0.5196, accuracy is 0.77
iteration 1000, log loss is 0.4933, accuracy is 0.774
iteration 1200, log loss is 0.4559, accuracy is 0.792
iteration 1400, log loss is 0.3832, accuracy is 0.864
iteration 1600, log loss is 0.3248, accuracy is 0.898
iteration 1800, log loss is 0.2847, accuracy is 0.916
iteration 2000, log loss is 0.2549, accuracy is 0.92
iteration 2200, log loss is 0.2329, accuracy is 0.934
iteration 2400, log loss is 0.2167, accuracy is 0.942
iteration 2600, log loss is 0.2045, accuracy is 0.946
iteration 2800, log loss is 0.1949, accuracy is 0.948
iteration 3000, log loss is 0.1873, accuracy is 0.95
iteration 3200, log loss is 0.1809, accuracy is 0.952
iteration 3400, log loss is 0.1755, accuracy is 0.952
iteration 3600, log loss is 0.1708, accuracy is 0.952
iteration 3800, log loss is 0.1667, accuracy is 0.952
iteration 4000, log loss is 0.1630, accuracy is 0.954
iteration 4200, log loss is 0.1597, accuracy is 0.954
iteration 4400, log loss is 0.1568, accuracy is 0.954
iteration 4600, log loss is 0.1542, accuracy is 0.954
iteration 4800, log loss is 0.1517, accuracy is 0.954
```



Plot the predicted answers, with mistakes in yellow

In [30]:
```
pred1 = (y_pred>=.5)
pred0 = (y_pred<.5)

fig, ax = plt.subplots(figsize=(8, 8))
# true predictions
ax.plot(x_mat[pred1 & (y==1),0],x_mat[pred1 & (y==1),1], 'ro', label='true positives')
ax.plot(x_mat[pred0 & (y==0),0],x_mat[pred0 & (y==0),1], 'bx', label='true negatives')
# false predictions
ax.plot(x_mat[pred1 & (y==0),0],x_mat[pred1 & (y==0),1], 'yx', label='false positives', markersize=15)
ax.plot(x_mat[pred0 & (y==1),0],x_mat[pred0 & (y==1),1], 'yo', label='false negatives', markersize=15, alpha=.6)
ax.set(title='Truth vs Prediction')
ax.legend(bbox_to_anchor=(1, 0.8), fancybox=True, shadow=True, fontsize='x-large');
```