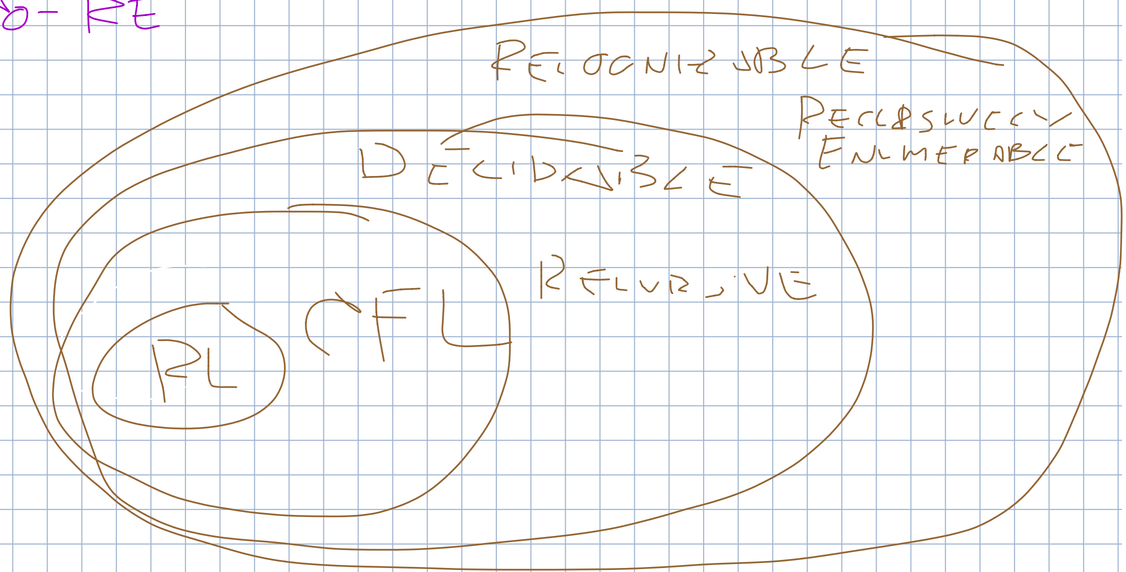
Mapping Reductions

- A function $f : \Sigma_1^* \to \Sigma_2^*$ is called a **mapping reduction** from A to B iff
  - For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$.
  - f is a computable function.
- Intuitively, a mapping reduction from A to B says that a computer can transform any instance of A into an instance of B such that the answer to B is the answer to A.

Why Mapping Reducibility Matters

- **Theorem**: If $B \in \mathbf{R}$ and $A \leq_{\mathrm{M}} B$, then $A \in \mathbf{R}$.
- **Theorem**: If $B \in \mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $A \in \mathbf{RE}$.
- **Theorem**: If $B \in \text{co-}\mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $A \in \text{co-}\mathbf{RE}$.
- *Intuitively:* $A \leq_{\mathrm{M}} B$ means "A is not harder than B."

co-RE


RECOGNIZABLE
RECURSIVELY ENUMERABLE
DECIDABLE
RECURSIVE
CFL
RL

Why Mapping Reducibility Matters

- **Theorem**: If $A \notin \mathbf{R}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \mathbf{R}$.
- **Theorem**: If $A \notin \mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \mathbf{RE}$.
- **Theorem**: If $A \notin \text{co-}\mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \text{co-}\mathbf{RE}$.
- *Intuitively:* $A \leq_{\mathrm{M}} B$ means "B is at at least as hard as A."

Why Mapping Reducibility Matters

If this one is "easy" (R, RE, co-RE)…

$$A \leq_{\mathrm{M}} B$$

… then this one is "easy" (R, RE, co-RE) too.

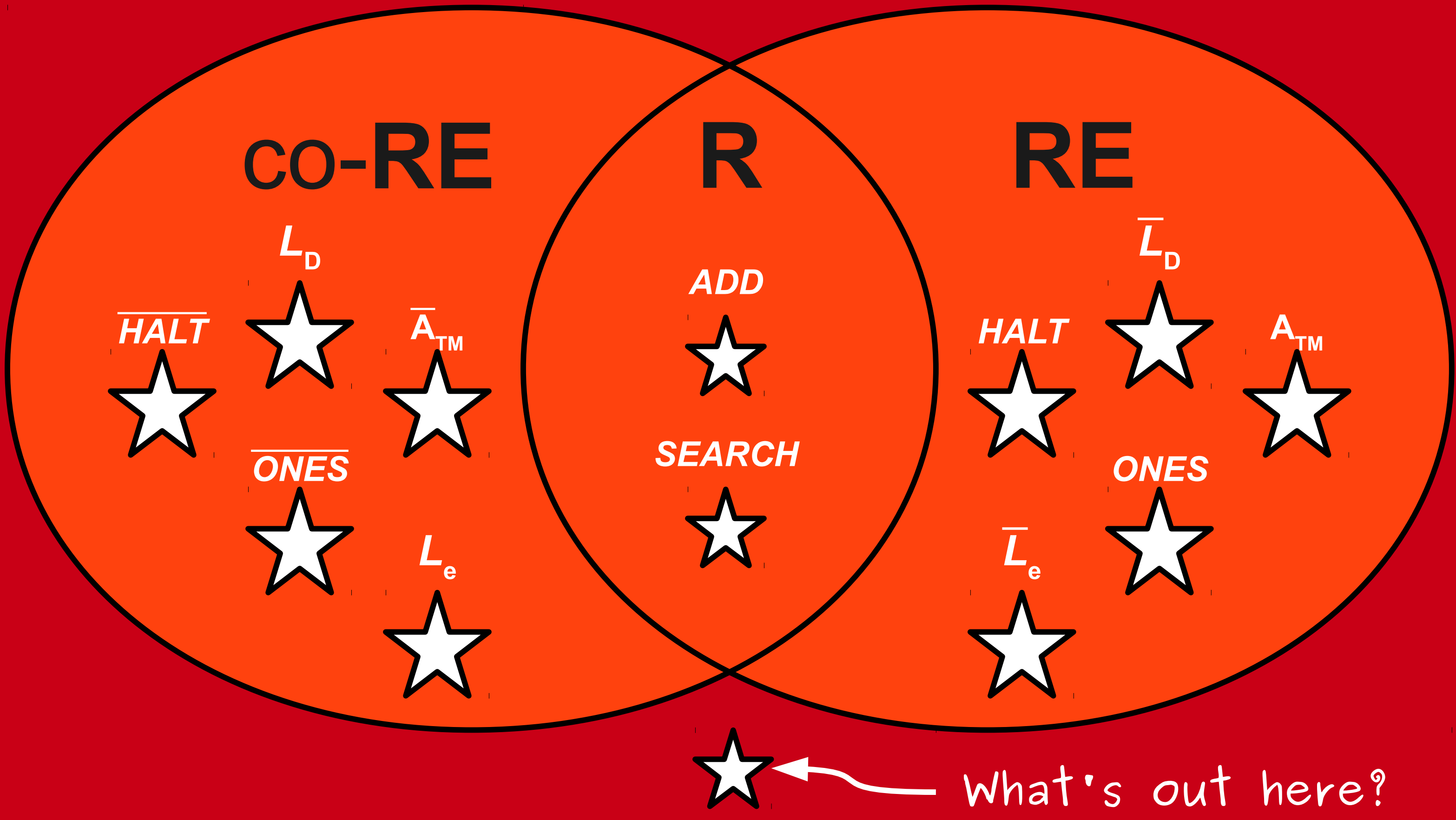
Why Mapping Reducibility Matters

If this one is "hard" (not **R**, not **RE**, or not co-**RE**)…

$$A \leq_M B$$

… then this one is "hard" (not **R**, not **RE**, or not co-**RE**) too.

The Limits of Computability



What problems can be solved by a computer?

What problems can be solved **efficiently** by a computer?

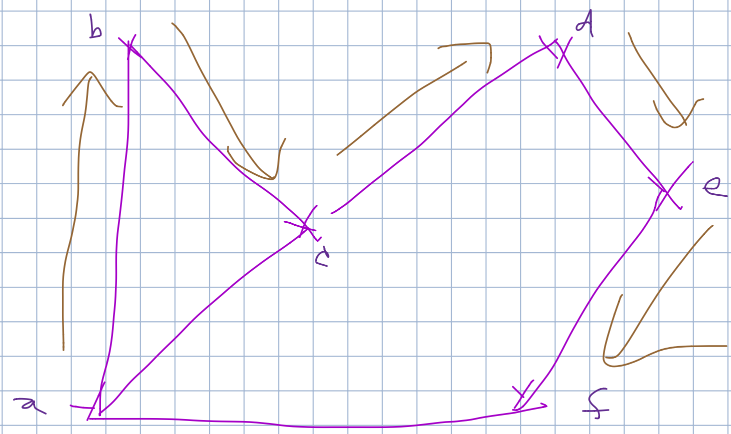
Where We've Been

- The class **R** represents problems that can be solved by a computer.
- The class **RE** represents problems where “yes” answers can be verified by a computer.
- The class co-**RE** represents problems where “no” answers can be verified by a computer.
- The mapping reduction can be used to find connections between problems.

Where We're Going

- The class **P** represents problems that can be solved *efficiently* by a computer.
- The class **NP** represents problems where “yes” answers can be verified *efficiently* by a computer.
- The class co-**NP** represents problems where “no” answers can be verified *efficiently* by a computer.
- The *polynomial-time* mapping reduction can be used to find connections between problems.

The Traveling Salesman Problem - NP



a -> b -> c -> d -> e -> f

P ⇒ Polynomial Time Solvable

$n = 1$	$n^2 = 1$
2	$n^2 = 4$
$\vdots$	
100	$n^2 = 10000$

$n \to$ Size of problem (e.g. # cities)

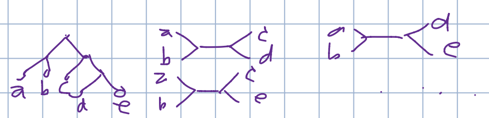
$n^i \to i$ fixed

$n^2 \in [$

CONSTANT

$a^i \, b^j \, c^k \to$ constants

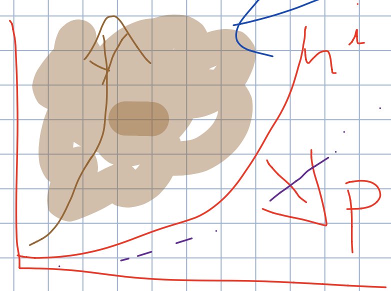
size of problem



$n^4 \to n^3 \to n^{2.3}$

$n^{1.521} \to n^{1.384}$

Binary Search → $O(\log n)$



NP

↳ Non-Determ Polynomial

A Decidable Problem

- **Presburger arithmetic** is a logical system for reasoning about arithmetic.
  - $\forall x.\ x + 1 \neq 0$
  - $\forall x.\ \forall y.\ (x + 1 = y + 1 \rightarrow x = y)$
  - $\forall x.\ x + 0 = x$
  - $\forall x.\ \forall y.\ (x + y) + 1 = x + (y + 1)$
  - $\forall x.\ ((P(0) \wedge \forall y.\ (P(y) \rightarrow P(y + 1))) \rightarrow \forall x.\ P(x)$
- Given a statement, it is decidable whether that statement can be proven from the laws of Presburger arithmetic.
- Any Turing machine that decides whether a statement in Presburger arithmetic is true or false has to move the tape head at least $2^{2^{cn}}$ times on some inputs of length n (for some fixed constant c).

For Reference

- Assume $c = 1$.

$$2^{2^0} = 2$$

$$2^{2^1} = 4$$

$$2^{2^2} = 16$$

$$2^{2^3} = 256$$

$$2^{2^4} = 65536$$

$$2^{2^5} = 18446744073709551616$$

$$2^{2^6} = 340282366920938463463374607431768211456$$

The Limits of Decidability

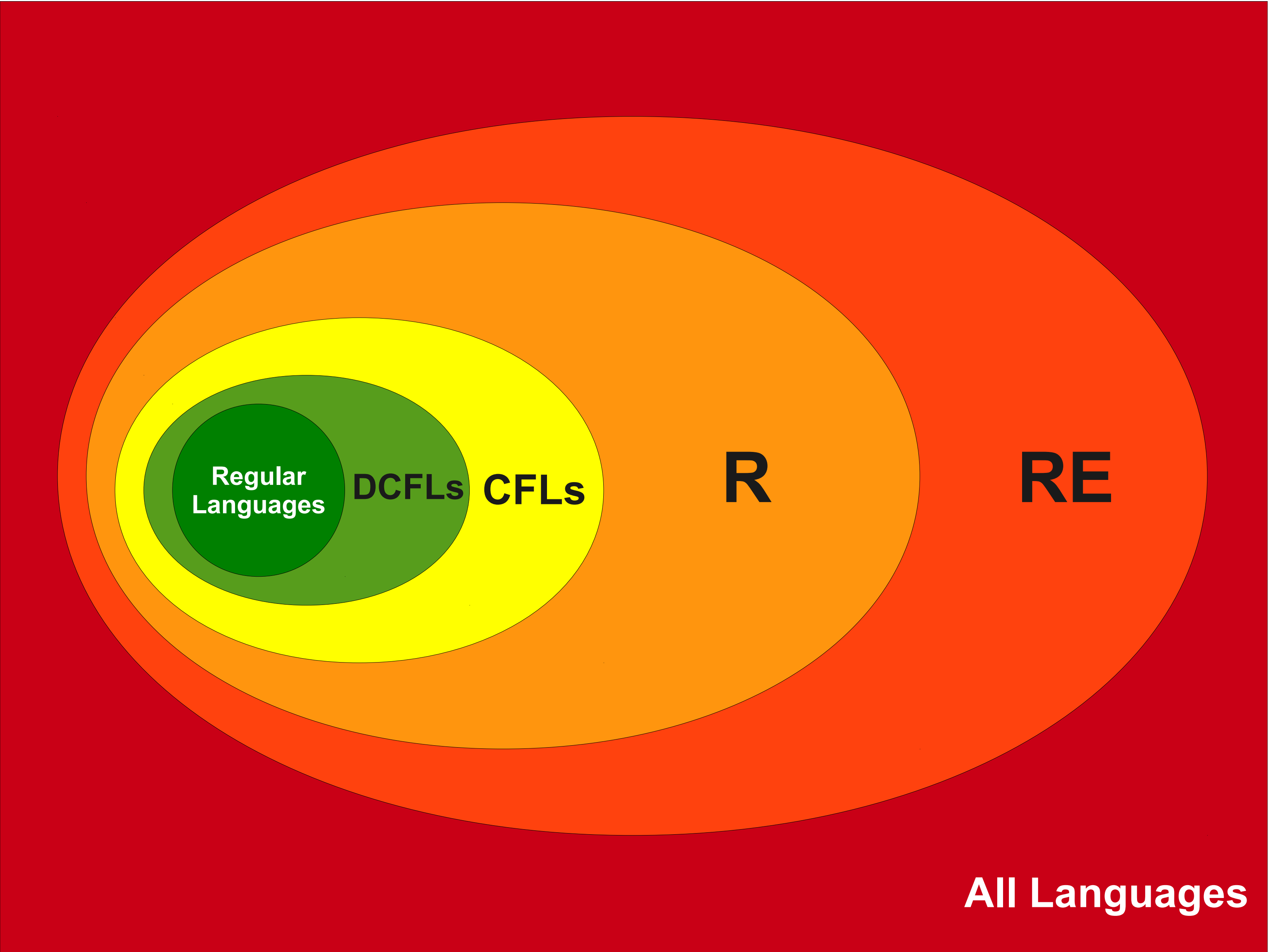
- The fact that a problem is decidable does not mean that it is *feasibly* decidable.
- In **computability theory**, we ask the question

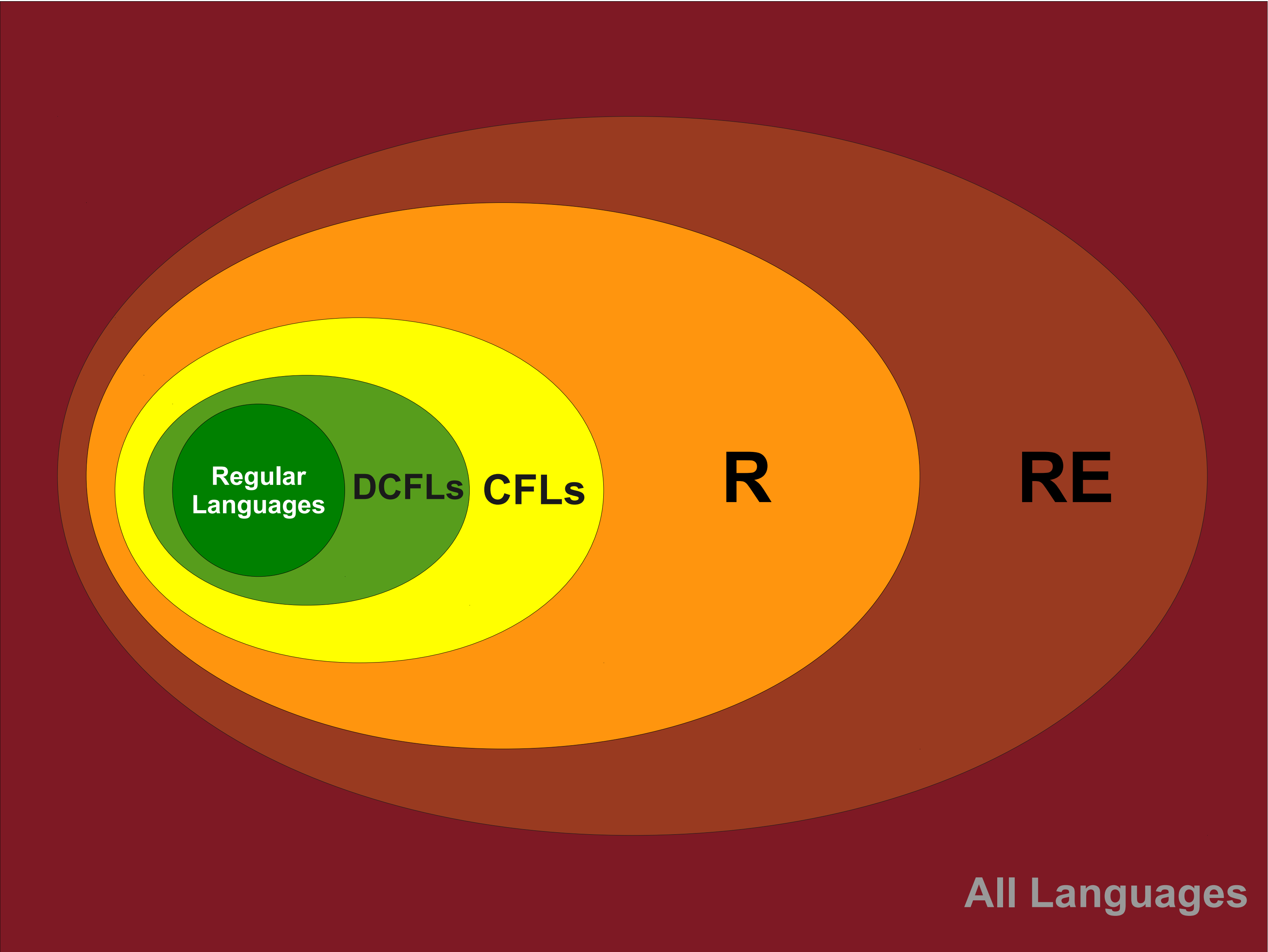
  Is it **possible** to solve problem L?

- In **complexity theory**, we ask the question

  Is it possible to solve problem L **efficiently**?

- In the remainder of this course, we will explore this question in more detail.


Regular Languages
DCFLs
CFLs
R
RE
All Languages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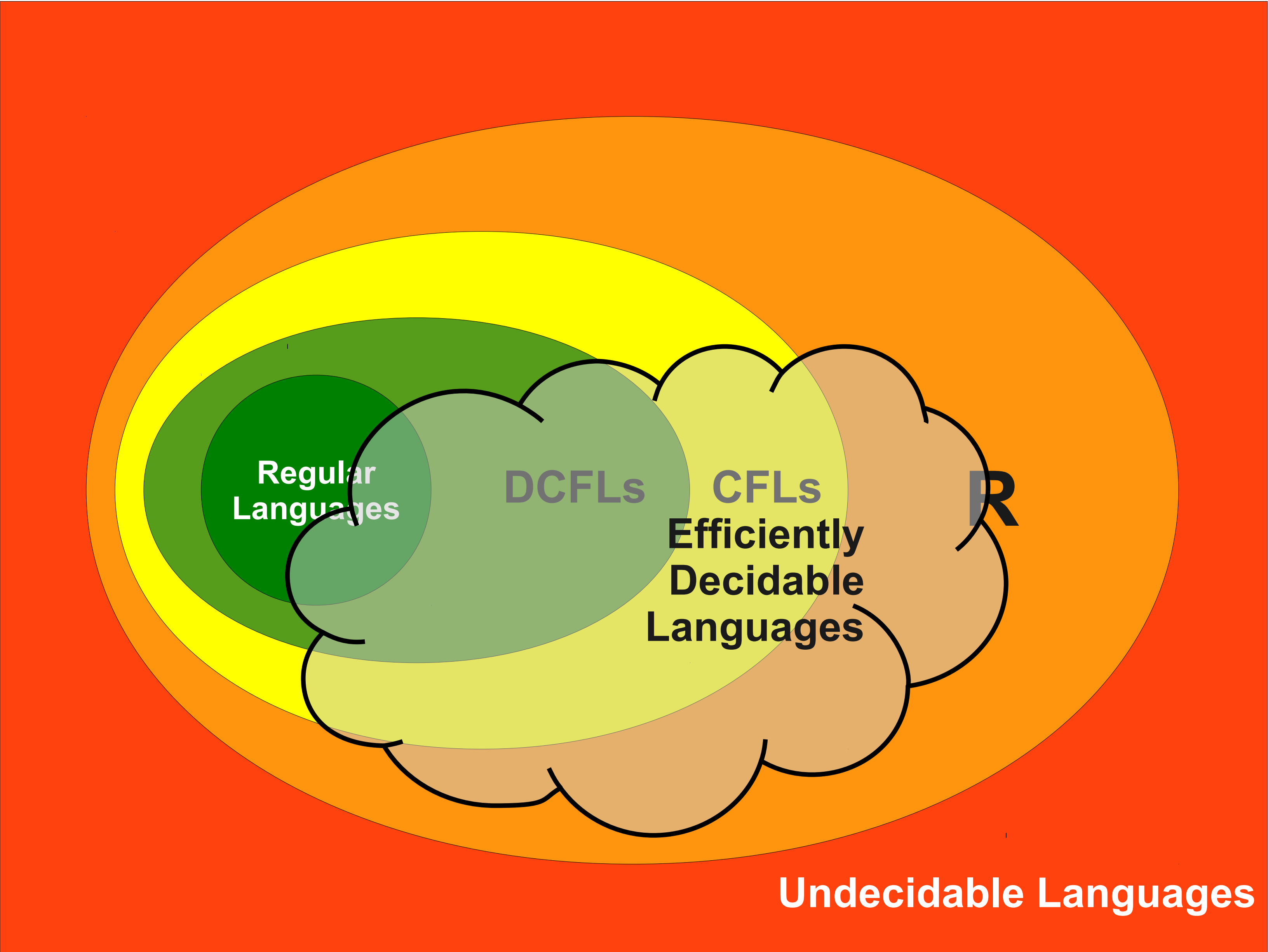

Regular Languages
DCFLs
CFLs
Efficiently Decidable Languages
R
Undecidable Languages

The Setup

- In order to study computability, we needed to answer these questions:
  - What is “computation?”
  - What is a “problem?”
  - What does it mean to “solve” a problem?
- To study complexity, we need to answer these questions:
  - What does “complexity” even mean?
  - What is an “efficient” solution to a problem?

Measuring Complexity

- Suppose that we have a decider D for some language L.
- How might we measure the complexity of D?

Measuring Complexity

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  - Number of states.
  - Size of tape alphabet.
  - Size of input alphabet.
  - Amount of tape required.
  - Number of steps required.
  - Number of times a given state is entered.
  - Number of times a given symbol is printed.
  - Number of times a given transition is taken.
  - (Plus a whole lot more...)

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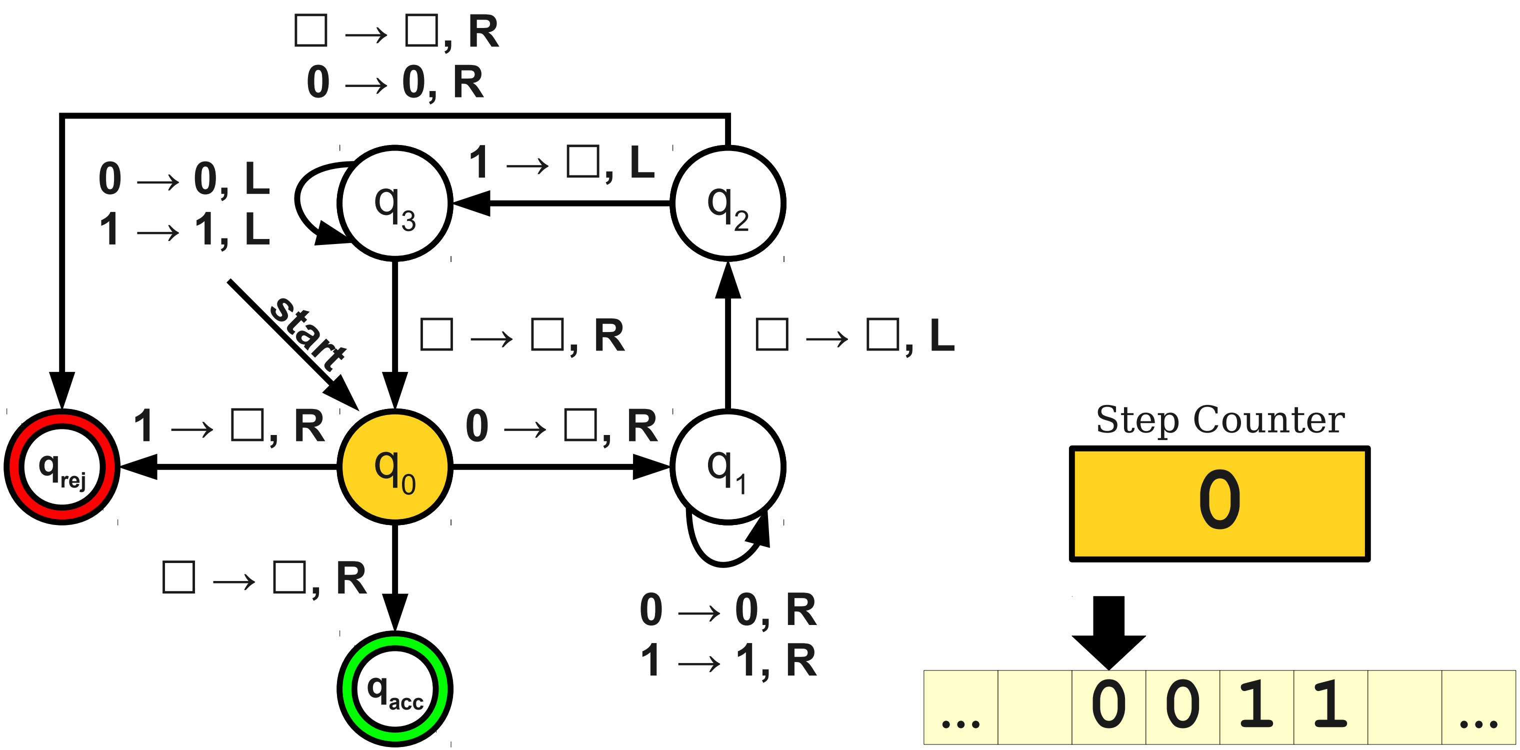
Time Complexity

- A **step** of a Turing machine is one event where the TM takes a transition.



Time Complexity

- The number of steps a TM takes on some input is sensitive to
  - The structure of that input.
  - The length of the input.
- How can we come up with a consistent measure of a machine's runtime?

Time Complexity

- The **time complexity** of a TM M is a function (typically denoted $f(n)$) that measures the *worst-case* number of steps M takes on any input of length n.
  - By convention, n denotes the length of the input.
  - If M loops on some input of length k, then $f(k) = \infty$.
- The previous TM has a time complexity that is (roughly) proportional to n^2 / 2.

  $O(n^2/2)$

  $\to O(n^2)$

  - Difficult and utterly unrewarding exercise: compute the *exact* time complexity of the previous TM.

A Slight Problem

- Consider the following TM over $\Sigma = \{0, 1\}$ for the language *BALANCE* $= \{ w \in \Sigma^* \mid w$ has the same number of 0s and 1s $\}$:
  - M = "On input w:
    - Scan across the tape until a 0 or 1 is found.
    - If none are found, accept.
    - If one is found, continue scanning until a matching 1 or 0 is found.
    - If none is found, reject.
    - Otherwise, cross off that symbol and repeat."

ALSO
SPACE

- What is the time complexity of M?

A Loss of Precision

- When considering *computability*, using high-level TM descriptions is perfectly fine.
- When considering *complexity*, high-level TM descriptions make it nearly impossible to precisely reason about the actual time complexity.
- What are we to do about this?

The Best We Can

M = "On input w:

- Scan across the tape until a `0` or `1` is found.
- If none are found, accept.
- If one is found, continue scanning until a matching `1` or `0` is found.
- If none are found, reject.
- Otherwise, cross off that symbol and repeat."

At most n steps.

At most 1 step.

At most n more steps.

At most 1 step

At most n steps to get back to the start of the tape.

At most $n/2$ loops

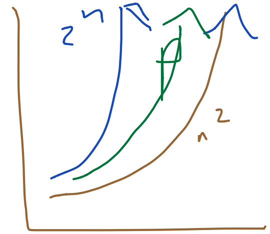
+

At most $3n + 2$ steps.

× At most $n/2$ loops.

At most $3n^2 / 2 + n$ steps.

An Easier Approach



- In complexity theory, we rarely need an exact value for a TM's time complexity.
- Usually, we are curious with the long-term growth rate of the time complexity.
- For example, if the time complexity is $3n + 5$, then doubling the length of the string roughly doubles the worst-case runtime.
- If the time complexity is $2^n - n^2$, since 2^n grows much more quickly than n^2, for large values of n, increasing the size of the input by 1 doubles the worst-case running time.

Big-O Notation

- Ignore *everything* except the dominant growth term, including constant factors.
- Examples:
  - $4n + 4 = \mathbf{O}(\boldsymbol{n})$
  - $137n + 271 = \mathbf{O}(\boldsymbol{n})$
  - $n^2 + 3n + 4 = \mathbf{O}(\boldsymbol{n^2})$
  - $2^n + n^3 = \mathbf{O}(\mathbf{2}^{\boldsymbol{n}})$
  - $137 = \mathbf{O(1)}$
  - $n^2 \log n + \log^5 n = \mathbf{O}(\boldsymbol{n^2} \mathbf{\log} \boldsymbol{n})$

Big-O Notation, Formally

- Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$.
- Then $f(n) = O(g(n))$ iff there exist constants $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

For any $n \geq n_0$, $f(n) \leq cg(n)$

- Intuitively, as n gets "large" (greater than n_0), $f(n)$ is bounded from above by some multiple (determined by c) of $g(n)$.

Properties of Big-O Notation

- **Theorem**: If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$.
  - Intuitively: If you run two programs one after another, the big-O of the result is the big-O of the sum of the two runtimes.
- **Theorem**: If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n)f_2(n) = O(g_1(n)g_2(n))$.
  - Intuitively: If you run one program some number of times, the big-O of the result is the big-O of the program times the big-O of the number of iterations.
- This makes it substantially easier to analyze time complexity, though we do lose some precision.

Life is Easier with Big-O

M = "On input w:

- Scan across the tape until a `0` or `1` is found.
- If none are found, accept.
- If one is found, continue scanning until a matching `1` or `0` is found.
- If none is found, reject.
- Otherwise, cross off that symbol and repeat."

	O(n) steps
	O(1) steps
	O(n) steps
	O(1) steps
+	O(n) steps
	O(n) steps
×	O(n) loops
	O(n^2) steps

O(n) loops

Next Time

- **P**
  - What problems can be decided efficiently?
- **Polynomial-Time Reductions**
  - Constructing efficient algorithms.
- **NP**
  - What can we *verify* quickly?