Reducibility

Charlotte → Heathrow ← Reducable

$\leq$

Charleston → London

If problem A is no harder than problem B we can say $A \leq B$ and if by solving problem B we can solve problem A, we say A is reducable to B.

1). If B is decidable and $A \leq B$, then A is also decidable

2). If A is undecidable and $A \leq B$, then B is also undecidable

$A_{TM} = \{\langle M, w \rangle \mid M$ is a TM and M accepts $w\}$

- Undecidable (Diagonalization)

$HALT_{TM} = \{\langle M, w \rangle \mid M$ is a TM and M halts on $w\}$

If $A_{TM} \leq HALT_{TM}$ → HALT is also undecidable

$ACCEPT_{TM} \leq_T Halt_{TM}$

↑ Turing reduction

Transitivity ✓

$A \leq B$ & $B \leq C \Rightarrow A \leq C$ TRUE

Turing Reducibility

THE HALTING problem

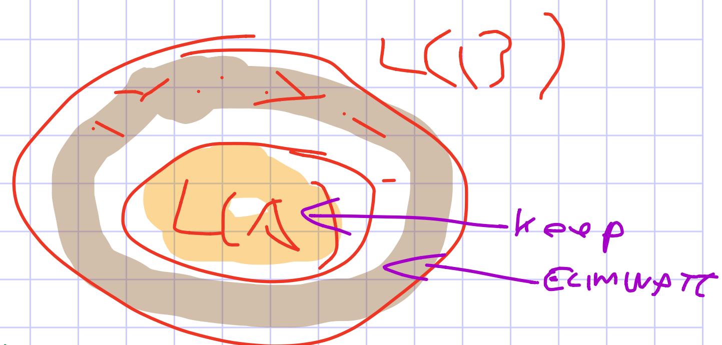
$ACCEPT_{TM} \leq_T Halt_{TM}$

1). H IS DECIDABLE

2). Show A can be reduced to H

3) A IS UNDECIDABLE, H MUST ALSO BE UNDECIDABLE

GIVEN LANGUAGE A (L(A)) & LANGUAGE B (L(B)). IF ANY STRING W IN LANGUAGE A IS CONVERTED INTO A STRING IN L(B) W_B, ANY STRING IN L(B) THAT IS NOT IN L(A) IS CONVERTED INTO A STRING NOT IN L(B)



1) ASSUME TM_H DECIDES $HALT_{TM}$

2). To DECIDE A, CREATE TM_A THAT WILL BE SIMULATE ON TM_H

3). RUN TM_H ON <M, w>

   a). IF TM_H REJECTS ⇒ REJECT

   b). IF TM_H ACCEPTS

      i) SIMULATE TM_A ON TM_H UNTIL H HALTS

      ii) IF TM_H ACCEPTS → ACCEPT

      iii) IF TM_A REJECTS ⇒ REJECT

4). Clearly, A_{TM} CAN BE REDUCED TO H_{TM} AND IF TM_H DECIDES HALT THEN TM_A DECIDED ACCEPT.

   BUT ⇒ SINCE $ACCEPT_{TM}$ IS UNDECIDABLE, $HALT_{TM}$ MUST ALSO BE UNDECIDABLE

Reducation via Computational History

ACCEPTING COMPUTATIONAL HISTORY → FOR M ON w THE SEQ OF CONFIGURATIONS $C_1, C_2 \ldots C_n$ where C_1 IS THE INITIAL CONFIG AND $C_m \to C_{m+1}$ logically, C_n IS AN ACCEPTING CONFIGURATION

REJECTING COMPUTATIONAL HISTORY → "

IS AN REJECTING CONFIGURATION

A DETERMINISTIC TM HAS AT MOST 1 COMPUTATIONAL HISTORY

A NON DETERMINISTIC TM HAS multiple REJECTING HISTORY AND NO MORE THAN 1 ACCEPTING HISTORY

IF WE CAN prove THE C.H. FOR some problem B IS FINITE (DECIDABLE) AND IF $A \leq B$ THE WE KNOW THAT A IS ALSO DECIDABLE.

Linear Bounded Automata

A TM where the tape head is BOUNDED ON BOTH ENDS.
- Tape has finite length

$A_{LBA} = \{\langle M, w\rangle \mid M$ IS AN LBA THAT ACCEPTS STRING w

q states
n tape positions
g Tape symbols

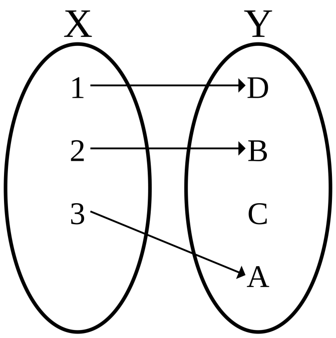
$\Big\}$ $g^n q$ ← upper bound on # of configurations

1). SIMULATE M ON w NO MORE THAN $g^n q$ STEPS.
→ IF HALTS → ACCEPT OR REJECT Appropriately
→ IF IT HAS NOT HALTED AFTER $g^n q$ STEPS → REJECT

$A_{LBA} \leq A_{TM}$

NEXT TIME:
$E_{LBA} = \{\langle M\rangle \mid M$ IS AN LBA AND $L(M) = \emptyset\}$??

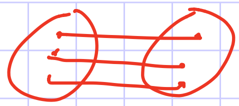
Mapping Reducability



$X \le_m Y$

INJECTIVE $f(a) \ne f(b)$ whenever $a \ne b$

ONE-TO-ONE } BOTH
ONTO



$A = \{x \mid x$ IS AN EVEN INTEGER $< 100\}$

$B = \{y \mid y$ IS AN ODD INTEGER $\}$

$f(x) = x + 1$

$A \le_m B$

X	Y
2	3
4	5
98	99
	101
	102

Example

Rice's Theorem

$A_{TM} \Rightarrow$ UNDECIDABLE $\Rightarrow$ RECURSIVELY ENUMERABLE
TURING RECOGNIZABLE

LET P BE ANY NON-TRIVIAL PROPERTY OF THE LANGUAGE OF A TM. THEN P IS UNDECIDABLE

A PROPERTY IS TRIVIAL IF IT'S EITHER EMPTY (NOT SATISFIED BY ANY LANGUAGE) OR IT'S ALL RECOGNIZABLE LANGUAGES.

$E_{TM} \Rightarrow$ UNDECIDABLE

$L(1) = \{\langle M\rangle \mid M$ IS A TM AND $1100 \in L(M)\}$
↳ NON TRIVIAL
↳ UNDECIDABLE

$ALL_{TM} = \{\langle M\rangle \mid M$ IS A TM AND $L(M) = \Sigma^*\}$