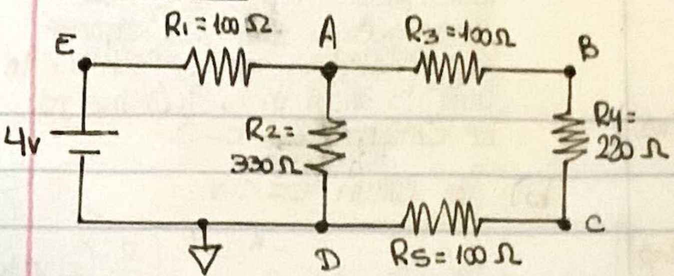
HOMEWORK 2:

QUESTION 01:



NODE A:

$i_1 \rightarrow \bullet \rightarrow i_2$, $\downarrow i_3$

$$i_1 = i_2 + i_3$$

$$\frac{V_E - V_A}{100} = \frac{V_A - V_B}{100} + \frac{V_A - V_D}{330}$$

$$V_E - V_A = V_A - V_B + \frac{10}{33}(V_A - \cancelto{0}{V_D})$$

$$4 - V_A = V_A - V_B + \frac{10}{33} V_A$$

$$\boxed{\frac{76}{33} V_A - V_B = 4}$$

NODE B:

$i_1 \rightarrow \bullet B$, $\downarrow i_2$

$$i_1 = i_2$$

$$\frac{V_A - V_B}{100} = \frac{V_B - V_C}{220}$$

$$V_A - V_B = \frac{5}{11}(V_B - V_C)$$

$$V_A - V_B = \frac{5}{11} V_B - \frac{5}{11} V_C$$

$$\boxed{V_A - \frac{16}{11} V_B + \frac{5}{11} V_C = 0}$$

NODE C:

$\downarrow i_1$, $i_2 \leftarrow \bullet C$

$$i_1 = i_2$$

$$\frac{V_B - V_C}{220} = \frac{V_C - V_D}{100}$$

$$V_B - V_C = \frac{11}{5}(V_C - \cancelto{0}{V_D})$$

$$\boxed{V_B - \frac{16}{5} V_C = 0}$$

Summarizing:

$$\begin{pmatrix} 76/33 & -1 & 0 \\ 1 & -16/11 & 5/11 \\ 0 & 1 & -16/5 \end{pmatrix} \begin{pmatrix} V_A \\ V_B \\ V_C \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

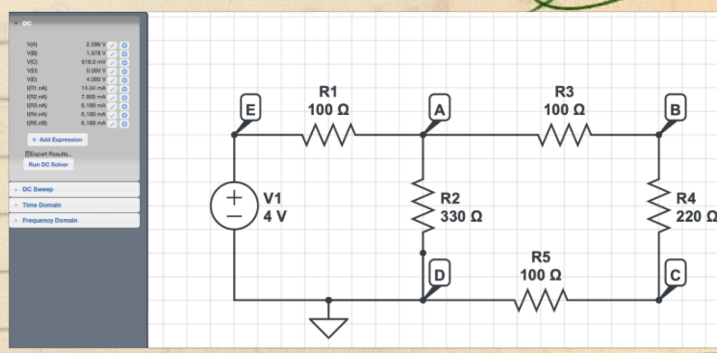
A.

$$\rightarrow \begin{pmatrix} V_A \\ V_B \\ V_C \end{pmatrix} = \begin{pmatrix} 231/89 \\ 176/89 \\ 55/89 \end{pmatrix} = \begin{pmatrix} 2.6 \\ 1.98 \\ 0.62 \end{pmatrix} V$$

B.

$V_E = 4V$
$V_D = 0V$

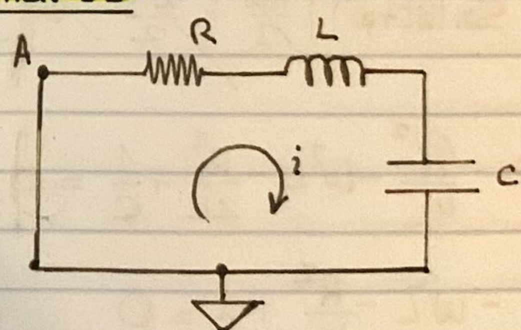
- $I_{R_1} = \frac{V_E - V_A}{100} = \frac{4 - 2.6}{100} = 14\,mA$
- $I_{R_2} = \frac{V_A - V_D}{330} = \frac{2.6}{330} = 7.88\,mA$
- $I_{R_3} = I_{R_4} = I_{R_5} = \frac{V_A - V_B}{100} = \frac{2.6 - 1.98}{100} = 6.2\,mA$



QUESTION 02:

A.

At $t=0$:



LOOP:

$$\cancel{V_A} - Ri - L\frac{di}{dt} - \frac{q}{C} = \cancel{V_A}$$

$$Ri + L\frac{di}{dt} + \frac{q}{C} = 0 \quad \text{but } i = \frac{dq}{dt}$$

$$\boxed{L\frac{dq^2}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0}$$

- $L\frac{dq^2}{dt^2} \rightarrow$ Voltage drop across the inductor due to the self-induced emf.
- $R\frac{dq}{dt} \rightarrow$ Voltage drop due to the resistance R.
- $q/C \rightarrow$ Voltage drop due to the accumulation of charge in the capacitor.

B. $q(t) = Q_0 e^{-\frac{R}{2L}t}\cos(\omega t+\phi)$

$$\frac{dq(t)}{dt} = -\frac{Q_0 R}{2L} e^{-\frac{R}{2L}t}\cos(\omega t+\phi) - Q_0\omega e^{-\frac{R}{2L}t}\sin(\omega t+\phi)$$

$$= -Q_0 e^{-\frac{R}{2L}t}\left[\frac{R}{2L}\cos(\omega t+\phi) + \omega\sin(\omega t+\phi)\right]$$

$$\frac{d^2q(t)}{dt^2} = \frac{Q_0 R}{2L} e^{-\frac{R}{2L}t}\left[\frac{R}{2L}\cos(\omega t+\phi) + \omega\sin(\omega t+\phi)\right] - Q_0 e^{-\frac{R}{2L}t}\left[-\frac{\omega R}{2L}\sin(\omega t+\phi) + \omega^2\cos(\omega t+\phi)\right]$$

$$\rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$= \frac{\cancel{Q_0}R}{2}\cancel{e^{-\frac{R}{2L}t}}\left[\frac{R}{2L}\cos(\omega t+\phi) + \omega\sin(\omega t+\phi)\right] - \cancel{Q_0}L\cancel{e^{-\frac{R}{2L}t}}\left[-\frac{\omega R}{2L}\sin(\omega t+\phi) + \omega^2\cos(\omega t+\phi)\right]$$

$$-\frac{\cancel{Q_0}R^2}{2L}\cancel{e^{-\frac{R}{2L}t}}\cos(\omega t+\phi) - \cancel{Q_0}\omega R\cancel{e^{-\frac{R}{2L}t}}\sin(\omega t+\phi) + \frac{\cancel{Q_0}}{C}\cancel{e^{-\frac{R}{2L}t}}\cos(\omega t+\phi) = 0$$

$$= \cos(\omega t+\phi)\left[\frac{R^2}{4L} - \omega^2 L - \frac{R^2}{2L} + \frac{1}{C}\right] + \sin(\omega t+\phi)\left[\cancel{\frac{R\omega}{2}} + \cancel{\frac{\omega R}{2}} - \cancel{\omega R}\right] = 0$$

$$\rightarrow \frac{R^2}{4L} - \omega^2 L - \frac{R^2}{2L} + \frac{1}{C} = 0$$

$$-\omega^2 L - \frac{R^2}{4L} + \frac{1}{C} = 0$$

$$\omega^2 L = \frac{1}{C} - \frac{R^2}{4L}$$

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

$$\boxed{\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}$$

$q(t) = Q_0 e^{-R/2L\, t}\cos(\omega t+\phi)$ is solution if and only if $\omega = \sqrt{1/LC - (R/2L)^2}$

C.

a) Because at $t=0 \rightarrow q(0) \neq 0$, which means that $q/C \neq 0$. As a consequence, there is charge accumulated on the capacitor. In order to allow this, $q(t)$ has to be continuous at $t=0$.

b) the current equation:

$$I(t) = -Q_0 e^{-R/2L\, t}\left[\frac{R}{2L}\cos(\omega t+\phi) + \omega\sin(\omega t+\phi)\right]$$

We can see that the phase constant ϕ allows the system to have an initial charge $q(0)$ and current $I(0)$ as expected in an underdamped circuit. It has to be continuous because then $\frac{dq}{dt} = I$ and $\frac{d^2q}{dt^2} = \frac{dI}{dt}$ can be also continuous and the equation still working.

c) $q(0) = Q_0\cos\phi$

d) $I(0) = -Q_0\left[\frac{R}{2L}\cos\phi + \omega\sin\phi\right]$

Question 03