Poroelasticity Implementation

PyLith Version 3

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COMPUTATIONAL INFRASTRUCTURE for GEODYNAMICS

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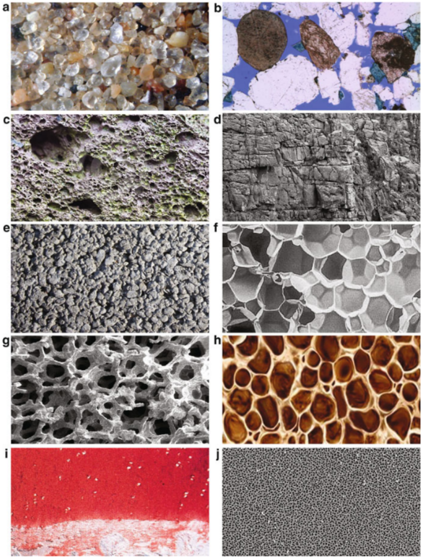
What is Poroelasticity?

Poroelasticity is the study of the interaction between fluid flow and solid deformation in a porous medium.



1. Sand
2. Sandstone
3. Volcanic rock
4. Fractured rock
5. Pervious concrete
6. Polyurethane doam
7. Metal foam
8. Bone
9. Articular cartilage
10. Nanoporous alumnina

Biot Formulation

1. Conservation of Momentum

$$\rho_b \ddot{\vec{u}} = \vec{f}(t) + \nabla \cdot \boldsymbol{\sigma}(\vec{u}, p)$$

- Drawn directly from linear elasticity, with stress tensor modified to account for fluid pressure: $\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\epsilon} - \alpha \boldsymbol{I} p$
- Bulk density defined as $\rho_b = (1 - \phi)\rho_s + \phi \rho_f$

2. Conservation of Mass

$$\dot{\zeta} + \nabla \cdot \vec{q}(p) = \gamma(\vec{x}, t)$$

- Specific discharge is defined by Darcy's Law: $\vec{q} = -\frac{\boldsymbol{k}}{\mu_f} \cdot \left(\nabla p - \vec{f}_f\right)$.
- Variation of fluid content is defined as $\zeta = \alpha \epsilon_v + \frac{p}{M}$
- Biot Modulus is defined as: $M = \frac{K_f}{\phi} + \frac{K_s}{\alpha - \phi}$

Governing Assumptions

for Quasistatic, Isotropic Linear Poroelasticity

- Infinitesimal strain formulation
- Undrained condition
- Linear elastic solid matrix
- Slightly compressible fluid
- Inertial term is negligible ($\rho_b \ddot{\vec{u}} = 0$)
- Solid phase mass is constant, fluid phase mass is conserved.

Quasistatic Formulation

Implicit Time Stepping

- We want to solve equations in which the weak form can be expressed as:

$$F(t, s, \dot{s}) = G(t, s)$$

$$F(t, s, \dot{s}) = \vec{0}$$

$$F(t, s, \dot{s}) = \cancel{G(t, s)} \xrightarrow{} \vec{0}$$

  Thus, all terms are shifted to the Left Hand Side.

- We include a volumetric strain relation,

$$\nabla \cdot \vec{u} = \epsilon_v$$

  and the volumetric strain term, ϵ_v, to the solution vector to aid with stability close to incompressibility.

Material	Bulk Rheology	Description
Elasticity	IsotropicLinearElasticity	Isotropic, linear elasticity
	IsotropicLinearMaxwell	Isotropic, linear Maxwell viscoelasticity
	IsotropicLinearGenMaxwell	Isotropic, generalized Maxwell viscoelasticity
	IsotropicPowerLaw	Isotropic, power-law viscoelasticity
	IsotropicDruckerPrager	Isotropic, Drucker-Prager elastoplasticity

Poroelasticity has ONE rheology (at the moment).

Auxiliary Fields

for Quasistatic Linear Isotropic Poroelasticity

Origin	Variable	Description	Position	Notes
Material	ρ_b	Rock Density	0	
	ρ_f	Fluid Density	1	
	μ_f	Fluid Viscosity	2	
	ϕ	Porosity	3	
	$\vec{f}_b$	Body Force	$+1$	
	$\vec{g}$	Gravity	$+1$	
	γ	Fluid Source	$+1$	
Rheology	$\boldsymbol{\sigma}_R$	Reference Stress	NumAux - 7	
	$\boldsymbol{\epsilon}_R$	Reference Strain	NumAux - 6	
	G	Shear Modulus	NumAux - 5	
	K_d	Drained Bulk Modulus	NumAux - 4	
	α	Biot Coefficient	NumAux - 3	
	M	Biot Modulus	NumAux - 2	$\frac{K_f}{\phi} + \frac{K_s}{\alpha - \phi}$
	$\boldsymbol{k}$	Permeability	NumAux - 1	
Input	K_s	Solid Grain Bulk Modulus	-	
	K_f	Fluid Bulk Modulus	-	

Residuals

for Quasistatic, Isotropic Linear Poroelasticity

$$F^u(t,s,\dot{s}) = \int_\Omega \vec{\psi}^u_{trial} \cdot \underbrace{\vec{f}(\vec{x},t)}_{\vec{f}^u_0} + \nabla\vec{\psi}^u_{trial} : \underbrace{-\boldsymbol{\sigma}(\vec{u},p_f)}_{\boldsymbol{f}^u_1}\, d\Omega + \int_{\Gamma_\tau} \vec{\psi}^u_{trial} \cdot \underbrace{\vec{\tau}(\vec{x},t)}_{\vec{f}^u_0}\, d\Gamma,$$

$$F^p(t,s,\dot{s}) = \int_\Omega \psi^p_{trial} \underbrace{\left[\frac{\partial\zeta(\vec{u},p_f)}{\partial t} - \gamma(\vec{x},t)\right]}_{f^p_0} + \nabla\psi^p_{trial} \cdot \underbrace{-\vec{q}(p_f)}_{\vec{f}^p_1}\, d\Omega + \int_{\Gamma_q} \psi^p_{trial} \underbrace{(q_0(\vec{x},t))}_{f^p_0}\, d\Gamma,$$

$$F^{\epsilon_v}(t,s,\dot{s}) = \int_\Omega \psi^{\epsilon_v}_{trial} \cdot \underbrace{(\nabla\cdot\vec{u} - \epsilon_v)}_{f^{\epsilon_v}_0}\, d\Omega.$$

Brown refers to Material, Green to Rheology

Jacobians

for Quasistatic, Isotropic Linear Poroelasticity

$$J_F^{uu} = \frac{\partial F^u}{\partial u} + t_{shift}\frac{\partial F^u}{\partial \dot{u}} = \int_\Omega \psi^u_{trial\,i,k} \underbrace{(-C_{ikjl})}_{J_{f3}^{uu}} \psi^u_{basis\,j,l}\, d\Omega$$

$$J_F^{up} = \frac{\partial F^u}{\partial p} + t_{shift}\frac{\partial F^u}{\partial \dot{p}} = \int_\Omega \psi^u_{trial\,i,j} \underbrace{(\alpha\delta_{ij})}_{J_{f2}^{up}} \psi^p_{basis}\, d\Omega$$

$$J_F^{u\epsilon_v} = \frac{\partial F^u}{\partial \epsilon_v} + t_{shift}\frac{\partial F^u}{\partial \dot{\epsilon_v}} = \int_\Omega \nabla\vec{\psi}^u_{trial} : \frac{\partial}{\partial \epsilon_v}\ \left[-\left(2\mu\boldsymbol{\epsilon} + \lambda\boldsymbol{I}\epsilon_v - \alpha\boldsymbol{I}p\right)\right] d\Omega = \int_\Omega \psi^u_{trial\,i,j} \underbrace{(-\lambda\delta_{ij})}_{J_{f2}^{u\epsilon_v}} \psi^{\epsilon_v}_{basis} d\Omega$$

$$J_F^{pp} = \frac{\partial F^p}{\partial p} + t_{shift}\frac{\partial F^p}{\partial \dot{p}} = \int_\Omega \psi^p_{trial,k} \underbrace{\left(-\frac{\boldsymbol{k}}{\mu_f}\delta_{kl}\right)}_{J_{f3}^{pp}} \psi^p_{basis,l}\, d\Omega + \int_\Omega \psi^p_{trial} \underbrace{\left(t_{shift}\frac{1}{M}\right)}_{J_{f0}^{pp}} \psi^p_{basis}\, d\Omega$$

$$J_F^{p\epsilon_v} = \frac{\partial F^p}{\partial \epsilon_v} + t_{shift}\frac{\partial F^p}{\partial \dot{\epsilon_v}} = \int_\Omega \psi^p_{trial} \underbrace{(t_{shift}\alpha)}_{J_{f0}^{p\epsilon_v}} \psi^{\epsilon_v}_{basis}\, d\Omega$$

$$J_F^{\epsilon_v u} = \frac{\partial F^{\epsilon_v}}{\partial u} + t_{shift}\frac{\partial F^{\epsilon_v}}{\partial \dot{u}} = \int_\Omega \psi^{\epsilon_v}_{trial} \nabla\cdot\vec{\psi}^u_{basis}\, d\Omega = \int_\Omega \psi^{\epsilon_v}_{basis} \underbrace{(\delta_{ij})}_{J_{f1}^{\epsilon_v u}} \psi^u_{basis\,i,j}\, d\Omega$$

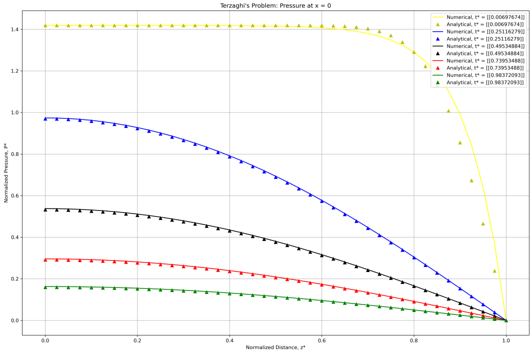
$$J_F^{\epsilon_v\epsilon_v} = \frac{\partial F^{\epsilon}_v}{\epsilon_v} + t_{shift}\frac{\partial F^{\epsilon_v}}{\partial \dot{\epsilon_v}} = \int_\Omega \psi^{\epsilon_v}_{basis} \underbrace{(-1)}_{J_{f0}^{\epsilon_v\epsilon_v}} \psi^{\epsilon_v}_{basis}\, d\Omega$$

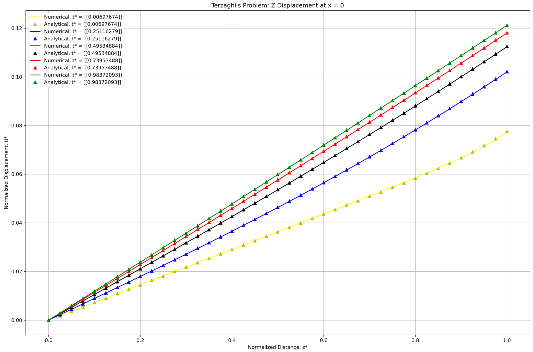
Parameters

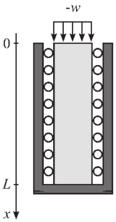
for Quasistatic Isotropic Linear Poroelasticity

Category	Symbol	Description
Unknowns	$\vec{u}$	Displacement field
	p	Pressure field (corresponds to pore fluid pressure)
	ϵ_v	Volumetric (trace) strain
Derived quantities	$\boldsymbol{\sigma}$	Cauchy stress tensor
	$\boldsymbol{\epsilon}$	Cauchy strain tensor
	ζ	Variation of fluid content (variation of fluid vol. per unit vol. of PM), $\alpha\epsilon_v + \frac{p}{M}$
	ρ_b	Bulk density, $(1-\phi)\,\rho_s + \phi\rho_f$
	$\vec{q}$	Darcy flux, $-\frac{\boldsymbol{k}}{\mu_f} \cdot \left(\nabla p - \vec{f}_f\right)$
	M	Biot Modulus, $\frac{K_f}{\phi} + \frac{K_s}{\alpha - \phi}$
Common constitutive parameters	ρ_f	Fluid density
	ρ_s	Solid (matrix) density
	ϕ	Porosity
	$\boldsymbol{k}$	Permeability
	μ_f	Fluid viscosity
	K_s	Solid grain bulk modulus
	K_f	Fluid bulk modulus
	K_d	Drained bulk modulus
	α	Biot coefficient, $1 - \frac{K_d}{K_s}$
Source terms	$\vec{f}$	Body force per unit volume, for example: $\rho_b\vec{g}$
	$\vec{f}_f$	Fluid body force, for example: $\rho_f\vec{g}$
	γ	Source density; rate of injected fluid per unit volume of the porous solid

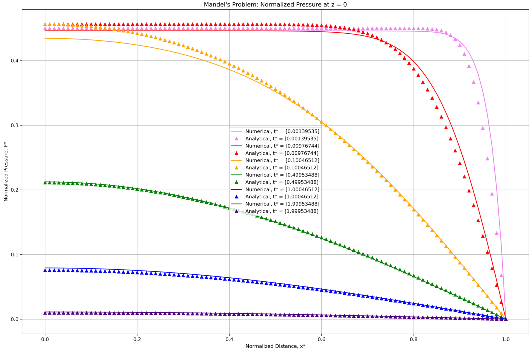
Terzaghi’s Problem Test Results

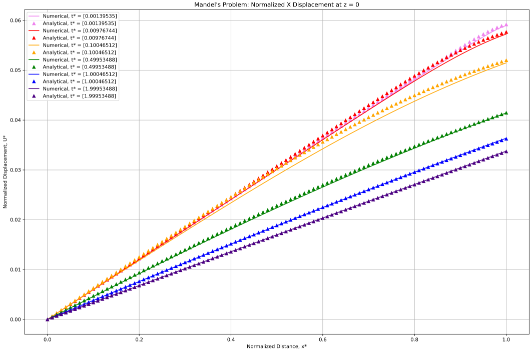


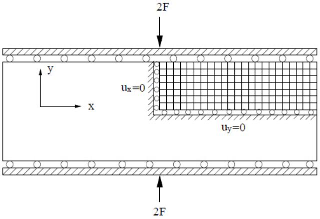




Mandel's Problem Test Results







Next Steps

Ordered by Difficulty Level

- Straightforward
  - Clean and test dynamic poroelasticity for functionality
  - Update functions for auxiliary variables
  - Conversion scripts for poroelastic benchmark cases (e.g. SPE10)
- Medium
  - Analytical tests for dynamic poroelasticity
  - Well model combined with point source
- More difficult
  - Multiphase model
  - Fully poroelastic faults

Dynamic Poroelasticity

Explicit Time Stepping

- Explicit time stepping with the PETSc TS requires $F(t, s, \dot{s}) = \dot{s}$.
- Normally $F(t, s, \dot{s})$ contains the inertial term ($\rho\ddot{u}$).
- Therefore, when using explicit time stepping we transform our equation into the form:

$$
\begin{aligned}
F^*(t, s, \dot{s}) = \dot{s} &= G^*(t, s) \\
\dot{s} &= M^{-1} G(t, s).
\end{aligned}
$$

- Terms must be rewritten to ensure that the LHS consists only of time derivatives and coefficients.
- Velocity is introduced as an unknown, again resulting in a three field solution

Dynamic Poroelasticity

Strong Formulation

For compatibility with PETSc TS algorithms, we want to turn the second order equation elasticity equation into two first order equations. We introduce the velocity as a unknown, $\vec{v} = \frac{\partial u}{\partial t}$, which leads to a slightly different three field problem,

$$\vec{s}^T = \begin{pmatrix} \vec{u} & p & \vec{v} \end{pmatrix}$$

$$\frac{\partial \vec{u}}{\partial t} = \vec{v} \text{ in } \Omega$$

$$\frac{1}{M}\frac{\partial p}{\partial t} = \gamma(\vec{x},t) - \alpha\left(\nabla \cdot \dot{\vec{u}}\right) - \nabla \cdot \vec{q}\,(p) = 0 \text{ in } \Omega$$

$$\rho_b \frac{\partial \vec{v}}{\partial t} = \vec{f}(\vec{x},t) + \nabla \cdot \boldsymbol{\sigma}(\vec{u},p) \text{ in } \Omega$$

$$\boldsymbol{\sigma} \cdot \vec{n} = \vec{\tau}(\vec{x},t) \text{ on } \Gamma_\tau$$

$$\vec{u} = \vec{u}_0(\vec{x},t) \text{ on } \Gamma_u$$

$$\vec{q} \cdot \vec{n} = q_0(\vec{x},t) \text{ on } \Gamma_q$$

$$p = p_0(\vec{x},t) \text{ on } \Gamma_p$$

Dynamic Poroelasticity

Weak Formulation

Using trial functions $\vec{\psi}^{u}_{trial}$, ψ^{p}_{trial}, and $\vec{\psi}^{v}_{trial}$, and incorporating the Neumann boundary conditions, the weak form may be written as:

$$\int_\Omega \vec{\psi}^{u}_{trial} \cdot \left(\frac{\partial \vec{u}}{\partial t} \right) d\Omega = \int_\Omega \vec{\psi}^{u}_{trial} \cdot (\vec{v})\, d\Omega$$

$$\int_\Omega \psi^{p}_{trial} \left(\frac{1}{M} \frac{\partial p}{\partial t} \right) d\Omega = \int_\Omega \psi^{p}_{trial} \left[\gamma(\vec{x}, t) - \alpha \left(\nabla \cdot \dot{\vec{u}} \right) \right] + \nabla \psi^{p}_{trial} \cdot \vec{q}(p)\ d\Omega + \int_{\Gamma_q} \psi^{p}_{trial} (-q_0(\vec{x}, t))\ d\Gamma$$

$$\int_\Omega \vec{\psi}^{v}_{trial} \cdot \left(\rho_b \frac{\partial \vec{v}}{\partial t} \right) d\Omega = \int_\Omega \vec{\psi}^{v}_{trial} \cdot \vec{f}(\vec{x}, t) + \nabla \vec{\psi}^{v}_{trial} : -\boldsymbol{\sigma}(\vec{u}, p_f)\, d\Omega + \int_{\Gamma_\tau} \vec{\psi}^{u}_{trial} \cdot \vec{\tau}(\vec{x}, t)\, d\Gamma.$$

Residual Functions

for Dynamic Isotropic Linear Poroelasticity

$$G^u(t,s) = \int_\Omega \vec{\psi}^u_{trial} \cdot \left(\underbrace{\vec{v}}_{\vec{g}^u_0} \right) d\Omega$$

$$G^p(t,s) = \int_\Omega \psi^p_{trial} \left[\underbrace{\gamma\left(\vec{x},t\right) - \alpha\left(\nabla \cdot \dot{\vec{u}}\right)}_{g^p_0} \right] + \nabla \psi^p_{trial} \cdot \underbrace{\vec{q}\left(p\right)}_{\vec{g}^p_1} d\Omega + \int_{\Gamma_q} \psi^p_{trial} \left(\underbrace{q_0\left(\vec{x},t\right)}_{g^p_0} \right) d\Gamma,$$

$$G^v(t,s) = \int_\Omega \vec{\psi}^v_{trial} \cdot \underbrace{\vec{f}\left(\vec{x},t\right)}_{\vec{g}^v_0} + \nabla \vec{\psi}^v_{trial} : \underbrace{-\boldsymbol{\sigma}\left(\vec{u},p\right)}_{\boldsymbol{g}^v_1} d\Omega + \int_{\Gamma_\tau} \vec{\psi}^v_{trial} \cdot \underbrace{\vec{\tau}\left(\vec{x},t\right)}_{\vec{g}^v_0} d\Gamma,$$

Jacobian Functions

for Dynamic Isotropic Linear Poroelasticity

These are the pointwise functions associated with M_u, M_p, and M_v for computing the lumped LHS Jacobian. We premultiply the RHS residual function by the inverse of the lumped LHS Jacobian while s tshift remains on the LHS with $\dot{s}$. As a result, we use LHS Jacobian pointwise functions, but set s tshift = 1 . The LHS Jacobians are:

$$M_u = J_F^{uu} = \frac{\partial F^u}{\partial u} + s_{tshift}\frac{\partial F^u}{\partial \dot{u}} = \int_\Omega \psi_{trial\,i}^u \underbrace{s_{tshift}\delta_{ij}}_{J_{f0}^{uu}} \psi_{basis\,j}^u \, d\Omega$$

$$M_p = J_F^{pp} = \frac{\partial F^p}{\partial p} + t_{shift}\frac{\partial F^p}{\partial \dot{p}} = \int_\Omega \psi_{trial}^p \underbrace{\left(s_{tshift}\frac{1}{M}\right)}_{J_{f0}^{pp}} \psi_{basis}^p \, d\Omega$$

$$M_v = J_F^{vv} = \frac{\partial F^v}{\partial v} + t_{shift}\frac{\partial F^v}{\partial \dot{v}} = \int_\Omega \psi_{trial\,i}^v \underbrace{\rho_b(\vec{x}) s_{tshift}\delta_{ij}}_{J_{f0}^{vv}} \psi_{basis\,j}^v \, d\Omega$$