Noisy Signals
Rochester Institute of Technology

PHYS-316 Advanced Lab*

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Goals

In this experiment you will learn methods to measure the amount of voltage noise in a signal produced by thermal fluctuations in a resistor, a phenomenon known as Johnson noise. The TeachSpin Noise Fundamentals Apparatus NF1-A will be used to investigate how this noise depends on bandwidth and resistance. You will determines the Boltzmann constant, possibly to within a few percent.

Introduction

Noise is an ubiquitous part of experimental physics. Although the lab focuses on Johnson noise as the noise source, many of the concepts discussed apply to measuring any noisy signal.

Background and Theory

We define a noisy signal as one that has random fluctuations from its true value over time. In this lab, you will learn how to quantify and describe the voltage noise across an ordinary resistor at room temperature.

We typically model a resistor that has no current flowing through it as having no voltage drop across it, by Ohm's Law. While this is true on average, any resistor at a temperature $T > 0$ K also has random voltage fluctuations across its leads caused by thermal agitation of the electrons inside the resistor. This randomly fluctuating voltage is known as *Johnson noise*.

One way to quantify the amount of noise in a random signal is to find the variance, which is the time average of the squared signal. A resistor with no current flowing through it has a time-averaged voltage drop of zero: $\langle V_J(t)\rangle = 0$. However, statistical mechanics shows that the variance $\left\langle V_J^2(t)\right\rangle$ of the voltage fluctuations due to Johnson noise $V_J(t)$ is given by the Nyquist Formula:

$$\left\langle V_J^2(t)\right\rangle = 4k_B RT\Delta f, \tag{1}$$

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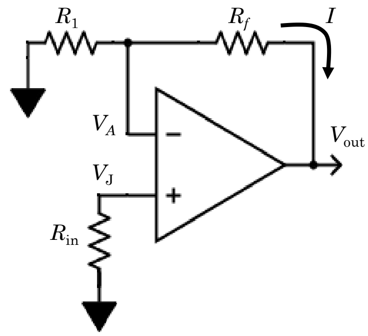

Figure 1: Pre-amplifier schematic

where T is the absolute temperature, R is the resistance, k_B is the Boltzmann constant, and Δf is the bandwidth of the noisy signal (or range of frequencies in which the noise is present) in Hz. The root-mean-square (RMS) voltage is related to the variance by

$$V_{J,\text{rms}} = \sqrt{\langle V_J^2(t)\rangle}. \tag{2}$$

Apparatus

The Johnson noise voltage across a resistor at room temperature is a very small quantity. Amplifier circuits must be used to make the signal large enough that it can be measured using ordinary multimeters or oscilloscopes. Both the Low-Level Electronics (LLE) box and High-Level Electronics (HLE) box contain amplifier or gain circuits. In addition to amplifiers, the NF1A contains circuits for filtering, squaring, and time-averaging voltage signals.

Even if not explicitly instructed to, you must demonstrate that you have measured the input and output of every module in use, and describe the action of each on your signals. Scope screenshots showing both the input and output on screen are the best way for you to gain (pun intended) this understanding.

Low-Level Electronics (LLE)

The resistor R_{in}, which serves as the input source of the Johnson noise signal, is located in the LLE. The noise source is connected to a low-noise op-amp gain circuit, called the pre-amplifier. A schematic is shown in Figure 1. The simplest (but still very useful) model of an op-amp gives two simple "golden rules" for op-amp behavior: (1) the output voltage V_{out} attempts to do whatever is necessary to keep a zero voltage difference between the inverting (negative) input and the non-inverting (positive) input ($V_A - V_J = 0$) and (2) the inverting and non-inverting terminals draw no current.

Using Figure 1, and a current I in the feedback path, use the op-amp "golden rules" to show that

$$V_{\text{out}} = \left(1 + \frac{R_f}{R_1}\right) V_J. \tag{3}$$

For the pre-amplifier shown in Figure 1, R_1 is a fixed 200-Ω resistor. R_f is an adjustable resistor and controls the gain of the pre-amplifier. After the pre-amplifier, the signal goes through a fixed gain amplifier of ×100. This second amplifier leads directly into the output of the LLE as is shown by the schematic in Figure 2.

The overall behavior of the LLE circuit is modeled by gain G_1 given by

$$G_1 = 100\left(1 + \frac{R_\mathrm{f}}{200\ \Omega}\right). \tag{4}$$

The output of the LLE, V_{LLE}, is measured from the "Output" jack in the upper-right of Figure 2. This output relates to the input Johnson noise via

$$V_{\mathrm{LLE}}(t) = G_1 V_J(t). \tag{5}$$

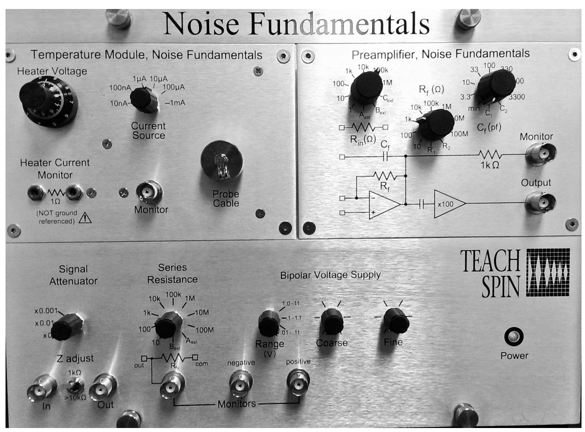

Figure 2: The TeachSpin Low-Level Electronics

High-Level Electronics (HLE)

The HLE box consists of five independent signal processing modules: two adjustable filters, an adjustable gain stage, a multiplier, and the output/averaging circuit. The modules can be used individually or in any series combination. Its use in this lab is to do signal processing on the output signal V_{LLE}. Refer to Figure 3 as you read about the function of each module.

Filter modules

Electronic filters enhance or restrict particular temporal frequencies in the signals they pass. The two Filter modules can be used as low-pass (which pass frequencies lower than a selected frequency), high-pass (which pass frequencies higher than a selected frequency), or band pass filters (which do both). The cut-off frequency (for low pass and high pass) and center frequency (for band pass) can be adjusted on this apparatus. The filter input can be AC or DC coupled but `all switches should be set to ''AC'' for this experiment`.

Figure 3: The TeachSpin High-Level Electronics

Gain module

Electronic gain is a multiplicative factor applied to a signal (and its noise). The Gain module has three settings: two $\times 1/\times 10$ switches and one fine adjustable $\times 10$ - $\times 100$ knob. The total gain is the product of the two switches and the knob. The maximum possible gain is $\times 10000$. Why is it typically preferable to filter a noisy signal before amplifying it?

Multiplier module

The Multiplier module allows you to square the input signal (in A×A mode) or take the product of two input signals (in A×B mode). The module then divides this squared voltage by 10 V to avoid saturating the apparatus. In A×A mode, the output of this module is $V_{Multiplier}(t) = \frac{V_A^2(t)}{10\text{ V}}$. The Multiplier module output voltage can be directly measured at the Monitor jack. The Multiplier module output is also internally connected to the Output module.

Output module

The Output module takes its input internally from the output of the Multiplier module. It averages the multiplier signal over a time equal to the adjustable time constant (which ranges from 0.01 s to 3.0 s). The averaged signal can be observed on the analog dial readout or measured at "OUT" using a DMM or oscilloscope: $V_{Output} = \langle V_{Multiplier}(t) \rangle$.

Measuring voltage noise using an oscilloscope

As a first method of measuring and quantifying noise, you will directly measure the voltage fluctuations as a function of time, using an oscilloscope.

- There is no power switch on either the LLE or HLE. You must plug in the HLE power supply and make sure the "Preamp Power Output" connects the LLE box to the HLE box.

- On the LLE box, set the Johnson noise resistance to $R_{\text{in}} = 100$ kΩ and the adjustable pre-amp resistor to $R_f = 1$ MΩ. Split the LLE output and send the signal to CH 1 of the oscilloscope and to the HLE second filter module's input. Set the Corner Frequency to 10 kHz. Measure the Low-Pass Filter output on CH 2 of the oscilloscope. These settings produce a large enough noisy signal to be seen on the oscilloscope without additional amplification.

- How does the oscilloscope trace of the noisy signal change as you widely vary the time scale on the oscilloscope?

- Save a screenshot of the scope display. Compare the RMS voltage of the pre-filter and post-filter signal. Describe the impact of the Low-Pass filter on your signal.

Characterizing the HLE modules

Another way to measure the noisy signal is to use the HLE analog circuit modules: the *Filter module*, the *Multiplier module*, and the *Output module*. The Multiplier module squares the input voltage and divides by 10 V, while the Output module averages its input voltage over a particular time constant. The Multiplier module is internally connected to the Output module, which creates a circuit whose output is a (roughly) constant voltage that is proportional to the variance in the input signal. You must first study the action of each module separately.

- Disconnect the LLE output from the HLE. Keep the same Low-pass filter setting at 10 kHz.

- Using a function generator as the input, simultaneously measure the input (on CH 1) and output (on CH 2) of the Low-pass filter. Set the generator to produce a 5-kHz sine wave. Save a screenshot and discuss the action of the filter on the signal.

- What happens to the filter as you change the amplitude and the frequency of the input sine wave signal? `Do not exceed 3 V RMS input voltage, or your signal may be clipped.` Explore the region around the cutoff frequency of the filter. When you are finished exploring, return the sine wave to 5 kHz.

- Bypass the Filter module and send the function generator output directly to "IN A" of the Multiplier module; also measure this on CH 1. Set the switch to "A$\times$A" mode. Switch CH 2 to measure the "Monitor" output of the Multiplier module. Save a screenshot and discuss the action of the Multiplier module on the signal. Calculate the expected output RMS voltage and compare to that shown on the scope.

- Switch CH 2 to measure the "OUT" of the Output module. Save a screenshot and discuss the action of the Output module on the signal. Does the signal agree with your expected output RMS voltage? Does changing the time-constant over its range impact the output here?

Using Analog Electronics to Measure Voltage Noise

Now that you have some experience with the individual behaviors of the Filter, Multiplier, and Output modules, you will use them in series to estimate the variance in the voltage noise fluctuations from the output of the low pass filter.

- Disconnect the function generator from the HLE box. Re-connect the LLE output to the Low-pass filter input using the same settings as before.

- Split the Filter output to CH 1 of the scope and to the Multiplier "IN A" jack. Measure the Output "OUT" on CH 2. Set the horizontal/time scale of the scope to 25 ms/div.

- What is the variance in the Johnson noise as measured by the HLE analog modules? Following Eq. 2, calculate $V_{J,\mathrm{rms}} = \sqrt{(10\ \mathrm{V})(V_{\mathrm{OUT}})}$ and compare with the scope's RMS reading on CH 1.

- Does the variance obtained through this series of analog circuits in the HLE agree with the measurement from the oscilloscope?

Filter Effects

The Gain function $G(f)$ of a filter is defined as:

$$G(f) = \frac{V_{\mathrm{out}}(f)}{V_{\mathrm{in}}(f)}, \tag{6}$$

where $V_{\mathrm{in}}(f)$ and $V_{\mathrm{out}}(f)$ represent the RMS voltages of the oscillating input and output signals at the frequency f.

Equivalent Noise Bandwidth

As shown in Eq. 1, the variance of the Johnson noise voltage fluctuations depends on Δf, which is the range of frequencies where noise is present. For a simple model of a filter, the cutoff is perfectly sharp, and is modeled as a step function. If a high-pass filter (cutoff f_{HP}) and a low pass filter (cutoff f_{LP}) are connected in series then the pass band has a nominal bandwidth $\Delta f = f_{\mathrm{LP}} - f_{\mathrm{HP}}$. But for the filter circuits used above, the cutoffs are not like a step function, and are smoothed out. In this case, we must take into account the actual gain function $G(f)$ for the filter in order to calculate a better estimate of Δf, known as the Equivalent Noise Bandwidth (ENBW), which is given by

$$\mathrm{ENBW} = \int_0^\infty G^2(f)\, df \ = \Delta f. \tag{7}$$

Numerical integration results for a variety of combinations of High-pass and Low-pass filters are provided in Table 1. Note that in some cases, the nominal bandwidth of $\Delta f = f_{\mathrm{LP}} - f_{\mathrm{HP}}$ gives a negative value ($f_{\mathrm{HP}} > f_{\mathrm{LP}}$), while the ENBW yields a positive value.

Mathematical Model of the HLE

Before we quantify Johnson Noise, we first need to create a mathematical model of how the apparatus manipulates the signal. The LLE output will be fed through two filters, gain amplification, signal-squaring and time-averaging circuits.

Filters, Gain, Multiplier, and Output modules

We have previously used all modules of the HLE save the Gain module, which is introduced here. The output voltage V_{LLE} will be sent through the High-pass filter, the Low-pass filter,

ENBW	$f_{LP} = 0.33$ kHz	1 kHz	3.3 kHz	10 kHz	33 kHz	100 kHz
$f_{HP} = 10$ Hz	355	1,100	3,654	11,096	36,643	111,061
30 Hz	333	1,077	3,632	11,074	36,620	111,039
100 Hz	258	1,000	3,554	10,996	36,543	110,961
300 Hz	105	784	3,332	10,774	36,321	110,739
1000 Hz	9	278	2,576	9,997	35,543	109,961
3000 Hz	0.4	28	1,051	7,839	33,324	107,740

Table 1: Theoretical Equivalent Noise Bandwidth (in Hz) for the combination High-pass and Low-pass filter. All possible combinations of the cut-off frequencies (f_{HP} and f_{LP}) are shown. Actual equipment has a 4% uncertainty in these tabulated ENBW values.

the Gain, the Multiplier, and the Output modules. This final Output signal will be read using the DMM.

Do the filters amplify ($G > 1$) the signal? Look at your data from last time. What is the gain of the filters in the pass band? If you have not done so already, fit a constant value to the pass band to measure the gain G_{Filter} of your combination filter.

The combined filter output signal is then $V_{\text{Filter}}(t) = G_{\text{Filter}} V_{\text{LLE}}(t)$.

The Gain module can amplify the signal by a minimum of $\times 10$ and a maximum of $\times 10{,}000$. We will call this additional gain G_2. The output signal of the Gain module is given by

$$V_{\text{Gain}}(t) = G_2 G_{\text{Filter}} G_1 V_J(t). \tag{8}$$

The action of the Multiplier and the Output modules together is to square the signal, divide it by 10 V, and time average it, outputting a roughly constant DC Voltage:

$$V_{\text{Output}}(t) = \left\langle \frac{(G_2 G_{\text{Filter}} G_1 V_J(t))^2}{10\ V} \right\rangle \tag{9}$$

In Eq. 9, the three constant gain factors come out of the time average. Therefore, the final signal measured by the DMM is:

$$V_{\text{DMM}} = V_{\text{Output}} = \frac{(G_2 G_{\text{Filter}} G_1)^2}{10\ V} \left\langle V_J^2(t) \right\rangle \tag{10}$$

Eq. 10 can be used to infer the Johnson noise variance $\left\langle V_J^2(t) \right\rangle$ from the DMM measurement.

Example calculation using the apparatus model

Using all relevant Equations developed heretofore, and the representative values:

- $V_{\text{DMM}} = 0.537$ V
- $R_f = 1$ kΩ

- $G_2 = 3000$
- $G_{\text{Filter}} = 1$
- $\Delta f = 10$ kHz
- $T = 300$ K

find the resistance R_{in} of the Johnson noise source resistor.

Quantitative measurement of Johnson noise

On the LLE, set $R_{\text{in}} = 10$ kΩ and $R_{\text{f}} = 1$ kΩ. What pre-amplifier gain G_1 does this yield?

Next, filter the signal through the combination High- and Low-pass filter. Set $f_{\text{HP}} = 0.1$ kHz and $f_{\text{LP}} = 100$ kHz. How big is our Δf in this case? Compare the nominal $\Delta f = f_{\text{LP}} - f_{\text{HP}}$ to the Equivalent Noise Bandwidth. You will use the ENBW in later analysis.

Collect a screenshot from the oscilloscope showing $V_{\text{LLE}}(t)$ on CH 1 and $V_{\text{Filter}}(t)$ on CH 2. Confirm that the Filter modules are behaving as expected.

G2=1000

Connect the Filter output to the Gain module. Adjust the gain G_2 such that the output of the Gain module is in the 3 V RMS range (any larger will saturate the Multiplier module and cause signal clipping). Split the output of the Gain module so that one signal goes to scope CH 2 and the other goes to the Multiplier module.

Collect a screenshot from the oscilloscope showing $V_{\text{LLE}}(t)$ on CH 1 and $V_{\text{Gain}}(t)$ on CH 2. Confirm that the Gain module is behaving as expected.

On the Output module, set the Meter Scale to 0-2 V for the greatest resolution on the meter readout. If the input voltage from the Gain module is in the 3 V RMS range, the analog meter should read between 0.6 V and 1.2 V. Measured: 0.92 V

- What impact does the time constant have on the DMM readout? Decide on a time constant to use. Seems more steady 3 seconds time constant
- What is your measured value of $\left\langle V_{\text{J}}^2(t) \right\rangle$? 0.26 nV = (0.92/10)(600x1000x3.44/11)^2
  G_filter = 1
- Is this on the same order of magnitude of what can be expected using the Nyquist Formula (Eq. 1)? Measure room temperature using the Fluke thermometer.
  295.1 K
- If you measured a noise signal greater than that expected, identify possible additional sources of noise in your circuits. R_N = 1000

Johnson Noise Dependence on Resistance

Correcting for internal noise

Now that you have seen how the electronics work and how to make measurements of the Johnson noise variance $\left\langle V_J^2(t) \right\rangle$, you will investigate the accuracy of those measurements. Two major questions arise when considering the accuracy:

1. Were systematic errors minimized?
2. Are there additional noise sources other than the source resistor R_{in}?

Regarding question (1), accurate noise measurements depend on the linear operation of the filters and amplifiers. These elements, in general, only have a specific range in which they can function linearly and we must make sure that our signal never leaves this range. In the previous section, we kept the input voltage to the Multiplier module under 3 V RMS in order to ensure ideal operation of the Multiplier. When no clipping or other non-ideal behavior occurs, the measurements are the most accurate.

Regarding question (2), we are measuring the total noise that the entire system has which includes all other resistors, resistive elements, and outside noise sources. These additional noise sources, which happen after the source resistor, can't be distinguished from the Johnson noise because their signals are just as random as the noise from the source resistor. But we can measure the additional noise sources and correct for them later.

If the total noise signal is modeled as the sum of contributions from Johnson noise $V_J(t)$ and all other additional noise sources $V_N(t)$, then the amplitude of the final signal after all the amplifiers is given by

$$V_{\text{Gain}}(t) = G\{V_J(t) + V_N(t)\}, \tag{11}$$

where $G = G_2 G_{\text{Filter}} G_1$. The mean-square (variance) of this output is then

$$\begin{aligned} \left\langle V_{\text{Gain}}^2(t) \right\rangle &= G^2 \left\langle \{V_J(t) + V_N(t)\}^2 \right\rangle \\ &= G^2 \left\{ \left\langle V_J^2(t) + 2V_J(t)V_N(t) + V_N^2(t) \right\rangle \right\} \end{aligned}$$

The cross term $\langle 2V_J(t)V_N(t)\rangle$ goes to zero because the two noise signals are uncorrelated which allows $\langle 2V_J(t)V_N(t)\rangle = 2\langle V_J(t)\rangle \langle V_N(t)\rangle$ and each has a time average of $V_J(t) = V_N(t) = 0$. The noise sources are uncorrelated because they arise from different circuit elements. The mean square output voltage can be restated as

$$\left\langle V_{\text{Gain}}^2(t) \right\rangle = G^2 \left\{ \left\langle V_J^2(t) \right\rangle + \left\langle V_N^2(t) \right\rangle \right\} \tag{12}$$

To find the internal noise $\left\langle V_N^2(t) \right\rangle$, it would make sense to set the source resistor to R_{in}=0 Ω so that we can see only noise from the rest of the circuit. The best way to get a value of $\left\langle V_N^2(t) \right\rangle$ is to make a plot of $\left\langle V_{\text{J}}^2 + V_{\text{N}}^2 \right\rangle$ versus resistance R_{in} and find the y-intercept. This y-intercept will be the value for $\left\langle V_N^2(t) \right\rangle$.

Procedure

To determine the internal noise, collect V_{DMM} with uncertainty as a function of R_{in}, at values 1 Ω, 10 Ω, 100 Ω, 1 kΩ, 10 kΩ, and 100 kΩ. You must set G_1=600 and vary G_2 in order to keep V_{DMM} as large as possible but still under ∼1.2 V. For each value of R_{in}, collect four data points having different G_2 values. The data that you collect should consist of R_{in}, G_2, and $\langle V_{\text{DMM}} \rangle$.

Analysis

- For each data point, use Eq. 10 to find the total noise $\left\langle V_{\text{J}}^2 + V_{\text{N}}^2 \right\rangle$.
- Average the four results for each value of R_{in}.
- Plot $\left\langle V_J^2 + V_N^2 \right\rangle$ versus R_{in} and find the y-intercept.
- What is your value for $\left\langle V_N^2(t) \right\rangle$?
- For each R_{in}, calculate the Johnson noise $\left\langle V_J^2(t) \right\rangle$ with the background noise $\left\langle V_N^2(t) \right\rangle$ subtracted.
- In the event that your data has one or more outliers that skew the fit, you may find that one or more of your background-subtracted data points is now negative. If this happens, you must re-visit the measurement of the offending point(s). If any outliers persists, craft a fit that excludes the outlier, and see if this results in physical (positive) background-subtracted final data points. Compare fit χ^2 values and choose the best-quality fit that retains the most possible data points.

Use the data that you took in the previous part to examine the Johnson noise variance $\left\langle V_J^2(t) \right\rangle$ versus resistance R_{in}. The data spans many orders of magnitude in $\left\langle V_J^2(t) \right\rangle$ and R_{in}, so plot this on a log-log plot. Look back to Eq. 1 to see that Nyquist's formula predicts a first-order power-law dependence on the resistance R_{in}.

- Plot $\ln \left\langle V_J^2(t) \right\rangle$ versus $\ln R_{\text{in}}$.
- Using a power law fit, confirm that the Johnson noise measurements have a linear dependence on R.
- Using the extracted intercept and its uncertainty from this fit, calculate the value of k_B with uncertainty and compare to the known value.