ENPM667 Control of Robotic Systems

Project 1

Geometric Tracking Control of a Quadrotor UAV on SE(3)

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1 Abstract

It is a known fact that quadrotor UAVs are in general under-actuated and nonlinear system and it is a challenge to control them, especially in case of aggressive maneuvers. Our goal in this project is to study the nonlinear geometric control approach to control a quadrotor. Geometric control theory is the study on how geometry of the state space influences control problems. In control systems engineering, the underlying geometric features of a dynamic system are often not considered carefully. Differential geometric control techniques utilize these geometric properties for control system design and analysis. The objective is to express both the system dynamics and control inputs on nonlinear manifolds instead of local charts. Based on the geometric properties of the system dynamic, this differential geometric based approach is used to model and control the system. Also, the intent is to design a controller that gives us global stability properties i.e. the system can recover from any initial state. The configuration of the quadrotor system described on smooth nonlinear geometric configuration spaces has been briefly discussed, and analyzed with the principles of differential geometry. This allows us to avoid any kind of singularities that would otherwise arise on local charts. Further, it is possible to construct an almost-globally defined nonlinear geometric controller by defining the error function on the same spaces utilizing the geometric properties. Finally, simulations demonstrate the stability and abilities of the nonlinear geometric controller.

2 Introduction

A Quadrotor (QR) is a type of Unmanned Aerial Vehicle (UAV) that has received an increasing amount of attention recently with many applications being actively investigated. Possible applications include search and rescue, surveillance, reliable supply of food and medicines in emergency situations and object manipulation in construction and transportation. It has already proven itself useful for many tasks like multi-agent missions, mapping, explorations, transportation and entertainment such as acrobatic performances.

Figure 1: QRM (Quad Rotor Mini) CAD model

For most of the linear controllers, the dynamics are linearized around an equilibrium point and a set of linear differential equations describe the system dynamics. The reason that linear control near an equilibrium state is commonly applied, is partly to avoid difficulties that come with modeling and controlling the non-linearities of the system. However, linear control limits the system to small angle movements, as the optimization will not allow large angles that deviate too much from the linearized point. This approach of modeling and control will not be sufficient for applications that require fast aggressive maneuvers. Nonlinear control systems are often governed by nonlinear differential equations and are able to represent the dynamics in a more realistic manner. Nonlinear Geometric Control is a nonlinear model based control technique based on a modeling approach involving the concepts of differential geometry. This results in a globally defined coordinate-free dynamical model, which prevents issues regarding singularities(like gimbal lock in Euler angle representation), and enables the design of controllers that offer almost-global convergence properties. Different aspects involving the modeling and control for the quadrotor must be investigated, for it can be expected that the non-linearity will have a great influence in the representation of the dynamics and the stability, accuracy and type of the control design. It is possible to investigate which advantages or disadvantages this nonlinear approach has compared to a linear approach, in terms of stability and performance.

3 Quadrotor Dynamics Model

For the derivation of the equations of motions, traditional modeling methods often parameterize the rotations in a local coordinate system. This can be done with Euler Angles, and despite this parametrization might result in singularities, this is a commonly used method to describe rotations. The definition of Euler angles is not unique and a sequence of rotations is not commutative. Therefore, Euler angles are never expressed in terms of the external frame, or in terms of the co-moving rotated body frame, but in a mixture.

The main disadvantages of Euler angles are that some functions have singularities and they are a less accurate measure for the integration of incremental changes in attitude over time, compared to other methods. To avoid these problems, in geometric mechanics rotations are expressed as rotation matrices to provide a global representation of the attitude of a rigid body.

$$SO(3) \triangleq \left\{ R \in \mathbb{R}^{3\times3} | RR^T = I_{3X3}, det(R) = 1 \right\} \tag{1}$$

Where SO(3) is the group of all rotations about the origin of a 3-D Euclidean space, which preserves the origin, Euclidean distance and orientation. The elements of Lie Algebra $so(3)$, a property associated with SO(3), are the elements of the tangent space of SO(3) at the identity element. Spaces SO(3) and $so(3)$ can be better understood by getting a geometric intuition by figure 2.

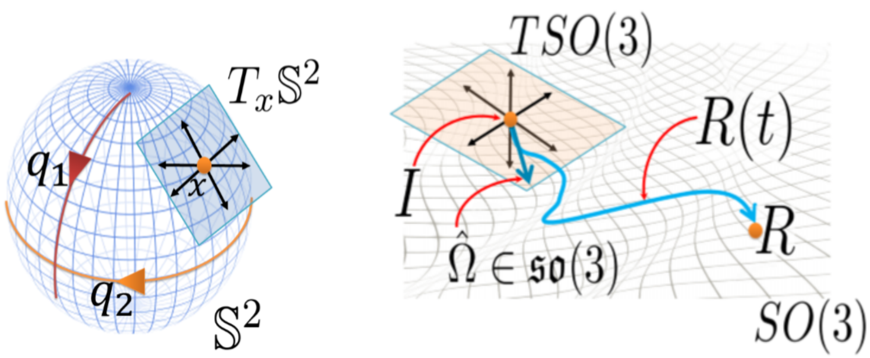

Figure 2: Tangent Spaces of different manifolds [6]

The Quadrotor described as a rigid body with six degrees of freedom, driven by the system inputs: the total upward force F and the moments $M = \left[M_\phi \;\; M_\theta \;\; M_\psi\right]^T$ around the body axes. We know from the concept of general fluid dynamics that $f_i \propto \Omega_i^2$ i.e.

$$f_i = k_f \Omega_i^2 \tag{2}$$

Similarly

$$M_i = k_M \Omega_i^2 \tag{3}$$

here k_f and k_M are taken as constants which depends on propeller's blades, diameter, pitch, material

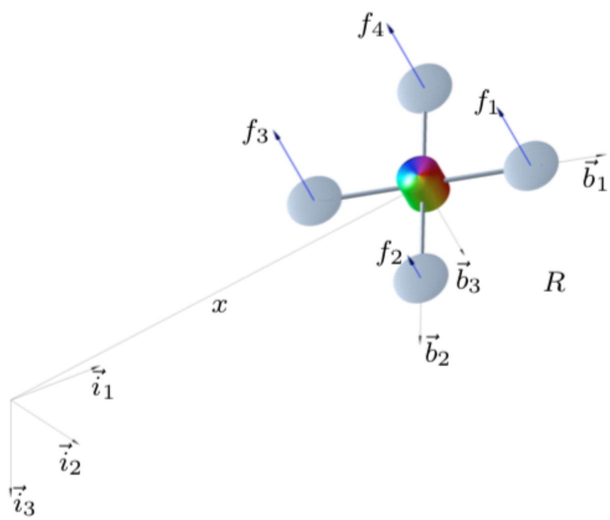

Figure 3: Quadrotor model representationl

and air viscosity but in reality these are not constant and they change with the change in velocity of the quadrotor.

So Net force: $f = \sum_{i=1}^{4} f_i - mg$ here f_i is the thrust generated by the i^{th} propellor and it acts aong -b_3 direction and Net Moment $M = \sum_{i=1}^{4} (r_i \times f_i) + M_i$

To write Newton-Euler equation of Quadrotor the angular velocity in the inertial frame is represented as

$$^{A}\Omega^{B} = pb_1 + qb_2 + rb_3 \tag{4}$$

where p, q and r are angular velocities along b_1,b_2 and b_3 axis respectively. Equating all the forces in Inertial frame

$$m\ddot{r} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} - R \begin{bmatrix} 0 \\ 0 \\ \underbrace{f_1 + f_2 + f_3 + f_4}_{u_1} \end{bmatrix} \tag{5}$$

where R is rotation matrix to take it from body frame to the world frame and $u_1 = f_1 + f_2 + f_3 + f_4$ is the sum of all the forces applied by propeller. From Newton's second law in an inertial frame of reference , the time derivative of the angular momentum "l" equals the applied torque

$$\frac{dl_{\text{in}}}{dt} \stackrel{\text{def}}{=} \frac{d}{dt}(J_{\text{in}}\Omega) = M_{\text{in}} \tag{6}$$

where J_{in} is the moment of inertia tensor calculated in the inertial frame.In a rotating reference frame, the time derivative must be replaced with

$$\left(\frac{dl}{dt}\right)_{\text{rot}} + (\Omega \times l) = M \tag{7}$$

Putting the value of $l = J\Omega$, we obtain the following, the Euler's rotation equation in the body frame

$$J\dot{\Omega} + \Omega \times (J\Omega) = M, \tag{8}$$

$$\dot{R} = R\hat{\Omega} \tag{9}$$

Putting the value of M as the total moments along b_1, b_2 and b_3 axis and taking the term $\omega \times (J\omega)$ to R.H.S.

$$J \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} d(f_4 - f_2) \\ d(f_1 - f_3) \\ -M_1 + M_2 - M_3 + M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times J \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{10}$$

Here d is the distance from center of mass to the center of each rotor. Two of the moments are opposite in direction as two motors are spinning in the opposite direction and are diagonally separated to each other. Let $c_\tau f = \frac{k_M}{k_f} = \frac{M_i}{f_i}$ is a constant and also known as air drag. So by re-arranging we can write

$$J \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -d & 0 & d \\ d & 0 & -d & 0 \\ -c_\tau f & c_\tau f & -c_\tau f & c_\tau f \end{bmatrix}}_{u_2} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times J \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{11}$$

Here u_1 and u_2 are the inputs that will be controlled in order to control the trajectory and equilibrium of the quadrotor.So to define the controller inputs i.e matrix "u" we can write

$$u = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -d & 0 & d \\ d & 0 & -d & 0 \\ -c_\tau f & c_\tau f & -c_\tau f & c_\tau f \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} thrust(f) \\ Moment_x(M_x) \\ Moment_y(M_y) \\ Moment_z(M_z) \end{bmatrix} \tag{12}$$

4 Geometric tracking control on SE(3)

In this section the nonlinear geometric control and concepts of geometric properties that are used for analysis and control design have been discussed. The configuration spaces of the system dynamics were expressed on nonlinear manifolds so error functions and geometric mappings are also defined on these same nonlinear manifolds allowing calculation of the error between current and desired states. Many control systems are developed for the standard form of ordinary differential equations

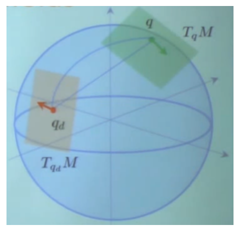

Figure 4: Tangent manifolds on SO(3) space [3]

$x = f(x, u)$, where x is the state and u the control input. It is assumed that the state and the control input lie in Euclidean spaces, and the system equations are defined in terms of smooth functions between Euclidean spaces. However, for many mechanical systems, the configuration space can only be expressed locally as a Euclidean space. A nonlinear space is required to express the configuration space globally. Geometric control theory is the study on how geometry of the state space influences control problems. In control systems engineering, the underlying geometric features of a dynamic system are often not considered carefully. Differential geometric control techniques utilize these geometric properties for control system design and analysis. The objective is to express both the system dynamics and control inputs on nonlinear manifolds instead of local charts. As opposed to locally defined linear control, nonlinear geometric control can be defined almost globally, avoiding singularities that would occur in the representation of large angles and complex maneuvering. Here the controller is designed to follow a defined trajectory of the location of the center of mass, $x_d(t)$, and the desired direction of the body-fixed axis, $\vec{b_1}_d$(t).

The translational dynamics of a quadrotor UAV is controlled by the total thrust $f \in Re_3$, where

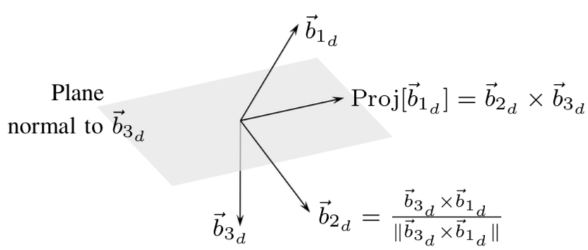

Figure 5: Definition of the desired heading direction [4]

the magnitude of the total thrust f is directly controlled, and the direction of the total thrust Re_3 is along the third body-fixed axis b_3. For a given translational command $x_d(t)$, we select the total thrust f , and the desired direction of the third body-fixed axis $\vec{b_3}_d$ to stabilize the translational dynamics.

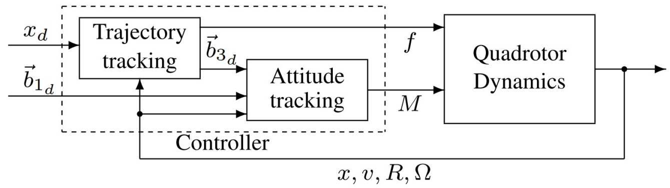

Figure 6: Controller Structure

Once $\vec{b_{3_d}}$ is defined,there is one more degree of freedom remaining in selecting the desired attitude R_d which accounts for heading direction of the quadrotor and it is determined by by the desired direction of $\vec{b_{1_d}}$(t). So we assumed that $\vec{b_{1_d}}$ is not parallel to $\vec{b_{3_d}}$ so we project $\vec{b_{1_d}}$ on $\vec{b_{3_d}}$ and calculated $\vec{b_{2_d}} = \vec{b_{3_d}} \times \vec{b_{1_d}}$ and similarly calculated Proj$\left[\vec{b_{1_d}}\right] = \vec{b_{2_d}} \times \vec{b_{3_d}}$ and calculatd their norms in order to define the desired attitude i.e. R_d.
So the controller will have four-dimensional control inputs which are designed to follow a three-dimensional translational command, and a one-dimensional heading direction, so that our quadrotor follows $x(t) \rightarrow x_d(t)$ and Proj$\left[\vec{b_1}(\mathrm{t})\right] \rightarrow$ Proj$\left[\vec{b_{1_d}}(\mathrm{t})\right]$ as t$\rightarrow \infty$, here Proj$[.]$ denotes normalised projection on the plane perpendicular to $\vec{b_{3_d}}$. so overall controller structure is defined in Figure 6.

4.1 Tracking Errors

Since the quadrotor UAV has four inputs, it is possible to achieve asymptotic output tracking for at most four quadrotor UAV outputs. The quadrotor UAV has three translational and three rotational degrees of freedom. As it is not possible to achieve asymptotic output tracking of both attitude and position of the quadrotor UAV there are two ight modes that have been presented, namely

1. Attitude controlled flight mode
2. Position controlled flight mode

While a quadrotor UAV is under-actuated, a complex flight maneuver can be dened by specifying a concatenation of ight modes together with conditions for switching between them.

4.1.1 Position Tracking Errors

Here we define the position tracking error for x, v. So the position and velocity error are given by

$$\begin{aligned} e_x &= x - x_d \\ e_v &= v - v_d \end{aligned} \tag{13}$$

4.1.2 Attitude Tracking Errors

The attitude error for R, Ω are defined in this section. It is known that R is the rotation matrix to describe the quadrotor attitude, and R_d is the desired rotation matrix. To describe the relative rotation from the body frame to the desired frame, an attitude error is defined as $R_d^T R$. Note that $R_d^T R$ is again a rotation matrix itself. Based on this attitude error, the tracking error function Ψ on SO(3) is chosen to be

$$\Psi\left[R, R_d\right] = \frac{1}{2} tr\left[I - R_d^T R\right] \tag{14}$$

the error function Ψ is chosen because it holds the following properties

1. Ψ is locally positive definite about R and R_d within the region where the rotation angle between R and R_d is less than 180°.

2. Ψ is bounded by the following conditions

$$\frac{1}{2}\left\|e_R\right\|^2 \leq \Psi(R, R_d) \leq \frac{1}{2-\Psi}\left\|e_R\right\|^2 \tag{15}$$

where Ψ is a positive constant such that $\Psi(R, R_d) < \Psi < 2$ and e_R is the tracking error defined later in this section.

For a given R_d, this set can be represented by the sublevel set $L_2 = R \in SO\big(3\big) \mid \Psi\big(R, Rd\big) < 2$, which almost covers $\mathrm{SO}\big(3\big)$. The variation of a rotation matrix can be expressed as $\delta R = R\hat{\eta}$ for $\eta \in \mathbb{R}^{3\times 3}$, so that the derivative of the error function is given by

$$\mathbf{D}_R\Psi\big(R, R_d\big) \cdot R\hat{\eta} = -\frac{1}{2}tr\big[R_d^T R\hat{\eta}\big] \tag{16}$$

Using the property of hat map that $tr[A\hat{x}] = \frac{1}{2}tr[\hat{x}(A - A^T)] = -x^T(A - A^T)^\vee$

$$\mathbf{D}_R\Psi\big(R, R_d\big) \cdot R\hat{\eta} = \frac{1}{2}\big(R_d^T R - R_d^T R_d\big)^\vee \cdot \eta \tag{17}$$

$SO(3)$ geometrically is a manifold(a Lie group) while $so(3)$ is the Lie algebra of $SO(3)$, $so(3)$ is a vector space with a bilinear antisymmetric bracket that satisfies the Jacobi identity. [1]

$$e_R = \frac{1}{2}\big(R_d^T R - R_d^T R_d\big)^\vee \tag{18}$$

It is important to note that the velocities i.e. $\dot{R}$ and $\dot{R}_d$ cannot be compared directly as they both are in different space lets say $\dot{R}$ has a tangent space $T_R SO(3)$ and $\dot{R}_d$ has a tangent space $T_{R_D}SO(3)$. So to de this we have to define a velocity error with a transport map which will allow us to do comparison of these two spaces . So the velocity error $\dot{e}$ along R which defines the velocity error between R and R_d is

$$\begin{aligned}\dot{e} &= \dot{R} - \dot{R}_d(R_d^T R) \\ &= R\hat{\Omega} - R_d\hat{\Omega}_d R_d^T R \\ &= R\big(\Omega - R^T R_d\Omega_d\big)^\wedge\end{aligned} \tag{19}$$

So the angular velocity tracking error is defined as

$$e_\Omega = \Omega - R^T R_d \Omega_d \tag{20}$$

The tracking error e_R and e_Ω are used to design attitude control for the Quadrotor.

$$\frac{\mathrm{d}}{\mathrm{d}x}(R_d^T R) = (R_d^T R)\hat{e}_\Omega \tag{21}$$

4.2 Tracking Controller

The tracking controller is deigned to control the quadrotor position and attitude by tracking a smooth desired $R_d(t)$. For these smooth tracking commands $x_d(t), \vec{b_1}_d(t)$ and some positive constants the paper defined

$$\vec{b_3}_d = -\frac{-k_x e_x - k_v e_v - mge_3 + m\ddot{x}_d}{\|-k_x e_x - k_v e_v - mge_3 + m\ddot{x}_d\|} \tag{22}$$

where it is assumed that

$$\|-k_x e_x - k_v e_v - mge_3 + m\ddot{x}_d\| \neq 0 \tag{23}$$

It was earlier assumed that $\vec{b_{1_d}}$ is not parallel to $\vec{b_{3_d}}$. The desired attitude is given by $R_d = [\vec{b_{2_d}} \times \vec{b_{3_d}}, \vec{b_{2_d}}, \vec{b_{3_d}}]$ and it was assumed that

$$\|-mge_3 + m\ddot{x}_d\| < B \tag{24}$$

for a given positive constant B. The control thrust is chosen as

$$f = -\big(-k_x e_x - k_v e_v - mge_3 + m\ddot{x}_d\big) \cdot Re_3 \tag{25}$$

Using equation 9 the derivative of e_R is calculated as follows

$$\begin{aligned} \dot{e_R} &= \frac{1}{2}(R_d^T + R\hat{e_\Omega} + \hat{e_\Omega} R^T R_d)^\vee \\ &= \frac{1}{2}(tr[R^T R_d]I - R^T R_d)e_\Omega \\ &\equiv C(R_d^T R)e_\Omega \end{aligned} \tag{26}$$

where $\left\|C(R_d^T R)\right\| \leq 1$ such that $\|\dot{e_R}\| \leq \|e_\Omega\|$ such that $R_d^T R \in SO(3)$, guaranteeing that $\dot{e_R}$ will be bounded, if and only i e_Ω is bounded. And using equation 20 to derive the derivative of angular velocity tracking error e_Ω $\hat{\Omega}_d \Omega_d = 0$,

$$\dot{e_\Omega} = \dot{\Omega} + \hat{\Omega} R^T R_d \Omega_d - R^T R_d \dot{\Omega}_d \tag{27}$$

From equation 9 we can obtain the equation of desired attitude as

$$\dot{R}_d = R_d \hat{\Omega}_d \;\; and \;\; \hat{\Omega}_d = R_d^T \dot{R}_d \tag{28}$$

The desired angular acceleration $\dot{\Omega}_d$ from the following equation

$$\begin{aligned} \dot{\hat{\Omega}} &= (\dot{R_d^T} \dot{R}_d) + (R_d^T \ddot{R}_d) \\ &= (R_d \hat{\Omega}_d)^T (R_d \hat{\Omega}_d) + (R_d^T \ddot{R}_d) \\ &= -\hat{\Omega}_d \hat{\Omega}_d + R_d^T \ddot{R}_d \end{aligned} \tag{29}$$

So

$$\dot{\Omega}_d = (-\hat{\Omega}_d \hat{\Omega}_d + R_d^T \ddot{R}_d)^\vee \tag{30}$$

Putting equation (8) in (27) we get

$$\dot{e_\Omega} = J^{-1}(-\Omega \times J\Omega + M) + \hat{\Omega} R^T R_d \Omega_d - R^T R_d \dot{\Omega}_d \tag{31}$$

From this, the control input M is defined, and consists of a proportional term, a derivative term and a canceling term, as follows

$$M = -k_R e_R - k_\Omega e_\Omega + \Omega \times J\Omega - J(\hat{\Omega} R^T R_d \Omega_d - R^T R_d \dot{\Omega}_d) \tag{32}$$

The equation for f and M will be defined as the control inputs. The moment M resembles the attitude tracking on SO(3). So by this equation the controller exponentially stabalizes the zero equilibrium of the attitude tracking error. Similarly the thrust equation corresponds for the translational dynamics for the uadrotor on $\mathbb{R}^3$. So The translational tracking error will converge to zero as long as e_R is almost zero.

As we know e_R may not be zero at any time .As this attitude tracking error increases, the direction of the control input term fRe_3 of the dynamics equation will deviates from the desired direction $R_d e_3$ and will make the dynamics of the system unstable. Succinctly, the system which is designed in the paper is when there is no attitude tracking error, the position tracking error tends to zero hence achieving asymptotic stability for the dynamics equation of the system.

4.3 Exponential Asymptotic Stability

We first show exponential stability of the attitude dynamics in the sub-level set

$$L_1 = \{R_d, R \in SO(3) | \Psi(R, R_d) < 2\} \tag{33}$$

and based on this result, we show exponential stability of the complete dynamics in the smaller sub-level set $L_1 = \{R_d, R \in SO(3) | \Psi(R, R_d) < 1\}$

Proposition 1: In this proposition we define Exponential Stability of Attitude Dynamics. So lets assume the initial condition to be

$$\Psi\big(R(0), R_d(0)\big) < 2 \tag{34}$$

$$\left\|e_\Omega(0))\right\|^2 < \frac{2}{\lambda_{max}(J))} k_R(2 - \Psi(R(0), R_d(0))) \tag{35}$$

where $\lambda_{min}(J)$ denotes the minimum eigenvalue of the inertia matrix J. Then, the zero equilibrium of the attitude tracking error e_R, e_Ω is exponentially stable. In addition to this there exist constants $\alpha_2, \beta_2 > 0$ such that

$$\Psi(R(t), R_d(t)) \leq min\left\{2, \alpha_2 e^{-\beta_2 t}\right\} \tag{36}$$

This proposition is only valid if the initial attitude error must be less than 180°. Therefore, the region of attraction for the attitude almost covers $SO(3)$, and the region of attraction for the angular velocity can be increased by choosing a larger gain k_R in equation (34).

Proposition 2: To achieve the Exponential stability of the system dynamics. we define the initial condition as

$$\Psi(R(0), R_d(0)) \leq \psi_1 < 1 \tag{37}$$

We computed the error for the dynamics of the system for e_R, e_Ω and define the Lyapunov function. With the help of Lyapunov function we bounded the control parameters to make sure our tracking erors are bounded.Using the controlled equation of moment from *Proposition 1* and general dynamics equation we compute time derivative of Je_Ω

$$J\dot{e}_\Omega = \{Je_\Omega + d\}^\wedge e_\Omega - k_R e_R - k_\Omega e_\Omega + \mathbb{W}_R \tilde{\theta}_R \tag{38}$$

here $d = 2J - (tr[J]I)R^T R_d \Omega_d \in \mathbb{R}^3$ and $\tilde{\theta}_R = \theta_R - \bar{\theta}_R$ and $\mathbb{W}_R$ corresponds to the modeling error and uncertainties in the the rotational dynamics. The important point to note is the term $\left[\{Je_\Omega + d\}^\wedge\right]$ is normal to e_Ω hence simplifies the Lyapunov analysis. So the Lyapunov function described in the paper([5]) was

$$V_2 = \frac{1}{2}e_\Omega \cdot Je_\Omega + k_R\Psi\left(R, R_d\right) + c_2 e_R \cdot Je_\Omega + \frac{1}{2\gamma_R}\left\|\tilde{\theta}_R\right\|^2 \tag{39}$$

from equation (15) the above Lyapunov function V_2 is bounded as

$$z_2^T M_{21} z_2 + \frac{1}{2\gamma_R}\left\|\tilde{\theta}_R\right\|^2 \leq V_2 \leq z_2^T M_{22} z_2 + \frac{1}{2\gamma_R}\left\|\tilde{\theta}_R\right\|^2 \tag{40}$$

where $z_2 = [\|e_R\|, \|e_\gamma\|]^2 \in \mathbb{R}^2$, and the matrices M_{12}, M_{22} are given by,

$$M_{21} = \frac{1}{2}\begin{bmatrix} k_R & -c_2\lambda_M \\ -c_2\lambda_M & \lambda_M \end{bmatrix}, M_{22} = \frac{1}{2}\begin{bmatrix} \frac{2k_R}{2-\Psi_2} & c_2\lambda_M \\ c_2\lambda_M & \lambda_M \end{bmatrix} \tag{41}$$

The upper-bound of (40) is satisfied in the following domain:

$$D_2 = \left\{(R, \Omega) \in SO(3) \times \mathbb{R}^3 | \Psi(R, R_d) < \psi_2 < 2\right\} \tag{42}$$

From equation (38) and the fact that $\dot{\Psi} = e_R \cdot e\Omega, \|\dot{e_R}\| \leq \|e_\Omega\|$

$$V_2 = -k_\Omega \left\|e_\Omega\right\|^2 + e_\Omega^T \mathbb{W}_R \theta_R + c_2\dot{e_R} \cdot Je_\Omega + c_2 e_R \cdot J\dot{e_\Omega} + \frac{1}{\gamma_R}(\tilde{\theta})^T(\dot{\tilde{\theta}}) \tag{43}$$

We know $\dot{\tilde{\theta}} = -\dot{\tilde{\theta}}_R$ and putting the value of $J\dot{e_\Omega}$ we get

$$\begin{aligned} \dot{V}_2 = -k_\Omega \left\|e_\Omega\right\|^2 + c_2\dot{e}_R \cdot Je_\Omega - c_2 k_R \left\|e_R\right\|^2 + c_2 e_R \cdot \left(\left(Je_\Omega + d\right)^\wedge e_\Omega - k_\Omega e_\Omega\right) \\ + \tilde{\theta}_R^T \mathbb{W}_R^T\left(e_\Omega + c_2 e_R\right) - \frac{1}{\gamma_R}\tilde{\theta}_R^T\dot{\tilde{\theta}}_R \end{aligned} \tag{44}$$

By substituting the adaptive law i.e. $\dot{\tilde{\theta}}_R = \gamma_R \mathbb{W}_R^T\left(e_\Omega + c_2 e_R\right)$ we get

$$\dot{V}_2 = -k_\Omega \left\|e_\Omega\right\|^2 + c_2\dot{e}_R \cdot Je_\Omega - c_2 k_R \left\|e_R\right\|^2 + c_2 e_R \cdot \left(\left(Je_\Omega + d\right)^\wedge e_\Omega - k_\Omega e_\Omega\right) \tag{45}$$

Since $\|e_R\| \leq 1, \|\dot{e_R}\| \leq 1$ and $\|d\| \leq B_2$ we can define

$$V_2 \leq -z_2^T W_2 z_2 \tag{46}$$

where the marix $W_2 \in \mathbb{R}^{2\times 2}$ is given by

$$W_2 = \begin{bmatrix} \frac{c_2 k_R}{\lambda_{max}(J)} & -\frac{c_2 k_\Omega}{2\lambda_{min}(J)} \\ -\frac{c_2 k_\Omega}{2\lambda_{min}(J)} & k_\Omega - c_2 \end{bmatrix}, \tag{47}$$

Similarly we can derive the value of W_{12} and W_1 as,

$$W_1 = \begin{bmatrix} \frac{c_1 k_x}{m} & \frac{-c_1 k_v}{2m}(1+\alpha) \\ \frac{-c_1 k_v}{2m}(1+\alpha) & k_v(1-\alpha) - c_1 \end{bmatrix} \tag{48}$$

$$W_{12} = \begin{bmatrix} k_x e_{v_{max}} + \frac{c_1}{m}B & 0 \\ B & 0 \end{bmatrix}, \tag{49}$$

where $\alpha = \sqrt{\psi_1(2-\psi_1)}$, $e_{v_{max}} = max\left\{\|e_v(0)\|, \frac{B}{k_v(1-\alpha)}\right\}$, $c_1, c_2 \in \mathbb{R}$. For any positive constants k_x, k_v, we choose positive constants c_1, c_2, k_R, k_Ω such that:

$$c_1 < min\left\{k_v(1-\alpha), \frac{4mk_xk_v(1-\alpha)}{k_v^2(1+\alpha)^2 + 4mk_x}, \sqrt{k_x m}\right\} \tag{50}$$

$$c_2 < min\left\{k_\Omega, \frac{4k_\Omega k_R \lambda_{min}(J)^2}{k_\Omega^2 \lambda_{max}(J) + 4k_R \lambda_{min}(J)^2}, \sqrt{k_R \lambda_{min}(J)}, \sqrt{\frac{2}{2-\psi_1} k_R \lambda_{max}(J)}\right\}, \tag{51}$$

$$\lambda_{min}(W_2) > \frac{4\left\|W_{12}\right\|^2}{\lambda_{min}(W_1)} \tag{52}$$

With this the the equilibrium of the tracking error is exponentially stable. The region of attraction is characterized by (33) we get

$$\left\|e_\Omega(0)\right\|^2 < \frac{2}{\lambda_{max}(J)} k_R \left(1 - \Psi(R(0), R_d(0))\right) \tag{53}$$

5 Properties of the controller

The unique properties of this controller it is derived straightforward in $SE(3)$ using Euler angles and rotation matrix. With the help of that it avoid the singularities and complexities in the local coordinates in $SO(3)$. It avoids the use of quaternions as it could have ambiguities when representing attitude dynamics. As any quaternion described in attitude feedback controller could return different control inputs as it depends on the preffered quaternion vectors. Without these considerations the controller could display amorphous esults where the quadrotor unessentially rotates through large angles. So the use of Rotation matrices and in not only designing the controller equation but also in the stability analysis eradicate those complications.

Another uniqueness to this controller design is the choice of thrust f as the input. The controller is designed to follow the trajectory by mantaining the stability in the dynamics of the system by having the direction of third body fixed axis b_{3_d} as the feedback control of the system. This consideration is obvious as every coloum of the rotation matrix represents the body fixed axis associated with it. So another convenience in choosing this method of controller design that it has a clear-cut physical interpretation.

6 Simulation

The simulation for this system was carried out on Matlab 2016 software. The simulation done in order to observe the closed loop stability of the tracking command. The simulations were done on those cases and parameters provided in the "Numerical example" section of the paper. In order to simulate we have used a Geometric-toolbox created by Avinash Siravuru [2].
The parameters of quadrotor are chosen as per given in the paper.

$$J = [0.0820, 0.0845, 0.1377]\, kgm^2 \quad , \quad m = 4.34kg$$

$$d = 0.315m \quad , \quad c_\tau f = 8.004 \times 10^{-4} m$$

There were two cases that the author has considered. In Case 1 the quadrotor follows a spiral like movement in which the desired state is time dependent hence testing the behaviour in continuously changing trajectory after every point and for Case 2, The quadrotor is initially in inverted position and the aim is to recover form initial state to default state (with $\Omega = [0, 0, 0]$). To do so the UAV follows a smooth trajectory and in both the cases we observed the error reaching zero hence achieving our goal to achieve global exponential attractiveness.

6.1 Case : 1

In this maneuver the quadrotor follows an eliptical helix trajectory while rotating the heading direction at a predefined rate. So the initial conditions are chosen as per provided in the paper.

$$x\,(0) = [\;0\;,\;0\;,\;0\;] \quad , \quad v\,(0) = [\;0\;,\;0\;,\;0\;]$$

$$R(0) = I \quad , \quad \Omega(0) = [\;0\;,\;0\;,\;0\;]$$

The desired trajectory $x_d(t)$ is given as

$$x_d(t) = [\;0.4t\;,\;0.4sin(\pi t)\;,\;0.6cos(\pi t)\;]$$

$$\vec{b}_{1_d}(t) = [\;cos(\pi t)\;,\;sin(\pi t)\;,\;0\;]$$

The results of the simulations are shown below

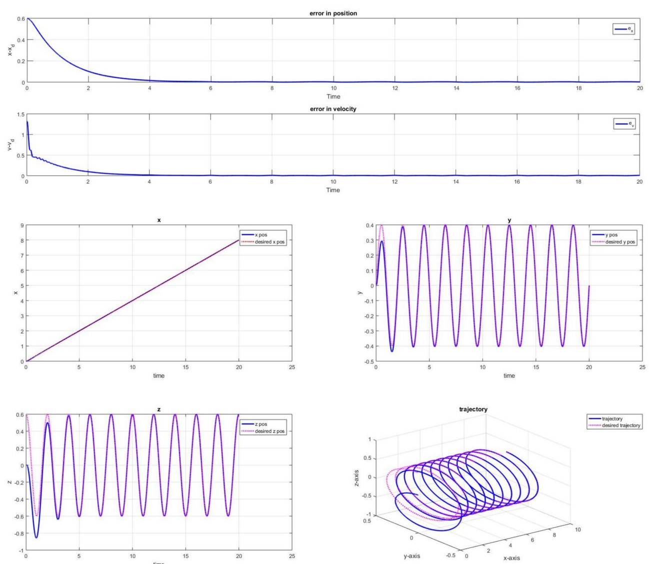

Figure 7: (a) position and velocity error (b) Trajectory tracking

Here we can observe the position error starts from 0.6 and velocity error starts from 1.3. This gratifies the Preposition 2 as we calculate the Tracking error $\Psi(0)$ is less than 0.15. This refer the Proposition 2, and exponential stability is achieved as shown in Figure 4, This justify that the quadrotor with this designed controller can follow complex trajectory that involve large angle rotations and nontrivial translations accurately.

6.2 Case : 2

In this case the quadrotor recovers itself from being upside down in initial conditions.

$$x\,(0) = [\,0\;,\;0\;,\;0\,] \quad , \quad v\,(0) = [\,0\;,\;0\;,\;0\,]$$

$$R(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.9995 & -0.0314 \\ 0 & 0.0314 & -0.9995 \end{bmatrix} \quad , \quad \Omega(0) = [\,0\;,\;0\;,\;0\,]$$

The desired trajectory is as follows

$$x_d(t) = [\,0\;,\;0\;,\;0\,] \quad , \quad \vec{b}_{1_d}(t) = [\,1\;,\;0\;,\;0\,]$$

The result of simulation is shown below

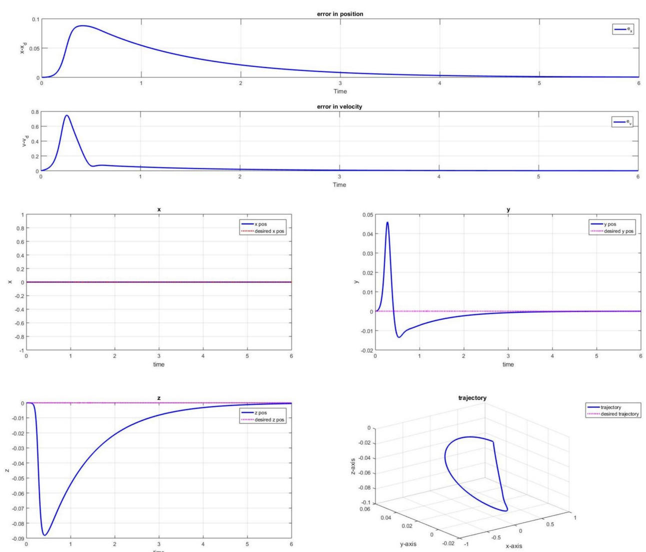

Figure 8: (a) position and velocity error (b) Trajectory tracking

The initial attitude error is 178°, which gives initial attitude error $\Psi(0) = 1.995$. This refers to Proposition 3, Which proves the earlier claimed fact that this controller achieves the global exponential attractiveness. So the region of attraction of the designed controller covers SO(3), with the help of that he quadrotor retrieve from being initially upside down.

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