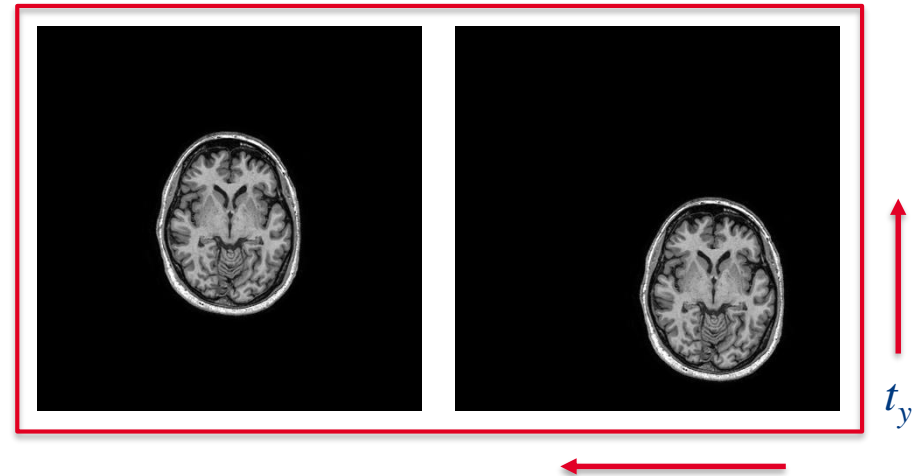


# Image registration: Introduction

Maureen van Eijnatten

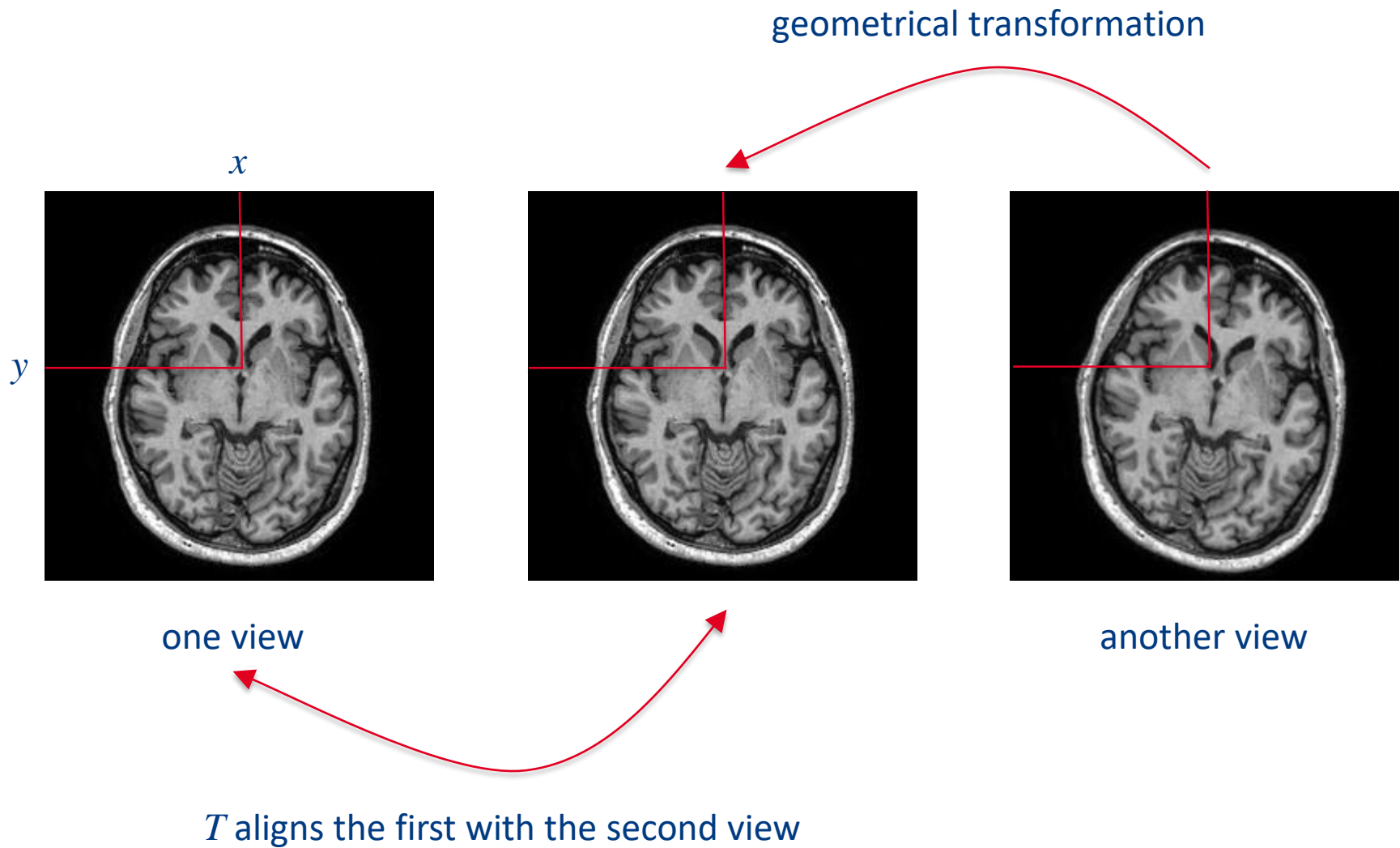


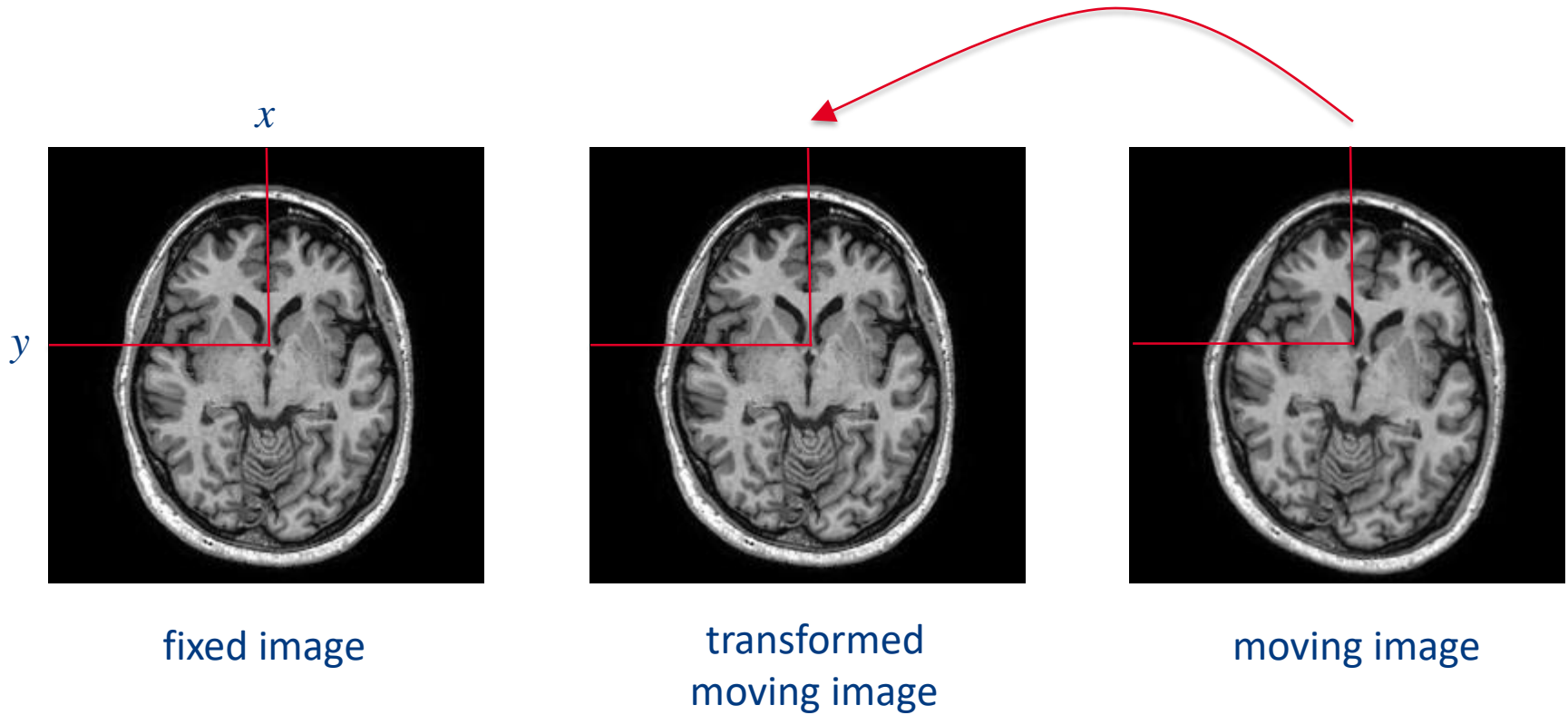
r/pics post: “I went to Milan to create a frame for this photo. Live frame.”

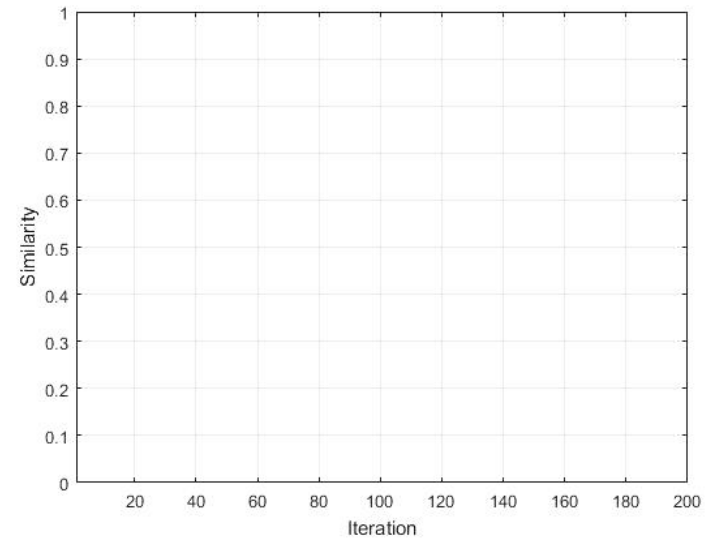
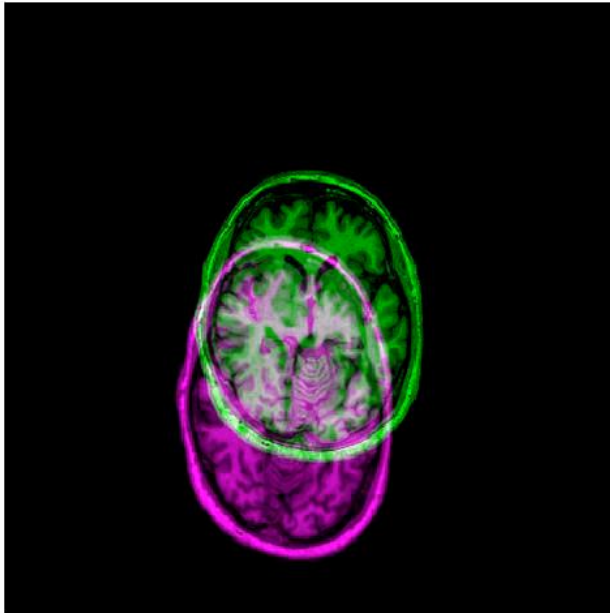


Image registration: determination of a **geometrical transformation** that aligns **one view** of an object with **another view** of that object or another object.









You will implement this functionality during the image registration project!



## Causes of misalignment:

- Different positioning of patient in scanner
- Movements of organs due to physiology
- Movements of patient
- Distortions caused by imaging system
- Changes caused by interventions (e.g. surgery, chemotherapy) in between acquisition of the images

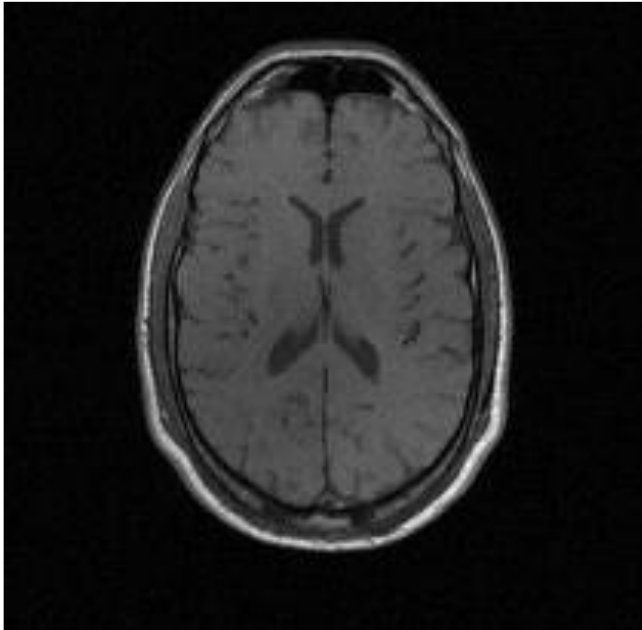
## Applications of image registration:

- Combining information from different sources
- Comparison: differences in (groups of) subjects
- Comparison: monitoring changes in a single subject
- Segmentation
- Motion correction
- Image-guided treatment
- Atlas, model of average anatomy

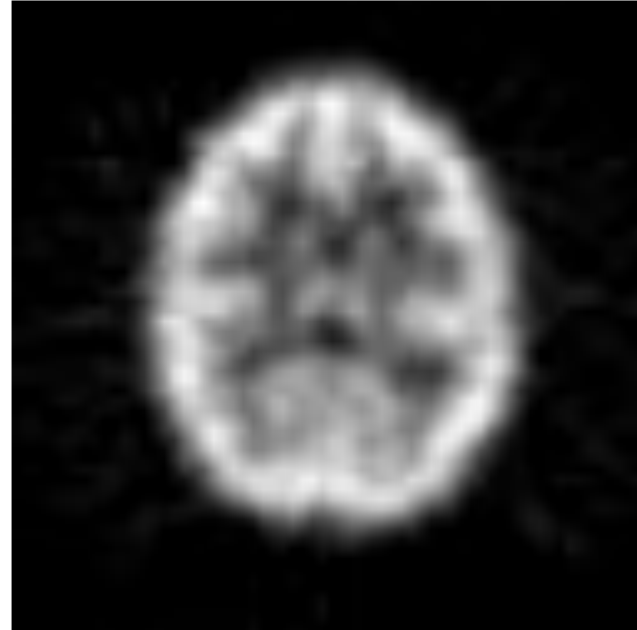


## Applications of image registration:

- **Combining information from different sources**
- Comparison: differences in (groups of) subjects
- Comparison: monitoring changes in a single subject
- Segmentation
- Motion correction
- Image-guided treatment
- Atlas, model of average anatomy



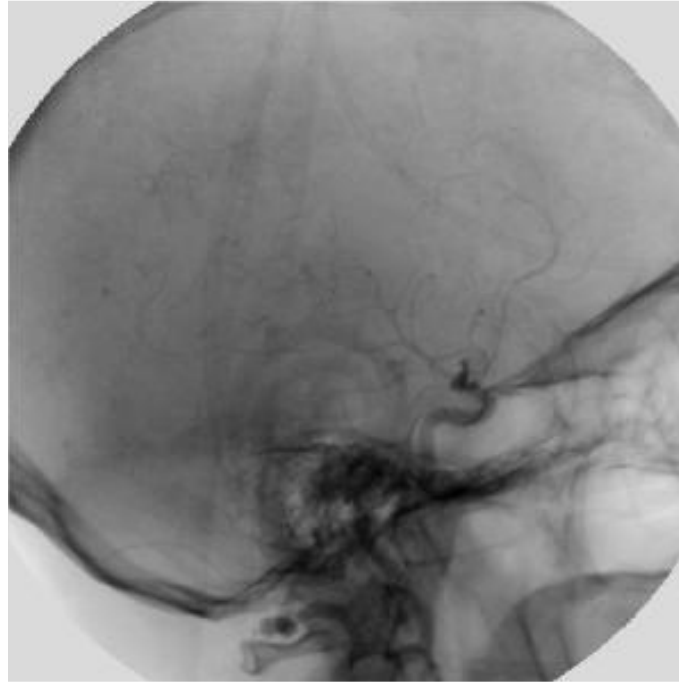
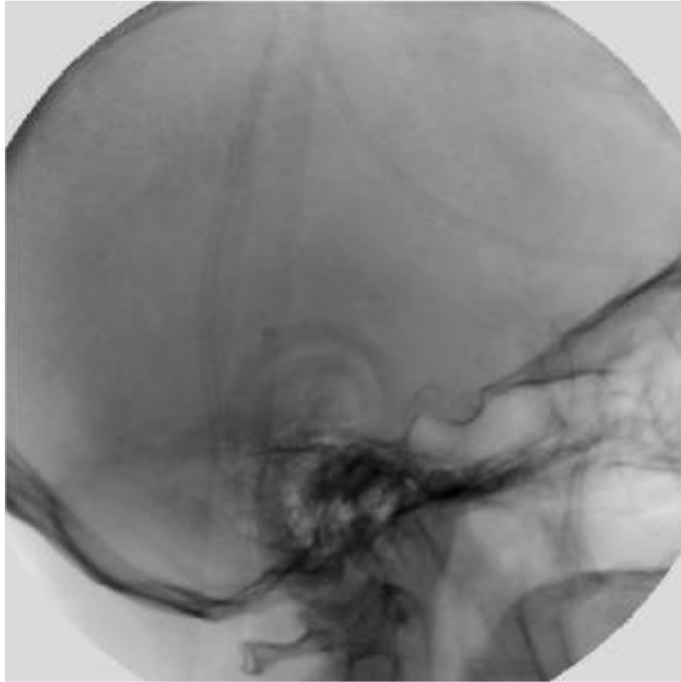
MRI, information about form



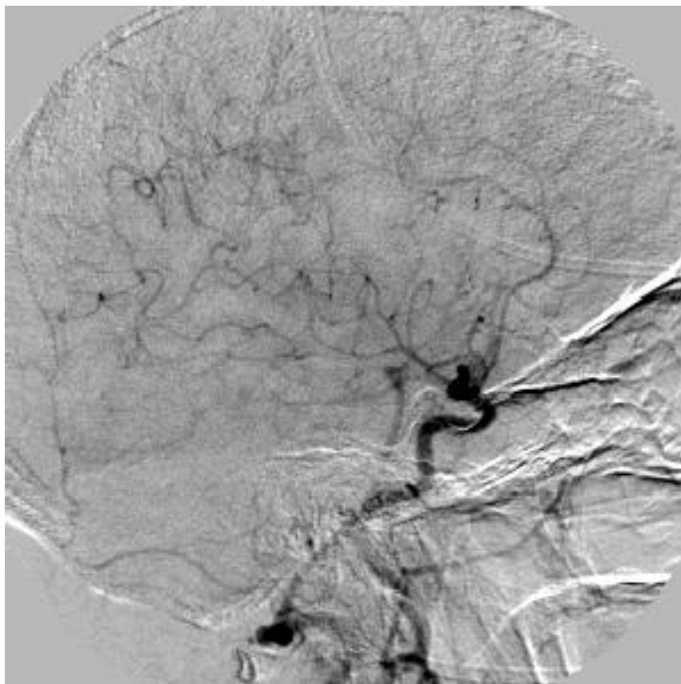
PET, information about function

## Applications of image registration:

- Combining information from different sources
- Comparison: differences in (groups of) subjects
- **Comparison: monitoring changes in a single subject**
- Segmentation
- Motion correction
- Image-guided treatment
- Atlas, model of average anatomy



## Digital subtraction angiography



Without registration



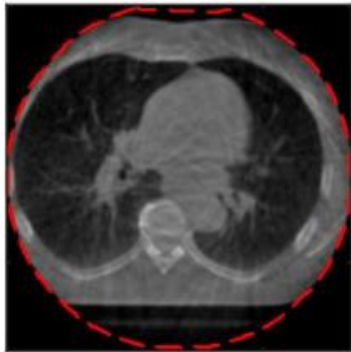
With registration

Digital subtraction angiography

## Applications of image registration:

- Combining information from different sources
- Comparison: differences in (groups of) subjects
- Comparison: monitoring changes in a single subject
- Segmentation
- Motion correction
- **Image-guided treatment**
- Atlas, model of average anatomy

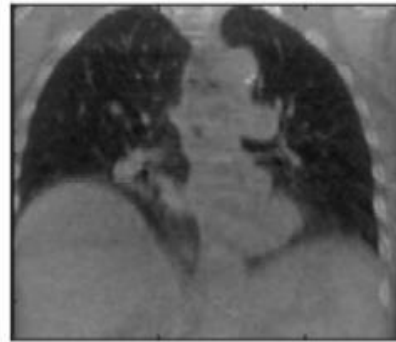
CBCT



CT



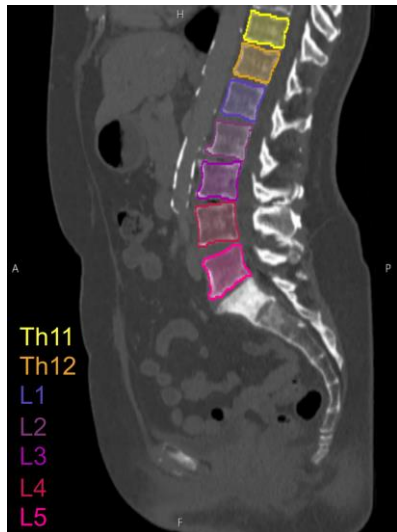
CBCT



CT



Cone-beam CT for *adaptive radiotherapy*



Abdominopelvic CT of patients  
with metastasis in the spine:  
*treatment-response assessment*



## Classification of image registration:

- Image dimensionality: 2D, 3D, 3D + time...
- Registration basis: point sets, intensity...
- Geometrical transformations: rigid, affine, nonlinear...
- Degree of interaction: automatic, semi-automatic
- Optimization procedure: closed-form solution, iterative
- Modalities: multi-modal, intra-modal
- Subject: inter-patient, intra-patient, atlas
- Object: head, vertebra, liver...

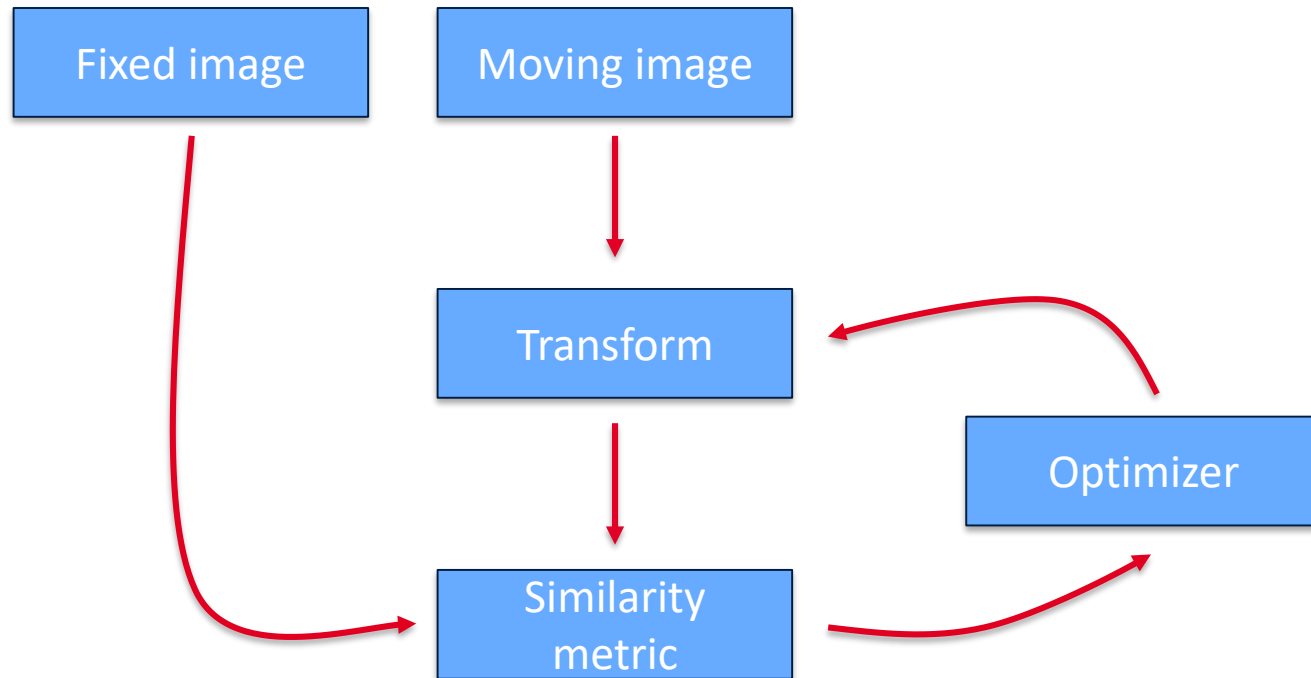
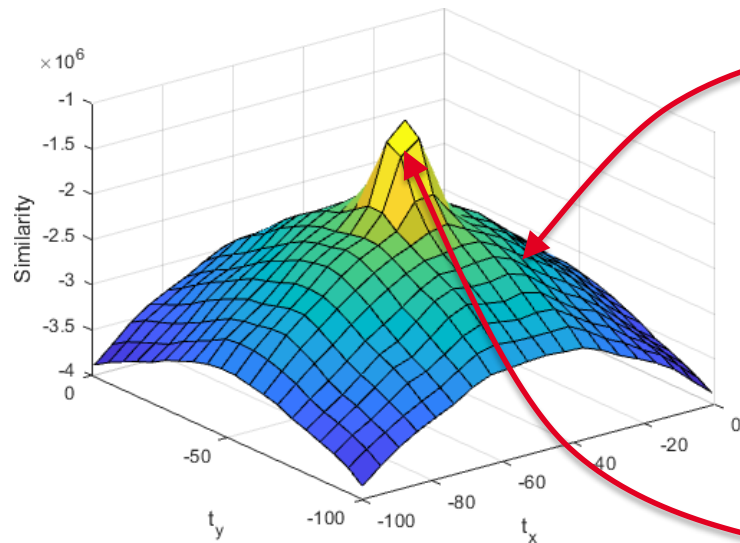
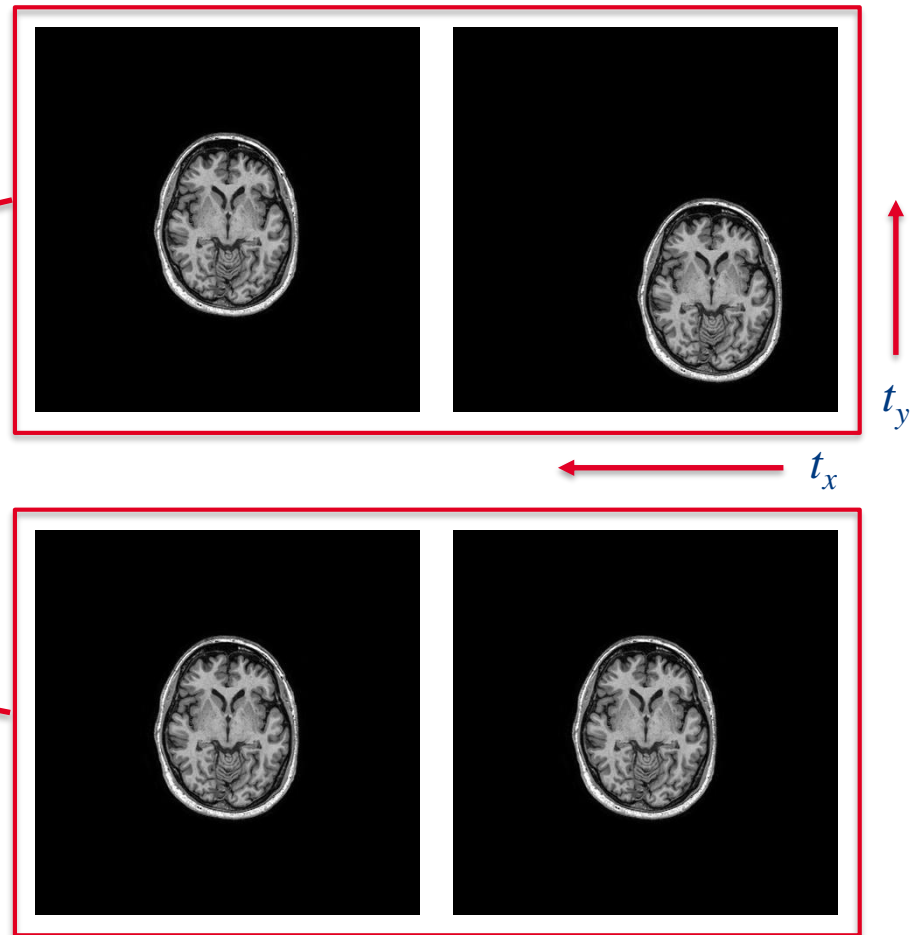


Image registration overview



Similarity as a function of translation



## Medical image registration in 8DC00 in a nutshell:

- **Week 1, day1:** Course introduction, introduction into medical image registration
- **Week 1, day2:** Image transformation, point-based image registration, review of linear algebra, least-squares
- **Week 2, day1:** Intensity-based image similarity metrics, correlation, review of probability theory, mutual information
- **Week 2, day2:** Gradient descent, intensity-based registration
- **Week 3, day 1:** Recap day (no lecture)
- **Week 3, day 2:** Applications of image registration in daily radiotherapy practice, guest lecture by *Dr. Cornel Zachiu* (Dept. of Radiotherapy, UMC Utrecht)

## Study materials:

- Primary: lecture slides, exercises
- Recommended reading – relevant sections from:  
[Fitzpatrick, J.M., Hill, D.L. and Maurer Jr, C.R., Image registration.](#)

# Review of linear algebra

Kolter, Z. Do, C., Linear Algebra  
Review and Reference (link on  
GitHub)

## Topics to review:

- Matrix-vector, vector-matrix products
- Transpose
- Norms
- Orthogonality
- Determinant

Week	Date	Lecturer	Topics
1	1 Sept.	Maureen	Course introduction; Software demo; Image registration (1)
	3 Sept.	Maureen	<b>Image registration (2); Geometrical transformations</b>
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# Scalars

- A scalar is a single number
- Integers, real numbers, rational numbers, etc.
- We denote it with italic font:

$a, n, x$

# Vectors

- A vector is a 1-D array of numbers:

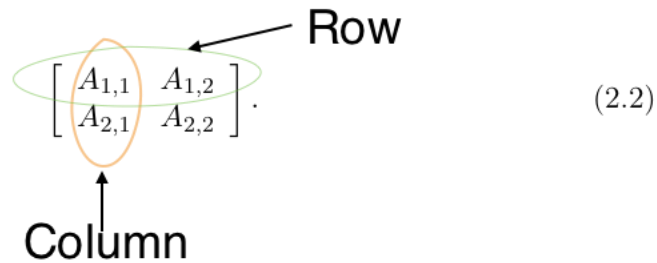
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \quad (2.1)$$

- Can be real, binary, integer, etc.
- Example notation for type and size:

$$\mathbb{R}^n$$

# Matrices

- A matrix is a 2-D array of numbers:



The diagram shows a 2x2 matrix  $\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$ . A green horizontal oval encircles the top row, with an arrow pointing to it from the word "Row". A red vertical oval encircles the first column, with an arrow pointing to it from the word "Column".

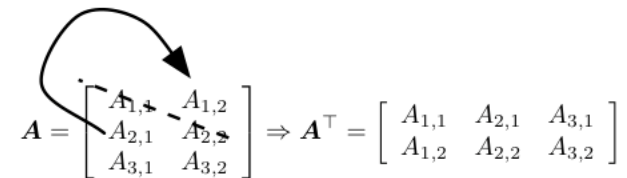
$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \quad (2.2)$$

- Example notation for type and shape:

$$\mathbf{A} \in \mathbb{R}^{m \times n}$$

# Matrix Transpose

$$(\mathbf{A}^\top)_{i,j} = A_{j,i}. \quad (2.3)$$



$$\mathbf{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix} \Rightarrow \mathbf{A}^\top = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

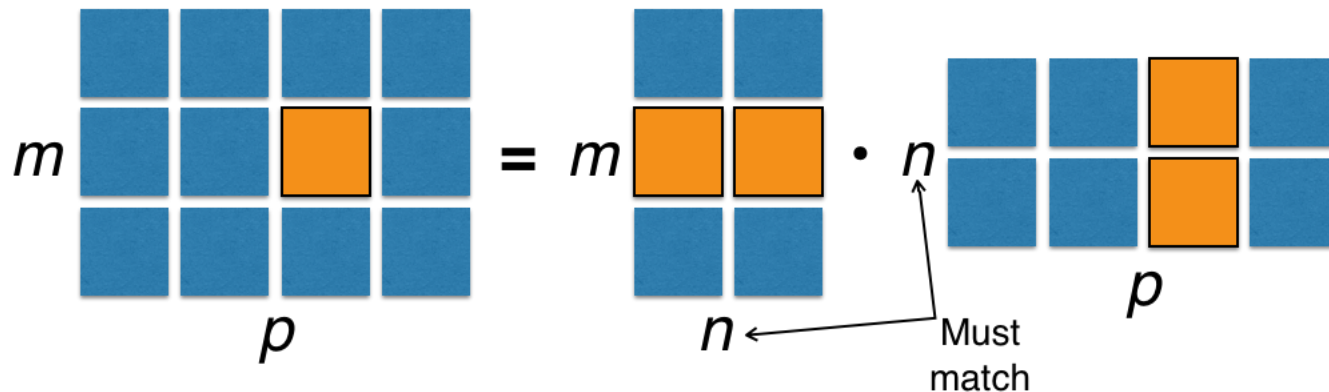
Figure 2.1: The transpose of the matrix can be thought of as a mirror image across the main diagonal.

$$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top. \quad (2.9)$$

# Matrix (Dot) Product

$$C = AB. \quad (2.4)$$

$$C_{i,j} = \sum_k A_{i,k} B_{k,j}. \quad (2.5)$$



# Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 2.2: *Example identity matrix*: This is  $\mathbf{I}_3$ .

$$\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{I}_n \mathbf{x} = \mathbf{x}. \quad (2.20)$$

# Systems of Equations

$$Ax = b \tag{2.11}$$

expands to

$$A_{1,:}x = b_1 \tag{2.12}$$

$$A_{2,:}x = b_2 \tag{2.13}$$

$$\dots \tag{2.14}$$

$$A_{m,:}x = b_m \tag{2.15}$$



# Matrix Inversion

- Matrix inverse:  
$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n. \quad (2.21)$$

- Solving a system using an inverse:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad (2.22)$$

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad (2.23)$$

$$\mathbf{I}_n\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \quad (2.24)$$

- Numerically unstable, but useful for abstract analysis

# Norms

- Functions that measure how “large” a vector is
- Similar to a distance between zero and the point represented by the vector
  - $f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} = \mathbf{0}$
  - $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$  (the *triangle inequality*)
  - $\forall \alpha \in \mathbb{R}, f(\alpha \mathbf{x}) = |\alpha|f(\mathbf{x})$

# Special Matrices and Vectors

- Unit vector:

$$\|x\|_2 = 1. \quad (2.36)$$

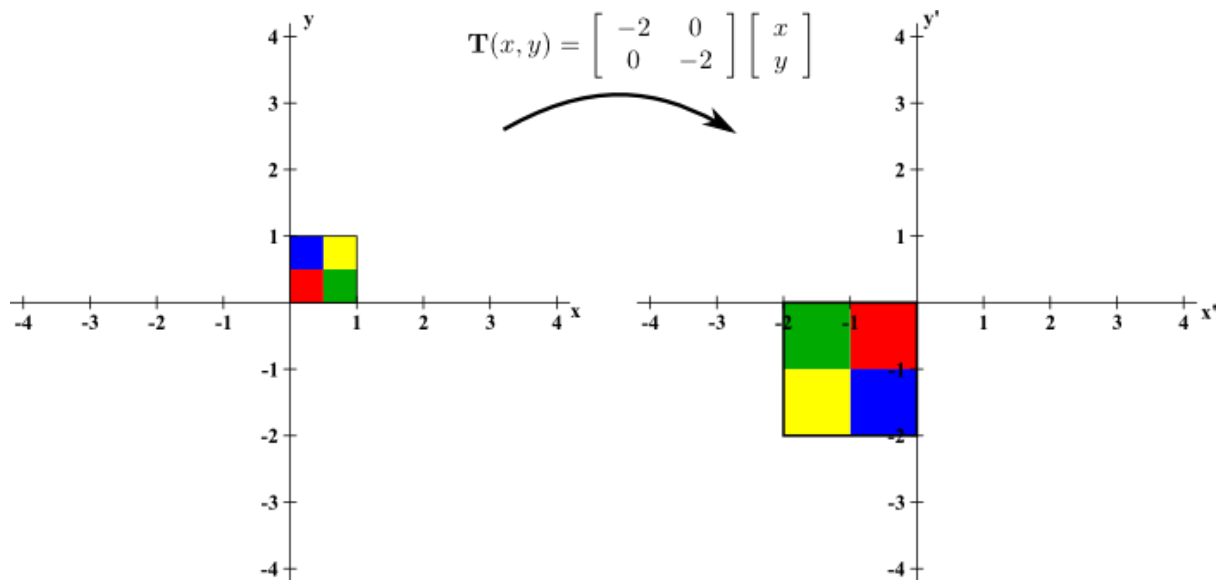
- Symmetric Matrix:

$$A = A^\top. \quad (2.35)$$

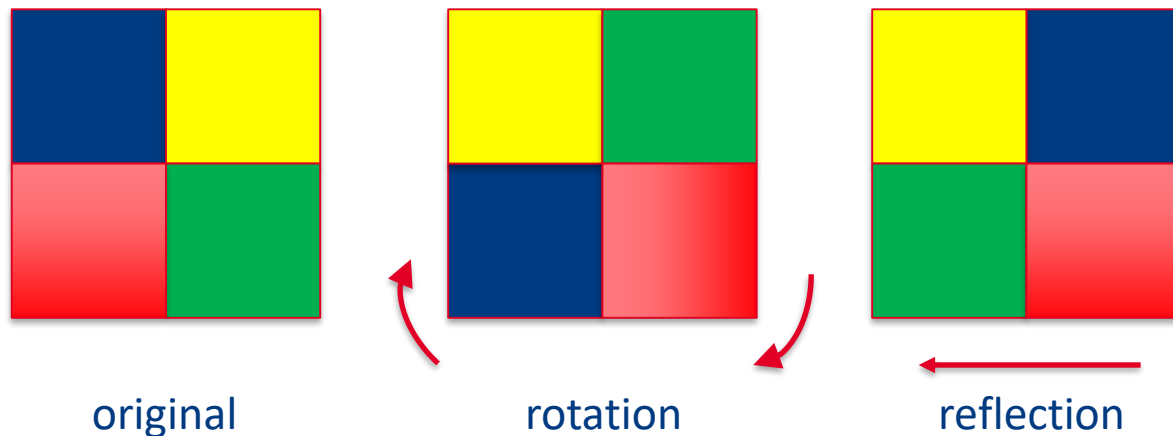
- Orthogonal matrix:

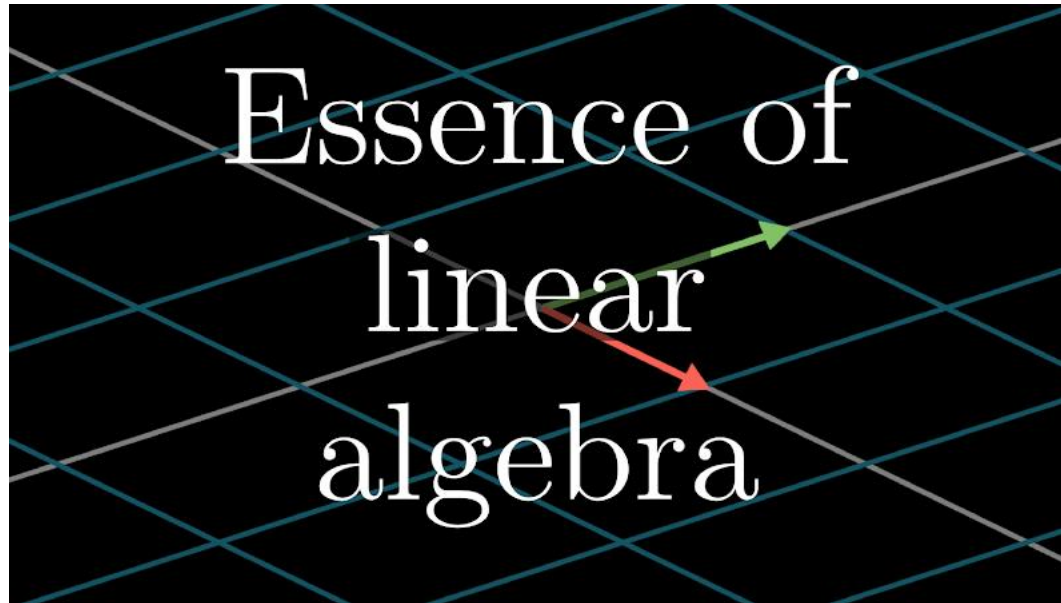
$$\begin{aligned} A^\top A &= A A^\top = I. \\ A^{-1} &= A^\top \end{aligned} \quad (2.37)$$

**Determinant** of a transformation matrix **T**: the signed area of a unit square shape after transforming with **T**. The sign reflects whether the orientation has changed.



**Determinant** of a transformation matrix  $\mathbf{T}$ : the signed **area** of a unit square shape after transforming with  $\mathbf{T}$ . The sign reflects whether the orientation has changed.

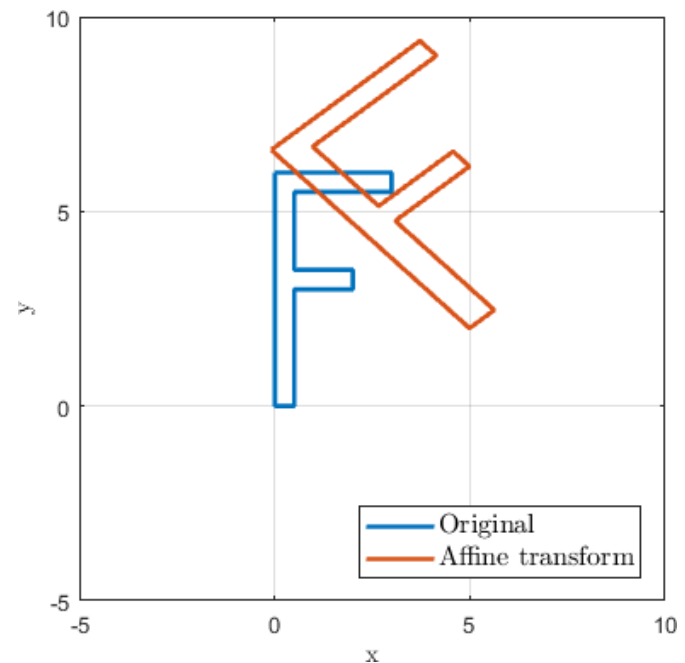




Essence of linear algebra, 3Blue1Brown channel

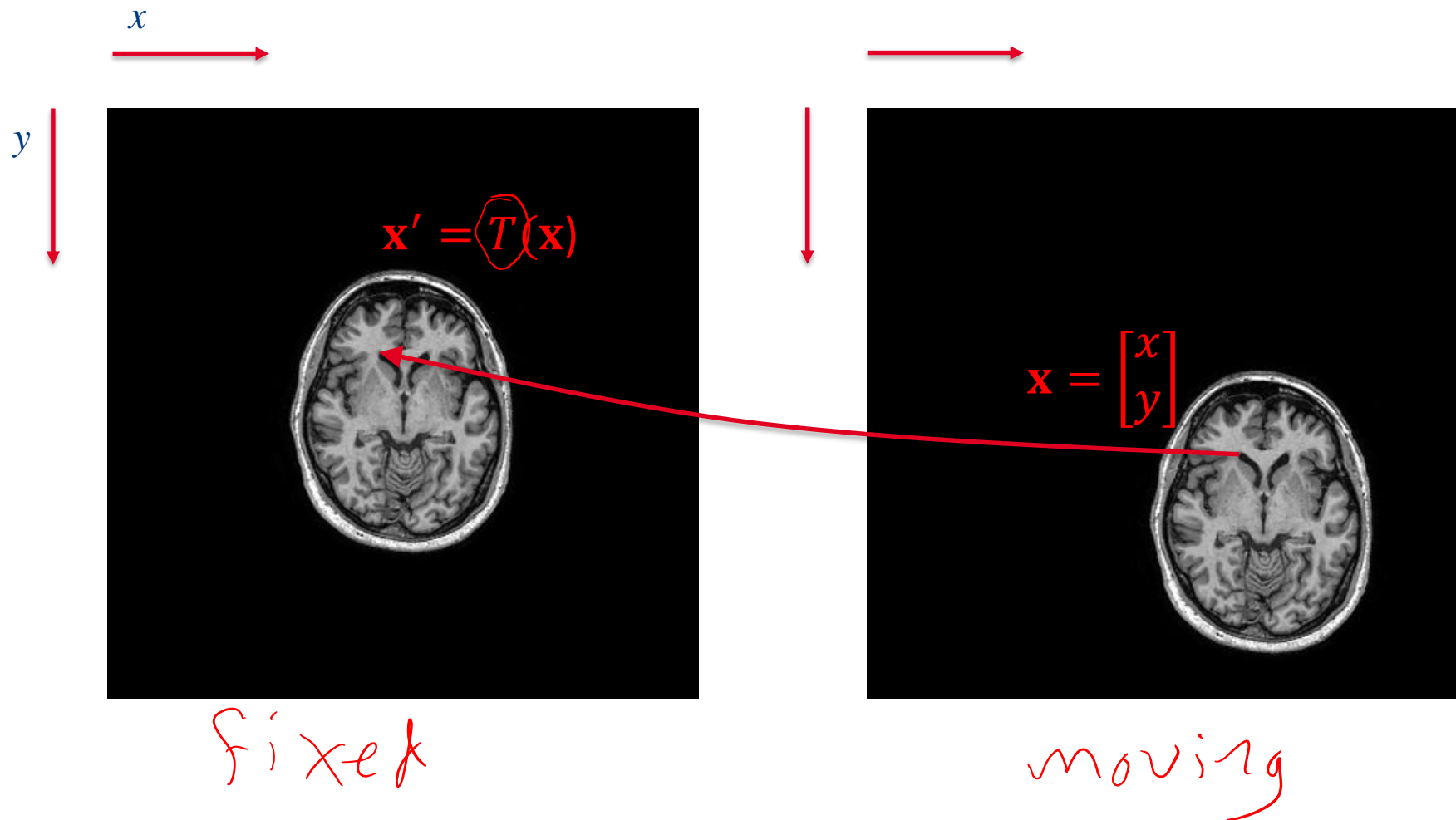
# Geometrical transformations

Maureen van Eijnatten

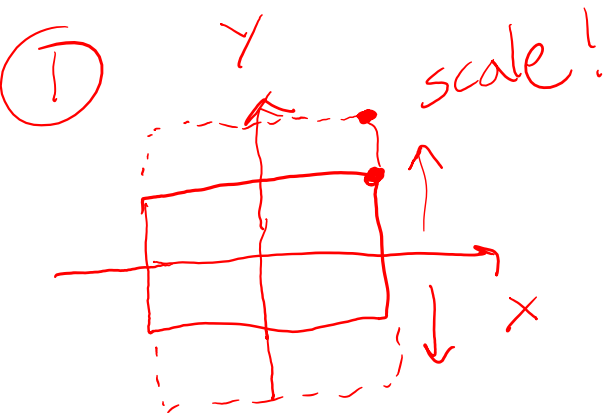




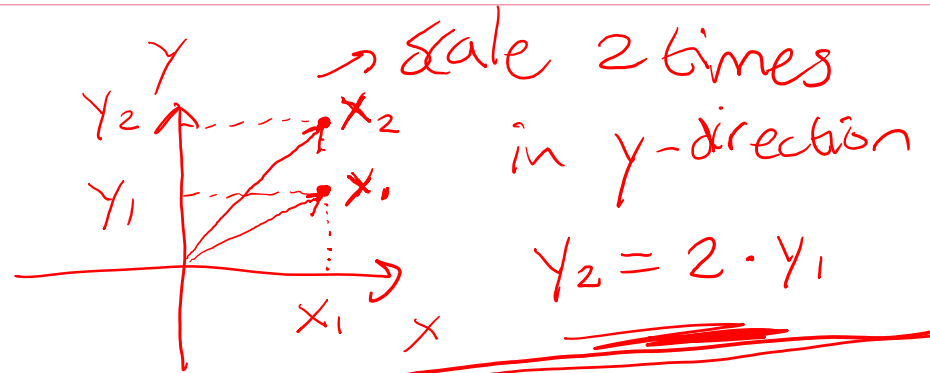
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All examples will be for 2D geometrical shapes and images, but they can be easily generalized to 3D.



②



③  $\vec{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$   $\vec{x}_2 = \begin{bmatrix} x_1 \\ 2 \cdot y_1 \end{bmatrix}$

$x_2 = 1 \cdot x_1 + 0 \cdot y_1$   
 $y_2 = 0 \cdot x_1 + 2 \cdot y_1$

④  $f(\vec{x}_1) = \vec{x}_2$  what is  $f(\cdot)$ ?

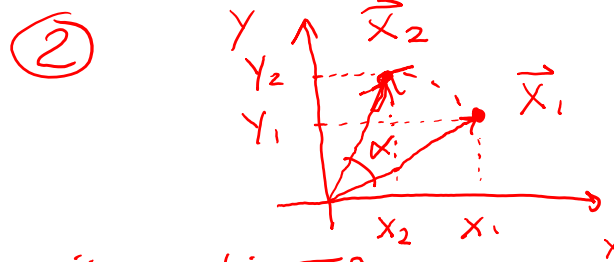
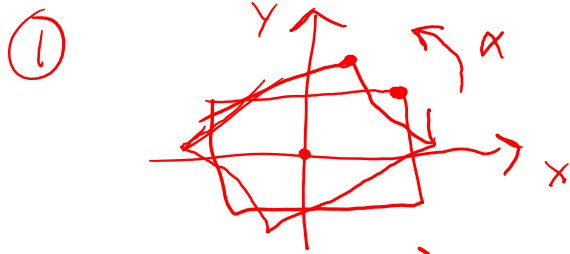
⑤  $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

matrix-vector product

⑥ apply  $f(\cdot)$  (= transformation) to all points of the object.

# Rotation ???

$\alpha$  = angle of rotation!



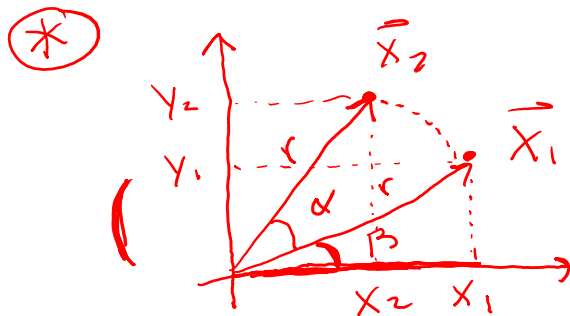
③  $\vec{x}_2 = T \vec{x}_1$  What is matrix T?

$$\vec{x}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}_1$$

$$\begin{aligned} x_2 &= a x_1 + b y_1 \\ y_2 &= c x_1 + d y_1 \end{aligned}$$

T =

$$\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$



$$r = |\vec{x}_1| = |\vec{x}_2|$$

$$\begin{aligned} x_1 &= r \cdot \cos \beta \\ y_1 &= r \cdot \sin \beta \end{aligned}$$

$$\begin{aligned} x_2 &= r \cos(\alpha + \beta) \\ y_2 &= r \sin(\alpha + \beta) \end{aligned}$$

$$\begin{aligned} x_2 &= r \cos(\alpha + \beta) = r (\cos \alpha \cos \beta - \sin \alpha \sin \beta) \\ &= r \cos \alpha \cos \beta - r \sin \alpha \sin \beta \end{aligned}$$

$$= \cos \alpha \cdot x_1 - \sin \alpha \cdot y_1$$

$$\begin{aligned} y_2 &= r \sin(\alpha + \beta) = r (\sin \alpha \cos \beta + \cos \alpha \sin \beta) \\ &= \sin \alpha \cdot x_1 + \cos \alpha \cdot y_1 \end{aligned}$$

## Translation:

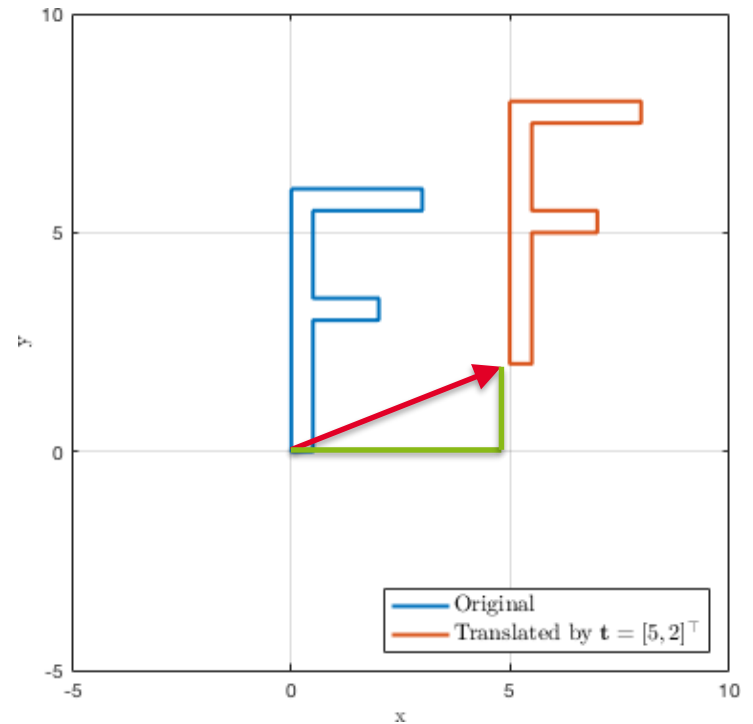
$$x = x + t_x$$

$$y = y + t_y$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



## Distance between two points in 2D:

$$D(\mathbf{x}, \mathbf{x}') = \sqrt{(x - x')^2 + (y - y')^2} = \sqrt{(\mathbf{x} - \mathbf{x}')^\top (\mathbf{x} - \mathbf{x}')}^{\frac{1}{2}}$$

## Rotation:

$$x' = \cos(\phi)x - \sin(\phi)y$$

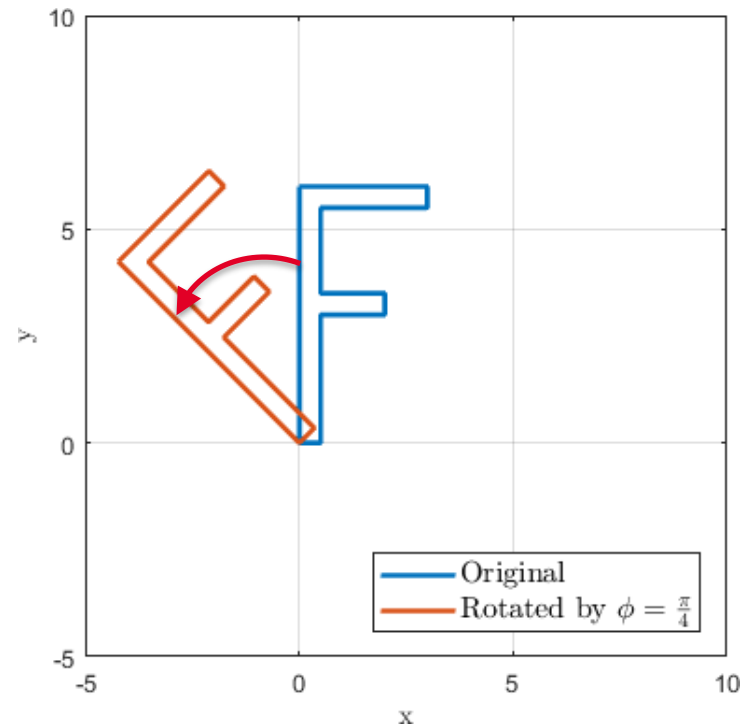
$$y' = \sin(\phi)x + \cos(\phi)y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\rightarrow T \rightarrow R$

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$



Not every matrix can be considered a rotation matrix.

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Rotation matrices:

- Are orthogonal:

$$\mathbf{R}\mathbf{R}^{-1} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$$

- Have determinant equal to 1:

$$\det(\mathbf{R}) = 1$$



## Scaling:

$$x' = s_x x$$

$$y' = s_y y$$

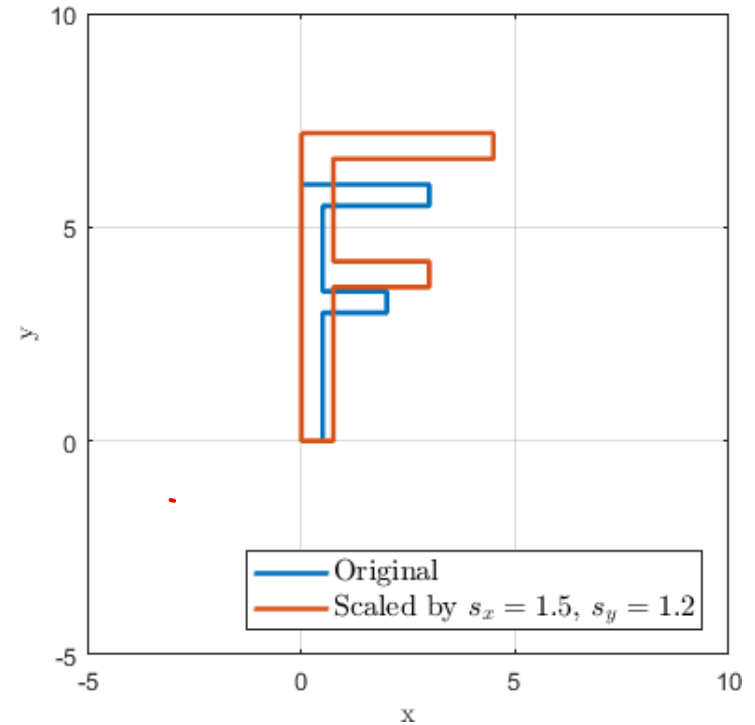
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{S}\mathbf{x}$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$f(\cdot) \rightarrow 5$



## Shearing:

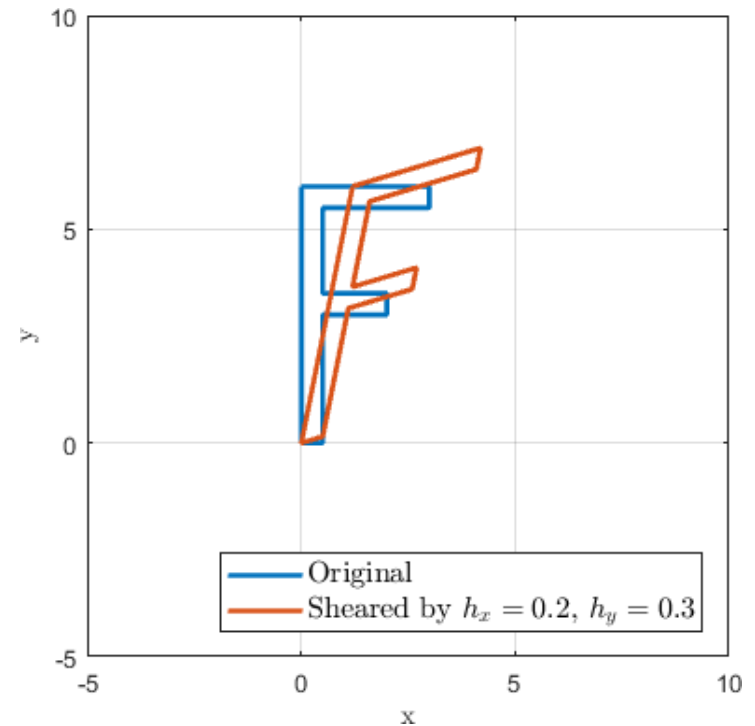
$$x' = x + h_x y$$

$$y' = h_y x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\mathbf{H} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix}$$



## Reflection:

### Horizontal:

$$x' = -x$$

$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

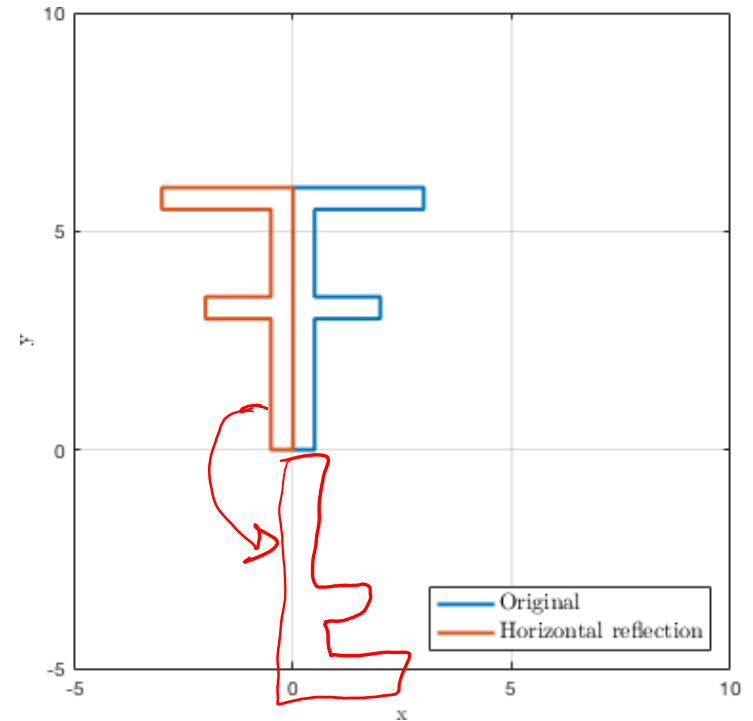
### Vertical:

$$x' = x$$

$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Composition of transformations:

Rotation + translation (rigid):

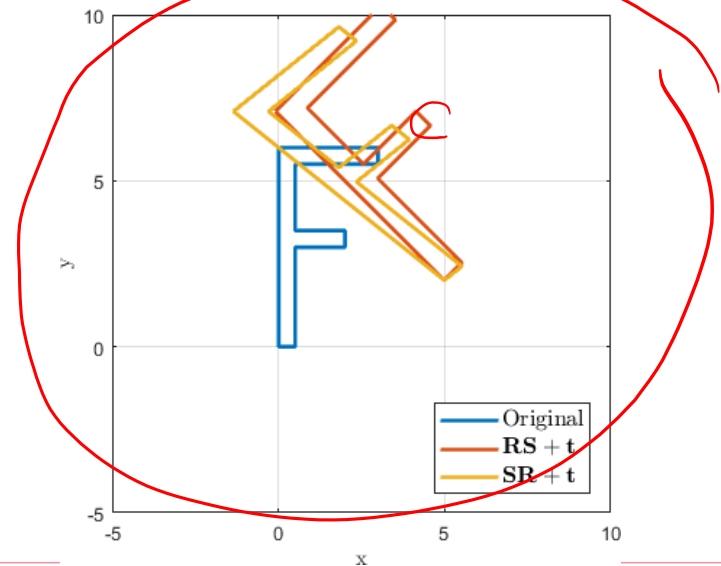
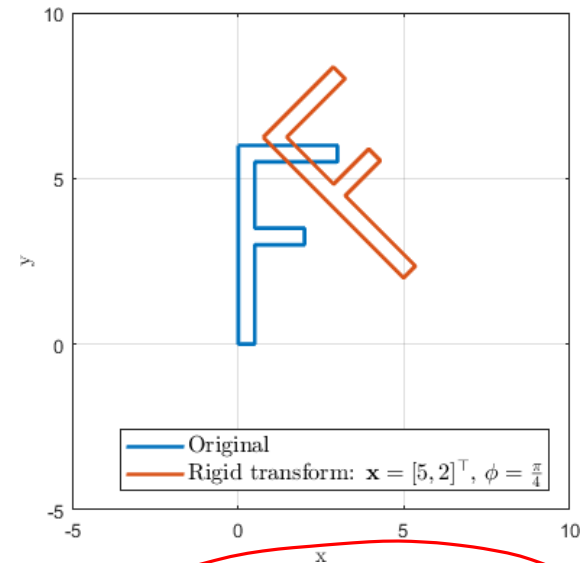
$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

Transformations can be combined by multiplying the transformation matrices.

Rotation, scaling + translation:

$$\mathbf{x}' = \mathbf{R}\mathbf{S}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \mathbf{S}\mathbf{R}\mathbf{x} + \mathbf{t}$$



Note that matrix multiplication is not commutative:

$$\mathbf{T}_1 \mathbf{T}_2 \mathbf{x} \neq \mathbf{T}_2 \mathbf{T}_1 \mathbf{x}$$

First scaling, then rotation, then translation:

$$\mathbf{x}' = \mathbf{R} \mathbf{S} \mathbf{x} + \mathbf{t}$$

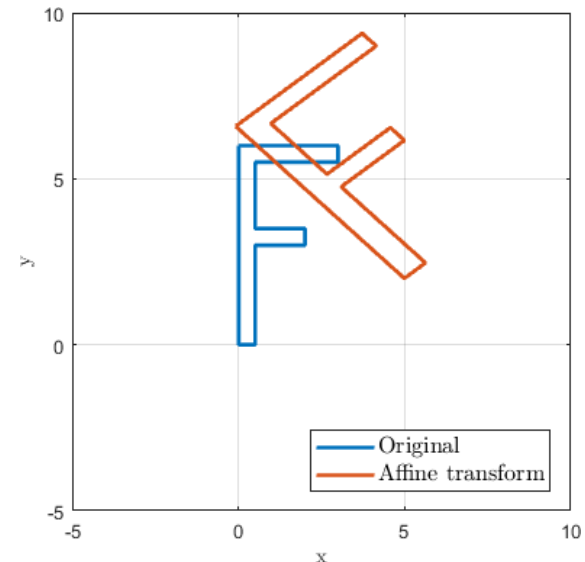
First rotation, then scaling, then translation:

$$\mathbf{x}' = \mathbf{S} \mathbf{R} \mathbf{x} + \mathbf{t}$$

Affine transformation (no restriction on the transformation parameters):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\underline{\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}}$$



It can be considered as a composition of any combination of rotations, scalings, shearings, reflections + translations.

Note that the affine transformation has **6 parameters**:  $2 \times 2$  transformation matrix and  $2 \times 1$  translation vector.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

6

The combination of rotation, scaling, shearing, reflection + translation has **9 parameters**: 1 rotation angle, 2 scaling parameters, 2 shearing parameters, 2 reflection parameters and  $2 \times 1$  translation vector.

$$[\phi \quad s_x \quad s_y \quad h_x \quad h_y \quad r_x \quad r_y \quad t_x \quad t_y]$$

However, the first 7 parameters are not independent.

The first parameterization is more compact, the second more human-readable.

Affine transformation in 2D has only **6 degrees of freedom**.

In medical image registration, **reflections do not usually occur**, and it can be very problematic if two images are incorrectly registered with a reflection (e.g. can cause a surgical procedure to be performed on the wrong side of the body).

Thus, reflections should be excluded from affine registration.

$$\begin{bmatrix} \phi & s_x & s_y & h_x & h_y & t_x & t_y \end{bmatrix}$$

When using the unrestricted transformation matrix, a check for reflection can be made by examining  $\det(\mathbf{A})$ . If a reflection has occurred  $\det(\mathbf{A}) < 0$ .

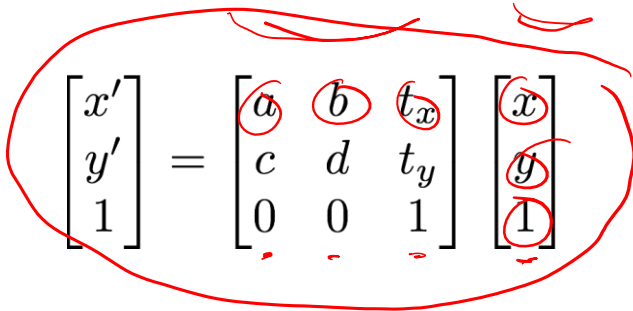
$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}$$





A transformation matrix and a translation vector can be combined when using homogeneous coordinates:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

⌋ This largely simplifies the notation and implementation of complex transformations.

Example:

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix}$$

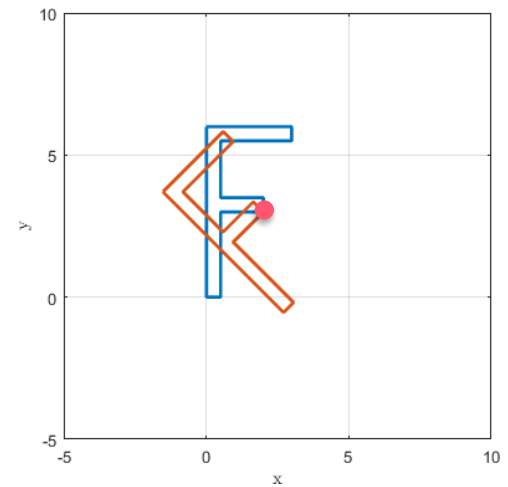
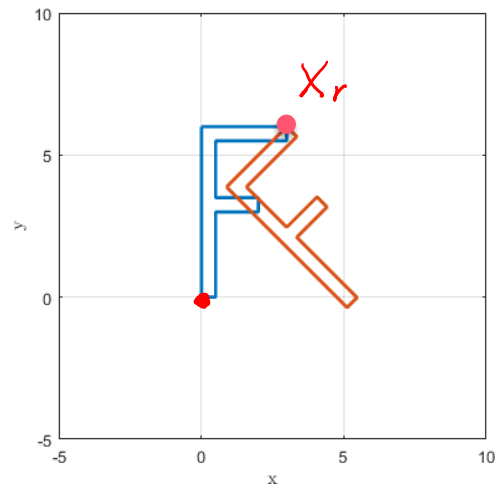
$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_1 \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x,1} + t_{x,2} \\ 0 & 1 & t_{y,1} + t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

Example – rotation around an arbitrary point  $\mathbf{x}_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$ :

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Example – rotation around an arbitrary point  $\mathbf{x}_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$ :



Inverse transformation can be achieved by taking the inverse of the transformation matrix:

$$\mathbf{T} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

## Affine transformation in 3D:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How many rotation angles in 3D?

How many degrees of freedom?

## Non-linear transformations:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = ax + by + t_x$$

$$y' = cx + dy + t_y$$

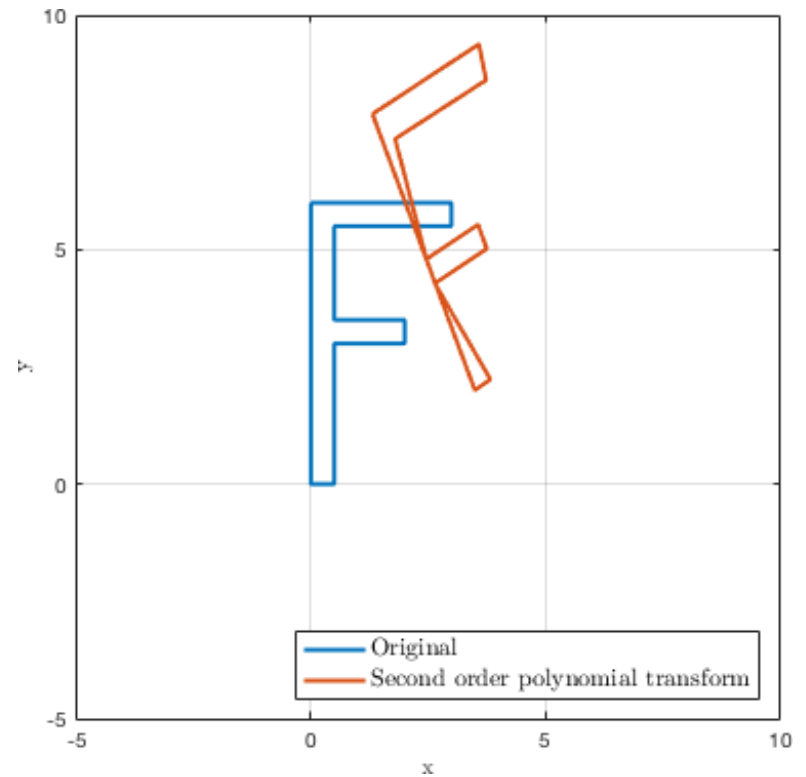
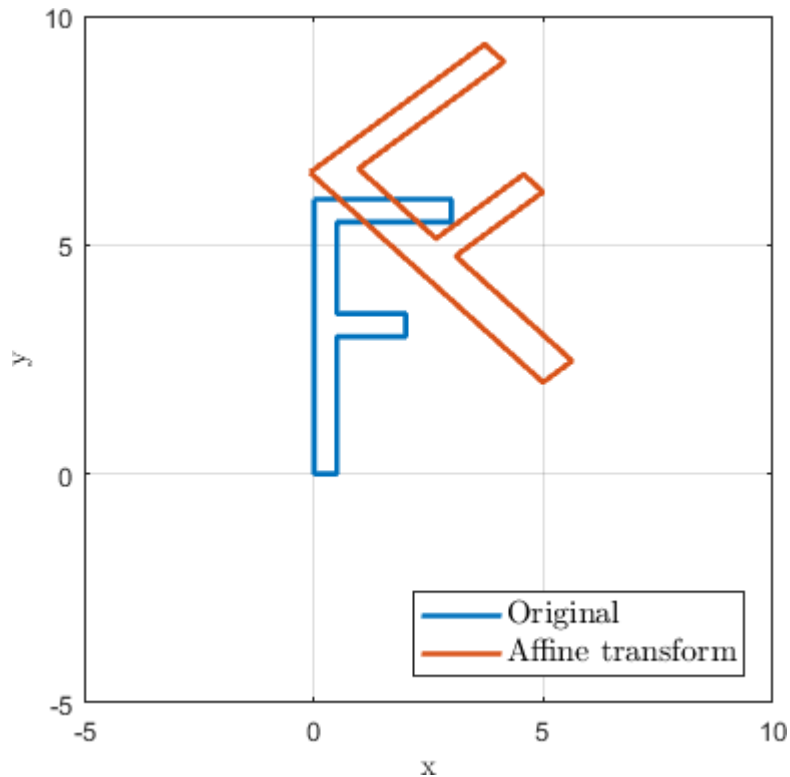
← Linear polynomial

$$x' = ax + by + t_x + u_1x^2 + u_2y^2 + u_3xy \dots$$

$$y' = cx + dy + t_y + v_1x^2 + v_2y^2 + v_3xy \dots$$

← Higher order polynomial

## Non-linear transformations:

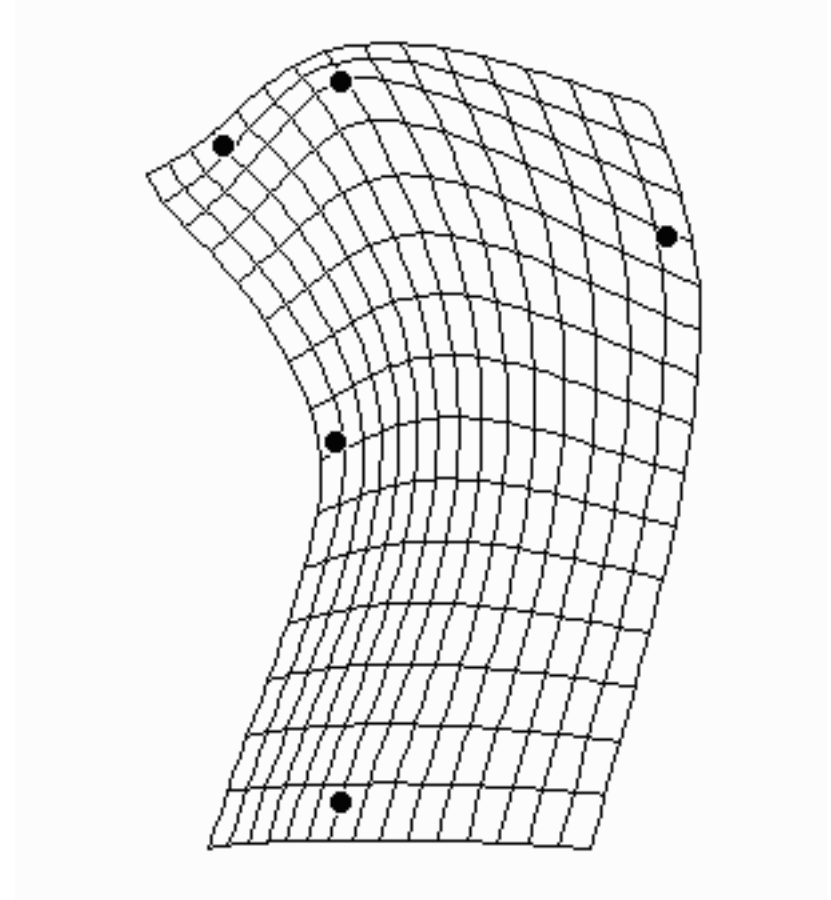


## Non-linear transformations: thin-plate spline

$$x' = ax + by + t_x + \sum_{i=1}^N u_i r_i^2 \ln r_i^2$$

$$y' = cx + dy + t_y + \sum_{i=1}^N v_i r_i^2 \ln r_i^2$$

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2$$





# Thank you

Next: Image  
transformation, point-  
based registration

