Recursive procedure to calculate the commutator

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1 Terminology

Let's fix an arbitrary associative algebra $\mathcal{A}$. We will call elements of this algebra letters. Now consider the tensor algebra $T(\mathcal{A})$. We will call the elements of this tensor algebra words and denote the length of a word w by $l(w)$ - it is the number of letters the word consists of. To a word from tensor algebra we can associate an element of the universal enveloping algebra $U(\mathfrak{gl}(\mathcal{A}))$

$$e_{ij}(w) = \sum e_{ik_1}(w_1)e_{k_1k_2}(w_2)\cdots e_{k_{n-1}j}(w_n)$$

$$w = w_1 \otimes w_2 \cdots \otimes w_n$$

We will call this sum a train. When we have two trains, we can compute their commutator in the universal enveloping algebra. Using stability lemma, we can express the commutator using one of the four formulas described below. We will use a lower index to denote that expansion of the commutator. For example $[e_{ij}(w), e_{kl}(\tilde{w})]_1$ means that the commutator was expanded using the first formula. Also, although mathematically it means nothing, we will need to be able to emphasize whether the two parts of the formula should be exchanged (see the **N.B.** comments in section 3.1). We will make the index bold to denote that, e.g. $[e_{ij}(w), e_{kl}(\tilde{w})]_{\mathbf{1}}$

2 Preliminary observations

Recall the four formulas

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum \varphi_1 e_{kj}(z')e_{il}(z'') + \psi_1 e_{ij}(z')e_{kl}(z'') \tag{1}$$

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum \varphi_2 e_{kj}(z')e_{il}(z'') + \psi_2 e_{kl}(z')e_{ij}(z'') \tag{2}$$

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum \varphi_3 e_{il}(z')e_{kj}(z'') + \psi_3 e_{ij}(z')e_{kl}(z'') \tag{3}$$

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum \varphi_4 e_{il}(z')e_{kj}(z'') + \psi_4 e_{kl}(z')e_{ij}(z'') \tag{4}$$

A simple observation is that if we permute the indices on the l.h.s., we just have to do the same permutation on the r.h.s. and the identity will, of course, still be valid. Let's use it to expand the following identity.

$$[e_{ij}(w), e_{kl}(\tilde{w})] = -[e_{kl}(\tilde{w}), e_{ij}(w)]$$

If we use the first formula to express the l.h.s. and the fourth formula for the r.h.s. using the permutation

$$\begin{pmatrix} i & j & k & l \\ k & l & i & j \end{pmatrix}$$

we get

$$\sum \varphi_1 e_{kj}(z')e_{il}(z'')+\psi_1 e_{ij}(z')e_{kl}(z'') = -\sum \varphi_4 e_{kj}(z')e_{il}(z'')+\psi_4 e_{ij}(z')e_{kl}(z'')$$

This means, that whenever we want to compute the first formula for $w, \tilde{w}$ we can instead compute the fourth formula for $\tilde{w}, w$ and take it with a minus sign (and vice versa we can compute the fourth formula via the first one). We have the same relationship between the second and third formulas, which is proved exactly as above - by applying permutation to the indices

$$\sum \varphi_2 e_{kj}(z')e_{il}(z'')+\psi_2 e_{kl}(z')e_{ij}(z'') = -\sum \varphi_3 e_{kj}(z')e_{il}(z'')+\psi_3 e_{kl}(z')e_{ij}(z'')$$

This observation allows us to only consider the case where $l(w) \geq l(\tilde{w})$ when we want to compute the commutator. Indeed, if the first word was shorter than the second, and we wanted to compute formula x, we could swap the words, compute the formula $5 - x$ and negate the coefficients.

3 Recursive step

3.1 Derivations

Assuming that we are not in the base case (where both words have length one), which is handled in the appendix and that $l(w) \geq l(\tilde{w})$, we have $l(w) \geq 2$. Hence we can split w into two non-empty subwords, $w = w_1 \otimes w_2$. Next, we expand the train corresponding to w and use the Leibniz rule.

$$\begin{aligned}
[e_{ij}(w), e_{kl}(\tilde{w})] = [e_{ij}(w_1 \otimes w_2), e_{kl}(\tilde{w})] = \sum_a [e_{ia}(w_1)e_{aj}(w_2), e_{kl}(\tilde{w})] = \\
= \sum_a [e_{ia}(w_1), e_{kl}(\tilde{w})]e_{aj}(w_2) + e_{ia}(w_1)[e_{aj}(w_2), e_{kl}(\tilde{w})]
\end{aligned}$$

Next, we expand the commutator in the first summand using the fourth equation and in the second - using the third. For the first we get

$$[e_{ia}(w_1), e_{kl}(\tilde{w})] = \sum \varphi_4 e_{il}(z')e_{ka}(z'') + \psi_4 e_{kl}(z')e_{ia}(z'')$$

Hence

$$\begin{aligned}
&\sum_a [e_{ia}(w_1), e_{kl}(\tilde{w})]e_{aj}(w_2) = \\
&\quad = \sum_a \left(\sum \varphi_4 e_{il}(z')e_{ka}(z'') + \psi_4 e_{kl}(z')e_{ia}(z'') \right) e_{aj}(w_2) = \\
&\quad\quad = \sum \varphi_4 e_{il}(z')e_{kj}(z'' \otimes w_2) + \psi_4 e_{kl}(z')e_{ij}(z'' \otimes w_2)
\end{aligned}$$

Similarly we get

$$\sum_a e_{ia}(w_1)[e_{aj}(w_2), e_{kl}(\tilde{w})] =$$

$$= \sum_a e_{ia}(w_1) \left(\sum \varphi_3 e_{al}(z') e_{kj}(z'') + \psi_3 e_{aj}(z') e_{kl}(z'') \right) =$$

$$= \sum \varphi_3 e_{il}(w_1 \otimes z') e_{kj}(z'') + \psi_3 e_{ij}(w_1 \otimes z') e_{kl}(z'')$$

Hence we have expressed the commutator as

$$[e_{ij}(w), e_{kl}(\tilde{w})] = \sum \alpha e_{il}(y') e_{kj}(y'') + \beta e_{kl}(y') e_{ij}(y'') + \gamma e_{ij}(y') e_{kl}(y'')$$

Finally, we need to convert each of the terms in the resulting sum so that it has the indices corresponding either φ or ψ part of the formula we want to express. The easy part is when we want to compute the third or fourth formula, which is why we will do the formulas in reverse order.

Fourth formula

We only need to convert the terms of the form $e_{ij}(y')e_{kl}(y'')$ We use the following technique:

$$e_{ij}(y') e_{kl}(y'') = e_{kl}(y'') e_{ij}(y') + [e_{ij}(y'), e_{kl}(y'')]$$

Expanding the commutator using the fourth formula clearly gives us the terms with indices corresponding to the fourth formula. As an example, we will expand the commutator here, but will not do it for other formulas as it seems verbose.

$$\sum \alpha e_{il}(y') e_{kj}(y'') + \beta e_{kl}(y') e_{ij}(y'') + \gamma e_{ij}(y') e_{kl}(y'') =$$

$$= \sum \alpha e_{il}(y') e_{kj}(y'') + \beta e_{kl}(y') e_{ij}(y'') + \gamma e_{kl}(y'') e_{ij}(y') +$$

$$+ \gamma \left(\sum \varphi_4(y', y'', z', z'') e_{il}(z') e_{kj}(z'') + \psi_4(y', y'', z', z'') e_{kl}(z') e_{ij}(z'') \right)$$

Third formula

We only need to convert the $e_{kl}(y')e_{ij}(y'')$ term.

$$e_{kl}(y') e_{ij}(y'') = e_{ij}(y'') e_{kl}(y') + [e_{kl}(y'), e_{ij}(y'')]$$

The permutation application exercise for this case has already been done in the preliminary observations, so using the second formula gives

$$[e_{kl}(y'), e_{ij}(y'')]_2 = \sum \varphi_2 e_{il}(z') e_{kj}(z'') + \psi_2 e_{ij}(z') e_{kl}(z'')$$

Second formula

We need to convert the terms $e_{il}(y')e_{kj}(y'')$ and $e_{ij}(y')e_{kl}(y'')$

$$e_{il}(y') e_{kj}(y'') = e_{kj}(y'') e_{il}(y') + [e_{il}(y'), e_{kj}(y'')]$$

Guessing which formula to use is a matter of looking up the right permutaion. Each of the four formulas introduces two permutations - one for φ part and one for ψ. We know that we need the following two permutations:

$$\begin{pmatrix} i & l & k & j \\ k & j & i & l \end{pmatrix}$$

and

$$\begin{pmatrix} i & l & k & j \\ k & l & i & j \end{pmatrix}$$

Actually, this is a terrible abuse of notation, because, for example, what we really mean in the first case is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

acting on the tuple (i, l, k, j) and not on the set of four elements! So, replacing the abusive versions with proper numerical notation for each of the four formulas (there will be two permutations associated with each formula) and then writing the two desired permutations numerically, we simply match them. In this case, it turns out that the second formula provides the appropriate permutations. Indeed, applying the permutation

$$\begin{pmatrix} i & j & k & l \\ i & l & k & j \end{pmatrix}$$

to the second formula, we obtain

$$\begin{aligned}[e_{il}(y'), e_{kj}(y'')]_2 = \sum \varphi_2 e_{kl}(z') e_{ij}(z'') + \psi_2 e_{kj}(z') e_{il}(z'') = \\ = \sum \psi_2 e_{kj}(z') e_{il}(z'') + \varphi_2 e_{kl}(z') e_{ij}(z'')\end{aligned}$$

N.B. Note how the φ and ψ part were initially not in the usual order for formula 2! To emphasize it, I have swapped the φ and ψ parts in the last line above. As mentioned in the terminology section, we use bold index $[e_{il}(y'), e_{kj}(y'')]_{\mathbf{2}}$ to indicate it.

We need to express one more term for the second formula. From now on we are just using the techniques described above, so it should be clear without explanations.

$$e_{ij}(y') e_{kl}(y'') = e_{kl}(y'') e_{ij}(y') + [e_{ij}(y'), e_{kl}(y'')]$$

Clearly we just apply the second formula to $[e_{ij}(y'), e_{kl}(y'')]$

First formula

Here we need to convert $e_{il}(y') e_{kj}(y'')$ and $e_{kl}(y') e_{ij}(y'')$

$$e_{il}(y') e_{kj}(y'') = e_{kj}(y'') e_{il}(y') + [e_{il}(y'), e_{kj}(y'')]$$

Applying the the permutation

$$\begin{pmatrix} i & j & k & l \\ i & l & k & j \end{pmatrix}$$

to the fourth formula, we obtain

$$
\begin{aligned}
[e_{il}(y'), e_{kj}(y'')]_4 = \sum \varphi_4 e_{ij}(z') e_{kl}(z'') + \psi_4 e_{kj}(z') e_{il}(z'') = \\
= \sum \psi_4 e_{kj}(z') e_{il}(z'') + \varphi_4 e_{ij}(z') e_{kl}(z'')
\end{aligned}
$$

N.B. Here the φ and ψ parts are swapped as in the second formula.
For the other term,

$$
e_{kl}(y') e_{ij}(y'') = e_{ij}(y'') e_{kl}(y') + [e_{kl}(y'), e_{ij}(y'')]
$$

We notice that we applied the fourth formula to the commutator above in the preliminary observations and obtained the first formula

$$
[e_{kl}(y'), e_{ij}(y'')]_4 = \sum \varphi_4 e_{kj}(z') e_{il}(z'') + \psi_4 e_{ij}(z') e_{kl}(z'')
$$

3.2 Summary

Here are the steps to expand the commutator for $w, \tilde{w}$ using formula x.
Step 1
If $l(w) < l(\tilde{w})$ swap w and $\tilde{w}$, expand using formula $5 - x$ and then negate coefficients.
Step 2
Assuming $l(w) \geq 2$ split $w = w_1 \otimes w_2$
Step 3
Expand the commutators for the splitted subwords using third and fourth formulas and merge the trains

$$
\begin{aligned}
[e_{ij}(w), e_{kl}(\tilde{w})] = \sum \varphi_4 e_{il}(z') e_{kj}(z'' \otimes w_2) + \psi_4 e_{kl}(z') e_{ij}(z'' \otimes w_2) + \\
+ \sum \varphi_3 e_{il}(w_1 \otimes z') e_{kj}(z'') + \psi_3 e_{ij}(w_1 \otimes z') e_{kl}(z'') = \\
= \sum \alpha e_{il}(y') e_{kj}(y'') + \beta e_{kl}(y') e_{ij}(y'') + \gamma e_{ij}(y') e_{kl}(y'')
\end{aligned}
$$

Step 4
This step depends on x, i.e. the formula we want to compute.
Formula 1

$$
\begin{aligned}
\sum \alpha e_{il}(y') e_{kj}(y'') + \beta e_{kl}(y') e_{ij}(y'') + \gamma e_{ij}(y') e_{kl}(y'') = \\
= \sum \alpha \left(e_{kj}(y'') e_{il}(y') + [e_{il}(y'), e_{kj}(y'')]_{\mathbf{4}} \right) + \\
+ \beta \left(e_{ij}(y'') e_{kl}(y') + [e_{kl}(y'), e_{ij}(y'')]_4 \right) + \gamma e_{ij}(y') e_{kl}(y'')
\end{aligned}
$$

Formula 2

$$
\begin{aligned}
\sum \alpha e_{il}(y') e_{kj}(y'') + \beta e_{kl}(y') e_{ij}(y'') + \gamma e_{ij}(y') e_{kl}(y'') = \\
= \sum \alpha \left(e_{kj}(y'') e_{il}(y') + [e_{il}(y'), e_{kj}(y'')]_{\mathbf{2}} \right) + \beta e_{kl}(y') e_{ij}(y'') + \\
+ \gamma \left(e_{kl}(y'') e_{ij}(y') + [e_{ij}(y'), e_{kl}(y'')]_2 \right)
\end{aligned}
$$

Formula 3

$$\begin{aligned}&\sum \alpha e_{il}(y')e_{kj}(y'') + \beta e_{kl}(y')e_{ij}(y'') + \gamma e_{ij}(y')e_{kl}(y'') = \\ =&\sum \alpha e_{il}(y')e_{kj}(y'') + \beta (e_{ij}(y'')e_{kl}(y') + [e_{kl}(y'), e_{ij}(y'')]_2) + \gamma e_{ij}(y')e_{kl}(y'')\end{aligned}$$

Formula 4

$$\begin{aligned}&\sum \alpha e_{il}(y')e_{kj}(y'') + \beta e_{kl}(y')e_{ij}(y'') + \gamma e_{ij}(y')e_{kl}(y'') = \\ =&\sum \alpha e_{il}(y')e_{kj}(y'') + \beta e_{kl}(y')e_{ij}(y'') + \gamma \left(e_{kl}(y'')e_{ij}(y') + [e_{ij}(y'), e_{kl}(y'')]_4\right)\end{aligned}$$

4 Appendix

4.1 Base case

TBD!