PSWF[1]-Radon approach to reconstruction from band-limited Hankel transform

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December 6, 2023

[1] Prolate Spheroidal Wave Functions

Abstract

- New formulas for reconstructions from band-limited Hankel transform of integer and half-integer order
- PSWF-Radon approach to super-resolution in multidimensional Fourier analysis
- Numerical examples to illustrate super-resolution

Presentation Plan

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Bessel Functions (of the first kind)

Integral representation

Bessel function of *order* $\nu \in \mathbb{R}$

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(\nu\tau - x\sin\tau)\,d\tau - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-x\sinh t - \nu t}dt \quad (1.1)$$

Bessel Functions of Integer Order

For $\nu \in \mathbb{Z}$

- $J_\nu(x) = \frac{1}{\pi} \int_0^\pi \cos(\nu\tau - x \sin \tau)\, d\tau = \frac{1}{2\pi} \int_{-\pi}^\pi e^{i(\nu\tau - x \sin \tau)}\, d\tau$

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- $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$

Bessel Functions of Half-integer Order

For $\nu \in \frac{1}{2}\mathbb{N}$ the ordinary Bessel function is related to the spherical Bessel function by

$$j_{\nu-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} J_\nu(x) \tag{1.2}$$

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Rayleigh's formula for the spherical Bessel functions

$$j_n(x) = (-x)^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x} \qquad n \in \mathbb{Z}_{\geq 0} \tag{1.3}$$

Hankel Transform

Definition

Hankel transform of *order* $\nu \geq -\frac{1}{2}$

$$\begin{aligned} &\mathcal{H}_\nu : \mathcal{L}^2(\mathbb{R}_+) \to \mathcal{L}^2(\mathbb{R}_+) \\ &\mathcal{H}_\nu[f](x) := \int_0^\infty f(y) J_\nu(xy)\sqrt{xy}\, dy, \qquad x \geq 0 \end{aligned} \tag{1.4}$$

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Remark

It is an involution, i.e. $\mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$

Hankel Transform and the 2D Laplacian

Theorem

If $\lim_{r\to\infty} rf'(r) = \lim_{r\to\infty} rf(r) = 0$ *then*

$$\mathcal{H}_\nu\left[\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{\nu^2}{r^2}\right)f(r)\right](p) = -p^2\mathcal{H}_\nu[f](p) \tag{1.5}$$

Remark

Consider the 2-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

Its radial part is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

Hankel transform in PDEs

We will consider the following example

Axisymmetric Steady-State Heat Equation with a Symmetric Source

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = -q(r), \qquad 0 < r < \infty, \quad 0 < z < \infty \tag{1.6}$$

$$u(r, 0) = 0 \tag{1.7}$$

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Solution

Apply the zeroth Hankel transform over r and use the theorem from the previous slide. Denoting $\tilde{u}(p, z) := \mathcal{H}_0[u]$ and $\tilde{q}(p) := \mathcal{H}_0[q](p)$, we have

$$\tilde{u}_{zz} - p^2 \tilde{u} = -\tilde{q} \tag{1.8}$$

$$\tilde{u}(p, 0) = 0 \tag{1.9}$$

Hankel transform in PDEs

Solution (continued)

The bounded solution of this boundary value problem is

$$\tilde{u}(p, z) = \frac{\tilde{q}(p)}{p^2}\left(1 - e^{-pz}\right) \tag{1.10}$$

Taking the inverse Hankel transform (which is the Hankel transform itself) we get the answer to the original problem

$$u(r, z) = \mathcal{H}_0\left[\frac{\mathcal{H}_0[q](p)}{p}\left(1 - e^{-pz}\right)\right](r, z) \tag{1.11}$$

Presentation Plan

Problem Statement

Problem

Let $\sigma, r > 0$ *be given. Find* $f \in \mathcal{L}^2(\mathbb{R}_+)$ *from* $w = \mathcal{H}_\nu[f]$ *given on* $[0, r]$ *(possibly with some noise), under a priori assumption that* $\operatorname{supp} f \subset [0, \sigma]$.

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Remark (noiseless case)

The solution is unique, because $\mathcal{H}_\nu[f]$ is analytic if f has compact support.

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We will only consider the problem in case ν is integer or half-integer.

Naive Approach

$$f \approx f_{\mathrm{naive}} := \mathcal{H}_{\nu}^{-1}\left[w^{\mathrm{ext}}\right] \text{ on } [0,\sigma], \tag{2.1}$$

where

$$w^{\mathrm{ext}}(x) := \begin{cases} w(x), & \text{for } x \in [0,r], \\ 0, & \text{otherwise.} \end{cases}$$

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- Stable and accurate reconstruction for sufficiently large r
- *diffraction limit*: small details (especially less than π/r) are blurred

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Definition

Super-resolution: techniques that allow reconstruction beyond this diffraction limit. This is the main purpose of our work.

Operator $\mathcal{F}_c$

Definition

$$\mathcal{F}_c : \mathcal{L}^2([-1,1]) \to \mathcal{L}^2([-1,1])$$
$$\mathcal{F}_c[f](x) := \mathcal{F}_c[f](x) := \int_{-1}^{1} e^{icxy} f(y)dy \tag{2.2}$$

$c > 0$ is the *bandwidth parameter*

Prolate Spheroidal Wave Functions

SVD decomposition for $\mathcal{F}_c$

$$\mathcal{F}_c[f](x) = \sum_{j=0}^{\infty} \mu_{j,c}\psi_{j,c}(x) \int_{-1}^{1} \psi_{j,c}(y) f(y) dy$$

$$\mathcal{F}_c^{-1}[g](y) = \sum_{j\in\mathbb{N}} \frac{1}{\mu_{j,c}} \psi_{j,c}(y) \int_{-1}^{1} \psi_{j,c}(x) g(x) dx$$

Definition

The eigenfunctions $\{\psi_{j,c},\ j = 0, 1, 2 \ldots\}$ of $\mathcal{F}_c$ are *prolate spheroidal wave functions* (PSWFs)

Remark

Their eigenvalues obey $0 < |\mu_{j+1,c}| < |\mu_{j,c}|$

Exact formula for integer ν

Let $T_n,\ n = 0, 1, 2\ldots$ denote the Chebyshev polynomial of the first kind

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Theorem (Cormack-type formula for integer ν)

Let $r, \sigma > 0$, $c = r\sigma$, and $f \in \mathcal{L}^2(\mathbb{R}_+)$ be supported in $[0, \sigma]$. Then, for $\nu \in \mathbb{Z}$, the Hankel transform $\mathcal{H}_\nu[f]$ on $[0, r]$ uniquely determines f by the formula

$$f(y) = -\frac{2i^\nu}{\sigma}\sqrt{y}\,\frac{d}{dy}\int_y^\sigma \frac{yT_{|\nu|}\left(\frac{x}{y}\right)}{x(x^2-y^2)^{\frac{1}{2}}}\mathcal{F}_c^{-1}[g_{r,\nu}](x/\sigma)dx, \qquad y \in [0, \sigma]$$

$$g_{r,\nu}(x) := \begin{cases} \frac{1}{\sqrt{r|x|}}\mathcal{H}_\nu[f](r|x|), & \text{if } x \geq 0, \\ (-1)^\nu \frac{1}{\sqrt{r|x|}}\mathcal{H}_\nu[f](r|x|), & \text{otherwise.} \end{cases}$$

Exact formula for half-integer ν

Let $P_n,\ n = 0, 1, 2 \dots$ denote the Legendre polynomial

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$$f(y) = \frac{\sqrt{2\pi} i^{2\nu-1}}{\sigma} \frac{d^2}{dy^2} \int_y^\sigma \frac{y^2}{x^2} P_{2\nu-1}\left(\frac{x}{y}\right) \mathcal{F}_c^{-1}[g_{r,\nu}](x/\sigma) dx, \qquad y \in [0, \sigma]$$

$$g_{r,\nu}(x) := \begin{cases} \frac{1}{r|x|}\mathcal{H}_\nu[f](r|x|), & \text{if } x \geq 0, \\ (-1)^{2\nu-1}\frac{1}{r|x|}\mathcal{H}_\nu[f](r|x|), & \text{otherwise.} \end{cases}$$

Plane-wave expansion

$$e^{i\mathrm{pq}} = \sum_{l \in \mathbb{Z}} i^l J_l(|\mathrm{p}||\mathrm{q}|) e^{il(\phi_\mathrm{p} - \phi_\mathrm{q})} \qquad (d = 2)$$

$$e^{i\mathrm{pq}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(|\mathrm{p}||\mathrm{q}|) Y^*_{lm}(\theta_\mathrm{q}, \phi_\mathrm{q}) Y_{lm}(\theta_\mathrm{p}, \phi_\mathrm{p}) \qquad (d = 3)$$

Connection between the Fourier and Hankel Transforms

- $d = 2$

$$\mathcal{F}[f(|\mathrm{q}|)e^{il\phi_x}](|\mathrm{p}|, \phi_\mathrm{p}) = \frac{1}{2\pi} i^l e^{il\phi_\mathrm{p}} H_l[f](|\mathrm{p}|) \tag{2.3}$$

- $d = 3$

$$\mathcal{F}[f(|\mathrm{q}|)Y_{lm}(\theta_\mathrm{q}, \phi_\mathrm{q})](|\mathrm{p}|, \theta_\mathrm{p}, \phi_\mathrm{p}) = \frac{i^l}{2\pi^2|\mathrm{p}|}\sqrt{\frac{\pi}{2}}Y_{lm}(\theta_\mathrm{p}, \phi_\mathrm{p})H_{l+\frac{1}{2}}[|\mathrm{q}|f(|\mathrm{q}|)] \tag{2.4}$$

Projection Theorem

- Classical

$$\mathcal{F}[v](p) = \hat{v}(p) := \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{ipq} v(q) dq, \qquad p \in \mathbb{R}^d \tag{2.5}$$

$$\mathcal{R}_\theta[u](y) := \int_{q \in \mathbb{R}^d \,:\, q\theta = y} u(q) dq, \qquad y \in \mathbb{R} \tag{2.6}$$

$$\mathcal{F}[u](s\theta) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} e^{ist} \mathcal{R}[u](t,\theta) dt, \quad s \in \mathbb{R},\ \theta \in \mathbb{S}^{d-1} \tag{2.7}$$

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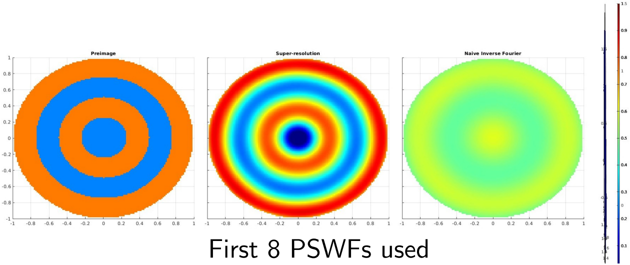
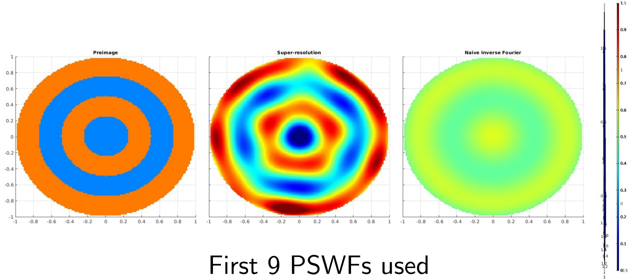
- Band-limited analogue

$$\hat{v}(rx\theta) = \left(\frac{\sigma}{2\pi}\right)^d \mathcal{F}_c \left[\mathcal{R}_\theta[v_\sigma]\right](x), \qquad \text{for } d \geq 2, \tag{2.8}$$

20% Noise Reconstruction

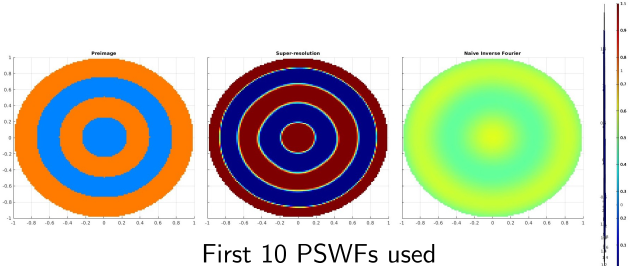


First 8 PSWFs used

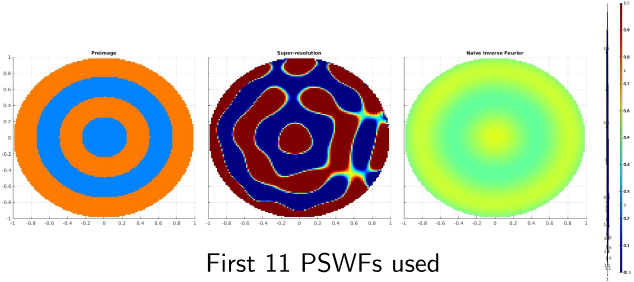


First 9 PSWFs used

20% Noise Reconstruction



First 10 PSWFs used



First 11 PSWFs used

Q&A

Thank you for attention!

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