# $1^{st}$ Topic

# Infinite Series

**[Definition, Partial Sum, Geometric series test, Positive term series]**

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(13 Solved problems and 05 Home assignments)

## IMPORTANCE OF INFINITE SERIES:

**Differential Equations are frequently solved by using infinite series. Fourier series, Fourier-Bessel series, etc. expansions involve infinite series. Transcendental functions (trigonometric, exponential, logarithmic, hyperbolic, etc.) can be expressed conveniently in terms of infinite series. Many problems that cannot be solved in terms of elementary (algebraic and transcendental) functions can also be solved in terms of infinite series.**

**So, in other words, infinite series occur so frequently in all types of engineering problems that the necessity of studying their convergence or divergence is very important. Unless a series employed in an investigation is convergent, it may lead to illogical conclusion.**

**Hence, it is essential that the students of engineering begin by acquiring an intelligent grasp of this subject.**

## INFINITE SERIES:

**Definition:** Let $\{u_n\}$ be a sequence of real numbers.

i.e. $\{u_1, u_2, u_3, u_4, ..........., u_n, ...........\infty\}$ be a sequence of real numbers.

Then, the expression $u_1 + u_2 + .........+ u_n + ..........\infty$ [i.e. the sum of the terms of the sequence, which are infinite in number] is called an **infinite series** and is denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by $\sum u_n$.

**$n^{th}$ Term:** The term $u_n$ in an infinite series is called the $n^{th}$ term of the series.

## Finite series:

If the number of terms in the series is limited, then the series is called a ***finite series.***

## Infinite series:

If the number of terms in the series is infinite, then the series is called an ***infinite series.***

## $n^{th}$ Partial sum:

The sum of the first n terms of the series is called its ***nth partial sum*** of $\sum u_n$ and is denoted by $S_n$.

i.e., $S_n = u_1 + u_2 + .........+ u_n$.

Clearly, $S_n$ is a function of n and as n increases indefinitely, then the following three cases arise:

**(i)** If $S_n$ tends to a finite limit as $n \to \infty \Rightarrow \sum u_n$ is ***convergent.***

**(ii)** If $S_n$ tends to $+\infty$ or $-\infty$ as $n \to \infty \Rightarrow \sum u_n$ is ***divergent.***

**(iii)** If $S_n$ does not tends to a unique limit as $n \to \infty \Rightarrow \sum u_n$ is ***oscillatory or non-convergent.***

**Remark:** Any series which diverge or oscillate are said to be non-convergent series.

### Now let us examine the behavior of the well known series:

**Q.No.1.:** Examine the behavior of the series: $1 + \frac{1}{2} + \frac{1}{2^2} + ......+ \frac{1}{2^{n-1}} + \frac{1}{2^n} + ......$ .

**Sol.:** Given series is $1 + \frac{1}{2} + \frac{1}{2^2} + ......+ \frac{1}{2^{n-1}} + \frac{1}{2^n} + ......$

Here $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + ......  + n^{th}$ term. $\left[\because \text{it is a G.P. whose first term is } 1 \text{ and common ratio is } 1/2. \therefore S_n = \frac{a(1-r^n)}{1-r}\right]$

$$\Rightarrow S_n = \frac{1\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} \Rightarrow S_n = 2\left(1-\frac{1}{2^n}\right).$$

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2(1-0) = 2.$$

$\Rightarrow$ The given series converges to 2**.**

**Q.No.2.:** Examine the behavior of the series: $1+2+3+.......+n+........\infty$ **.**

**Sol.:** Given series is $1+2+3+.......+n+........\infty$.

Here $S_n = 1+2+3+.........+n = \frac{n(n+1)}{2}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} \to \infty.$$

$\Rightarrow$ The given series is divergence**.**

**Q.No.3.:** Examine the behavior of the series: $1-1+1-1+......... = \sum_{n=1}^{\infty}(-1)^{n-1}$.

**Sol.:** Given series is $1-1+1-1+......... = \sum_{n=1}^{\infty}(-1)^{n-1}$.

Here $S_n = 1-1+1-1+..........\text{upto } n \text{ terms} = 0, n = 2m$ (even)

$= 1, n = 2m+1$ (odd).

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = 0,$$

$$= 1.$$

Clearly, in this case, $S_n$ does not tend to a unique limit as $n \to \infty$.

Hence, the given series is oscillatory and oscillates between 0 and 1.

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# GEOMETRIC SERIES TEST:

**Examine the behaviour of geometric series.**

**or**

**Show that the series** $1+r+r^2+r^3+............\infty$

**(i)** **converges if** $|r|<1$,

**(ii)** **diverges if** $r\geq 1$, **and**

**(iii)** **oscillates if** $r\leq -1$

**Proof:** Here $S_n = 1+r+r^2+r^3+.......+r^{n-1} = \dfrac{1-r^n}{1-r} = \dfrac{1}{1-r} - \dfrac{r^n}{1-r}$. (i)

**Case (i):** When $|r|<1$ :

Now since $S_n = \dfrac{1}{1-r} - \dfrac{r^n}{1-r}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{1-r} - \frac{r^n}{1-r}\right] = \frac{1}{1-r} = \text{a definite quantity.} \qquad \left[\because \lim_{n\to\infty} r^n = 0 \text{ for } |r|<1\right]$$

$\Rightarrow$ The given series is convergent if $|r|<1$.

**Case (ii): (a)** When $r>1$:

Now since $S_n = \dfrac{1}{1-r} - \dfrac{r^n}{1-r} \Rightarrow S_n = \dfrac{r^n}{r-1} - \dfrac{1}{r-1}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{r^n}{r-1} - \frac{1}{r-1}\right] \to \infty. \qquad \left[\because \lim_{n\to\infty} r^n \to \infty \text{ for } r>1\right]$$

$\Rightarrow$ The given series is divergent if $r>1$.

**(b)** When $r = 1$, then $S_n = 1+1+1+.........n \text{ terms} = n$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n \to \infty.$$

$\Rightarrow$ The given series is divergent if $r = 1$.

Hence, the given series is divergent if $r\geq 1$.

**Case (iii): (a)** When $r = -1$, then $S_n = 1 - 1 + 1 - 1 + ........... + (-1)^{n-1} = 0$ if n is even,

$= 1$ if n is odd.

Taking limit on both sides, we get

$\lim\limits_{n\to\infty} S_n = 0,$

$= 1.$

$\Rightarrow$ $S_n$ does not tend to a unique limit as $n \to \infty$.

$\Rightarrow$ The given series is oscillatory and oscillates finitely.

**(b)** When $r < -1$. Let $r = -R$ so that $R > 1$. Then $r^n = (-R)^n = (-1)^n R^n$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1 - r^n}{1 - r}\right] = \lim_{n\to\infty}\left[\frac{1 - (-1)^n R^n}{1 + R}\right]$$

$\to -\infty$ if n is even $[R^n \to \infty$ as $n \to \infty \because R > 1]$

$+\infty$ if n is odd.

$\Rightarrow$ $S_n$ does not tend to a unique limit as $n \to \infty$.

Hence, the given series is oscillatory and oscillates infinitely.

This completes the proof.

## Positive term series:

**Definition:** A series in which all the terms, after some particular term, are positive is called a positive term series.

e.g. $-5 - 3 - 1 + 1 + 3 + 5 + 7 + ..................$

### Now let us solve some more problems:

**Q.No.4.:** Examine the behavior of the series: $1 + 3 + 5 + 7 + ..........\infty$.

**Sol.:** Given series is $1 + 3 + 5 + 7 + ...... + (2n - 1) + .....\infty = \sum u_n$.

Here $S_n = 1 + 3 + 5 + 7 + ...... + (2n - 1) = \sum(2n - 1) = 2\sum n - n$

$$= \frac{2n(n+1)}{2} - n = n^2 + n - n = n^2.$$

Taking limit on both sides, we obtain

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} n^2 \to \infty$.

Hence, the given series is divergent.

**Q.No.5.:** Show that the series $1^2 + 2^2 + 3^2 + \dots\dots\dots\dots + n^2 + \dots\dots\dots$ diverges to $\infty$.

**Sol.:** Given series is $1^2 + 2^2 + 3^2 + \dots\dots\dots\dots + n^2 + \dots\dots\dots$

Here $S_n = 1^2 + 2^2 + 3^2 + \dots\dots\dots\dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Taking limit on both sides, we obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} \to \infty .$$

$\Rightarrow S_n$ tends to $\infty$ as $n \to \infty$.

$\Rightarrow$ The given series is divergent and diverges to $\infty$.

**Q.No.6.:** Examine the behavior of the series:

$$1^2 + 3^2 + 5^2 + \dots\dots\dots\dots + (2n-1)^2 + \dots\dots\dots\infty .$$

**Sol,:** Given $\sum u_n = 1^2 + 3^2 + 5^2 + \dots\dots\dots\dots + (2n-1)^2 + \dots\dots\dots\infty$

$$\therefore S_n = 1^2 + 3^2 + 5^2 + \dots\dots\dots\dots + (2n-1)^2$$

$$= \sum (2n-1)^2 = \sum \left[4n^2 + 1 - 4n\right] = 4\sum n^2 + \sum 1 - 4\sum n$$

$$= 4\left[\frac{n(n+1)(2n+1)}{6}\right] + n - 4\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{2}{3} n(n+1)(2n+1) + n - 2n(n+1).$$

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \frac{2}{3} n(n+1)(2n+1) + n - 2n(n+1) \to \infty .$$

$\Rightarrow$The given series is divergent.

**Q.No.7.:** Examine the behavior of the series: $6 - 10 + 4 + 6 - 10 + 4 + 6 + \dots\dots\dots\dots\infty$.

**Sol.:** Given $\sum u_n = 6 - 10 + 4 + 6 - 10 + 4 + 6 + \dots\dots\dots\dots\infty$.

Here $S_n = 0$, if $n = 3m$,

$= 6$, if $n = 3m + 1$,

$= -4$, if $n = 3m + 2$.

Taking limit on both sides, we obtain

$\lim\limits_{n\to\infty} S_n = 0$,

$= 6$,

$= -4$.

$\Rightarrow S_n$ does not tend to a unique limit as $n \to \infty$.

$\Rightarrow$The given series $\sum u_n$ is said to be oscillatory.

**Q.No.8.:** Examine the behavior of the series:

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+..........+\frac{1}{n(n+1)}+...........\infty .$$

**Sol.:** Given series is $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+..........+\frac{1}{n(n+1)}+...........\infty$.

Here $u_n = \frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}$.

Putting 1, 2, 3, ………….., n.

$u_1 = \frac{1}{1}-\frac{1}{2}$,

$u_2 = \frac{1}{2}-\frac{1}{3}$,

$u_3 = \frac{1}{3}-\frac{1}{4}$,

……………..,

$u_n = \frac{1}{n}-\frac{1}{n+1}$.

Adding, we get $S_n = u_1 + u_2 + .......... + u_n = \left(1-\frac{1}{n+1}\right)$.

Taking limit on both sides, we obtain

$\lim\limits_{n\to\infty} S_n = 1-0 = 1$.

$\Rightarrow S_n$ tends to a finite limit as $n \to \infty$.

$\Rightarrow$ The given series is convergent and converges to 1.

**Q.No.9.:** Examine the behavior of the series: $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+............\infty$.

**Sol.:** Given $\sum u_n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+................\infty$.

Clearly, this is a geometric series, in which first term is 1 and common ratio is $r=\frac{1}{2} \Rightarrow |r|=\frac{1}{2}<1$.

$\therefore$ By geometric series test, the given series is convergent.

**Q.No.10.:** Examine the behavior of the series: $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+............\infty$.

**Sol.:** Given $\sum u_n = 1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+............\infty$.

Clearly, this is a geometric series, in which first term is 1 and common ratio is $r=-\frac{1}{3} \Rightarrow |r|=\frac{1}{3}<1$.

$\therefore$ By geometric series test, the given series is convergent.

**Q.No.11.:** Determine how many terms are needed to compute the sum S of the geometric series $1+r+r^2+r^3+............\infty$ with an error less than 0.01, when r = 0.25, r = 0.5, r = 0.9 ?

**Sol.:** Here $S_n = 1+r+r^2+r^3+.......+r^{n-1} = \frac{1-r^n}{1-r} = \frac{1}{1-r}-\frac{r^n}{1-r}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{1-r}-\frac{r^n}{1-r}\right] = \frac{1}{1-r} = \text{a definite quantity.}$$

**(i) When r = 0.25:** Given $\frac{r^n}{1-r}<0.01 \Rightarrow \frac{(0.25)^n}{1-0.25}=\frac{(0.25)^n}{0.75}<0.01 \Rightarrow (0.25)^n<0.0075$

$\Rightarrow n\log(0.25)<\log(0.0075) \Rightarrow n(-0.6021)<(-2.125)$

$\Rightarrow n(0.6021)>(2.125) \Rightarrow n>\frac{2.125}{0.6021} \Rightarrow n>3.53$

Thus 4 terms are needed to compute the sum S of the geometric series $1+r+r^2+r^3+............\infty$ with an error less than 0.01.

**(ii) When r = 0.5:** Given $\frac{r^n}{1-r}<0.01 \Rightarrow \frac{(0.5)^n}{1-0.5}=\frac{(0.5)^n}{0.5}<0.01 \Rightarrow (0.5)^n<0.005$

$\Rightarrow n\log(0.5) < \log(0.005) \Rightarrow n(-0.3010) < (-2.3010)$

$\Rightarrow n(0.3010) > (2.3010) \Rightarrow n > \dfrac{2.3010}{0.3010} \Rightarrow n > 7.64$

Thus 8 terms are needed to compute the sum S of the geometric series $1+r+r^2+r^3+\ldots\ldots\ldots\ldots\infty$ with an error less than $0.01$.

**(iii) When r = 0.9:** Given $\dfrac{r^n}{1-r} < 0.01 \Rightarrow \dfrac{(0.9)^n}{1-0.9} = \dfrac{(0.9)^n}{0.1} < 0.01 \Rightarrow (0.9)^n < 0.001$

$\Rightarrow n\log(0.9) < \log(0.001) \Rightarrow n(-0.0458) < (-3)$

$\Rightarrow n(0.0458) > (3) \Rightarrow n > \dfrac{3}{0.3010} \Rightarrow n > 65.50$

Thus 66 terms are needed to compute the sum S of the geometric series $1+r+r^2+r^3+\ldots\ldots\ldots\ldots\infty$ with an error less than $0.01$.

**Q.No.12.:** Determine how many terms are needed to compute the sum S of the geometric series $\dfrac{1}{2}-\dfrac{1}{4}+\dfrac{1}{8}-..+\ldots\ldots\ldots\ldots$ with an error less than $0.01$?

**Sol.:** Since $S_n = a+ar+ar^2+ar^3+\ldots\ldots.+ar^{n-1} = \dfrac{a(1-r^n)}{1-r} = \dfrac{a}{1-r} - \dfrac{ar^n}{1-r}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{a}{1-r} - \frac{ar^n}{1-r}\right] = \frac{a}{1-r} = \text{a definite quantity.}$$

Here $r = -\dfrac{1}{2} = -0.5, a = \dfrac{1}{2} = 0.5$ **:** To find n, s.t. $\dfrac{ar^n}{1-r} < 0.01$.

Calculate 'n', number of terms to compute the sum S of the geometric series with an error less than $0.01$.

$\dfrac{(0.5)(-0.5)^n}{1+0.5} = \dfrac{(0.5)(-0.5)^n}{1.5} < 0.01 \Rightarrow (-0.5)^n < 0.03$.

Squaring both sides, we get

$(-0.5)^{2n} < 0.0009 \Rightarrow (-1)^{2n}(0.5)^{2n} < 0.0009 \Rightarrow (0.5)^{2n} < 0.0009$

$\Rightarrow 2n\log(0.5) < \log(0.0009) \Rightarrow 2n(-0.30) < (-3.046)$

$$\Rightarrow 2n(0.30) > (3.046) \Rightarrow n > \frac{3.046}{0.60} \Rightarrow n > 5.07$$

Thus 6 terms are needed to compute the sum S of the geometric series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - .. + ............$ with an error less than 0.01.

**Q.No.13.:** Determine how many terms are needed to compute the sum S of the geometric series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + .. - ............$ with an error less than 0.01?

**Sol.:** Since $S_n = 1 + r + r^2 + r^3 + ....... + r^{n-1} = \frac{(1 - r^n)}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{1-r} - \frac{r^n}{1-r}\right] = \frac{1}{1-r} = \text{a definite quantity.}$$

Here $r = -\frac{1}{3} = -0.33$ : To find n, s.t. $\frac{r^n}{1-r} < 0.01$.

Calculate 'n', number of terms to compute the sum S of the geometric series with an error less than 0.01.

$$\frac{(-0.3333)^n}{1 + 0.3333} = \frac{(-0.3333)^n}{1.3333} < 0.01 \Rightarrow (-0.3333)^n < 0.013333.$$

Squaring both sides, we get

$$(-0.3333)^{2n} < 0.00017689 \Rightarrow (-1)^{2n}(0.3333)^{2n} < 0.00017777 \Rightarrow (0.3333)^{2n} < 0.00017777$$

$$\Rightarrow 2n\log(0.3333) < \log(0.00017777) \Rightarrow 2n(-0.4771) < (-3.750)$$

$$\Rightarrow 2n(0.4771) > (3.750) \Rightarrow n > \frac{3.750}{0.9542} \Rightarrow n > 3.930$$

Thus 4 terms are needed to compute the sum S of the geometric series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + .. - ............$ with an error less than 0.01.

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# Let's summarize

- **Importance of infinite series**

- **Definition of infinite series, $n^{th}$ term**
- **Definition of $n^{th}$ partial sum, finite series, infinite series**
- **Discussion of convergence, divergence and oscillatory cases**
- **Examples of various cases**
- **Behaviour of Geometric series**
- **And few solved problems**

# Home Assignments

**Q.No.1.:** A ball is dropped from a height b feet from a flat surface. Each time the ball hits the ground after falling a distance h it rebounds a distance rh, where $0 < r < 1$.

Find the total distance the ball travels if b = 4 ft and r = 3/4.

**Q.No.2.:** Determine how many terms are needed to compute the sum S of the geometric series $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+..-............$ with an error less than 0.01?

**Hint:** $\sin x = 1-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+..-............$ . Here x = 1.

**Q.No.3.:** Determine how many terms are needed to compute the sum S of the geometric series $1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+..-............$ with an error less than 0.01?

**Q.No.4.:** Determine how many terms are needed to compute the sum S of the geometric series $1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-..+............$ with an error less than 0.01?

**Hint:** $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-..+............$ . Here x = 1.

**Q.No.5.:** Determine how many terms are needed to compute the sum S of the geometric series $1-\frac{1}{3}+\frac{1}{6}-\frac{1}{12}+\frac{1}{24}-..+............$ with an error less than 0.01?

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# Thank you

## Next Topics

### Tests for convergence and divergence of a series

1. $n^{th}$ term Test,
2. Comparison Test,
3. p-series Test

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# $2^{nd}$ Topic

## Infinite Series

**[$n^{th}$ term Test, Comparison Test, p-series Test]**


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**(17 Solved problems and 01 Home assignment)**

## TESTS FOR CONVERGENCE AND DIVERGENCE OF A SERIES

1. **$n^{th}$ term Test**
2. **Comparison Test**
3. **p-series Test**

**Before we use an infinite series for computational or other purposes we must know whether it converges or diverges. In most cases that arise in engineering mathematics, this question may be answered by applying one of the various tests for convergence and divergence, which we shall now consider. These tests, therefore, are of great practical interest.**

### 1. $n^{th}$ term Test: [Preliminary test for divergence]

**Statement:** If $\lim\limits_{n\to\infty} u_n \neq 0$, then an infinite series $\sum u_n$ is said to be divergent.

**Example:** $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+.........+\frac{n}{n+1}+......\infty$

Here $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{n}{n+1} = \lim\limits_{n\to\infty}\frac{n}{n\left(1+\frac{1}{n}\right)} = \lim\limits_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)} = 1 \neq 0$

By the above preliminary test, the given series diverges.

## NECESSARY CONDITION FOR CONVERGENCE:

Necessary condition for convergence of a series $\sum u_n$ is that, its $n^{th}$ term $u_n$ approaches zero as n becomes infinite.

Show that if a positive term series $\sum u_n$ is convergent, then $\lim\limits_{n\to\infty} u_n = 0$.

(The converse of it may or may not be true).

**Important note:** This is not a test for convergence.

**Proof:** Let $S_n = u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots + u_n$.

**Given:** $\sum u_n$ is convergent.

**To show:** $\lim\limits_{n\to\infty} u_n = 0$.

Since $\sum u_n$ is convergent $\Rightarrow \lim\limits_{n\to\infty} S_n = \text{a finite quantity, } k \text{ (say)}$.

Also $\lim\limits_{n\to\infty} S_{n-1} = k$. But $u_n = S_n - S_{n-1}$.

$\therefore \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} (S_n - S_{n-1}) = 0$.

This completes the proof.

**Remark: It is important to note that the converse of this result is not always true.**

**Example:** Let us consider the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} + \ldots\ldots\infty$.

Here $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Here $S_n = \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}}\right) > \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots\ldots + \frac{1}{\sqrt{n}}\right) = \frac{n}{\sqrt{n}} = \sqrt{n}$.

$\therefore \lim\limits_{n\to\infty} S_n > \lim\limits_{n\to\infty} \sqrt{n} \to \infty$.

Thus, the series is divergent even despite the fact that $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Hence $\lim\limits_{n\to\infty} u_n = 0$ is a ***necessary* but *not sufficient*** condition for convergence of $\sum u_n$.

## 2. Comparison Test:

**Statement:** Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that from and after some particular term, $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = k$ (a non-zero, finite quantity), then $\sum u_n$ and $\sum v_n$ either both converge or both diverge, or in other words, both the series behave alike.

**Remark:** To select auxiliary series (or Harmonic series) $\sum v_n = \sum \frac{1}{n^p}$, it should be noted that p = difference in degree of n in denominator and numerator of $u_n$.

## 3. p-series Test:

## (Behaviour of Auxiliary Series or Harmonic Series)

**Statement:** Show that the series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots = \sum \frac{1}{n^p}$

**(i)** converges if $p > 1$ and

**(ii)** diverges if $p \le 1$.

**Proof:** Let $S = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$

**Case I:** When $p > 1$.

Let the terms of the given series be grouped in such a manner that the first, second, third, ... ... groups contains 1 term, 2 terms, 4 terms... ...respectively.

$$\therefore S = \left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots\ldots\ldots\ldots .$$

Now $\frac{1}{1^p} = 1$, $\left(\frac{1}{2^p} + \frac{1}{3^p}\right) < \left(\frac{1}{2^p} + \frac{1}{2^p}\right) = \frac{2}{2^p} = \left(\frac{1}{2}\right)^{p-1}$,

$$\left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) < \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) = \frac{4}{4^p} = \frac{2^2}{2^{2p}} = \left(\frac{1}{2}\right)^{2p-2}$$, ... .. and so on.

Adding, we get $S = \left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots\ldots\ldots\ldots .$

$$< 1 + \left(\frac{1}{2}\right)^{p-1} + \left(\frac{1}{2}\right)^{2p-2} + \left(\frac{1}{2}\right)^{3p-3} + \ldots\ldots .$$

$$< 1 + \left[\left(\frac{1}{2}\right)^{p-1}\right] + \left[\left(\frac{1}{2}\right)^{p-1}\right]^2 + \left[\left(\frac{1}{2}\right)^{p-1}\right]^3 + \ldots\ldots$$

The series on the RHS is a geometric series with common ratio $\left(\frac{1}{2}\right)^{p-1} < 1$ (when $p > 1$).

Thus, by geometric series test, the series on the RHS is convergent.

Hence, the given series is also convergent, for $p > 1$.

**Case II:** When $p = 1$. Then $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots.$ .

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots \quad .$$

$$\because \left(\frac{1}{3} + \frac{1}{4}\right) > \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}, \quad \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = \frac{4}{8} = \frac{1}{2} \ldots \ldots \text{ and so on.}$$

$$\therefore S = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\right) > 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots\right) \to \infty \; .$$

The series, on the RHS (ignoring the 1st term) is a geometric series with common ratio unity.

Thus, by geometric series test, the series on the RHS is divergent.

Hence, the given series is also divergent, for $p = 1$.

**Case III:** When $p < 1$. Then $S = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$

$$\because \frac{1}{1^p} = 1, \frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}, \ldots\ldots\ldots \text{ and so on .}$$

Then $S > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots$ .

The series on the RHS is divergent (by Case II).

Hence, the given series is also divergent, for $p < 1$ .

This completes the proof.

**Now let us use these tests for the examination of the behavior of the following infinite series:**

**Q.No.1.** Discuss the behaviour of the series $\sum_{n=1}^{\infty} \frac{n+3}{n^3 - n + 1}$.

**Sol.:** Here $u_n = \frac{n+3}{n^3 - n + 1}$. Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1 + \frac{3}{n}}{1 - \frac{1}{n^2} + \frac{1}{n^3}} = 1 \text{ (a finite, non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

**Q.No.2.:** Discuss the behaviour of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$.

**Sol.:** Here $u_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$. Take $v_n = \frac{1}{\sqrt{n}}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{2} \text{ (a finite, non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{1/2}} \approx \sum \frac{1}{n^p}$ (here $p = 1/2 < 1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.3.:** Discuss the behaviour of the series $\sum_{n=0}^{\infty} \frac{2n^3 + 5}{4n^5 + 1}$.

**Sol.:** Here $u_n = \frac{2n^3 + 5}{4n^5 + 1}$. Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2+\frac{5}{n^3}}{4+\frac{1}{n^5}} = \frac{2}{4} = \frac{1}{2} \text{ (a finite, non-zero number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

**Q.No.4.:** Discuss the behaviour of the series $\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}$.

**Sol.:** Here $u_n = \frac{n^2-1}{n^2+1}$. Take $v_n = \frac{1}{n^0} = 1$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1 \text{ (a non-zero, finite number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^0} \approx \sum \frac{1}{n^p}$ (here $p = 0 < 1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.5.:** Discuss the behaviour of the series $\sum \frac{n+\sqrt{n}}{n^2-n}$.

**Sol.** Here $u_n = \frac{n+\sqrt{n}}{n^2-n}$. Take $v_n = \frac{1}{n}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{n}} = 1 \text{ (a non-zero, finite number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n}$ (here p = 1) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.6.:** Discuss the behaviour of the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ......\infty$ .

**Sol.:** Here $u_n = \frac{2n-1}{n(n+1)(n+2)}$ . Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2 \text{ (a finite non-zero number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

**Q.No.7.:** Discuss the behaviour of the series $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + ......\infty$ .

**Sol.:** Here $u_n = \frac{(n+1)}{n^p}$. Take $v_n = \frac{1}{n^{p-1}}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1 \text{(a finite, non-zero number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{p-1}}$ is convergent for $(p-1) > 1$ i.e. $p > 2$,

and divergent for $(p-1) \le 1$ i.e. $p \le 2$. [by p-series test]

Hence $\sum u_n$ is convergent for $p > 2$, and divergent for $p \le 2$.

**Q.No.8.:** Discuss the behaviour of the series

$$\textbf{(i)}\ \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots,\quad \textbf{(ii)}\ \frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ldots\ldots.$$

**Sol.: (i)** Here $u_n=\dfrac{1}{n(n+1)}$. Take $v_n=\dfrac{1}{n^2}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1 \text{ (a non-zero, finite number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n=\sum\dfrac{1}{n^2}$ (here $p=2>1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

**(ii)** Here $u_n=\dfrac{n}{(2n-1)(2n+1)}$. Take $v_n=\dfrac{1}{n}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}=\frac{1}{4} \text{ (a non-zero, finite number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n=\sum\dfrac{1}{n}$ (here $p=1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.9.:** Discuss the behaviour of the series $1+\dfrac{2^2+1}{2^3+1}+\dfrac{3^2+1}{3^3+1}+\dfrac{4^2+1}{4^3+1}+\ldots\ldots.$

**Sol.** Here $u_n=\dfrac{n^2+1}{n^3+1}$. Take $v_n=\dfrac{1}{n}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}}=1 \text{ (a non-zero, finite number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n}$ (here p = 1) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.10.:** Discuss the behaviour of the series $\frac{1.2}{3^2.4^2} + \frac{3.4}{5^2.6^2} + \frac{5.6}{7^2.8^2} + \ldots\ldots$.

**Sol.:** Here $u_n = \frac{(2n-1)(2n)}{(2n+1)^2(2n+2)^2}$. Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)2}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2} = \frac{4}{16} = \frac{1}{4} \text{ (a non-zero, finite number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

**Q.No.11.:** Discuss the behaviour of the series $1 + \frac{2}{1+2^2} + \frac{3}{2+3^2} + \frac{4}{3+4^2} + \ldots\ldots$ .

**Sol.:** Here $u_n = \frac{n}{(n-1)+n^2} = \frac{n}{n^2+n-1}$. Take $v_n = \frac{1}{n}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}-\frac{1}{n^2}\right)} = 1 \text{ (a non-zero, finite number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n}$ (here p = 1) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.12.:** Discuss the behaviour of the series $\sum\left[\left(n^3+1\right)^{1/3}-n\right]$.

**Sol.:** Here

$$u_n=\left(n^3+1\right)^{1/3}-n=n\left[\left(1+\frac{1}{n^3}\right)^{1/3}-1\right]=n\left[\left(1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^6}+\ldots\ldots\right)-1\right]$$

$\Rightarrow u_n=\left[\dfrac{1}{3n^2}-\dfrac{1}{9n^5}+\ldots\ldots\right]$. Take $v_n=\dfrac{1}{n^2}$.

$\therefore \lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=\lim\limits_{n\to\infty}\left(\dfrac{1}{3}-\dfrac{1}{9n^3}+\ldots\ldots\right)=\dfrac{1}{3}$ (a finite, non-zero number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n=\sum\dfrac{1}{n^2}$ (here $p=2>1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

**Q.No.13.:** Discuss the behaviour of the series $\sum\limits_{n=1}^{\infty}\left[\sqrt{\left(n^2+1\right)}-n\right]$.

**Sol.:** Here $u_n=\sqrt{\left(n^2+1\right)}-n$. Rationalizing, we get

$$u_n=\left[\sqrt{\left(n^2+1\right)}-n\right]\times\frac{\sqrt{\left(n^2+1\right)}+n}{\sqrt{\left(n^2+1\right)}+n}=\frac{1}{\sqrt{n^2+1}+n}.$$

Take $v_n=\dfrac{1}{n}$. $\therefore \lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=\lim\limits_{n\to\infty}\dfrac{1}{\left[\sqrt{1+\frac{1}{n}}+1\right]}=\dfrac{1}{2}$. (a non-zero, finite number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n=\sum\dfrac{1}{n}\approx\sum\dfrac{1}{n^p}$ (here $p=1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.14.:** Discuss the behaviour of the series $\sum\left[(n+1)^{1/3}-n^{1/3}\right]$.

**Sol.:** Here

$$u_n=(n+1)^{1/3}-n^{1/3}=n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3}-1\right]=n^{1/3}\left[\left(1+\frac{1}{3}\cdot\frac{1}{n}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^2}+......\right)-1\right]$$

$$\Rightarrow u_n=n^{1/3}\left[\frac{1}{3n}-\frac{1}{9n^2}+......\right].\text{ Take } v_n=\frac{1}{n^{2/3}}.$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{9n}+......\right)=\frac{1}{3}\text{ (a non-zero, finite number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n=\sum\frac{1}{n^{2/3}}$ (here $p=\frac{2}{3}<1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.15.:** Discuss the behaviour of the series $\sum_{n=2}^{\infty}\left[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}\right]$.

**Sol.:** Here $u_n=\sqrt{(n^4+1)}-\sqrt{(n^4-1)}$. Rationalizing, we get

$$u_n=\left[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}\right]\times\frac{\sqrt{(n^4+1)}+\sqrt{(n^4-1)}}{\sqrt{(n^4+1)}+\sqrt{(n^4-1)}}=\frac{2}{\sqrt{(n^4+1)}+\sqrt{(n^4-1)}}.$$

Take $v_n=\frac{1}{n^2}$. $\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}=\frac{2}{2}=1$ (a non-zero, finite number) .

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n=\sum\frac{1}{n^2}$ (here $p=2>1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

**Q.No.16.:** Discuss the behaviour of the series $\sum \sin\frac{1}{n}$.

**Sol.:** Here $u_n = \sin\frac{1}{n}$. Take $v_n = \frac{1}{n}$. $\therefore \lim\limits_{n\to\infty}\frac{u_n}{v_n} = \lim\limits_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}$.

Put $\frac{1}{n} = x$ $\therefore$ as $n \to \infty \Rightarrow x \to 0$

$\therefore \lim\limits_{n\to\infty}\frac{u_n}{v_n} = \lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ (a non-zero, finite number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum\frac{1}{n}$ (here p = 1) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**or**

Here $u_n = \sin\frac{1}{n} = \left[\frac{1}{n} - \frac{1}{3!}.\frac{1}{n^3} + \frac{1}{5!}.\frac{1}{n^5} - ......\right]$. Take $v_n = \frac{1}{n}$.

$\therefore \lim\limits_{n\to\infty}\frac{u_n}{v_n} = \lim\limits_{n\to\infty}\left[1 - \frac{1}{6}.\frac{1}{n^2} + \frac{1}{120}.\frac{1}{n^4} - ......\right] = 1$ (a non-zero, finite number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum\frac{1}{n}$ (here p = 1) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

**Q.No.17.:** Discuss the behaviour of the series $\sum\frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$.

**Sol.:** Here $u_n = \frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}\left[\left(\frac{1}{n}\right) + \frac{1}{3}.\left(\frac{1}{n}\right)^3 + \frac{2}{15}.\left(\frac{1}{n}\right)^5 + ......\right]$

$$\Rightarrow u_n = \frac{1}{n\sqrt{n}}\left[1 + \frac{1}{3}.\left(\frac{1}{n}\right)^2 + \frac{2}{15}.\left(\frac{1}{n}\right)^4 + ......\right]. \text{ Take } v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[1+\frac{1}{3}.\left(\frac{1}{n}\right)^2+\frac{2}{15}.\left(\frac{1}{n}\right)^4+......\right] = 1 \text{ (a non-zero, finite number) .}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{3/2}}$ (here $p = \frac{3}{2} > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

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# Let's summarize

- $n^{th}$ term Test
- Necessary condition for convergence
- Comparison Test
- p-series Test
- And few solved problems

# Home Assignments

**Q.No.1:** Give an example of an infinite series, which is divergent but $\lim_{n\to\infty} u_n = 0$.

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**Thank you**

# NEXT TOPIC

## 4. D'Alembert's Ratio Test:

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# 3rd Topic

## Infinite Series

**[D'Alembert's Ratio Test]**

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**Banasthali**

**(Last updated on 26-07-2017)**

**(40 Solved problems and 00 Home assignments)**

## 4. D'Alembert's Ratio Test:

**Statement:** If $\sum u_n$ be a positive term series s.t. from and after some particular term,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k \text{, then } \sum u_n$$

**(i)** converges if $k < 1$ and

**(ii)** diverges if $k > 1$ **.**

This test fails when limit does not exist or equal to 1.

Here $\frac{u_{n+1}}{u_n}$ measures the rate or growth of the terms of the series.

**Note:** This test was developed by Jean Le-Rond D'Alembert (1717–1783) a French Mathematician.

**Proof:** Let the series from and after the particular term be

$$u_1 + u_2 + .................. + u_n + .............$$

**Case I:** When $k < 1$.

Then by definition of limit, a positive number $\lambda(k < \lambda < 1)$ can be found s.t.

$$\frac{u_{n+1}}{u_n} < \lambda \ \forall n$$

$$\Rightarrow \frac{u_2}{u_1} < \lambda, \frac{u_3}{u_2} < \lambda, \frac{u_4}{u_3} < \lambda, \ldots\ldots\ldots\ldots\ldots\ldots, \text{and so on}.$$

Now $S_n = u_1 + u_2 + \ldots\ldots\ldots\ldots\ldots\ldots + u_{n-1} + u_n$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \ldots\ldots\ldots\ldots + \frac{u_n}{u_1}\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \ldots\ldots\ldots\ldots\ldots\right)$$

$$< u_1\left(1 + \lambda + \lambda^2 + \ldots\ldots\ldots\ldots\ldots\ldots\lambda^{n-1}\right) = u_1 \frac{1-\lambda^n}{1-\lambda}$$

$\lim\limits_{n\to\infty} S_n < \lim\limits_{n\to\infty} u_1 \frac{1-\lambda^n}{1-\lambda} = \frac{u_1}{1-\lambda}$, a finite quantity $(\because \lambda < 1) \Rightarrow \sum u_n$ is convergent.

**Case II:** When $k > 1$.

Then, by definition of limit, $\frac{u_{n+1}}{u_n} > 1$

$$\Rightarrow \frac{u_2}{u_1} > 1, \frac{u_3}{u_2} > 1, \frac{u_4}{u_3} > 1, \ldots\ldots\ldots\ldots\ldots\ldots, \text{and so on}.$$

Now $S_n = u_1 + u_2 + \ldots\ldots\ldots\ldots\ldots\ldots + u_{n-1} + u_n$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \ldots\ldots\ldots\ldots + \frac{u_n}{u_1}\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \ldots\ldots\ldots\ldots\ldots\right)$$

$$> u_1\left(1 + 1 + 1 + \ldots\ldots\ldots\ldots\ldots\ldots 1\right) = nu_1$$

$\lim\limits_{n\to\infty} S_n > \lim\limits_{n\to\infty} nu_1 \to +\infty$, $\sum u_n$ is divergent.

This completes the proof.

**Remark:** Ratio test fails when k = 1. So when k = 1, in that case comparison test is helpful in determining the behaviour of the series.

**Counter example:**

For the convergent series, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ : $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n^2}{(1+n)^2} = 1$

Also for the divergent series, $\sum_{n=1}^{\infty} \frac{1}{n}$: $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{(1+n)} = 1$

Thus ratio test cannot be used to distinguish between convergent and divergent series when k = 1

**Now let us examine the behavior of the following infinite series:**

**Q.No.1.:** Discuss the behaviour of infinite series $\sum \frac{n!}{n^n}$.

**Sol.:** Here $u_n = \frac{n!}{n^n}$ and so $u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \times \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1. \quad \text{[since e=2.71828...]}$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

**Q.No.2.:** Discuss the behaviour of infinite series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots$.

**Sol.:** Here $u_n = \frac{n^p}{n!}$ and so $u_{n+1} = \frac{(n+1)^p}{(n+1)!}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p . \frac{1}{(n+1)} = (1)^p.0 = 0 < 1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

**Q.No.3.:** Discuss the behaviour of infinite series $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \ldots\infty \ (x > 0)$.

**Sol.:** Here $u_n = \frac{x^n}{(2n-1)(2n)}$ and so $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)(2)}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}.x = x.$$

**Case 1:** The given series is convergent if $x < 1$.

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When $x = 1$. $u_n = \frac{1}{(2n-1)(2n)}$. Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(2-\frac{1}{n}\right)2} = \frac{1}{4} \text{ (a finite non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x = 1$.

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

**Q.No.4.:** Discuss the behaviour of infinite series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots\ldots\ldots\infty$.

**Sol.:** Here $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and so $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$.

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{2n}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{2n-2}} = \lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}\frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}\right)}x^2 = x^2.$$

**Case 1:** The given series is convergent if $x^2 < 1$.

**Case 2:** The given series is divergent if $x^2 > 1$.

**Case 3:** When $x^2 = 1$. $u_n = \frac{1}{(n+1)\sqrt{n}}$. Take $v_n = \frac{1}{n^{3/2}}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)} = 1 \text{ (a finite non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{3/2}} \approx \sum \frac{1}{n^p}$ ( here p = 3/2 > 1) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x^2 = 1$.

Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

**Q.No.5.:** Discuss the behaviour of infinite series $\sum \frac{x^n}{(2n)!}$, $(x > 0)$.

**Sol.:** Here $u_n = \frac{x^n}{(2n)!}$ and so $u_{n+1} = \frac{x^{n+1}}{(2n+2)!}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+2)!} \times \frac{(2n)!}{x^n} = \lim_{n\to\infty} \frac{x}{(2n+2)(2n+1)} = 0 < 1.$$

Hence $\sum u_n$ is convergent for all x (by D'Alembert's Ratio Test).

**Q.No.6.:** Discuss the behaviour of infinite series $\sum \frac{n!2^n}{n^n}$.

**Sol.:** Here $u_n = \frac{n!2^n}{n^n}$ and so $u_{n+1} = \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!2^n} = \lim_{n\to\infty} 2.\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

**Q.No.7:** Discuss the behaviour of infinite series

$$x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \ldots\ldots. \ (x > 0)$$

**Sol.:** Here $u_n = x^n(\log n)^q$ for $n \ge 2$ and so $u_{n+1} = x^{n+1}[\log(n+1)]^q$.

$$\therefore \frac{u_{n+1}}{u_n} = x\left[\frac{\log(n+1)}{\log n}\right]^q = x\left[\frac{\log n\left(1+\frac{1}{n}\right)}{\log n}\right]^q = x\left[\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right]^q$$

$$=x\left[\frac{\log n+\frac{1}{n}-\frac{\left(\frac{1}{n}\right)^2}{2}+\frac{\left(\frac{1}{n}\right)^3}{3}+\ldots\ldots}{\log n}\right]^q = x\left[\frac{\left(\log n+\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots\right)}{\log n}\right]^q$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}x\left[1+\frac{1}{n\log n}-\frac{1}{2n^2\log n}+\frac{1}{3n^3\log n}-\ldots\ldots\right]^q=x$$

**Case 1:** The given series is convergent if $x<1$.

**Case 2:** The given series is divergent if $x>1$.

**Case 3:** When x = 1. $u_n=(\log n)^q \Rightarrow \lim_{n\to\infty}(\log n)^q \to \infty \neq 0$.

Hence given series is divergent if x = 1. [ by nth term test]

Hence the given series converges if $x<1$ and diverges if $x\geq 1$.

**Q.No.8.:** If $\alpha>0, \beta>0$; Show that the series

$$1+\frac{(\alpha+1)}{(\beta+1)}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}\ldots\ldots$$

**(i)** converges if $\beta>\alpha$ and

**(ii)** diverges if $\beta\leq\alpha$.

**Sol.:** Omitting first term, $u_n=\dfrac{(\alpha+1)(2\alpha+1)(3\alpha+1)\ldots\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)\ldots\ldots(n\beta+1)}$

$$u_{n+1}=\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)\ldots\ldots(n\alpha+1)[(n+1)\alpha+1]}{(\beta+1)(2\beta+1)(3\beta+1)\ldots\ldots(n\beta+1)[(n+1)\beta+1]}$$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n\alpha+\alpha+1}{n\beta+\beta+1}=\lim_{n\to\infty}\frac{n\left(\alpha+\frac{\alpha}{n}+\frac{1}{n}\right)}{n\left(\beta+\frac{\beta}{n}+\frac{1}{n}\right)}=\frac{\alpha}{\beta}$$

**Case 1:** The given series is convergent if $\frac{\alpha}{\beta}<1$ i.e. $\alpha<\beta$ i.e $\beta>\alpha$

**Case 2:** The given series is divergent if $\frac{\alpha}{\beta}>1$ i.e. $\alpha>\beta$ i.e. $\beta<\alpha$

**Case 3:** When $\frac{\alpha}{\beta} = 1$ i.e. when $\alpha = \beta$. In that case, nth term is given by

$$u_n = \frac{(\alpha+1)(2\alpha+1)(3\alpha+1).....(n\alpha+1)}{(\alpha+1)(2\alpha+1)(3\alpha+1).....(n\alpha+1)} = 1$$

$$\therefore \lim_{n\to\infty} u_n = 1 \neq 0.$$

Hence given series is divergent if $\alpha = \beta$. [ by nth term test]

Hence the given series converges if $\beta > \alpha$ and diverges if $\beta \leq \alpha$.

**Q.No.9.:** Discuss the behaviour of infinite series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, where $x > 0$.

**Sol.:** Here $u_n = \frac{x^n}{n!}$ and $\therefore u_{n+1} = \frac{x^{n+1}}{(n+1)!}$.

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$.

Hence the given series $\sum u_n$ is convergent. [by D'Alembert's ratio test]

**Q.No.10.:** Discuss the behaviour of infinite series $\sum_{n=1}^{\infty} \frac{n+1}{n^3}.x^n$, where $x > 0$.

**Sol.:** Here $u_n = \frac{n+1}{n^3}.x^n$ and $\therefore u_{n+1} = \frac{n+2}{(n+1)^3}.x^{n+1}$.

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+2}{(n+1)^3}.x^{n+1} \times \frac{n^3}{n+1}.\frac{1}{x^n} = x$.

**Case 1:** The given series is convergent if $x < 1$. [by D'Alembert's ratio test]

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When x = 1. $= u_n = \frac{n+1}{n^3}$. Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^3\left(1+\frac{1}{n}\right)}{n^3} = 1 \text{ (a finite non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$ ( here $p = 2 > 1$ ) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x = 1$.

Hence the given series $\sum u_n$ converges if $x \le 1$ and diverges if $x > 1$.

**Q.No.11.:** Discuss the behaviour of infinite series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

**Sol.:** Here $u_n = \frac{n}{3^n}$ and $\therefore u_{n+1} = \frac{n+1}{3^{n+1}}$.

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+1}{3^{n+1}} \times \frac{3^n}{n} = \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right) = \frac{1}{3} < 1$.

Hence the given series $\sum u_n$ is convergent. [by D'Alembert's ratio test]

**Q.No.12.:** Discuss the behaviour of infinite series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

**Sol.:** Here $u_n = \frac{n^2}{3^n}$ and $\therefore u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$.

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2} = \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{3} < 1$.

Hence the given series $\sum u_n$ is convergent. [by D'Alembert's ratio test].

**Q.No.13.:** Discuss the behaviour of infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$.

**Sol.:** Here $u_n = \frac{1}{n}$. The given series is divergent (here $p = 1$). [by p-series test]

Or

[2nd method]

Let $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots \quad .= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots$ .

$\because \left(\frac{1}{3} + \frac{1}{4}\right) > \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}, \ \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = \frac{4}{8} = \frac{1}{2} \ldots \ldots$ and so on.

$$\therefore S > 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots\right) \to \infty .$$

The series, on the RHS (ignoring the Ist term) is a geometric series with common ratio unity, hence divergent, by geometric series test.

Hence, the given series is divergent.

**Q.No.14.:** Discuss the behaviour of infinite series $\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+......+\frac{1}{n!}+.....$

**Sol.:** Here $u_n=\frac{1}{n!}$ and $\therefore u_{n+1}=\frac{1}{(n+1)!}$.

Now $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{n!}{(n+1)!}=\lim\limits_{n\to\infty}\frac{1}{(n+1)}=0<1$.

Hence the given series $\sum u_n$ is convergent. [by D'Alembert's ratio test]

**Q.No.15.:** Discuss the behaviour of infinite series

$$\left(\frac{2-1}{3-1}\right)^{\frac{1}{2}}+\left(\frac{2^2-1}{3^2-1}\right)^{\frac{1}{2}}+......+\left(\frac{2^n-1}{3^n-1}\right)^{\frac{1}{2}}+.......$$

**Sol.:** Here $u_n=\sqrt{\frac{2^n-1}{3^n-1}}$ and $\therefore u_{n+1}=\sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}}$.

Now $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}}\times\sqrt{\frac{3^n-1}{2^n-1}}$

$$=\lim_{n\to\infty}\frac{\sqrt{2^n}}{\sqrt{3^n}}\sqrt{\frac{2-\frac{1}{2^n}}{3-\frac{1}{3^n}}}\times\frac{\sqrt{3^n}}{\sqrt{2^n}}\sqrt{\frac{1-\frac{1}{3^n}}{1-\frac{1}{2^n}}}=\sqrt{\frac{2}{3}}<1.$$

Hence, the given series $\sum u_n$ is convergent. [by D'Alembert's ratio test]**.**

**Q.No.16.:** Discuss the behaviour of infinite series $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+...\infty\ (x>0)$**.**

**Sol.:** Here $u_n=\frac{x^n}{n(n+1)}$ and so $u_{n+1}=\frac{x^{n+1}}{(n+1)(n+2)}$.

$$\therefore\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)(n+2)}\times\frac{n(n+1)}{x^n}=\lim_{n\to\infty}x.\frac{n}{(n+2)}=\lim_{n\to\infty}x.\frac{1}{\left(1+\frac{2}{n}\right)}=x.$$

**Case 1:** The given series is convergent if $x<1$.

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When $x = 1$. $u_n = \dfrac{1}{n(n+1)}$. Take $v_n = \dfrac{1}{n^2}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = 1 \text{ (a finite non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \dfrac{1}{n^2} \approx \sum \dfrac{1}{n^p}$ ( here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x = 1$.

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

**Q.No.17.:** Discuss the behaviour of infinite series $1 + \dfrac{2}{5}x + \dfrac{6}{9}x^2 + \ldots. + \dfrac{2^n - 2}{2^n + 1}x^{n-1} + \ldots$

($x > 0$).

**Sol.:** Here $u_n = \dfrac{2^n - 2}{2^n + 1}x^{n-1}$ and so $u_{n+1} = \dfrac{2^{n+1} - 2}{2^{n+1} + 1}x^n$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n+1} - 2}{2^{n+1} + 1}x^n \times \frac{2^n + 1}{2^n - 2}.\frac{1}{x^{n-1}} = \lim_{n\to\infty} \frac{2^n\left(2 - \frac{2}{2^n}\right)}{2^n\left(2 + \frac{1}{2^n}\right)}.\frac{2^n\left(1 + \frac{1}{2^n}\right)}{2^n\left(1 - \frac{2}{2^n}\right)}.x = x.$$

**Case 1:** The given series is convergent if $x < 1$.

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When $x = 1$, then $u_n = \dfrac{2^n - 2}{2^n + 1}$.

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n - 2}{2^n + 1} = \lim_{n\to\infty} \frac{2^n\left(1 - \frac{2}{2^n}\right)}{2^n\left(1 + \frac{1}{2^n}\right)} = 1 \ne 0$$

Hence the given series is divergent when $x = 1$ .[by nth term test]

Hence the given series converges if $x < 1$ diverges if $x \ge 1$.

**Q.No.18.:** Discuss the behaviour of infinite series $\sum_{n=2}^{\infty}\left[\sqrt{\left(n^4+1\right)}-\sqrt{\left(n^4-1\right)}\right]x^n$, $(x>0)$.

**Sol.:** Here $u_n\left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]x^n$. Rationalising, we get

$$u_n=\left[\sqrt{\left(n^4+1\right)}-\sqrt{\left(n^4-1\right)}\right]\times\frac{\sqrt{\left(n^4+1\right)}+\sqrt{\left(n^4-1\right)}}{\sqrt{\left(n^4+1\right)}+\sqrt{\left(n^4-1\right)}}.x^n=\frac{2}{\sqrt{\left(n^4+1\right)}+\sqrt{\left(n^4-1\right)}}.x^n.$$

and so $u_{n+1}=\dfrac{2}{\sqrt{(n+1)^4+1}+\sqrt{(n+1)^4-1}}.x^{n+1}$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{(n+1)^4+1}+\sqrt{(n+1)^4-1}}.x=x$$

**Case 1:** The given series is convergent if $x<1$.

**Case 2:** The given series is divergent if $x>1$.

**Case 3:** When x = 1. Then $u_n=\dfrac{2}{\sqrt{\left(n^4+1\right)}+\sqrt{\left(n^4-1\right)}}$

Take $v_n=\dfrac{1}{n^2}$. $\therefore \lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=\lim\limits_{n\to\infty}\dfrac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}=\dfrac{2}{2}=1$ (a non-zero, finite number)

.

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n=\sum\dfrac{1}{n^2}$ (here $p=2>1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

Hence the given series converges if $x\le 1$ diverges if $x>1$.

**Q.No.19.:** Discuss the behaviour of infinite series $\sum\left[(n+1)^{1/3}-n^{1/3}\right]x^n$, $(x>0)$.

**Sol.:** Here $u_n=\left[(n+1)^{1/3}-n^{1/3}\right]x^n=n^{1/3}\left[\left(1+\dfrac{1}{n}\right)^{1/3}-1\right]x^n$

$$= n^{1/3}\left[\left(1+\frac{1}{3}.\frac{1}{n}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^2}+\ldots\right)-1\right]x^n$$

$$\Rightarrow u_n = n^{1/3}\left[\frac{1}{3n}-\frac{1}{9n^2}+\ldots\ldots\right]x^n = n^{2/3}\left[\frac{1}{3}-\frac{1}{9n}+\ldots\ldots\right]x^n$$

and so $u_{n+1} = (n+1)^{2/3}\left[\frac{1}{3}-\frac{1}{9(n+1)}+\ldots\ldots\right]x^{n+1}$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n^{2/3}\left(1+\frac{1}{n}\right)\left[\frac{1}{3}-\frac{1}{9(n+1)}+\ldots\right]}{n^{2/3}\left[\frac{1}{3}-\frac{1}{9n}+\ldots\ldots\right]}.x = x$$

**Case 1:** The given series is convergent if $x < 1$.

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When x = 1. Then

$$u_n = (n+1)^{1/3} - n^{1/3} = n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3}-1\right] = n^{1/3}\left[\left(1+\frac{1}{3}.\frac{1}{n}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^2}+\ldots\ldots\right)-1\right]$$

$\Rightarrow u_n = n^{1/3}\left[\frac{1}{3n}-\frac{1}{9n^2}+\ldots\ldots\right]$. Take $v_n = \frac{1}{n^{2/3}}$.

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{9n}+\ldots\ldots\right) = \frac{1}{3}$ (a non-zero, finite number) .

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words , both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{2/3}}$ (here p = 2/3 <1) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

Hence the given series converges if $x < 1$ diverges if $x \geq 1$.

**Q.No.20.:** Discuss the behaviour of infinite series $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$, $(x > 0)$.

**Sol.:** Here $u_n = \dfrac{x^n}{n(n+1)}$ and $\therefore u_{n+1} = \dfrac{x^{n+1}}{(n+1)(n+2)}$.

Now $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{x^{n+1}}{(n+1)(n+2)}\times\dfrac{n(n+1)}{x^n} = \lim\limits_{n\to\infty}\dfrac{n}{n+2}.x = \lim\limits_{n\to\infty}\dfrac{1}{\left(1+\dfrac{2}{n}\right)}x = x$.

**Case 1:** The given series is convergent if $x < 1$.

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When x = 1. Then $u_n = \dfrac{1}{n(n+1)}$

Let $v_n = \dfrac{1}{n^2}$ . $\therefore \lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = \lim\limits_{n\to\infty}\dfrac{1}{\left(1+\dfrac{1}{n}\right)} = 1$ (a finite, non-zero number) .

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \dfrac{1}{n^2} \approx \sum \dfrac{1}{n^p}$ ( here p = 2 > 1) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when x = 1.

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

**Q.No.21.:** Discuss the behaviour of infinite series $\dfrac{1}{2\sqrt{1}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + .........\infty$.

**Sol.:** Here $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$ and so $u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{n+1}}$ .

$\therefore \lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{x^{2n}}{(n+2)\sqrt{n+1}}\times\dfrac{(n+1)\sqrt{n}}{x^{2n-2}} = \lim\limits_{n\to\infty}\dfrac{n\left(1+\dfrac{1}{n}\right)}{n\left(1+\dfrac{2}{n}\right)}\dfrac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\dfrac{1}{n}}\right)}x^2 = x^2$.

**Case 1:** The given series is convergent if $x^2 < 1$.

**Case 2:** The given series is divergent if $x^2 > 1$.

**Case 3:** When $x^2 = 1$. $u_n = \dfrac{1}{(n+1)\sqrt{n}}$ . Take $v_n = \dfrac{1}{n^{3/2}}$.

$\therefore \lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = 1$ (a finite non-zero number) .

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{3/2}} \approx \sum \frac{1}{n^p}$ (here $p = \frac{3}{2} > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x^2 = 1$ .

Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

**Q.No.22.:** Discuss the behaviour of infinite series $\frac{2}{1^2}x + \frac{3}{2^2}x^2 \frac{4}{3^2}x^3 + \frac{5}{4^2}x^4 + ......\infty$,

$(x > 0)$.

**Sol.:** Here $u_n = \frac{n+1}{n^2}.x^n$ and $\therefore u_{n+1} = \frac{n+2}{(n+1)^2}.x^{n+1}$.

Now $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n+2}{(n+1)^2}.x^{n+1} \times \frac{n^2}{n+1}.\frac{1}{x^n} = x$ .

**Case 1:** The given series is convergent if $x < 1$. [by D'Alembert's ratio test]

**Case 2**: The given series is divergent if $x > 1$.

**Case 3**: When $x = 1$. $u_n = \frac{n+1}{n^2}$ . Take $v_n = \frac{1}{n}$.

$\therefore \lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)}{n^2} = 1$ (a finite non-zero number) .

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n} \approx \sum \frac{1}{n^p}$ (here $p = 1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent when $x = 1$.

Hence the given series $\sum u_n$ converges if $x < 1$ and diverges if $x \ge 1$.

**Q.No.23.:** Discuss the behaviour of infinite series $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+.....,(x>0)$**.**

**Sol.:** Here $u_n=\frac{x^n}{n}$ and $\therefore u_{n+1}=\frac{x^{n+1}}{(n+1)}$.

$$\text{Now } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)}\times\frac{n}{x^n}=\lim_{n\to\infty}\frac{n}{n+1}.x=x.$$

**Case 1:** The given series is convergent if $x<1$. [by D'Alembert's ratio test]

**Case 2:** The given series is divergent if $x>1$.

**Case 3:** When $x=1$

Then $u_n=\frac{1}{n}$. But $\sum u_n=\sum\frac{1}{n}$ (here $p=1$) is divergent. [by p-series test]

Thus $\sum u_n$ is also divergent when $x=1$.

Hence the given series $\sum u_n$ converges if $x<1$ and diverges if $x\geq 1$**.**

**Q.No.24.:** Discuss the behaviour of infinite series $\frac{1}{1!}+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+.....$

**Sol.:** Here $u_n=\frac{2n-1}{n!}$ and $\therefore u_{n+1}=\frac{2n+1}{(n+1)!}$.

$$\text{Now } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{2n+1}{(n+1)!}\times\frac{n!}{2n-1}=\lim_{n\to\infty}\frac{n\left(2+\frac{1}{n}\right)}{n\left(1+\frac{1}{n}\right)(2n-1)}=0<1.$$

Hence the given series $\sum u_n$ is convergent.

**Q.No.25.:** Discuss the behaviour of infinite series by the expansion of $e^x$.

**Sol.:** Now $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+......$

Here $u_n=\frac{x^n}{n!}$ and $\therefore u_{n+1}=\frac{x^{n+1}}{(n+1)!}$. (by ignoring the first term)

$$\text{Now } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n}=\lim_{n\to\infty}\frac{x}{n+1}=0<1.$$

Hence the given series $\sum u_n$ is convergent. [by D'Alembert's ratio test].

**Q.No.26.:** Discuss the behaviour of infinite series $\sum \frac{n^3+1}{n^n+1}$.

**Sol.:** Here $u_n = \frac{n^3+1}{n^n+1}$ and $\therefore u_{n+1} = \frac{(n+1)^3+1}{(n+1)^{n+1}+1}$. Now

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)^3+1}{(n+1)^{n+1}+1}\times\frac{n^n+1}{n^3+1} = \lim_{n\to\infty}\frac{n^3\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}{n^n\left[(n+1)\left(1+\frac{1}{n}\right)^n+\frac{1}{n^n}\right]}\times\frac{n^n\left(1+\frac{1}{n^n}\right)}{n^3\left(1+\frac{1}{n^3}\right)} = 0 < 1$$

Hence the given series $\sum u_n$ is convergent. [by D'Alembert's ratio test].

**Q.No.27.:** Discuss the behaviour of infinite series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}}x^n$, $(x > 0)$.

**Sol.:** Here $u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}}.x^n$ and $\therefore u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}}.x^{n+1}$.

Now $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}}x^{n+1}\times\frac{\sqrt{n^2+1}}{\sqrt{n}}.\frac{1}{x^n} = x$.

**Case 1:** The given series is convergent if $x < 1$. [by D'Alembert's ratio test]

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When x = 1. Then $u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}}$. Take $v_n = \frac{1}{\sqrt{n}}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n^2+1}}.\frac{\sqrt{n}}{1} = 1 \text{ (a finite non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{1/2}} \approx \sum \frac{1}{n^p}$ (here $p = \frac{1}{2} < 1$) is divergent. [by p-series test]

Thus $\sum u_n$ is also divergent when x = 1.

Hence the given series $\sum u_n$ converges if $x < 1$ and diverges if $x \geq 1$.

**Q.No.28.:** Examine the behaviour of the infinite series $\sum_{n=1}^{\infty}\frac{n!}{3^n}$.

**Sol.:** Here $u_n = \frac{n!}{3^n}$ and $\therefore u_{n+1} = \frac{(n+1)!}{3^{n+1}}$.

Now $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)!}{3^{n+1}}\times\frac{3^n}{n!} = \lim_{n\to\infty}\frac{1}{3}(n+1) \to \infty$.

Hence the given series $\sum u_n$ is divergent. [by D'Alembert's ratio test]

**Q.No.29.:** Find whether the series $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+......, (x>0)$.

is convergent or divergent for positive values of $x$.

**Sol.:** Here $u_n = \frac{n+1}{n}.x^{n-1}$ and $u_{n+1} = \frac{n+2}{n+1}.x^n$

Now $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n+2}{n+1}x^n \times \frac{n}{n+1}.\frac{1}{x^{n-1}} = \lim_{n\to\infty}\frac{n^2\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}.x = x$.

**Case 1:** The given series is convergent if $x<1$. [by D'Alembert's ratio test]

**Case 2:** The given series is divergent if $x>1$.

**Case 3:** When x = 1, then $u_n = \frac{n+1}{n}$, Take $v_n = \frac{1}{n^0} = 1$.

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n} = 1$ (a non-zero, finite number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum\frac{1}{n^0} \approx \sum\frac{1}{n^p}$ ( here $p=0<1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent, when x = 1.

Hence the given series $\sum u_n$ converges if $x<1$ and diverges if $x\ge 1$.

**Q.No.30.** Examine the behaviour of the infinite series

$$\frac{1}{3}+\frac{x}{36}+\frac{x^2}{243}+\frac{x^3}{1296}+\ldots\ldots+\frac{x^{n-1}}{3^n.n^2}+\ldots \quad (x>0).$$

**Sol.:** Here $u_n = \dfrac{x^{n-1}}{3^n.n^2}$ and $u_{n+1} = \dfrac{x^n}{3^{n+1}.(n+1)^2}$.

Now $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{x^n}{3^{n+1}.(n+1)^2}\times\dfrac{3^n.n^2}{x^{n-1}} = \lim\limits_{n\to\infty}\dfrac{n^2.x}{3n^2\left(1+\frac{1}{n}\right)^2} = \dfrac{x}{3}$.

**Case 1:** The given series is convergent if $\frac{x}{3}<1$ i.e. if $x<3$ [by D'Alembert's ratio test]

**Case 2:** The given series is divergent if $\frac{x}{3}>1$ i.e. $x>3$.

**Case 3:** When x = 3, then $u_n = \dfrac{3^{n-1}}{3^n.n^2} = \dfrac{1}{3n^2}$. Take $v_n = \dfrac{1}{n^2}$

$\therefore \lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = \lim\limits_{n\to\infty}\dfrac{1}{3n^2}\times\dfrac{n^2}{1} = \dfrac{1}{3}$ (a non-zero, finite number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum\dfrac{1}{n^2}$ (here $p=2>1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent, when x = 3.

Hence the given series converges if $x\le 3$ diverges if $x>3$.

**Q.No.31.:** Discuss the behaviour of infinite series $x+2x^2+3x^3+4x^4+\ldots\ldots..\infty$

$(x>0)$.

**Sol.:** Here $u_n = nx^n$ and so $u_{n+1} = (n+1)x^{n+1}$.

$\therefore \lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{(n+1)x^{n+1}}{nx^n}.x = \lim\limits_{n\to\infty}\dfrac{(n+1)}{n}.x = \lim\limits_{n\to\infty}\dfrac{n\left(1+\frac{1}{n}\right)}{n}.x = x$.

**Case 1:** The given series is convergent if $x<1$.

**Case 2:** The given series is divergent if $x>1$.

**Case 3:** When x = 1. $u_n = n$.

$\sum u_n = 1 + 2 + 3 + 4 + \dots\dots\infty$.

Here $S_n = 1 + 2 + 3 + 4 + \dots\dots + n = \frac{n(n+1)}{2}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} \to \infty.$$

Hence $\sum u_n$ is also divergent when x = 1 .

Hence the given series converges if $x < 1$ and diverges if $x \ge 1$.

**Q.No.32.:** Examine the behaviour of the infinite series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \dots\dots + \frac{x^n}{n^2+1} + \dots\dots\dots\infty \quad . (x > 0).$$

**Sol.:** Here $u_n = \frac{x^n}{n^2+1}$ and so $u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$ .

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(x+1)^2+1} . \frac{n^2+1}{x^n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n^2}\right)}{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}} . x = x$$

**Case 1:** The given series is convergent if $x < 1$.

**Case 2:** The given series is divergent if $x > 1$.

**Case 3:** When x = 1. This test fails.

$u_n = \frac{1}{n^2+1}$ . Let $v_n = \frac{1}{n^2}$ .

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n^2}\right)} = 1 \text{ (a finite, non-zero number)} .$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$ ( here p = 2 > 1) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when x = 1.

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

**Q.No.33.:** Examine the behaviour of the infinite series $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+......\infty$ **.**

$(x>0)$.

**Sol.:** Here $u_n=\frac{n}{1+2^n}$ and so $u_{n+1}=\frac{n+1}{1+2^{n+1}}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{n}{1+2^n}\cdot\frac{1+2^{n+1}}{n+1}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}\cdot\frac{\frac{1}{2^n}+2}{\frac{1}{2^n}+1}=2>1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test)**.**

**Q.No.34.:** Examine the behaviour of the infinite series $\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ **.**

**Sol.:** Let $S=\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)=\sum_{n=1}^{\infty}\frac{n^2}{2^n}+\sum_{n=1}^{\infty}\frac{1}{n^2}=S_1+S_2$.

Where $S_1=\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ and $S_2=\sum_{n=1}^{\infty}\frac{1}{n^2}$.

The series $S_2$ is convergent by p-series test.

For $S_1$, $u_n=\frac{n^2}{2^n}$ and $u_{n+1}=\frac{(n+1)^2}{2^{n+1}}$.

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{2^n}{n^2}\cdot\frac{(n+1)^2}{2^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2}=\frac{1}{2}<1.$$

Hence the series $S_1$ is also converges.

So the series $S=S_1+S_2$ is also convergent series.

**Q.No.35.:** Examine the behaviour of the infinite series $\sum_{n=1}^{\infty}\frac{n^3+a}{2^n+a}$ **.**

**Sol.:** $u_n=\frac{n^3+a}{2^n+a}$ and $u_{n+1}=\frac{(n+1)^3+a}{2^{n+1}+a}$**.**

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^3+a}{2^{n+1}+a}\cdot\frac{2^n+a}{n^3+a}=\lim_{n\to\infty}\left[\frac{\left\{\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}\right\}}{\left(2+\frac{a}{2^n}\right)}\cdot\frac{\left(1+\frac{a}{2^n}\right)}{\left(1+\frac{a}{n^3}\right)}\right]=\frac{1}{2}<1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test).

**Q.No.36.:** Examine the behaviour of the infinite series $\sum_{n=1}^{\infty}\frac{n^3-n+1}{n!}$ .

**Sol.:** Here $u_n=\frac{n^3-n+1}{n!}$ and $u_{n+1}=\frac{(n+1)^3-n}{(n+1)!}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^3-n}{(n+1)!}\cdot\frac{n!}{n^3-n+1}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^3-\frac{1}{n^2}}{(n+1)\left(1-\frac{1}{n^2}+\frac{1}{n^3}\right)}=0<1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test).

**Q.No.37.:** Examine the behaviour of the infinite series $\frac{2}{1}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ldots\ldots\ldots\infty$ .

**Sol.:** Here $u_n=\frac{2.5.8.11\ldots\ldots\ldots(3n-1)}{1.5.9.13\ldots\ldots\ldots\ldots(4n-3)}$ and $u_{n+1}=\frac{2.5.8.11\ldots\ldots\ldots\ldots(3n-1)(3n+2)}{1.5.9.13\ldots\ldots\ldots\ldots(4n-3)(4n+1)}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{3n+2}{4n+1}=\lim_{n\to\infty}\frac{3+\frac{2}{n}}{4+\frac{1}{n}}=\frac{3}{4}<1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test).

**Q.No.38.:** Examine the behaviour of the infinite series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots\ldots\ldots\infty\ .$$

**Sol.:** Here $u_n=\left(\frac{1.2.3\ldots\ldots\ldots n}{3.5.7\ldots\ldots\ldots\ldots(2n-1)}\right)^2$ and $u_{n+1}=\left(\frac{1.2.3\ldots\ldots\ldots n.(n+1)}{3.5.7\ldots\ldots\ldots\ldots(2n-1)(2n+1)}\right)^2$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left[\frac{n+1}{2n+1}\right]^2=\lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right]^2=\frac{1}{4}<1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test).

**Q.No.39.:** Examine the behaviour of the infinite series $\frac{4}{18}+\frac{4.12}{18.27}+\frac{4.12.20}{18.27.36}+........\infty$ **.**

**Sol.:** Here $u_n=\frac{4.12.20.........(8n-4)}{18.27.36...........(9n+9)}$ and $u_{n+1}=\frac{4.12.20.........(8n-4)(8n+4)}{18.27.36...........(9n+9)(9n+18)}$.

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{8n+4}{9n+18}=\lim_{n\to\infty}\frac{8+\frac{4}{n}}{9+\frac{18}{n}}=\frac{8}{9}<1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test).

**Q.No.40.:** Examine the behaviour of the infinite series $\sum_{n=1}^{\infty}\frac{1}{x^n+x^{-n}}$**.**

**or**

Examine the behaviour of the infinite series

$$\frac{x^1}{x^2+1}+\frac{x^2}{x^4+1}+\frac{x^3}{x^6+1}+\frac{x^4}{x^8+1}+..............\infty.$$

**Sol.:** Here $u_n=\frac{1}{x^n+x^{-n}}=\frac{1}{x^n+\frac{1}{x^n}}=\frac{x^n}{x^{2n}+1}$ and so $u_{n+1}=\frac{x^{n+1}}{x^{2n+2}+1}$ .

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{x^{2n+2}+1}\times\frac{x^{2n}+1}{x^n}=\lim_{n\to\infty}\frac{x^{2n}\left(1+\frac{1}{x^n}\right)}{x^{2n+2}\left(1+\frac{1}{x^{2n+2}}\right)}x=\lim_{n\to\infty}\frac{\left(1+\frac{1}{x^n}\right)}{x\left(1+\frac{1}{x^{2n+2}}\right)}.$$

**Case 1:** When $x>1$, then $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{x^n}\right)}{x\left(1+\frac{1}{x^{2n+2}}\right)}=\frac{1}{x}<1.$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test).

**Case 2:** When $x < 1$, then

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{x^{2n+2}+1} \times \frac{x^{2n}+1}{x^n} = x < 1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's ratio test).

**Case 3:** When $x = 1$, then

$u_n = \dfrac{1^n}{1^{2n}+1}$. In this case series becomes

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots\ldots\ldots\ldots$$

This series is geometric series whose first term is $\frac{1}{2}$ and common ratio is unity. Hence by geometric series test the series is divergent when $x = 1$.

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## Let's summarize

- **D'Alembert's Ratio Test**

# Home Assignments

# Thank you

## NEXT LECTURE

**5. Cauchy's Root Test (or Radical Test)**

**6. Cauchy's Integral Test**

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# $4^{th}$ Topic

# Infinite Series

**[Cauchy's Root Test (or Radical Test) and Cauchy's Integral Test]**

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**(30 Solved problems and 00 Home assignment)**



## 5. Cauchy's Root Test (Radical Test):

**Statement:** If $\sum u_n$ be a positive term series s.t. from and after some particular

term, $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = k$, then $\sum u_n$

**(i)** converges if $k < 1$ and

**(ii)** diverges if $k > 1$.

**Remarks:**

**(i)** This test is useful when power of every factor of $u_n$ is n or a multiple of n and $u_n$ does not involve $n!$.

**(ii)** Cauchy's root test fails when $k = 1$ and in that case p-series or comparison test may be used.

**e.g.** Let $u_n = \frac{1}{n^p} = n^{-p}$. $\therefore \lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(n^{-p}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(n^{\frac{1}{n}}\right)^{-p} = 1^{-p} = 1$.

Here Cauchy's root test fails because $k = 1$.

But we know that $\sum u_n = \sum \frac{1}{n^p}$ is convergent if $p > 1$ and (ii) divergent if $p \le 1$.

**Now let us examine the behavior of the following infinite series by using this test:**

**Q.No.1.:** Test the convergence of the infinite series $\sum \frac{1}{n^n}$.

**Sol.:** Here $u_n = \frac{1}{n^n}$. Then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$.

Hence by Cauchy's root test, the given series is convergent.

**Q.No.2.:** Test the convergence of the infinite series $\sum \frac{1}{(\log n)^n}$.

**Sol.:** Here $u_n = \frac{1}{(\log n)^n}$. Then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$. $\left[\because \lim_{x\to\infty} \log x = -\infty\right]$.

Hence by Cauchy's root test, the given series is convergent.

**Q.No.3.:** Test the convergence of the infinite series $\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

**Sol.:** Here $u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$. Then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$.

Hence by Cauchy's root test, the given series is convergent.

**Q.No.4.:** Discuss the behaviour of the infinite series $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$.

**Sol.:** Here $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$. Then $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}\right] = \frac{1}{e} < 1$.

Hence by Cauchy's root test, the given series is convergent.

**Q.No.5.:** Discuss the behaviour of the infinite series $\sum \frac{[(n+1)x]^n}{n^{n+1}}$.

**Sol.:** Here $u_n = \frac{[(n+1)x]^n}{n^{n+1}}$.

Then $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\frac{[(n+1)x]^n}{n^n.n}\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\frac{[(n+1)x]}{n.n^{\frac{1}{n}}}\right] = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right).\frac{x}{n^{\frac{1}{n}}} = x.$

Hence by Cauchy's root test, the given series is convergent if $x < 1$ and divergent if $x > 1$ and this test fails when $x = 1$.

When $x = 1$, then $u_n = \frac{[(n+1)]^n}{n^{n+1}}$. Take $v_n = \frac{1}{n}$.

$\therefore \lim\limits_{n\to\infty}\frac{u_n}{v_n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (a finite non-zero number) .

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n}$ (here $p = 1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent when $x = 1$. Hence $\sum u_n$ converges if $x < 1$ and diverges if $x \geq 1$.

**Q.No.6.:** Discuss the behaviour of the infinite series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots\ldots\ldots .$$

**Sol.:** In this case, let us neglect the first term of the series. Then $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$.

Then $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)^n x^n\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right]x = x.$

Hence by Cauchy's root test, the given series is convergent if $x<1$ and divergent if $x>1$ and this test fails when $x=1$. Now when $x=1$, then

$$u_n=\left(\frac{n+1}{n+2}\right)^n \therefore \lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}=\frac{e}{e^2}=\frac{1}{e}\neq 0.$$

Hence $\sum u_n$ is also divergent. $\left[\begin{array}{l}\text{by nth term test i.e. if } \lim\limits_{n\to\infty}u_n\neq 0,\\ \text{then the infinite series is said to be divergent}\end{array}\right]$

Hence $\sum u_n$ (i) converges if $x<1$ and (ii) diverges if $x\geq 1$.

**Q.No.7.:** Test the convergence of the infinite series $\sum\frac{(n-\log n)^n}{2^n.n^n}$.

**Sol.:** Here $u_n=\frac{(n-\log n)^n}{2^n.n^n}$. Then

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=\lim_{n\to\infty}\left[\frac{(n-\log n)^n}{2^n.n^n}\right]^{\frac{1}{n}}=\lim_{n\to\infty}\left[\frac{(n-\log n)}{2.n}\right]=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{\log n}{n}\right)=\frac{1}{2}(1-0)=\frac{1}{2}<1$$

Hence by Cauchy's root test, the given series is convergent.

**Q.No.8.:** Discuss the behaviour of the infinite series $\sum\left(1+\frac{1}{n}\right)^{-n^2}$.

**Sol.:** Here $u_n=\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$. Then $\lim\limits_{n\to\infty}u_n^{\frac{1}{n}}=\lim\limits_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1$.

Hence by Cauchy's root test, the given series is convergent.

**Q.No.9.:** Discuss the behaviour of the infinite series whose nth terms are

**(i)** $\left(1-\frac{1}{n}\right)^{n^2}$ and **(ii)** $\left(1+\frac{1}{n}\right)^{n^2}$.

**Sol.:** (i) Here $u_n=\left(1-\frac{1}{n}\right)^{n^2}$.

Then $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim\limits_{n\to\infty}\frac{1}{\left(1+\frac{1}{-n}\right)^{-n}} = \lim\limits_{t\to\infty}\frac{1}{\left(1+\frac{1}{t}\right)^t} = \frac{1}{e} < 1$. [where t = -n]

Hence by Cauchy's root test, the given series is convergent.

(ii) Here $u_n = \left(1+\frac{1}{n}\right)^{n^2}$.

Then $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n.n}\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$.

Hence by Cauchy's root test, the given series is divergent.

**Q.No.10.:** Prove that $\frac{1+2}{2.1} + \left(\frac{2+2}{2.2}\right)^2 + \left(\frac{3+2}{2.3}\right)^3 + \ldots$ is convergent**.**

**Sol.:** Here $u_n = \left(\frac{n+2}{2n}\right)^n$. Then $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\left(\frac{n+2}{2n}\right)^n\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\frac{n+2}{2n}\right) = \frac{1}{2} < 1$.

Hence by Cauchy's root test, the given series is convergent.

**Q.No.11.:** Discuss the behaviour of the infinite series $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\ldots$ where $x > 0$ **.**

**Sol.:** Here $u_n = \frac{x^n}{(n+1)^n}$. [by omitting first term]

Then $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\frac{x^n}{(n+1)^n}\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\left(\frac{x}{n+1}\right)^n\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\frac{x}{n+1}\right) = 0 < 1$.

Hence by Cauchy's root test, the given series is convergent.

**Q.No.12.:** Discuss the behaviour of the infinite series ies $\sum \frac{2^{3n}}{3^{2n}}$ **.**

**Sol.:** Here $u_n = \frac{2^{3n}}{3^{2n}}$.

Then $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\dfrac{2^{3n}}{3^{2n}}\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\left(\dfrac{2^3}{3^2}\right)^n\right]^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\dfrac{8}{9}\right) = \dfrac{8}{9} < 1.$

Hence by Cauchy's root test, the given series is convergent.

**Q.No.13.:** Discuss the behaviour of the infinite series $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \dfrac{4}{2^4} + \ldots$ **.**

**Sol.:** Here $u_n = \dfrac{n}{2^n}$. Then $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left(\dfrac{n}{2^n}\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{n^{\frac{1}{n}}}{2} = \dfrac{1}{2} < 1.$ $\because \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

Hence by Cauchy's root test, the given series is convergent.

**Q.No.14.:** Discuss the behaviour of the infinite series $\sum\limits_{n=1}^{\infty}\left(\dfrac{n}{n+1}\right)^{n^2}.x^n$ **.**

**Sol.:** Here $u_n = \left(\dfrac{n}{n+1}\right)^{n^2}.x^n$ .

Then $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = \lim\limits_{n\to\infty}\left[\left(\dfrac{n}{n+1}\right)^{n^2}.x^n\right]^{\frac{1}{n}}$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^n.x\right] = \lim_{n\to\infty}\left[\left\{\frac{1}{\left(1+\frac{1}{n}\right)^n}.x\right\}\right] = \frac{x}{e} < 1.$$

Hence by Cauchy's root test,

the given series is convergent if $\dfrac{x}{e} < 1$ and divergent if $\dfrac{x}{e} > 1$

i.e. The given series is convergent if $x < e$ and divergent if $x > e$

and this test fails when $x = e$. Now, when $x = e$, then

$$u_n = \left(\frac{n}{n+1}\right)^{n^2}.e^n \quad \therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{e^n}{\left(1+\frac{1}{n}\right)^{n.n}} = \lim_{n\to\infty}\frac{e^n}{e^n} = 1 \neq 0.$$

Hence $\sum u_n$ is also divergent.(By nth term test i.e. An infinite series $\sum u_n$ is said to be divergent if $\lim\limits_{n\to\infty} u_n \neq 0$).

Hence $\sum u_n$ (i) converges if $x < e$ and (ii) diverges if $x \geq e$.

**Q.No.15.:** Discuss the behaviour of the infinite series $\sum\limits_{n=1}^{\infty} \frac{[(n+1)x]^n}{n^n+1}$.

**Sol.:** Here $u_n = \frac{[(n+1)x]^n}{n^n+1}$.

Then $$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\frac{[(n+1)x]^n}{(n^n+1)}\right]^{1/n} = \lim_{n\to\infty}\frac{[(n+1)x]}{(n^n+1)^{1/n}} = \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n^n}\right)^{1/n}}\right]x = x.$$

$$\left[\because \lim_{n\to\infty} n^n = \infty\right]$$

Hence by Cauchy's root test, the given series is convergent if $x < 1$ and divergent if $x > 1$ and this test fails when $x = 1$.

When $x = 1$, then $u_n = \frac{[(n+1)]^n}{n^n+1}$.

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n\left(1+\frac{1}{n^n}\right)} = e \neq 0. \left[\because \lim_{n\to\infty} n^n = \infty\right]$$

Hence $\sum u_n$ is also divergent. $\left[\begin{array}{l}\text{by nth term test i.e. if } \lim\limits_{n\to\infty} u_n \neq 0, \\ \text{then the infinite series is said to be divergent}\end{array}\right]$

Hence $\sum u_n$ converges if $x < 1$ and diverges if $x \geq 1$.

**Q.No.16.:** Discuss the behaviour of the infinite series $\sum\limits_{n=1}^{\infty} \frac{e^n}{n+1}$.

**Sol.:** Here $u_n = \frac{e^n}{n+1}$.

Then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left[\frac{e^n}{n+1}\right]^{\frac{1}{n}} = \lim_{n\to\infty}\frac{e}{(n+1)^{\frac{1}{n}}} = e > 1.$ $\quad \because \lim_{n\to\infty}(n+1)^{\frac{1}{n}} = 1$

Hence by Cauchy's root test, the given series is divergent**.**

**Q.No.17.:** Discuss the behaviour of the infinite series $\sum\frac{(\sqrt{2}-1)^n}{n^2+1}$ **.**

**Sol.:** Here $u_n = \frac{(\sqrt{2}-1)^n}{n^2+1}$ . Then

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{(\sqrt{2}-1)^n}{n^2+1}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{\sqrt{2}-1}{(n^2+1)^{\frac{1}{n}}} = \sqrt{2}-1 = (1.4142-1) < 1. \quad \left[\because \lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} = 1\right]$$

Hence by Cauchy's root test, the given series is convergent**.**

**Q.No.18.:** Discuss the behaviour of the infinite series $\sum\frac{x^n}{n^n}$ **.**

**Sol.:** Here $u_n = \frac{x^n}{n^n}$. Then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{x^n}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{x}{n} = 0 < 1.$

Hence by Cauchy's root test, the given series is convergent**.**

**Q.No.19.:** Discuss the behaviour of the infinite series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \ldots$$

**Sol.:** Here $u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$ . Then

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left\{\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right\}^{-n}\right]^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{n}\right)\right]^{-1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n-1\right]^{-1} = 1.(e-1)^{-1}$$

$$= \frac{1}{e-1} < 1 \ .$$

Hence by Cauchy's root test, the given series is convergent.

## 6. Cauchy's Integral Test:

**Statement:** A positive term series f(1) + f(2) + f(3)+... ... ...+ f(n) +......, where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^\infty f(x)dx$ is finite or infinite.

**or**

If for $x \geq 1$, f(x) is a non-negative, monotonic decreasing function of x such that $f(n) = u_n$ for all positive integral values of n, then the series $\sum u_n$ and the integral $\int_1^\infty f(x)dx$ converge or diverge together.

**Q.No.1.:** Apply Cauchy's integral test to discuss the behaviour of the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}, p > 0.$$

**Sol.:** Here $u_n = \frac{1}{n^p} = f(n)$. Changing n to x, we get $f(x) = \frac{1}{x^p}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall\ x \geq 1$.

$\therefore$ Cauchy's integral test is applicable.

**Case 1:** When $p \neq 1$, then $I_n = \int_1^n f(x)dx = \int_1^n \frac{1}{x^p}dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^n = \frac{1}{1-p}\left[n^{1-p}-1\right]$ .

**(a)** If $p > 1$; then $\lim_{n\to\infty} I_n = \frac{1}{1-p}\left[\lim_{n\to\infty} n^{1-p}-1\right] = \frac{1}{1-p}[0-1] = \frac{1}{p-1}$ (finite)

$\Rightarrow \int_1^\infty f(x)dx$ is convergent, hence by Cauchy's integral test the given series is convergent .

**(b)** If $p < 1$; then $\lim\limits_{n\to\infty} I_n = \dfrac{1}{1-p}\left[\lim\limits_{n\to\infty} n^{1-p} - 1\right] = \dfrac{1}{1-p}[\infty - 1] = \infty$.

$\Rightarrow \int\limits_1^\infty f(x)dx$ diverges to $\infty$, hence by Cauchy's integral test the given series is divergent.

**Case 2:** When $p = 1$. Putting $p = 1$ in (1), we get $f(x) = \dfrac{1}{x}$.

Then $I_n = \int\limits_1^n f(x)dx = \int\limits_1^n \dfrac{1}{x}dx = [\log x]_1^n = \log n - \log 1 = \log n$.

$\therefore \lim\limits_{n\to\infty} I_n \to \infty$.

$\Rightarrow \int\limits_1^\infty f(x)dx$ diverges to $\infty$.

Hence by Cauchy's integral test, the given series is divergent.

Thus the given series is convergent if $p > 1$ and divergent if $p \le 1$.

This completes the proof.

**Q.No.2.:** Apply Cauchy's integral test to discuss the behaviour of the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}.$$

**Sol.:** Here $u_n = \dfrac{1}{n \log n} = f(n)$. Changing n to x, we get $f(x) = \dfrac{1}{x \log x}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \ge 2$.

$\therefore$ Cauchy's integral test is applicable.

Then $I_n = \int\limits_2^n f(x)dx = \int\limits_2^n \dfrac{1}{x \log x}dx = \int_2^n \dfrac{\frac{1}{x}}{\log x}dx = [\log(\log x)]_2^n = \log(\log n) - \log(\log 2)$

$$\therefore \lim_{n\to\infty} I_n = \lim_{n\to\infty}[\log(\log n) - \log(\log 2)] \to \infty - \log(\log 2)$$

$\therefore \lim\limits_{n\to\infty} I_n \to \infty$. $\left[\because \int \dfrac{f'(x)}{f(x)}dx = \log f(x)\right]$

$\Rightarrow \int\limits_1^\infty f(x)dx$ diverges to $\infty$. Hence by Cauchy's integral test, the given series is divergent.

**Q.No.3:** State the Cauchy's integral test and use it to test the behaviour of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}, \quad \alpha \geq 0.$$

**Sol.: Statement:** A positive term series $f(1) + f(2) + f(3) + \dots \dots \dots + f(n) + \dots\dots$,

where $f(n)$ decreases as n increases, converges or diverges according as

the integral $\int_1^{\infty} f(x)dx$ is finite or infinite.

Now the given series is $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}, \quad \alpha \geq 0,$

Here $\frac{1}{n(\log n)^{\alpha}} = f(x)$. Changing n into x, we get $f(x) = \frac{1}{x(\log x)^{\alpha}}$

$\Rightarrow f(x)$ is positive and monotonically decreasing $\forall x \geq 2$.

$\therefore$ Cauchy's integral test is applicable.

**Case1.:** If $\alpha \neq 1$

Then $I_n = \int_2^n f(x)dx = \int_2^n \frac{1}{x(\log x)^{\alpha}} dx = \int_2^n (\log x)^{-\alpha}\left(\frac{1}{x}\right)dx$

$$= \left[\frac{(\log x)^{-\alpha+1}}{-\alpha+1}\right]_2^n = \frac{1}{1-\alpha}\left[(\log n)^{1-\alpha} - (\log 2)^{1-\alpha}\right].$$

**(i)** If $\alpha > 1$, then taking limit on both sides, we get

$$\lim_{n\to\infty} I_n = \frac{1}{1-\alpha}\left[0 - (\log 2)^{1-\alpha}\right] = \text{finite}. \quad \left[\begin{array}{l}\because \alpha > 1 \Rightarrow (1-\alpha) \text{ is negative} \\ \text{and} (\infty)^{-ve} = 0\end{array}\right]$$

$\Rightarrow \int_1^{\infty} f(x)dx$ is convergent and $\therefore$ the series is convergent by Cauchy's integral test.

**(ii)** If $\alpha < 1$, then taking limit on both sides, we get

$$\lim_{n\to\infty} I_n = \frac{1}{1-\alpha}\left[\infty - (\log 2)^{1-\alpha}\right] \to \infty \quad \left[\begin{array}{l}\because \alpha < 1 \Rightarrow (1-\alpha) \text{ is positive} \\ \text{and} \lim\limits_{n\to\infty} (\log n)^{-\alpha} \to \infty\end{array}\right]$$

$\Rightarrow \int_1^{\infty} f(x)dx$ diverges to $\infty$ and $\therefore$ the series is divergent by Cauchy's integral test.

**Case 2.:** If $\alpha = 1$, then

$u_n = \dfrac{1}{n\log n} = f(n)$. Changing n to x, we get $f(x) = \dfrac{1}{x\log x}$ .... .... .... (1).

Here $f(x)$ is positive and monotonically decreasing $\forall x \le 2$. Then

$$I_n = \int_2^n f(x)dx = \int_2^n \frac{1}{x\log x}dx = \int_2^n \frac{\frac{1}{x}}{\log x} = [\log(\log x)]_2^n = \log(\log n) - \log(\log 2).$$

$$\therefore \lim_{n\to\infty} I_n = \lim_{n\to\infty}[\log(\log n) - \log(\log 2)] \to \infty - \log(\log 2)$$

$$\therefore \lim_{n\to\infty} I_n \to \infty. \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$\Rightarrow \int_1^\infty f(x)dx$ diverges to $\infty$, hence by Cauchy's integral test the given series is divergent.

**Q.No.4.:** Discuss the behaviour of the infinite series: $\sum \dfrac{1}{n^2+1}$.

**Sol.:** Here $u_n = \dfrac{1}{n^2+1} = f(n)$. Changing n into x, we get $f(x) = \dfrac{1}{x^2+1}$.

For $x \ge 1$, f(x) is positive and monotonic decreasing.

$\therefore$ Cauchy's integral test is applicable.

$$\text{Now } \int_1^\infty f(x)dx = \int_1^\infty \frac{dx}{x^2+1} = \left[\tan^{-1}x\right]_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} = \text{finite}$$

$\Rightarrow \int_1^\infty f(x)dx$ converges and hence by integral test, $\sum u_n$ also converges.

**Q.No.5.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum \frac{1}{2n+3}.$$

**Sol.:** Here $u_n = \dfrac{1}{2n+3} = f(n)$. Changing n to x, we get $f(x) = \dfrac{1}{2x+3}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \ge 1$. Then

$$I_n = \int_1^n f(x)dx = \int_1^n \frac{1}{2x+3}dx = \left[\frac{\log(2x+3)}{2}\right]_1^n = \frac{\log(2n+3) - \log 5}{2}.$$

$\therefore \lim_{n\to\infty} I_n \to \infty$.

$\Rightarrow \{I_n\}$ diverges to $\infty$, hence by Cauchy's integral test the given series is divergent**.**

**Q.No.6.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum \frac{1}{n(n+1)}.$$

**Sol.:** Here $u_n = \dfrac{1}{n(n+1)} = f(n)$. Changing n to x, we get $f(x) = \dfrac{1}{x(x+1)}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

$$I_n = \int_1^n f(x)dx = \int_1^n \frac{1}{x(x+1)}dx = \int_1^n \left[\frac{1}{x} - \frac{1}{x+1}\right]dx = [\log x - \log(x+1)]_1^n = \left[\log\left(\frac{x}{x+1}\right)\right]_1^n$$

$$= \log\left(\frac{n}{n+1}\right) - \log\frac{1}{2}$$

$\therefore \lim_{n\to\infty} I_n = \log 2$.

$\Rightarrow \{I_n\}$ converges to log2 , hence by Cauchy's integral test the given series is convergent**.**

**Q.No.7.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum \frac{1}{\sqrt{n}}.$$

**Sol.:** Here $u_n = \dfrac{1}{\sqrt{n}} = f(n)$. Changing n to x, we get $f(x) = \dfrac{1}{\sqrt{x}}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

$$I_n = \int_1^n f(x)dx = \int_1^n \frac{1}{\sqrt{x}}dx = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^n = \left[2\sqrt{x}\right]_1^n = 2\sqrt{n} - 2.$$

$\therefore \lim_{n\to\infty} I_n \to \infty$.

$\Rightarrow \{I_n\}$ diverges to $\infty$, hence by Cauchy's integral test the given series is divergent**.**

**Q.No.8.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum \frac{2n^3}{n^4+3}.$$

**Sol.:** Here $u_n = \frac{2n^3}{n^4+3} = f(n)$. Changing n to x, we get $f(x) = \frac{2x^3}{x^4+3}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

$$I_n = \int_1^n f(x)dx = \int_1^n \frac{2x^3}{x^4+3}dx = \left[\frac{1}{2}\log(x^4+3)\right]_1^n = \frac{1}{2}\log\left(n^4+3\right) - \frac{1}{2}\log 4$$

$$= \frac{1}{2}\log\left(\frac{n^4+3}{4}\right).$$

$\therefore \lim_{n\to\infty} I_n \to \infty$.

$\Rightarrow \{I_n\}$ diverges to $\infty$, hence by Cauchy's integral test the given series is divergent.

**Q.No.9.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum \frac{n}{(n^2+1)^2}.$$

**Sol.:** Here $u_n = \frac{n}{(n^2+1)^2} = f(n)$. Changing n to x, we get $f(x) = \frac{x}{(x^2+1)^2}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

$$I_n = \int_1^n f(x)dx = \int_1^n \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\int_1^n \frac{2x}{(x^2+1)^2}dx$$

Let $x^2+1 = t \Rightarrow 2xdx = dt$.

As $x \to 1$ we have $t \to 2$ and as $x \to n$ we have $t \to n^2+1$.

$$\therefore I_n = \frac{1}{2}\int_2^{n^2+1} \frac{dt}{t^2} = \frac{1}{2}\left[-\frac{1}{t}\right]_2^{n^2+1} = \frac{1}{2}\left[-\frac{1}{n^2+1}+\frac{1}{2}\right].$$

$\therefore \lim_{n\to\infty} I_n = \frac{1}{4}$.

$\Rightarrow \{I_n\}$ converges to $\frac{1}{4}$, hence by Cauchy's integral test the given series is convergent.

**Q.No.10.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum \frac{n}{n\sqrt{n^2-1}}.$$

**Sol.:** Here $u_n = \frac{n}{n\sqrt{n^2-1}} = f(n)$. Changing n to x, we get $f(x) = \frac{x}{x\sqrt{x^2-1}}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

$$I = \int_1^\infty f(x)dx = \int_1^\infty \frac{x}{(x^2+1)^{1/2}}dx$$

Let $x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta d\theta$.

As $x \to 1$ we have $\theta \to 0$ and as $x \to \infty$ we have $\theta \to \frac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

$\Rightarrow$ I converges to $\frac{\pi}{2}$, hence by Cauchy's integral test the given series is convergent**.**

**Q.No.11.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum ne^{-n^2}.$$

**Sol.:** Here $u_n = ne^{-n^2} = f(n)$. Changing n to x, we get $f(x) = xe^{-x^2}$ .... .... .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

Let $xe^{-x^2} = a \ \therefore \frac{da}{dx} = e^{-x^2}(-2x) = -2a$.

Also when x = 1, we have $a = \frac{1}{e}$ and when $x = \infty$, we have a = 0.

$$I = \int_1^\infty f(x)dx = \int_{1/e}^{0} \frac{a}{-2a}da = \int_0^{1/e} \frac{da}{2} = \frac{1}{2}[a] = \frac{1}{2e}.$$

$\Rightarrow$ I converges to $\frac{1}{2e}$, hence by Cauchy's integral test the given series is convergent**.**

**Q.No.12.:** Apply Cauchy's integral test to discuss the behaviour of the series

$$\sum_{n=3}^{\infty} \frac{1}{n\log n\left[\log(\log n)\right]^p}.$$

**Sol.:** Here $u_n = \dfrac{1}{n \log n [\log(\log n)]^p} = f(n)$.

Changing n to x, we get $f(x) = \dfrac{1}{x \log x [\log(\log x)]^p}$. (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 3$. Then

$$I = \int_3^{\infty} \frac{1}{x \log x [\log(\log x)]^p} dx . \quad \text{(i)}$$

Put $\log(\log x) = y \quad \therefore \dfrac{1}{x \log x} dx = dy$.

At $x = 3$, then $y = \log(\log 3)$.

At $x \to \infty$, then $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \log(\log x) \to \infty$.

Putting in (i), we get

$$I = \int_{\log(\log 3)}^{\infty} \frac{1}{y^p} dy = \int_{\log(\log 3)}^{\infty} y^{-p} dy = \left[ \frac{y^{-p+1}}{-p+1} \right]_{\log(\log 3)}^{\infty} . \quad \text{(ii)}$$

(i) At $p = 1$. Then $I = \displaystyle\int_{\log(\log 3)}^{\infty} \frac{dy}{y^p} = [\log y]_{\log(\log 3)}^{\infty} \to \infty$.

Thus, the given series is divergent.

(ii) If $p < 1$. $I = \left[ \dfrac{y^{-p+1}}{-p+1} \right]_{\log(\log 3)}^{\infty} \to \infty \therefore$ Given series is divergent.

(iii) If $p > 1$, assumes some finite quantity. $\therefore$ Given series is convergent.

## Let's summarize

- **Cauchy's Root Test (or Radical Test)**
- **Cauchy's Integral Test**

**Thank you**

# NEXT LECTURE

## HIGHER RATIO TESTS

7. Raabe's Test

8. Logarithmic Test

9. Gauss's Test

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# $5^{th}$ Topic

# Infinite Series

## HIGHER RATIO TESTS

**[Raabe's Test, Logarithmic Test, Gauss's Test]**

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**(Last updated on 26-07-2017)**

**(14 Solved problems and 00 Home assignment)**

**7. Raabe's Test**

**8. Logarithmic Test**

**9. Gauss's Test**

## 7. RAABE'S TEST*:

**Statement:** If $\sum u_n$ be a positive term series s.t. from and after some particular term, $\lim\limits_{n\to\infty} n\left[\dfrac{u_n}{u_{n+1}}-1\right]=k$ or $\lim\limits_{n\to\infty} n\left[1-\left|\dfrac{u_{n+1}}{u_n}\right|\right]=k$,

then $\sum u_n$ **(i)** converges if $k>1$ and

**(ii)** diverges if $k<1$.

**Remarks:** **(i)** Raabe's test fails when $k=1$.

**(ii)** This test is very useful when D'Alembert's ratio test fails.

*** Joseph Ludwing Raabe (1801-1859) was born in Zuosch. He was the first Mathematician who proved the various delicate tests of convergence.**

# 8. LOGARITHMIC TEST:

**Statement:** If $\sum u_n$ be a positive term series s.t. from and after some particular term, $\lim_{n\to\infty}\left[n\log\frac{u_n}{u_{n+1}}\right] = k$,

then $\sum u_n$ **(i)** converges if $k > 1$ and

**(ii)** diverges if $k < 1$.

**Remarks:** **(i)** Logarithmic test fails when $k = 1$.

**(ii)** This test is very useful when D'Alembert's ratio test fails.

# 9. GAUSS'S TEST:

**Statement:** If $\sum u_n$ be a positive term series s.t. from and after some particular term,

$$\frac{u_n}{u_{n+1}} = 1 + \frac{k}{n} + \frac{\{\alpha_n\}}{n^p} \text{ or } \left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{k}{n} + \frac{\{\alpha_n\}}{n^2},$$

where k is independent of n, $p > 1$,

$\{\alpha_n\}$ is a bounded sequence when $n \to \infty$,

then $\sum u_n$ **(i)** converges if $k > 1$ and

**(ii)** diverges if $k \le 1$.

**Remark:** **(i)** This test is very useful when Raabe's test fails.

## Imp. Remarks:

(i) If $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ involves e, then we can apply **Logarithmic test.**

(ii) If $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply **Raabe's or Gauss's test.**

**Now let us examine the behavior of the following infinite series, where we have use RAABE'S TEST:**

**Q.No.1.:** Discuss the behaviour of infinite series $\sum \frac{4.7.10.......(3n+1)}{1.2.3.......n} x^n$.

**Sol.:** Here $\frac{u_n}{u_{n+1}} = \frac{4.7.......(3n+1)}{1.2.......n} x^n \times \frac{1.2....n.(n+1)}{4.7....(3n+1).(3n+4)} \frac{1}{x^{n+1}} = \frac{(n+1)}{(3n+4)} \cdot \frac{1}{x}$

$\Rightarrow \lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{3x}$. Hence, by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{1}{3x} > 1$ and (ii) diverges if $\frac{1}{3x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < \frac{1}{3}$ and (ii) diverges if $x > \frac{1}{3}$.

But this test fails if $x = \frac{1}{3}$. Now if $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's or Gauss's test. Now let us try the Raabe's test. When $x = \frac{1}{3}$, then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{1}{n}\right]\left[1+\frac{4}{3n}\right]^{-1} = \left[1+\frac{1}{n}\right]\left[1-\frac{4}{3n}+\frac{16}{9n^2}......\right] = 1-\frac{1}{3n}+\frac{4}{9n^2}-......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[-\frac{1}{3}+\frac{4}{9n}-.....\right] = -\frac{1}{3} < 1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x = \frac{1}{3}$.

Hence the given series $\sum u_n$ (i) converges if $x < \frac{1}{3}$ and (ii) diverges if $x \ge \frac{1}{3}$.

**Q.No.2.:** Discuss the behaviour of infinite series

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+.........\infty .$$

**Sol.:** Here (by omitting first term) $u_n = \frac{3.6.9.....(3n)}{7.10.13.......(3n+4)} x^n$

and $\therefore u_{n+1} = \frac{3.6.9.......(3n)(3n+3)}{7.10.13.......(3n+4)(3n+7)} x^{n+1}$.

Now $\frac{u_n}{u_{n+1}} = \frac{3.6.9.......(3n)}{7.10.13........(3n+4)} x^n \times \frac{7.10.13......(3n+4)(3n+7)}{3.6.9.....(3n).(3n+3)} \frac{1}{x^{n+1}} = \frac{(3n+7)}{(3n+3)} \cdot \frac{1}{x}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{7}{3n}\right)}{\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x} = \frac{1}{x}.$$

Hence by D'Alembert's ratio test:

$\sum u_n$ (i) converges if $\frac{1}{x} > 1$ and (ii) diverges if $\frac{1}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x > 1$.

But this test fails if $x = 1$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test.

When $x = 1$, then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{7}{3n}\right]\left[1+\frac{1}{n}\right]^{-1} = \left[1+\frac{7}{3n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}......\right] = 1+\frac{4}{3n}-\frac{4}{3n^2}-......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[\frac{4}{3}-\frac{4}{3n}-.....\right] = \frac{4}{3} > 1.$$

$\Rightarrow$ By Raabe's test, the given series converges when $x = 1$.

Hence the given series $\sum u_n$ (i) converges if $x \le 1$ and (ii) diverges if $x > 1$.

**Q.No.3.:** Discuss the behaviour of infinite series $\frac{x}{3}+\frac{1.2}{3.5}x^2+\frac{1.2.3}{3.5.7}x^3+......\infty \ (x > 0)$.

**Sol.:** Here $u_n = \frac{1.2.3....n}{3.5.7....(2n+1)}x^n$ and so $u_{n+1} = \frac{1.2.3....n.(n+1)}{3.5.7....(2n+1)(2n+3)}x^{n+1}$

Now $\frac{u_n}{u_{n+1}} = \frac{1.2.3....n}{3.5.7....(2n+1)}x^n \times \frac{3.5.7....(2n+1)(2n+3)}{1.2.3....n.(n+1)}\frac{1}{x^{n+1}} = \frac{(2n+3)}{(n+1)}\frac{1}{x} = \frac{2\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{n}\right)}\frac{1}{x}.$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{2\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{n}\right)}\frac{1}{x} = \frac{2}{x}.$$

Hence by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{2}{x} > 1$ and (ii) diverges if $\frac{2}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < 2$ and (ii) diverges if $x > 2$.

But this test fails if $x = 2$. Now if $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test.

When $x = 2$ then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{3}{2n}\right]\left[1+\frac{1}{n}\right]^{-1} = \left[1+\frac{3}{2n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}......\right] = 1+\frac{1}{2n}-\frac{1}{2n^2}-......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{2n}-.....\right] = \frac{1}{2} < 1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x = 1$.

Hence the given series $\sum u_n$ (i) converges if $x < 2$ and (ii) diverges if $x \geq 2$.

**Q.No.4.:** Discuss the behaviour of infinite series $1 + 3x + 5x^2 + 7x^3 + ....$ $(x > 0)$.

**Sol.:** Here (by omitting first term) $u_n = (2n+1)x^n$ and so $u_{n+1} = (2n+3)x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)x^n}{(2n+3)x^{n+1}} = \frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{3}{2n}\right)}\frac{1}{x} \qquad \therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{3}{2n}\right)}\frac{1}{x} = \frac{1}{x}.$$

Hence by D'Alembert's ratio test

$\sum u_n$ (i)converges if $\frac{1}{x} > 1$ and (ii) diverges if $\frac{1}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x > 1$.

But this test fails if $x = 1$. Now if $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test.

When $x = 1$, then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{1}{2n}\right]\left[1+\frac{3}{2n}\right]^{-1} = \left[1+\frac{1}{2n}\right]\left[1-\frac{3}{2n}+\frac{9}{4n^2}......\right] = 1-\frac{1}{n}+\frac{3}{2n^2}-......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[-1+\frac{3}{2n}-.....\right] = -1 < 1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x = 1$.

Hence the given series $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x \geq 1$.

**or**

When $x = 1$, then $u_n = (2n+1)$

Now $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(2n+1) \to \infty \neq 0$.

Hence by $n^{th}$ term test, this series is divergent when $x = 1$.

**Q.No.5.:** Discuss the behaviour of infinite series

$$1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+...\infty \;(x>0).$$

**Sol.:** Here (by omitting first term) $u_n = \frac{1.3.5.....(2n-1)}{2.4.6.......(2n)}x^n$

and $\therefore u_{n+1} = \frac{1.3.5.......(2n-1)(2n+1)}{2.4.6.......(2n)(2n+2)}x^{n+1}$.

Now $\frac{u_n}{u_{n+1}} = \frac{1.3.5.....(2n-1)}{2.4.6.....(2n)}x^n \times \frac{2.4.6......(2n)(2n+2)}{1.3.5......(2n-1).(2n+1)}\frac{1}{x^{n+1}} = \frac{(2n+2)}{(2n+1)}.\frac{1}{x}$

$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{2n}\right)}.\frac{1}{x} = \frac{1}{x}$. Hence by D'Alembert's ratio test

$\sum u_n$ (i)converges if $\frac{1}{x} > 1$ and (ii) diverges if $\frac{1}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x > 1$.

But this test fails if $x = 1$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test. When $x = 1$ then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{1}{n}\right]\left[1+\frac{1}{2n}\right]^{-1} = \left[1+\frac{1}{n}\right]\left[1-\frac{1}{2n}+\frac{1}{(2n)^2}-...\right] = 1+\frac{1}{2n}-\frac{1}{4n^2}+...$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{4n}-.....\right]=\frac{1}{2}<1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x=1$.

Hence the given series $\sum u_n$ (i) converges if $x<1$ and (ii) diverges if $x\geq 1$.

**Q.No.6.:** Discuss the behaviour of infinite series

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+.........\infty . \qquad (x>0)$$

**Sol.:** Here (by omitting first term) $u_n=\frac{3.6.9.....(3n)}{7.10.13.......(3n+4)}x^n$ and

$$\therefore u_{n+1}=\frac{3.6.9.......(3n)(3n+3)}{7.10.13.......(3n+4)(3n+7)}x^{n+1}.$$

Now $\frac{u_n}{u_{n+1}}=\frac{3.6.9.......(3n)}{7.10.13........(3n+4)}x^n\times\frac{7.10.13......(3n+4)(3n+7)}{3.6.9.....(3n).(3n+3)}\frac{1}{x^{n+1}}=\frac{(3n+7)}{(3n+3)}.\frac{1}{x}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(1+\frac{7}{3n}\right)}{\left(1+\frac{1}{n}\right)}.\frac{1}{x}=\frac{1}{x}.$$

Hence by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{1}{x}>1$ and (ii) diverges if $\frac{1}{x}<1$.

i.e. $\sum u_n$ (i) converges if $x<1$ and (ii) diverges if $x>1$.

But this test fails if $x=1$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test. When $x=1$ then

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{7}{3n}\right]\left[1+\frac{1}{n}\right]^{-1}=\left[1+\frac{7}{3n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}......\right]=1+\frac{4}{3n}-\frac{4}{3n^2}-......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty}\left[\frac{4}{3}-\frac{4}{3n}-.....\right]=\frac{4}{3}>1.$$

$\Rightarrow$ By Raabe's test, the given series converges when $x=1$.

Hence the given series $\sum u_n$ (i) converges if $x\leq 1$ and (ii) diverges if $x>1$.

**Q.No.7.:** Discuss the behaviour of infinite series $\frac{x}{1}+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+...\infty$

$x>1$.

**Sol.:** Here (by omitting first term) $u_n=\frac{1.3.5.....(2n-1)}{2.4.6.......(2n)}.\frac{x^{2n+1}}{(2n+1)}$, $n\geq 1$.

and $\therefore u_{n+1}=\frac{1.3.5.......(2n-1)(2n+1)}{2.4.6.......(2n)(2n+2)}.\frac{x^{2n+3}}{(2n+3)}$. Now

$$\frac{u_n}{u_{n+1}}=\frac{1.3.5.....(2n-1)}{2.4.6.....(2n)}.\frac{x^{2n+1}}{(2n+1)}\times\frac{2.4.6......(2n)(2n+2)}{1.3.5......(2n-1).(2n+1)}.\frac{(2n+3)}{x^{2n+3}}=\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}.\frac{1}{x^2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}.\frac{1}{x^2}=\frac{1}{x^2}.$$

Hence by D'Alembert's ratio test

$\sum u_n$ (i)converges if $\frac{1}{x^2}>1$ and (ii) diverges if $\frac{1}{x^2}<1$.

i.e. $\sum u_n$ (i) converges if $x^2<1$ and (ii) diverges if $x^2>1$.

But this test fails if $x^2=1$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test. When $x^2=1$ then

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{1}{n}\right]\left[1+\frac{3}{2n}\right]\left[1+\frac{1}{2n}\right]^{-2}=\left[1+\frac{1}{n}\right]\left[1+\frac{3}{2n}\right]\left[1-\frac{2}{2n}+(...)\frac{1}{n^2}...\right]=1+\frac{3}{2n}+(...)\frac{1}{n^2}-...$$

$$\Rightarrow \lim_{n\to\infty}n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty}\left[\frac{3}{2}+(...)\frac{1}{n}-.....\right]=\frac{3}{2}>1.$$

$\Rightarrow$ By Raabe's test, the given series converges when $x^2=1$.

Hence the given series $\sum u_n$ (i) converges if $x^2\leq 1$ and (ii) diverges if $x^2>1$.

**Q.No.8.:** Discuss the behaviour of infinite series $1+\frac{(1!)^2}{2!}x^2+\frac{(2!)^2}{4!}x^4+\frac{(3!)^2}{6!}x^6+...\infty$

$x>0$.

**Sol.:** Here (by omitting first term) $u_n = \frac{(n!)^2}{(2n)!}x^{2n}$ and so $u_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!}x^{2n+2}$.

Now

$$\frac{u_n}{u_{n+1}} = \frac{(n!)^2}{(2n)!}x^{2n} \times \frac{(2n+2)!}{[(n+1)!]^2}\frac{1}{x^{2n+2}} = \frac{(2n+2)(2n+1)}{(n+1)^2}.\frac{1}{x^2} = \frac{2(2n+1)}{n+1}.\frac{1}{x^2} = \frac{4\left(1+\frac{1}{2n}\right)}{\left(1+\frac{1}{n}\right)}.\frac{1}{x^2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{4\left(1+\frac{1}{2n}\right)}{\left(1+\frac{1}{n}\right)^1}.\frac{1}{x^2} = \frac{4}{x^2}.$$

Hence by D'Alembert's ratio test

$\sum u_n$ (i)converges if $\frac{4}{x^2} > 1$ and (ii) diverges if $\frac{4}{x^2} < 1$.

i.e. $\sum u_n$ (i) converges if $x^2 < 4$ and (ii) diverges if $x^2 > 4$.

But this test fails if $x^2 = 4$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test. When $x^2 = 4$ then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{1}{2n}\right]\left[1+\frac{1}{n}\right]^{-1} = \left[1+\frac{1}{2n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}......\right] = 1-\frac{1}{2n}+\frac{1}{2n^2}-......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[-\frac{1}{2}+\frac{1}{2n}-.....\right] = -\frac{1}{2} < 1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x^2 = 4$.

Hence the given series $\sum u_n$ (i) converges if $x^2 < 4$ and (ii) diverges if $x^2 \ge 4$.

**Q.No.9.:** Discuss the behaviour of infinite series

$$1+\frac{a}{b}x+\frac{a(a+1)}{b(b+1)}x^2+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3+....\infty \quad (a, b > 0,\ x > 0).$$

**Sol.:** Here (by omitting first term) $u_n = \frac{a(a+1)(a+2)......(a+n-1)}{b(b+1)(b+2)......(b+n-1)}x^n$ and so

$$u_{n+1} = \frac{a(a+1)(a+2)......(a+n-1)(a+n)}{b(b+1)(b+2)......(b+n-1)(b+n)} x^{n+1}.$$

Now $\frac{u_n}{u_{n+1}} = \frac{(b+n)}{(a+n)}\frac{1}{x} = \frac{\left(1+\frac{b}{n}\right)}{\left(1+\frac{a}{n}\right)}\frac{1}{x} \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{b}{n}\right)}{\left(1+\frac{a}{n}\right)}.\frac{1}{x} = \frac{1}{x}.$

Hence by D'Alembert's ratio test, $\sum u_n$ (i) converges if $\frac{1}{x} > 1$ and (ii) diverges if $\frac{1}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x > 1$.

But this test fails if $x = 1$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test.

Now let us try the Raabe's test.

When $x = 1$, then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{b}{n}\right]\left[1+\frac{a}{n}\right]^{-1} = \left[1+\frac{b}{n}\right]\left[1-\frac{a}{n}+\frac{a^2}{n^2}......\right] = 1+(b-a)\frac{1}{n}+(.....)\frac{1}{n^2}+......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[(b-a)+(.....)\frac{1}{n}+.....\right] = (b-a).$$

$\Rightarrow$ By Raabe's test, the given series converges if $(b-a) > 1$ and diverges if $(b-a) < 1$ and Raabe's test fails if $(b-a) = 1$. When $(b-a) = 1$

We get $u_n = \frac{a}{a+n}$ . Take $v_n = \frac{1}{n}$.

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{a}{a+n}\times\frac{n}{1} = \lim_{n\to\infty}\frac{a}{\frac{a}{n}+1} = a$ (a non-zero, finite number) . [$\because a > 0$]

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n}$ (here $p = 1$) is divergent. [by p-series test]

Thus $\sum u_n$ is also divergent, when $(b-a) = 1$.

Hence $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x > 1$.

When $x = 1$, the given series converges if $(b-a) > 1$ and diverges if $(b-a) \le 1$

**Now let us examine the behavior of the following infinite series, where we have use LOGARITHMIC TEST:**

**Q.No.1.:** Discuss the behaviour of infinite series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + .........\infty$.

**Sol.:** Here $\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}x^{n+1}} = \frac{n^n}{(n+1)^n x} = \frac{1}{\left(1+\frac{1}{n}\right)^n}.\frac{1}{x}$ .

$\Rightarrow \lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{ex}$. Hence by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{1}{ex} > 1$ and (ii) diverges if $\frac{1}{ex} < 1$.

i.e. $\sum u_n$ (i) converges if $x < \frac{1}{e}$ and (ii) diverges if $x > \frac{1}{e}$. But this test fails if $x = \frac{1}{e}$.

Now if $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}}$ involves e, then we can apply Logarithmic test.

Now let us try Logarithmic test.

When $x = \frac{1}{e}$, then $\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$$\therefore \log\frac{u_n}{u_{n+1}} = \log_e e - n\log\left(1+\frac{1}{n}\right) = 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - ......\right] = \frac{1}{2n} - \frac{1}{3n^2} + ... \ .$$

Hence $\lim\limits_{n\to\infty}\left[n\log\frac{u_n}{u_{n+1}}\right] = \frac{1}{2} < 1$.

$\Rightarrow$ By Logarithmic test, the given series diverges when $x = \frac{1}{e}$.

Hence the given series $\sum u_n$ (i) converges if $x < \frac{1}{e}$ and (ii) diverges if $x \ge \frac{1}{e}$.

**Q.No.2.:** Discuss the behaviour of infinite series $\frac{(a+x)}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+......\infty$.

$(x>0)$.

**Sol.:** Here $u_n=\frac{(a+nx)^n}{n!}$ and $\therefore u_{n+1}=\frac{[a+(n+1)x]^{n+1}}{(n+1)!}$.

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(a+nx)^n}{n!}\times\frac{(n+1)!}{[a+(n+1)x]^{n+1}}=\lim_{n\to\infty}\frac{n^n\left(x+\frac{a}{n}\right)^n}{n^n\left(1+\frac{1}{n}\right)^n\left(x+\frac{a}{n+1}\right)^{n+1}}=\frac{1}{ex}.$$

Hence by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{1}{ex}>1$ and (ii) diverges if $\frac{1}{ex}<1$.

i.e. $\sum u_n$ (i) converges if $x<\frac{1}{e}$ and (ii) diverges if $x>\frac{1}{e}$. But this test fails if $x=\frac{1}{e}$.

Now if $\frac{u_n}{u_{n+1}}$ involves e, then we can apply Logarithmic test. Now let us try Logarithmic test.

When $x=\frac{1}{e}$, $\frac{u_n}{u_{n+1}}=\frac{\left(\frac{1}{e}+\frac{a}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n\left(\frac{1}{e}+\frac{a}{n+1}\right)^{n+1}}=e\frac{\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{ae}{n+1}\right)^{n+1}}$.

$$\therefore \log\frac{u_n}{u_{n+1}}=\log e+n\log\left(1+\frac{ae}{n}\right)-n\log\left(1+\frac{1}{n}\right)-(n+1)\log\left(1+\frac{ae}{n+1}\right)$$

$$=1+n\left[\frac{ae}{n}-\frac{a^2e^2}{2n^2}+...\right]-n\left[\frac{1}{n}-\frac{1}{2n^2}+...\right]-(n+1)\left[\frac{ae}{n+1}-\frac{a^2e^2}{2(n+1)^2}+......\right]$$

$$=\left[-\frac{a^2e^2}{2}+\frac{1}{2}+\frac{a^2e^2}{2\left(\frac{n+1}{n}\right)}\right].\frac{1}{n}.$$

Hence $\lim_{n\to\infty}\left[n\log\frac{u_n}{u_{n+1}}\right]=\frac{1}{2}<1.$

$\Rightarrow$ By Logarithmic test, the given series diverges when $x = \frac{1}{e}$.

Hence the given series $\sum u_n$ (i) converges if $x < \frac{1}{e}$ and (ii) diverges if $x \geq \frac{1}{e}$.

**Q.No.3.:** Discuss the behaviour of infinite series $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + .....\infty$

$(x > 0)$.

**Sol.:** Here (by omitting first term) $u_n = \frac{n!}{(n+1)^n}x^n$ and $\therefore u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$.

Now $\frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)^n}x^n \times \frac{(n+2)^{n+1}}{(n+1)!}\frac{1}{x^{n+1}} = \left(1 + \frac{1}{n+1}\right)^{n+1}\frac{1}{x}$

$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^{n+1}\frac{1}{x} = \frac{e}{x}$. Hence by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{e}{x} > 1$ and (ii) diverges if $\frac{e}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < e$ and (ii) diverges if $x > e$.

But this test fails if $x = e$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ involve e, then we can apply Logarithmic test.

Now let us try the Logarithmic test.

When $x = e$, then $\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n+1}\right)^{n+1}\frac{1}{e}$

$$n\log\frac{u_n}{u_{n+1}} = n\log\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{e} = n\left[(n+1)\log\left(1 + \frac{1}{n+1}\right) - \log_e e\right]$$

$$= n\left[(n+1)\left\{\frac{1}{n+1} - \frac{1}{2}.\frac{1}{(n+1)^2} + \frac{1}{3}.\frac{1}{(n+1)^3} - ...\right\} - 1\right] = n\left[-\frac{1}{2}.\frac{1}{n+1} + \frac{1}{3}.\frac{1}{(n+1)^2} - ...\right]$$

$$= -\frac{1}{2}\frac{n}{n\left(1 + \frac{1}{n}\right)} + \frac{1}{3}\frac{n}{n^2\left(1 + \frac{1}{n}\right)^2} - ... = -\frac{1}{2}\frac{1}{\left(1 + \frac{1}{n}\right)} + \frac{1}{3}\frac{1}{n\left(1 + \frac{1}{n}\right)^2} - ... .$$

$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = -\frac{1}{2} < 1$.

$\Rightarrow$ By Logarithmic test, the given series diverges when $x = e$.

Hence the given series $\sum u_n$ (i) converges if $x < e$ and (ii) diverges if $x \geq e$.

**Now let us examine the behavior of the following infinite series, where we have use GAUSS'S TEST:**

**Q.No.1.:** Discuss the behaviour of infinite series $\sum \left[\frac{1.3.5...(2n-1)}{2.4.6...(2n)}\right]^2$.

**or**

Examine the behaviour of the series $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + ..................$ .

**Sol.:** Here $u_n = \left[\frac{1.3.5....(2n-1)}{2.4.6...(2n)}\right]^2$ and so $u_{n+1} = \left[\frac{1.3.5....(2n-1)(2n+1)}{2.4.6...(2n)(2n+2)}\right]^2$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left[\frac{2n+2}{2n+1}\right]^2 = \lim_{n\to\infty} \left[\frac{2+\frac{2}{n}}{2+\frac{1}{n}}\right]^2 = 1.$$

This shows that D'Alembert's ratio test fails. Now let us apply Raabe's test.

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{(2n+2)^2}{(2n+1)^2} - 1\right] = \lim_{n\to\infty} \frac{4n^2+3n}{(2n+1)^2} = 1$$

This shows that Raabe's test fails. Now let us apply Gauss's test.

$$\frac{u_n}{u_{n+1}} = \left[\frac{2n+2}{2n+1}\right]^2 = \left[1+\frac{1}{n}\right]^2\left[1+\frac{1}{2n}\right]^{-2} = \left[1+\frac{2}{n}+\frac{1}{n^2}\right]\left[1-\frac{2}{2n}+\frac{3}{4n^2}-\frac{1}{2n^3}+\frac{5}{16n^4}-......\right]$$

$$= 1+\frac{1}{n}-\frac{1}{n^2}\left[\frac{1}{4}-\frac{1}{4n^2}+...........\right].$$

or

$$\text{Also } \frac{u_{n+1}}{u_n} = \left[\frac{2n+1}{2n+2}\right]^2 = \left[1+\frac{1}{2n}\right]^2\left[1+\frac{1}{n}\right]^{-2} = \left[1+\frac{1}{n}+\frac{1}{4n^2}\right]\left[1-\frac{2}{n}+\frac{3}{n^2}-\frac{4}{n^3}+\frac{5}{n^4}......\right]$$

$$= 1-\frac{1}{n}+\frac{5}{4}.\frac{1}{n^2}-\frac{3}{2}.\frac{1}{n^3}-\frac{7}{4}.\frac{1}{n^4}+...... = 1-\frac{1}{n}+\frac{1}{n^2}\left[\frac{5}{4}-\frac{3}{2n}+\frac{7}{4n^2}+......\right]$$

Then by Gauss's test, which states that, if $\sum u_n$ be a positive term series s.t. from and after some particular term $\frac{u_n}{u_{n+1}} = 1 + \frac{k}{n} + \frac{\alpha_n}{n^p}$ or $\left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{k}{n} + \frac{\alpha_n}{n^2}$, where k is independent of n, $p > 1$ $\{\alpha_n\}$ is a bounded sequence when $n \to \infty$, then $\sum u_n$

(i) converges if $k > 1$ and (ii) diverges if $k \le 1$.

Here $k = 1$ and, $\alpha_n = \frac{5}{4} - \frac{3}{2n} + \frac{7}{4n^n} + ......$ and $\alpha_n \to \frac{5}{4}$ as $n \to \infty$.

Thus $\alpha_n$ is bounded as $n \to \infty$.

Hence the given series is divergent.

**Q.No.2.:** Discuss the behaviour of infinite series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + .....\infty.\ (x > 0)$$

**Sol.:** Here (by omitting first term)

$$u_n = \frac{\alpha(\alpha+1)(\alpha+2)...(\alpha+n-1)\beta(\beta+1)(\beta+2)...........(\beta+n-1)}{1.2.3....n.\gamma(\gamma+1)(\gamma+2)......(\gamma+n-1)}x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)(\alpha+2)...(\alpha+n-1)(\alpha+n)\beta(\beta+1)(\beta+2)........(\beta+n-1)(\beta+n)}{1.2.3....n.(n+1)\gamma(\gamma+1)(\gamma+2)......(\gamma+n-1)(\gamma+n)}x^{n+1}$$

Now $$\frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}.\frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}.\frac{1}{x}.$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}.\frac{1}{x} = \frac{1}{x}.$$

Hence by D'Alembert's ratio test, $\sum u_n$ (i) converges if $\frac{1}{x} > 1$ and (ii) diverges if $\frac{1}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x > 1$.

But this test fails if $x = 1$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Gauss's test. When $x = 1$, then

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{1}{n}\right]\left[1+\frac{\gamma}{n}\right]\left[1+\frac{\alpha}{n}\right]^{-1}\left[1+\frac{\beta}{n}\right]^{-1}=\left[1+\frac{1}{n}\right]\left[1+\frac{\gamma}{n}\right]\left[1-\frac{\alpha}{n}+\ldots\right]\left[1-\frac{\beta}{n}+\ldots\ldots\right]$$

$$=1+(1+\gamma-\alpha-\beta)\frac{1}{n}+(\ldots\ldots)\frac{1}{n^2}+\ldots\ldots .$$

Then by Gauss's test, if $\sum u_n$ be a positive term series s.t. from and after some particular term $\frac{u_n}{u_{n+1}}=1+\frac{k}{n}+\frac{\alpha_n}{n^p}$ where k is independent of n, $p>1$, $\{\alpha_n\}$ is a bounded sequence when $n\to\infty$, then $\sum u_n$ (i) converges if $k>1$ and (ii) diverges if $k\le 1$.

Here $k=(1+\gamma-\alpha-\beta)$

then $\sum u_n$ (i) converges if $(1+\gamma-\alpha-\beta)>1$ and (ii) diverges if $(1+\gamma-\alpha-\beta)\le 1$.

i.e. $\sum u_n$ (i) converges if $\gamma>(\alpha+\beta)$ and (ii) diverges if $\gamma\le(\alpha+\beta)$ when $x=1$.

## Let's summarize

- **Raabe's Test**
- **Logarithmic Test**
- **Gauss's Test**

# Thank you

## NEXT LECTURE

### Alternating series

**Alternating Series, Absolute Series, Leibnitz's Rule,**

**Absolutely convergent series, Conditionally convergent series**

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# 6th Topic

# Infinite Series

## Alternating series

[Alternating Series, Absolute Series, Leibnitz's Rule, Absolutely convergent series, Conditionally convergent series]

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(20 Solved problems and 00 Home assignment)

**Alternating Series:** A series in which the terms are alternately positive and negative is called an *alternating series*.

**Absolute Series:** The series obtained from the alternating series by taking each term as positive is known as *absolute series*.

Absolute series of $\sum u_n$ is denoted by $\sum |u_n|$.

**Note:** If we denote an alternating series by $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$,

then absolute series of $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is denoted by $\sum u_n$.

# Leibnitz's Rule

# Leibnitz's Rule

## [Behaviour Test for Alternating Series]

**Statement:** An alternating series $u_1 - u_2 + u_3 - u_4 + .... + (-1)^{n-1}u_n + .... = \sum_{n=1}^{\infty}(-1)^{n-1}u_n$

converges if **(i)** each term is numerically less than its preceding term,

i.e. Mathematically $u_{n+1} < u_n$ for $n \geq 1$, and

**(ii)** $\lim_{n\to\infty} u_n = 0$.

If $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \sum(-1)^{n-1}u_n$ is oscillatory.

**Another form of this statement:**

If $\{u_n\}$ is monotonically decreasing sequence of positive terms and converges to zero, then $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ is convergent.

**Proof:** The given series is $u_1 - u_2 + u_3 - u_4 + .... + (-1)^{n-1}u_n + .... = \sum_{n=1}^{\infty}(-1)^{n-1}u_n$.

Given (i) each term is numerically less than its preceding term,

$$\text{i.e. } u_1 > u_2 > u_3 > ...... > u_n > u_{n+1} > ...... \qquad (1)$$

$$\text{(ii) } \lim_{n\to\infty} u_n = 0. \qquad (2)$$

Now consider the sum of 2n terms. It can be written as

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + ...... + (u_{2n-1} - u_{2n}). \qquad (3)$$

Also $S_{2n} = \{u_1 - (u_2 - u_3) - (u_4 - u_5) - ...... - u_{2n}\} < u_1$. (4)

Also $S_{2n+2} - S_{2n} = (S_{2n} + u_{2n+1} - u_{2n+2}) - S_{2n} = (u_{2n+1} - u_{2n+2}) > 0.$[by (1)] (5)

Since the expressions within the brackets in (3) and (4) are all positive. [by (1)]

Therefore, (3) $\Rightarrow$ The sequence $\{S_{2n}\}$ is positive,

(4) $\Rightarrow$ The sequence $\{S_{2n}\}$ is bounded above and always remains less than $u_1$.

Also (5) $\Rightarrow$ The sequence $\{S_{2n}\}$ is monotonically increasing.

Since we know, every monotonically increasing sequence, which is bounded above, converges. Therefore, the sequence $\{S_{2n}\}$ converges.

Let us suppose $\lim\limits_{n\to\infty} S_{2n} = s$ (finite). (6)

Thus, given alternating series is convergent, if we consider initially sum of even terms.

### To show: The uniqueness of this limit

Since $\lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty}(S_{2n} + u_{2n+1}) = \lim\limits_{n\to\infty} S_{2n} + \lim\limits_{n\to\infty} u_{2n+1} = s + 0 = s$. [by (2) and (6)]

$\left[\text{Since, } \lim\limits_{n\to\infty} u_n = 0, \text{ then by definition of limit, we get } \lim\limits_{n\to\infty} u_{2n} = 0 \Rightarrow \lim\limits_{n\to\infty} u_{2n+1} = 0\right]$

This shows that $\lim\limits_{n\to\infty} S_n$ tends to the same limit, whether n is even or odd.

Hence the given series is convergent.

### To show: The oscillatory case

If $\lim\limits_{n\to\infty} u_n \neq 0$, then $\lim\limits_{n\to\infty} S_{2n} \neq \lim\limits_{n\to\infty} S_{2n+1}$.

$\Rightarrow$ The given series is oscillatory. This completes the proof.

***********************************

**Absolutely convergent series:** If $\sum |u_n|$ is convergent,

then $\sum u_n$ is said to be *absolutely convergent*.

**Conditionally convergent series:** If $\sum u_n$ is convergent and $\sum |u_n|$ is divergent,

then $\sum u_n$ is *conditionally convergent*.

**Theorem: Show that an absolutely convergent series is necessarily convergent but not conversely.**

**Proof:** Let $\sum u_n$ be an absolutely convergent series $\Rightarrow \sum |u_n|$ is convergent.

Also we know $u_1 + u_2 + u_3 + .... + u_n + ... \leq |u_1| + |u_2| + |u_3| + ...... + |u_n| + ...$ .

The series on the R.H.S. is convergent.

Hence, the given series is also convergent.

***The converse of this result is not true.***

Now we can prove this result with the help of the following example:

**Example:** Let us consider this series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+......$

The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{1}{n}$, which is an alternating series.

$\because (n+1) > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n$ .

Thus $u_{n+1} \le u_n$ for $n \ge 1$

i.e. each term is numerically less than its preceding term and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

Also $\sum|u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...... = \sum\frac{1}{n}$.

Hence by p-series test, this absolute series is divergent.

This result shows that a convergent series is not necessarily absolutely convergent.

Hence, an absolutely convergent series is necessarily convergent but not conversely.

**Now let us examine the behavior of the following infinite series using Leibnitz's rule:**

**Q.No.1.:** Discuss the behaviour of the infinite series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.......$

**Sol.:** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{1}{n}$, which is an alternating series.

$\because (n+1) > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n$ .

Thus $u_{n+1} \le u_n$ for $n \ge 1$ i.e. each term is numerically less than its preceding term,

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

**Q.No.2.:** Examine the character of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.n}{2n-1}$.

**Sol.:** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{n}{2n-1}$, which is an alternating series .

Here $u_n = \dfrac{n}{2n-1}$ and $u_{n+1} = \dfrac{n+1}{2n+1}$.

Now $u_n - u_{n+1} = \dfrac{n}{2n-1} - \dfrac{n+1}{2n+1} = \dfrac{1}{4n^2-1} \geq 0$ for $n \geq 1$.

Thus $u_{n+1} \leq u_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{n}{2n-1} = \dfrac{1}{2} \neq 0$.

Hence, by Leibnitz's rule, the given alternating series is oscillatory.

**Q.No.3.:** Discuss the behaviour of the infinite series $\sum\limits_{n=1}^{\infty} \dfrac{\cos n\pi}{n^2+1}$.

**Sol.:** The given series is $\sum\limits_{n=1}^{\infty} \dfrac{\cos n\pi}{n^2+1} = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{n^2+1} = \sum\limits_{n=1}^{\infty} (-1)^n u_n$ which is an alternating series. Now $u_n - u_{n+1} = \dfrac{1}{n^2+1} - \dfrac{1}{(n+1)^2+1} = \dfrac{(2n+1)}{n^4+2n^3+3n^2+2n+2} > 0$ for $n \geq 1$.

Thus $u_{n+1} \leq u_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{n^2+1} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

**Q.No.4.:** Discuss the behaviour of the infinite series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots\ldots\infty \ (0 < x < 1).$$

**Sol.:** The given series is $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{1+x^n} = \sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$, which is an alternating series.

Now $u_n - u_{n+1} = \dfrac{x^n}{1+x^n} - \dfrac{x^{n+1}}{1+x^{n+1}} = \dfrac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} > 0$ for $n \geq 1$. $[\because (0 < x < 1)]$

Thus $u_{n+1} \leq u_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{x^n}{1+x^n} = 0$. $[\because (0 < x < 1)]$.

Hence, by Leibnitz's rule, the given alternating series is convergent**.**

**Q.No.5.:** Discuss the behaviour of the infinite series

$$\left(\frac{1}{2}-\frac{1}{\log 2}\right)-\left(\frac{1}{2}-\frac{1}{\log 3}\right)+\left(\frac{1}{2}-\frac{1}{\log 4}\right)-\left(\frac{1}{2}-\frac{1}{\log 5}\right)+\ldots\ldots\infty.$$

**Sol.:** The given series is $\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{2}-\frac{1}{\log n}\right)=\sum_{n=1}^{\infty}(-1)^n u_n$ which is an alternating series.

Now $u_n-u_{n+1}=\left(\frac{1}{2}-\frac{1}{\log n}\right)-\left(\frac{1}{2}-\frac{1}{\log(n+1)}\right)=\left(\frac{1}{\log(n+1)}-\frac{1}{\log n}\right)<0$ for $n\geq 1$

$\because n+1>n\Rightarrow\log(n+1)>\log n\Rightarrow\frac{1}{\log(n+1)}<\frac{1}{\log n}\Rightarrow\left(\frac{1}{\log(n+1)}-\frac{1}{\log n}\right)<0$.

Thus $u_{n+1}\geq u_n$ for $n\geq 1$ i.e. each term is numerically greater than its preceding term,

and $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{\log n}\right)=\frac{1}{2}\neq 0$.

Hence, by Leibnitz's rule, the given alternating series is oscillatory**.**

**Q.No.6.:** Discuss the behaviour of the infinite series $\frac{1}{6}-\frac{2}{11}+\frac{3}{16}-\frac{4}{21}+\frac{5}{26}-\ldots\ldots\infty$**.**

**Sol.:** The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{5n+1}=\sum_{n=1}^{\infty}(-1)^{n-1}u_n$, which is an alternating series.

Now $u_n-u_{n+1}=\frac{n}{5n+1}-\frac{n+1}{5(n+1)+1}=\frac{-1}{(5n+1)(5n+6)}<0$ for $n\geq 1$.

Thus $u_{n+1}\geq u_n$ for $n\geq 1$ i.e. each term is numerically greater than its preceding term,

and $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{n}{5n+1}=\frac{1}{5}\neq 0$.

Hence, by Leibnitz's rule, the given alternating series is oscillatory**.**

**Q.No.7:** Discuss the behaviour of the infinite series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\ldots$ **.**

**Sol.:** The given series is $\sum(-1)^{n-1}u_n=\sum(-1)^{n-1}\frac{1}{\sqrt{n}}$, which is an alternating series .

$\Rightarrow u_n=\frac{1}{\sqrt{n}}$ and so $u_{n+1}=\frac{1}{\sqrt{n+1}}$ .

$\Rightarrow u_{n+1}<u_n$ for $n\geq 1$. $\left[\because\sqrt{n+1}>\sqrt{n}\Rightarrow\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}\right]$

Thus $u_{n+1} \le u_n$ for $n \ge 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

**Q.No.8:** Discuss the behaviour of the infinite series

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots + (-1)^{n-1}\frac{n}{n+1} + \ldots .$$

**Sol.:** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{n}{n+1}$, which is an alternating series .

$\Rightarrow u_n = \frac{n}{n+1}$ and so $u_{n+1} = \frac{n+1}{n+2}$.

Since $\because u_n - u_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+3} = \frac{-1}{(n+1)(n+3)} < 0$ for $n \ge 1$.

Thus $u_{n+1} > u_n$ for $n \ge 1$ i.e. each term is numerically greater than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \ne 0$.

Hence, by Leibnitz's rule, the given alternating series is not convergent and hence it oscillates.

**Q.No.9:** Discuss the behaviour of the infinite series $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots.\infty$ .

**Sol.:** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{n+1}{n}$, which is an alternating series .

$\Rightarrow u_n = \frac{n+1}{n}$ and so $u_{n+1} = \frac{n+2}{n+1}$.

Since $\because u_n - u_{n+1} = \frac{n+1}{n} - \frac{n+2}{n+1} = \frac{1}{n(n+1)} > 0$ for $n \ge 1$.

Thus $u_{n+1} \le u_n$ for $n \ge 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n+1}{n} = 1 \ne 0$.

Hence, by Leibnitz's rule, the given alternating series is not convergent and hence it oscillates.

**Q.No.10:** Discuss the behaviour of the infinite series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.....\infty$ **.**

**Sol.:** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{1}{n^2}$, which is an alternating series .

$\Rightarrow u_n = \frac{1}{n^2}$ and so $u_{n+1} = \frac{1}{(n+1)^2}$ .

Since $\because (n+1) > n \Rightarrow (n+1)^2 > n^2 \Rightarrow \frac{1}{(n+1)^2} < \frac{1}{n^2} \Rightarrow u_{n+1} < u_n$ for $n \geq 1$..

Thus $u_{n+1} \leq u_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n^2} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

**Q.No.11:** Prove that $\sum_{n=1}^{\infty}(-1)^n.\frac{2^{3n}}{3^{2n}}$ is convergent.

**Sol.:** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{2^{3n}}{3^{2n}}$, which is an alternating series .

$\Rightarrow u_n = \frac{2^{3n}}{3^{2n}}$ and so $u_{n+1} = \frac{2^{3n+3}}{3^{2n+2}}$ . $\therefore \frac{u_{n+1}}{u_n} = \frac{2^{3n}.2^3}{3^{2n}.3^2}\times\frac{3^{2n}}{2^{3n}} = \frac{8}{9} < 1$ for $n \geq 1$.

Thus $u_{n+1} \leq n_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{2^{3n}}{3^{2n}} = \lim_{n\to\infty}\left(\frac{8}{9}\right)^n = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent**.**

**Q.No.12:** Discuss the behaviour of the infinite series

**(i)** $\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}$ **(ii)** $\sum_{n=2}^{\infty}\frac{(-1)^n.\sqrt{n}}{\log n}$ .

**Sol.: (i)** The given series is $\sum(-1)^n u_n = \sum(-1)^n\frac{1}{\log n}$, which is an alternating series .

$\Rightarrow u_n = \frac{1}{\log n}$ and so $u_{n+1} = \frac{1}{\log(n+1)}$ .

Since $\because (n+1) > n \Rightarrow \log(n+1) > \log n \Rightarrow \frac{1}{\log(n+1)} < \frac{1}{\log n} \Rightarrow u_{n+1} < u_n$ for $n \geq 1$..

Thus $u_{n+1} \leq u_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\log n} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

**(ii)** The given series is $\sum(-1)^n u_n = \sum(-1)^n \frac{\sqrt{n}}{\log n}$, which is an alternating series .

Here $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{\log n} = \lim\limits_{n\to\infty} \frac{\frac{1}{2} n^{-\frac{1}{2}}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{1}{2}\sqrt{n} \to \infty \neq 0$ .[by using L'Hospital's rule]

We know that, if either of the two conditions is not satisfied then the alternating series cannot be convergent.

Hence, by Leibnitz's rule, the given alternating series is oscillatory or non-convergent**.**

**Now let us show the following series are absolutely convergent:**

**Remember:**

**Absolutely convergent series:** If $\sum|u_n|$ is convergent,

then $\sum u_n$ is said to be *absolutely convergent.*

**Q.No.1.:**Prove that the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...... = \sum(-1)^{n-1}\frac{1}{n^2}$ is absolutely convergent**.**

**Sol.:** Since we know, if the absolute series $\sum|u_n|$ is convergent then the alternating series $\sum u_n$ is said to be *absolutely convergent*.

Here $\sum|u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...... = \sum\frac{1}{n^2}$.

Hence by p-series test, this absolute series is convergent.

So by def. of absolutely convergent series, the given alternating series is absolutely convergent**.**

**Q.No.2.:** Prove that the series $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\frac{\sin 4x}{4^3}+..........$ converges absolutely.

**Sol.:** The given series is $\sum_{n=1}^{\infty} u_n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n^3}$.

The absolute series of given alternating series is $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\left|\frac{\sin nx}{n^3}\right|$.

Now since $|u_n|=\left|\frac{\sin nx}{n^3}\right|=\frac{|\sin nx|}{n^3}\leq\frac{1}{n^3}\quad\forall n$

and we know that $\sum\frac{1}{n^3}$ is convergent (by p-series test, p>1).

Therefore when higher series is convergent then smaller series is also convergent.

Thus the absolute series $\sum|u_n|$ is convergent.

Hence the given alternating series converges absolutely.

**Q.No.3:** Are the following series absolutely convergent?

**(i)** $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+......$ .

**(ii)** $1-2x+3x^2-4x^3+.....\ \ (x<1)$.

**Sol.: (i)**The given series is $\sum(-1)^{n-1}u_n=\sum(-1)^{n-1}\frac{1}{n\sqrt{n}}$, which is an alternating series.

Since we know, if the absolute series $\sum|u_n|$ is convergent then the alternating series $\sum u_n$ is said to be *absolutely convergent*.

Now the absolute series of the given alternating series is

$$\sum|u_n|=1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+......=\sum\frac{1}{n\sqrt{n}}.$$

Hence by p-series test, this absolute series is convergent.

So by def. of absolutely convergent series, the given alternating series is absolutely convergent.

**(ii)** The given series is $\sum(-1)^n u_n=\sum(-1)^n(n+1)x^n$, which is an alternating series .

Since we know, if the absolute series $\sum|u_n|$ is convergent then the alternating series

$\sum u_n$ is said to be *absolutely convergent*. Now the absolute series of the given alternating series is $\sum |u_n| = 1 + 2x + 3x^2 + 4x^3 + ...... = \sum (n+1)x^n$.

Here $|u_n| = (n+1)x^n$ and so $|u_{n+1}| = (n+2)x^{n+1}$.

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{(n+2)x^{n+1}}{(n+1)x^n} = x < 1 \text{ (given)}.$$

Hence, the given absolute series is convergent.

So by def. of absolutely convergent series, the given alternating series is absolutely convergent.

**Now let us show the following series are conditionally convergent:**

**Remember:**

**Conditionally convergent series:** If $\sum u_n$ is convergent and $\sum |u_n|$ is divergent, then $\sum u_n$ is *conditionally convergent.*

**Q.No.1.:** Prove that the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...... = \sum (-1)^{n-1}\frac{1}{n}$ is conditionally convergent**.**

**Sol.:** Since we know, $\sum u_n$ is convergent and $\sum |u_n|$ is divergent, then $\sum u_n$ is *conditionally convergent.*

**(i)** The given series is $\sum (-1)^{n-1} u_n = \sum (-1)^{n-1}\frac{1}{n}$, which is an alternating series.

$$\because (n+1) > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n .$$

Thus $u_{n+1} \le u_n$ for $n \ge 1$ i.e. each term is numerically less than its preceding term,

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0$.

Hence by Leibnitz's rule, the given alternating series is convergent.

**(ii)** Here $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...... = \sum \frac{1}{n}$.

Hence by p-series test, this absolute series is divergent.

So by def. of conditionally convergent series, the given alternating series is conditionally convergent.

**Q.No.2:** Prove that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ is conditionally convergent**.**

**Sol.:** Since we know, $\sum u_n$ is convergent and $\sum |u_n|$ is divergent, then $\sum u_n$ is *conditionally convergent.*

**(i)** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{n}{n^2+1}$, which is an alternating series .

Here $u_n = \frac{n}{n^2+1}$ and so $u_{n+1} = \frac{n+1}{(n+1)^2+1}$.

$$\therefore u_n - u_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} = \frac{n^2+n-1}{(n^2+1)\left[(n+1)^2+1\right]} > 0 \text{ for } n \ge 1.$$

Thus $u_{n+1} \le u_n$ for $n \ge 1$ i.e. each term is numerically less than its preceding term,

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{n^2+1} = 0$.

Hence by Leibnitz's rule, the given alternating series is convergent.

**(ii)** Here $\sum |u_n| = \sum \frac{n}{n^2+1}$. Take $v_n = \frac{1}{n}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^2}\right)} = 1 \text{ (a non-zero, finite number) .}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n}$ (here $p = 1$) is divergent. [by p-series test]

Hence the given absolute series is also divergent.

So by def. of conditionally convergent series, the given alternating series is conditionally convergent**.**

**Q.No.3.:** Prove that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$ is conditionally convergent.

**Sol.:** Since we know, $\sum u_n$ is convergent and $\sum |u_n|$ is divergent, then $\sum u_n$ is *conditionally convergent.*

**(i)** The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{1}{\sqrt{n}}$, which is an alternating series .

Here $u_n = \frac{1}{\sqrt{n}}$ and so $u_{n+1} = \frac{1}{\sqrt{n+1}}$.

$\because \sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \Rightarrow u_{n+1} < u_n$ for $n \geq 1$..

Thus $u_{n+1} \leq u_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0$.

Hence by Leibnitz's rule, the given alternating series is convergent**.**

**(ii)** Here $\sum |u_n| = \sum \frac{1}{\sqrt{n}}$.

Hence by p-series test, the given series is divergent $\left[\text{here } p = \frac{1}{2}\right]$.

So by def. of conditionally convergent series, the given alternating series is conditionally convergent.

**Now let us examine the behaviour of the some important series:**

**Q.No.1:** Prove that the series $\sum(-1)^{n+1}\left[\frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}\right]$ is divergent, although the terms are alternately positive and negative and $\lim\limits_{n\to\infty} u_n = 0$**.**

**Sol.:** Let $v_n = \frac{1}{\sqrt{n}}$ and $w_n = \frac{(-1)^{n-1}}{n}$.Then the given series become $\sum(-1)^{n+1}[v_n + w_n]$

**To show:** The given series is divergent, although the terms are alternately positive and negative and $\lim\limits_{n\to\infty} u_n = 0$.

Let us examine the behaviour, separately.

Now (i) $\sum(-1)^{n+1}v_n = \sum(-1)^{n+1}\frac{1}{\sqrt{n}}$, which is an alternating series .

$\Rightarrow v_n = \frac{1}{\sqrt{n}}$ and so $v_{n+1} = \frac{1}{\sqrt{n+1}}$ .

$\because \sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \Rightarrow v_{n+1} < v_n$ for $n \geq 1$..

Thus $v_{n+1} \leq v_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Hence by Leibnitz's rule, the given alternating series is convergent.

**(ii)** $\sum(-1)^{n+1} w_n = \sum(-1)^{n+1}(-1)^{n-1}\frac{1}{n} = \sum\frac{1}{n}$

Hence by p-series test, the given series is divergent [here $p = 1$].

The given series

$$\sum(-1)^{n+1}[v_n + w_n] = \sum\left[(-1)^{n+1}v_n + (-1)^{n+1}w_n\right] = \sum(-1)^{n+1}\frac{1}{\sqrt{n}} + \sum\frac{1}{n}$$

= convergent series + divergent series.

Hence the given series $\sum(-1)^{n+1}\left[\frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}\right]$ is divergent, although the terms are alternately positive and negative and $\lim_{n\to\infty} u_n = 0$.

**Q.No.2:** Test whether the series $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \ldots$ is convergent or not**.**

**Sol.:** Since we know, an absolute convergent series is necessarily convergent.

Here $\sum|u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \ldots\ldots = \sum\frac{1}{n^2}$ .

Thus by p-series test, this absolute series is convergent.

Hence the given series is convergent.

# Let's summarize

- Alternating Series
- Absolute Series
- Leibnitz's Rule
- Absolutely convergent series
- Conditionally convergent series

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Thank you

## NEXT TOPIC

**Power series, Interval of convergence**

**Convergence of logarithmic series**

**Convergence of exponential series**

**Convergence of Binomial series**

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# 7th Topic

# Infinite Series

**[Power series, Interval of convergence]**

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## Power Series:

A series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots$ (i)

where the $a_i$'s are independent of x, is called a **power series** in x. Such a series may converge for some or for all values of x.

## Interval of convergence:

In the power series (i), $u_n = a_n x^n$ and $u_{n+1} = a_{n+1} x^{n+1}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \left( \frac{a_{n+1}}{a_n} \right) . x \quad .$$

If $\lim_{n\to\infty} \left( \frac{a_{n+1}}{a_n} \right) = \ell$, then by D'Alembert's ratio test,

the series (i) converges, when $\ell x$ is numerically less than 1.

i.e. when $|x| < \frac{1}{\ell}$ and diverges for all other values.

Thus, the power series (i) has an interval $-\frac{1}{\ell} < x < \frac{1}{\ell}$, within which it converges and diverges for all values of x outside this interval.

Such an interval is called the "***interval of convergence***" of the power series.

**Remarks:**

If $\sum a_n x^n$ has radius of convergence is R (assumed finite), then radius of convergence of $\sum a_n x^{2n}$ is $\sqrt{R}$.

## Convergence of logarithmic series:

**Q.No.:1.:** State with reasons, the values of x for which the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots \text{ converges.}$$

**or**

Show that the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots\infty \text{ is convergent for } -1 < x \le 1.$$

**Sol.:** Here $u_n = (-1)^{n-1}\frac{x^n}{n}$ and $u_{n+1} = (-1)^n \frac{x^{n+1}}{n+1}$.

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[-\frac{n}{n+1}x\right] = -x.\lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}\right] = -x.$$

Then, by D'Alembert's ratio test, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

Now let us examine the series for $x = \pm 1$.

Putting $x = 1$, the series becomes, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots$.

The given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{1}{n}$, which is an alternating series.

$$\because (n+1) > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n.$$

Thus $u_{n+1} \le u_n$ for $n \ge 1$ i.e. each term is numerically less than its preceding term, and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

Putting $x = -1$, the series becomes, $-\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ......\right]$.

The given series becomes $-\sum u_n = -\sum \frac{1}{n}$,

Thus, by p-series test (here p=1), the given series is divergent when $x = -1$.

Hence, the given series converges for $-1 < x \leq 1$.

i.e. Interval of convergence is $(-1,\ 1]$.

**Q.No.:2.:** State with reasons, the values of x for which the series

$$\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + ......\text{ converges.}$$

**Sol.:** Here $u_n = (-1)^{n+1} \frac{x^{n+1}}{n+1}$ and $u_{n+1} = (-1)^{n+2} \frac{x^{n+2}}{n+2}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[-\frac{n+1}{n+2}x\right] = -x.\lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right] = -x .$$

Then, by D'Alembert's ratio test, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

Now let us examine the series for $x = \pm 1$.

Putting x = 1, the series becomes, $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + ......$ .

The given series is $\sum (-1)^{n+1} u_n = \sum (-1)^{n+1} \frac{1}{n+1}$, which is an alternating series.

$$\because (n+2) > (n+1) \Rightarrow \frac{1}{n+2} < \frac{1}{n+1} \Rightarrow u_{n+1} < u_n .$$

Thus $u_{n+1} \leq u_n$ for $n \geq 1$ i.e. each term is numerically less than its preceding term,

and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{n+1} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

Putting $x = -1$, the series becomes, $\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+......\right]$.

The given series is $\sum u_n = \sum \frac{1}{n+1}$,

Here $u_n = \frac{1}{n+1}$.Take $v_n = \frac{1}{n}$. $\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)} = 1$ (a non-zero, finite number) .

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words , both the series behave alike .

But $\sum v_n = \sum \frac{1}{n}$ (here p = 1) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

Hence, the given series converges for $-1 < x \le 1$.

i.e. Interval of convergence is $(-1, 1)$.

## Convergence of exponential series:

**Q.No.:3.:** Show that the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...\infty \text{ is convergent for all values of x.}$$

**Proof:** Here $u_n = \frac{x^{n-1}}{(n-1)!}$ and $\therefore u_{n+1} = \frac{x^n}{n!}$.

$$\text{Now } \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^n}{n!}\times\frac{(n-1)!}{x^{n-1}} = \lim_{n\to\infty}\frac{x}{n} = 0 < 1.$$

Hence, by D'Alembert's ratio test, the given series is convergent for all values of x.

## Convergence of Binomial series:

**Q.No.:4.:** Show that the series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + ......+ \frac{n(n-1)...[n-(r-1)]}{r!}x^r + ......\infty \text{ converges for } |x| < 1.$$

**Proof:** Here $u_r = \frac{n(n-1)...[n-(r-2)]}{(r-1)!}x^{r-1}$ , $r \ge 1$

and $u_{r+1} = \frac{n(n-1)...[n-(r-2)][n-(r-1)]}{r!}x^r$

$$\therefore \lim_{r\to\infty}\frac{u_{r+1}}{u_r} = \lim_{r\to\infty}\left[\frac{n-(r-1)}{r}x\right] = \lim_{r\to\infty}\left[\frac{n+1}{r}-1\right]x = -x \text{ for } r > (n+1)$$

Then, by D'Alembert's ratio test, the series converges for $|x| < 1$.

**Q.No.5:** Discuss the behaviour of the infinite series $\sum \frac{(-1)^n x^n}{n}$ .

**Sol.:** Here $|u_n| = \frac{x^n}{n}$ and so $|u_{n+1}| = \frac{x^{n+1}}{n+1}$.

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{x^{n+1}}{n+1}\times\frac{n}{x^n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}x = x .$$

Hence, the given series is convergent if $|x| < 1$ and divergent if $|x| > 1$.

This test fails when $x = 1$.

When $x = 1$, the given series is $\sum(-1)^{n-1}u_n = \sum(-1)^{n-1}\frac{1}{n}$, which is an alternating series.

Since $(n+1) > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n$ .

Thus $u_{n+1} \le u_n$ for $n \ge 1$ i.e. each term is numerically less than its preceding term,

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0$.

Hence, by Leibnitz's rule, the given alternating series is convergent.

**Q.No.6.:** Prove that the series $\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \ldots\ldots$

converges if $x < 0$ and also if $x > 2$.

or

Determine for what values of *x*, the following series are convergent:

$$\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \ldots\ldots$$

**Sol.:** Here $u_n = \frac{1}{n(1-x)^n}$ and $\therefore u_{n+1} = \frac{1}{(n+1)(1-x)^{n+1}}$.

Now $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\dfrac{1}{(n+1)(1-x)^{n+1}}\times\dfrac{n(1-x)^n}{1}=\lim\limits_{n\to\infty}\dfrac{n}{n\left(1+\dfrac{1}{n}\right)}\cdot\dfrac{1}{1-x}=\dfrac{1}{1-x}$.

Then, by D'Alembert's ratio test, the given series is convergent if $\left|\dfrac{1}{1-x}\right|<1$.

i.e. if $-1<\dfrac{1}{1-x}<1$ $\quad\Rightarrow -1<\dfrac{1}{1-x}$ and $\dfrac{1}{1-x}<1$

$\Rightarrow \dfrac{1}{1-x}>-1$ and $1<1-x$ $\quad\Rightarrow 1-x<-1$ and $0<-x$

$\Rightarrow -x<-2$ and $0<-x$ $\quad\Rightarrow x>2$ and $0>x$

$\Rightarrow x>2$ and $x<0$

i.e. if $x<0$ and also if $x>2$

Test for convergence at the end points x = 0 and x = 2.

For x = 0, the given series reduces to

$$1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}+\ldots.$$

which is a divergent harmonic series with p = 1.

For x = 2, the given series becomes

$$-1+\frac{1}{2}-\frac{1}{3}+\ldots\ldots+\frac{(-1)^n}{n}+\ldots.$$

This alternating series is convergent by Leibnitz's rule.

Thus the given series converges for $x \geq 2$ and $x < 0$.

# Next Topic

# "Differential Calculus"

# Partial Differentiation

# Thank you

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