

PSWF-Radon approach to reconstruction from band-limited Hankel transform

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Abstract

We give new formulas for reconstructions from band-limited Hankel transform of integer or half-integer order. Our formulas rely on the PSWF-Radon approach to super-resolution in multidimensional Fourier analysis. This approach consists of combining the theory of classical one-dimensional prolate spheroidal wave functions with the Radon transform theory. We also use the relation between Fourier and Hankel transforms and Cormack-type inversion of the Radon transform. Our numerical examples illustrate super-resolution even with a considerable noise.

Keywords: Hankel transform, Fourier transform, Radon transform, prolate spheroidal wave functions, super-resolution

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1 Introduction

We consider the Hankel transform $\mathcal{H}_\nu : \mathcal{L}^2(\mathbb{R}_+) \rightarrow \mathcal{L}^2(\mathbb{R}_+)$ of order ν defined by

$$\mathcal{H}_\nu[f](x) := \int_0^\infty f(y) J_\nu(xy) \sqrt{xy} dy, \quad x \geq 0, \quad (1.1)$$

where J_ν is the Bessel function of the first kind and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. We recall that $\mathcal{H}_\nu^{-1} = \mathcal{H}_\nu$, that is, the Hankel transform defined as in (1.1) coincides with its own inverse.

We consider the following inverse problem.

Problem 1.1. Let $\sigma, r > 0$ be given. Find $f \in \mathcal{L}^2(\mathbb{R}_+)$ from $w = \mathcal{H}_\nu[f]$ given on $[0, r]$ (possibly with some noise), under a priori assumption that $\text{supp } f \subset [0, \sigma]$.

Problem 1.1 is uniquely solvable for the noiseless case in view of the analyticity of $\mathcal{H}_\nu[f]$ on $(0, +\infty)$ under our assumptions. On the other hand, this problem is severely ill-posed in the sense of Hadamard in view of the super-exponential decay of the eigenvalues of the band-limited Hankel transform; see, for example, [25, Lemma 1].

In the present work, we consider Problem 1.1 with integer or half-integer ν in connection with important particular cases of the problem of reconstructing unknown function supported in the ball $B_\sigma \subset \mathbb{R}^d$ from its Fourier transform given on the ball B_r under a priori assumptions of the cylindrical or spherical symmetry. Here, for $\rho \geq 0$, we let

$$B_\rho := \{q \in \mathbb{R}^d : |q| \leq \rho\}.$$

A possible approach for solving Problem 1.1 is based on the following approximation

$$f \approx f_{\text{naive}} := \mathcal{H}_\nu^{-1} [w^{\text{ext}}] \quad \text{on } [0, \sigma], \quad (1.2)$$

where

$$w^{\text{ext}}(x) := \begin{cases} w(x), & \text{for } x \in [0, r], \\ 0, & \text{otherwise.} \end{cases}$$

Formula (1.2) leads to a stable and accurate reconstruction for sufficiently large r . However, similarly to the reconstruction from the Fourier transform given on the ball B_r , this naive approach has the *diffraction limit*: small details (especially less than π/r) are blurred. The term *super-resolution* refers to the techniques that allow reconstruction beyond this diffraction limit.

In the present work we give new exact formulas for solving Problem 1.1 with integer or half-integer ν following the PSWF-Radon approach of [21–23] in Fourier analysis in dimension $d = 2, 3$. Our numerical experiments show that the regularised version of these formulas lead to super-resolution in Problem 1.1 even for the noisy data.

In connection with the PSWF theory, we consider the integral operator \mathcal{F}_c on $\mathcal{L}^2([-1, 1])$, defined by

$$\mathcal{F}_c[f](x) := \int_{-1}^1 e^{icxy} f(y) dy, \quad (1.3)$$

where $c > 0$ is the *bandwidth parameter*. The eigenfunctions $(\psi_{j,c})_{j \in \mathbb{N}}$ of \mathcal{F}_c are *prolate spheroidal wave functions* (PSWFs), where $\mathbb{N} := \{0, 1, \dots\}$. Recall that the operator \mathcal{F}_c and its inverse admit the singular value decompositions:

$$\mathcal{F}_c[f](x) = \sum_{j \in \mathbb{N}} \mu_{j,c} \psi_{j,c}(x) \int_{-1}^1 \psi_{j,c}(y) f(y) dy, \quad (1.4)$$

$$\mathcal{F}_c^{-1}[g](y) = \sum_{j \in \mathbb{N}} \frac{1}{\mu_{j,c}} \psi_{j,c}(y) \int_{-1}^1 \psi_{j,c}(x) g(x) dx. \quad (1.5)$$

In addition, we can assume that $0 < |\mu_{j+1,c}| < |\mu_{j,c}|$ for all $j \in \mathbb{N}$. For more details on the PSWF theory, see [1, 7, 8, 21, 24, 37, 39, 40].

First, we consider the case of integer ν .

Theorem 1.1. *Let $r, \sigma > 0$, $c = r\sigma$, and $f \in \mathcal{L}^2(\mathbb{R}_+)$ be supported in $[0, \sigma]$. Then, for integer ν , the Hankel transform $\mathcal{H}_\nu[f]$ on $[0, r]$ uniquely determines f by the formula*

$$f(y) = -4(-i)^\nu \sqrt{y} \frac{d}{dy} \int_y^\sigma \frac{y T_{|\nu|}\left(\frac{x}{y}\right)}{x(x^2 - y^2)^{-\frac{1}{2}}} \mathcal{F}_c^{-1}[g_\nu](x/\sigma) dx, \quad y \in [0, \sigma], \quad (1.6)$$

where $T_{|\nu|}$ denote the Chebyshev polynomial of the first kind and

$$g_{r,\nu}(x) := \begin{cases} \frac{1}{\sqrt{r|x|}} \mathcal{H}_\nu[f](r|x|), & \text{if } x\nu \geq 0, \\ (-1)^\nu \frac{1}{\sqrt{r|x|}} \mathcal{H}_\nu[f](r|x|), & \text{otherwise.} \end{cases} \quad (1.7)$$

Note that the function $g_{r,\nu}$ in Theorem 1.1 has no singularity at 0 under our assumptions, in view of the definition of the Hankel transform in (1.1).

Another approach to Problem 1.1 [\[MI comments: bla-bla\]](#) can be solved exactly using circular prolate spheroidal wave functions (CPSWFs) $(\psi_{j,c}^\nu)_{j \in \mathbb{N}}$

[25] - Karoui, [26] - Kolobov

2 Proof of Theorem

3 Numerical examples

We consider the *Fourier transform* \mathcal{F} defined by the formula

$$\mathcal{F}[v](p) = \hat{v}(p) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ipq} v(q) dq, \quad p \in \mathbb{R}^d, \quad (3.1)$$

where v is a complex-valued test function on \mathbb{R}^d , $d \geq 1$.

For any $\rho > 0$, let

$$B_\rho := \{q \in \mathbb{R}^d : |q| < \rho\}. \quad (3.2)$$

We consider the following problem.

Problem 3.1. Find $v \in \mathcal{L}^2(\mathbb{R}^d)$, where $\text{supp } v \subset B_\sigma$, from $\hat{v} = \mathcal{F}[v]$ given on the ball B_r (possibly with some noise), for fixed $r, \sigma > 0$.

Problem 1.1 arises in different areas such as Fourier analysis, linearised inverse scattering and image processing, and has been extensively studied in the literature. Solving Problem 1.1 is complicated considerably by the fact that it is ill-posed in the sense of Hadamard (for example, when the noisy data is taken from $\mathcal{L}^2(B_r)$) and, moreover, it is exponentially unstable. Nevertheless, there exist several techniques to approach this problem theoretically and numerically. For more background on Problem 1.1 see, for example, [3, 6, 11, 13, 19–21, 27, 30] and references therein. In addition, for general background on ill-posed inverse problems see [16, 38].

where \mathcal{F}^{-1} is the standard inverse Fourier transform and w is such that $w|_{B_r}$ coincides with the data of Problem 1.1 and $w|_{\mathbb{R}^d \setminus B_r} \equiv 0$. Formula (1.2) leads to a stable and accurate reconstruction for sufficiently large r . However, it has the well-known diffraction limit: small details (especially less than π/r) are blurred. A new approach for *super-resolution* in comparison with the resolution of (1.2) was recently suggested in [21]; see also [19, Section 6.3]. In the present work, we study numerically the approach of [21] and demonstrate its efficiency.

For convenience, we consider the scaling of v with respect to the size of its support:

$$v_\sigma(q) := v(\sigma q), \quad q \in \mathbb{R}^d. \quad (3.3)$$

Note that $\text{supp } v_\sigma \subset B_1$. Let

$$c := r\sigma. \quad (3.4)$$

Then, the data in Problem 1.1 (for the case without noise) can be presented as follows:

$$\hat{v}(rx) = \frac{\sigma}{2\pi} \mathcal{F}_c[v_\sigma](x), \quad \text{for } d = 1, \quad (3.5)$$

$$\hat{v}(rx\theta) = \left(\frac{\sigma}{2\pi}\right)^d \mathcal{F}_c[\mathcal{R}_\theta[v_\sigma]](x), \quad \text{for } d \geq 2, \quad (3.6)$$

where $x \in [-1, 1]$, $\theta \in \mathbb{S}^{d-1}$; see [21, Theorem 1.1 and Section 4.1]. Here, the operators \mathcal{F}_c and \mathcal{R}_θ are defined by

$$\mathcal{F}_c[f](x) := \int_{-1}^1 e^{icxy} f(y) dy, \quad x \in [-1, 1], \quad (3.7)$$

$$\mathcal{R}_\theta[u](y) := \int_{q \in \mathbb{R}^d : q\theta = y} u(q) dq, \quad y \in \mathbb{R}, \quad (3.8)$$

where f is a test function on $[-1, 1]$ and u is a test function on \mathbb{R}^d .

Recall that $\mathcal{R}_\theta[u] \equiv \mathcal{R}[u](\cdot, \theta)$, where \mathcal{R}_θ is defined by (3.8) and \mathcal{R} is the classical *Radon transform*; see, for example, [28, 32]. In fact, presentation (3.6) follows from the projection theorem going back to [9]:

$$\mathcal{F}[u](s\theta) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} e^{ist} \mathcal{R}[u](t, \theta) dt, \quad s \in \mathbb{R}, \theta \in \mathbb{S}^{d-1}. \quad (3.9)$$

At present, the theory and applications of the Radon transform \mathcal{R} are developed in very detail; see, for example, [28, 33] and references therein. In particular, formula (3.9) and Fourier analysis lead to different inversion methods for \mathcal{R} . In [21] and the present work we also use (3.9) (or, more precisely, (3.6)) for inversion of \mathcal{F} in the framework of Problem 1.1.

The operator \mathcal{F}_c defined by (3.7) is a variant of band-limited Fourier transform. This operator is one of the key objects of the theory of *prolate spheroidal wave functions* (PSWFs) going back to [29, 37]. In particular, the operator \mathcal{F}_c has the following *singular value decomposition* (SVD) in $\mathcal{L}^2([-1, 1])$:

$$\mathcal{F}_c[f](x) = \sum_{j \in \mathbb{N}} \mu_{j,c} \psi_{j,c}(x) \int_{-1}^1 \psi_{j,c}(y) f(y) dy, \quad (3.10)$$

where $(\psi_{j,c})_{j \in \mathbb{N}}$ are the prolate spheroidal wave functions (PSWFs) and the corresponding eigenvalues $\{\mu_{j,c}\}_{j \in \mathbb{N}}$ satisfy $0 < |\mu_{j+1,c}| < |\mu_{j,c}|$ for all $j \in \mathbb{N}$. Here and throughout the paper, we set $\mathbb{N} := \{0, 1, 2, \dots\}$.

The functions $(\psi_{j,c})_{j \in \mathbb{N}}$ are certain of families of wave functions introduced by Niven in [29] for solving the Helmholtz equation in prolate spheroidal coordinates. Originally, $(\psi_{j,c})_{j \in \mathbb{N}}$ are defined as the eigenfunctions of the spectral problem

$$-\frac{d}{dx} \left[(1 - x^2) \frac{d\psi}{dx} \right] + c^2 x^2 \psi = \chi \psi, \quad \psi \in C^2([-1, 1]).$$

It is well known that the eigenvalues $(\chi_{j,c})_{j \in \mathbb{N}}$ of this problem are real, positive, and simple and the eigenfunctions $(\psi_{j,c})_{j \in \mathbb{N}}$ are real-valued and form an orthonormal basis in $\mathcal{L}^2([-1, 1])$.

The fact that $(\psi_{j,c})_{j \in \mathbb{N}}$ are the eigenfunctions of the finite Fourier transform \mathcal{F}_c defined by (3.7) was pointed out by Slepian and Pollak in [37] as a special case of more general integral relations satisfied by Niven's wave functions of [29]. As mentioned in [37],

"These functions ... possess properties that make them ideally suited for the study of certain questions regarding the relationship between functions and their Fourier transforms."

At present, the theory and numerics of PSWFs $(\psi_{j,c})_{j \in \mathbb{N}}$ are well developed, see, for example, [1, 7, 8, 21, 24, 39, 40] and references therein. Surprisingly, to our knowledge, these results were not used in the context of Problem 1.1 before the recent work [21].

The approach for solving Problem 1.1 suggested in [21] is based on presentations (3.5), (3.6), inversion of \mathcal{F}_c , and inversion of \mathcal{R} . The inversion of \mathcal{R} is given using standard results of the Radon transform theory. The inversion of \mathcal{F}_c is given using the singular value decomposition (3.10). In the framework of this approach, the operator \mathcal{F}_c^{-1} is approximated by the finite-rank operator $\mathcal{F}_{n,c}^{-1}$ (see (??) for precise definition), where n is the rank. In fact, the number n is a regularisation parameter and its choice is crucial for both theoretical results and numerical applications.

Note also that the truncated SVD technique as a regularization method is similar to the long-used method of summing Fourier-type series with approximate coefficients, where the series sum is approximated by a sum of finite (not too large) number of its first terms; see [38]. In the context of Radon-type transforms this regularization technique has been recently studied in [4, 5, 14, 34, 35].

Our numerical implementation of the approach of [21] includes, in particular, different principles for choosing the aforementioned regularisation parameter n such as *residual minimisation* and the *Morozov discrepancy principle*. One of the most interesting points of our results lies in examples of super-resolution, that is, recovering details of size less than π/r , where r is the band-limiting radius of Problem 1.1. We also obtain a better reconstruction in the sense of \mathcal{L}^2 -norm than the conventional reconstruction based on formula (1.2).

The paper is structured as follows. In Section ??, we recall the aforementioned reconstruction formulas of [21]. In Section ??, we discuss numerical principles for choosing the regularisation parameter n . Numerical examples are presented in Section ??. In

conclusion, we summarise the main points in Section ??.

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