COMMUTATION RELATIONS IN YANGIAN-TYPE ALGEBRAS

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Abstract. In paper [Ols22], G.I. Olshanski showed that there are four stable formulas for the commutator in the universal enveloping algebra $U(\mathfrak{gl}(N,\Omega))$ (where $\Omega = \mathbb{C}^L$). In a recent paper [OS23] Nikita Safonkin and G.I. Olshanski generalized these results for an arbitrary algebra Ω.

This paper addresses one of the problems posed in section 8 of [Ols22], namely finding a more explicit presentation of the formula. The question is of interest, because the formulas are used in a construction which generalizes that of Yangians $Y_d = Y(\mathfrak{gl}(d,\mathbb{C}))$.

In this work a recursive procedure to compute the formulas is derived. Several results regarding the structure of the formulas made through computer experiments are formulated and proved.

Contents

1. Preliminaries

The Yangian of the Lie algebra $\mathfrak{gl}(N,\mathbb{C})$ can be defined as the associative algebra with generators $t_{ij}^{(m)}$, where $1 \leq i,j \leq d, m = 1,2\ldots$ and defining quadratic-linear relations [MNO96]:

$$[t_{ij}^{(m)}, t_{kl}^{(n)}] = \sum_{r=0}^{\min(m,n)} t_{kj}^{(r)} t_{il}^{(m+n-r-1)} - t_{kj}^{(m+n-r-1)} t_{il}^{(r)}$$

where by convention $t_{ab}^{(0)} = \delta_{ab}$. The above relations can be written in terms of R-matrix formalism. An analogue of R-matrix formalism has not been found for the general case we are about to discuss. That is one of the reasons why it is difficult to work with.
The generalization of the Yangian construction is determined by an arbitrary associative algebra Ω. The conventional Yangian is obtained for $\Omega = \mathbb{C}$. The commutator relations are taken from the universal enveloping algebra $U(\mathfrak{gl}(N,\Omega))$. To a word $w = w_1 \otimes w_2 \cdots \otimes w_n$ from tensor algebra over Ω we can associate an element of the universal enveloping algebra $U(\mathfrak{gl}(N,\Omega))$

$$e_{ij}(w) = \sum_{k_1,\ldots,k_{n-1}=1}^{N} E_{ik_1}(w_1) E_{k_1k_2}(w_2) \cdots E_{k_{n-1}j}(w_n) \qquad 1 \leq i,j \leq N$$

In [Ols22] lemma 7.1 it is proved that the commutators of the above elements can be expressed by one of the four formulas below

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum_{z',z''} \varphi_1 e_{kj}(z') e_{il}(z'') + \psi_1 e_{ij}(z') e_{kl}(z'') \tag{1}$$

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum_{z',z''} \varphi_2 e_{kj}(z') e_{il}(z'') + \psi_2 e_{kl}(z') e_{ij}(z'') \tag{2}$$

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum_{z',z''} \varphi_3 e_{il}(z') e_{kj}(z'') + \psi_3 e_{ij}(z') e_{kl}(z'') \tag{3}$$

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum_{z',z''} \varphi_4 e_{il}(z') e_{kj}(z'') + \psi_4 e_{kl}(z') e_{ij}(z'') \tag{4}$$

The above formulas are stable, in a sense that they do not depend on N. Under additional restriction that the lengths $l(z') + l(z'')$ of the φ-part are in $n-1, n-3\ldots$, where $n = l(w) + l(\tilde{w})$ is the total length of original words, and the lengths $l(z') + l(z'')$ are in $n-2, n-4\ldots$, the above four formulas are uniquely determined. These formulas can be used as defining relations in the generalized construction of Yangian. In this work we show that the formulas obey certain combinatorial restrictions and satisfy certain identities, thus partially achieving the goal of an explicit description of the formulas.

2. Main algorithm

First we introduce some notation along with obvious identities, then we state the algorithm to compute the formulas, and afterwards we prove it.

2.1. **Notation.** Let's fix an arbitrary associative algebra Ω. We will call elements of this algebra letters. Now consider the tensor algebra $T(\Omega)$. We will call the elements of this tensor algebra words and denote the length of a word w by $l(w)$ - it is the number of letters the word consists of. To avoid writing redundant symbols, instead of $e_{ij}(w)$ we can just write w_{ij} and treat it as an element of $T(\Omega)$ to which indices were added. Moreover, we can consider the operation of adding indices ij as a map

$$\begin{aligned} ij : T(\Omega) &\to U(\mathfrak{gl}(N,\Omega)) \\ (w)_{ij} &\mapsto e_{ij}(w) \end{aligned}$$

Let's extend this notion to quadratic terms that arise in formulas. Let $T(\Omega)\boxtimes T(\Omega)$ denote the tensor product of two copies of $T(\Omega$ - we use the symbol $\boxtimes$ instead of conventional $\otimes$ to distinguish it from the tensor product within $T(\Omega)$. Then we can consider the map

$$\begin{aligned} {}^{ij}_{kl} : T(\Omega)\boxtimes T(\Omega) &\to U(\mathfrak{gl}(N,\Omega)) \\ (w\boxtimes \tilde{w})^{ij}_{kl} &\mapsto w_{ij}\tilde{w}_{kl} \end{aligned}$$

Given the uniqueness of the φ and ψ parts, we obtain well-defined maps

$$\Phi_i : T(\Omega)\boxtimes T(\Omega) \to T(\Omega)\boxtimes T(\Omega)$$

and

$$\Psi_i : T(\Omega)\boxtimes T(\Omega) \to T(\Omega)\boxtimes T(\Omega)$$

such that their composition with the appropriate index maps described above will give the expression corresponding to the φ and ψ respectively:

$$[x,y]^{ij}_{kl} = \Phi_1(x,y)^{kj}_{il} + \Psi_1(x,y)^{ij}_{kl} \tag{1'}$$

$$[x,y]^{ij}_{kl} = \Phi_2(x,y)^{kj}_{il} + \Psi_2(x,y)^{kl}_{ij} \tag{2'}$$

$$[x,y]^{ij}_{kl} = \Phi_3(x,y)^{il}_{kj} + \Psi_3(x,y)^{ij}_{kl} \tag{3'}$$

$$[x,y]^{ij}_{kl} = \Phi_4(x,y)^{il}_{kj} + \Psi_4(x,y)^{kl}_{ij} \tag{4'}$$

Here we are abusing the notation: $[x,y]^{ij}_{kl}$ denotes $[x_{ij}, y_{kl}]$. In other words, we apply the tensor product of the index maps

$$\begin{aligned} T(\Omega)\boxtimes T(\mathcal{A}) &\to U(\mathfrak{gl}(N,\Omega))\otimes U(\mathfrak{gl}(N,\Omega)) \\ x\boxtimes y &\mapsto x_{ij}\otimes y_{kl} \end{aligned}$$

and then take the commutator. Thus we can regard the commutator with specified indices as a map

$$\begin{aligned} [\cdot]^{ij}_{kl} : T(\Omega)\boxtimes T(\Omega) &\to U(\mathfrak{gl}(N,\Omega)) \\ [x\boxtimes y]^{ij}_{kl} &\mapsto [x_{ij}, y_{kl}] \end{aligned}$$

Each of the four formulas allows us to split this map as

$$\begin{tikzcd} & (T(\Omega)\boxtimes T(\Omega))^{\oplus 2} \arrow[dr] & \\ T(\Omega)\boxtimes T(\Omega) \arrow[ur, "\Phi_i\oplus\Psi_i"] \arrow[rr, "{[\cdot]^{ij}_{kl}}"] & & U(\mathfrak{gl}(N,\Omega)) \end{tikzcd}$$

Observing that

$$[x_{ij}, y_{kl}] = -[y_{kl}, x_{ij}]$$

and expanding l.h.s. using first formula, and the r.h.s. using the fourth, we obtain

$$\begin{aligned} \Phi_1(x,y) &= -\Phi_4(y,x) \\ \Psi_1(x,y) &= -\Psi_4(y,x) \end{aligned}$$

Similarly

$$\begin{aligned} \Phi_2(x,y) &= -\Phi_3(y,x) \\ \Psi_2(x,y) &= -\Psi_3(y,x) \end{aligned}$$

We will also need one more operation $^\circ$, which swaps the two components of $T(\Omega)\boxtimes T(\Omega)$:

$$(x\boxtimes y)^\circ = y\boxtimes x$$

Since

$$[x,y]^{ij}_{kl} = (x\boxtimes y)^{ij}_{kl} - (y\boxtimes x)^{kl}_{ij}$$

i.e.

$$(x\boxtimes y)^{ij}_{kl} = (y\boxtimes x)^{kl}_{ij} + [x,y]^{ij}_{kl}$$

we have the identity

$$\omega^{ij}_{kl} = (\omega^\circ)^{kl}_{ij} + [\omega]^{ij}_{kl} \qquad \forall\, \omega\in T(\Omega)\boxtimes T(\Omega)$$

2.2. **Algorithm.** To expand the commutator $[x,y]$ using one of the formulas $1'-4'$, either $l(x)=l(y)=1$ and we are in the base case (explained below). Otherwise we can assume WLOG that $l(x)>1$, using the relationships. Next we split $x=x'x''$ and use the appropriate formulas which recursively require to know the formulas for shorter words.

$$\begin{cases}\Phi_1(x'x'',y) = & (\Phi_4(x',y)x''+x'\Phi_3(x'',y))^\circ + \Phi_4\left(\Psi_4(x',y)x''\right)\\ & \qquad +\Psi_4\left(\Phi_4(x',y)x''+x'\Phi_3(x'',y)\right)\\ \Psi_1(x'x'',y) = & x'\Psi_3(x'',y) + \Phi_4\left(\Phi_4(x',y)x''+x'\Phi_3(x'',y)\right)\\ & \qquad +\left(\Psi_4(x',y)x''\right)^\circ + \Psi_4\left(\Psi_4(x',y)x''\right)\end{cases} \qquad \text{(for } 1'\text{)}$$

$$\begin{cases}\Phi_2(x'x'',y) = & (\Phi_4(x',y)x''+x'\Phi_3(x'',y))^\circ + \Phi_2\left(x'\Psi_3(x'',y)\right)\\ & \qquad +\Psi_2\left(\Phi_4(x',y)x''+x'\Phi_3(x'',y)\right)\\ \Psi_2(x'x'',y) = & \Psi_4(x',y)x'' + \Phi_2\left(\Phi_4(x',y)x''+x'\Phi_3(x'',y)\right)\\ & \qquad +\left(x'\Psi_3(x'',y)\right)^\circ + \Psi_2\left(x'\Psi_3(x'',y)\right)\end{cases} \qquad \text{(for } 2'\text{)}$$

$$\begin{cases}\Phi_3(x'x'',y) = \Phi_4(x',y)x''+x'\Phi_3(x'',y)+\Phi_2\left(\Psi_4(x',y)x''\right)\\ \Psi_3(x'x'',y) = x'\Psi_3(x'',y)+\left(\Psi_4(x',y)x''\right)^\circ+\Psi_2\left(\Psi_4(x',y)x''\right)\end{cases} \qquad \text{(for } 3'\text{)}$$

$$\begin{cases}\Phi_4(x'x'',y) = \Phi_4(x',y)x''+x'\Phi_3(x'',y)+\Phi_4\left(x'\Psi_3(x'',y)\right)\\ \Psi_4(x'x'',y) = \Psi_4(x',y)x''+\left(x'\Psi_3(x'',y)\right)^\circ+\Psi_4\left(x'\Psi_3(x'',y)\right)\end{cases} \qquad \text{(for } 4'\text{)}$$

And here are the formulas in case we want to split the second word instead of the first

$$\begin{cases}\Phi_1(x,y'y'') = \Phi_1(x,y')y''+y'\Phi_2(x,y'')+\Phi_4\left(y'\Psi_2(x,y'')\right)\\ \Psi_1(x,y'y'') = \Psi_1(x,y')y''+\left(y'\Psi_2(x,y'')\right)^\circ+\Psi_4\left(y'\Psi_2(x,y'')\right)\end{cases} \qquad \text{(for } 1'\text{)}$$

$$\begin{cases}\Phi_2(x,y'y'') = \Phi_1(x,y')y''+y'\Phi_2(x,y'')+\Phi_2\left(\Psi_1(x,y')y''\right)\\ \Psi_2(x,y'y'') = y'\Psi_2(x,y'')+\left(\Psi_1(x,y')y''\right)^\circ+\Psi_2\left(\Psi_1(x,y')y''\right)\end{cases} \qquad \text{(for } 2'\text{)}$$

$$\begin{cases}\Phi_3(x,y'y'') = & (\Phi_1(x,y')y''+y'\Phi_2(x,y''))^\circ + \Phi_2\left(y'\Psi_2(x,y'')\right)\\ & \qquad +\Psi_2\left(\Phi_1(x,y')y''+y'\Phi_2(x,y'')\right)\\ \Psi_3(x,y'y'') = & \Psi_1(x,y')y'' + \Phi_2\left(\Phi_1(x,y')y''+y'\Phi_2(x,y'')\right)\\ & \qquad +\left(y'\Psi_3(x,y'')\right)^\circ + \Psi_2\left(y'\Psi_2(x,y'')\right)\end{cases} \qquad \text{(for } 3'\text{)}$$

$$\begin{cases}\Phi_4(x,y'y'') = & (\Phi_1(x,y')y''+y'\Phi_2(x,y''))^\circ + \Phi_4\left(\Psi_1(x,y')y''\right)\\ & \qquad +\Psi_4\left(\Phi_1(x,y')y''+y'\Phi_3(x,y'')\right)\\ \Psi_4(x,y'y'') = & y'\Psi_2(x,y'') + \Phi_4\left(\Phi_1(x,y')y''+y'\Phi_2(x,y'')\right)\\ & \qquad +\left(\Psi_1(x,y')y''\right)^\circ + \Psi_4\left(\Psi_1(x,y')y''\right)\end{cases} \qquad \text{(for } 4'\text{)}$$

2.2.1. *Base case.*

$$[x,y]^{ij}_{kl} = [e_{ij}(x), e_{kl}(y)] = e_{il}(xy)\delta_{kj} - e_{kj}(yx)\delta_{il}$$

Rewriting it as

$$\delta_{kj}e_{il}(xy) - e_{kj}(yx)\delta_{il} = (\varepsilon \boxtimes xy - yx \boxtimes \varepsilon)^{kj}_{il}$$

we recognize the Φ part of formulas 1 and 2 (where ε is the identity of $T(\Omega)$). Rewriting as

$$e_{il}(xy)\delta_{kj} - \delta_{il}e_{kj}(yx) = (xy \boxtimes \varepsilon - \varepsilon \boxtimes xy)^{il}_{kj}$$

we obtain the Φ part of formulas 3 and 4.

2.3. **Proof of the algorithm.** We will prove the case when the first word is split. If we apply symmetries to both sides of identities when the first word is split, we will obtain the formulas for the case when the second word is split.
Assume $l(x) > 1$, so that $x = x'x''$. Applying Leibniz rule, we have

$$[x_{ij}, y_{kl}] = [(x'x'')_{ij}, y_{kl}] = \sum_a [x'_{ia}x''_{aj}, y_{kl}] =$$
$$= \sum_a [x'_{ia}, y_{kl}]x''_{aj} + x'_{ia}[x''_{aj}, y_{kl}]$$

Next, we expand the commutator in the first summand using the fourth equation and in the second - using the third.

$$\sum_a [x'_{ia}, y_{kl}]x''_{aj} = \sum_a \Phi_4(x', y)^{il}_{ka}x''_{aj} + \Psi_4(x', y)^{kl}_{ia}x''_{aj} =$$
$$(\Phi_4(x', y)x'')^{il}_{kj} + (\Psi_4(x', y)x'')^{kl}_{ij}$$

Similarly we get

$$\sum_a x'_{ia}[x''_{aj}, y_{kl}] = \sum_a x'_{ia}\Phi_3(x'', y)^{al}_{kj} + x'_{ia}\Psi_3(x'', y)^{aj}_{kl} =$$
$$(x'\Phi_3(x'', y))^{il}_{kj} + (x'\Psi_3(x'', y))^{ij}_{kl}$$

Therefore

$$[x'x'', y]^{ij}_{kl} = (\Phi_4(x', y)x'' + x'\Phi_3(x'', y))^{il}_{kj} + (\Psi_4(x', y)x'')^{kl}_{ij} + (x'\Psi_3(x'', y))^{ij}_{kl}$$

Finally, we need to convert each of the terms above so that it has the indices corresponding to one of the formulas $1' - 4'$. The easy part is when we want to compute the third or fourth formula, which is why we will do them first.

Fourth formula. Applying the identity to $\omega = x'\Psi_3(x'', y)$ we obtain

$$(x'\Psi_3(x, y))^{ij}_{kl} = \left((x'\Psi_3(x, y))^\circ\right)^{kl}_{ij} + [x'\Psi_3(x, y)]^{ij}_{kl} =$$
$$\left((x'\Psi_3(x, y))^\circ\right)^{kl}_{ij} + \Phi_4\left(x'\Psi_3(x, y)\right)^{il}_{kj} + \Psi_4\left(x'\Psi_3(x, y)\right)^{kl}_{ij}$$

Therefore the recursive procedure for the fourth formula is

$$\begin{aligned}
&[x'x'', y]^{ij}_{kl} = \\
&\qquad (\Phi_4(x', y)x'' + x'\Phi_3(x'', y)) + \Phi_4\left(x'\Psi_3(x'', y)\right))^{il}_{kj} \\
&\qquad\qquad + \left(\Psi_4(x', y)x'' + (x'\Psi_3(x'', y))^\circ + \Psi_4\left(x'\Psi_3(x'', y)\right)\right)^{kl}_{ij}
\end{aligned}$$

In other words

$$\begin{cases}
\Phi_4(x'x'', y) = \Phi_4(x', y)x'' + x'\Phi_3(x'', y) + \Phi_4\left(x'\Psi_3(x'', y)\right) \\
\Psi_4(x'x'', y) = \Psi_4(x', y)x'' + (x'\Psi_3(x'', y))^\circ + \Psi_4\left(x'\Psi_3(x'', y)\right)
\end{cases}$$

Third formula. This time we must convert $(\Psi_4(x',y)x'')^{kl}_{ij}$ term.

$$(\Psi_4(x',y)x'')^{kl}_{ij} = \left((\Psi_4(x',y)x'')^\circ\right)^{ij}_{kl} + [(\Psi_4(x',y)x'')]^{kl}_{ij}$$

Applying the second formula we can rewrite the second term above as

$$[(\Psi_4(x',y)x'')]^{kl}_{ij} = \Phi_2\left(\Psi_4(x',y)x''\right)^{il}_{kj} + \Psi_2\left(\Psi_4(x',y)x''\right)^{ij}_{kl}$$

Thus

$$\begin{cases} \Phi_3(x'x'',y) = \Phi_4(x',y)x'' + x'\Phi_3(x'',y) + \Phi_2\left(\Psi_4(x',y)x''\right) \\ \Psi_3(x'x'',y) = x'\Psi_3(x'',y) + (\Psi_4(x',y)x'')^\circ + \Psi_2\left(\Psi_4(x',y)x''\right) \end{cases}$$

Second formula. This time we need to convert two terms: $(\Phi_4(x',y)x'' + x'\Phi_3(x'',y))^{il}_{kj}$ and $(x'\Psi_3(x'',y))^{ij}_{kl}$. For the first term we have

$$\begin{aligned} &(\Phi_4(x',y)x'' + x'\Phi_3(x'',y))^{il}_{kj} = \\ &\qquad\qquad \left((\Phi_4(x',y)x'' + x'\Phi_3(x'',y))^\circ\right)^{kj}_{il} + [\Phi_4(x',y)x'' + x'\Phi_3(x'',y)]^{il}_{kj} \end{aligned}$$

Expanding the commutator using the second formula we have

$$\begin{aligned} [\Phi_4(x',y)x'' + x'\Phi_3(x'',y)]^{il}_{kj} &= \Phi_2\left(\Phi_4(x',y)x'' + x'\Phi_3(x'',y)\right)^{kl}_{ij} + \\ &\qquad \Psi_2\left(\Phi_4(x',y)x'' + x'\Phi_3(x'',y)\right)^{kj}_{il} \end{aligned}$$

Now the second term

$$(x'\Psi_3(x'',y))^{ij}_{kl} = \left((x'\Psi_3(x'',y))^\circ\right)^{kl}_{ij} + [x'\Psi_3(x'',y)]^{ij}_{kl}$$

Applying the second formula (yes, here too) to the commutator, we get

$$[x'\Psi_3(x'',y)]^{ij}_{kl} = \Phi_2\left(x'\Psi_3(x'',y)\right)^{kj}_{il} + \Psi_2\left(x'\Psi_3(x'',y)\right)^{kl}_{ij}$$

Collecting the terms we get

$$\begin{cases} \Phi_2(x'x'',y) = & (\Phi_4(x',y)x'' + x'\Phi_3(x'',y))^\circ + \Phi_2\left(x'\Psi_3(x'',y)\right) \\ & \qquad +\Psi_2\left(\Phi_4(x',y)x'' + x'\Phi_3(x'',y)\right) \\ \Psi_2(x'x'',y) = & \Psi_4(x',y)x'' + \Phi_2\left(\Phi_4(x',y)x'' + x'\Phi_3(x'',y)\right) \\ & \qquad + (x'\Psi_3(x'',y))^\circ + \Psi_2\left(x'\Psi_3(x'',y)\right) \end{cases}$$

First formula. Here we need to convert $(\Phi_4(x',y)x'' + x'\Phi_3(x'',y))^{il}_{kj}$ and $(\Psi_4(x',y)x'')^{kl}_{ij}$. As above

$$\begin{aligned} &(\Phi_4(x',y)x'' + x'\Phi_3(x'',y))^{il}_{kj} = \\ &\qquad\qquad \left((\Phi_4(x',y)x'' + x'\Phi_3(x'',y))^\circ\right)^{kj}_{il} + [\Phi_4(x',y)x'' + x'\Phi_3(x'',y)]^{il}_{kj} \end{aligned}$$

but here we expand the commutator using the fourth formula

$$\begin{aligned} &[\Phi_4(x',y)x'' + x'\Phi_3(x'',y)]^{il}_{kj} = \\ &\qquad \Phi_4\left(\Phi_4(x',y)x'' + x'\Phi_3(x'',y)\right)^{ij}_{kl} + \Psi_4\left(\Phi_4(x',y)x'' + x'\Phi_3(x'',y)\right)^{kj}_{il} \end{aligned}$$

For the other term

$$(\Psi_4(x',y)x'')^{kl}_{ij} = \left((\Psi_4(x',y)x'')^\circ\right)^{ij}_{kl} + [(\Psi_4(x',y)x'')]^{kl}_{ij}$$

we expand the commutator using the fourth formula too

$$[(\Psi_4(x',y)x'')]^{kl}_{ij} = \Phi_4\left(\Psi_4(x',y)x''\right)^{kj}_{il} + \Psi_4\left(\Psi_4(x',y)x''\right)^{ij}_{kl}$$

Collecting the terms we obtain

$$\begin{cases} \Phi_1(x'x'',y) = & (\Phi_4(x',y)x''+x'\Phi_3(x'',y))^\circ + \Phi_4\left(\Psi_4(x',y)x''\right) \\ & \qquad +\Psi_4\left(\Phi_4(x',y)x''+x'\Phi_3(x'',y)\right) \\ \Psi_1(x'x'',y) = & x'\Psi_3(x'',y)+\Phi_4\left(\Phi_4(x',y)x''+x'\Phi_3(x'',y)\right) \\ & \qquad +\left(\Psi_4(x',y)x''\right)^\circ+\Psi_4\left(\Psi_4(x',y)x''\right) \end{cases}$$

3. Main results

3.1. **Polynomial algebra identity.** Suppose that our algebra is the polynomial ring, i.e. $\Omega = k\,[x_1,\ldots,x_n]$. Then the symmetric group S_n acts on Ω by permuting the variables and hence on $U(\mathfrak{gl}_N(\Omega))$

Proposition 3.1.1. *Take*

$$x = x_1\otimes x_2\cdots\otimes x_m,\, y = x_{m+1}\otimes x_{m+2}\cdots\otimes x_{m+n}\in T(k\,[x_1,\ldots,x_{n+m}])$$

then

$$\sum_{\sigma\in S_{n+m}}(-1)^\sigma\sigma([x,y])=0$$

Proof. Induction on $n+m$. The base case $n=m=1$ follows from commutativity:

$$[x,y]^{ij}_{kl} = (\varepsilon\boxtimes xy - yx\boxtimes\varepsilon)^{kj}_{il} = (\varepsilon\boxtimes yx - xy\boxtimes\varepsilon)^{kj}_{il} = [y,x]^{ij}_{kl}$$

Due to antisymmetry of the commutator, it suffices to do the inductive step in case $m>1$. Split the first word

$$x = x'x''$$

We won't need explicit representation, so we will drop the indices. Apply the Leibniz rule

$$\sum_{\sigma\in S_{n+m}}(-1)^\sigma\sigma([x'x'',y]) = \sum_{\sigma\in S_{n+m}}(-1)^\sigma\sigma([x',y]x'') + \sum_{\sigma\in S_{n+m}}(-1)^\sigma\sigma(x'[x'',y])$$

We will show that the second sum vanishes, the first is done analogously. Let $k=l(x')$ and note that

$$S_{n+m-k}\hookrightarrow S_{n+m}$$

Where S_{n+m-k} permutes $x_{k+1}\ldots x_{n+m}$. Decompose

$$S_{n+m} = \bigsqcup \sigma_l S_{n+m-k}$$

into left cosets and expand the sum

$$\sum(-1)^\sigma\sigma(x'[x'',y]) = \sum_{\substack{l=1,2\ldots\\ \tau\in S_{n+m-k}}}(-1)^{\sigma_l\tau}\sigma_l\tau\left(x'[x'',y]\right)=$$

$$=\sum_l(-1)^{\sigma_l}\sigma_l\left(\sum_{\tau\in S_{n+m-k}}(-1)^\tau\tau(x'[x'',y])\right)=$$

$$=\sum_l(-1)^{\sigma_l}\sigma_l\left(x'\sum_{\tau\in S_{n+m-k}}(-1)^\tau\tau([x'',y])\right)=0$$

because the inner sum vanishes by induction hypothesis. □

3.2. **Preservation of letter order.** Throughout this subsection let the ground algebra be the free algebra, i.e. $\Omega = \mathbb{C}\langle x_1, \ldots, x_n\rangle$. The results in this subsection place combinatorial constraints on which non-zero terms can appear in the Ψ part of the formula. For convenience, let us introduce some terminology: take, for example, $\Phi_1(x, y)$. It is a sum of terms of the form $z' \boxtimes z''$ with non-zero coefficients. We will call z' the left part of each term and z'' the right part.

Proposition 3.2.1.

1. *For $\Psi_1(x, y)$ the letters of y appear only on the right side of each term*
2. *For $\Psi_2(x, y)$ the letters of y appear only on the left side of each term*
3. *For $\Psi_3(x, y)$ the letters of x appear only on the left side of each term*
4. *For $\Psi_4(x, y)$ the letters of x appear only on the right side of each term*

In the proposition above one easily sees that statements 1 and 4 as well as 2 and 3, are equivalent to each other due to symmetries. So it suffices to just prove the 3rd and the 4th. We will be proving by induction on $n = l(x) + l(y)$, assuming all the four statements hold whenever $l(a) + l(b) < n$. The base case $l(a) = l(b) = 1$ is trivial, because it can be seen from base case formulas that the Ψ part is zero.

Proof. Induction step for the 3rd statement. Recall the formula for Ψ_3:

$$\Psi_3(x'x'', y) = x'\Psi_3(x'', y) + (\Psi_4(x', y)x'')^\circ + \Psi_2\left(\Psi_4(x', y)x''\right)$$

We will show for each of the three summands that the letters of x are contained on the left side. For $x'\Psi_3(x'', y)$ we have the letters of x' concatenated to the left side and the letters of x'' are on the left side by induction. Similarly for $\Psi_4(x', y)x''$ the letters of x'' are concatenated on right and letters of x' appear on the right by induction. So when we apply the swap operation $\circ$, the letters of x will all be on the left for $(\Psi_4(x', y)x'')^\circ$. And when we apply Ψ_2, using the second statement by induction (which is possible, as Ψ strictly lowers the total length), the letters which were on the right will be on the left. Since letters of x were contained in the right side, after Ψ_2 they will appear on the left, so we have also shown the property for $\Psi_2\left(\Psi_4(x', y)x''\right)$.
Induction for the 4th statement is proved in exactly the same way. The formula is

$$\Psi_4(x'x'', y) = \Psi_4(x', y)x'' + (x'\Psi_3(x'', y))^\circ + \Psi_4\left(x'\Psi_3(x'', y)\right)$$

For term $\Psi_4(x', y)x''$, x' is on the right by induction and x'' is concatenated on the right. For $x'\Psi_3(x'', y)$ we showed above that x is on the left, so in $(x'\Psi_3(x'', y))^\circ$ it's on the right. Finally, applying Ψ_4 moves the left to the right, so since x was contained on the left in each term of $x'\Psi_3(x'', y)$, it will be on the right in $\Psi_4\left(x'\Psi_3(x'', y)\right)$ $\square$

Taking the same example as above, we are considering a term $\Phi_1(x, y)$. It has the form $z' \boxtimes z''$. Write out the letters of $z' \otimes z''$ consecutively, so we will get $z_1, \ldots, z_p$. Each letter z_l may contain several letters of $x \otimes y$, because two letters may be multiplied as elements of Ω. Let's say that two letters a and b have been *glued* with each in the term $z' \boxtimes z''$, if they both appear within one letter z_l (not necessarily directly adjacent though). Also, for convenience of the formulation of the following statement, let's say that two letters a, b are *not glued* if they aren't glued in any of the terms. When we say that more than two letters are not glued with each other, we mean that those letters aren't glued with each other pairwise.

Proposition 3.2.2.

1. *For $\Psi_1(x, y)$ the letters of y are not glued with each other*
2. *For $\Psi_2(x, y)$ the letters of y are not glued with each other*
3. *For $\Psi_3(x, y)$ the letters of x are not glued with each other*

(4) *For $\Psi_4(x,y)$ the letters of x are not glued with each other*

Again, due to symmetries, it suffices to prove only the 3rd and the 4th statements. We will also be using induction in the same way as above. The base case $l(x) = l(y) = 1$ is trivial, because there is only one letter both in x and in y.

Proof.

$$\Psi_3(x'x'',y) = x'\Psi_3(x'',y) + (\Psi_4(x',y)x'')^\circ + \Psi_2\left(\Psi_4(x',y)x''\right)$$
$$\Psi_4(x'x'',y) = \Psi_4(x',y)x'' + (x'\Psi_3(x'',y))^\circ + \Psi_4\left(x'\Psi_3(x'',y)\right)$$

As above, we will prove the property for each summand. Since we are proving the same property for 3rd and 4th formulas, it is more concise to treat them at once (because some summands are essentially the same). For $x'\Psi_3(x'',y)$, x' is concatenated, so its letters are not glued to anythihg. Letters of x'' are not glued to each other by induction. The swap operation preserves this property, so it is still true for $(x'\Psi_3(x'',y))^\circ$. The same reasoning shows that letters of x are not glued with each other in $\Psi_4(x',y)x''$ and $(\Psi_4(x',y)x'')^\circ$.
In the proof of the previous proposition, we have shown that the letters of x will appear on the right side of $\Psi_4(x',y)x''$, and as we have just shown, the letters of x aren't glued with each other. So using the 2nd statement under the induction hypothesis, we obtain that the letters of x aren't glued in $\Psi_2\left(\Psi_4(x',y)x''\right)$.
Similarly, the letters of x are on the left and not glued with each other for $x'\Psi_3(x'',y)$. Applying 4th statement by induction, we have that letters of x aren't glued in $\Psi_4\left(x'\Psi_3(x'',y)\right)$ □

Let's continue with the example of $\Phi_1(x,y)$ above. We take a term $z' \boxtimes z''$ and we write out the letters of z', z'' to obtain $z_1, \ldots, z_p$. Since we are working over the free algebra, it makes sense to talk about the order of the letters of x, y as they appear in $z_1, \ldots, z_p$, despite some of the letters being glued (though in view of the above proposition, the gluing will not even be a problem for the claim we are about to state). Let's say that the letters of y appear in their original order, if they appear in their original order (i.e. as $y_1, y_2 \ldots$) for all terms.

Proposition 3.2.3.

(1) *For $\Psi_1(x,y)$ the letters of y appear in their original order*
(2) *For $\Psi_2(x,y)$ the letters of y appear in their original order*
(3) *For $\Psi_3(x,y)$ the letters of x appear in their original order*
(4) *For $\Psi_4(x,y)$ the letters of x appear in their original order*

Proof. Again, using induction, the base case being trivial (because there is only one possible order on one letter). Only need to prove 3rd and 4th due to symmetries, and as in the last proposition, since the property we are trying to prove is the same for 3rd and 4th statements, we will prove it simultaneously for all terms that appear in the recursive relations for Ψ_3, Ψ_4. Here are the expressions:

$$\Psi_3(x'x'',y) = x'\Psi_3(x'',y) + (\Psi_4(x',y)x'')^\circ + \Psi_2\left(\Psi_4(x',y)x''\right)$$
$$\Psi_4(x'x'',y) = \Psi_4(x',y)x'' + (x'\Psi_3(x'',y))^\circ + \Psi_4\left(x'\Psi_3(x'',y)\right)$$

The letters of x' appear in their original order in $x'\Psi_3(x'',y)$, as do letters of x'' by induction hypothesis. Since x' is concatenated on the left, the letters of x' are in correct order w.r.t. letters of x''. So altogether, letters of x are in original order. In the proof of proposition 3.2.1, we have shown that all letters of x are on the left for $x'\Psi_3(x'',y)$, so after swap operation they will still be in the same order (as the swap operation preserves order within either side). So letters of x will be in original order for $(x'\Psi_3(x'',y))^\circ$.

The same reasoning applies to $\Psi_4(x',y)x''$ and $(\Psi_4(x',y)x'')^\circ$.
From proof of proposition 3.2.1, x is contained on the right for $\Psi_4(x',y)x''$. Using 2nd statement from induction hypothesis, we see that if we take Ψ_2, the letters of the right side of each term will come in original order. Since letters of x are on the right side of every term, their order will be preserved, that is letters of x come in original order for $\Psi_2(\Psi_4(x',y)x'')$.
Similarly, since letters of x are on the left for $x'\Psi_3(x'',y)$ and come in original order, using 4th statement by induction, letters of x will be in original order for $\Psi_4(x'\Psi_3(x'',y))$. $\square$

3.2.1. *Summary.* Let's make a summary of the three propositions we just proved, that give combinatorial constraints on the Ψ part of our formulas.

(1) For $\Psi_1(x,y)$ the letters of y appear on the right side, in their original order and are not glued together
(2) For $\Psi_2(x,y)$ the letters of y appear on the left side, in their original order and are not glued together
(3) For $\Psi_3(x,y)$ the letters of x appear on the left side, in their original order and are not glued together
(4) For $\Psi_4(x,y)$ the letters of x appear on the right side, in their original order and are not glued together

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