SPATIAL AWARENESS

Coordinates, transformations and uncertainty

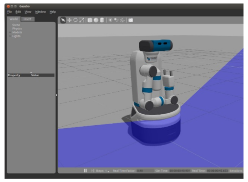
Teresa Vidal-Calleja

ROBOTIC SYSTEMS - MODEL BASED

Sensors

Perception
(physics-driven models,
geometric representations)

Actuators

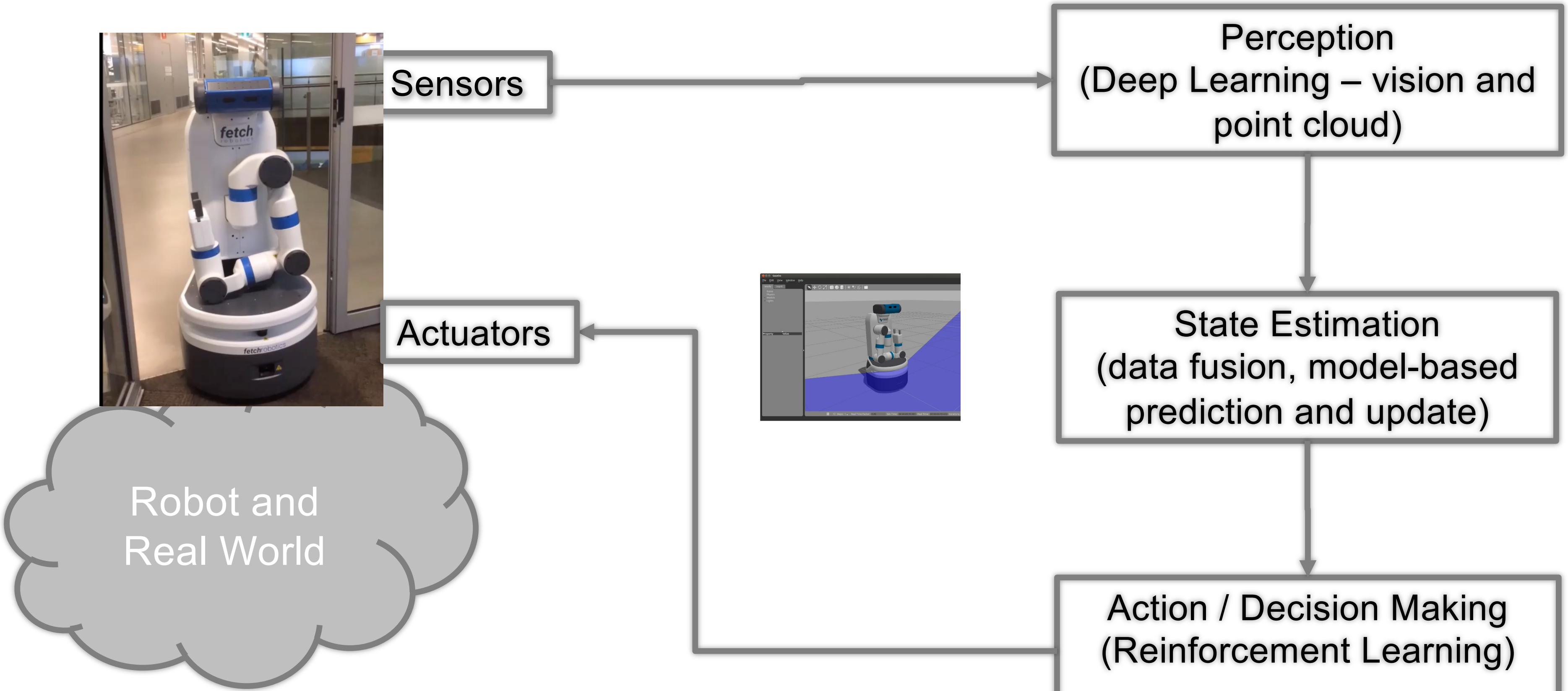


State Estimation
(data fusion, model-based
prediction and update)

Robot and
Real World

Action / Decision Making
(model-based control, path
planning, forward/inverse
kinematics)

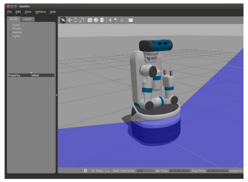
ROBOTIC SYSTEMS – MODEL/DATA DRIVEN



ROBOTIC SYSTEMS – DATA DRIVEN

Sensors

Perception / Estimation
(Deep Learning – vision and point cloud)

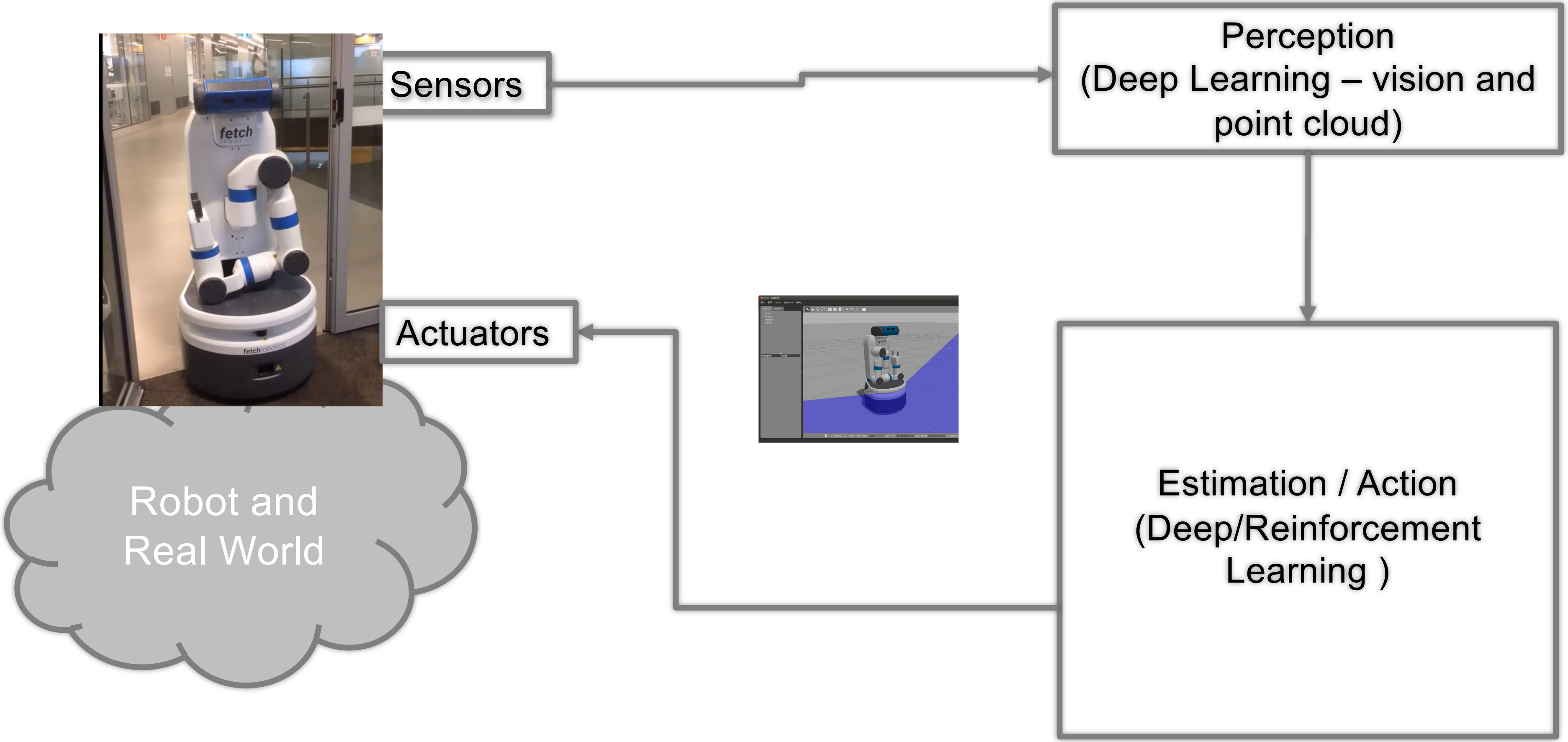


Actuators

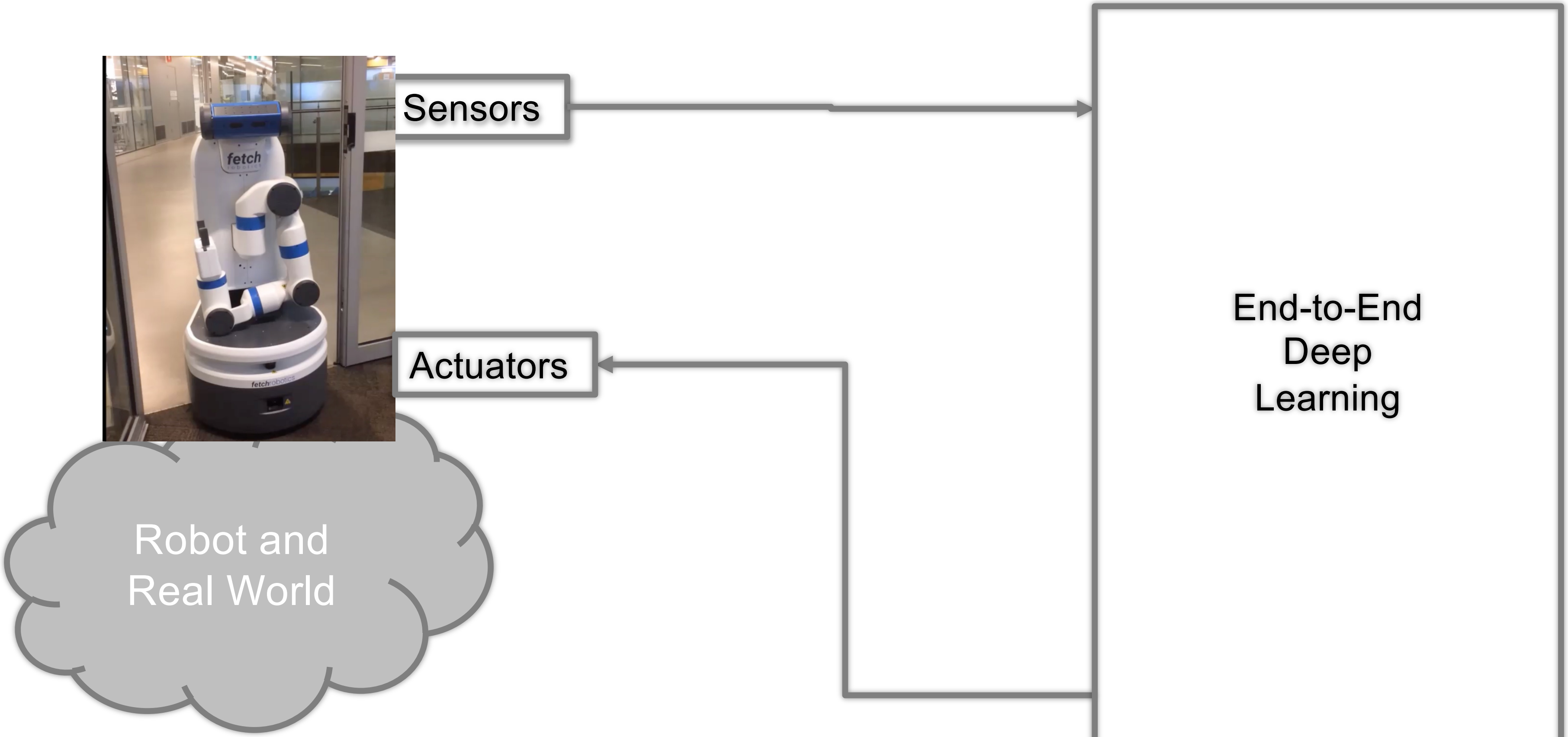
Robot and
Real World

Action / Decision Making
(Reinforcement Learning)

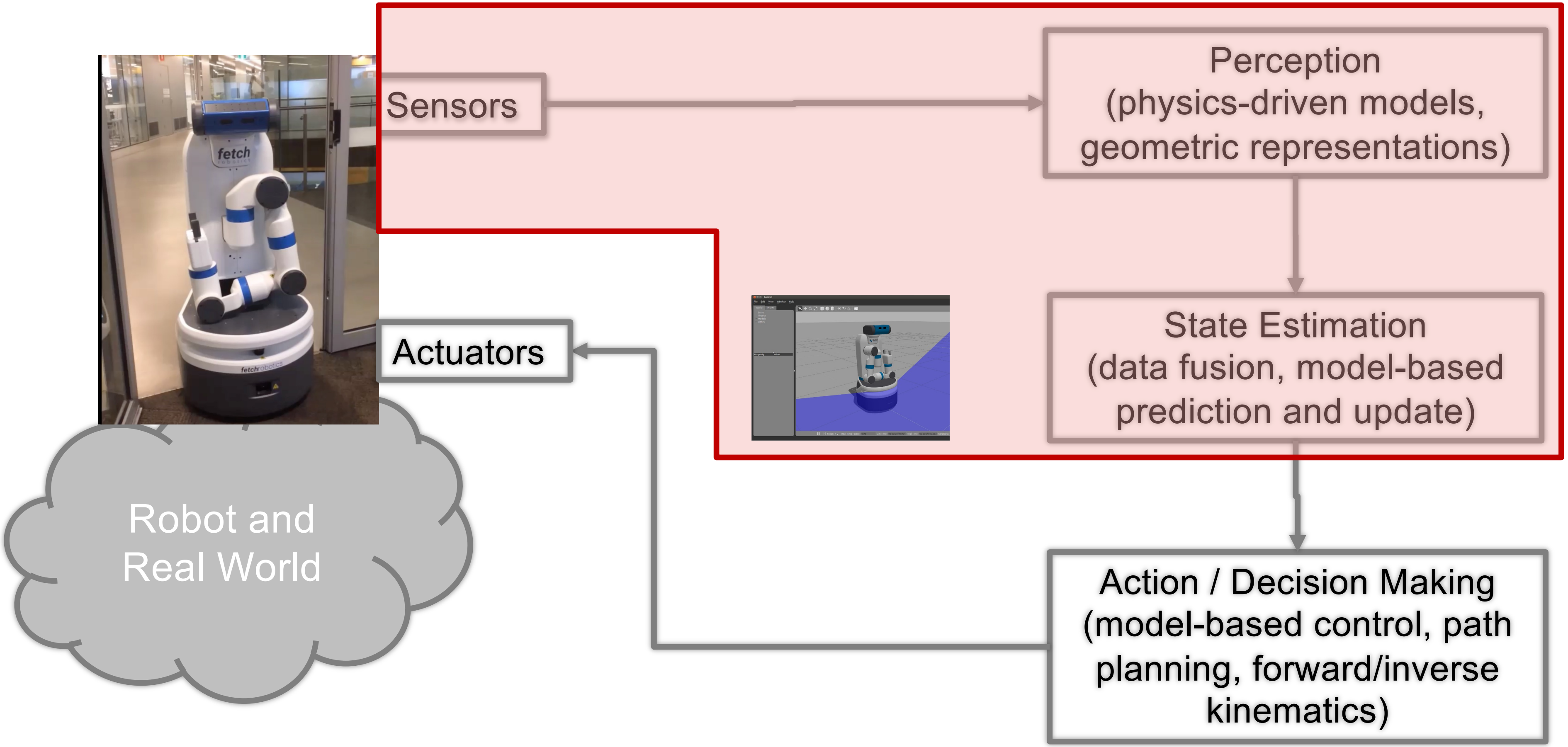
ROBOTIC SYSTEMS – DATA DRIVEN



ROBOTIC SYSTEMS – DATA DRIVEN



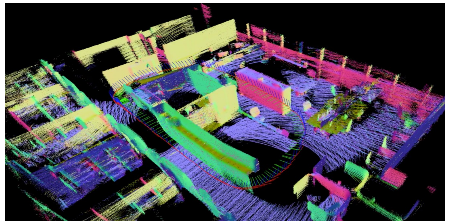
ROBOTIC SYSTEMS - MODEL BASED



SPATIAL AWARENESS

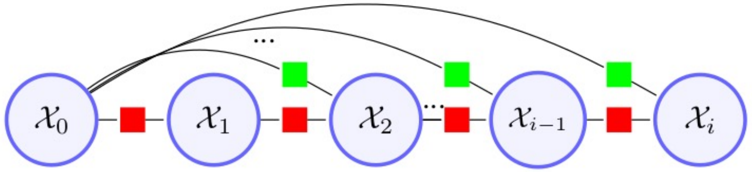
Robot Pose – position and orientation

- 2D position and heading of a vehicle
- angular position of each joint in a robot arm
- 3D position and orientation of an aerial vehicle or the tool at the end of the robot arm



Spatial representation of the world

Estimation algorithms to maintain the state of the robot and/or world and to filter the sensor noise

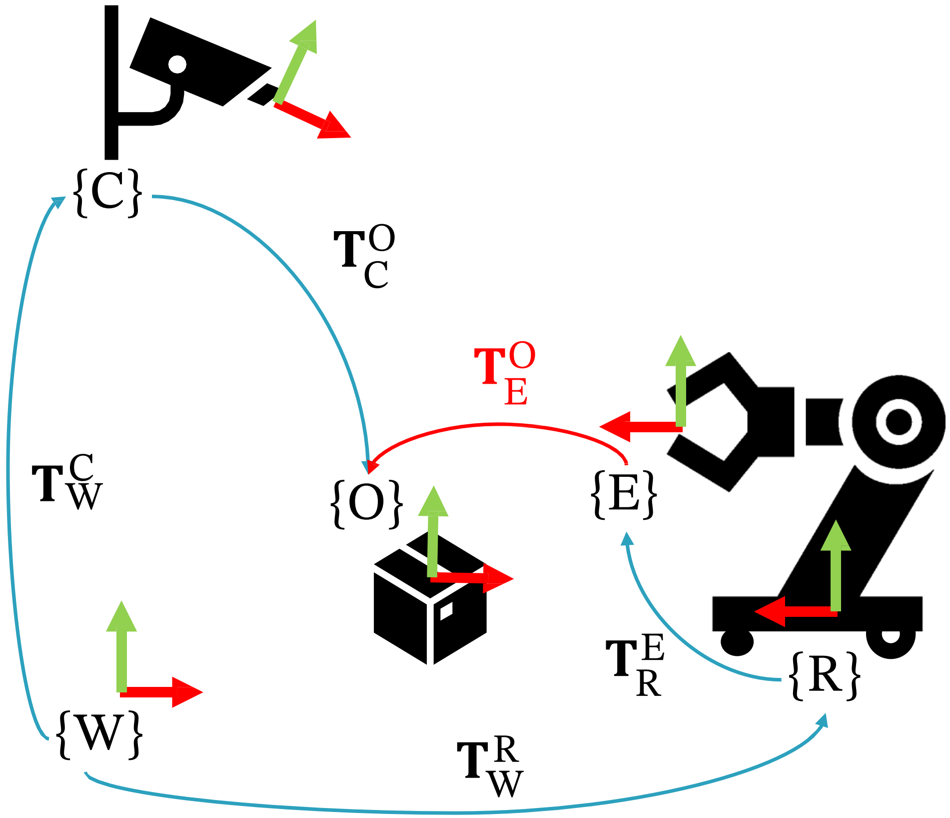


through Sensors (and models)

SPATIAL AWARENESS

We need to

- define the coordinate reference frames and find the transformation between fixed coordinate frames (calibration)
- have a systematic way represent poses (and world)
- infer the current robot pose (estimation) and/or understand the world through sensor observations and models



ROBOT SENSING

Proprioceptive sensors measure the state of the robot itself

- the angle of the joints on a robot arm
- the number of wheel revolutions on a mobile robot
- the accelerations and angular velocities of drones and ground robots

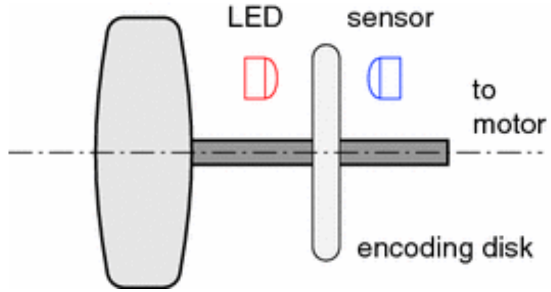
Exteroceptive sensors measure the state of the world with respect to the robot

- a simple contact switch on a vacuum cleaner robot to detect collision
- a GPS receiver that measures distances to an orbiting satellite constellation
- a compass that measures the direction of the Earth's magnetic field relative to the robot's heading
- a depth sensor that can sense distance and direction to the obstacles in the field of view (fov)
- a vision sensor that provide images that can be processed to understand the world in the fov

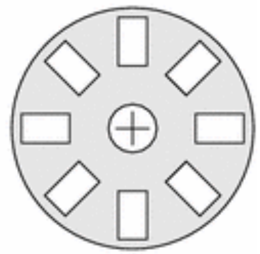
SENSING POSITION

Optical encoders (proprioceptive)

- used to measure the angular position of each joint in a robot arm
- they can also be used to measure the rotation of each wheel of a vehicle for odometry
- highly noisy



Optical wheel encoder

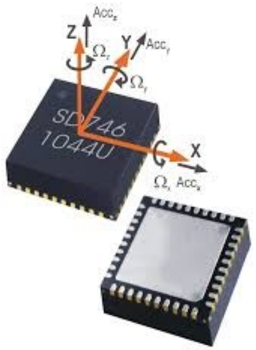


Encoding disk

SENSING POSE

Inertial Measurement Unit (proprioceptive)
- measures the proper linear accelerations and angular velocities with accelerometers and gyroscopes
- highly noisy
- require bias estimation and gravity alignment

Motion Capture Systems (proprioceptive/exteroceptive)
- optical positioning (mainly indoors) through infrared cameras and markers with infra-red retro reflective material
- Very accurate



IMU

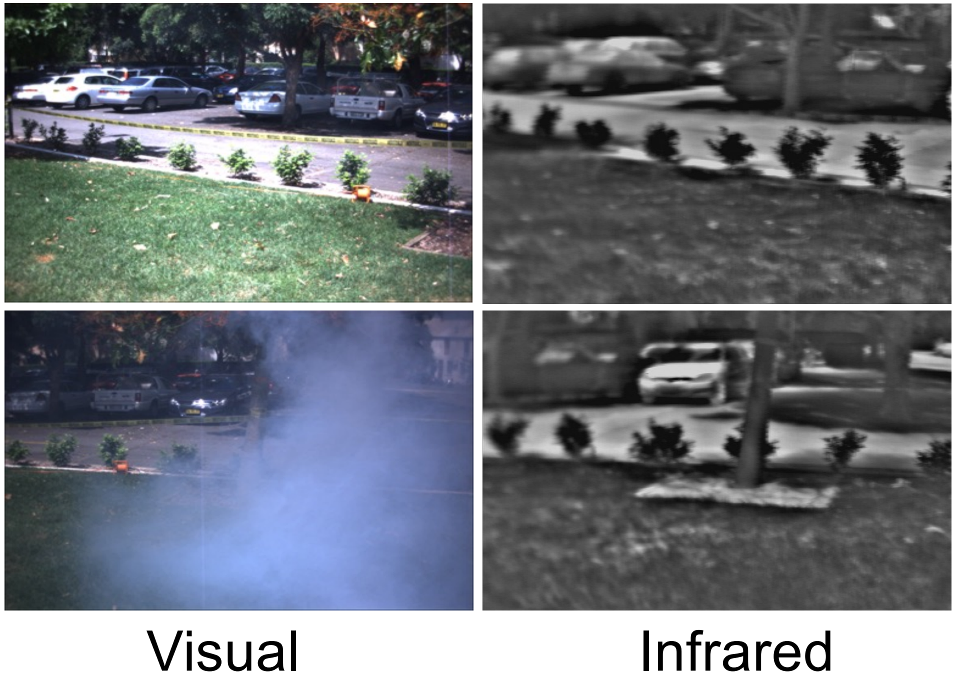
Vicon System

SENSING WITH VISION

Cameras (exteroceptive)

- Monocular / Stereo
- Monochrome / RGB
- Infrared / Thermal / Multispectral
- Depth

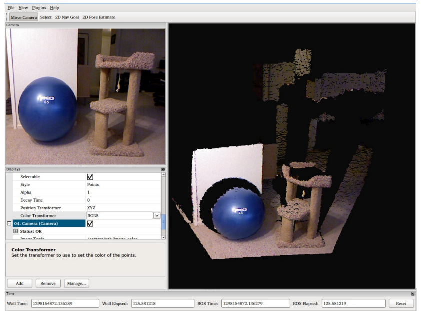
- Requires image processing
- monocular is subject to scale
- can provide spatial and appearance information
- noise depends on algorithms



SENSING WITH DEPTH SENSORS

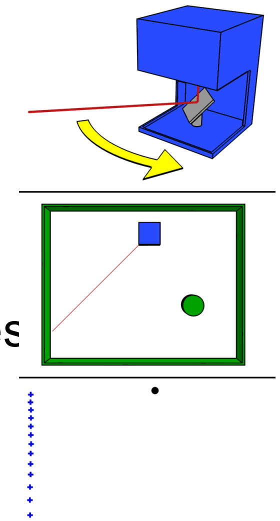
Depth Cameras (exteroceptive)

- Measures 3D information
- Similar to Stereo/ can be combined with Stereo
- Structure Light or Time of Flight
- highly noisy in depth (0.6 – 5m)
- can provide spatial and appearance information



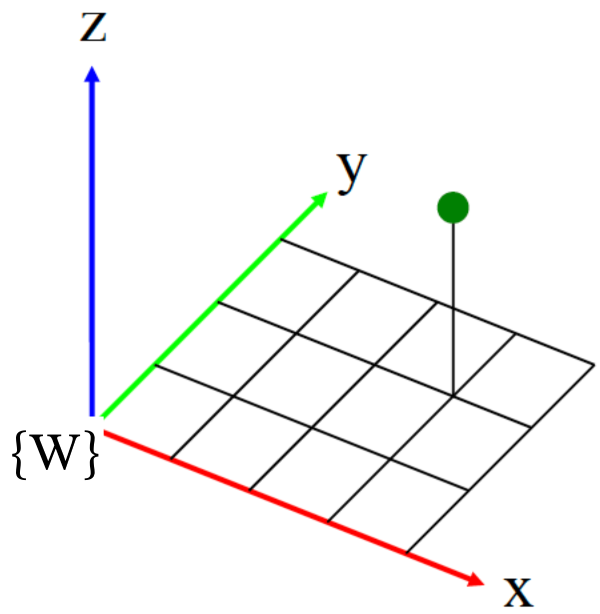
Lidars(exteroceptive)

- Scanning (rotating elements – mirror)
- returns distance and bearings to the neares objects
- range 1-100m
- can be accurate



REPRESENTING POSE - POSITION

Position in space is represented using a vector for the x, y and z coordinates of a point:

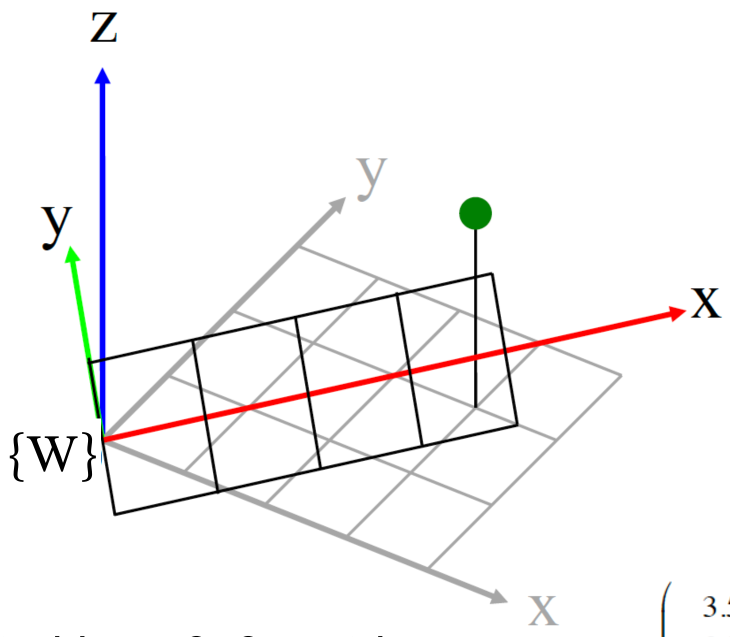


Define the origin and axes conventions

credit T. Drummond

REPRESENTING POSE - ORIENTATION

If the axes point in different directions, then the point has different coordinates:



This rotation can be represented by a 3x3 matrix:

$$\begin{pmatrix} 3.54 \\ -0.707 \\ 2 \end{pmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

credit T. Drummond

ROTATION

Rotations in 3D are represented by 3x3 matrices

But not just any 3x3 matrix – what are the rules?

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Lengths must be preserved: $|v| = |Rv|$

Angles must be preserved: $v_1 \cdot v_2 = (Rv_1) \cdot (Rv_2)$

PRESERVING LENGTHS

$$|v| = |Rv|$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_{11} \\ r_{21} \\ r_{31} \end{pmatrix} \quad \text{so} \quad \left| \begin{matrix} r_{11} \\ r_{21} \\ r_{31} \end{matrix} \right| = 1$$

This means that every column of R must be unit length

PRESERVING ANGLES

$$v_1 \cdot v_2 = (Rv_1) \cdot (Rv_2)$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_{11} \\ r_{21} \\ r_{31} \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r_{12} \\ r_{22} \\ r_{32} \end{pmatrix}$$

$$\begin{pmatrix} r_{11} \\ r_{21} \\ r_{31} \end{pmatrix} \cdot \begin{pmatrix} r_{12} \\ r_{22} \\ r_{32} \end{pmatrix} = 0$$

Different columns of R are orthogonal

R^T R – ORTHOGONAL MATRIX

What is $R^T R$?

$$RR^T = \begin{bmatrix} \boxed{r_{11} \quad r_{21} \quad r_{31}} \\ r_{12} \quad r_{22} \quad r_{32} \\ r_{13} \quad r_{23} \quad r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}$$

Because this is unit length

R^T R

What is $R^T R$?

$$RR^T = \begin{bmatrix} \boxed{r_{11} \quad r_{21} \quad r_{31}} \\ r_{12} \quad r_{22} \quad r_{32} \\ r_{13} \quad r_{23} \quad r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ & & \\ & & \end{bmatrix}$$

Because these are orthogonal

credit T. Drummond

R^T R

What is $R^T R$?

$$R^T R = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $R^T = R^{-1}$

This means that R^T is also a rotation (it rotates back)

So *rows* of R are also unit length and orthogonal

These matrices form a group called SO(3) – Special Orthogonal Group

ROTATING ABOUT THE AXES

The matrices that represent rotations about the x y and z axes are:

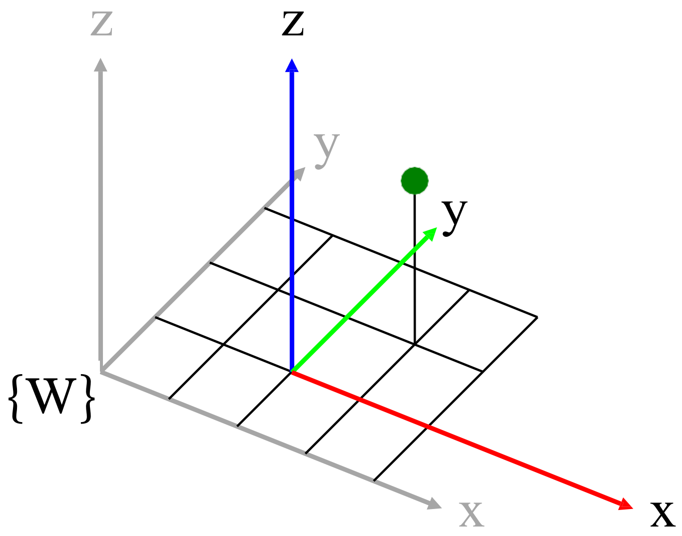
$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & -\sin(\theta_x) \\ 0 & \sin(\theta_x) & \cos(\theta_x) \end{bmatrix} \qquad R_y = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos(\theta_z) & -\sin(\theta_z) & 0 \\ \sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

credit T. Drummond

TRANSLATION

Move the origin to a different place:

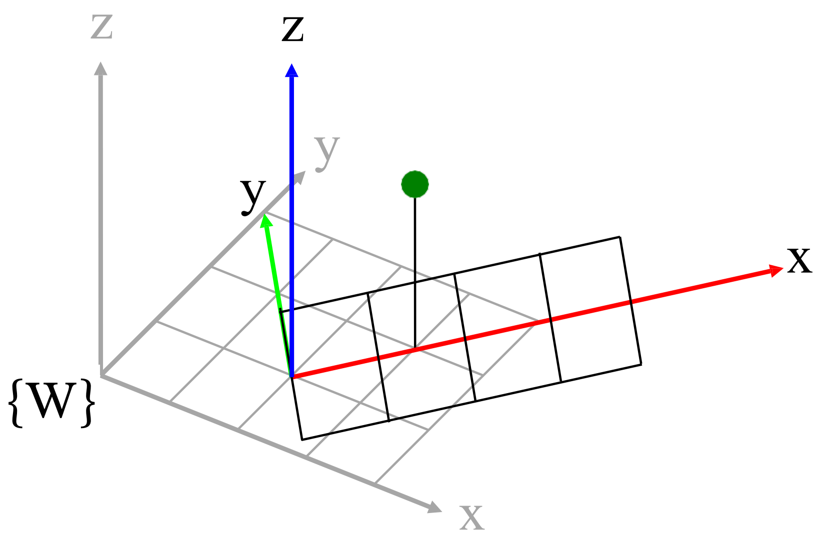


This shift (translation) can be represented by adding a vector:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

BOTH TRANSLATION AND ROTATION

We can move the origin *and* rotate the frame



This can be represented by a rotation matrix and adding a vector:

$$\begin{pmatrix} 1.414 \\ 0 \\ 2 \end{pmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2.121 \\ 0.707 \\ 0 \end{pmatrix}$$

credit T. Drummond

HOMOGENEOUS COORDINATES

Position vectors can be extended to 4 dimensions by adding a 1 at the end

This representation of position is called homogeneous coordinates

This lets us use a matrix to apply a translation:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x+2 \\ y+3 \\ z+4 \\ 1 \end{pmatrix}$$

credit T. Drummond

HOMOGENEOUS COORDINATES

Position vectors can be extended to 4 dimensions by adding a 1 at the end

This representation of position is called homogeneous coordinates

This lets us use a matrix to apply a translation:

This is the translation that gets added to x y z

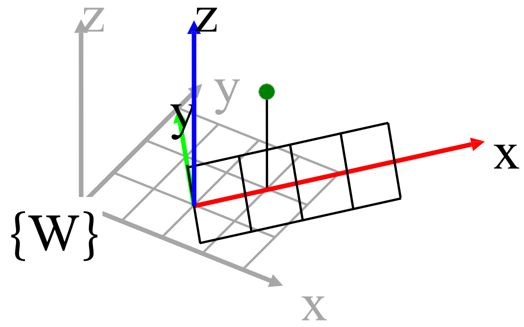
$$\begin{bmatrix} 1 & 0 & 0 & \boxed{2} \\ 0 & 1 & 0 & \boxed{3} \\ 0 & 0 & 1 & \boxed{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x+2 \\ y+3 \\ z+4 \\ 1 \end{pmatrix}$$

credit T. Drummond

HOMOGENEOUS COORDINATES

Translations and rotations can be applied at the same time in a single matrix

Using the previous example



$$\begin{pmatrix} 1.414 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 & -2.121 \\ -0.707 & 0.707 & 0 & 0.707 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

HOMOGENEOUS COORDINATES

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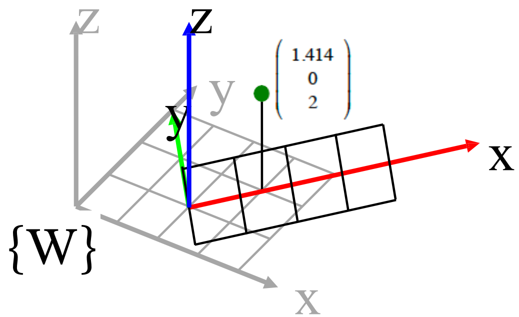


The rotation goes here

The translation goes here

$$\begin{pmatrix} 1.414 \\ 0 \\ 2 \\ \boxed{1} \end{pmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 & -2.121 \\ -0.707 & 0.707 & 0 & 0.707 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \\ \boxed{1} \end{pmatrix}$$

Add a 1 to the end of the vectors

HOMOGENEOUS TRANSFORMATION MATRIX

$\mathbf{R} \in$ SO(3) → Special Orthogonal Group

$\mathbf{t} \in \mathbb{R}^3$ → Set of Real Values (3 dimensions)

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & x \\ r_{21} & r_{22} & r_{23} & y \\ r_{31} & r_{32} & r_{33} & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3)$$

SE is the Special Euclidean Group

Describes both rotation and translation in 3-dimensional Euclidean space

INVERSE OF THE TRANSFORMATION MATRIX

Transform from {1} to {2}:

$$\mathbf{T}_1^2 = \begin{bmatrix} \mathbf{R}_1^2 & \mathbf{t}_1^2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

Then its opposite, {2} to {1} is:

$$\boxed{\mathbf{T}_2^1 = \begin{bmatrix} \left(\mathbf{R}_1^2\right)^{\mathrm{T}} & -\left(\mathbf{R}_1^2\right)^{\mathrm{T}} \mathbf{t}_1^2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}$$

Such that:

$$\mathbf{T}_2^1 = \left(\mathbf{T}_1^2\right)^{-1}$$

Multiply the 2 transforms:

$$\mathbf{T}_1^2 \mathbf{T}_2^1 = \begin{bmatrix} \mathbf{R}_1^2 & \mathbf{t}_1^2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} \left(\mathbf{R}_1^2\right)^{\mathrm{T}} & -\left(\mathbf{R}_1^2\right)^{\mathrm{T}} \mathbf{t}_1^2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_1^2 \left(\mathbf{R}_1^2\right)^{\mathrm{T}} & -\mathbf{R}_1^2 \left(\mathbf{R}_1^2\right)^{\mathrm{T}} \mathbf{t}_1^2 + \mathbf{t}^2{}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & -\mathbf{t}_1^2 + \mathbf{t}^2{}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \quad ✓$$

COORDINATE FRAMES

The rotation matrix and the translation vector describe the change in coordinate frame

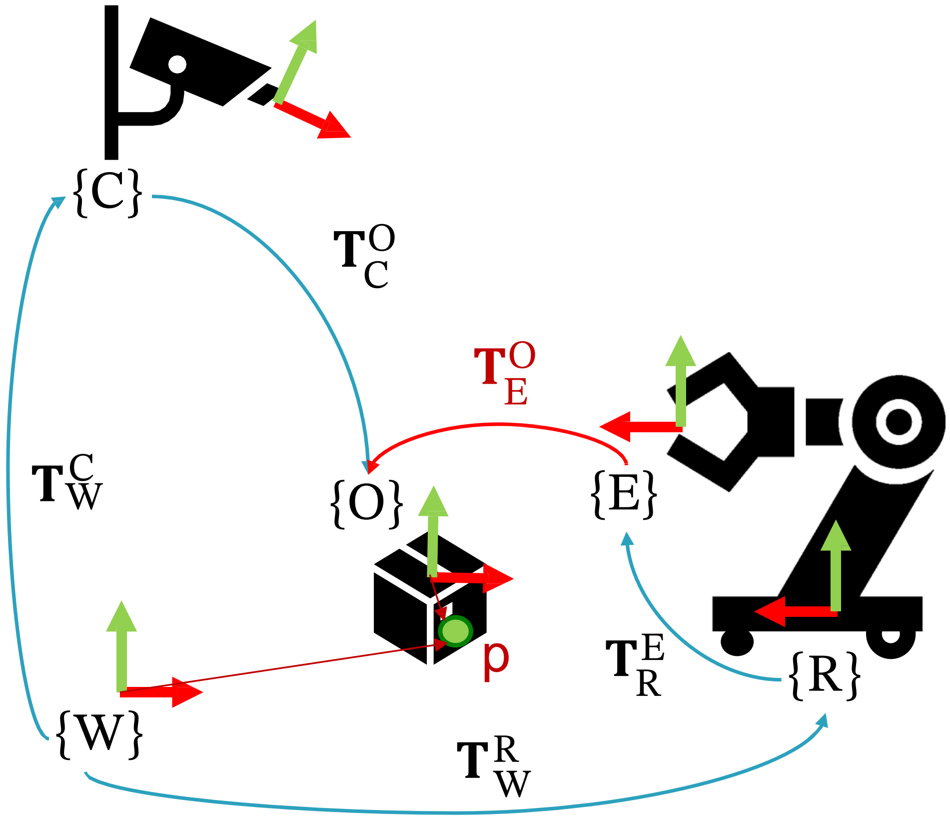
We can transform points from one coordinate frame to another

$$p_W = \mathbf{T}_W^C \; \mathbf{T}_C^O \; p_O$$

We can find a coordinate transformation through the kinematic chain

$$\mathbf{T}_E^O = \mathbf{T}_E^R \; \mathbf{T}_R^W \; \mathbf{T}_W^C \; \mathbf{T}_C^O$$

$$\mathbf{T}_E^O = (\mathbf{T}_R^E)^{-1} (\mathbf{T}_W^R)^{-1} \mathbf{T}_W^C \mathbf{T}_C^O$$



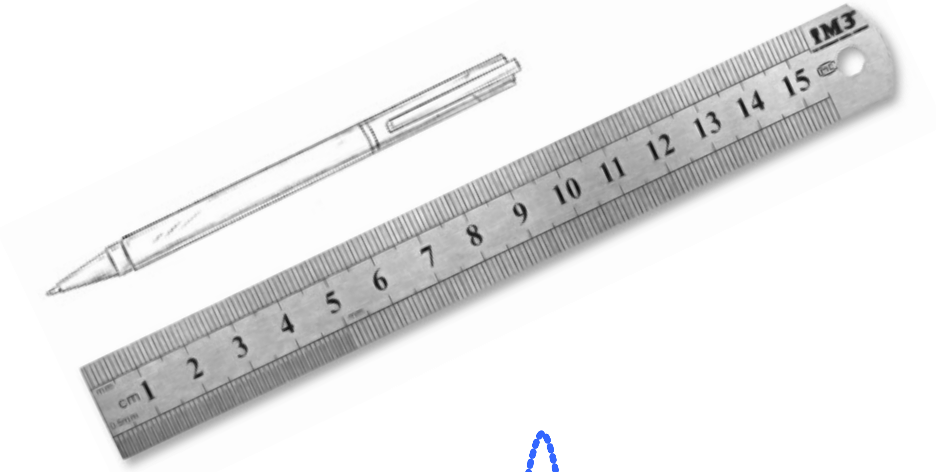
SENSOR UNCERTAINTIES

Sensors are noisy!

Suppose we are measuring a pen using ruler

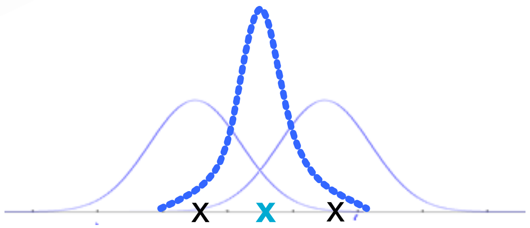
One measurement gave 10.1cm

Another measurement gave 10.3cm



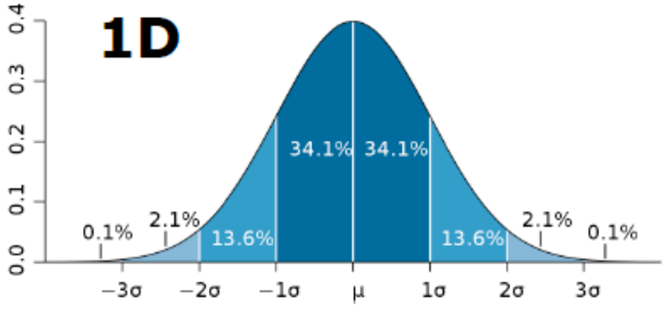
What value would you take for the length of the pen?

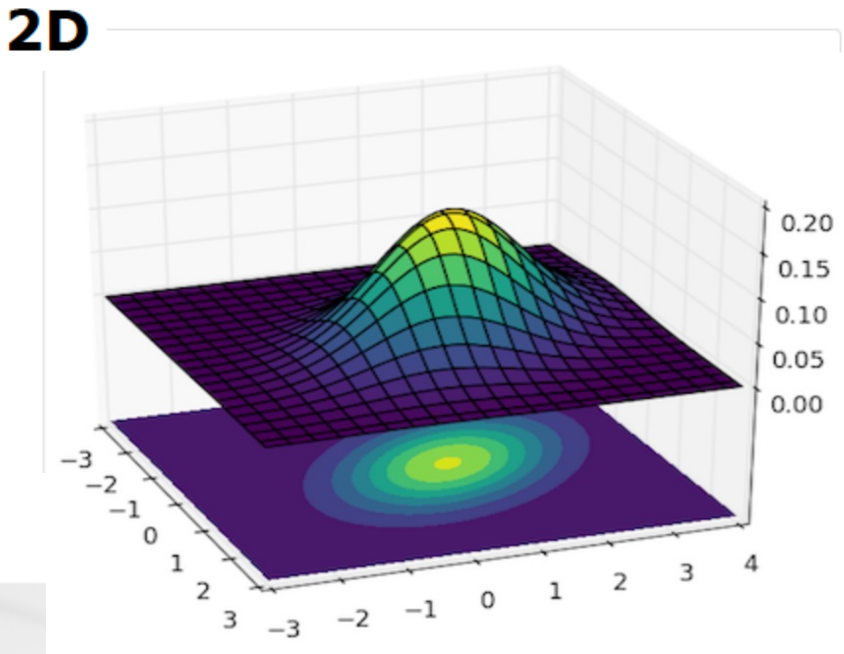
$$x = \frac{10.1+10.3}{2} = 10.2$$

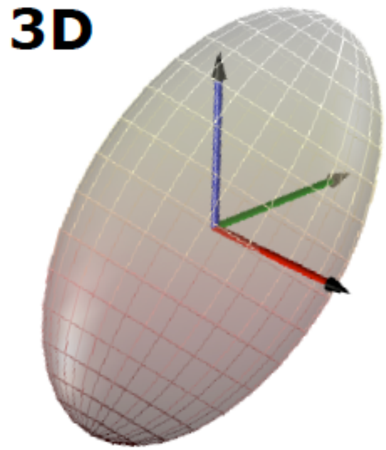


NORMAL DISTRIBUTIONS

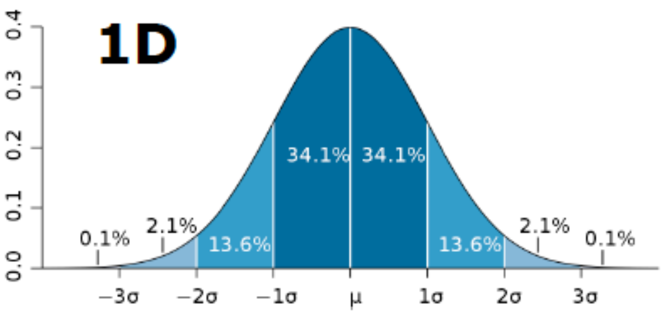
Measurement uncertainties are usually represented with normal distributions (often with mean zero)







UNIVARIATE (1D) GAUSSIAN PROBABILITY DENSITY FUNCTION



$$p(x) \sim N(\mu, \sigma^2):$$

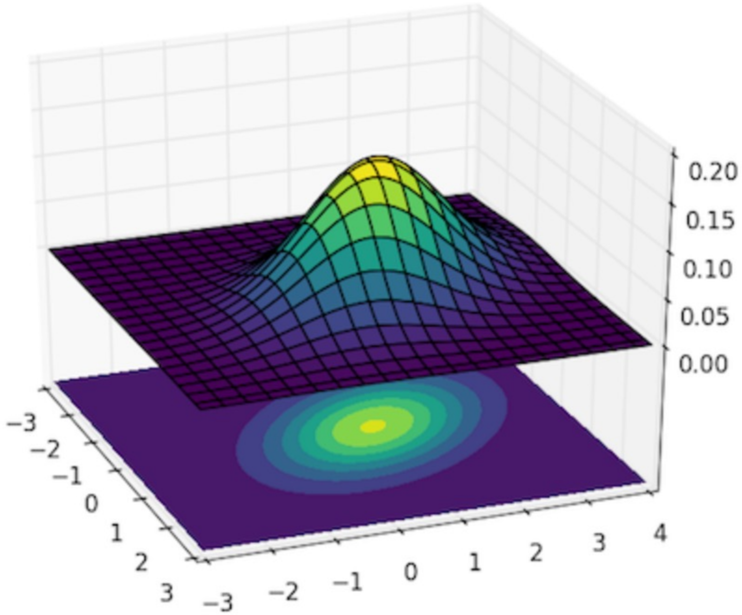
Mean

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Variance

Scaling to sum to 1

MULTIVARIATE (2D) GAUSSIAN PROBABILITY DENSITY FUNCTION



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

$$\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$$

Cross correlation

Scaling to sum to 1

Mean

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu) \right)$$

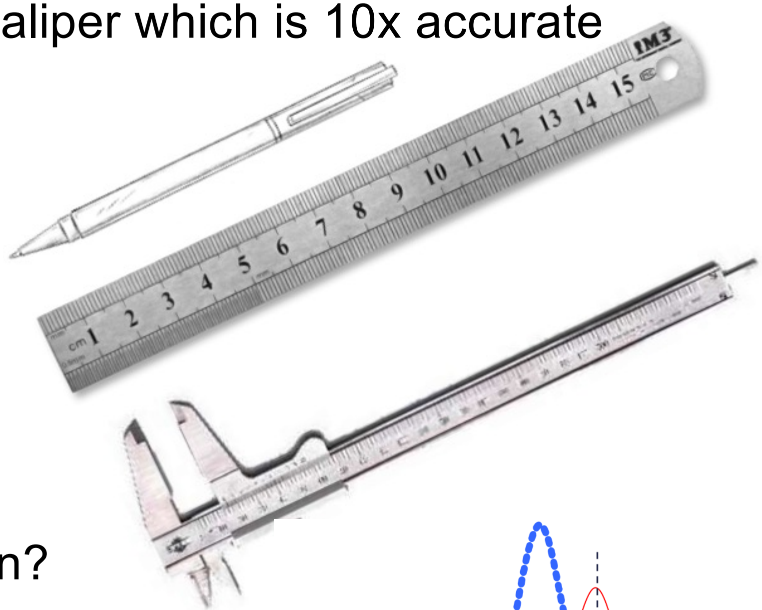
Covariance Matrix

COMBINING MEASUREMENTS

Suppose we now have and a ruler and a Vernier Caliper which is 10x accurate

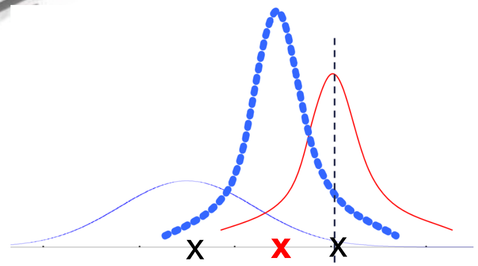
Ruler measurement gave 10.1cm

Vernier measurement gave 10.3cm



What value would you take for the length of the pen?

$$x = \frac{1*10.1+10*10.3}{1+10} = 10.27$$



weighted average

COMBINING MEASUREMENTS

Because the measurements are independent the joint probability is the product

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^{\ 2}) \\ X_2 \sim N(\mu_2, \sigma_2^{\ 2}) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2^{\ 2}}{\sigma_1^{\ 2} + \sigma_2^{\ 2}} \mu_1 + \frac{\sigma_1^{\ 2}}{\sigma_1^{\ 2} + \sigma_2^{\ 2}} \mu_2, \quad \frac{1}{\sigma_1^{\ -2} + \sigma_2^{\ -2}} \right)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}} \right)$$