Linear Rotation

14th October 2013

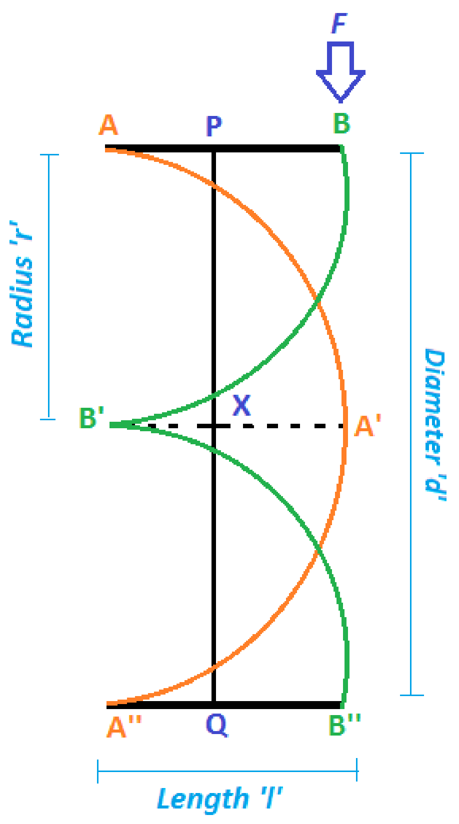
Have you ever observed certain motions of objects which execute two or more types of motions simultaneously? One such type is the motion of an object moving linearly forward and rotating along its centre simultaneously.

LINEAR ROTATION

The locus of two end points of a linear object which is traced out as it moves forward in linear motion and rotates about its centre along its path executes a unique form of motion known as Linear Rotation or Linear Rotational Motion.

The path traced out by the two-ended object is as follows.
NOTE: The two points move simultaneously, but not in sync. They move one after another.
The figure represents two points A & B of the object, and a force 'F' is applied at point B. With respect to the force applied, B moves first and then 'A' follows.

In the figure, AP = PB = A′X = XB′ = A″Q = QB″



The centre 'P', when considered as a point, moves in linear motion in one dimension. The point 'A' takes a parabolic path and point 'B' resembles a cycloid produced by a circle of varying radius (Will be discussed later in detail).

Terms & Definitions:

Cycles
The number of rotations completed by a linearly rotating object is called its cycle. Represented by **'n'**

Linear Distance
The linear distance covered by a linearly rotating object is its Linear Distance Represented by the letter, **d**

Linear Distance = PQ units = d

Object Length
The length of the object from one end-point to the other is the object length. Represented by **'l'**

Object Length = AB = l

Arc Distance
The total distance covered by each end-point (arc length) is called Arc Distance
Represented by **'s'**

Arc Distance = A=A′=A″=B=B′=B″ = s

Focus
The point at the centre of the linear object is called Focus

*In the fig. above, **P** (or **Q** or **X**) is the Focus of the object*

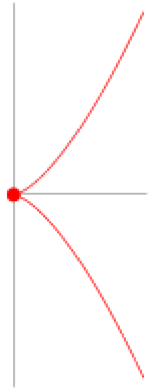
Vertex
The centre of the parabolic curve traced out by the end-point opposite to where the Force was applied.

*In the fig. above, **A′** is the Vertex*

Cusp

A cusp is a point at which two branches of a curve meet such that the tangents of each branch are equal.

In the fig. below, the red point is the Cusp



Properties of objects in Linear Rotation:

Some observed properties of objects with two end-points in linear rotation are given below:

NOTE: These have been concluded after experimental observations and thus do not require any scientific proofs;

1. Objects in LR travel on a plane in two dimensions.
2. The Linear Distance of an object in LR is directly proportional to the force applied (for a given length of the object).
3. The Linear Distance of an object in LR is inversely proportional to the object's length (for a constant force applied on it).
4. The Linear Distance of an object in LR for a given length will vary with varying force.
5. The Linear Distance of an object in LR for a constant force applied will vary with varying lengths.
6. The Linear Distance of an object will remain the same when a constant force is applied on any two points lying opposite to each other, provided both the points are equidistant from the centre.
7. The centre, when considered as a separate point, moves in linear motion.

8. The end points, when taken as two separate points revolving around the centre and moving along the circumference of a circle, apparently executes circular motion, but since they’re the end points attached to a rigid body, it is considered to rotate about its centre (axis of rotation), thus executing rotational motion.
9. When a force is applied at one of the points, the object executes LR and later comes to a stop after a certain period of time due to friction, air resistance and also due to the work done by the object.
10. When force is applied exactly at the centre, the object executes linear motion only and the two end points move together along with the centre, where all the points in the body move at the same velocity and acceleration.

11. While an object stops due to friction, it first stops moving linearly and later stops rotating; this is because the opposing energy required to stop an object from moving forward is lesser than the energy required in stopping an object from rotating. Moreover, since the area covered by a rotating object at a particular point is less, friction acting on the body will be less, which makes the object rotate for a longer time.

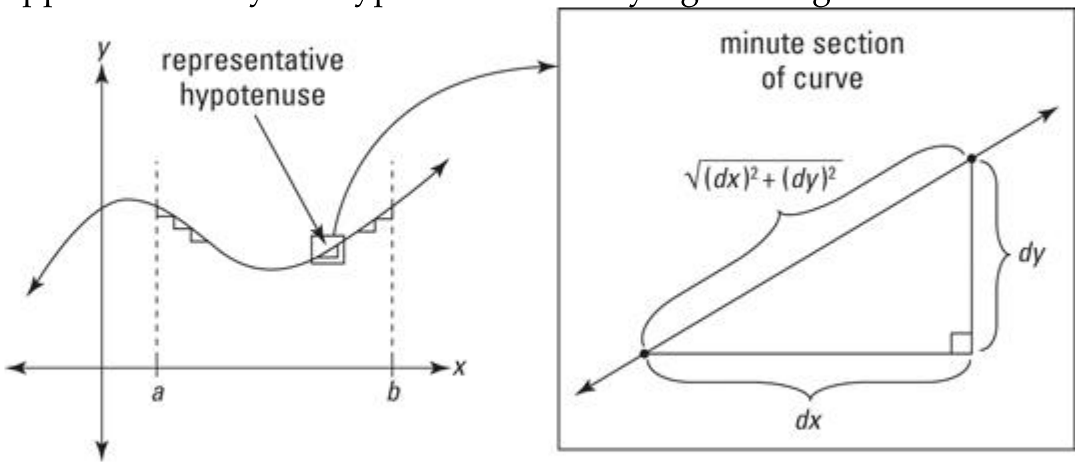
Important Resources –I

Before we proceed into the subject, it is required for us to have knowledge about certain concepts. Let us revise these derivations.

I. <u>Arc length formula for any curve</u>

When you use integration to calculate arc length, what you’re doing (sort of) is dividing a length of curve into infinitesimally small sections, figuring the length of each small section, and then adding up all the little lengths.

The following figure shows how each section of a curve can be approximated by the hypotenuse of a tiny right triangle.



The Pythagorean Theorem is the key to the arc length formula.

You can imagine that as you zoom in further and further on a curve, dividing the curve into more and more sections, the minute sections get straighter and straighter and the hypotenuses of the right triangles better and better approximate the curve. That's why — when this process of adding up smaller and smaller sections is taken to the limit — you get the precise length of the curve.
So, all you have to do is add up all the hypotenuses along the curve between your start and finish points. The lengths of the legs of each infinitesimal triangle are *dx* and *dy*, and thus the length of the hypotenuse — given by the Pythagorean Theorem —

$$\sqrt{(dx)^2 + (dy)^2}$$

Integrating the equation from a to b, we get

$$\int_a^b \sqrt{(dx)^2 + (dy)^2}$$

First, factor out a $(dx)^2$ under the square root and simplify:

$$\int_a^b \sqrt{(dx)^2\left[1 + \frac{(dy)^2}{(dx)^2}\right]} = \int_a^b \sqrt{(dx)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

Now you can take the square root of $(dx)^2$ – that's dx, of course – and bring it outside the radical, and voilà, you've got the formula.

The arc length along a curve, $y = f(x)$, from a to b, is given by the following integral:

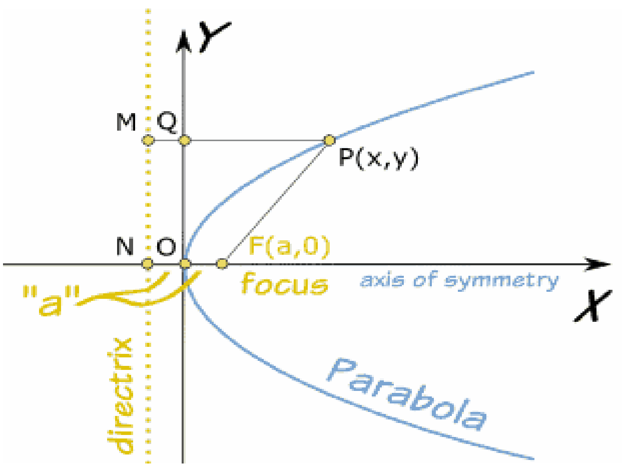
$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Therefore, the arc length, s, for a Cartesian curve *y=f(x)* from *a* to *b* is given by:

$$\boxed{s = \int_a^b \sqrt{1+f'(x)}\,dx}$$

Now we shall use this Arc Length Formula and derive another important equation to the find the arc length of a parabola, $\boldsymbol{y^2 = 4ax}$ **,** measured from the vertex.

II. Length of the arc of the parabola $y^2 = 4ax$, measured from the vertex



GIVEN: $\boldsymbol{y^2 = 4ax}$

Differentiating w.r.t $\boldsymbol{x}$, we get

$$= 2y.\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{2a}$$

If s denotes the arc length of the parabola measured from the vertex O to any point $P(x, y)$ then s *increases* as y *increases*

∴ Required length of the arc is

$$s = \int_0^y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\Rightarrow \int_0^y \sqrt{1 + \frac{y^2}{4a^2}}\, dy$$

$$\Rightarrow \int_0^y \sqrt{\frac{1}{4a^2}(4a^2 + y^2)}\, dy$$

$$\Rightarrow \int_0^y \sqrt{\frac{1}{4a^2}(4a^2 + y^2)}\, dy$$

$$\Rightarrow \frac{1}{2a}\int_0^y \sqrt{4a^2 + y^2}\, dy$$

WKT $\boldsymbol{\sqrt{x^2 + a^2}.dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x^2 + \sqrt{x^2 + a^2} + C}$

Applying the above identity, we get

$$\Rightarrow \frac{1}{2a}\left[\frac{y}{2}\left(\sqrt{y^2 + 4a^2}\right) + \frac{4a^2}{2}\,\log\left(y + \sqrt{y^2 + 4a^2}\right)\right]_0^y$$

$$\Rightarrow \frac{1}{2a}\left[\frac{y}{2}\left(\sqrt{y^2+4a^2}\right)+\frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)-\frac{4a^2}{2}\log 2a\right]$$

$$\Rightarrow \frac{1}{2a}\left[\frac{y}{2}\left(\sqrt{y^2+4a^2}\right)+\frac{4a^2}{2}\log\left(\frac{y+\sqrt{y^2+4a^2}}{2a}\right)\right]$$

$$\Rightarrow \frac{1}{4a}\left[y\left(\sqrt{y^2+4a^2}\right)+4a^2\log\left(\frac{y+\sqrt{y^2+4a^2}}{2a}\right)\right]$$

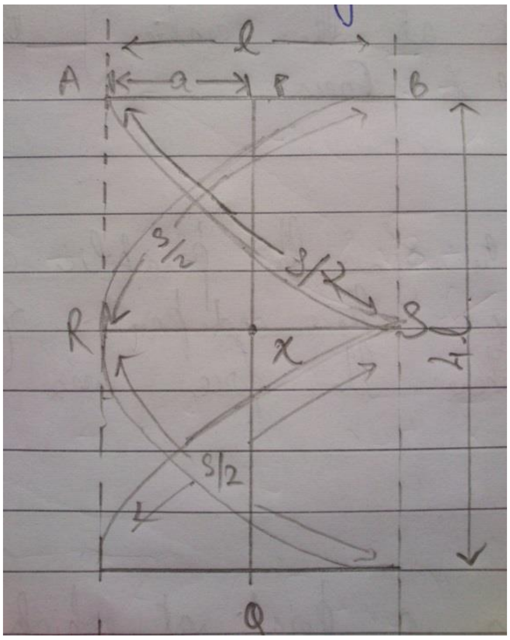
The above equation is the arc length for the upper half of the parabola. The Total arc length *(upper & lower half)* is twice the obtained equation.

∴ Length of the arc of a parabola is given by

$$\boxed{s=\frac{1}{2a}\left[y\left(\sqrt{y^2+4a^2}\right)+4a^2\log\left(\frac{y+\sqrt{y^2+4a^2}}{2a}\right)\right]}$$

Now that we have the required knowledge, we can proceed to dig deep into the subject of Linear Rotation.

<u>Arc Distance Formula</u>



Let us now derive a relation between the Arc distance and the length of an object moving in linear rotation.

<u>NOTE:</u>

The arc distance varies directly with the object length for a constant force applied on it.