Advanced topics on LOGIC and SEMANTICS

Conjunctive Normal Form

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- Conjunctive normal form (CNF) formulas:

$$(A \vee B \vee \neg C) \wedge (B \vee D) \wedge (\neg A) \wedge (B \vee C)$$

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  - A, B, and $\neg$ C are literals, each of which is a variable or the negation of a variable.
  - Each clause is a requirement that must be satisfied and can be satisfied in multiple ways
  - Every sentence in propositional logic can be written in CNF

Converting to CNF

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3. Distribute **or** over **and**

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

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4. Every sentence can be converted to CNF, but it may grow exponentially in size

CNF Conversion Example

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2. Drive in negations

$$(\neg A \wedge \neg B) \vee (\neg C \vee D)$$

3. Distribute

$$(\neg A \vee \neg C \vee D) \wedge (\neg B \vee \neg C \vee D)$$

Propositional Resolution

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- Resolution rule:

$$\frac{\alpha \vee \beta \qquad \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

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  - Convert all sentences to CNF
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  - Apply resolution rule until either
    - Derive false (a contradiction)
    - Can't apply any more

Propositional Resolution

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- Resolution refutation:
  - Convert all sentences to CNF
  - Negate the desired conclusion (converted to CNF)
  - Apply resolution rule until either
    - Derive false (a contradiction)
    - Can't apply any more
- Resolution refutation is sound and complete
  - If we derive a contradiction, then the conclusion follows from the axioms
  - If we can't apply any more, then the conclusion cannot be proved from the axioms.

Propositional Resolution Example

Prove R

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation

Propositional Resolution Example

Prove R

1	P v Q
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	P v Q	Given

Propositional Resolution Example

Prove R

1	P v Q
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	P v Q	Given
2	$\neg$ P v R	Given

Propositional Resolution Example

Prove R

1	P v Q
2	$P \rightarrow R$
3	$Q \rightarrow R$

Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given

Propositional Resolution Example

Prove R

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2	$P \rightarrow R$
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Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
4	¬ R	Negated conclusion

Propositional Resolution Example

Prove R

1	P v Q
2	P → R
3	Q → R

Step	Formula	Derivation
1	P v Q	Given
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Propositional Resolution Example

Prove R

1	P v Q
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Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
4	¬ R	Negated conclusion
5	Q v R	1,2

Propositional Resolution Example

Prove R

1	P v Q
2	P → R
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Step	Formula	Derivation
1	P v Q	Given
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3	¬ Q v R	Given
4	¬ R	Negated conclusion
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6	¬ P	2,4

Propositional Resolution Example

Prove R

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Step	Formula	Derivation
1	P v Q	Given
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3	¬ Q v R	Given
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5	Q v R	1,2
6	¬ P	2,4
7	¬ Q	3,4

Propositional Resolution Example

Prove R

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Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
4	¬ R	Negated conclusion
5	Q v R	1,2
6	¬ P	2,4
7	¬ Q	3,4
8	R	5,7

Propositional Resolution Example

Prove R

1	P v Q
2	P → R
3	Q → R

Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
4	¬ R	Negated conclusion
5	Q v R	1,2
6	¬ P	2,4
7	¬ Q	3,4
8	R	5,7
9	•	4,8

Propositional Resolution Example

Prove R

1	P v Q
2	P → R
3	Q → R

false v R

¬ R v false

false v false

Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
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5	Q v R	1,2
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Propositional Resolution Example

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1	P v Q	Given
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The Power of False

Prove Z

1	P
2	¬ P

Step	Formula	Derivation

The Power of False

Prove Z

1	P
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Step	Formula	Derivation
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The Power of False

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The Power of False

Prove Z

1	P
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1	P	Given
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4	•	1,2

Note that $(P \wedge \neg P) \rightarrow Z$ is valid

The Power of False

Prove Z

1	P
2	¬ P

Step	Formula	Derivation
1	P	Given
2	¬ P	Given
3	¬ Z	Negated conclusion
4	•	1,2

Note that $(P \wedge \neg P) \rightarrow Z$ is valid

Any conclusion follows from a contradiction – and so strict logic systems are very brittle.

Example Problem

Convert to CNF

Prove R

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

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Prove R

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Convert to CNF

- $\neg(\neg P \vee Q) \vee Q$
- $(P \wedge \neg Q) \vee Q$
- $(P \vee Q) \wedge (\neg Q \vee Q)$
- $(P \vee Q)$

Example Problem

Prove R

1	$(P \rightarrow Q) \rightarrow Q$
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Convert to CNF

- $\neg(\neg P \vee Q) \vee Q$
- $(P \wedge \neg Q) \vee Q$
- $(P \vee Q) \wedge (\neg Q \vee Q)$
- $(P \vee Q)$

- $\neg(\neg P \vee P) \vee R$
- $(P \wedge \neg P) \vee R$
- $(P \vee R) \wedge (\neg P \vee R)$

Example Problem

Prove R

1	$(P \rightarrow Q) \rightarrow Q$
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Convert to CNF

- $\neg(\neg P \vee Q) \vee Q$
- $(P \wedge \neg Q) \vee Q$
- $(P \vee Q) \wedge (\neg Q \vee Q)$
- $(P \vee Q)$

- $\neg(\neg P \vee P) \vee R$
- $(P \wedge \neg P) \vee R$
- $(P \vee R) \wedge (\neg P \vee R)$

- $\neg(\neg R \vee S) \vee \neg(\neg S \vee Q)$
- $(R \wedge \neg S) \vee (S \wedge \neg Q)$
- $(R \vee S) \wedge (\neg S \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee \neg Q)$
- $(R \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee \neg Q)$

Resolution Proof Example

Prove R

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg (S \rightarrow Q)$

1	$P \vee Q$	
2	$P \vee R$	
3	$\neg P \vee R$	
4	$R \vee S$	
5	$R \vee \neg Q$	
6	$\neg S \vee \neg Q$	
7	$\neg R$	Neg

Resolution Proof Example

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1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg (S \rightarrow Q)$

1	$P \vee Q$	
2	$P \vee R$	
3	$\neg P \vee R$	
4	$R \vee S$	
5	$R \vee \neg Q$	
6	$\neg S \vee \neg Q$	
7	$\neg R$	Neg
8	S	4,7
9	$\neg Q$	6,8
10	P	1,9
11	R	3,10
12	•	7,11

Proof Strategies

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- Unit preference: prefer a resolution step involving a unit clause (clause with one literal).
  - Produces a shorter clause – which is good since we are trying to produce a zero-length clause, that is, a contradiction.

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- Unit preference: prefer a resolution step involving a unit clause (clause with one literal).
  - Produces a shorter clause – which is good since we are trying to produce a zero-length clause, that is, a contradiction.
- Set of support: Choose a resolution involving the negated goal or any clause derived from the negated goal.
  - We're trying to produce a contradiction that follows from the negated goal, so these are "relevant" clauses.
  - If a contradiction exists, one can find one using the set-of-support strategy.

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First-Order Resolution -- Conversion to Clausal Form

First-Order Resolution

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$$\frac{\forall x.\ P(x) \rightarrow Q(x) \qquad P(A)}{Q(A)}$$

uppercase letters: constants

lowercase letters: variables

First-Order Resolution

$$\frac{\forall x.\ P(x) \rightarrow Q(x) \qquad P(A)}{Q(A)}$$

Syllogism:

All men are mortal

Socrates is a man

Socrates is mortal

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First-Order Resolution

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uppercase letters: constants

lowercase letters: variables

$$\frac{\forall x.\ \neg P(x) \vee Q(x) \qquad P(A)}{Q(A)}$$

Equivalent by definition of implication

First-Order Resolution

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$$\frac{\forall x.\ \neg P(x) \vee Q(x) \qquad P(A)}{Q(A)}$$

Equivalent by definition of implication

$$\frac{\neg P(A) \vee Q(A) \qquad P(A)}{Q(A)}$$

Substitute A for x, still true

then

Propositional resolution

First-Order Resolution

$$\frac{\forall x.\ P(x) \rightarrow Q(x) \qquad P(A)}{Q(A)}$$

Syllogism:

All men are mortal

Socrates is a man

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$$\frac{\forall x.\ \neg P(x) \vee Q(x) \qquad P(A)}{Q(A)}$$

Equivalent by definition of implication

Two new things:

- converting FOL to clausal form
- resolution with variable substitution

$$\frac{\neg P(A) \vee Q(A) \qquad P(A)}{Q(A)}$$

Substitute A for x, still true

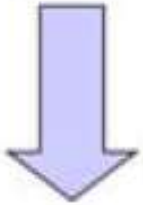
then

Propositional resolution

Clausal Form

- like CNF in outer structure
- no quantifiers

$$\forall x.\ \exists y.\ P(x) \rightarrow R(x, y)$$



$$\neg P(x) \vee R(x, F(x))$$

Converting to Clausal Form

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1. Eliminate arrows

$$\alpha \leftrightarrow \beta \Rightarrow (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$$

$$\alpha \rightarrow \beta \Rightarrow \neg\alpha \vee \beta$$

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2. Drive in negation

$$\neg(\alpha \vee \beta) \Rightarrow \neg\alpha \wedge \neg\beta$$

$$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$$

$$\neg\neg\alpha \Rightarrow \alpha$$

$$\neg\forall x.\ \alpha \Rightarrow \exists x.\ \neg\alpha$$

$$\neg\exists x.\ \alpha \Rightarrow \forall x.\ \neg\alpha$$

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$$\neg(\alpha \vee \beta) \Rightarrow \neg\alpha \wedge \neg\beta$$

$$\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$$

$$\neg\neg\alpha \Rightarrow \alpha$$

$$\neg\forall x.\ \alpha \Rightarrow \exists x.\ \neg\alpha$$

$$\neg\exists x.\ \alpha \Rightarrow \forall x.\ \neg\alpha$$

3. Rename variables apart

$$\forall x.\ \exists y.\ (\neg P(x) \vee \exists x.\ Q(x, y)) \Rightarrow$$

$$\forall x_1.\ \exists y_2.\ (\neg P(x_1) \vee \exists x_3.\ Q(x_3, y_2))$$

Skolemization

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   - substitute new name for each existential var

$$\exists x.\ P(x) \Rightarrow P(\text{Fred})$$

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$$\exists x.\ P(x) \Rightarrow P(\text{Fred})$$

$$\exists x, y.\ R(x, y) \Rightarrow R(\text{Thing1}, \text{Thing2})$$

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$$\exists x, y.\ R(x, y) \Rightarrow R(\text{Thing1}, \text{Thing2})$$

$$\exists x.\ P(x) \wedge Q(x) \Rightarrow P(\text{Fleep}) \wedge Q(\text{Fleep})$$

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$$\exists x.\ P(x) \wedge \exists x.\ Q(x) \Rightarrow P(\text{Frog}) \wedge Q(\text{Grog})$$

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$$\exists x.\, P(x) \Rightarrow P(\text{Fred})$$

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$$\exists x.\, P(x) \wedge \exists x.\, Q(x) \Rightarrow P(\text{Frog}) \wedge Q(\text{Grog})$$

$$\exists y.\, \forall x.\, \text{Loves}(x, y) \Rightarrow \forall x.\, \text{Loves}(x, \text{Englebert})$$

Skolemization

4. Skolemize
   - substitute new name for each existential var

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$$\exists x, y.\, R(x, y) \Rightarrow R(\text{Thing1}, \text{Thing2})$$

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   - substitute new function of all universal vars in outer scopes

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   - substitute new name for each existential var

$$\exists x.\, P(x) \Rightarrow P(\text{Fred})$$

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   - substitute new function of all universal vars in outer scopes

$$\forall x.\, \exists y.\, \text{Loves}(x, y) \Rightarrow \forall x.\, \text{Loves}(x, \text{Beloved}(x))$$

Skolemization

4. Skolemize
   - substitute new name for each existential var

$$\exists x.\, P(x) \Rightarrow P(\text{Fred})$$
$$\exists x, y.\, R(x, y) \Rightarrow R(\text{Thing1}, \text{Thing2})$$
$$\exists x.\, P(x) \wedge Q(x) \Rightarrow P(\text{Fleep}) \wedge Q(\text{Fleep})$$
$$\exists x.\, P(x) \wedge \exists x.\, Q(x) \Rightarrow P(\text{Frog}) \wedge Q(\text{Grog})$$
$$\exists y.\, \forall x.\, \text{Loves}(x, y) \Rightarrow \forall x.\, \text{Loves}(x, \text{Englebert})$$

   - substitute new function of all universal vars in outer scopes

$$\forall x.\, \exists y.\, \text{Loves}(x, y) \Rightarrow \forall x.\, \text{Loves}(x, \text{Beloved}(x))$$
$$\forall x.\, \exists y.\, \forall z.\, \exists w.\, P(x, y, z) \wedge R(y, z, w) \Rightarrow$$
$$P(x, F(x), z) \wedge R(F(x), z, G(x, z))$$

Convert to Clausal Form: Last Steps

5. Drop universal quantifiers

$$\forall x.\ \text{Loves}(x, \text{Beloved}(x)) \Rightarrow \text{Loves}(x, \text{Beloved}(x))$$

Convert to Clausal Form: Last Steps

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$$\forall x.\ \text{Loves}(x, \text{Beloved}(x)) \Rightarrow \text{Loves}(x, \text{Beloved}(x))$$

6. Distribute or over and; return clauses

$$P(z) \vee (Q(z,w) \wedge R(w,z)) \Rightarrow$$
$$\{\{P(z), Q(z,w)\}, \{P(z), R(w,z)\}\}$$

Convert to Clausal Form: Last Steps

5. Drop universal quantifiers

$$\forall x.\ \text{Loves}(x, \text{Beloved}(x)) \Rightarrow \text{Loves}(x, \text{Beloved}(x))$$

6. Distribute or over and; return clauses

$$P(z) \vee (Q(z,w) \wedge R(w,z)) \Rightarrow \{\{P(z), Q(z,w)\}, \{P(z), R(w,z)\}\}$$

7. Rename the variables in each clause

$$\{\{P(z), Q(z,w)\}, \{P(z), R(w,z)\}\} \Rightarrow \{\{P(z_1), Q(z_1,w_1)\}, \{P(z_2), R(w_2,z_2)\}\}$$

Example: Converting to clausal form

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a. John owns a dog
$\exists x.\ D(x) \wedge O(J,x)$

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D(Fido) $\wedge$ O(J, Fido)

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a. John owns a dog
$\exists$ x. D(x) $\wedge$ O(J,x)
D(Fido) $\wedge$ O(J, Fido)

b. Anyone who owns a dog is a lover-of-animals
$\forall$ x. ($\exists$ y. D(y) $\wedge$ O(x,y)) $\rightarrow$ L(x)

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$\exists$ x. D(x) $\wedge$ O(J,x)
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$\forall$ x. ($\exists$ y. D(y) $\wedge$ O(x,y)) $\rightarrow$ L(x)
$\forall$ x. ($\neg\exists$ y. (D(y) $\wedge$ O(x,y)) v L(x)

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$\exists$ x. D(x) $\wedge$ O(J,x)
D(Fido) $\wedge$ O(J, Fido)

b. Anyone who owns a dog is a lover-of-animals
$\forall$ x. ($\exists$ y. D(y) $\wedge$ O(x,y)) $\rightarrow$ L(x)
$\forall$ x. ($\neg\exists$ y. (D(y) $\wedge$ O(x,y)) v L(x)
$\forall$ x. $\forall$ y. $\neg$(D(y) $\wedge$ O(x,y)) v L(x)
$\forall$ x. $\forall$ y. $\neg$ D(y) v $\neg$ O(x,y) v L(x)

Example: Converting to clausal form

a. John owns a dog
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c. Lovers-of-animals do not kill animals
$\forall$ x. L(x) $\rightarrow$ ($\forall$ y. A(y) $\rightarrow$ $\neg$ K(x,y))

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More converting to clausal form

d. Either Jack killed Tuna or curiosity killed Tuna
K(J,T) v K(C,T)

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C(T)

More converting to clausal form

d. Either Jack killed Tuna or curiosity killed Tuna
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e. Tuna is a cat
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f. All cats are animals
¬ C(x) v A(x)

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Thank You

First-Order Resolution -- Substitution and Unification

First-Order Resolution

$$\frac{\forall x.\ P(x) \rightarrow Q(x) \qquad P(A)}{Q(A)}$$

Syllogism:

All men are mortal

Socrates is a man

Socrates is mortal

uppercase letters: constants

lowercase letters: variables

$$\frac{\forall x.\ \neg P(x) \vee Q(x) \qquad P(A)}{Q(A)}$$

Equivalent by definition of implication

$$\frac{\neg P(A) \vee Q(A) \qquad P(A)}{Q(A)}$$

Substitute A for x, still true

then

Propositional resolution

The key is finding the correct substitutions for the variables.

Substitutions

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P(x, f(y), B) : an atomic sentence

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Substitution instances	Substitution $\{v_1/t_1, \ldots, v_n/t_n\}$	Comment

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P(C, f(A), B)	{x/C, y/A}	Ground instance

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P(x, f(A), B)	{y/A}	
P(g(z), f(A), B)	{x/g(z), y/A}	
P(C, f(A), B)	{x/C, y/A}	Ground instance

Applying a substitution:

P(x, f(y), B) {y/A} = P(x,f(A),B)

P(x, f(y), B) {y/A, x/y} = P(A, f(A), B)

Unification

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- Expressions ω_1 and ω_2 are unifiable iff there exists a substitution s such that ω_1 s = ω_2 s

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{y/x}	x	x
{x/y}	y	y
{x/f(f(A)), y/f(f(A))}	f(f(A))	f(f(A))

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{x/y}	y	y
{x/f(f(A)), y/f(f(A))}	f(f(A))	f(f(A))
{x/A, y/A}	A	A

Most General Unifier

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g is a most general unifier of ω_1 and ω_2 iff for all unifiers s, there exists s′ such that ω_1 s = (ω_1 g) s′ and ω_2 s = (ω_2 g) s′

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ω_1	ω_2	MGU
P(x)	P(A)	{x/A}

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P(f(x), y, g(x))	P(f(x), x, g(x))	{y/x} or {x/y}

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P(f(x), y, g(y))	P(f(x), z, g(x))	{y/x, z/x}

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P(f(x), y, g(y))	P(f(x), z, g(x))	{y/x, z/x}
P(x, B, B)	P(A, y, z)	{x/A, y/B, z/B}

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ω_1	ω_2	MGU
P(x)	P(A)	{x/A}
P(f(x), y, g(x))	P(f(x), x, g(x))	{y/x} or {x/y}
P(f(x), y, g(y))	P(f(x), z, g(x))	{y/x, z/x}
P(x, B, B)	P(A, y, z)	{x/A, y/B, z/B}
P(g(f(v)), g(u))	P(x, x)	{x/g(f(v)), u/f(v)}

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ω_1	ω_2	MGU
P(x)	P(A)	{x/A}
P(f(x), y, g(x))	P(f(x), x, g(x))	{y/x} or {x/y}
P(f(x), y, g(y))	P(f(x), z, g(x))	{y/x, z/x}
P(x, B, B)	P(A, y, z)	{x/A, y/B, z/B}
P(g(f(v)), g(u))	P(x, x)	{x/g(f(v)), u/f(v)}
P(x, f(x))	P(x, x)	No MGU!

Unification Algorithm

```
unify(Expr x, Expr y, Subst s){
```

Unification Algorithm

```
unify(Expr x, Expr y, Subst s){
  if s = fail, return fail
```

Unification Algorithm

```
unify(Expr x, Expr y, Subst s){
  if s = fail, return fail
  else if x = y, return s
```

Unification Algorithm

```
unify(Expr x, Expr y, Subst s){
  if s = fail, return fail
  else if x = y, return s
  else if x is a variable, return unify-var(x, y, s)
  else if y is a variable, return unify-var(y, x, s)
```

Unification Algorithm

```
unify(Expr x, Expr y, Subst s){
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  else if x = y, return s
  else if x is a variable, return unify-var(x, y, s)
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  else if x is a predicate or function application,
```

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  else if y is a variable, return unify-var(y, x, s)
  else if x is a predicate or function application,
       if y has the same operator,
            return unify(args(x), args(y), s)
```

Unification Algorithm

```
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  if s = fail, return fail
  else if x = y, return s
  else if x is a variable, return unify-var(x, y, s)
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  else if x is a predicate or function application,
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            return unify(args(x), args(y), s)
       else  return fail
```

Unification Algorithm

```
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  else if y is a variable, return unify-var(y, x, s)
  else if x is a predicate or function application,
       if y has the same operator,
            return unify(args(x), args(y), s)
       else  return fail
  else                          ; x and y have to be lists
       return unify(rest(x), rest(y),
                    unify(first(x), first(y), s))
}
```

Unify-var subroutine

Substitute in for var and x as long as possible, then add new binding

```
unify-var(Variable var, Expr x, Subst s){
```

Unify-var subroutine

Substitute in for var and x as long as possible, then add new binding

```
unify-var(Variable var, Expr x, Subst s){
  if var is bound to val in s,
        return unify(val, x, s)
```

Unify-var subroutine

Substitute in for var and x as long as possible, then add new binding

```
unify-var(Variable var, Expr x, Subst s){
  if var is bound to val in s,
        return unify(val, x, s)
  else if x is bound to val in s,
        return unify-var(var, val, s)
```

Unify-var subroutine

Substitute in for var and x as long as possible, then add new binding

```
unify-var(Variable var, Expr x, Subst s){
  if var is bound to val in s,
       return unify(val, x, s)
  else if x is bound to val in s,
       return unify-var(var, val, s)
  else if var occurs anywhere in (x s), return fail
```

Unify-var subroutine

Substitute in for var and x as long as possible, then add new binding

```
unify-var(Variable var, Expr x, Subst s){
  if var is bound to val in s,
        return unify(val, x, s)
  else if x is bound to val in s,
        return unify-var(var, val, s)
  else if var occurs anywhere in (x s), return fail
  else return add({var/x}, s)
}
```

Some Examples

ω_1	ω_2	MGU
A(B, C)	A(x, y)	{x/B, y/C}
A(x, f(D,x))	A(E, f(D,y))	{x/E, y/E}
A(x, y)	A(f(C,y), z)	{x/f(C,y),y/z}
P(A, x, f(g(y)))	P(y, f(z), f(z))	{y/A,x/f(z),z/g(y)}
P(x, g(f(A)), f(x))	P(f(y), z, y)	none
P(x, f(y))	P(z, g(w))	none

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Thank You

First-Order Resolution -- Resolution with Variables

Resolution with Variables

Resolution with Variables

$$\frac{\alpha \vee \varphi \quad \neg\varphi \vee \beta}{(\alpha \vee \beta)\theta} \quad \text{MGU}(\varphi, \psi) = \theta$$

Resolution with Variables

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$$\frac{P(x) \vee Q(x, y) \quad \neg P(A) \vee R(B, z)}{}$$

$\theta = \{x/A\}$

Resolution with Variables

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$$\frac{P(x) \vee Q(x, y) \quad \neg P(A) \vee R(B, z)}{(Q(x, y) \vee R(B, z))\theta}$$

$\theta = \{x/A\}$

Resolution with Variables

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$$\frac{P(x) \vee Q(x, y) \quad \neg P(A) \vee R(B, z)}{(Q(x, y) \vee R(B, z))\theta}$$

$$Q(A, y) \vee R(B, z)$$

$\theta = \{x/A\}$

Resolution with Variables

$$\frac{\alpha \vee \varphi \quad \neg\varphi \vee \beta}{(\alpha \vee \beta)\theta} \quad \text{MGU}(\varphi, \psi) = \theta$$

$$\frac{P(x) \vee Q(x, y) \quad \neg P(A) \vee R(B, x)}{}$$

$$\frac{P(x) \vee Q(x, y) \quad \neg P(A) \vee R(B, z)}{(Q(x, y) \vee R(B, z))\theta}$$

$$Q(A, y) \vee R(B, z)$$

$\theta = \{x/A\}$

Resolution with Variables

$$\frac{\alpha \vee \varphi \quad \text{MGU}(\varphi, \psi) = \theta \qquad \neg\varphi \vee \beta}{(\alpha \vee \beta)\theta}$$

$$\frac{\forall x, y.\ \ P(x) \vee Q(x, y) \qquad \forall x.\ \ \neg P(A) \vee R(B, x)}{}$$

$$\frac{\forall x, y.\ \ P(x) \vee Q(x, y) \qquad \forall z.\ \ \neg P(A) \vee R(B, z)}{(Q(x, y) \vee R(B, z))\theta}$$

$$Q(A, y) \vee R(B, z)$$

$\theta = \{x/A\}$

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$$\frac{\forall x, y.\ \ P(x) \vee Q(x, y) \qquad \forall z.\ \ \neg P(A) \vee R(B, z)}{(Q(x, y) \vee R(B, z))\theta}$$

$$Q(A, y) \vee R(B, z)$$

$\theta = \{x/A\}$

$$\frac{P(x_1) \vee Q(x_1, y_1) \qquad \neg P(A) \vee R(B, x_2)}{(Q(x_1, y_1) \vee R(B, x_2))\theta}$$

$$Q(A, y_1) \vee R(B, x_2)$$

$\theta = \{x_1/A\}$

Curiosity Killed the Cat

Curiosity Killed the Cat

1	D(Fido)	a
2	O(J,Fido)	a
3	¬ D(y) v ¬ O(x,y) v L(x)	b
4	¬ L(x) v ¬ A(y) v ¬ K(x,y)	c
5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f

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2	O(J,Fido)	a
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4	¬ L(x) v ¬ A(y) v ¬ K(x,y)	c
5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f
8	¬ K(C,T)	Neg

Curiosity Killed the Cat

1	D(Fido)	a
2	O(J,Fido)	a
3	¬ D(y) v ¬ O(x,y) v L(x)	b
4	¬ L(x) v ¬ A(y) v ¬ K(x,y)	c
5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f
8	¬ K(C,T)	Neg
9	K(J,T)	5,8

Curiosity Killed the Cat

1	D(Fido)	a
2	O(J,Fido)	a
3	¬ D(y) v ¬ O(x,y) v L(x)	b
4	¬ L(x) v ¬ A(y) v ¬ K(x,y)	c
5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f
8	¬ K(C,T)	Neg
9	K(J,T)	5,8
10	A(T)	6,7 {x/T}

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5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f
8	¬ K(C,T)	Neg
9	K(J,T)	5,8
10	A(T)	6,7 {x/T}
11	¬ L(J) v ¬ A(T)	4,9 {x/J, y/T}

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1	D(Fido)	a
2	O(J,Fido)	a
3	¬ D(y) v ¬ O(x,y) v L(x)	b
4	¬ L(x) v ¬ A(y) v ¬ K(x,y)	c
5	K(J,T) v K(C,T)	d
6	C(T)	e
7	¬ C(x) v A(x)	f
8	¬ K(C,T)	Neg
9	K(J,T)	5,8
10	A(T)	6,7 {x/T}
11	¬ L(J) v ¬ A(T)	4,9 {x/J, y/T}
12	¬ L(J)	10,11

Curiosity Killed the Cat

1	D(Fido)	a
2	O(J,Fido)	a
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13	¬ D(y) v ¬ O(J,y)	3,12 {x/J}

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13	¬ D(y) v ¬ O(J,y)	3,12 {x/J}
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Proving validity

- How do we use resolution refutation to prove something is valid?

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- Normally, we prove a sentence is entailed by the set of axioms
- Valid sentences are entailed by the empty set of sentences
- To prove validity by refutation, negate the sentence and try to derive contradiction.

Proving validity: example

- Syllogism

$$\left(\forall x.\, P(x) \rightarrow Q(x)\right) \wedge P(A) \rightarrow Q(A)$$

Proving validity: example

- Syllogism

$$\left(\forall x.\, P(x) \rightarrow Q(x)\right) \wedge P(A) \rightarrow Q(A)$$

- Negate and convert to clausal form

$$\neg\left(\left(\forall x.\, P(x) \rightarrow Q(x)\right) \wedge P(A) \rightarrow Q(A)\right)$$

$$\neg\left(\neg\left(\forall x.\, \neg P(x) \vee Q(x)\right) \vee \neg P(A) \vee Q(A)\right]$$

$$\left(\forall x.\, \neg P(x) \vee Q(x)\right) \wedge P(A) \wedge \neg Q(A)$$

$$\left(\neg P(x) \vee Q(x)\right) \wedge P(A) \wedge \neg Q(A)$$

Proving validity: example

- Do proof

1.	$\neg P(x) \vee Q(x)$	
2.	$P(A)$	
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4.		
5.		

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References

- Flach, P. (2012). Machine Learning: The Art and Science of Algorithms that Make Sense of Data. Cambridge University Press
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Miscellaneous Logic Topics

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- Factoring
- Green's trick
- Equality
- Completeness and decidability

Binary Resolution

- Binary resolution matches one literal from each clause
- Binary resolution isn't complete

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$$\neg P(v) \vee \neg P(w)$$

- We should!
- But all we can get is $P(x) \vee \neg P(w)$
- And from there all we can do is get back to one of the original clauses

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$$\frac{Q(y) \vee P(x, y) \vee P(v, A)}{(Q(y) \vee P(x, y))\{x / v, y / A\}}$$

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- Binary resolution plus factoring is complete

Green's Trick

- Use resolution to get answers to existential queries

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$\exists x.\ \text{Mortal}(x)$

1.	$\neg\text{Man}(x) \vee \text{Mortal}(x)$	
2.	$\text{Man}(\textit{Socrates})$	
3.	$\neg\text{Mortal}(x) \vee \text{Answer}(x)$	
4.		
5.		

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3.	$\neg\text{Mortal}(x) \vee \text{Answer}(x)$	
4.	$\text{Mortal}(\textit{Socrates})$	1,2
5.	$\text{Answer}(\textit{Socrates})$	3,5

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$$\forall x,y.\mathrm{Eq}(x,y) \rightarrow \mathrm{Eq}(y,x)$$

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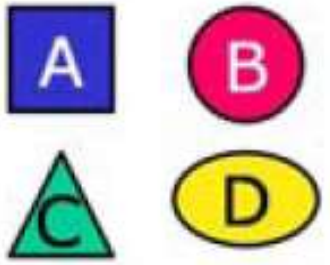
- For every predicate, allow substitutions

$$\forall x,y.\mathrm{Eq}(x,y) \rightarrow (\mathrm{P}(x) \rightarrow \mathrm{P}(y))$$

Proof Example

- Let's go back to our old geometry domain and try to prove what the hat of A is



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- Axioms in FOL (plus equality axioms)

$\text{Above}(A, C)$

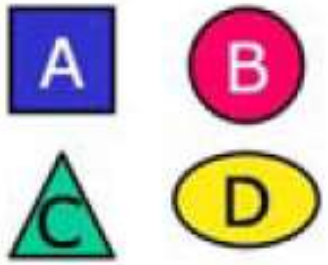
$\text{Above}(B, D)$

$\neg\exists x.\ \text{Above}(x, A)$

$\neg\exists x.\ \text{Above}(x, B)$

$\forall x, y.\ \text{Above}(x, y) \rightarrow \text{hat}(y) = x$

$\forall x.\ (\neg\exists y.\ \text{Above}(y, x)) \rightarrow \text{hat}(x) = x$



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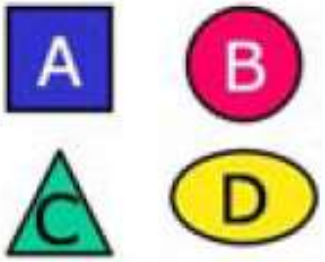

- Desired conclusion: $\exists x.\ \text{hat}(A) = x$
- Use Green's trick to get the binding of x

The Clauses

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.		
11.		
12.		

The Query

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.	~Eq(hat(A), x) v Answer(x)	

The Proof

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.	~Eq(hat(A), x) v Answer(x)	conclusion
11.	Above(sk(A), A) v Answer(A)	6, 10 {x/A}
12.	Answer(A)	11, 3 {x/sk(A)}

Hat of D

1.	Above(A, C)	
2.	Above(B, D)	
3.	~Above(x, A)	
4.	~Above(x, B)	
5.	~Above(x, y) v Eq(hat(y), x)	
6.	Above(sk(x), x) v Eq(hat(x), x)	
7.	Eq(x, x)	
8.	~Eq(x, y) v ~Eq(y, z) v Eq(x, z)	
9.	~Eq(x, y) v Eq(y, x)	
10.	~Eq(hat(D), x) v Answer(x)	conclusion
11.	~Above(x,D) v Answer(x)	5, 10 {x1/x}
12.	Answer(B)	11, 2 {x/B}

Who is Jane's Lover?

- Jane's lover drives a red car
- Fred is the only person who drives a red car
- Who is Jane's lover?

1.	Drives(lover(Jane))	
2.	~Drives(x) v Eq(x,Fred)	
3.	~Eq(lover(Jane),x) v Answer(x)	
4.	Eq(lover(Jane), Fred)	1,2 {x/lover(Jane)}
5.	Answer(Fred)	3,4 {x/Fred}

Completeness and Decidability

- Complete: If KB entails S, then we can prove S from KB

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- Complete: If KB entails S, then we can prove S from KB

- Gödel's Completeness Theorem: *There exists a complete proof system for FOL*
- Robinson's Completeness Theorem: *Resolution refutation is a complete proof system for FOL*

- *FOL is semi-decidable:* if the desired conclusion follows from the premises then eventually resolution refutation will find a contradiction.
  - If there's a proof, we'll halt with it
  - If not, maybe we'll halt, maybe not

Adding Arithmetic

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- Gödel's Incompleteness Theorem: *There is no consistent, complete proof system for FOL + Arithmetic.*
- Either there are sentences that are true, but not provable or there are sentences that are provable, but not true.

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  P = "P is not provable."

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- Arithmetic gives you the ability to construct code-names for sentences within the logic.

  P = "P is not provable."

  - If P is true: it's not provable (incomplete)
  - If P is false: it's provable (inconsistent)

References

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LOGIC
in the
Real World

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- Encode information formally in web pages

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- Encode information formally in web pages
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- Airfare pricing

Airfare Pricing

- Ignore, for now, finding the best itinerary
- Given an itinerary, what's the least amount we can pay for it?
- Can't just add up prices for the flight legs; different prices for different flights in various combinations and circumstances

Fare Restrictions

- Passenger under 2 or over 65
- Passenger accompanying someone paying full fare
- Doesn't go through an expensive city
- No flights during rush hour
- Stay over Saturday night
- Layovers are legal
- Round-the-world itinerary that doesn't backtrack
- Regular two phase round-trip
- No flights on another airline
- This fare would not be cheaper than the standard price

Ontology

Ontology is the science of something and of nothing, of being and not-being, of the thing and the mode of the thing, of substance and accident.

Leibniz

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Leibniz

The Role of Ontological Engineering in B2B Net Markets

ONTOLOGY.ORG
ENABLING VIRTUAL BUSINESS

Ontology

- What kinds of things are there in the world?
- What are their properties and relations?

Ontology is the science of something and of nothing, of being and not-being, of the thing and the mode of the thing, of substance and accident.

Leibniz

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Airfare Domain Ontology

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- passenger
- flight
- city
- airport
- terminal
- flight segment (list of flights, to be flown all in one "day")
- itinerary (a passenger and list of flight segments)

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- list
- number

Representing Properties

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- Object P is red
  - Red(P)
  - Color(P, Red)
  - color(P) = Red
  - Property(P, Color, Red)

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$$\exists c.\ \forall b.\ \text{In}(b, S) \rightarrow \text{Color}(b, c)$$

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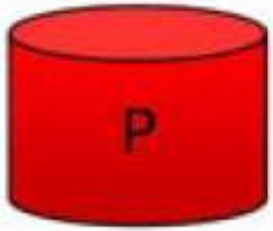

- All the blocks in stack S are the same color

$$\exists c.\ \forall b.\ \text{In}(b, S) \rightarrow \text{Color}(b, c)$$

- All the blocks in stack S have the same properties

$$\forall p.\ \exists v.\ \forall b.\ \text{In}(b, S) \rightarrow \text{Property}(b, p, v)$$

Basic Relations

Basic Relations

- Age(passenger, number)
- Nationality(passenger, country)
- Wheelchair(passenger)
- Origin(flight, airport)
- Destination(flight, airport)
- Departure_Time(flight, number)
- Arrival_Time(flight, number)
- Latitude(city, number)
- Longitude(city, number)
- In_Country(city, country)
- In_City(airport, city)
- Passenger(itinerary, passenger)
- Flight_Segments(itinerary, passenger, segments)
- Nil
- cons(object,list) => list

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- cons(object,list) => list

Age(Fred, 47)
Nationality(Fred, US)
~Wheelchair(Fred)

Defined Relations

- Define complex relations in terms of basic ones
- Like using subroutines

$$\forall i.\ \mathrm{P}(i) \wedge \mathrm{Q}(i) \rightarrow \mathrm{Qualifies}\,37(i)$$

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- Implication rather than equivalence
  - easier to specify definitions in pieces

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Defined Relations

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- Like using subroutines

$$\forall i.\ P(i) \wedge Q(i) \rightarrow \text{Qualifies}\,37(i)$$

- Implication rather than equivalence
  - easier to specify definitions in pieces

$$\forall i.\ R(i) \wedge S(i) \rightarrow \text{Qualifies}\,37(i)$$

  - can't use the other direction

$$\text{Qualifies}\,37(i) \rightarrow ?$$

  - if you need it, write the equivalence

$$\forall i.\ (P(i) \wedge Q(i)) \vee (R(i) \wedge S(i)) \leftrightarrow \text{Qualifies}\,37(i)$$

Infant Fare

$$\forall i, a, p.\ \text{Passenger}(i, p) \wedge \text{Age}(p, a) \wedge a < 2 \rightarrow \text{InfantFare}(i)$$

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- First form is typical of rule-based systems
- Second form (only!) can be made into an equivalence

Infant Fare

$$\forall i, a, p.\ \text{Passenger}(i, p) \wedge \text{Age}(p, a) \wedge a < 2 \rightarrow \text{InfantFare}(i)$$

$$\forall i\ (\ \exists p, a.\ \text{Passenger}(i, p) \wedge \text{Age}(p, a) \wedge a < 2) \rightarrow \text{InfantFare}(i)$$

- First form is typical of rule-based systems
- Second form (only!) can be made into an equivalence
- What about a < 2 ?

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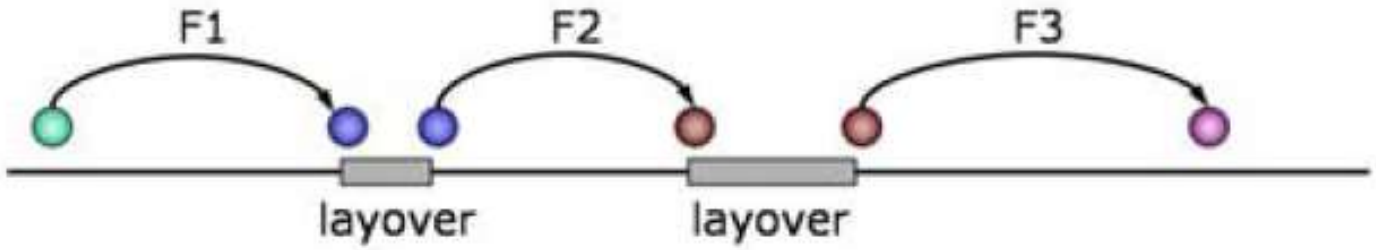
- First form is typical of rule-based systems
- Second form (only!) can be made into an equivalence
- What about a < 2 ?
  - axiomatize arithmetic
  - build it in to theorem prover

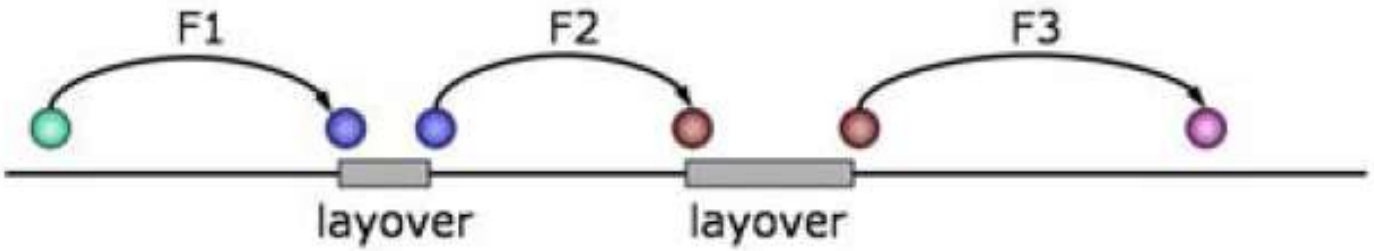
$$P(a) \vee (3 > 2) \vee (a > 1 + 2) \Rightarrow P(a) \vee (a > 3)$$

Well-Formed Segment



Well-Formed Segment

- Departure and arrival airports match up correctly
- Layovers aren't too short
- Layovers aren't too long



Lists in Logic

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- $\forall x$. LengthOne(cons(x, Nil))

$$\forall l, x.\ l = \text{cons}(x, \text{Nil}) \rightarrow \text{LengthOne}(l)$$
$$\forall l.\ (\exists x.\ l = \text{cons}(x, \text{Nil})) \rightarrow \text{LengthOne}(l)$$

Well-Formed Segment: Base Case

Define recursively, going down the list of flights

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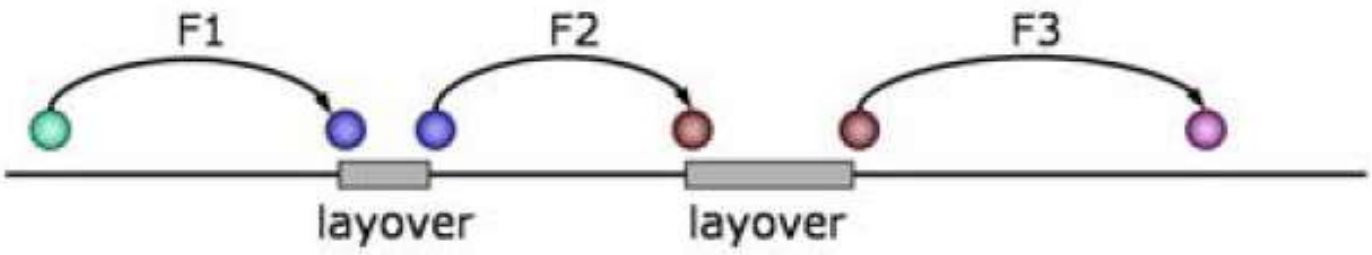
Any segment with 1 flight is well-formed

$\forall f$. WellFormed(cons(f, Nil))

Well-Formed Segment: Recursion

A flight segment with at least two flights is well-formed if

- first two flights are contiguous
- layover time between first two flights is legal
- rest of the flight segment is well-formed



Well-Formed Segment: Recursion

A flight segment with at least two flights is well-formed if

- first two flights are contiguous
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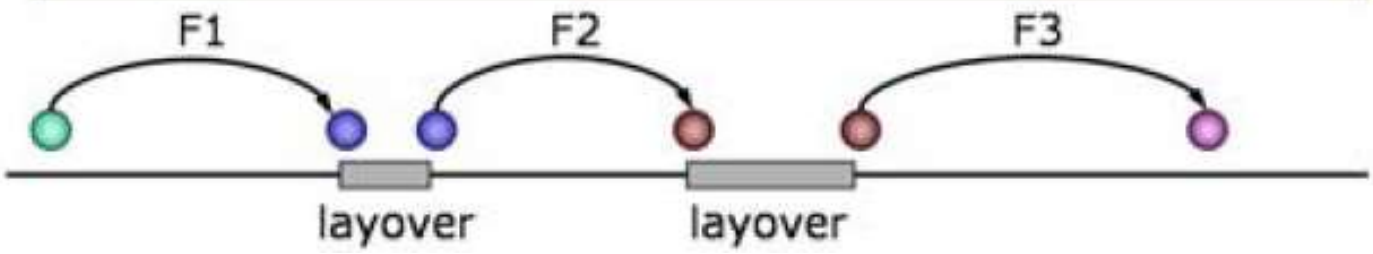
$$\forall f_1, f_2, r.\ \text{Contiguous}(f_1, f_2) \wedge \text{LegalLayover}(f_1, f_2) \wedge \text{WellFormed}(\text{cons}(f_2, r)) \rightarrow \text{WellFormed}(\text{cons}(f_1, \text{cons}(f_2, r)))$$



Helper Relations

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- Flights are contiguous if the arrival airport of the first is the same as the departure airport of the second

$$\forall f_1, f_2.\ (\exists c.\ \text{Destination}(f_1, c) \wedge \text{Origin}(f_2, c)) \rightarrow \text{Contiguous}(f_1, f_2)$$

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- Layovers are legal if they're not too short and not too long

$$\forall f_1, f_2.\ \text{LayoverNotTooShort}(f_1, f_2) \wedge \text{LayoverNotTooLong}(f_1, f_2) \rightarrow \text{LayoverLegal}(f_1, f_2)$$

Not Too Short

- A layover is not too short if it's more than 30 minutes long

$$\forall f_1, f_2.\ (\exists t_1, t_2.\ \text{ArrivalTime}(f_1, t_1) \wedge \text{DepartureTime}(f_2, t_2) \wedge (t_2 - t_1 > 30)) \rightarrow \text{LayoverNotTooShort}(f_1, f_2)$$

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- These are like the rules the airlines use, but it could involve all of common sense to know how long to allow someone to change planes

Not Too Long

- A layover is not too long if it's less than three hours

$$\forall f_1, f_2.\ (\exists t_1, t_2.\ \text{ArrivalTime}(f_1, t_1) \wedge \text{DepartureTime}(f_2, t_2) \wedge (t_2 - t_1 < 180)) \rightarrow \text{LayoverNotTooLong}(f_1, f_2)$$

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- A layover is also not too long if there was no other way to make the next leg of your journey sooner

$$\forall f_1, f_2.\ (\exists o, d, t_2.\ \text{Origin}(f_2, o) \wedge \text{Destination}(f_2, d) \wedge \text{DepartureTime}(f_2, t_2) \wedge \neg\exists f_3, t_3.\ (\text{Origin}(f_3, o) \wedge \text{Destination}(f_3, d) \wedge \text{DepartureTime}(f_3, t_3) \wedge (t_3 < t_2) \wedge \text{LayoverNotTooShort}(f_1, f_3))) \rightarrow \text{LayoverNotTooLong}(f_1, f_2)$$

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Thank You