Mathematical physics

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1 Assignment 1

1.1 Problem 1

Since

$$G(t) = 0, t < 0$$

we will look for a solution in the form of

$$\vartheta(t)Z(t) \tag{1.1.1}$$

Substituting (1.1.1) into the equation

$$(\partial_t + \gamma)G(t) = \delta(t) \tag{1.1.2}$$

and recalling that $\dot{\vartheta} = \delta$, we get

$$\begin{aligned} &(\partial_t + \gamma)Z = 0, \\ &Z(0) = 1 \end{aligned} \tag{1.1.3}$$

Solving (1.1.3), we get

$$G(t) = \vartheta(t)e^{-\gamma t} \tag{1.1.4}$$

Problem 2

The Green function in this case is

$$G(t) = \frac{\vartheta(t)}{\omega}\sin(\omega t)$$

Differentiating $\dot{x}_0$, we get

$$\ddot{x}_0 = \frac{v}{\tau} I_{[0,\tau]}$$

Where I is the characteristic function. Hence, the solution is

$$\varphi = \int_{\mathbb{R}} -G(t-s)\frac{\ddot{x}_0(s)}{l} ds = -\frac{v}{\omega l \tau} \int_0^{\tau} \sin(\omega(t-s))\vartheta(t-s) ds$$

Evaluating the integral we obtain

$$\varphi = \frac{v}{\omega^2 l \tau}(\cos(\omega(t-\tau)_+) - \cos(\omega t_+))$$

If we take $\tau \to 0$, clearly $\varphi = 0, t < 0$ and if $t > 0$, we are basically differentiating w.r.t. τ, so

$$\varphi = -\frac{v}{\omega l}\sin(\omega t)$$

In combining, we get

$$\varphi = -\frac{v\vartheta(t)}{\omega l}\sin(\omega t)$$

Problem 3

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2} = \int_{-\infty}^{+\infty} \frac{dx}{(x+ia)^2(x-ia)^2} = 2\pi i Res_{ia} \frac{dx}{(x+ia)^2(x-ia)^2}$$

Calculating the residues we get

$$I_1 = -\frac{4\pi i}{(2ia)^3} = \frac{\pi}{2a^3}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ipx}dx}{x^2+a^2} = \int_{-\infty}^{+\infty} \frac{e^{ipx}dx}{(x+ia)(x-ia)} = 2\pi i Res_{ia} \frac{e^{ipx}dx}{(x+ia)(x-ia)}$$

Calculating the residues we get

$$I_2 = \frac{2\pi i e^{-pa}}{2ia} = \frac{\pi e^{-pa}}{a}$$

2 Assignment 2

2.1 Problem 1

$$(\frac{d^4}{dt^4} + 4\nu^2 \frac{d^2}{dt^2} + 3\nu^4)G(t) = \delta(t) \tag{2.1}$$

As it was shown in the lecture notes, the Green's function for (2.1) can be searched for in the form of

$$\begin{cases} G(t) = \vartheta(t)Z(t) \\ (\frac{d^4}{dt^4} + 4\nu^2 \frac{d^2}{dt^2} + 3\nu^4)Z(t) = 0 \\ Z(0) = Z'(0) = Z''(0) = 0, Z'''(0) = 1 \end{cases} \tag{2.2}$$

The solution of the homogenous version of equation (2.1) is a standard procedure which boils down to factorizing

$$t^4 + 4is\nu^2 t^2 + 3\nu^4 = 0 \tag{2.3}$$

The general solution is given by

$$Z(t) = A\sin(\sqrt{3}\nu t) + B\cos(\sqrt{3}\nu t) + C\sin(\nu t) + D\cos(\nu t) \tag{2.4}$$

Substituting the initial conditions from (2.2), we obtain the Green's function.

2.1.1 Answer

$$G(t) = \frac{\vartheta(t)}{\sqrt{3}(\sqrt{3}-1)}(\sqrt{3}\sin(\nu t) - \sin(\sqrt{3}\nu t)) \tag{2.5}$$

2.2 Problem 2

Now the characteristic equation

$$(t^2 + \nu^2)^2 \tag{2.1}$$

has multiple roots, so the general solution of homogenous equation is now

$$Z(t) = (A + Bt)\sin(\nu t) + (C + Dt)\cos(\nu t) \tag{2.2}$$

The initial conditions are the same as in the first problem. It remains to solve an algebraic system.

2.2.1 Answer

$$G(t) = \vartheta(t)(\frac{\sin(\nu t)}{2\nu^3} - \frac{t\cos(\nu t)}{2\nu^2}) \tag{2.3}$$

2.3 Problem 3

As it was shown in the lecture,

$$G(t) = \vartheta(t)\exp(-t\hat{\Gamma}) \tag{2.1}$$

Because $hat\Gamma$ is a Jordan block, we have a formula for calculating its exponent.

2.3.1 Answer

$$G(t) = \vartheta(t)e^{-\lambda t}\begin{pmatrix} 1 & -t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & -t & \frac{t^2}{2} \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

2.4 Problem 4

As discussed in the lecture, when integrating along a closed contour which consists of a semicircle, we must have $t\,\mathrm{Im}(\omega) < 0$ for the integral along the arch to vanish as the radius increases. Hence

$$G(t) = \begin{cases} 2\pi i \sum_{\mathrm{Im}(\omega)>0} \mathrm{Res}, t < 0 \\ -2\pi i \sum_{\mathrm{Im}(\omega)<0} \mathrm{Res}, t > 0 \end{cases} \tag{2.1}$$

The minus sign in the second case is because we are integrating clockwise. Since all the critical points are of the first order, residues are easily calculated multiplying by a corresponding factor.

$$\mathrm{Res}_{\omega=-\omega_0} = \frac{e^{i\omega_0 t}}{2\omega_0}, \mathrm{Res}_{\omega=\omega_0} = -\frac{e^{-i\omega_0 t}}{2\omega_0} \tag{2.2}$$

We thus have that if both roots are above the real axis, then

$$G(t) = -\vartheta(-t)\frac{\sin(\omega_0)}{\omega_0} \tag{2.3}$$

If, on the other hand both are below, then

$$G(t) = \vartheta(t)\frac{\sin(\omega_0)}{\omega_0} \tag{2.4}$$

If the left one, i.e. $-\omega_0$ is above and the right one (ω_0) below, then

$$G(t) = \frac{i\vartheta(-t)e^{i\omega_0 t}}{2\omega_0} + \frac{i\vartheta(t)e^{-i\omega_0 t}}{2\omega_0} \tag{2.5}$$

Finally, if the left one is below and the right above,

$$G(t) = -\frac{i\vartheta(-t)e^{-i\omega_0 t}}{2\omega_0} - \frac{i\vartheta(t)e^{i\omega_0 t}}{2\omega_0} \tag{2.6}$$

3 Assignment 3

3.1 Problem 1

We have to find what the external perturbation was. We have

$$\begin{aligned}
&K(t-s) = e^{-\lambda t}\\
&\varphi(t) = t^n\\
&\tilde{K}(p) = \int\limits_0^{+\infty} e^{-(\lambda+p)t}dt = \frac{1}{\lambda+p}\\
&\tilde{\varphi}(p) = \int\limits_0^{+\infty} e^{-pt}t^n dt = (-\frac{d}{dp})^n \int\limits_0^{+\infty} e^{-pt}dt = (-\frac{d}{dp})^n \frac{1}{p} = \frac{n!}{p^{n+1}}
\end{aligned} \tag{3.1}$$

We know, that the Laplace transform of the perturbation is given by

$$\tilde{f}(p) = \frac{\tilde{\varphi}(p)}{\tilde{K}(p)}$$

Hence

$$\begin{aligned}
f(t) &= \Sigma \operatorname{res} \frac{n!e^{pt}}{(\lambda+p)p^{n+1}} =\\
&= \frac{(-1)^{n+1}n!e^{-\lambda t}}{\lambda^{n+1}} + \frac{d^n}{n!dp^n}\frac{n!e^{pt}}{\lambda+p}|_{p=0} =\\
&= \frac{(-1)^{n+1}n!e^{-\lambda t}}{\lambda^{n+1}} + \Sigma_{k=0}^n C_n^k (-1)^k \frac{k!t^{n-k}}{\lambda^{k+1}}
\end{aligned} \tag{3.2}$$

Simplifying, we get

$$f(t) = \frac{(-1)^n n!}{\lambda^{n+1}}\left(\Sigma_{k=0}^n \frac{(-\lambda t)^k}{k!} - e^{-\lambda t}\right) \tag{3.3}$$

3.2 Problem 2

Green's function is given by

$$G(t,s) = \vartheta(t-s)\exp(-\int\limits_s^t \frac{a}{\xi}d\xi) \tag{3.1}$$

Calculating the integral, we get

$$G(t,s) = \vartheta(t-s)\left(\frac{s}{t}\right)^a \tag{3.2}$$

3.3 Problem 3

Let

$$\Delta_\varepsilon(t) = \frac{2t^2\varepsilon}{\pi(t^2+\varepsilon^2)^2} \tag{3.1}$$

Then

$$\int\limits_{\mathbb{R}} \Delta_\varepsilon(t)dt = 2\pi i\,\mathrm{res}_{t=i\varepsilon}\frac{2t^2\varepsilon}{\pi(t^2+\varepsilon^2)^2} = 1 \tag{3.2}$$

On the other hand, if we integrate outside the neighbourhood of 0

$$\int \frac{2t^2\varepsilon}{\pi(t^2+\varepsilon^2)^2} < \varepsilon\int\frac{t^2}{t^4}dt \to 0$$

Hence

$$\lim_{\varepsilon\to 0}\Delta_\varepsilon(t) = \delta(t)$$

We reason similarly for

$$\Delta_\varepsilon(t) = \frac{1}{\sqrt{\pi\varepsilon}}\exp(-\frac{t^2}{\varepsilon}) \tag{3.3}$$

The integral over $\mathbb{R}$ is a standard Gaussian integral, and equal to $1\forall\varepsilon$. On the other hand, if we integrate outside a small neighbourhood, clearly for sufficiently large ε, $\exp(\frac{t^2}{\varepsilon}) > \frac{t^2}{\varepsilon}$, but the integral $\int\frac{dt}{t^2}$ converges, so $\int\Delta_\varepsilon(t)$ will be proportional to $\sqrt{\varepsilon}$, hence tend to zero. Again

$$\lim_{\varepsilon\to 0}\Delta_\varepsilon(t) = \delta(t)$$

Finally,

$$\Delta_n(t) = \frac{1-\cos(nt)}{\pi n t^2} = \frac{2\sin^2(nt/2)}{\pi n t^2} \tag{3.4}$$

This time it is obvious that outside any neighbourhood the integral tends to zero, because we have n in the denominator times a bounded integral. To integrate $\Delta_n(t)$, we make a substitution, and it turns into a constant,

$$\frac{1}{2\pi}\int\frac{1-\cos(2x)}{x^2}dx = \frac{1}{2\pi}\operatorname{Re}\int\frac{1-\exp(2ix)}{x^2}dx$$

We need to change the contour slightly though, and integrate just above zero, so that this integral converges - this won't affect the original integral, because it the integrand is analytic. Integrating near zero will give minus half the residue. If we close the contour in the upper semiplane, we will get zero. Hence,

$$\frac{1}{2\pi}\operatorname{Re}\int\frac{1-\exp(2ix)}{x^2}dx = \operatorname{Re}\frac{1}{2\pi}\frac{1}{2}2\pi i\operatorname{res}_{x=0}\frac{1-\exp(2ix)}{x^2}dx = 1$$

3.4 Problem 4

This problem basically boils down to calculating all the residues of a function in the complex plane (because we can move the contour infinitely to the left, and only residues will remain).

$$\begin{aligned}
\operatorname{res}\frac{e^{pt}\nu}{p^2+\nu^2} &= \nu(\frac{e^{i\nu t}}{2i\nu}-\frac{e^{-i\nu t}}{2i\nu}) = \sin(\nu t) \\
\operatorname{res}\frac{e^{pt}p}{p^2+\nu^2} &= \frac{i\nu e^{i\nu t}}{2i\nu}+\frac{i\nu e^{-i\nu t}}{2i\nu} = \cos(\nu t)
\end{aligned} \tag{3.1}$$

Setting $\nu = i\mu$, we easily calculate the next to using what we already have

$$\begin{aligned}
&\frac{\nu}{p^2-\nu^2} \to i\sin(\mu t) = \sinh(\nu t) \\
&\frac{p}{p^2-\nu^2} \to \cosh(\nu t)
\end{aligned} \tag{3.2}$$

The last transform we have to make is $\frac{1}{\sqrt{p+\alpha}}$. Here we can't directly calculate the residue, becuase we have a branch point. The integral about a loop near α is clearly zero, because the circumference is proportional to ε, while we only have $\sqrt{\varepsilon}$ in the denominator. In order to determine the sign of the root, we must extend the function analytically. Note that since the root is in the denominator there will be an extra minus sign. Only two integrals along the real axis remain.

$$
\begin{aligned}
&\frac{2\pi i}{\sqrt{p+\alpha}} \to -(e^{-i\pi/2} - e^{i\pi/2}) \int_{-\infty}^{-\alpha} \frac{e^{pt}dp}{\sqrt{|p+\alpha|}} = \\
&2i \int_{-\infty}^{0} \frac{e^{(s-\alpha)t}ds}{\sqrt{|s|}} = 2ie^{-\alpha t} \int_{0}^{+\infty} \frac{e^{-pt}dp}{\sqrt{p}} = \\
&2ie^{-\alpha t} \int_{0}^{+\infty} 2e^{-\xi^2 t} d\xi = 2ie^{-\alpha t}\sqrt{\frac{\pi}{t}}
\end{aligned}
\tag{3.3}
$$

Hence

$$
\frac{1}{\sqrt{p+\alpha}} \to \frac{e^{-\alpha t}}{\sqrt{\pi t}} \tag{3.4}
$$

3.5 Problem 5

Since $f(t) = e^{-\frac{t^2}{\tau^2}}$, clearly $f \to 0, t \to \pm\infty$. Hence the solution must approach the homogenous as $t \to \pm\infty$, that is $\varphi \to A_\pm \sin(\omega_0 t + \vartheta_\pm)$ The solution is given by

$$
\varphi(t) = A_- \sin(\omega_0 t + \vartheta_-) + \int_{-\infty}^{t} \frac{\sin(\omega_0(t-s))}{\omega_0} f(s)ds \tag{3.1}
$$

Letting $t = nT, n \in \mathbb{Z}$, as $n \to +\infty$, we get

$$
A_- \sin(\vartheta_-) = A_+ \sin(\vartheta_+)
$$

because the integral is zero by antisymmetry If $t = (n+1/2)T$, then we get

$$
A_- \cos(\vartheta_-) + \int_{-\infty}^{+\infty} \frac{\cos(\omega_0 s)}{\omega_0} f(s)ds = A_+ \cos(\vartheta_+)
$$

So we basically need to calculate the integral

$$
\int_{-\infty}^{+\infty} \frac{\cos(\omega_0 s)}{\omega_0} e^{-\frac{t^2}{\tau^2}} ds = \int_{-\infty}^{+\infty} F\frac{e^{i\omega_0 s}}{\omega_0} e^{-\frac{t^2}{\tau^2}} ds
$$

This is basically the Fourier transform of $f(t)$. Since the Fourier transform of $\exp(-t^2/2)$ is itself times $\sqrt{2\pi}$, we can easily calculate the transform of the stretched function. It is

$$
\frac{F\tau\sqrt{\pi}}{\omega_0} e^{-(\frac{\omega_0 \tau}{2})^2}
$$

Hence we have the following system

$$\begin{cases} A_- \sin(\vartheta_-) = A_+ \sin(\vartheta_+) \\ A_- \cos(\vartheta_-) + \frac{F\tau\sqrt{\pi}}{\omega_0} e^{-(\frac{\omega_0 \tau}{2})^2} = A_+ \cos(\vartheta_+) \end{cases}$$

This can be easily solved for A_+ by squaring and adding the right sides, and then it remains to arcsine of the first equation.

4 Assignment 4

4.1 Problem 1

Find the Green's function for operator

$$\hat{L} = \left(\frac{d^2}{dx^2} + 1\right)\frac{d^2}{dx^2}$$

Note that the kernel of this operator consists of

$$\sin(x), \cos(x), x, 1$$

therefore the zero modes are

$$\sin(x), \cos(x), 1$$

Writing the trigonometric Fourier series for the delta function, we get

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n\geqslant 1}\cos(nx)$$

Subtracting the zero modes, we get the equation for the Green's function

$$\hat{L}G(x) = \hat{L}\frac{1}{\pi}\sum_{n\geqslant 2} A_n \sin(nx) + B_n \cos(nx) =$$
$$\delta(x) - \frac{1}{2\pi} - \frac{1}{\pi}\cos(x) = \frac{1}{\pi}\sum_{n\geqslant 2}\cos(nx) \quad (4.1)$$

Since $\hat{L}\cos(nx) = n^2(n^2-1)\cos(nx)$, we get

$$n^2(n^2-1)B_n = 1, n \geqslant 2$$

Thus

$$G(x) = \frac{1}{\pi}\sum_{n\geqslant 2}\frac{1}{n^2(n^2-1)}\cos(nx)$$

It is defined on the subspace of functions which adheres to the pereodic boundary conditians and perpendicular to the subspace spanned by $\sin(x), \cos(x), 1$

$$\frac{1}{\pi}\sum_{n\geqslant 2}\frac{1}{n^2(n^2-1)}\cos(nx) = \frac{1}{2\pi}\,\mathrm{Re}\sum_{|n|\geqslant 2}\mathrm{res}\,\frac{1}{n^2(n^2-1)}e^{inx}\pi\left(\cot(\pi n)-i\right)$$

The sum of residues is equal to integral along the real axis in both directions (which is zero because we can extend the contour to infinity) minus three poles at $-1,0,1$. Hence

$$G(x) = -\frac{1}{2}\,\mathrm{Re}\sum_{n=-1,0,1}\mathrm{res}\,\frac{1}{n^2(n^2-1)}e^{inx}\left(\cot(\pi n)-i\right) = \frac{6-2\pi^2-3x^2+15\cos(x)+6x\sin(x)}{12\pi} \tag{4.2}$$

4.2 Problem 2

We need to solve Laplace equation for a sphere-symmetrical charge distribution in 3D. Since $G(r)=\frac{1}{4\pi r}$, we have

$$f(r) = -\int dV'\frac{\varphi(\mathbf{r}')}{4\pi(\mathbf{r}-\mathbf{r}')} = -\int 2\pi r'^2\sin(\vartheta)d\vartheta dr'\frac{\varphi(r')}{4\pi(\mathbf{r}-\mathbf{r}')} = -\int\limits_0^{+\infty} r'^2\frac{\varphi(r')}{2}dr'\int\limits_0^{\pi}\frac{\sin(\vartheta)d\vartheta}{r^2+r'^2-2rr'\cos(\vartheta)} \tag{4.1}$$

$$\int\limits_0^{\pi}\frac{\sin(\vartheta)d\vartheta}{r^2+r'^2-2rr'\cos(\vartheta)} = \int\limits_{-1}^{1}\frac{dt}{r^2+r'^2-2rr't} = -\frac{1}{2rr'}\ln(r^2+r'^2-2rr't)\Big|_{-1}^{1} = \frac{\ln(\frac{r+r'}{r-r'})}{rr'} \tag{4.2}$$

Thus

$$f(r) = -\int\limits_0^{+\infty}\frac{\varphi(r')\ln(\frac{r+r'}{r-r'})}{2r}r'dr' \tag{4.3}$$

4.3 Problem 3

We need to find the solution to Laplace equation with boundary condition $f(x,0) = e^{ix}$ Substituting into Poisson's integral formula, we get

$$f(x,y) = \int d\xi \frac{ye^{i\xi}}{\pi((\xi - x)^2 + y^2)} \tag{4.1}$$

This is easily integrated by extending into complex plain - the exponent will vanish in the positive semiplane (we integrate over a positive semicircle). There is one pole at $x + iy$ within the chosen loop - it is of the first order, so we just differentiate the denominator. Hence

$$f(x,y) = 2\pi i \frac{ye^{i(x+iy)}}{2\pi i y} = e^{ix-y}$$

4.4 Problem 4

We need to solve the equation

$$\left(\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{2}{r} + \varepsilon \right) \Phi = 0$$

Multiplying by r and taking the Laplace transform, we obtain

$$\frac{d}{dp}(p^2 + \varepsilon)\tilde{\Phi}(p) = (2 - 2p)\tilde{\Phi}(p)$$

Let $y = (p^2 + \varepsilon)\tilde{\Phi}(p)$, then

$$y' = \frac{2-2p}{p^2+\varepsilon} y \Rightarrow y = A \exp\left(\int \frac{2-2p}{p^2+\varepsilon} dp \right) = \frac{A}{p^2 - \alpha^2} \left(\frac{p-\alpha}{p+\alpha} \right)^{\frac{1}{\alpha}}$$

where $\varepsilon = -\alpha^2$

$$\tilde{\Phi}(p) = \frac{A}{(p^2-\alpha^2)^2} \left(\frac{p-\alpha}{p+\alpha} \right)^{\frac{1}{\alpha}}$$

Since we have $\frac{1}{\alpha} = n = 2$,

$$\tilde{\Phi}(p) = \frac{A}{(p^2 - \frac{1}{4})^2} \left(\frac{p - \frac{1}{2}}{p + \frac{1}{2}} \right)^2 = \frac{A}{(p + \frac{1}{2})^4}$$

It remains to take the inverse Laplace transform, which boils down to calculating the residue at $-\frac{1}{2}$.

$$R(r) \propto \frac{r^3}{6} e^{-\frac{r}{2}}$$

5 Assignment 6

5.1 Problem 1

Find the area of an n-dimensional sphere by considering

$$I = \int e^{-r^2} d^n\mathbf{r}$$

Since the integrand doesn't depend on the angles, we can integrate over the sphere

$$I = \int_0^{+\infty} e^{-r^2} S_{n-1}(r)dr = S_{n-1}(1)\int_0^{+\infty} e^{-r^2} r^{n-1}dr \tag{5.1}$$

because the $n-1$ dimensional volume is proportional to r^{n-1}. To evaluate the integral, we can change the variable $t = r^2$

$$\int_0^{+\infty} e^{-r^2} r^{n-1}dr = \frac{1}{2}\int_0^{+\infty} t^{\frac{n}{2}-1}e^{-t}dt = \frac{\Gamma(n/2)}{2} \tag{5.2}$$

On the other hand, writing the integral in Cartesian coordinates, we get

$$I = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^n = \pi^{n/2} \tag{5.3}$$

Comparing the two expressions, we get

$$S_{n-1}(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \implies S_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$$

Hence, using the recursive property, and applying the inversion formula, we obtain

$$\begin{cases} S_{2k+1} = \frac{2\pi^{k+1}}{k!} \\ S_{2k} = \frac{2\pi^k}{(k-1/2)!} \end{cases}$$

6 Problem 2

Find $\int_0^\infty duu^{z-1}e^{iu}$ By Jordan's lemma we can close the contour to the imaginary axis if $\operatorname{Re} z < 1$. Hence

$$\int\limits_0^\infty duu^{z-1}e^{iu} = \int\limits_0^{i\infty} duu^{z-1}e^{iu} = i^z\int\limits_0^\infty dtt^{z-1}e^{-t} = e^{z\ln(i)}\Gamma(z) = e^{\frac{i\pi z}{2}}\Gamma(z)$$

Since the integral converges when $\operatorname{Re} z > 0$, the formula works when

$$0 < \operatorname{Re} z < 1$$

although it is possible to extend the domain analytically

7 Problem 3

$$\int_0^{\pi/2} \cos^a(\varphi)\sin^b(\varphi)d\varphi = \int_0^{\pi/2} \cos^{a-1}(\varphi)\sin^b(\varphi)d\sin(\varphi) =$$

$$\int_0^1 (1-t^2)^{(a-1)/2} t^b dt = (1/2)\int_0^1 (1-x)^{(a+1)/2-1} x^{(b+1)/2-1} dx =$$

$$B((a+1)/2,(b+1)/2) \quad (7.1)$$

8 Problem 4

Logarithming the inversion formula and integrating, we get

$$2\int_0^1 \ln(\Gamma(z))dz = \int_0^1 \ln(\Gamma(1-z)) + \ln(\Gamma(z))dz = \ln(\pi) - \int_0^1 \ln(\sin(\pi z))dz =$$

$$\ln(\pi) - (1/\pi)\int_0^{\pi} \ln(t)dt = \ln(\pi) + (1/\pi)\pi\ln(2) = \ln(\frac{\pi}{2}) \quad (8.1)$$

Thus,

$$\int_0^1 \ln(\Gamma(z))dz = \frac{1}{2}\ln(\frac{\pi}{2})$$

9 Problem 5

Using the recursive formula for polygamma, we get

$$\psi(n+1+z) = 1/(n+z) + 1/(n-1+z) + \ldots + 1/z + \psi(z)$$

Call $S_n(z) = \sum_{k=0}^{n}(1/(k+1) - 1/(k+z)$ From the previous relationship, we get $S_n(z) = \psi(n+2) - \psi(1) - (\psi(n+1+z) - \psi(z)) = \psi(n+2) - \psi(n+1+z) + \psi(z) - \psi(1)$

Since $\psi(z) = \ln(z) + o(1), z \to \infty$

$$\psi(n+2) - \psi(n+1+z) \to \ln(\frac{n+2}{n+1+z}) \to 0$$

Hence

$$S(z) = \psi(z) - \psi(1)$$

10 Assignment 7

10.1 Problem 1

Evaluate Airy functions and their first derivatives at 0

$$\mathrm{Ai}(0) = 1/\pi \int_0^\infty du \cos(u^3/3) = 1/\pi \,\mathrm{Re} \int_0^\infty du e^{iu^3/3}$$

To calculate the last integral, we can close the contour in a sector with angle $\pi/6$, above the real axis. Then

$$\int_0^\infty du e^{iu^3/3} = e^{i\pi/6} \int_0^\infty du e^{-u^3/3} = e^{i\pi/6} \int_0^\infty e^{-t/3}/3t^{2/3} dt =$$
$$e^{i\pi/6}/3^{2/3} \int_0^\infty s^{1/3-1} e^{-s} ds = \frac{e^{i\pi/6}\Gamma(1/3)}{3^{2/3}} \tag{10.1}$$

Extracting the real part, we get

$$\mathrm{Ai}(0) = \frac{\Gamma(1/3)}{2\pi 3^{1/6}}$$

All the necessary calculations to obtain Bi(0) were done

$$\mathrm{Bi}(0) = 1/\pi \int_0^\infty (e^{-u^3/3} + \sin(u^3/3)) du =$$
$$1/\pi \left(\frac{\Gamma(1/3)}{3^{2/3}} + \frac{\Gamma(1/3)}{2 \times 3^{2/3}} \right) = \frac{3^{1/3}\Gamma(1/3)}{2\pi} \tag{10.2}$$

Differentiating under the integral by x, we get

$$\mathrm{Ai}'(0) = 1/\pi \int_0^\infty -u \sin(u^3/3) du = -1/\pi \,\mathrm{Im} \int_0^\infty u e^{iu^3/3} du$$

The calculation of the integral is as above

$$\int_0^\infty u e^{iu^3/3} du = e^{i\pi/6} \int_0^\infty u e^{-u^3/3} du =$$
$$e^{i\pi/6} \int_0^\infty e^{-t/3}/3t^{1/3} dt = e^{i\pi/6}/3^{1/3} \int_0^\infty s^{1-2/3} e^{-s} ds =$$
$$\frac{e^{i\pi/6}\Gamma(2/3)}{3^{1/3}} \tag{10.3}$$

Thus

$$\mathrm{Ai}'(0) = -\frac{\Gamma(2/3)}{2\pi 3^{1/3}}$$

Again, with these calculations we get $\mathrm{Bi}'(0)$ immediately

$$\mathrm{Bi}'(0) = \int_0^\infty (ue^{-u^3/3} + u\cos(u^3/3))du = 1/\pi\left(\frac{\Gamma(2/3)}{3^{1/3}} + \frac{\sqrt{3}\Gamma(2/3)}{2\times 3^{1/3}}\right) =$$

$$(2+\sqrt{3})\frac{\Gamma(2/3)}{2\pi 3^{1/3}} \quad (10.4)$$

10.2 Problem 2

Find the asymptotic behaviour of Airy function at infinity.

Recall that

$$\mathrm{Bi}(x) = 1/\pi \int_0^\infty (e^{xu-u^3/3} + \sin(xu+u^3/3))du$$

Clearly, as $x \to +\infty$, only the first term will contribute, because the phase of $\sin(xu+u^3/3$ will oscillate infinitely fast. Vice versa, as $x \to -\infty$, the first term tends to zero exponentially because the argument in the exponent is diminished. If $x \to +\infty$, the argument of the exponent has one maximum point

$$S'(u_0) = (xu_0 - u_0^3/3)' = 0 \implies u_0 = x^{1/2}$$

Hence,

$$\mathrm{Bi}(x) \approx 1/\pi\sqrt{2\pi/-S''(u_0)}e^{S(u_0)} = 1/\pi\sqrt{2\pi/2x^{1/2}}e^{2/3x^{3/2}} = e^{2/3x^{3/2}}/\sqrt{\pi}x^{1/4}$$

If $x \to -\infty$, then

$$S'(u_0) = (xu_0 + u_0^3/3)' = 0 \implies u_0 = |x|^{1/2}$$

Hence, applying the stationary-phase method

$$\mathrm{Bi}(x) \approx 1/\pi\int \sin(xu+u^3/3)du \approx 1/\pi\sqrt{2\pi/S''(u_0)}\sin(S(u_0)+\pi/4) =$$

$$1/\pi\sqrt{2\pi/2|x|^{1/2}}\sin(-2/3|x|^{3/2}+\pi/4) =$$

$$\cos(2/3|x|^{3/2}+\pi/4)/\sqrt{\pi}|x|^{1/4} \quad (10.1)$$

11 Assignment 8

11.1 Problem 1

Calculate the Laplace transform of J_1 We will use the integral representation of the Bessel function.

$$\begin{aligned}\tilde{J}_1(a) = \int_0^\infty \exp(-az)dz \oint \exp(\frac{z}{2}(t-1/t))\frac{dt}{2\pi i t^2} = \\ \oint \frac{dt}{2\pi i t^2}\int_0^\infty \exp(z(\frac{1}{2}(t-1/t)-a))dz = \\ \oint \frac{2dt}{2\pi i t^2}\frac{1}{2a-t+1/t} = \\ \frac{2}{2\pi i}\oint \frac{dt}{t(1+2at-t^2)} = \\ 2 + \frac{2}{2at-2t^2}\Big|_{t=a-\sqrt{a^2+1}} = \\ 2 - \frac{1}{a^2 - a\sqrt{a^2+1}+1}\end{aligned} \tag{11.1}$$

11.2 Problem 2

Evaluate

$$\int_0^\infty \frac{J_2(z)}{z^2}dz$$

Using the recurrence formula

$$\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z)$$

obtained by differentiating the generating function, we get

$$\frac{J_n}{z^2} = \frac{1}{2n}\left\{\frac{J_{n-2}}{2(n-1)} + \frac{nJ_n}{(n-1)(n+1)} + \frac{J_{n+2}}{2(n+1)}\right\}$$

In our case,

$$\frac{J_2}{z^2} = \frac{1}{4}\left\{\frac{J_0}{2} + \frac{2J_2}{3} + \frac{J_4}{6}\right\} \tag{11.1}$$

Using integral representation we can integrate the Bessel function. For even n, the Bessel function is even, so we have

$$\int\limits_0^\infty J_n(z)dz = \frac{1}{4\pi}\int\limits_{-\pi}^{\pi} e^{-in\vartheta}d\vartheta \int\limits_{-\infty}^{\infty} e^{iz\sin\vartheta}dz =$$
$$\frac{1}{4\pi}\int\limits_{-\pi}^{\pi} e^{-in\vartheta}2\pi\delta(\sin\vartheta)d\vartheta = \frac{e^{in\pi}+e^{-in\pi}}{2} = 1 \quad (11.2)$$

Integrating both sides of (11.1), we get

$$\int\limits_0^\infty \frac{J_2(z)}{z^2}dz = \frac{1}{4}(1/2+1/3+1/6) = \frac{1}{4}$$

11.3 Problem 3

Find the Fourier transform of $f(x) = e^{-p^2x^2}$ in Bessel functions $J_0(qx)$ We will use the integral representation

$$\hat{f}(x) = 1/2\pi \int\limits_{-\pi}^{\pi} d\vartheta \int\limits_0^\infty xe^{iq\sin\vartheta x - p^2x^2}dx$$

$$\int\limits_0^\infty xe^{iq\sin\vartheta x-p^2x^2}dx = e^{-(q\sin\vartheta/2p)^2}\int\limits_0^\infty xe^{-(px-iq\sin\vartheta/2p)^2}dx$$

$$\int\limits_0^\infty xe^{-(px-iq\sin\vartheta/2p)^2}dx = 1/p\int\limits_0^\infty (px-iq\sin\vartheta/2p)e^{-(px-iq\sin\vartheta/2p)^2}dx+$$
$$iq\sin\vartheta/2p^2\int\limits_0^\infty e^{-(px-iq\sin\vartheta/2p)^2}dx$$

The first integral is easily evaluated

$$
\begin{aligned}
1/p \int_0^\infty (px - iq\sin\vartheta/2p) x e^{-(px - iq\sin\vartheta/2p)^2} dx = \\
1/p^2 \int_0^\infty (px - iq\sin\vartheta/2p) e^{-(px - iq\sin\vartheta/2p)^2} d(px - iq\sin\vartheta/2p) = \\
1/2p^2 e^{-(px - iq\sin\vartheta/2p)^2}\Big|_0^\infty = \\
e^{(q\sin\vartheta/2p)^2}/2p^2
\end{aligned}
$$

Therefore its contribution to $\hat{f}$ is $1/2p^2$.

11.4 Problem 4

Find big zeros of Bessel function. We have

$$J_n(z) \approx \sqrt{2/\pi z}\cos(z - \pi n/2 - \pi/4), z \to \infty$$

So we have

$$\gamma_k - \pi n/2 - \pi/4 = \pi/2 + \pi k \implies \gamma_k = \pi/4 + \pi n/2 + \pi k$$

12 Assignment 9

12.1 Problem 1 (3.4.3)

$$
\begin{aligned}
(n+1)\frac{P_n(x)P_{n+1}(y) - P_n(y)P_{n+1}(x)}{y-x} + (2n+2)P_{n+1}(x)P_{n+1}(y) = \\
\frac{(n+1)P_n(x)P_{n+1}(y) - (n+1)P_n(y)P_{n+1}(x) + (y-x)(2n+2)P_{n+1}(x)P_{n+1}(y)}{y-x} = \\
\frac{((n+1)P_n(x) - (2n+2)xP_{n+1}(x))P_{n+1}(y) - ((n+1)P_n(y) - (2n+2)yP_{n+1}(y)P_{n+1}(x))}{y-x} = \\
\frac{-(n+2)P_{n+2}(x)P_{n+1}(y) + (n+2)P_{n+2}(y)P_{n+1}(x)}{y-x} = \\
(n+2)\frac{P_{n+1}(x)P_{n+2}(y) - P_{n+1}(y)P_{n+2}(x)}{y-x}
\end{aligned}
\tag{12.1}
$$

12.2 Problem 2 (3.4.7)

Prove that if 3.56 for $n-1$, then 3.56 for n due to 3.51 and 3.54. By 3.51 we have

$$(x^2-1)P'_{n+1} = (x^2-1)/(n+1)((2n+1)(xP_n)' - nP'_{n-1})$$

By 3.54 we have

$$(xP_n)' = (n+1)P_n + P'_{n-1}$$

so since we assumed 3.56 holds for $n-1$

$$(x^2-1)P'_{n+1} = (x^2-1)/(n+1)((2n+1)(n+1)P_n - (n-1)P'_{n-1}) = \\ (x^2-1)(2n+1)P_n - (n-1)^2/(n+1)(xP_{n-1} - P_{n-2}) \quad (12.1)$$

Using 3.51 again, we get 3.56

12.3 Problem 3

Find $\int_{-1}^{1} P_{n-1}P_{n+1}x^2dx$ Since P_n is orthogonal to all polynomials of smaller degree, we have

$$(P_{n+1}, x^2P_{n-1}) = a_{n-1}/a_{n+1}(P_{n+1}, P_{n+1})$$

where $a_n = 2n!/2^n n!^2$ is the leading coefficient of P_n As we know, $(P_{n+1}, P_{n+1}) = 2/(2n+3)$, so the answer is

$$\frac{2(n-1)!}{2^{n-1}(n-1)!^2}\frac{2^{n+1}(n+1)!^2}{2(n+1)!}\frac{2}{2n+3} = \frac{2n(n+1)}{(2n+3)(2n+1)(2n-1)}$$

12.4 Problem 4

Integrate $\int_{-\infty}^{+\infty} e^{-x^2}H_{2n}(xy)dx$

$$\sum_n \frac{t^n}{n!}\int\limits_{-\infty}^{+\infty} e^{-x^2}H_n(xy)dx = \int\limits_{-\infty}^{+\infty} e^{-x^2}e^{-t^2+2txy}dx = \\ e^{t^2(y^2-1)}\int\limits_{-\infty}^{+\infty} e^{-(x-ty)^2}dx = \sqrt{\pi}\sum\frac{t^{2n}(y^2-1)^n}{n!} \quad (12.1)$$

Comparing the coefficients, we obtain

$$1/2n! \int_{-\infty}^{+\infty} e^{-x^2} H_{2n}(xy)dx = \sqrt{\pi}\frac{(y^2-1)^n}{n!} \implies$$

$$\int_{-\infty}^{+\infty} e^{-x^2} H_{2n}(xy)dx = \sqrt{\pi}2^n(y^2-1)^n(2n-1)!!$$

12.5 Problem 5

Find

$$\int_{-\infty}^{+\infty} x^n e^{-x^2} H_n(xy)dx$$

As above

$$\sum_n \frac{t^n}{n!} \int_{-\infty}^{+\infty} x^n e^{-x^2} H_n(xy)dx = \int_{-\infty}^{+\infty} e^{-(tx)^2+2tyx^2} e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-(t^2-2ty+1)x^2} dx$$

Calculating the Gaussian integral, we get Legendre generating function

$$\frac{\sqrt{\pi}}{\sqrt{t^2-2ty+1}} = \sqrt{\pi}\sum P_n(y)t^n \implies \int_{-\infty}^{+\infty} x^n e^{-x^2} H_n(xy)dx = n!\sqrt{\pi}P_n(y)$$

12.6 Problem 6

Find

$$\int_0^{\infty} e^{-x^2} \cosh(\beta x) H_{2n}(x)dx$$

Let's first evaluate

$$\int_{-\infty}^{\infty} e^{-x^2} e^{\beta x} H_n(x)dx$$

Taking the sum, we get

$$\sum t^n/n! \int\limits_{-\infty}^{\infty} e^{-x^2} e^{\beta x} H_n(x)dx = \int\limits_{-\infty}^{\infty} e^{-t^2+2tx-x^2+\beta x}dx =$$

$$\sqrt{\pi}e^{-t^2+(t+\beta/2)^2} = \sqrt{\pi}\exp(\frac{\beta^2}{4} - \beta t) \implies$$

$$\int\limits_{-\infty}^{\infty} e^{-x^2} e^{\beta x} H_n(x)dx = \sqrt{\pi}\exp(\frac{\beta^2}{4})\beta^n/n! \quad (12.1)$$

It remains to notice, that the integral of coshine is symmetric, so we can integrate symmetrically and then half the result. Integrating the exponents separately is easy - we use the derived formula. Because we have $2n$, which is even, both exponents will give the same result, so we might as well integrate one and not half the total. Hence

$$\int\limits_{0}^{\infty} e^{-x^2}\cosh(\beta x)H_{2n}(x)dx = \frac{\sqrt{\pi}\exp(\frac{\beta^2}{4})\beta^{2n}}{2 \times 2n!}$$

Lets find another integral integral, using the generating function approach

$$\sum s^n/n! \int\limits_{-\infty}^{+\infty} (x+a)^n e^{-x^2}dx = \int\limits_{-\infty}^{+\infty} e^{sx+sa-x^2}dx = e^{s(a-s/4)}\sqrt{\pi}$$

Expanding the function, we get

$$e^{s(a-s/4)} = \sum_{k,l,m} \delta_{k,l+m}(-1)^l s^{k+l} a^m/4^l m! l! = \sum_n s^n \sum_{n-2l\geqslant 0} \frac{(-1)^l a^{n-2l}}{4^l(n-2l)!l!}$$

Therefore

$$\int\limits_{-\infty}^{+\infty} (x+a)^n e^{-x^2}dx = \sqrt{\pi}n! \sum_{n-2l\geqslant 0} \frac{(-1)^l a^{n-2l}}{4^l(n-2l)!l!}$$

Recalling the series for Hermite polynomials,

$$H_n(x) = n! \sum_{n-2l\geqslant 0} \frac{(-1)^l(2x)^{n-2l}}{(n-2l)!l!} = 2^n n! \sum_{n-2l\geqslant 0} \frac{(-1)^l x^{n-2l}}{4^l(n-2l)!l!}$$

We immediately obtain

$$\int\limits_{-\infty}^{+\infty} (x+a)^n e^{-x^2}dx = \sqrt{\pi}2^{-n}H_n(a)$$

Problem 1

Solve the wave equation $\left(\partial_t^2 - c^2\nabla\right)u = 0$ if

$$\begin{cases} u\left(t,x\right)\big|_{t=0} = e^{-r^2} \\ \partial_t u\left(t,x\right)\big|_{t=0} = 0 \end{cases}$$

Taking the Fourier transform, we obtain

$$\begin{cases} v\left(|k|\right) := u\left(0,k\right) = \mathcal{F}\left(e^{-r^2}\right) = \pi^{\frac{3}{2}} e^{-\frac{k^2}{4}} \\ \partial_t u\left(t,k\right)\big|_{t=0} = 0 \end{cases}$$

Since $u\left(t,k\right) = \omega_1\left(k\right)e^{ickt} + \omega_2\left(k\right)e^{-ickt}$, we have

$$\begin{cases} \omega_1 + \omega_2 = \pi^{\frac{3}{2}} e^{-\frac{k^2}{4}} \\ icq\omega_1 - icq\omega_2 = 0 \end{cases}$$

Solving these equations, we get

$$\omega_1 = \omega_2 \implies u\left(t,k\right) = \pi^{\frac{3}{2}} \frac{e^{-\frac{k^2}{4}}}{2}\left(e^{ic|k|t} + e^{-ic|k|t}\right)$$

Calculating the inverse Fourier transform, we obtain

$$\int \frac{v\left(r'\right)}{\left(2\pi\right)^3} \frac{e^{icr't} + e^{-icr't}}{2} e^{irr'\cos\vartheta} 2\pi r'^2 \sin\vartheta dr' d\vartheta$$

This will be the 'shifted' inverse Fourier transform and is equal to

$$\frac{\exp\left(-\left(ct-r\right)^2\right) + \exp\left(-\left(ct+r\right)^2\right)}{2} = e^{-r^2-c^2t^2}\cosh\left(2ctr\right)$$

Problem 2

Solve $\left[\partial_t^2 + \hat{\omega}^2\left(-i\nabla\right)\right]u = \cos\left(\omega t\right)\delta\left(\mathbf{r}\right)$ - the wave equation for a point source

As derived in the lecture notes, the solution can befound by integrating with the Green's funciton

$$u\left(t,\mathbf{r}\right) = \frac{1}{4\pi c^2}\int \frac{d^3\mathbf{r_1}}{|\mathbf{r} - \mathbf{r_1}|}\cos\left(\omega\left(t - |\mathbf{r} - \mathbf{r_1}|\right)\right)\delta\left(\mathbf{r_1}\right) = \frac{\cos\omega\left(t - \frac{r}{c}\right)}{4\pi r c^2}$$

which, as expected, is a spherical wave

Problem 3

Find the Green's function for $\partial_t^2 + \nabla^4$ at $r = 0$ Taking the Fourier transform, we obtain

$$\left(\partial_t^2 + k^4\right) \widetilde{G}(t,k) = \delta(t)$$

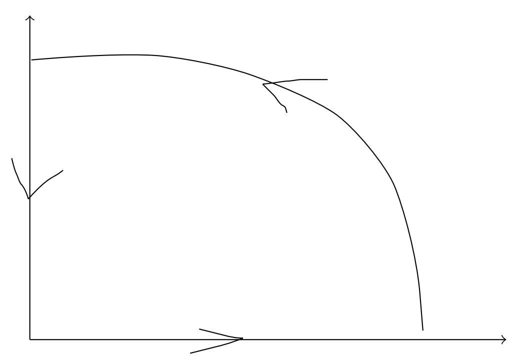
The solution of this ODE is easily calculated, $\widetilde{G} = \vartheta(t) \frac{\sin k^2 t}{k^2}$ Hence

$$G(t,x) = \int \frac{e^{ikx}}{(2\pi)^3} \vartheta(t) \frac{\sin k^2 t}{k^2} d^3k$$

setting $x = 0$ the integral is easily calculated in polar coordinates

$$G(t,0) = \frac{\vartheta}{(2\pi)^3} \int_0^\infty \frac{\sin k^2 t}{k^2} 4\pi k^2 dk = \frac{\vartheta}{4\pi^2} \int_0^\infty \frac{\sin \xi t}{\sqrt{\xi}} d\xi = \frac{\vartheta}{4\pi^2} \operatorname{Im} \int_0^\infty \frac{e^{i\xi t}}{\sqrt{\xi}} d\xi$$

This is integrated in the complex plane



Hence

$$G(t,0) = \frac{\vartheta}{4\pi^2} \operatorname{Im} \int_0^\infty \frac{e^{-\xi t}}{\sqrt{i\xi}} i d\xi = \frac{\vartheta}{4\pi^2\sqrt{2}} \int_0^\infty 2e^{-\xi^2 t} d\xi = \frac{\vartheta(t)}{4\pi^2\sqrt{2}} \sqrt{\frac{\pi}{t}}$$

Problem 4

Solve $(\partial_t + v\partial_x)\Psi = \frac{A}{\cosh^2(x)}$ with zero initial conditians Using the integral provided in the lecture notes, we find the general solution

$$\Psi = b(x - vt) + \int_{-\infty}^t \frac{A}{\cosh^2(x + v\tau - vt)} d\tau = b(x - vt) + \frac{A}{v} \tanh(x)$$

At $t = 0$ we have

$$b(x) + \frac{A}{v} \tanh(x) = 0 \implies b = -\frac{A}{v} \tanh(x)$$

Hence the solution is $\frac{A}{v} (\tanh(x) - \tanh(x - vt))$

Problem 5

Solve the Helmholtz equation $\left(\Delta+\kappa^2\right) f=\partial_z \delta\left(r\right)$ Integration by parts shows that $\partial_z \delta\left(r\right)=-\delta\left(r\right) \partial_z \delta\left(r\right)$ Let $r_{12}=|r-r_{1|}$. According to the lecture notes,

$$f\left(r\right)=\int d r_1 \delta\left(r_1\right) \partial_z \frac{\cos \kappa r_{12}}{4 \pi r_{12}}=\int d r_1 \delta\left(r_1\right)\left(\frac{-\kappa \sin \kappa r_{12}}{4 \pi r_{12}}-\frac{\cos \kappa r_{21}}{4 \pi r_{12}^2}\right) \frac{z_{12}}{r_{12}}=-\frac{z}{r}\left(\frac{\kappa \sin \kappa r}{4 \pi r}+\frac{\cos \kappa r}{4 \pi r^2}\right)$$