PROJECT 3

MGMTMFE 405
SPRING 2024

You will need to write codes for all the parts of the project. Make sure the codes work properly and understand the ideas behind each problem below. You may be asked to demonstrate how the codes work, by running them and interpreting the results. Code quality, speed, and accuracy will determine the grades.

LECTURE - 7 [Exotic Options, Variance Swaps, Jump-Diffusions]

1. Assume that the value of a collateral follows a jump-diffusion process:

$$\frac{dV_t}{V_t^-} = \mu dt + \sigma dW_t + \gamma dJ_t$$

where $\mu, \sigma,\ \gamma < 0$, and V_0 are given, J is a Poisson process, with intensity λ_1, independent of the Brownian Motion process W.
V_t^- is the value process before jump occurs at time t (if any).

Consider a collateralized loan, with a contract rate per period r and maturity T on the above-collateral, and assume the outstanding balance of that loan follows this process:

$$L_t = a - bc^{12t}$$

where $a > 0, b > 0\ , c > 1, and\ \ L_0$ are given. We have that $L_T = 0$.

Define the following stopping time: $\tau = min\{t \geq 0:\ V_t \leq q_t L_t\}$

Here, q_t is a known function of time. This stopping time is the first time when the relative value of the collateral (with respect to the outstanding loan balance) crosses a threshold, which will be viewed as the "optimal exercise boundary" of the option to default.

We assume the embedded default option will be exercised at time τ, if and only if $\tau \leq T$.

If the default option is exercised at time τ then the "payoff" to the borrower is: $(L_\tau - \epsilon V_\tau)^+$.

Notes: 1. If $\tau > T$ then there is no default option exercise.
2. ϵ Should be viewed as the recovery rate of the collateral, so $(\,1 - \epsilon)$ can be viewed as the legal and administrative expenses.

Assume J has intensity λ_1 , and J is independent of W. Assume the APR of the loan is $R = r_0 + \delta\lambda_2$ where r_0 is the "risk-free" rate, and δ is a positive parameter to measure the borrower's creditworthiness in determining the contract rate per period: r.

We have monthly compounding here, so the monthly rate is $r = R/12$.

Assume that $q_t = \alpha + \beta t$, where $\beta > 0,\ \alpha < V_0/L_0$ and $\beta = \frac{\epsilon - \alpha}{T}$.

Use r_0 for discounting cash flows. Use the following base-case parameter values:

$V_0 = \$20{,}000, L_0 = \$22{,}000, \mu = -0.1,\ \sigma = 0.2,\ \gamma = -0.4,\ \lambda_1 = 0.2,\ T = 5$ years, $r_0 = 0.055, \delta = 0.25, \lambda_2 = 0.4,\ \alpha = 0.7,\ \epsilon = 0.95$.
Notice that, $PMT = \frac{L_0.r}{\left[1 - \frac{1}{(1+r)^n)}\right]}$, where $r = R/12, n = T * 12$, and $a = \frac{PMT}{r}, b = \frac{PMT}{r(1+r)^n}\ , c = (1 + r)$.
Notice that, $q_T = \epsilon$.

Write the code as a function Proj3_2func.* that takes λ_1 and T as parameters, setting defaults if these parameters are not supplied, and outputs the default option value, the default probability, and the expected default option exercise time. Function specification:

function [D, Prob, Et] = Proj3_2func(lambda1, T)

(a) Estimate the **value of the default option** for the following ranges of parameters:
    λ_1 from 0.05 to 0.4 in increments of 0.05;
    T from 3 to 8 in increments of 1;

(b) Estimate the **default probability** for the following ranges of parameters:

    λ_1 from 0.05 to 0.4 in increments of 0.05;
    T from 3 to 8 in increments of 1;

(c) Find the **Expected option Exercise Time** of the default option, conditional on $\tau < T$. That is, estimate $E(\tau \mid \tau < T)$ for the following ranges of parameters:
    λ_1 from 0.05 to 0.4 in increments of 0.05;
    T from 3 to 8 in increments of 1;

Inputs: *seed*
Outputs:

i. Values: the default option D, the default probability Prob and the expected exercise time Et for parts (a), (b) and (c) with λ_1=0.2 and T=5.
ii. Graphs: For each of (a), (b) and (c) - a graph as a function of T with λ_1 from 0.05 to 0.4 in increments of 0.05.

2. Consider the following 2-factor model for a stock price process, under the risk-neutral measure:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t}\, S_t\, dW_t \\ dv_t = (\alpha + \beta v_t)dt + \gamma\sqrt{v_t}\, dB_t \end{cases}$$

where W_t and B_t are correlated Brownian Motion processes with $dW_t dB_t = \rho dt$.
Default parameter values: $v_0 = 0.1,\ \alpha = 0.45,\ \beta = -5.105,\ \gamma = 0.25,\ S_0 = \$100,\ r = 0.05,\ \rho = -0.75,\ K = \$100,\ T = 1$.

(a) Estimate the Price (P_1) of a Down-and-Out Put option with the barrier at $S_b^1(t) = 94$.

(b) Estimate the Price (P_2) of a Down-and-Out Put option with time-dependent barrier $S_b^2(t) = \frac{6}{T}t + 91$.

(c) Estimate the Price (P_3) of a Down-and-Out Put option with time-dependent barrier $S_b^3(t) = -\frac{6}{T}t + 97$.

Notes:

- All options in parts a, b, and c have payoffs similar to the European Put option; however, these options become void (the contract is cancelled), if the underlying security's price crosses it and goes below the barrier $S_b^i(t)$ at any time during the life of the option.
- You may use **any method** to price the securities – Monte Carlo Simulations, Binomial or Trinomial Tree Methods, the PDE approach, etc.
- If you use Monte Carlo simulations in this problem, use the "full truncation" method to simulate the volatility-process, v_{k+1}. The Euler discretization scheme, in that case, will be as follows:

$$S_{k+1} = S_k + rS_k\Delta_t + S_k\sqrt{v_k^+}\ (\Delta W_{k+1})$$

$$v_{k+1} = v_k + (\alpha + \beta v_k^+)\Delta_t + \gamma\sqrt{v_k^+}\ (\Delta B_{k+1})$$

where $v_k^+ = \max(v_k, 0)$.

Inputs: $K, T. \gamma$

Outputs: 1) prices P_1, P_2, P_3 for parts (a), (b), and (c).
2) Writeup: Compare the prices P_1, P_2, P_3 for parts (a), (b), and (c). and comment on them.

3. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following SDE (**CIR model**):

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

with $r_0 = 5\%, \sigma = 12\%,\ \kappa = 0.92,\ \bar{r} = 5.5\%$.

(a) Use Monte Carlo Simulation to find the price of a coupon-paying bond, with Face Value of \$1,000, paying semiannual coupons of \$30, maturing in $T = 4$ years:

$$P(0, C, T) = \mathbb{E}_0^*\left[\sum_{i=1}^{8} C_i * exp\left(-\int_0^{T_i} r(s)ds\right)\right]$$

where $C = \{C_i = \$30 \text{ for } i = 1,2, \dots ,7; \text{ and } C_8 = \$1{,}030\}$, and $T = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \{0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4\}$.

(b) Use Monte Carlo Simulation to find at time $t = 0$ the price $c_{MC}(t, T, S)$ of a European Call option, with strike price of $K = \$980$ and expiration in $T = 0.5$ years on a Pure Discount Bond that has Face Value of \$1,000 and matures in $S = 1$ year:

$$c_{MC}(t, T, S) = \mathbb{E}_t^*\left[exp\left(-\int_t^T r(u)du\right) * \max(P(T, S) - K, 0)\right]$$

(c) Use the *Implicit Finite-Difference Method* to find at time $t = 0$ the price $c_{PDE}(t, T, S)$ of a European Call option, with strike price of $K = \$980$ and expiration in $T = 0.5$ years on a Pure Discount Bond that has Face Value of \$1,000 and matures in $S = 1$ year. The PDE is given as follows

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 c}{\partial r^2} + \kappa(\bar{r} - r)\frac{\partial c}{\partial r} - rc = 0$$

with $c(T, T, S) = \max(P(T, S) - K, 0)$, and $P(T, S)$ is computed **explicitly**.

Inputs: $r_0, \sigma,\ \kappa,\ \bar{r}$ for all parts, and T, S only for parts (b) and (c)

Outputs:
(1) (a): Bond price, (b): Call option price, (c): Call option price.
2) Writeup: Compare the prices $c_{MC}(t, T, S)$ and $c_{PDE}(t, T, S)$ in parts (b) and (c) and comment on them.

4. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following system of SDEs (**G2++ model**):

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t^1 \\ dy_t = -by_t dt + \eta dW_t^2 \\ r_t = x_t + y_t + \phi_t \end{cases}$$

Default parameter values: $x_0 = y_0 = 0,\ \phi_0 = r_0 = 5.5\%,\ dW_t^1 dW_t^2 = \rho dt,\ \rho = 0.7, a = 0.1,\ b = 0.3, \sigma = 5\%,\ \eta = 9\%$. Assume $\phi_t = const = 5.5\%$ for any $t \geq 0$.
Use Monte Carlo Simulation to find at time $t = 0$ the price $p(t, T, S, K, \rho)$ of a European Put option, with strike price of $K = \$950$, expiration in $T = 0.5$ years on a Pure Discount Bond with Face value of \$1,000 that matures in $S = 1$ year. Compare it with the price found by the explicit formula and comment on it.

Inputs: T, S, K, ρ

Outputs:
1) Price $p(0, T, S, K, \rho)$
2) Writeup: Compare the prices by varying ρ in the range [-0.7, 0.7] in increments of 0.1.

LECTURE - 9, 10 [MBS]

5. Consider a 30-year MBS with a fixed weighted-average-coupon, $WAC = 8\%$. Monthly cash flows are starting in January of this year. The Notional Amount of the Pool is $\$100{,}000$.

   Use the CIR model of interest rates, $dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_{t,}$ with the following default parameters: $r_0 = 0.078, k = 0.6, \bar{r} = 0.08,\ \sigma = 0.12$.

   Consider the ***Numerix Prepayment Model*** in all problems below.

(a) Compute the price of the MBS. The code should be generic: the user is prompted for inputs and the program runs and gives the output.

(b) Compute the Option-Adjusted-Spread (***OAS***) if the Market Price of MBS is $\hat{P}$ =$98,000.

(c) Consider the MBS described above and the ***IO*** and ***PO*** tranches. Price the ***IO*** and ***PO*** tranches.

Inputs: $\bar{r},\ k,\ \sigma,\ \hat{P}$ (for part (b))

Outputs:
1) (a): Price of MBS; (b): OAS; (c): Prices of IO and PO.
2) Writeup: Compare the MBS prices by varying ρ in the range [-0.7, 0.7] in increments of 0.1.