PSWF-Radon approach to reconstruction from band-limited Hankel transform

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Abstract

We give new formulas for reconstructions from band-limited Hankel transform of integer or half-integer order. Our formulas rely on the PSWF-Radon approach to super-resolution in multidimensional Fourier analysis. This approach consists of combining the theory of classical one-dimensional prolate spheroidal wave functions with the Radon transform theory. We also the use relation between Fourier and Hankel transforms and Cormack-type inversion of the Radon transform. Our numerical examples illustrate super-resolution even with a considerable noise.

Keywords: Hankel transform, Fourier transform, Radon transform, prolate spheroidal wave functions, super-resolution
AMS subject classification: 42A38, 49K40

*Research supported by ARC DE200101045.

1 Introduction

Define Hankel tranform. History and backgrounds. Define PSWF. Define Radon. Define Cormac.

2 Proof of Theorem

3 Numerical examples

We consider the *Fourier transform* $\mathcal{F}$ defined by the formula

$$\mathcal{F}[v](p) = \hat{v}(p) := \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{ipq} v(q) dq, \qquad p \in \mathbb{R}^d, \tag{3.1}$$

where v is a complex-valued test function on $\mathbb{R}^d$, $d \geq 1$.

For any $\rho > 0$, let

$$B_\rho := \left\{ q \in \mathbb{R}^d : |q| < \rho \right\}. \tag{3.2}$$

We consider the following problem.

Problem 3.1. Find $v \in \mathcal{L}^2(\mathbb{R}^d)$, where $supp\, v \subset B_\sigma$, from $\hat{v} = \mathcal{F}[v]$ given on the ball B_r (possibly with some noise), for fixed $r, \sigma > 0$.

Problem 3.1 arises in different areas such as Fourier analysis, linearised inverse scattering and image processing, and has been extensively studied in the literature. Solving Problem 3.1 is complicated considerably by the fact that it is ill-posed in the sense of Hadamard (for example, when the noisy data is taken from $\mathcal{L}^2(B_r)$) and, moreover, it is exponentially unstable. Nevertheless, there exist several techniques to approach this problem theoretically and numerically. For more background on Problem 3.1 see, for example, [3, 6, 11, 12, 18–20, 22, 25] and references therein. In addition, for general background on ill-posed inverse problems see [15, 32].

The conventional approach for solving Problem 3.1 is based on the following approximation

$$v \approx v_{\text{naive}} := \mathcal{F}^{-1}[w] \text{ on } B_\sigma, \tag{3.3}$$

where $\mathcal{F}^{-1}$ is the standard inverse Fourier transform and w is such that $w|_{B_r}$ coincides with the data of Problem 3.1 and $w|_{\mathbb{R}^d \setminus B_r} \equiv 0$. Formula (3.3) leads to a stable and accurate reconstruction for sufficiently large r. However, it has the well-known diffraction

limit: small details (especially less than π/r) are blurred. A new approach for *super-resolution* in comparison with the resolution of (3.3) was recently suggested in [20]; see also [18, Section 6.3]. In the present work, we study numerically the approach of [20] and demonstrate its efficiency.

For convenience, we consider the scaling of v with respect to the size of its support:

$$v_\sigma(q) := v(\sigma q), \qquad q \in \mathbb{R}^d. \tag{3.4}$$

Note that supp $v_\sigma \subset B_1$. Let

$$c := r\sigma. \tag{3.5}$$

Then, the data in Problem 3.1 (for the case without noise) can be presented as follows:

$$\hat{v}(rx) = \tfrac{\sigma}{2\pi}\mathcal{F}_c\left[v_\sigma\right](x), \qquad \text{for } d = 1, \tag{3.6}$$

$$\hat{v}(rx\theta) = \left(\tfrac{\sigma}{2\pi}\right)^d \mathcal{F}_c\left[\mathcal{R}_\theta[v_\sigma]\right](x), \qquad \text{for } d \geq 2, \tag{3.7}$$

where $x \in [-1,1]$, $\theta \in \mathbb{S}^{d-1}$; see [20, Theorem 1.1 and Section 4.1]. Here, the operators $\mathcal{F}_c$ and $\mathcal{R}_\theta$ are defined by

$$\mathcal{F}_c[f](x) := \int_{-1}^{1} e^{icxy} f(y)dy, \qquad x \in [-1,1], \tag{3.8}$$

$$\mathcal{R}_\theta[u](y) := \int_{q\in\mathbb{R}^d\,:\,q\theta=y} u(q)dq, \qquad y \in \mathbb{R}, \tag{3.9}$$

where f is a test function on $[-1,1]$ and u is a test function on $\mathbb{R}^d$.

Recall that $\mathcal{R}_\theta[u] \equiv \mathcal{R}[u](\cdot,\theta)$, where $\mathcal{R}_\theta$ is defined by (3.9) and $\mathcal{R}$ is the classical *Radon transform*; see, for example, [23, 26]. In fact, presentation (3.7) follows from the projection theorem going back to [9]:

$$\mathcal{F}[u](s\theta) = \tfrac{1}{(2\pi)^d}\int_{-\infty}^{\infty} e^{ist}\mathcal{R}[u](t,\theta)dt, \qquad s \in \mathbb{R},\ \theta \in \mathbb{S}^{d-1}. \tag{3.10}$$

At present, the theory and applications of the Radon transform $\mathcal{R}$ are developed in very detail; see, for example, [23, 27] and references therein. In particular, formula (3.10) and Fourier analysis lead to different inversion methods for $\mathcal{R}$. In [20] and the present work we also use (3.10) (or, more precisely, (3.7)) for inversion of $\mathcal{F}$ in the framework of Problem 3.1.

The operator $\mathcal{F}_c$ defined by (3.8) is a variant of band-limited Fourier transform. This operator is one of the key objects of the theory of *prolate spheroidal wave functions* (PSWFs) going back to [24, 31]. In particular, the operator $\mathcal{F}_c$ has the following

singular value decomposition (SVD) in $\mathcal{L}^2([-1,1])$:

$$\mathcal{F}_c[f](x) = \sum_{j\in\mathbb{N}} \mu_{j,c}\psi_{j,c}(x) \int_{-1}^{1} \psi_{j,c}(y)f(y)dy, \tag{3.11}$$

where $(\psi_{j,c})_{j\in\mathbb{N}}$ are the prolate spheroidal wave functions (PSWFs) and the corresponding eigenvalues $\{\mu_{j,c}\}_{j\in\mathbb{N}}$ satisfy $0 < |\mu_{j+1,c}| < |\mu_{j,c}|$ for all $j \in \mathbb{N}$. Here and throughout the paper, we set $\mathbb{N} := \{0, 1, 2\ldots\}$.

The functions $(\psi_{j,c})_{j\in\mathbb{N}}$ are certain of families of wave functions introduced by Niven in [24] for solving the Helmholtz equation in prolate spheroidal coordinates. Originally, $(\psi_{j,c})_{j\in\mathbb{N}}$ are defined as the eigenfunctions of the spectral problem

$$-\frac{d}{dx}\left[(1-x^2)\frac{d\psi}{dx}\right] + c^2x^2\psi = \chi\psi, \qquad \psi \in C^2([-1,1]).$$

It is well known that the eigenvalues $(\chi_{j,c})_{j\in\mathbb{N}}$ of this problem are real, positive, and simple and the eigenfunctions $(\psi_{j,c})_{j\in\mathbb{N}}$ are real-valued and form an orthonormal basis in $\mathcal{L}^2([-1,1])$.

The fact that $(\psi_{j,c})_{j\in\mathbb{N}}$ are the eigenfunctions of the finite Fourier transform $\mathcal{F}_c$ defined by (3.8) was pointed out by Slepian and Pollak in [31] as a special case of more general integral relations satisfied by Niven's wave functions of [24]. As mentioned in [31],

> *"These functions ... possess properties that make them ideally suited for the study of certain questions regarding the relationship between functions and their Fourier transforms."*

At present, the theory and numerics of PSWFs $(\psi_{j,c})_{j\in\mathbb{N}}$ are well developed, see, for example, [1,7,8,20,21,33,34] and references therein. Surprisingly, to our knowledge, these results were not used in the context of Problem 1.1 before the recent work [20].

The approach for solving Problem 3.1 suggested in [20] is based on presentations (3.6), (3.7), inversion of $\mathcal{F}_c$, and inversion of $\mathcal{R}$. The inversion of $\mathcal{R}$ is given using standard results of the Radon transform theory. The inversion of $\mathcal{F}_c$ is given using the singular value decomposition (3.11). In the framework of this approach, the operator $\mathcal{F}_c^{-1}$ is approximated by the finite-rank operator $\mathcal{F}_{n,c}^{-1}$ (see (**??**) for precise definition), where n is the rank. In fact, the number n is a regularisation parameter and its choice is crucial for both theoretical results and numerical applications.

Note also that the truncated SVD technique as a regularization method is similar to the long-used method of summing Fourier-type series with approximate coefficients, where the series sum is approximated by a sum of finite (not too large) number of its first terms;

see [32]. In the context of Radon-type transforms this regularization technique has been recently studied in [4, 5, 13, 28, 29].

Our numerical implementation of the approach of [20] includes, in particular, different principles for choosing the aforementioned regularisation parameter n such as *residual minimisation* and the *Morozov discrepancy principle*. One of the most interesting points of our results lies in examples of super-resolution, that is, recovering details of size less than π/r, where r is the band-limiting radius of Problem 3.1. We also obtain a better reconstruction in the sense of $\mathcal{L}^2$-norm than the conventional reconstruction based on formula (3.3).

The paper is structured as follows. In Section **??**, we recall the aforementioned reconstruction formulas of [20]. In Section **??**, we discuss numerical principles for choosing the regularisation parameter n. Numerical examples are presented in Section **??**. In conclusion, we summarise the main points in Section **??**.

4 Acknowledgements

We thank the referees for valuable comments and suggestions that helped us to improve the presentation. We also thank F.O. Goncharov for useful discussions.

The work was initiated in the framework of the internship of G.V. Sabinin at the Centre de Mathématique Appliquées of Ecole Polytechnique under the supervision of R.G. Novikov in August-October 2021.

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