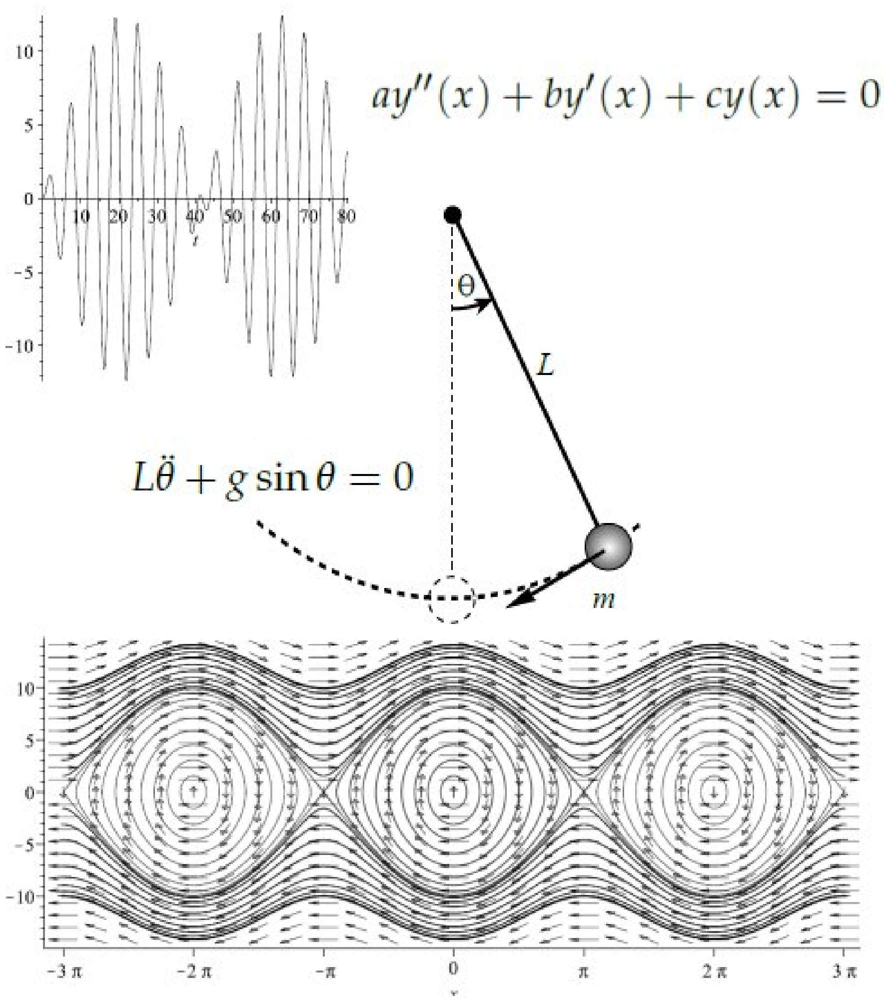
RUSSELL L. HERMAN

# A FIRST COURSE IN DIFFERENTIAL EQUATIONS FOR SCIENTISTS AND ENGINEERS

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A FIRST COURSE IN DIFFERENTIAL EQUATIONS FOR SCIENTISTS AND ENGINEERS by Russell Herman. Versions of these notes have been posted online since Summer 2014.
HTTP://PEOPLE.UNCW.EDU/HERMANR/MAT361/ODEBOOK

*June 2015*

# *Contents*

*Dedicated to those students who have endured previous versions of my notes.*

# *Prologue*

*"How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?." - Albert Einstein (1879-1955)*

## *Introduction*

THIS BOOK IS WRITTEN FOR AN UNDERGRADUATE COURSE on the introduction to differential equations typically taken by majors in mathematics, the physical sciences, and engineering. In this course we will investigate analytical, graphical, and approximate solutions of differential equations. We will study the theory, methods of solution and applications of ordinary differential equations. This will include common methods of finding solutions, such as using Laplace transform and power series methods.

Students should also be prepared to review their calculus, especially if they have been away from calculus for a while. Some of the key topics are reviewed in the appendix. In particular, students should know how to differentiate and integrate all elementary functions, including hyperbolic functions. They should review the methods of integration as the need arises, including methods of substitution and integration by parts. For the most part, we will just need material from Calculus I and II. Other topics from Calculus II that we will review are infinite series and introductory differential equations and applications.

Most students will have just come out of the calculus sequence knowing all about differentiation and integration. We hope that they have also seen plenty of applications. In this course, we will extend these applications to those connected with differential equations. Differential equations are equations involving an unknown function and its derivatives. If the function is a function of a single variable, then the equations are known as ordinary differential equations, the subject of this book. If the unknown function is a function of several independent variables, then the equation is a partial differential equation, which we will not deal with in this course. Finally, there may be several unknown functions satisfying several coupled differential equations. These systems of differential equations will be treated later in the course and are often the subject of a second course in differential equations.

In all cases we will be interested in specific solutions satisfying a set of initial conditions, or values, of the function and some of its derivatives at a given point of its domain. These are known as initial value problems. When such conditions are given at several points, then one is dealing with boundary value problems. Boundary value problems would be the subject of a second course in differential equations and in partial differential equations.

We will begin the study of differential equations with first order ordinary differential equations. These equations involve only derivatives of first order. Typical examples occur in population modeling and in free fall problems. There are a few standard forms which can be solved quite easily. In the second chapter we move up to second order equations. As the order increases, it becomes harder to solve differential equations analytically. So, we either need to deal with simple equations or turn to other methods of finding approximate solutions.

For second order differential equations there is a theory for linear second order differential equations and the simplest equations are constant coefficient second order linear differential equations. We will spend some time looking at these solutions. Even though constant coefficient equations are relatively simple, there are plenty of applications and the simple harmonic oscillator is one of these. The solutions make physical sense and adding damping and forcing terms leads to interesting solutions and additional methods of solving these equations.

Not all differential equations can be solved in terms of elementary functions. So, we turn to the numerical solution of differential equations using the solvable models as test beds for numerical schemes. This also allows for the introduction of more realistic models. Using Computer Algebra Systems (CAS) or other programming environments, we can explore these examples.

A couple hundred years ago there were no computers. So, mathematicians of the day sought series solutions of differential equations. These series solutions led to the discovery of now famous functions, such as Legendre polynomials and Bessel functions. These functions are quite common in applications and the use of power series solutions is a well known approach to finding approximate solutions by hand.

Another common technique for solving differential equation, both ordinary and partial, are transform methods. One of the simplest of these is the Laplace transform. This integral transform is used to transform the ordinary differentia equation to an algebraic equation. The solution of the algebraic equation is then used to uncover the solution to the differential equation. These techniques are often useful in systems theory or electrical engineering.

In recent decades the inclusion of technology in the classroom has allowed for the introduction of systems of differential equations into the typical course on differential equations. Solutions of linear systems of equations is an important tool in the study of nonlinear differential equations and nonlinear differential equations have been the subject of many research papers over the last several decades. We will look at systems of differential equations at the end of the book and discuss the stability of solutions in dynamical systems.

### *Technology and Tables*

As you progress through the course, you will often have to compute integrals and derivatives by hand. However, many readers know that some of the tedium can be alleviated by using computers, or even looking up what you need in tables. In some cases you might even find applets online that can quickly give you the answers you seek.

You also need to be comfortable in doing many computations by hand. This is necessary, especially in your early studies, for several reasons. For example, you should try to evaluate integrals by hand when asked to do them. This reinforces the techniques, as outlined earlier. It exercises your brain in much the same way that you might jog daily to exercise your body. The more comfortable you are with derivations and evaluations, the easier it is to follow future lectures without getting bogged down by the details, wondering how your professor got from step A to step D. You can always use a computer algebra system, or a Table of Integrals, to check on your work.

Problems can arise when depending purely on the output of computers, or other "black boxes." Once you have a firm grasp on the techniques and a feeling as to what answers should look like, then you can feel comfortable with what the computer gives you. Sometimes, Computer Algebra Systems (CAS) like Maple, can give you strange looking answers and sometimes even wrong answers. Also, these programs cannot do every integral or solve every differential equation that you ask them to do. Even some of the simplest looking expressions can cause computer algebra systems problems. Other times you might even provide wrong input, leading to erroneous results.

Another source of indefinite integrals, derivatives, series expansions, etc, is a Table of Mathematical Formulae. There are several good books that have been printed. Even some of these have typos in them, so you need to be careful. However, it may be worth the investment to have such a book in your personal library. Go to the library, or the bookstore, and look at some of these tables to see how useful they might be.

There are plenty of online resources as well. For example, there is the Wolfram Integrator at http://integrals.wolfram.com/ as well as the recent http://www.wolframalpha.com/. There is also a wealth of information at the following sites: http://www.sosmath.com/,
http://www.math2.org/, http://mathworld.wolfram.com/, and
http://functions.wolfram.com/.

While these resources are useful for problems which have analytical solutions, at some point you will need to realize that most problems in texts, especially those from a few decades ago, are mostly aimed at solutions which either have nice analytical solutions or have solutions that can be approximated using pencil and paper.

More and more you will see problems which need to be solved numerically. While most of this book (97%) stresses the traditional methods used for determining the exact or approximate behavior of systems based upon solid mathematical methods, there are times that an basic understanding of computational methods is useful. Therefore, we will occasionally discuss some numerical methods related to the subject matter in the text. In particular, we will discuss some methods of computational physics such as the numerical solution of differential equations and fitting data to curves. Applications will be discussed which can only be solved using these methods.

There are many programming languages and software packages which can be used to determine numerical solutions to algebraic equations or differential equations. For example, CAS (Computer Algebra Systems) such as Maple and Mathematica are available. Open source packages such as Maxima, which has been around for a while, Mathomatic, and the SAGE Project, do exist as alternatives. One can use built in routines and do some programming. The main features are that they can produce symbolic solutions. Generally, they are slow in generating numerical solutions.

For serious programming, one can use standard programming languages like FORTRAN, C and its derivatives. Recently, Python has become an alternative and much accepted resource as an open source programming language and is useful for doing scientific computing using the right packages.

Also, there is MATLAB. MATLAB was developed in the 1980's as a Matrix Laboratory and for a long time was the standard outside "normal" programming languages to handle non-symbolic solutions in computational science. Similar open source clones have appeared, such as Octave. Octave can run most MATLAB files and some of its own. Other clones of MATLAB are SciLab, Rlab, FreeMat, and PyLab.

In this text there are some snippets provided of Maple and MATLAB routines. Most of the text does not rely on these; however, the MATLAB snippets should be relatively readable to anyone with some knowledge of computer packages, or easy to pass to the open source clones, such as Octave. Maple routines are not so simple, but may be translatable to other packages with a little effort. However, the theory lying behind the use of any of these routines is described in the text and the text can be read without explicit understanding of the particular computer software.

## *Acknowledgments*

MOST, IF NOT ALL, OF THE IDEAS AND EXAMPLES are not my own. These notes are a compendium of topics and examples that I have used in teaching not only differential equations, but also in teaching numerous courses in physics and applied mathematics. Some of the notions even extend back to when I first learned them in courses I had taken.

I would also like to express my gratitude to the many students who have found typos, or suggested sections needing more clarity in the core set of notes upon which this book was based. This applies to the set of notes used in my mathematical physics course, applied mathematics course, *An Introduction to Fourier and Complex Analysis with Application to the Spectral Analysis of Signals*, and ordinary differential equations course, *A Second Course in Ordinary Differential Equations: Dynamical Systems and Boundary Value Problems*, all of which have some significant overlap with this book.

# 1
# *First Order Differential Equations*

*"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)*

## 1.1 Free Fall

In this chapter we will study some common differential equations that appear in physics. We will begin with the simplest types of equations and standard techniques for solving them We will end this part of the discussion by returning to the problem of free fall with air resistance. We will then turn to the study of oscillations, which are modeled by second order differential equations.

Free fall example.

Let us begin with a simple example from introductory physics. Recall that free fall is the vertical motion of an object solely under the force of gravity. It has been experimentally determined that an object near the surface of the Earth falls at a constant acceleration in the absence of other forces, such as air resistance. This constant acceleration is denoted by $-g$, where $g$ is called the acceleration due to gravity. The negative sign is an indication that we have chosen a coordinate system in which up is positive.

We are interested in determining the position, $y(t)$, of the falling body as a function of time. From the definition of free fall, we have

$$\ddot{y}(t) = -g. \tag{1.1}$$

Differentiation with respect to time is often denoted by dots instead of primes.

Note that we will occasionally use a dot to indicate time differentiation. This notation is standard in physics and we will begin to introduce you to this notation, though at times we might use the more familiar prime notation to indicate spatial differentiation, or general differentiation.

In Equation (1.1) we know $g$. It is a constant. Near the Earth's surface it is about 9.81 $\mathrm{m/s^2}$ or 32.2 $\mathrm{ft/s^2}$. What we do not know is $y(t)$. This is our first differential equation. In fact it is natural to see differential equations appear in physics often through Newton's Second Law, $F = ma$, as it plays an important role in classical physics. We will return to this point later.

So, how does one solve the differential equation in (1.1)? We do so by using what we know about calculus. It might be easier to see when we put in a particular number instead of $g$. You might still be getting used to the fact that some letters are used to represent constants. We will come back to the more general form after we see how to solve the differential equation.

Consider

$$\ddot{y}(t) = 5. \tag{1.2}$$

Recalling that the second derivative is just the derivative of a derivative, we can rewrite this equation as

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = 5. \tag{1.3}$$

This tells us that the derivative of $dy/dt$ is 5. Can you think of a function whose derivative is 5? (Do not forget that the independent variable is $t$.) Yes, the derivative of $5t$ with respect to $t$ is 5. Is this the only function whose derivative is 5? No! You can also differentiate $5t+1$, $5t+\pi$, $5t-6$, etc. In general, the derivative of $5t+C$ is 5, where $C$ is an arbitrary integration constant.

So, Equation (1.2) can be reduced to

$$\frac{dy}{dt} = 5t + C. \tag{1.4}$$

Now we ask if you know a function whose derivative is $5t+C$. Well, you might be able to do this one in your head, but we just need to recall the Fundamental Theorem of Calculus, which relates integrals and derivatives. Thus, we have

$$y(t) = \frac{5}{2}t^2 + Ct + D, \tag{1.5}$$

where $D$ is a second integration constant.

Equation (1.5) gives the solution to the original differential equation. That means that when the solution is placed into the differential equation, both sides of the differential equation give the same expression. You can always check your answer to a differential equation by showing that your solution satisfies the equation. In this case we have

$$\ddot{y}(t) = \frac{d^2}{dt^2}\left(\frac{5}{2}t^2 + Ct + D\right) = \frac{d}{dt}(5t+C) = 5.$$

Therefore, Equation (1.5) gives the general solution of the differential equation.

We also see that there are two arbitrary constants, $C$ and $D$. Picking any values for these gives a whole family of solutions. As we will see, the equation $\ddot{y}(t) = 5$ is a linear second order ordinary differential equation. The general solution of such an equation always has two arbitrary constants.

Let's return to the free fall problem. We solve it the same way. The only difference is that we can replace the constant 5 with the constant $-g$. So, we find that

$$\frac{dy}{dt} = -gt + C, \tag{1.6}$$

and

$$y(t) = -\frac{1}{2}gt^2 + Ct + D. \tag{1.7}$$

Once you get down the process, it only takes a line or two to solve.

There seems to be a problem. Imagine dropping a ball that then undergoes free fall. We just determined that there are an infinite number of solutions for the position of the ball at any time! Well, that is not possible. Experience tells us that if you drop a ball you expect it to behave the same way every time. Or does it? Actually, you could drop the ball from anywhere. You could also toss it up or throw it down. So, there are many ways you can release the ball before it is in free fall producing many different paths, $y(t)$. That is where the constants come in. They have physical meanings.

If you set $t = 0$ in the equation, then you have that $y(0) = D$. Thus, $D$ gives the initial position of the ball. Typically, we denote initial values with a subscript. So, we will write $y(0) = y_0$. Thus, $D = y_0$.

That leaves us to determine $C$. It appears at first in Equation (1.6). Recall that $\frac{dy}{dt}$, the derivative of the position, is the vertical velocity, $v(t)$. It is positive when the ball moves upward. We will denote the initial velocity $v(0) = v_0$. Inserting $t = 0$ in Equation (1.6), we find that $\dot{y}(0) = C$. This implies that $C = v(0) = v_0$.

Putting this all together, we have the physical form of the solution for free fall as

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0. \tag{1.8}$$

Doesn't this equation look familiar? Now we see that the infinite family of solutions consists of free fall resulting from initially dropping a ball at position $y_0$ with initial velocity $v_0$. The conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$ are called the initial conditions. A solution of a differential equation satisfying a set of initial conditions is often called a particular solution. Specifying the initial conditions results in a unique solution.

So, we have solved the free fall equation. Along the way we have begun to see some of the features that will appear in the solutions of other problems that are modeled with differential equation. Throughout the book we will see several applications of differential equations. We will extend our analysis to higher dimensions, in which we case will be faced with so-called partial differential equations, which involve the partial derivatives of functions of more that one variable.

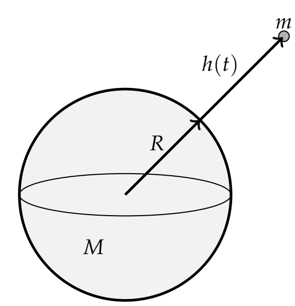


Figure 1.1: Free fall far from the Earth from a height $h(t)$ from the surface.

But are we done with free fall? Not at all! We can relax some of the conditions that we have imposed. We can add air resistance. We will visit this problem later in this chapter after introducing some more techniques. We can also provide a horizontal component of motion, leading to projectile motion.

Finally, we should also note that free fall at constant $g$ only takes place near the surface of the Earth. What if a tile falls off the shuttle far from the surface of the Earth? It will also fall towards the Earth. Actually, the tile also has a velocity component in the direction of the motion of the shuttle. So, it would not necessarily take radial path downwards. For now, let's ignore that component.

To look at this problem in more detail, we need to go to the origins of the acceleration due to gravity. This comes out of Newton's Law of Gravitation. Consider a mass $m$ at some distance $h(t)$ from the surface of the (spherical) Earth. Letting $M$ and $R$ be the Earth's mass and radius, respectively, Newton's Law of Gravitation states that

$$\begin{aligned} ma &= F \\ m\frac{d^2h(t)}{dt^2} &= -G\frac{mM}{(R+h(t))^2}. \end{aligned} \tag{1.9}$$

Here $G = 6.6730 \times 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ is the Universal Gravitational Constant, $M = 5.9736 \times 10^{24}$ kg and $R = 6371$ km are the Earth's mass and mean radius, respectively. For $h << R$, $GM/R^2 \approx g$.

Thus, we arrive at a differential equation

$$\frac{d^2h(t)}{dt^2} = -\frac{GM}{(R+h(t))^2}. \tag{1.10}$$

This equation is not as easy to solve. We will leave it as a homework exercise for the reader.

## 1.2 *First Order Differential Equations*

$n$-th order ordinary differential equation

Before moving on, we first define an $n$-th order ordinary differential equation. It is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first $n$ derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{1.11}$$

Here $y^{(n)}(x)$ represents the $n$th derivative of $y(x)$.

Initial value problem.

An initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say $x_0$:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{1.12}$$

Linear $n$th order differential equation

A linear $n$th order differential equation takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{1.13}$$

If $f(x) \equiv 0$, then the equation is said to be homogeneous, otherwise it is called nonhomogeneous.

Homogeneous and nonhomogeneous equations.

First order differential equation

Typically, the first differential equations encountered are first order equations. A first order differential equation takes the form

$$F(y', y, x) = 0. \tag{1.14}$$

There are two common first order differential equations for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

### 1.2.1 Separable Equations

A FIRST ORDER EQUATION IS SEPARABLE if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{1.15}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be autonomous.

The general solution to equation (1.15) is obtained in terms of two integrals:

Separable equations.

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\,dx + C,} \tag{1.16}$$

where $C$ is an integration constant. This yields a 1-parameter family of solutions to the differential equation corresponding to different values of $C$. If one can solve (1.16) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a particular solution.

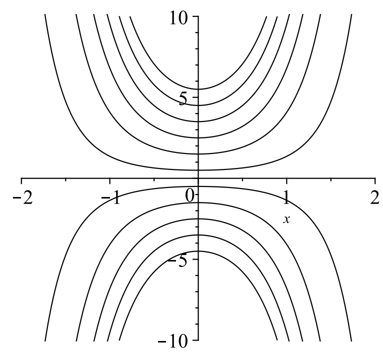



Figure 1.2: Plots of solutions from the 1-parameter family of solutions of Example 1.1 for several initial conditions.

**Example 1.1.** $y' = 2xy$, $y(0) = 2$.

*Applying (1.16), one has*

$$\int \frac{dy}{y} = \int 2x\,dx + C.$$

*Integrating yields*

$$\ln|y| = x^2 + C.$$

*Exponentiating, one obtains the general solution,*

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

*Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, $A$ is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 1.2.*

*Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial condition is $y(x) = 2e^{x^2}$.*

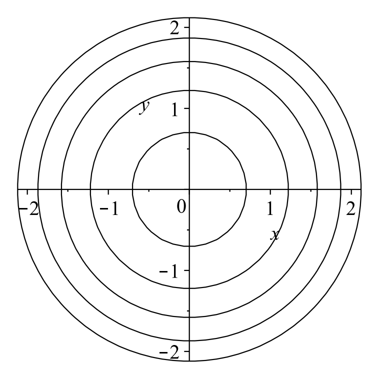



Figure 1.3: Plots of solutions of Example 1.2 for several initial conditions.

**Example 1.2.** $yy' = -x$. *Following the same procedure as in the last example, one obtains:*

$$\int y\,dy = -\int x\,dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

*Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure 1.3.*

### 1.2.2 Linear First Order Equations

THE SECOND TYPE OF FIRST ORDER EQUATION encountered is the linear first order differential equation in the standard form

$$y'(x) + p(x)y(x) = q(x). \tag{1.17}$$

In this case one seeks an integrating factor, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)q(x). \tag{1.18}$$

The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\, d\xi)$. One can derive $\mu(x)$ by expanding the derivative in Equation (1.18),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{1.19}$$

and comparing this equation to the one obtained from multiplying (1.17) by $\mu(x)$ :

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{1.20}$$

Note that these last two equations would be the same if the second terms were the same. Thus, we will require that

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation for $\mu(x)$ whose solution is the integrating factor:

Integrating factor.

$$\boxed{\mu(x) = \exp\left(\int^x p(\xi)\, d\xi\right).} \tag{1.21}$$

Equation (1.18) is now easily integrated to obtain the general solution to the linear first order differential equation:

$$\boxed{y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\, d\xi + C\right].} \tag{1.22}$$

**Example 1.3.** $xy' + y = x, \quad x > 0, y(1) = 0.$

*One first notes that this is a linear first order differential equation. Solving for $y'$, one can see that the equation is not separable. Furthermore, it is not in the standard form (1.17). So, we first rewrite the equation as*

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{1.23}$$

*Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor*

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

*Multiplying equation (1.23) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, the differential equation becomes*

$$(xy)' = x.$$

*Integrating, one obtains*

$$xy = \frac{1}{2}x^2 + C,$$

*or*

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

*Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is*

$$y(x) = \frac{1}{2}(x - \frac{1}{x}).$$

*We can verify that this is the solution. Since $y' = \frac{1}{2} + \frac{1}{2x^2}$, we have*

$$xy' + y = \frac{1}{2}x + \frac{1}{2x} + \frac{1}{2}\left(x - \frac{1}{x}\right) = x.$$

*Also, $y(1) = \frac{1}{2}(1 - 1) = 0$.*

**Example 1.4.** $(\sin x)y' + (\cos x)y = x^2$.

*Actually, this problem is easy if you realize that the left hand side is a perfect derivative. Namely,*

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

*But, we will go through the process of finding the integrating factor for practice.*

*First, we rewrite the original differential equation in standard form. We divide the equation by $\sin x$ to obtain*

$$y' + (\cot x)y = x^2 \csc x.$$

*Then, we compute the integrating factor as*

$$\mu(x) = \exp\left(\int^x \cot\xi \, d\xi\right) = e^{\ln(\sin x)} = \sin x.$$

*Using the integrating factor, the standard form equation becomes*

$$\frac{d}{dx}((\sin x)y) = x^2.$$

*Integrating, we have*

$$y \sin x = \frac{1}{3}x^3 + C.$$

*So, the solution is*

$$y(x) = \left(\frac{1}{3}x^3 + C\right)\csc x.$$

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields or numerical methods. We will return to a discussion of the qualitative behavior of differential equations later and numerical solutions of ordinary differential equations later in the book.

### 1.2.3 *Exact Differential Equations*

Some first order differential equations can be solved easily if they are what are called exact differential equations. These equations are typically written using differentials. For example, the differential equation

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0 \tag{1.24}$$

can be written in the form

$$M(x,y)dx + N(x,y)dy = 0.$$

This is seen by multiplying Equation (1.24) by $dx$ and noting from calculus that for a function $y = y(x)$, the relation between the differentials $dx$ and $dy$ is

$$dy = \frac{dy}{dx}\,dx.$$

Differential one-forms.

The expression $M(x,y)dx + N(x,y)dy$ is called a differential one-form. Such a one-form is called exact if there is a function $u(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = du.$$

Exact one-form.

However, from calculus we know that for any function $u(x,y)$,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

If $du = M(x,y)dx + N(x,y)dy$, then we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= M(x,y) \\ \frac{\partial u}{\partial y} &= N(x,y). \end{aligned} \tag{1.25}$$

Since

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

when these second derivatives are continuous, by Clairaut's Theorem, then we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

must hold if $M(x,y)dx + N(x,y)dy$ is to be an exact one-form.

In summary, we have found that

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is exact in the domain $D$ of the $xy$-plane for $M$, $N$, $M_y$, and $N_x$ continuous functions in $D$ if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

holds in the domain.

Condition for $M(x,y)dx + N(x,y)dy = 0$ to be exact.

Furthermore, if $du = M(x,y)dx + N(x,y)dy = 0$, then $u(x,y) = C$, for $C$ an arbitrary constant. Thus, an implicit solution can be found as

$$\int_{x_0}^{x} M(x,y)\,dx + \int_{y_0}^{y} N(x,y)\,dy = C.$$

We show this in the following example.

**Example 1.5.** *Show that $(x^3 + xy^2)\,dx + (x^2y + y^3)\,dy = 0$ is an exact differential equation and obtain the corresponding implicit solution*

*We first note that*

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = 2xy.$$

*Since these partial derivatives are the same, the differential equation is exact. So, we need to find the function $u(x,y)$ such that $du = (x^3 + xy^2)\,dx + (x^2y + y^3)\,dy$.*

*First, we note that $x^3 = d\left(\frac{x^4}{4}\right)$ and $y^3 = d\left(\frac{y^4}{4}\right)$. The remaining terms can be combined to find that*

$$\begin{aligned} xy^2\,dx + x^2y\,dy &= xy(y\,dx + x\,dy) \\ &= xy\,d(xy) \\ &= d\left(\frac{(xy)^2}{2}\right). \end{aligned} \tag{1.26}$$

*Combining these results, we have*

$$u = \frac{x^4}{4} + \frac{(xy)^2}{2} + \frac{y^4}{4} = C.$$

What if the one-form is not exact?

So ,what if $M(x,y)dx + N(x,y)dy$ is not exact? We can multiply the one-form by an integrating factor, $\mu(x)$, and try to make he resulting form exact. We let

$$du = \mu M dx + \mu N dy.$$

For the new form to be exact, we have to require that

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}.$$

Carrying out the differentiation, we have

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$$

Thus, the integrating factor satisfies a partial differential equation. If the integrating factor is a function of only $x$ or $y$, then this equation reduces to ordinary differential equations for $\mu$.

As an example, if $\mu = \mu(x)$, then the integrating factor satisfies

$$N\frac{d\mu}{dx} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right),$$

or

$$N\frac{d\ln\mu}{dx} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

**Example 1.6.** *Find the general solution to the differential equation* $(1+y^2)\,dx + xy\,dy = 0$.

*First, we not that this is not exact. We have* $M(x,y) = 1+y^2$ *and* $N(x,y) = xy$. *Then,*

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = y.$$

*Therefore, the differential equation is not exact.*

*Next, we seek the integrating factor. We let*

$$du = \mu(1+y^2)\,dx + \mu xy\,dy.$$

*For the new form to be exact, we have to require that*

$$xy\frac{\partial\mu}{\partial x} - (1+y^2)\frac{\partial\mu}{\partial y} = \mu\left(\frac{\partial(1+y^2)}{\partial y} - \frac{\partial xy}{\partial x}\right) = \mu y.$$

*If* $\mu = \mu(x)$, *then*

$$x\frac{d\mu}{dx} = \mu.$$

*This is easily solved as a separable first order equation. We find that* $\mu(x) = x$.

*Multiplying the original equation by* $\mu = x$, *we obtain*

$$0 = x(1+y^2)\,dx + x^2y\,dy = d\left(\frac{x^2}{2} + \frac{x^2y^2}{2}\right).$$

*Thus,*

$$\frac{x^2}{2} + \frac{x^2y^2}{2} = C$$

*gives the solution.*

The Bernoulli's were a family of Swiss mathematicians spanning three generations. It all started with Jacob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748). Jacob had a son, Nicolaus Bernoulli (1687-1759) and Johann (1667-1748) had three sons, Nicolaus Bernoulli II (1695-1726), Daniel Bernoulli (1700-1872), and Johann Bernoulli II (1710-1790). The last generation consisted of Johann II's sons, Johann Bernoulli III (1747-1807) and Jacob Bernoulli II (1759-1789). Johann, Jacob and Daniel Bernoulli were the most famous of the Bernoulli's. Jacob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler and Daniel Bernoulli, who is known for his work in hydrodynamics.

### *1.2.4 Bernoulli Equation*

There are several nonlinear first order equations whose solution can be obtained using special techniques. We conclude this section be looking at a few of these equations. We begin with the Bernoulli equation, named after Jacob Bernoulli (1655-1705). The Bernoulli equation is of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 0,1.$$

Note that when $n = 0,1$ the equation is linear and can be solved using an integrating factor. The key to solving this equation is using the transformation $z(x) = \frac{1}{y^{n-1}(x)}$ to make the equation for $z(x)$ linear. We demonstrate the procedure using an example.

**Example 1.7.** *Solve the Bernoulli equation $xy' + y = y^2 \ln x$ for $x > 0$.*

*In this example $p(x) = 1$, $q(x) = \ln x$, and $n = 2$. Therefore, we let $z = \frac{1}{y}$. Then,*

$$z' = -\frac{1}{y^2} y' = z^2 y'.$$

*Inserting $z = y^{-1}$ and $z' = z^2 y'$ into the differential equation, we have*

$$\begin{aligned} xy' + y &= y^2 \ln x \\ -x\frac{z'}{z^2} + \frac{1}{z} &= \frac{\ln x}{z^2} \\ -xz' + z &= \ln x \\ z' - \frac{1}{x} z = -\frac{\ln x}{x}. & \end{aligned} \tag{1.27}$$

*Thus, the resulting equation is a linear first order differential equation. It can be solved using the integrating factor,*

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \frac{1}{x}.$$

*Multiplying the differential equation by the integrating factor, we have*

$$\left(\frac{z}{x}\right)' = \frac{\ln x}{x^2}.$$

*Integrating, we obtain*

$$\begin{aligned} \frac{z}{x} &= -\int \frac{\ln x}{x^2} + C \\ &= \frac{\ln x}{x} + \int \frac{dx}{x^2} + C \\ &= \frac{\ln x}{x} + \frac{1}{x} + C. \end{aligned} \tag{1.28}$$

*Multiplying by $x$, we have $z = \ln x + 1 + Cx$. Since $z = y^{-1}$, the general solution to the problem is*

$$y = \frac{1}{\ln x + 1 + Cx}.$$

### 1.2.5 *Lagrange and Clairaut Equations*

Alexis Claude Clairaut (1713-1765) solved the differential equation

$$y = xy' + g(y').$$

This is a special case of the family of Lagrange equations,

$$y = xf(y') + g(y'),$$

named after Joseph Louis Lagrange (1736-1813). These equations also have solutions called singular solutions. Singular solution are solutions for which there is a failure of uniqueness to the initial value problem at every point on the curve. A singular solution is often one that is tangent to every solution in a family of solutions.

First, we consider solving the more general Lagrange equation. Let $p = y'$ in the Lagrange equation, giving

$$y = xf(p) + g(p). \tag{1.29}$$

Next, we differentiate with respect to $x$ to find

$$y' = p = f(p) + xf'(p)p' + g'(p)p'.$$

Lagrange equations, $y = xf(y') + g(y')$.

Here we used the Chain Rule. For example,

$$\frac{dg(p)}{dx} = \frac{dg}{dp}\frac{dp}{dx}.$$

Solving for $p'$, we have

$$\frac{dp}{dx} = \frac{p - f(p)}{xf'(p) + g'(p)}. \tag{1.30}$$

We have introduced $p = p(x)$, viewed as a function of $x$. Let's assume that we can invert this function to find $x = x(p)$. Then, from introductory calculus, we know that the derivatives of a function and its inverse are related,

$$\frac{dx}{dp} = \frac{1}{\frac{dp}{dx}}.$$

Applying this to Equation (1.30), we have

$$\begin{aligned} \frac{dx}{dp} &= \frac{xf'(p) + g'(p)}{p - f(p)} \\ x' - \frac{f'(p)}{p - f(p)}x &= \frac{g'(p)}{p - f(p)}, \end{aligned} \tag{1.31}$$

assuming that $p - f(p) \neq 0$.

As can be seen, we have transformed the Lagrange equation into a first order linear differential equation (1.31) for $x(p)$. Using methods from earlier in the chapter, we can in principle obtain a family of solutions

$$x = F(p, C),$$

where $C$ is an arbitrary integration constant. Using Equation (1.29), one might be able to eliminate $p$ in Equation (1.31) to obtain a family of solutions of the Lagrange equation in the form

$$\varphi(x, y, C) = 0.$$

If it is not possible to eliminate $p$ from Equations (1.29) and (1.31), then one could report the family of solutions as a parametric family of solutions with $p$ the parameter. So, the parametric solutions would take the form

$$\begin{aligned} x &= F(p, C), \\ y &= F(p, C) f(p) + g(p). \end{aligned} \tag{1.32}$$

We had also assumed the $p - f(p) \neq 0$. However, there might also be solutions of Lagrange's equation for which $p - f(p) = 0$. Such solutions are called singular solutions.

Singular solutions are possible for Lagrange equations.

**Example 1.8.** *Solve the Lagrange equation $y = 2xy' - y'^2$.*

*We will start with Equation (1.31). Noting that $f(p) = 2p$, $g(p) = -p^2$, we have*

$$\begin{aligned} x' - \frac{f'(p)}{p - f(p)} x &= \frac{g'(p)}{p - f(p)} \\ x' - \frac{2}{p - 2p} x &= \frac{-2p}{p - 2p} \\ x' + \frac{2}{p} x &= 2. \end{aligned} \tag{1.33}$$

*This first order linear differential equation can be solved using an integrating factor. Namely,*

$$\mu(p) = \exp\left( \int \frac{2}{p} \, dp \right) = e^{2 \ln p} = p^2.$$

*Multiplying the differential equation by the integrating factor, we have*

$$\frac{d}{dp}\left( xp^2 \right) = 2p^2.$$

*Integrating,*

$$xp^2 = \frac{2}{3} p^3 + C.$$

*This gives the general solution*

$$x(p) = \frac{2}{3} p + \frac{C}{p^2}.$$

*Replacing $y' = p$ in the original differential equation, we have $y = 2xp - p^2$. The family of solutions is then given by the parametric equations*

$$\begin{aligned} x &= \frac{2}{3} p + \frac{C}{p^2}, \\ y &= 2\left( \frac{2}{3} p + \frac{C}{p^2} \right) p - p^2 \\ &= \frac{1}{3} p^2 + \frac{2C}{p}. \end{aligned} \tag{1.34}$$

*The plots of these solutions is shown in Figure 1.4.*

*We also need to check for a singular solution. We solve the equation $p - f(p) = 0$, or $p = 0$. This gives the solution $y(x) = (2xp - p^2)_{p=0} = 0$.*

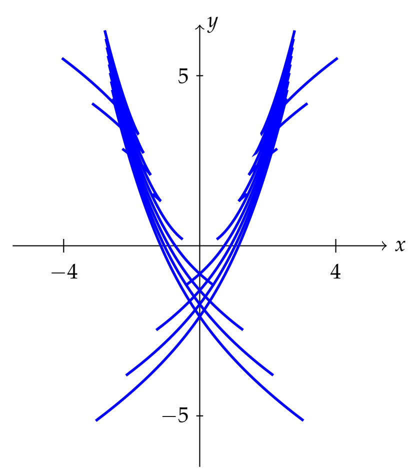



Figure 1.4: Family of solutions of the Lagrange equation $y = 2xy' - y'^2$.

The Clairaut differential equation is given by

$$y = xy' + g(y').$$

Letting $p = y'$, we have

$$y = xp + g(p).$$

Clairaut equations, $y = xy' + g(y')$.

This is the Lagrange equation with $f(p) = p$. Differentiating with respect to $x$,

$$p = p + xp' + g'(p)p'.$$

Rearranging, we find

$$x = -g'(p)$$

So, we have the parametric solution

$$\begin{aligned} x &= -g'(p), \\ y &= -pg'(p) + g(p). \end{aligned} \tag{1.35}$$

For the case that $y' = C$, it can be seen that $y = Cx + g(C)$ is a general solution solution.

**Example 1.9.** *Find the solutions of $y = xy' - y'^2$.*

*As noted, there is a family of straight line solutions $y = Cx - C^2$, since $g(p) = -p^2$. There might also by a parametric solution not contained n this family. It would be given by the set of equations*

$$\begin{aligned} x &= -g'(p) = 2p, \\ y &= -pg'(p) + g(p) = 2p^2 - p^2 = p^2. \end{aligned} \tag{1.36}$$

*Eliminating $p$, we have the parabolic curve $y = x^2/4$.*

*In Figure 1.5 we plot these solutions. The family of straight line solutions are shown in blue. The limiting curve traced out, much like string figures one might create, is the parametric curve.*

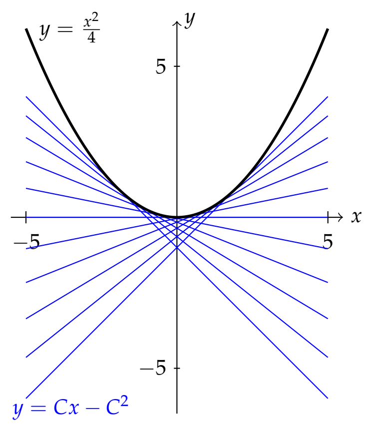



Figure 1.5: Plot of solutions to the Clairaut equation $y = xy' - y'^2$. The straight line solutions are a family of curves whose limit is the parametric solution.

### 1.2.6 Riccati Equation

JACOPO FRANCESCO RICCATI (1676-1754) STUDIED CURVES with some specified curvature. He proposed an equation of the form

$$y' + a(x)y^2 + b(x)y + c(x) = 0$$

around 1720. He communicated this to the Bernoulli's. It was Daniel Bernoulli who had actually solved this equation. As noted by Ranjan Roy (2011), Riccati had published his equation in 1722 with a note that D. Bernoulli giving the solution in terms of an anagram. Furthermore, when $a \equiv 0$, the Riccati equation reduces to a Bernoulli equation.

In Section seq:riccati, we will show that the Ricatti equation can be transformed into a second order linear differential equation. However, there are special cases in which we can get our hands on the solutions. For example, if $a$, $b$, and $c$ are constants, then the differential equation can be integrated directly. We have

$$\frac{dy}{dx} = -(ay^2 + by + c).$$

This equation is separable and we obtain

$$x - C = -\int \frac{dy}{ay^2 + by + c}.$$

When a differential equation is left in this form, it is said to be solved by quadrature when the resulting integral in principle can be computed in terms of elementary functions.[1]

[1] By elementary functions we mean well known functions like polynomials, trigonometric, hyperbolic, and some not so well know to undergraduates, such as Jacobi or Weierstrass elliptic functions.

If a particular solution is known, then one can obtain a solution to the Riccati equation. Let the known solution be $y_1(x)$ and assume that the general solution takes the form $y(x) = y_1(x) + z(x)$ for some unknown function $z(x)$. Substituting this form into the differential equation, we can show that $v(x) = 1/z(x)$ satisfies a first order linear differential equation.

Inserting $y = y_1 + z$ into the general Riccati equation, we have

$$\begin{aligned}
0 &= \frac{dy}{dx} + a(x)y^2 + b(x)y + c \\
&= \frac{dz}{dx} + az^2 + 2azy_1 + bz + \\
&\quad + \frac{dy_1}{dx} + ay_1^2 + by_1 + c \\
&= \frac{dz}{dx} + a(x)[2y_1 z + z^2] + b(x)z \\
-a(x)z^2 &= \frac{dz}{dx} + [2a(x)y_1 + b(x)]z. \qquad (1.37)
\end{aligned}$$

The last equation is a Bernoulli equation with $n = 2$. So, we can make it a linear equation with the substitution $z = \frac{1}{v}$, $z' = -\frac{z'}{v^2}$. Then, we obtain a differential equation for $v(x)$. It is given by

$$v' - (2a(x)y_1(x) + b(x))v = a(x).$$

**Example 1.10.** *Fond the general solution of the Riccati equation, $y' - y^2 + 2e^x y - e^{2x} - e^x = 0$, using the particular solution $y_1(x) = e^x$.*

*We let the sought solution take the form $y(x) = z(x) + e^x$. Then, the equation for $z(x)$ is found as*

$$\frac{dz}{dx} = z^2.$$

*This equation is simple enough to integrate directly to obtain $z = \frac{1}{C-x}$. Then, the solution to the problem becomes*

$$y(x) = \frac{1}{C - x} + e^x.$$

## 1.3 Applications

In this section we will look at some simple applications which are modeled with first order differential equations. We will begin with simple exponential models of growth and decay.

### 1.3.1 Growth and Decay

Some of the simplest models are those involving growth or decay. For example, a population model can be obtained under simple assumptions. Let $P(t)$ be the population at time $t$. We want to find an expression for the rate of change of the population, $\frac{dP}{dt}$. Assuming that there is no migration of population, the only way the population can change is by adding or subtracting individuals in the population. The equation would take the form

$$\frac{dP}{dt} = \text{Rate In } - \text{ Rate Out.}$$

The *Rate In* could be due to the number of births per unit time and the *Rate Out* by the number of deaths per unit time. The simplest forms for these rates would be given by

$$\text{Rate In } = bP \text{ and the } \text{ Rate Out } = mP.$$

Here we have denoted the birth rate as $b$ and the mortality rate as $m$. This gives the total rate of change of population as

$$\frac{dP}{dt} = bP - mP \equiv kP. \tag{1.38}$$

Equation (1.38) is a separable equation. The separation follows as we have seen earlier in the chapter. Rearranging the equation, its differential form is

$$\frac{dP}{P} = k\,dt.$$

Integrating, we have

$$\begin{aligned} \int \frac{dP}{P} &= \int k\,dt \\ \ln|P| &= kt + C. \end{aligned} \tag{1.39}$$

Next, we solve for $P(t)$ through exponentiation, Integrating, we have

$$\begin{aligned} |P(t)| &= e^{kt+C} \\ P(t) &= \pm e^{kt+C} \\ &= \pm e^{C} e^{kt} \\ &= Ae^{kt}. \end{aligned} \tag{1.40}$$

Here we reamed the arbitrary constant, $\pm e^{C}$, as $A$.

If the population at $t = 0$ is $P_0$, i.e., $P(0) = P_0$, then the solution gives $P(0) = Ae^0 = A = P_0$. So, the solution of the initial value problem is

$$P(t) = P_0 e^{kt}.$$

More generally, the initial value problem $dP/dt = kP$, $P(t_0) = P_0$ has the solution

$$P(t) = P_0 e^{k(t-t_0)}.$$

Malthusian population growth.

Equation (1.38) the familiar exponential model of population growth:

$$\frac{dP}{dt} = kP.$$

This is easily solved and one obtains exponential growth $(k > 0)$ or decay $(k < 0)$. This Malthusian growth model has been named after Thomas Robert Malthus (1766-1834), a clergyman who used this model to warn of the impending doom of the human race if its reproductive practices continued.

**Example 1.11.** *Consider a bacteria population of weight 20 g. If the population doubles every 20 minutes, then what is the population after 30 minutes? [Note: It is easier to weigh this population than to count it.]*

*One looks at the given information before trying to answer the question. First, we have the initial condition* $P_0 = 20$ *g. Since the population doubles every 20 minutes, then* $P(20) = 2P_0 = 40$. *Here we have take the time units as minutes. We are then asked to find* $P(30)$.

*We do not need to solve the differential equation. We will assume a simple growth model. Using the general solution,* $P(t) = 20e^{kt}$, *we have*

$$P(20) = 20e^{20k} = 40,$$

*or*

$$e^{20k} = 2.$$

*We can solve this for* $k$,

$$20k = \ln 2, \quad \Rightarrow k = \frac{\ln 2}{20} \approx 0.035.$$

*This gives an approximate solution,* $P(t) \approx 20e^{.035t}$. *Now we can answer the original question. Namely,* $P(30) \approx 57$.

*Of course, we could get an exact solution. With some simple manipulations, we have*

$$\begin{aligned} P(t) &= 20e^{kt} \\ &= 20e^{\left(\frac{\ln 2}{20}\right)t} \\ &= 20\left(e^{\ln 2}\right)^{\frac{t}{20}} \\ &= 20\left(2^{\frac{t}{20}}\right). \end{aligned} \tag{1.41}$$

*This answer takes the general form for population doubling,* $P(t) = P_0 2^{\frac{t}{\tau}}$, *where* $\tau$ *is the doubling rate.*

Radioactive decay problems.

Another standard growth-decay problem is radioactive decay. Certain isotopes are unstable and the nucleus breaks apart, leading to nuclear decay. The products of the decay may also be unstable and undergo further nuclear decay. As an example, Uranium-238 (U-238) decays into Thorium-234 (Th-234). Thorium-234 is unstable and decays into Protactinium (Pa-234). This in turn decays in many steps until lead (Pb-206) is produced as shown in Table 1.1. This lead isotope is stable and the decay process stops. While this is one form of radioactive decay, there are other types. For example, Radon 222 (Rn-222) gives up an alpha particle (helium nucleus) leaving Polonium (Po-218).

Table 1.1: U-238 decay chain.

| Isotope | Half-life |
|---|---|
| $U^{238}$ | $4.468x10^9$ years |
| $Th^{234}$ | 24.1 days |
| $Pa^{234m}$ | 1.17 minutes |
| $U^{234}$ | $2.47x10^5$ years |
| $Th^{230}$ | $8.0x10^4$ years |
| $Ra^{226}$ | 1602 years |
| $Rn^{222}$ | 3.823 days |
| $Po^{218}$ | 3.05 minutes |
| $Pb^{214}$ | 26.8 minutes |
| $Bi^{214}$ | 19.7 minutes |
| $Po^{214}$ | 164 microsec |
| $Pb^{210}$ | 21 years |
| $Bi^{210}$ | 5.01 days |
| $Po^{210}$ | 138.4 days |
| $Pb^{206}$ | Stable |

Given a certain amount of radioactive material, it does not all decay at one time. A measure of the tendency of a nucleus to decay is called the half-life. This is the time it takes for half of the material to decay. This is similar to the last example and can be understood using a simple example.

**Example 1.12.** *If 150.0 g of Thorium-234 decays to 137.6 g of Thorium-234 in three days, what is its half-life?*

*This is another simple decay process. If $Q(t)$ represents the quantity of unstable material, then $Q(t)$ satisfies the rate equation*

$$\frac{dQ}{dt} = kQ$$

*with $k < 0$. The solution of the initial value problem, as we have seen, is $Q(t) = Q_0 e^{kt}$.*

*Now, let the half-life be given by $\tau$. Then, $Q(\tau) = \frac{1}{2}Q_0$. Inserting this fact into the solution, we have*

$$\begin{aligned} Q(\tau) &= Q_0 e^{k\tau} \\ \frac{1}{2}Q_0 &= Q_0 e^{k\tau} \end{aligned}$$

$$\frac{1}{2} = e^{k\tau}. \tag{1.42}$$

*Noting that* $Q(t) = Q_0 \left(e^k\right)^t$*, we solve Equation (1.42) for*

$$e^k = 2^{-1/\tau}.$$

*Then, the solution can be written in the general form*

$$Q(t) = Q_0 2^{-\frac{t}{\tau}}.$$

*Note that the decay constant is* $k = -\frac{\ln 2}{\tau} < 0$.

*Returning to the problem, we are given*

$$Q(3) = 1502^{-\frac{3}{\tau}} = 137.6.$$

*Solving to* $\tau$,

$$\begin{aligned} 2^{-\frac{3}{\tau}} &= \frac{136.7}{150} \\ -3\ln 2 &= \ln .9173\tau \\ \tau &= -\frac{3\ln 2}{\ln .9173} = 24.09. \end{aligned} \tag{1.43}$$

*Therefore, the half-life is about* 24.1 *days.*

### *1.3.2 Newton's Law of Cooling*

IF YOU TAKE YOUR HOT CUP OF TEA, and let it sit in a cold room, the tea will cool off and reach room temperature after a period of time. The law of cooling is attributed to Isaac Newton (1642-1727) who was probably the first to state results on how bodies cool.[2] The main idea is that a body at temperature $T(t)$ is initially at temperature $T(0) = T_0$. It is placed in an environment at an ambient temperature of $T_a$. A simple model is given that the rate of change of the temperature of the body is proportional to the difference between the body temperature and its surroundings. Thus, we have

$$\frac{dT}{dt} \propto T - T_a.$$

The proportionality is removed by introducing a cooling constant,

$$\frac{dT}{dt} = -k(T - T_a), \tag{1.44}$$

where $k > 0$.

This differential equation can be solved by noting that the equation can be written in the form

$$\frac{d}{dt}(T - T_a) = -k(T - T_a).$$

This is now of the form of exponential decay of the function $T(t) - T_a$. The solution is easily found as

$$T(t) - T_a = (T_0 - T_a)e^{-kt},$$

[2] Newton's 1701 Law of Cooling is an approximation to how bodies cool for small temperature differences ($T - T_a \ll T$) and does not take into account all of the cooling processes. One account is given by C. T. O'Sullivan, Am. J. Phys (1990) p 956-960.

or

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

**Example 1.13.** *A cup of tea at $90^oC$ cools to $85^oC$ in ten minutes. If the room temperature is $22^oC$, what is its temperature after 30 minutes?*

*Using the general solution with $T_0 = 90^oC$,*

$$T(t) = 22 + (90-22)e^{-k} = 22 + 68e^{-kt},$$

*we then find $k$ using the given information, $T(10) = 85^oC$. We have*

$$\begin{aligned} 85 &= T(10) \\ &= 22 + 68e^{-10k} \\ 63 &= 68e^{-10k} \\ e^{-10k} &= \frac{63}{68} \approx 0.926 \\ -10k &= \ln 0.926 \\ k &= -\frac{\ln 0.926}{10} = 0.00764. \end{aligned} \tag{1.45}$$

*This gives the equation for this model as*

$$T(t) = 22 + 68e^{-0.00764t}.$$

*Now we can answer the question. What is $T(30)$?*

$$T(30) = 22 + 68e^{-0.00764(30)} = 76^oC.$$

### 1.3.3 Terminal Velocity

Now let's return to free fall. What if there is air resistance? We first need to model the air resistance. As an object falls faster and faster, the drag force becomes greater. So, this resistive force is a function of the velocity. There are a couple of standard models that people use to test this. The idea is to write $F = ma$ in the form

$$m\ddot{y} = -mg + f(v), \tag{1.46}$$

where $f(v)$ gives the resistive force and $mg$ is the weight. Recall that this applies to free fall near the Earth's surface. Also, for it to be resistive, $f(v)$ should oppose the motion. If the body is falling, then $f(v)$ should be positive. If it is rising, then $f(v)$ would have to be negative to indicate the opposition to the motion.

One common determination derives from the drag force on an object moving through a fluid. This force is given by

$$f(v) = \frac{1}{2}CA\rho v^2, \tag{1.47}$$

where $C$ is the drag coefficient, $A$ is the cross sectional area and $\rho$ is the fluid density. For laminar flow the drag coefficient is constant.

Unless you are into aerodynamics, you do not need to get into the details of the constants. So, it is best to absorb all of the constants into one to simplify the computation. So, we will write $f(v) = bv^2$. The differential equation including drag can then be rewritten as

$$\dot{v} = kv^2 - g, \tag{1.48}$$

where $k = b/m$. Note that this is a first order equation for $v(t)$. It is separable too!

Formally, we can separate the variables and integrate over time to obtain

$$t + K = \int^v \frac{dz}{kz^2 - g}. \tag{1.49}$$

(Note: We used an integration constant of $K$ since $C$ is the drag coefficient in this problem.) If we can do the integral, then we have a solution for $v$. In fact, we can do this integral. You need to recall another common method of integration, which we have not reviewed yet. Do you remember Partial Fraction Decomposition? It involves factoring the denominator in the integral. In the simplest case there are two linear factors in the denominator and the integral is rewritten:

This is the first use of Partial Fraction Decomposition. We will explore this method further in the section on Laplace Transforms.

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{b-a} \int \left[ \frac{1}{x-a} - \frac{1}{x-b} \right] dx \tag{1.50}$$

The new integral now has two terms which can be readily integrated.

In order to factor the denominator in the current problem, we first have to rewrite the constants. We let $\alpha^2 = g/k$ and write the integrand as

$$\frac{1}{kz^2 - g} = \frac{1}{k} \frac{1}{z^2 - \alpha^2}. \tag{1.51}$$

Now we use a partial fraction decomposition to obtain

$$\frac{1}{kz^2 - g} = \frac{1}{2\alpha k} \left[ \frac{1}{z - \alpha} - \frac{1}{z + \alpha} \right]. \tag{1.52}$$

Now, the integrand can be easily integrated giving

$$t + K = \frac{1}{2\alpha k} \ln \left| \frac{v - \alpha}{v + \alpha} \right|. \tag{1.53}$$

Solving for $v$, we have

$$v(t) = \frac{1 - Be^{2\alpha kt}}{1 + Be^{2\alpha kt}} \alpha, \tag{1.54}$$

where $B \equiv e^K$. $B$ can be determined using the initial velocity.

There are other forms for the solution in terms of a tanh function, which the reader can determine as an exercise. One important conclusion is that for large times, the ratio in the solution approaches $-1$. Thus, $v \to -\alpha = -\sqrt{\frac{g}{k}}$ as $t \to \infty$. This means that the falling object will reach a constant terminal velocity.

As a simple computation, we can determine the terminal velocity. We will take an 80 kg skydiver with a cross sectional area of about 0.093 $m^2$. (The skydiver is falling head first.) Assume that the air density is a constant 1.2 kg/$m^3$ and the drag coefficient is $C = 2.0$. We first note that

$$v_{\text{terminal}} = -\sqrt{\frac{g}{k}} = -\sqrt{\frac{2mg}{CA\rho}}.$$

So,

$$v_{\text{terminal}} = -\sqrt{\frac{2(70)(9.8)}{(2.0)(0.093)(1.2)}} = 78m/s.$$

This is about 175 mph, which is slightly higher than the actual terminal velocity of a sky diver with arms and feet fully extended. One would need a more accurate determination of $C$ and $A$ for a more realistic answer. Also, the air density varies along the way.

### *1.3.4 Orthogonal Trajectories of Curves*

There are many problems from geometry which have lead to the study of differential equations. One such problem is the construction of orthogonal trajectories. Give a a family of curves, $y_1(x;a)$, we seek another family of curves $y_2(x;c)$ such that the second family of curves are perpendicular the to given family. This means that the tangents of two intersecting curves at the point of intersection are perpendicular to each other. The slopes of the tangent lines are given by the derivatives $y_1'(x)$ and $y_2'(x)$. We recall from elementary geometry that the slopes of two perpendicular lines are related by

$$y_2'(x) = -\frac{1}{y_1'(x)}.$$

**Example 1.14.** *Find a family of orthogonal trajectories to the family of parabolae $y_1(x;a) = ax^2$.*

*We note that the new collection of curves has to satisfy the equation*

$$y_2'(x) = -\frac{1}{y_1'(x)} = -\frac{1}{2ax}.$$

*Before solving for $y_2(x)$, we need to eliminate the parameter $a$. From the give function, we have that $a = \frac{y}{x^2}$. Inserting this into the equation for $y_2'$, we have*

$$y'(x) = -\frac{1}{2ax} = -\frac{x}{2y}.$$

*Thus, to find $y_2(x)$, we have to solve the differential equation*

$$2yy' + x = 0.$$

*Noting that $(y^2)' = 2yy'$ and $(\frac{1}{2}x^2)' = x$,, this (exact) equation can be written as*

$$\frac{d}{dx}\left(y^2 + \frac{1}{2}x^2\right) = 0.$$

*Integrating, we find the family of solutions,*

$$y^2 + \frac{1}{2}x^2 = k.$$

*In Figure 1.6 we plot both families of orthogonal curves.*

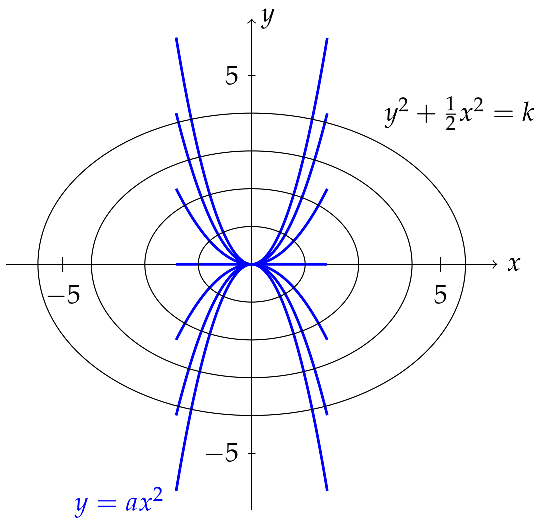



Figure 1.6: Plot of orthogonal families of curves, $y = ax^2$ and $y^2 + \frac{1}{2}x^2 = k$.

### 1.3.5 *Pursuit Curves*

ANOTHER APPLICATION THAT IS INTERESTING IS TO FIND the path that a body traces out as it moves towards a fixed point or another moving body. Such curses are know as pursuit curves. These could model aircraft or submarines following targets, or predators following prey. We demonstrate this with an example.

**Example 1.15.** *A hawk at point $(x, y)$ sees a sparrow traveling at speed $v$ along a straight line. The hawk flies towards the sparraw at constant speed $w$ but always in a direction along line of sight between their positions. If the hawk starts out at the point $(a, 0)$ at $t = 0$, when the sparrow is at $(0, 0)$, then what is the path the hawk needs to follow? Will the hawk catch the sparrow? The situation is shown in Figure 1.7. We pick the path of the sparrow to be along the $y-$axis. Therefore, the sparrow is at position $(0, vt)$.*

*First we need the equation of the line of sight between the points $(x, y)$ and $(0, vt)$. Considering that the slope of the line is the same as the slope of the tangent to the path, $y = y(x)$, we have*

$$y' = \frac{y - vt}{x}.$$

*The hawk is moving at a constant speed, $w$. Since the speed is related to the time through the distance the hawk travels. we need to find the arclength of the path between $(a, 0)$ and $(x, y)$. This is given by*

$$L = \int ds = \int_x^a \sqrt{1 + [y'(x)]^2}\, dx.$$

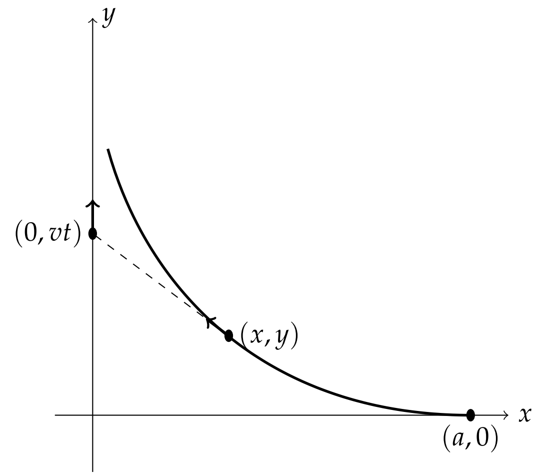



Figure 1.7: A hawk at point $(x, y)$ sees a sparrow at point $(0, vt)$ and always follows the straight line between these points.

*The distance is related to the speed, $w$, and the time, $t$, by $L = wt$.*

*Eliminating the time using $y' = \frac{y - vt}{x}$, we have*

$$\int_x^a \sqrt{1 + [y'(x)]^2}\,dx = \frac{w}{v}(y - xy').$$

*Furthermore, we can differentiate this result with respect to $x$ to get rid of the integral,*

$$\sqrt{1 + [y'(x)]^2} = \frac{w}{v} x y''.$$

*Even though this is a second order differential equation for $y(x)$, it is a first order separable equation in the speed function $z(x) = y'(x)$. Namely,*

$$xz' = \frac{v}{x}\sqrt{1 + z^2}.$$

*Separating variables, we find*

$$\int \frac{dz}{\sqrt{1 + z^2}} = \ln(z + \sqrt{1 + z^2}) \int \frac{dx}{x}.$$

*The integrals can be computed using standard methods from calculus. We can easily integrate the right had side,*

$$\int \frac{dx}{x} = \ln|x| + c_1.$$

*The left hand side takes a little extra work, or looking the value up in Tables or using a CAS package. Recall a trigonometric substitution is in order. [See the Appendix.] We let $z = \tan\theta$. Then $dz = \sec^2\theta\,d\theta$. The methods proceeds as follows:*

$$\begin{aligned}
\int \frac{dz}{\sqrt{1 + z^2}} &= \int \frac{sec^2\theta}{\sqrt{1 + \tan^2\theta}}\,d\theta \\
&= \int sec\theta\,d\theta \\
&= \ln(\tan\theta + \sec\theta) + c_2 \\
&= \ln(z + \sqrt{1 + z^2}) + c_2. \qquad (1.55)
\end{aligned}$$

*Putting these together, we have for $x > 0$,*

$$\ln(z + \sqrt{1+z^2}) = \frac{v}{w} \ln x + C.$$

*Using the initial condition $z = y' = 0$ and $x = a$ at $t = 0$,*

$$0 = \frac{v}{w} \ln a + C,$$

*or $C = -\frac{v}{w} \ln a$.*

*Using this value for $c$, we find*

$$\begin{aligned}
\ln(z + \sqrt{1+z^2}) &= \frac{v}{w} \ln x - \frac{v}{w} \ln a \\
\ln(z + \sqrt{1+z^2}) &= \frac{v}{w} \ln \frac{x}{a} \\
\ln(z + \sqrt{1+z^2}) &= \ln \left(\frac{x}{a}\right)^{\frac{v}{w}} \\
z + \sqrt{1+z^2} &= \left(\frac{x}{a}\right)^{\frac{v}{w}}.
\end{aligned} \tag{1.56}$$

*We can solve for $z = y'$, to find*

$$y' = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{\frac{v}{w}} - \left(\frac{x}{a}\right)^{-\frac{v}{w}}\right]$$

*Integrating,*

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + k.$$

*The integration constant, $k$, can be found knowing $y(a) = 0$. This gives*

$$\begin{aligned}
0 &= \frac{a}{2}\left[\frac{1}{1+\frac{v}{w}} - \frac{1}{1-\frac{v}{w}}\right] + k \\
k &= \frac{a}{2}\left[\frac{1}{1-\frac{v}{w}} - \frac{1}{1+\frac{v}{w}}\right] \\
&= \frac{avw}{w^2 - v^2}.
\end{aligned} \tag{1.57}$$

*The full solution for the path is given by*

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + \frac{avw}{w^2 - v^2}.$$

*Can the hawk catch the sparrow? This would happen if there is a time when $y(0) = vt$. Inserting $x = 0$ into the solution, we have $y(0) = \frac{avw}{w^2-v^2} = vt$. This is possible if $w > v$.*

## *Problems*

**1.** Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\dfrac{dy}{dx} = \dfrac{e^x}{2y}$.

b. $\dfrac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

c. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$.

d. $xy' = y(1-2y), \quad y(1) = 2$.

e. $y' - (\sin x)y = \sin x$.

f. $xy' - 2y = x^2, y(1) = 1$.

g. $\dfrac{ds}{dt} + 2s = st^2, \quad , s(0) = 1$.

h. $x' - 2x = te^{2t}$.

i. $\dfrac{dy}{dx} + y = \sin x$, $y(0) = 0$.

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3$, $y(1) = 4$.

**2.** For the following determine if the differential equation is exact. If it is not exact, find the integrating factor. Integrate the equations to obtain solutions.

a. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$.

b. $(x + y^2)\,dx - 2xy\,dy = 0$.

c. $(\sin xy + xy\cos xy)\,dx + x^2\cos xy\,dy = 0$.

d. $(x^2 + y)\,dx - x\,dy = 0$.

e. $(2xy^2 - 3y^3)\,dx + (7 - 3xy^2)\,dy = 0$.

**3.** Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

**4.** A ball is thrown upward with an initial velocity of 49 m/s from 539 m high. How high does the ball get and how long does in take before it hits the ground? [Use results from the simple free fall problem, $y'' = -g$.]

**5.** Consider the case of free fall with a damping force proportional to the velocity, $f_D = \pm kv$ with $k = 0.1$ kg/s.

a. Using the correct sign, consider a 50 kg mass falling from rest at a height of 100m. Find the velocity as a function of time. Does the mass reach terminal velocity?

b. Let the mass be thrown upward from the ground with an initial speed of 50 m/s. Find the velocity as a function of time as it travels upward and then falls to the ground. How high does the mass get? What is its speed when it returns to the ground?

**6.** An piece of a satellite falls to the ground from a height of 10,000 m. Ignoring air resistance, find the height as a function of time. [Hint: For free fall from large distances,

$$\ddot{h} = -\frac{GM}{(R+h)^2}.$$

Multiplying both sides by $\dot{h}$, show that

$$\frac{d}{dt}\left(\frac{1}{2}\dot{h}^2\right) = \frac{d}{dt}\left(\frac{GM}{R+h}\right).$$

Integrate and solve for $\dot{h}$. Further integrating gives $h(t)$.]

**7.** The problem of growth and decay is stated as follows: The rate of change of a quantity is proportional to the quantity. The differential equation for such a problem is

$$\frac{dy}{dt} = \pm ky.$$

The solution of this growth and decay problem is $y(t) = y_0 e^{\pm kt}$. Use this solution to answer the following questions if forty percent of a radioactive substance disappears in 100 years.

a. What is the half-life of the substance?

b. After how many years will 90% be gone?

**8.** Uranium 237 has a half-life of 6.78 days. If there are 10.0 grams of U-237 now, then how much will be left after two weeks?

**9.** The cells of a particular bacteria culture divide every three and a half hours. If there are initially 250 cells, how many will there be after ten hours?

**10.** The population of a city has doubled in 25 years. How many years will it take for the population to triple?

**11.** Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}$.

b. $y = 2xy' + \ln y'$.

c. $y' + 2xy = 2xy^2$.

d. $y' + 2xy = y^2 e^{x^2}$.

**12.** Find the general solution of the Riccati equation given the particular solution.

a. $xy' - y^2 + (2x+1)y = x^2 + 2x$, $y_1(x) = x$.

b. $y'e^{-x} + y^2 - 2ye^x = 1 - e^{2x}$, $y_1(x) = e^x$.

A function $F(x, y)$ is said to be homogeneous of degree $k$ if $F(tx, ty) = t^k F(x, y)$.

**13.** The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in this chapter. The function on the righthand side is a homogeneous function of degree zero. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ which can be solved. Use this substitution to solve the initial value problem for $y(x)$.

**14.** If $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, then $M/N$ can be written as a function of $y/x$. This suggests that a substitution of $y(x) = xz(x)$ into $M(x, y)\,dx + N(x, y)\,dy$ might simplify the equation. For the following problems use this method to find the family of solutions.

a. $(x^2 - xy + y^2)\,dx - xy\,dy = 0$.

b. $xy\,dx - (x^2 + y^2)\,dy = 0$.

c. $(x^2 + 2xy - 4y^2)\,dx - (x^2 - 8xy - 4y^2)\,dy = 0$.

**15.** Find the family of orthogonal curves to the given family of curves.

a. $y = ax$

b. $y = ax^2$.

c. $x^2 + y^2 = 2ax$.

**16.** The temperature inside your house is $70^o$F and it is $30^o$F outside. At 1:00 A.M. the furnace breaks down. At 3:00 A.M. the temperature in the house has dropped to $50^o$F. Assuming the outside temperature is constant and that Newton's Law of Cooling applies, determine when the temperature inside your house reaches $40^o$F.

**17.** A body is discovered during a murder investigation at 8:00 P.M. and the temperature of the body is $70^o$F. Two hours later the body temperature has dropped to $60^o$F in a room that is at $50^o$F. Assuming that Newton's Law of Cooling applies and the body temperature of the person was $98.6^o$F at the time of death, determine when the murder occurred.

**18.** Newton's Law of Cooling states that the rate of heat loss of an object is proportional to the temperature gradient, or

$$\frac{dQ}{dt} = hA\Delta T,$$

where $Q$ is the thermal energy, $h$ is the heat transfer coefficient, $A$ is the surface area of the body, and $\Delta T = T - T_a$. If $Q = CT$, where $C$ is the heat capacity, then we recover Equation (1.44) with $k = hA/C$.

However, there are modifications which include convection or radiation. Solve the following models and compare the solution behaviors.

a. Newton $T' = -k(T - T_a)$

b. Dulong-Petit $T' = -k(T - T_a)^{5/4}$

c. Newton-Stefan $T' = -k(T - T_a) - \epsilon\sigma(T^4 - T_a^4) \approx -k(T - T_a) - b(T - T_a)^2$.

# 2
# *Second Order Differential Equations*

*"Either mathematics is too big for the human mind or the human mind is more than a machine." - Kurt Gödel (1906-1978)*

## 2.1 *The Simple Harmonic Oscillator*

The next physical problem of interest is that of simple harmonic motion. Such motion comes up in many places in physics and provides a generic first approximation to models of oscillatory motion. This is the beginning of a major thread running throughout this course. You have seen simple harmonic motion in your introductory physics class. We will review SHM (or SHO in some texts) by looking at springs and pendula (the plural of pendulum). We will use this as our jumping board into second order differential equations and later see how such oscillatory motion occurs in AC circuits.

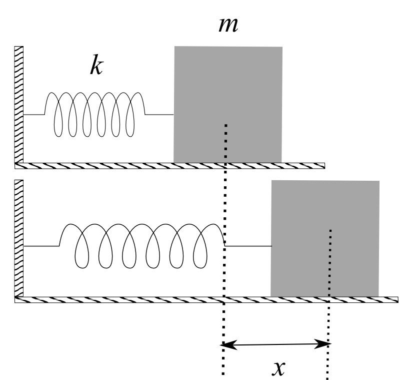


Figure 2.1: Spring-Mass system.

### 2.1.1 *Mass-Spring Systems*

We begin with the case of a single block on a spring as shown in Figure 2.1. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant. Here $x$ is the elongation, or displacement of the spring from equilibrium. When the displacement is positive, the spring force is negative and when the displacement is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface. A similar model can be provided for vertically oriented springs. However, you need to account for gravity to determine the location of equilibrium. Otherwise, the oscillatory motion about equilibrium is modeled the same.

From Newton's Second Law, $F = m\ddot{x}$, we obtain the equation for the motion of the mass on the spring:

$$m\ddot{x} + kx = 0. \tag{2.1}$$

Dividing by the mass, this equation can be written in the form

$$\ddot{x} + \omega^2 x = 0, \tag{2.2}$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

This is the generic differential equation for simple harmonic motion.

We will later derive solutions of such equations in a methodical way. For now we note that two solutions of this equation are given by

$$\begin{aligned} x(t) &= A\cos\omega t, \\ x(t) &= A\sin\omega t, \end{aligned} \tag{2.3}$$

where $\omega$ is the angular frequency, measured in rad/s, and $A$ is called the amplitude of the oscillation. .

The angular frequency is related to the frequency by

$$\omega = 2\pi f,$$

where $f$ is measured in cycles per second, or Hertz. Furthermore, this is related to the period of oscillation, the time it takes the mass to go through one cycle:

$$T = 1/f.$$

### 2.1.2 *The Simple Pendulum*

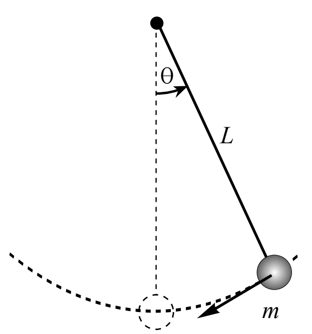



Figure 2.2: A simple pendulum consists of a point mass $m$ attached to a string of length $L$. It is released from an angle $\theta_0$.

THE SIMPLE PENDULUM consists of a point mass $m$ hanging on a string of length $L$ from some support. [See Figure 2.2.] One pulls the mass back to some starting angle, $\theta_0$, and releases it. The goal is to find the angular position as a function of time.

There are a couple of possible derivations. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque, $\tau = I\alpha$. We will use the former only to limit the amount of physics background needed.

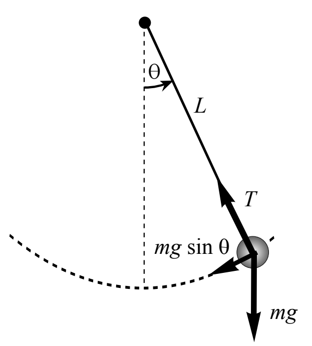



Figure 2.3: There are two forces acting on the mass, the weight $mg$ and the tension $T$. The net force is found to be $F = mg\sin\theta$.

There are two forces acting on the point mass. The first is gravity. This points downward and has a magnitude of $mg$, where $g$ is the standard symbol for the acceleration due to gravity. The other force is the tension in the string. In Figure 2.3 these forces and their sum are shown. The magnitude of the sum is easily found as $F = mg\sin\theta$ using the addition of these two vectors.

Now, Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg\sin\theta.$$

Next, we need to relate $x$ and $\theta$. $x$ is the distance traveled, which is the length of the arc traced out by the point mass. The arclength is related to the angle, provided the angle is measure in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

$$mL\ddot{\theta} = -mg\sin\theta.$$

Canceling the masses, this then gives us the nonlinear pendulum equation

Linear and nonlinear pendulum equation.

$$L\ddot{\theta} + g\sin\theta = 0. \tag{2.4}$$

We note that this equation is of the same form as the mass-spring system. We define $\omega = \sqrt{g/L}$ and obtain the equation for simple harmonic motion,

$$\ddot{\theta} + \omega^2\theta = 0.$$

The equation for a compound pendulum takes a similar form. We start with the rotational form of Newton's second law $\tau = I\alpha$. Noting that the torque due to gravity acts at the center of mass position $\ell$, the torque is given by $\tau = -mg\ell\sin\theta$. Since $\alpha = \ddot{\theta}$, we have $I\ddot{\theta} = -mg\ell\sin\theta$. Then, for small angles $\ddot{\theta} + \omega^2\theta = 0$, where $\omega = \frac{mg\ell}{I}$. For a simple pendulum, we let $\ell = L$ and $I = mL^2$, and obtain $\omega = \sqrt{g/L}$.

There are several variations of Equation (2.4) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin\theta \approx \theta$. Under this approximation (2.4) becomes

$$L\ddot{\theta} + g\theta = 0. \tag{2.5}$$

## 2.2 *Second Order Linear Differential Equations*

IN THE LAST SECTION WE SAW how second order differential equations naturally appear in the derivations for simple oscillating systems. In this section we will look at more general second order linear differential equations.

Second order differential equations are typically harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.6}$$

One can rewrite this equation using operator terminology. Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (2.6) becomes

$$Ly = f. \tag{2.7}$$

The solutions of linear differential equations are found by making use of the linearity of $L$. Namely, we consider the *vector space*[1] consisting of real-valued functions over some domain. Let $f$ and $g$ be vectors in this function space. $L$ is a *linear operator* if for two vectors $f$ and $g$ and scalar $a$, we have that

a. $L(f + g) = Lf + Lg$

b. $L(af) = aLf$.

[1] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

One typically solves (2.6) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (2.6) is simply given as $y = y_h + y_p$. This is true because of the linearity of $L$. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{2.8}$$

There are methods for finding a particular solution of a nonhomogeneous differential equation. These methods range from pure guessing, the Method of Undetermined Coefficients, the Method of Variation of Parameters, or Green's functions. We will review these methods later in the chapter.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, many now famous mathematicians and physicists have studied a variety of second order linear equations and they have saved us the trouble of finding solutions to the differential equations that often appear in applications. We will encounter many of these in the following chapters. We will first begin with some simple homogeneous linear differential equations.

Linearity is also useful in producing the general solution of a homogeneous linear differential equation. If $y_1$ and $y_2$ are solutions of the homogeneous equation, then the *linear combination* $y = c_1 y_1 + c_2 y_2$ is also a solution of the homogeneous equation. In fact, if $y_1$ and $y_2$ are *linearly independent*,[2] then $y = c_1 y_1 + c_2 y_2$ is the general solution of the homogeneous problem.

[2] A set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1 y_1(x) + \ldots + c_n y_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$.

For $n = 2$, $c_1 y_1(x) + c_2 y_2(x) = 0$. If $y_1$ and $y_2$ are linearly dependent, then the coefficients are not zero and $y_2(x) = -\frac{c_1}{c_2} y_1(x)$ and is a multiple of $y_1(x)$.

Linear independence can also be established by looking at the Wronskian of the solutions. For a second order differential equation the Wronskian is defined as

$$W(y_1, y_2) = y_1(x) y_2'(x) - y_1'(x) y_2(x). \tag{2.9}$$

The solutions are linearly independent if the Wronskian is not zero.

### 2.2.1 Constant Coefficient Equations

The simplest second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{2.10}$$

where $a$, $b$, and $c$ are constants.

Solutions to (2.10) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (2.10) leads to the characteristic equation

$$ar^2 + br + c = 0. \tag{2.11}$$

Namely, we compute the derivatives of $y(x) = e^{rx}$, to get $y(x) = re^{rx}$, and $y(x) = r^2e^{rx}$. Inserting into (2.10), we have

$$0 = ay''(x) + by'(x) + cy(x) = (ar^2 + br + c)e^{rx}.$$

Since the exponential is never zero, we find that $ar^2 + br + c = 0$.

The characteristic equation for $ay'' + by' + cy = 0$ is $ar^2 + br + c = 0$. Solutions of this quadratic equation lead to solutions of the differential equation.

The roots of this equation, $r_1$, $r_2$, in turn lead to three types of solutions depending upon the nature of the roots. In general, we have two linearly independent solutions, $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$, and the general solution is given by a linear combination of these solutions,

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Two real, distinct roots, $r_1$ and $r_2$, give solutions of the form

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

For two real distinct roots, we are done. However, when the roots are real, but equal, or complex conjugate roots, we need to do a little more work to obtain usable solutions.

**Example 2.1.** $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0.$

*The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:*

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

*Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.*

*One also needs to evaluate the first derivative*

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

*in order to attempt to satisfy the initial conditions. Evaluating $y$ and $y'$ at $x = 0$ yields*

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{2.12}$$

*These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.*

In the case when there is a repeated real root, one has only one solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second linearly independent solution? Since the solutions should be independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. (This process is called the Method of Reduction of Order. See Section 2.5.3)

Repeated roots, $r_1 = r_2 = r$, give solutions of the form

$$y(x) = (c_1 + c_2 x)e^{rx}.$$

For constant coefficient second order equations, we can write the equation as

$$(D-r)^2 y = 0,$$

where $D = \frac{d}{dx}$. We now insert $y_2(x) = v(x)e^{rx}$ into this equation. First we compute

For more on the Method of Reduction of Order, see Section 2.5.3.

$$(D-r)ve^{rx} = v'e^{rx}.$$

Then,

$$0 = (D-r)^2 ve^{rx} = (D-r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, then $v''(x)e^{rx} = 0$ for all $x$. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax+b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$. The general solution is then

$$y(x) = c_1 e^{rx} + c_2 xe^{rx}.$$

**Example 2.2.** $y'' + 6y' + 9y = 0$.

*In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. From the above discussion, we easily find the solution $y(x) = (c_1 + c_2 x)e^{-3x}$.*

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula (See Chapter 6 for more on complex variables)

$$e^{i\beta x} = \cos\beta x + i\sin\beta x. \tag{2.13}$$

Then, the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] \\ &\equiv e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned} \tag{2.14}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

Complex roots, $r = \alpha \pm i\beta$, give solutions of the form

$$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

**Example 2.3.** $y'' + 4y = 0$.

*The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$, and the general solution consists purely of sinusoidal functions, $y(x) = c_1\cos(2x) + c_2\sin(2x)$, since $\alpha = 0$ and $\beta = 2$.*

**Example 2.4.** $y'' + 2y' + 4y = 0$.

*The characteristic equation in this case is* $r^2 + 2r + 4 = 0$. *The roots are complex,* $r = -1 \pm \sqrt{3}i$ *and the general solution can be written as*

$$y(x) = \left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right] e^{-x}.$$

**Example 2.5.** $y'' + 4y = \sin x$.

*This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 2.3. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.*

*The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of* $y_p(x) = A \sin x$ *and determine what* $A$ *needs to be. Inserting this guess into the differential equation gives* $(-A + 4A) \sin x = \sin x$. *So, we see that* $A = 1/3$ *works. The general solution of the nonhomogeneous problem is therefore* $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x$.

The three cases for constant coefficient linear second order differential equations are summarized below.

**Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs**

1. **Real, distinct roots** $r_1, r_2$. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as $xe^{rx}$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$.

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i \sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$.

As we have seen, one of the most important applications of such equations is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass $m$ on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$m\ddot{x} + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are simple sine and cosine functions, leading to simple harmonic motion.

## 2.3 LRC Circuits

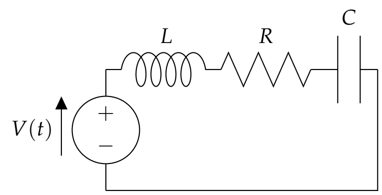


Figure 2.4: Series LRC Circuit.

Another typical problem often encountered in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 2.4. The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, store magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V = IR$.
2. Capacitor: $V = \frac{q}{C}$.
3. Inductor: $V = L\frac{dI}{dt}$.

Furthermore, we need to define the current as $I = \frac{dq}{dt}$. where $q$ is the charge in the circuit. Adding these potential drops, we set them equal to the voltage supplied by the voltage source, $V(t)$. Thus, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

Since both $q$ and $I$ are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

This is a second order equation for $q(t)$.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.5. This is not a problem that can be covered in the first year physics course. One can set up a system of second order equations and proceed to solve them. We will see how to solve such problems in the next chapter.

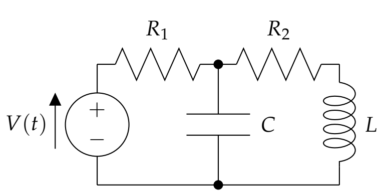


Figure 2.5: Parallel LRC Circuit.

### 2.3.1 Special Cases

In this section we will look at special cases that arise for the series LRC circuit equation. These include $RC$ circuits, solvable by first order methods and $LC$ circuits, leading to oscillatory behavior.

**Case I.** RC Circuits

We first consider the case of an RC circuit in which there is no inductor. Also, we will consider what happens when one charges a capacitor with a DC battery ($V(t) = V_0$) and when one discharges a charged capacitor ($V(t) = 0$) as shown in Figures 2.6 and 2.9.

For charging a capacitor, we have the initial value problem

Charging a capacitor.

$$R\frac{dq}{dt} + \frac{q}{C} = V_0, \quad q(0) = 0. \tag{2.15}$$

This equation is an example of a linear first order equation for $q(t)$. However, we can also rewrite it and solve it as a separable equation, since $V_0$ is a constant. We will do the former only as another example of finding the integrating factor.

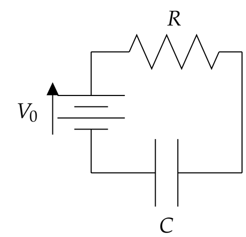


Figure 2.6: RC Circuit for charging.

We first write the equation in standard form:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}. \tag{2.16}$$

The integrating factor is then

$$\mu(t) = e^{\int \frac{dt}{RC}} = e^{t/RC}.$$

Thus,

$$\frac{d}{dt}\left(qe^{t/RC}\right) = \frac{V_0}{R}e^{t/RC}. \tag{2.17}$$

Integrating, we have

$$qe^{t/RC} = \frac{V_0}{R}\int e^{t/RC}\,dt = CV_0e^{t/RC} + K. \tag{2.18}$$

Note that we introduced the integration constant, $K$. Now divide out the exponential to get the general solution:

$$q = CV_0 + Ke^{-t/RC}. \tag{2.19}$$

(If we had forgotten the $K$, we would not have gotten a correct solution for the differential equation.)

Next, we use the initial condition to get the particular solution. Namely, setting $t = 0$, we have that

$$0 = q(0) = CV_0 + K.$$

So, $K = -CV_0$. Inserting this into the solution, we have

$$q(t) = CV_0(1 - e^{-t/RC}). \tag{2.20}$$

Now we can study the behavior of this solution. For large times the second term goes to zero. Thus, the capacitor charges up, asymptotically, to the final value of $q_0 = CV_0$. This is what we expect, because the current is no longer flowing over $R$ and this just gives the relation between the potential difference across the capacitor plates when a charge of $q_0$ is established on the plates.

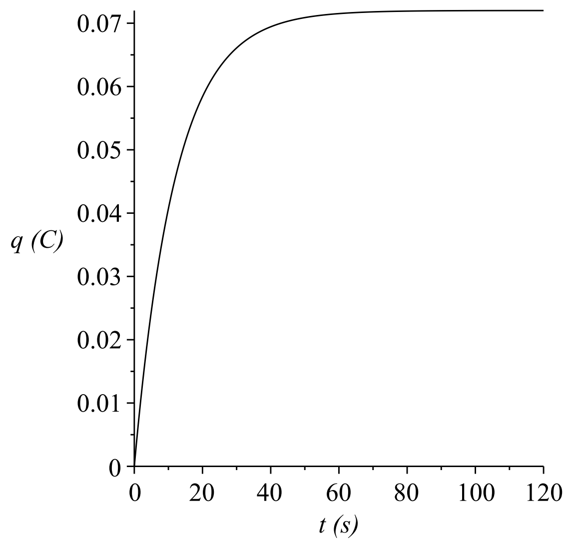



Figure 2.7: The charge as a function of time for a charging capacitor with $R = 2.00$ k$\Omega$, $C = 6.00$ mF, and $V_0 = 12$ V.

Let's put in some values for the parameters. We let $R = 2.00$ k$\Omega$, $C = 6.00$ mF, and $V_0 = 12$ V. A plot of the solution is given in Figure 2.7. We see that the charge builds up to the value of $CV_0 = 0.072$ C. If we use a smaller resistance, $R = 200\ \Omega$, we see in Figure 2.8 that the capacitor charges to the same value, but much faster.

Time constant, $\tau = RC$.

The rate at which a capacitor charges, or discharges, is governed by the time constant, $\tau = RC$. This is the constant factor in the exponential. The larger it is, the slower the exponential term decays. If we set $t = \tau$, we find that

$$q(\tau) = CV_0(1 - e^{-1}) = (1 - 0.3678794412\ldots)q_0 \approx 0.63q_0.$$

Thus, at time $t = \tau$, the capacitor has almost charged to two thirds of its final value. For the first set of parameters, $\tau = 12$s. For the second set, $\tau = 1.2$s.

Discharging a capacitor.

Now, let's assume the capacitor is charged with charge $\pm q_0$ on its plates. If we disconnect the battery and reconnect the wires to complete the circuit as shown in Figure 2.9, the charge will then move off the plates, discharging the capacitor. The relevant form of the initial value problem becomes

$$R\frac{dq}{dt} + \frac{q}{C} = 0, \quad q(0) = q_0. \tag{2.21}$$

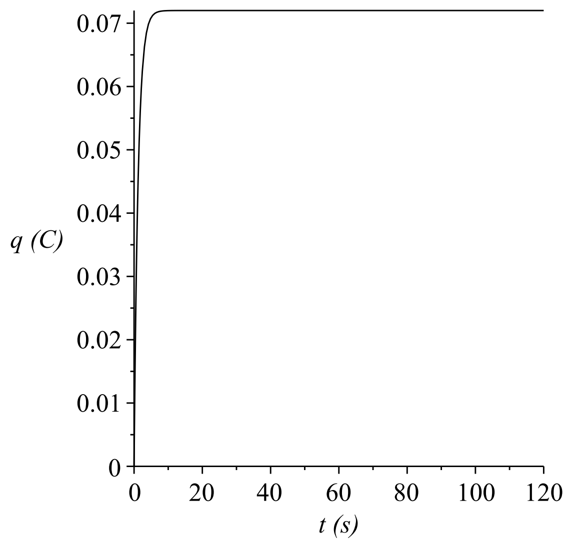



Figure 2.8: The charge as a function of time for a charging capacitor with $R = 200\ \Omega$, $C = 6.00$ mF, and $V_0 = 12$ V.

This equation is simpler to solve. Rearranging, we have

$$\frac{dq}{dt} = -\frac{q}{RC}. \tag{2.22}$$

This is a simple exponential decay problem, which one can solve using separation of variables. However, by now you should know how to immediately write down the solution to such problems of the form $y' = ky$. The solution is

$$q(t) = q_0 e^{-t/\tau}, \quad \tau = RC.$$

We see that the charge decays exponentially. In principle, the capacitor never fully discharges. That is why you are often instructed to place a shunt across a discharged capacitor to fully discharge it.

In Figure 2.10 we show the discharging of the two previous RC circuits. Once again, $\tau = RC$ determines the behavior. At $t = \tau$ we have

$$q(\tau) = q_0 e^{-1} = (0.3678794412\ldots)q_0 \approx 0.37q_0.$$

So, at this time the capacitor only has about a third of its original value.

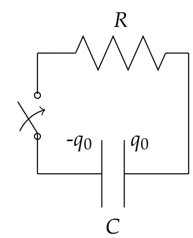



Figure 2.9: RC Circuit for discharging.

**Case II.** LC Circuits

LC Oscillators.

Another simple result comes from studying $LC$ circuits. We will now connect a charged capacitor to an inductor as shown in Figure 2.11. In this case, we consider the initial value problem

$$L\ddot{q} + \frac{1}{C}q = 0, \quad q(0) = q_0, \dot{q}(0) = I(0) = 0. \tag{2.23}$$

Dividing out the inductance, we have

$$\ddot{q} + \frac{1}{LC}q = 0. \tag{2.24}$$

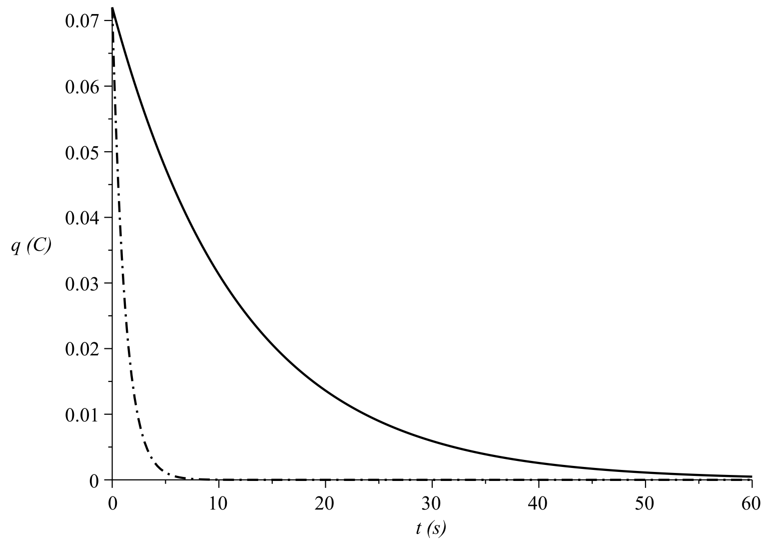



Figure 2.10: The charge as a function of time for a discharging capacitor with $R = 2.00$ k$\Omega$ (solid) or $R = 200$ $\Omega$ (dashed), and $C = 6.00$ mF, and $q_0 = 0.072$ C.

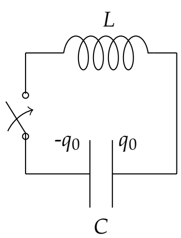



Figure 2.11: An LC circuit.

This equation is a second order, constant coefficient equation. It is of the same form as the ones for simple harmonic motion of a mass on a spring or the linear pendulum. So, we expect oscillatory behavior. The characteristic equation is

$$r^2 + \frac{1}{LC} = 0.$$

The solutions are

$$r_{1,2} = \pm \frac{i}{\sqrt{LC}}.$$

Thus, the solution of (2.24) is of the form

$$q(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \omega = (LC)^{-1/2}. \tag{2.25}$$

Inserting the initial conditions yields

$$q(t) = q_0 \cos(\omega t). \tag{2.26}$$

The oscillations that result are understandable. As the charge leaves the plates, the changing current induces a changing magnetic field in the inductor. The stored electrical energy in the capacitor changes to stored magnetic energy in the inductor. However, the process continues until the plates are charged with opposite polarity and then the process begins in reverse. The charged capacitor then discharges and the capacitor eventually returns to its original state and the whole system repeats this over and over.

The frequency of this simple harmonic motion is easily found. It is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \frac{1}{\sqrt{LC}}. \tag{2.27}$$

This is called the tuning frequency because of its role in tuning circuits.

**Example 2.6.** *Find the resonant frequency for $C = 10\mu F$ and $L = 100mH$.*

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{(10\times 10^{-6})(100\times 10^{-3})}} = 160Hz.$$

Of course, this is an ideal situation. There is always resistance in the circuit, even if only a small amount from the wires. So, we really need to account for resistance, or even add a resistor. This leads to a slightly more complicated system in which damping will be present.

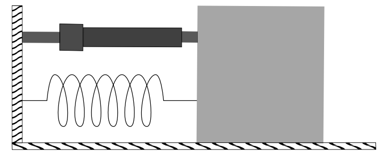


Figure 2.12: A spring-mass-damper system has a damper added which can absorb some of the energy of the oscillations and is modeled with a term proportional to the velocity.

## 2.4 *Damped Oscillations*

As we have indicated, simple harmonic motion is an ideal situation. In real systems we often have to contend with some energy loss in the system. This leads to the damping of the oscillations. This energy loss could be in the spring, in the way a pendulum is attached to its support, or in the resistance to the flow of current in an LC circuit. The simplest models of resistance are the addition of a term proportional to first derivative of the dependent variable. Thus, our three main examples with damping added look like:

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{2.28}$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \tag{2.29}$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0. \tag{2.30}$$

These are all examples of the general constant coefficient equation

$$ay''(x) + by'(x) + cy(x) = 0. \tag{2.31}$$

We have seen that solutions are obtained by looking at the characteristic equation $ar^2 + br + c = 0$. This leads to three different behaviors depending on the discriminant in the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.32}$$

We will consider the example of the damped spring. Then we have

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}. \tag{2.33}$$

For $b > 0$, there are three types of damping.

Damped oscillator cases: Overdamped, critically damped, and underdamped.

**I. Overdamped**, $b^2 > 4mk$

In this case we obtain two real root. Since this is Case I for constant coefficient equations, we have that

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

We note that $b^2 - 4mk < b^2$. Thus, the roots are both negative. So, both terms in the solution exponentially decay. The damping is so strong that there is no oscillation in the system.

**II. Critically Damped**, $b^2 = 4mk$

In this case we obtain one real root. This is Case II for constant coefficient equations and the solution is given by

$$x(t) = (c_1 + c_2 t)e^{rt},$$

where $r = -b/2m$. Once again, the solution decays exponentially. The damping is just strong enough to hinder any oscillation. If it were any weaker the discriminant would be negative and we would need the third case.

**III. Underdamped**, $b^2 < 4mk$

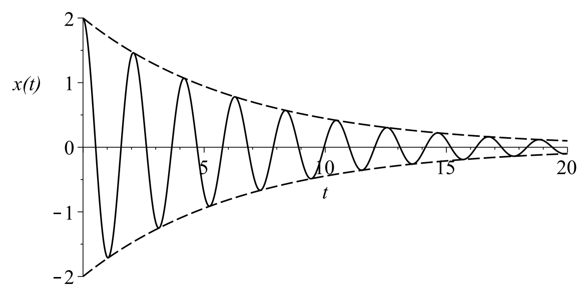



Figure 2.13: A plot of underdamped oscillation given by $x(t) = 2e^{0.15t}\cos 3t$. The dashed lines are given by $x(t) = \pm 2e^{0.15t}$, indicating the bounds on the amplitude of the motion.

In this case we have complex conjugate roots. We can write $\alpha = -b/2m$ and $\beta = \sqrt{4mk - b^2}/2m$. Then the solution is

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

These solutions exhibit oscillations due to the trigonometric functions, but we see that the amplitude may decay in time due the overall factor of $e^{\alpha t}$ when $\alpha < 0$. Consider the case that the initial conditions give $c_1 = A$ and $c_2 = 0$. (When is this?) Then, the solution, $x(t) = Ae^{\alpha t}\cos \beta t$, looks like the plot in Figure 2.13.

## 2.5 *Forced Systems*

All of the systems presented at the beginning of the last section exhibit the same general behavior when a damping term is present. An additional term can be added that might cause even more complicated behavior. In the case of LRC circuits, we have seen that the voltage source makes the system nonhomogeneous. It provides what is called a source term. Such terms can also arise in the mass-spring and pendulum systems. One can drive such systems by periodically pushing the mass, or having the entire system moved, or impacted by an outside force. Such systems are called forced, or driven.

Typical systems in physics can be modeled by nonhomogeneous second order equations. Thus, we want to find solutions of equations of the form

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.34}$$

As noted in Section 2.2, one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (2.6) is simply given as $y = y_h + y_p$.

So far, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous term to such equations we will need to find the particular solution to the nonhomogeneous equation.

We could guess a solution, but that is not usually possible without a little bit of experience. So, we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically developed the particular solution. We will come back to the Method of Variation of Parameters and we will also introduce the powerful machinery of Green's functions later in this section.

### 2.5.1 *Method of Undetermined Coefficients*

LET'S SOLVE A SIMPLE DIFFERENTIAL EQUATION highlighting how we can handle nonhomogeneous equations.

**Example 2.7.** *Consider the equation*

$$y'' + 2y' - 3y = 4. \tag{2.35}$$

*The first step is to determine the solution of the homogeneous equation. Thus, we solve*

$$y_h'' + 2y_h' - 3y_h = 0. \tag{2.36}$$

*The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution*

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

*The second step is to find a particular solution of (2.35). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to $x$, then we are left with a linear function after inserting $x$ and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.*

*Let's see. Inserting $y = A$ into (2.35), we obtain*

$$-3A = 4.$$

*Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.*

*Combining the two solutions, we have the general solution to the original nonhomogeneous equation (2.35). Namely,*

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

*Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine the arbitrary constants.*

**Example 2.8.** *What if we had a different source term? Consider the equation*

$$y'' + 2y' - 3y = 4x. \tag{2.37}$$

*The only thing that would change is the particular solution. So, we need a guess.*

*We know a constant function does not work by the last example. So, let's try* $y_p = Ax$*. Inserting this function into Equation (2.37), we obtain*

$$2A - 3Ax = 4x.$$

*Picking* $A = -4/3$ *would get rid of the* $x$ *terms, but will not cancel everything. We still have a constant left. So, we need something more general.*

*Let's try a linear function,* $y_p(x) = Ax + B$*. Then we get after substitution into (2.37)*

$$2A - 3(Ax + B) = 4x.$$

*Equating the coefficients of the different powers of* $x$ *on both sides, we find a system of equations for the undetermined coefficients:*

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{2.38}$$

*These are easily solved to obtain*

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{2.39}$$

*So, the particular solution is*

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

*This gives the general solution to the nonhomogeneous problem as*

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 2.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have the solution. This solution is then added to the general solution of the homogeneous differential equation.

| $f(x)$ | Guess |
|---|---|
| $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ | $A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$ |
| $ae^{bx}$ | $Ae^{bx}$ |
| $a\cos\omega x + b\sin\omega x$ | $A\cos\omega x + B\sin\omega x$ |

Table 2.1: Forms used in the Method of Undetermined Coefficients.

**Example 2.9.** *Solve*

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{2.40}$$

*According to the above, we would guess a solution of the form* $y_p = Ae^{-3x}$. *Inserting our guess, we find*

$$0 = 2e^{-3x}.$$

*Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?*

*The answer lies in the general solution of the homogeneous problem. Note that* $e^x$ *and* $e^{-3x}$ *are solutions to the homogeneous problem. So, a multiple of* $e^{-3x}$ *will not get us anywhere. It turns out that there is one further modification of the method. If the driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess* $y_p(x) = Axe^{-3x}$ *and differentiate this guess to obtain the derivatives* $y_p' = A(1-3x)e^{-3x}$ *and* $y_p'' = A(9x-6)e^{-3x}$.

*Inserting these derivatives into the differential equation, we obtain*

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x}.$$

*Comparing coefficients, we have*

$$-4A = 2.$$

*So,* $A = -1/2$ *and* $y_p(x) = -\frac{1}{2}xe^{-3x}$. *Thus, the solution to the problem is*

$$y(x) = \left(2 - \frac{1}{2}x\right)e^{-3x}.$$

**Modified Method of Undetermined Coefficients**

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by $x^k$, where $k$ is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

### 2.5.2 *Periodically Forced Oscillations*

A SPECIAL TYPE OF FORCING is periodic forcing. Realistic oscillations will dampen and eventually stop if left unattended. For example, mechanical clocks are driven by compound or torsional pendula and electric oscillators are often designed with the need to continue for long periods of time. However, they are not perpetual motion machines and will need a periodic injection of energy. This can be done systematically by adding periodic forcing. Another simple example is the motion of a child on a swing in the park. This simple damped pendulum system will naturally slow down to equilibrium (stopped) if left alone. However, if the child pumps energy into the swing at the right time, or if an adult pushes the child at the right time, then the amplitude of the swing can be increased.

There are other systems, such as airplane wings and long bridge spans, in which external driving forces might cause damage to the system. A well know example is the wind induced collapse of the Tacoma Narrows Bridge due to strong winds. Of course, if one is not careful, the child in the last example might get too much energy pumped into the system causing a similar failure of the desired motion.

The Tacoma Narrows Bridge opened in Washington State (U.S.) in mid 1940. However, in November of the same year the winds excited a transverse mode of vibration, which eventually (in a few hours) lead to large amplitude oscillations and then collapse.

While there are many types of forced systems, and some fairly complicated, we can easily get to the basic characteristics of forced oscillations by modifying the mass-spring system by adding an external, time-dependent, driving force. Such as system satisfies the equation

$$m\ddot{x} + b\dot{(x)} + kx = F(t), \tag{2.41}$$

where $m$ is the mass, $b$ is the damping constant, $k$ is the spring constant, and $F(t)$ is the driving force. If $F(t)$ is of simple form, then we can employ the Method of Undetermined Coefficients. Since the systems we have considered so far are similar, one could easily apply the following to pendula or circuits.

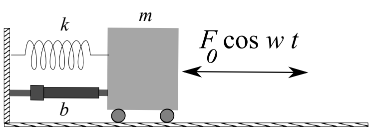



Figure 2.14: An external driving force is added to the spring-mass-damper system.

As the damping term only complicates the solution, we will consider the simpler case of undamped motion and assume that $b = 0$. Furthermore, we will introduce a sinusoidal driving force, $F(t) = F_0 \cos \omega t$ in order to study periodic forcing. This leads to the simple periodically driven mass on a spring system

$$m\ddot{x} + kx = F_0 \cos \omega t. \tag{2.42}$$

In order to find the general solution, we first obtain the solution to the homogeneous problem,

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. Next, we seek a particular solution to the nonhomogeneous problem. We will apply the Method of Undetermined Coefficients.

A natural guess for the particular solution would be to use $x_p = A \cos \omega t + B\ sin \omega t$. However, recall that the guess should not be a solution of the homogeneous problem. Comparing $x_p$ with $x_h$, this would hold if $\omega \neq \omega_0$. Otherwise, one would need to use the Modified Method of Undetermined Coefficients as described in the last section. So, we have two cases to consider.

Dividing through by the mass, we solve the simple driven system,

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.$$

**Example 2.10.** *Solve* $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$, *for* $\omega \neq \omega_0$.

*In this case we continue with the guess* $x_p = A \cos \omega t + B\ sin \omega t$. *Since there is no damping term, one quickly finds that* $B = 0$. *Inserting* $x_p = A \cos \omega t$ *into the differential equation, we find that*

$$\left(-\omega^2 + \omega_0^2\right) A \cos \omega t = \frac{F_0}{m} \cos \omega t.$$

*Solving for $A$, we obtain*

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}.$$

*The general solution for this case is thus,*

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t. \tag{2.43}$$

**Example 2.11.** *Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t$.*

*In this case, we need to employ the Modified Method of Undetermined Coefficients. So, we make the guess $x_p = t\left(A \cos \omega_0 t + B\, sin \omega_0 t\right)$. Since there is no damping term, one finds that $A = 0$. Inserting the guess in to the differential equation, we find that*

$$B = \frac{F_0}{2m\omega_0},$$

*or the general solution is*

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega} t \sin \omega t. \tag{2.44}$$

The general solution to the problem is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \begin{cases} \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t, & \omega \neq \omega_0, \\ \frac{F_0}{2m\omega_0} t \sin \omega_0 t, & \omega = \omega_0. \end{cases} \tag{2.45}$$

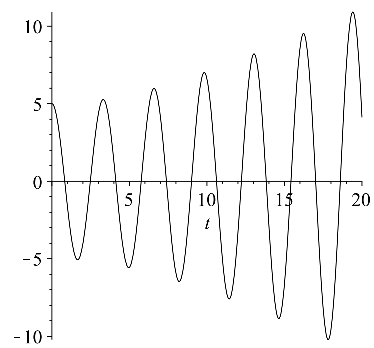


Figure 2.15: Plot of

$$x(t) = 5\cos 2t + \frac{1}{2} t \sin 2t,$$

Special cases of these solutions provide interesting physics, which can be explored by the reader in the homework. In the case that $\omega = \omega_0$, we see that the solution tends to grow as $t$ gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency. As the system is moving to the left, one pushes it to the left. If it is moving to the right, one is adding energy in that direction. This forces the amplitude of oscillation to continue to grow until the system breaks. An example of such an oscillation is shown in Figure 2.15.

In the case that $\omega \neq \omega_0$, one can rewrite the solution in a simple form. Let's choose the initial conditions that $c_1 = -F_0/(m(\omega_0^2 - \omega^2))$, $c_2 = 0$. Then one has (see Problem 11)

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}. \tag{2.46}$$

For values of $\omega$ near $\omega_0$, one finds the solution consists of a rapid oscillation, due to the $\sin \frac{(\omega_0 + \omega)t}{2}$ factor, with a slowly varying amplitude, $\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}$. The reader can investigate this solution.

This slow variation is called a beat and the beat frequency is given by $f = \frac{|\omega_0 - \omega|}{4\pi}$. In Figure 2.16 we see the high frequency oscillations are contained by the lower beat frequency, $f = \frac{0.15}{4\pi}$s. This corresponds to a period of $T = 1/f \approx 83.7$ Hz, which looks about right from the figure.

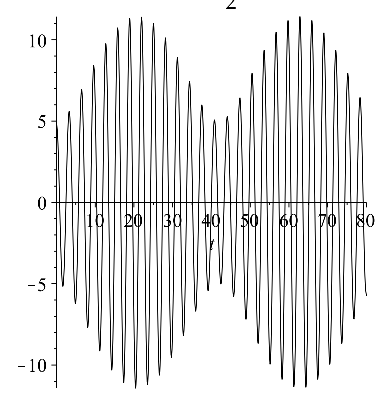


Figure 2.16: Plot of

$$x(t) = \frac{1}{249}\left(2045 \cos 2t - 800 \cos \frac{43}{20} t\right),$$

a solution of $\ddot{x} + 4x = 2\cos 2.15t$.

**Example 2.12.** *Solve $\ddot{x} + x = 2\cos \omega t$, $x(0) = 0$, $\dot{x}(0) = 0$, for $\omega = 1, 1.15$. For each case, we need the solution of the homogeneous problem,*

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

*The particular solution depends on the value of $\omega$.*

*For $\omega = 1$, the driving term, $2\cos\omega t$, is a solution of the homogeneous problem. Thus, we assume*

$$x_p(t) = At\cos t + Bt\sin t.$$

*Inserting this into the differential equation, we find $A = 0$ and $B = 1$. So, the general solution is*

$$x(t) = c_1\cos t + c_2\sin t + t\sin t.$$

*Imposing the initial conditions, we find*

$$x(t) = t\sin t.$$

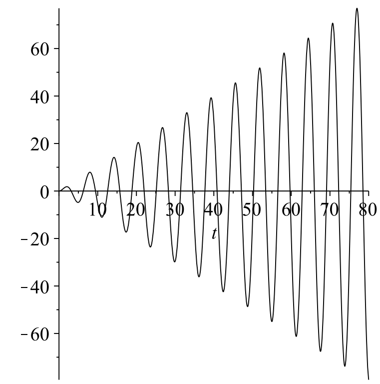



Figure 2.17: Plot of

$$x(t) = t\sin 2t,$$

a solution of $\ddot{x} + x = 2\cos t$.

*This solution is shown in Figure 2.17.*

*For $\omega = 1.15$, the driving term, $2\cos\omega 1.15t$, is not a solution of the homogeneous problem. Thus, we assume*

$$x_p(t) = A\cos 1.15t + B\sin 1.15t.$$

*Inserting this into the differential equation, we find $A = -\frac{800}{129}$ and $B = 0$. So, the general solution is*

$$x(t) = c_1\cos t + c_2\sin t - \frac{800}{129}\cos t.$$

*Imposing the initial conditions, we find*

$$x(t) = \frac{800}{129}\left(\cos t - \cos 1.15t\right).$$

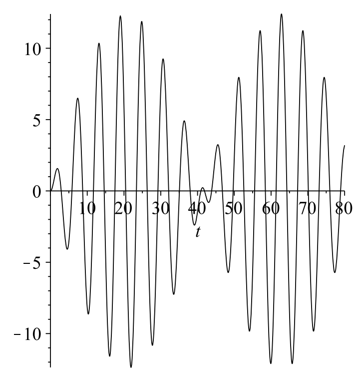



Figure 2.18: Plot of

$$x(t) = \frac{800}{129}\left(\cos t - \cos\frac{23}{20}t\right),$$

a solution of $\ddot{x} + x = 2\cos 1.15t$.

*This solution is shown in Figure 2.18. The beat frequency in this case is the same as with Figure 2.16.*

### 2.5.3 *Reduction of Order*

We have seen the the Method of Reduction of Order is useful in obtaining a second solution of a second order differential equation when one solution is known. It can also be used to solve a nonhomogeneous differential equation. First, we review the method by providing an example for homogeneous equations and then use it to solve nonhomogeneous differential equations.

**Example 2.13.** *Verify that $y_1(x) = xe^{2x}$ is a solution of $y'' - 4y' + 4y = 0$ and use the Method of Reduction of Order to find a second linearly independent solution.*

*We note that*

$$\begin{aligned} y_1'(x) &= (1+2x)e^{2x}, \\ y_1''(x) &= [2+2(1+2x)]e^{2x} = (4+4x)e^{2x}, \end{aligned}$$

*Substituting the $y_1(x)$ and its derivatives into the differential equation, we have*

$$\begin{aligned} y_1'' - 4y_1' + 4y_1 &= (4+4x)e^{2x} - 4(1+2x)e^{2x} + 4xe^{2x} \\ &= 0. \end{aligned} \tag{2.47}$$

*In order to find a second linearly independent solution, $y_2(x)$, we need a solution that is not a constant multiple of $y_1(x)$. So, we guess the form $y_2(x) = v(x)y_1(x)$. For this example, the function and its derivatives are given by*

$$\begin{aligned} y_2 &= vy_1. \\ y_2' &= (vy_1)', \\ &= v'y_1 + vy_1'. \\ y_2'' &= (v'y_1 + vy_1')', \\ &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

*Substituting $y_2$ and its derivatives into the differential equation, we have*

$$\begin{aligned} 0 &= y_2'' - 4y_2' + 4y_2 \\ &= (v''y_1 + 2v'y_1' + vy_1'') - 4(v'y_1 + vy_1') + 4vy_1 \\ &= v''y_1 + 2v'y_1' - 4v'y_1 + v[y_1'' - 4y_1' + 4y_1] \\ &= v''y_1 + 2v'y_1' - 4v'y_1 \\ &= v''xe^{2x} + 2v'(1+2x)e^{2x} - 4v'xe^{2x} \\ &= [v''x + 2v']e^{2x}. \end{aligned} \tag{2.48}$$

*Therefore, $v(x)$ satisfies the equation*

$$v''x + 2v' = 0.$$

*This is a first order equation for $v'(x)$, which can be seen by introducing $z(x) = v'(x)$, leading to the separable first order equation*

$$x\frac{dz}{dx} = -2z.$$

*This is readily solved to find $z(x) = \frac{A}{x^2}$. This gives*

$$z = \frac{dv}{dx} = \frac{A}{x^2}.$$

*Further integration leads to*

$$v(x) = -\frac{A}{x} + C.$$

*This gives*

$$\begin{aligned} y_2(x) &= \left(-\frac{A}{x} + C\right) xe^{2x} \\ &= -Ae^{2x} + Cxe^{2x}. \end{aligned}$$

*Note that the second term is the original $y_1(x)$, so we do not need this term and can set $C = 0$. Since the second linearly independent solution can be determined up to a multiplicative constant, we can set $A = -1$ to obtain the answer $y_2(x) = e^{2x}$. Note that this argument for obtaining the simple form is reason enough to ignore the integration constants when employing the Method of Reduction of Order.*

The Method of Reduction of Order is also useful for solving nonhomogeneous problems. In this case if we know one solution of the homogeneous problem, then we can use it to obtain a particular solution of the nonhomogeneous problem. For example, consider the nonhomogeneous differential equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.49}$$

Let's assume that $y_1(x)$ satisfies the homogeneous differential equation

$$a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x) = 0. \tag{2.50}$$

Then, we seek a particular solution, $y_p(x) = v(x)y_1(x)$. Its derivatives are given by

$$\begin{aligned} y_p' &= (vy_1)', \\ &= v'y_1 + vy_1'. \\ y_p'' &= (v'y_1 + vy_1')', \\ &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Substituting $y_p$ and its derivatives into the differential equation, we have

$$\begin{aligned} f &= ay_p'' + by_p' + cy_p \\ &= a(v''y_1 + 2v'y_1' + vy_1'') + b(v'y_1 + vy_1') + cvy_1 \\ &= av''y_1 + 2av'y_1' + bv'y_1 + v[ay_1'' + by_1' + cy_1] \\ &= av''y_1 + 2av'y_1' + bv'y_1 \end{aligned}$$

Therefore, $v(x)$ satisfies the second order equation

$$a(x)y_(x)v''(x) + [2a(x)y_1'(x) + b(x)y_1(x)]v'(x) = f(x).$$

Letting $z = v'$, we see that we have the linear first order equation for $z(x)$ :

$$a(x)y_(x)z'(x) + [2a(x)y_1'(x) + b(x)y_1(x)]z(x) = f(x).$$

**Example 2.14.** *Use the Method of Reduction of Order to solve $y'' + y = \sec x$.*

*Solutions of the homogeneous equation, $y'' + y = 0$ are $\sin x$ and $\cos x$. We can choose either to begin using the Method of Reduction of Order. Let's take $y_p = v\cos x$. Its derivatives are given by*

$$\begin{aligned} y_p' &= (v\cos x)', \\ &= v'\cos x - v\sin x. \\ y_p'' &= (v'\cos x - v\sin x)', \\ &= v''\cos x - 2v'\sin x - v\cos x. \end{aligned}$$

*Substituting into the nonhomogeneous equation, we have*

$$\begin{aligned} \sec x &= y_p'' + y_p \\ &= v''\cos x - 2v'\sin x - v\cos x + v\cos x \\ &= v''\cos x - 2v'\sin x \end{aligned}$$

*Letting $v' = z$, we have the linear first order differential equation*

$$(\cos x)z' - (2\sin x)z = \sec x.$$

*Rewriting the equation as,*

$$z' - (2\tan x)z = \sec^2 x.$$

*Multiplying by the integrating factor,*

$$\begin{aligned} \mu(x) &= -\exp\int^x 2\tan\xi\, d\xi \\ &= -\exp 2\ln|secx| \\ &= \cos^2 x, \end{aligned}$$

*we obtain*

$$(z\cos^2 x)' = 1.$$

*Integrating,*

$$v' = z = x\sec^2 x.$$

*This can be integrated using integration by parts (letting $U = x$ and $V = \tan x$):*

$$\begin{aligned} v &= \int x\sec^2 x\, dx \\ &= x\tan x - \int \tan x\, dx \\ &= x\tan x - \ln|secx|. \end{aligned}$$

*We now have enough to write out the solution. The particular solution is given by*

$$\begin{aligned} y_p &= vy_1 \\ &= (x\tan x - \ln|secx|)\cos x \\ &= x\sin x + \cos x\ln|\cos x|. \end{aligned}$$

*The general solution is then*

$$y(x) = c_1\cos x + c_2\sin x + x\sin x + \cos x\ln|\cos x|.$$

### *2.5.4 Method of Variation of Parameters*

A MORE SYSTEMATIC WAY to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little detailed and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.51}$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$ :

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Replacing the constants with functions, then we no longer have a solution to the homogeneous equation. Is it possible that we could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that we can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

We assume the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{2.52}$$

If this is to be a solution, then insertion into the differential equation should make the equation hold. To do this we will first need to compute some derivatives.

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{2.53}$$

Next we will need the second derivative. But, this will yield eight terms. So, we will first make a simplifying assumption. Let's assume that the last two terms add to zero:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{2.54}$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

Under the assumption the first derivative simplifies to

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x). \tag{2.55}$$

The second derivative now only has four terms:

$$y_p'(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{2.56}$$

Now that we have the derivatives, we can insert the guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) \quad = \quad & a(x)\left[c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right] \\ & + b(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] \\ & + c(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right]. \end{aligned} \tag{2.57}$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) \quad = \quad & c_1(x)\left[a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x)\right] \\ & + c_2(x)\left[a(x)y_2''(x) + b(x)y_2'(x) + c(x)y_2(x)\right] \\ & + a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{2.58}$$

Note that the first two rows vanish since $y_1$ and $y_2$ are solutions of the homogeneous problem. This leaves the equation

$$f(x) = a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right],$$

which can be rearranged as

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}. \tag{2.59}$$

In order to solve the differential equation $Ly = f$, we assume

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

for $Ly_{1,2} = 0$. Then, one need only solve a simple system of equations (2.60).

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

$$\boxed{\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{a(x)}. \end{aligned}} \tag{2.60}$$

System (2.60) can be solved as

$$c_1'(x) = -\frac{fy_2}{aW(y_1, y_2)},$$

$$c_1'(x) = \frac{fy_1}{aW(y_1, y_2)},$$

where $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian. We use this solution in the next section.

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just as easily start from this system and solve the system for each problem encountered.

**Example 2.15.** *Find the general solution of the nonhomogeneous problem: $y'' - y = e^{2x}$.*

*The general solution to the homogeneous problem $y_h'' - y_h = 0$ is*

$$y_h(x) = c_1e^x + c_2e^{-x}.$$

*In order to use the Method of Variation of Parameters, we seek a solution of the form*

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

*We find the unknown functions by solving the system in (2.60), which in this case becomes*

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \tag{2.61}$$

*Adding these equations we find that*

$$2c_1'e^x = e^{2x} \to c_1' = \frac{1}{2}e^x.$$

*Solving for $c_1(x)$ we find*

$$c_1(x) = \frac{1}{2}\int e^x\,dx = \frac{1}{2}e^x.$$

*Subtracting the equations in the system yields*

$$2c_2'e^{-x} = -e^{2x} \rightarrow c_2' = -\frac{1}{2}e^{3x}.$$

*Thus,*

$$c_2(x) = -\frac{1}{2}\int e^{3x}\,dx = -\frac{1}{6}e^{3x}.$$

*The particular solution is found by inserting these results into $y_p$:*

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (\frac{1}{2}e^x)e^x + (-\frac{1}{6}e^{3x})e^{-x} \\ &= \frac{1}{3}e^{2x}. \end{aligned} \tag{2.62}$$

*Thus, we have the general solution of the nonhomogeneous problem as*

$$y(x) = c_1e^x + c_2e^{-x} + \frac{1}{3}e^{2x}.$$

**Example 2.16.** *Now consider the problem: $y'' + 4y = \sin x$.*

*The solution to the homogeneous problem is*

$$y_h(x) = c_1\cos 2x + c_2\sin 2x. \tag{2.63}$$

*We now seek a particular solution of the form*

$$y_h(x) = c_1(x)\cos 2x + c_2(x)\sin 2x.$$

*We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $a(x) = 1$, $f(x) = \sin x$ in system (2.60):*

$$\begin{aligned} c_1'(x)\cos 2x + c_2'(x)\sin 2x &= 0 \\ -2c_1'(x)\sin 2x + 2c_2'(x)\cos 2x &= \sin x. \end{aligned} \tag{2.64}$$

*Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2\sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the $c_1'$ terms. Thus, we have*

$$c_2'(x) = \frac{1}{2}\sin x\cos 2x = \frac{1}{2}(2\cos^2 x - 1)\sin x.$$

*Inserting this into the first equation of the system, we have*

$$c_1'(x) = -c_2'(x)\frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\sin x\sin 2x = -\sin^2 x\cos x.$$

*These can easily be solved:*

$$c_2(x) = \frac{1}{2}\int(2\cos^2 x - 1)\sin x\,dx = \frac{1}{2}(\cos x - \frac{2}{3}\cos^3 x).$$

$$c_1(x) = -\int \sin^x\cos x\,dx = -\frac{1}{3}\sin^3 x.$$

*The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives*

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (-\frac{1}{3}\sin^3 x)\cos 2x + (\frac{1}{2}\cos x - \frac{1}{3}\cos^3 x)\sin x \\ &= \frac{1}{3}\sin x. \end{aligned} \tag{2.65}$$

*So, the general solution is*

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\sin x. \tag{2.66}$$

### 2.5.5 *Initial Value Green's Functions*

In this section we will investigate the solution of initial value problems involving nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \tag{2.67}$$

subject to the initial conditions

$$y(0) = y_0 \quad y'(0) = v_0.$$

Since we are interested in initial value problems, we will denote the independent variable as a time variable, $t$.

Equation (2.67) can be written compactly as

$$L[y] = f,$$

where $L$ is the differential operator

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t).$$

The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(t) = \int G(t,\tau) f(\tau)\, d\tau.$$

The function $G(t,\tau)$ is referred to as the kernel of the integral operator and is called the Green's function.

The history of the Green's function dates back to 1828, when George Green published work in which he sought solutions of Poisson's equation $\nabla^2 u = f$ for the electric potential $u$ defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced

George Green (1793-1841), a British mathematical physicist who had little formal education and worked as a miller and a baker, published *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in which he not only introduced what is now known as Green's function, but he also introduced potential theory and Green's Theorem in his studies of electricity and magnetism. Recently his paper was posted at arXiv.org, arXiv:0807.0088.

a function now identified as what Riemann later coined the "Green's function". In this section we will derive the initial value Green's function for ordinary differential equations. Later in the book we will return to boundary value Green's functions and Green's functions for partial differential equations.

In the last section we solved nonhomogeneous equations like (2.67) using the Method of Variation of Parameters. Letting,

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t), \tag{2.68}$$

we found that we have to solve the system of equations

$$\begin{aligned} c_1'(t)y_1(t) + c_2'(t)y_2(t) &= 0. \\ c_1'(t)y_1'(t) + c_2'(t)y_2'(t) &= \frac{f(t)}{q(t)}. \end{aligned} \tag{2.69}$$

This system is easily solved to give

$$\begin{aligned} c_1'(t) &= -\frac{f(t)y_2(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]} \\ c_2'(t) &= \frac{f(t)y_1(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}. \end{aligned} \tag{2.70}$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem, which is given by the determinant

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

When $y_1(t)$ and $y_2(t)$ are linearly independent, then the Wronskian is not zero and we are guaranteed a solution to the above system.

So, after an integration, we find the parameters as

$$\begin{aligned} c_1(t) &= -\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ c_2(t) &= \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau, \end{aligned} \tag{2.71}$$

where $t_0$ and $t_1$ are arbitrary constants to be determined from the initial conditions.

Therefore, the particular solution of (2.67) can be written as

$$y_p(t) = y_2(t)\int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{2.72}$$

We begin with the particular solution (2.72) of the nonhomogeneous differential equation (2.67). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y_p(t) = c_1y_1(t) + c_2y_2(t) + y_2(t)\int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{2.73}$$

However, an appropriate choice of $t_0$ and $t_1$ can be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1y_1(t) + c_2y_2(t)$. However, setting up the solution in this form will allow us to use $t_0$ and $t_1$ to determine particular solutions which satisfies certain homogeneous conditions. In particular, we will show that Equation (2.73) can be written in the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \int_0^t G(t, \tau) f(\tau)\, d\tau, \qquad (2.74)$$

where the function $G(t, \tau)$ will be identified as the Green's function.

The goal is to develop the Green's function technique to solve the initial value problem

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0. \qquad (2.75)$$

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$a(t)y_h''(t) + b(t)y_h'(t) + c(t)y_h(t) = 0, \quad y_h(0) = y_0, \quad y_h'(0) = v_0. \qquad (2.76)$$

We then assume that the particular solution satisfies the problem

$$a(t)y_p''(t) + b(t)y_p'(t) + c(t)y_p(t) = f(t), \quad y_p(0) = 0, \quad y_p'(0) = 0. \qquad (2.77)$$

Since the differential equation is linear, then we know that

$$y(t) = y_h(t) + y_p(t)$$

is a solution of the nonhomogeneous equation. Also, this solution satisfies the initial conditions:

$$y(0) = y_h(0) + y_p(0) = y_0 + 0 = y_0,$$
$$y'(0) = y_h'(0) + y_p'(0) = v_0 + 0 = v_0.$$

Therefore, we need only focus on finding a particular solution that satisfies homogeneous initial conditions. This will be done by finding values for $t_0$ and $t_1$ in Equation (2.72) which satisfy the homogeneous initial conditions, $y_p(0) = 0$ and $y_p'(0) = 0$.

First, we consider $y_p(0) = 0$. We have

$$y_p(0) = y_2(0) \int_{t_1}^0 \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(0) \int_{t_0}^0 \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \qquad (2.78)$$

Here, $y_1(t)$ and $y_2(t)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then, we have

$$y_p(0) = y_2(0) \int_{t_1}^0 \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau \qquad (2.79)$$

We can force $y_p(0) = 0$ if we set $t_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(t) = y_2'(t) \int_0^t \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1'(t) \int_{t_0}^t \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau, \tag{2.80}$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at $t = 0$, we have

$$y_p'(0) = -y_1'(0) \int_{t_0}^0 \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{2.81}$$

Assuming that $y_1'(0) \neq 0$, we can set $t_0 = 0$.

Thus, we have found that

$$\begin{aligned} y_p(x) &= y_2(t) \int_0^t \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t) \int_0^t \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ &= \int_0^t \left[ \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \right] f(\tau)\, d\tau. \end{aligned} \tag{2.82}$$

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(t) = \int_0^t G(t,\tau) f(\tau)\, d\tau, \tag{2.83}$$

where the initial value Green's function is defined as

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}.$$

We summarize

**Solution of IVP Using the Green's Function**

The solution of the initial value problem,

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

takes the form

$$y(t) = y_h(t) + \int_0^t G(t,\tau) f(\tau)\, d\tau, \tag{2.84}$$

where

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \tag{2.85}$$

is the Green's function and $y_1, y_2, y_h$ are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0, y_h(0) = y_0, y_h'(0) = v_0.$$

**Example 2.17.** *Solve the forced oscillator problem*

$$x'' + x = 2\cos t, \quad x(0) = 4, \quad x'(0) = 0.$$

*We first solve the homogeneous problem with nonhomogeneous initial conditions:*

$$x_h'' + x_h = 0, \quad x_h(0) = 4, \quad x_h'(0) = 0.$$

*The solution is easily seen to be* $x_h(t) = 4\cos t$.

*Next, we construct the Green's function. We need two linearly independent solutions,* $y_1(x)$, $y_2(x)$, *to the homogeneous differential equation satisfying different homogeneous conditions,* $y_1(0) = 0$ *and* $y_2'(0) = 0$. *The simplest solutions are* $y_1(t) = \sin t$ *and* $y_2(t) = \cos t$. *The Wronskian is found as*

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\sin^2 t - \cos^2 t = -1.$$

*Since* $a(t) = 1$ *in this problem, we compute the Green's function,*

$$\begin{aligned} G(t,\tau) &= \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \\ &= \sin t\cos\tau - \sin\tau\cos t \\ &= \sin(t-\tau). \end{aligned} \tag{2.86}$$

*Note that the Green's function depends on* $t - \tau$. *While this is useful in some contexts, we will use the expanded form when carrying out the integration.*

*We can now determine the particular solution of the nonhomogeneous differential equation. We have*

$$\begin{aligned} x_p(t) &= \int_0^t G(t,\tau)f(\tau)\,d\tau \\ &= \int_0^t (\sin t\cos\tau - \sin\tau\cos t)\,(2\cos\tau)\;d\tau \\ &= 2\sin t\int_0^t \cos^2\tau d\tau - 2\cos t\int_0^t \sin\tau\cos\tau d\tau \\ &= 2\sin t\left[\frac{\tau}{2} + \frac{1}{2}\sin 2\tau\right]_0^t - 2\cos t\left[\frac{1}{2}\sin^2\tau\right]_0^t \\ &= t\sin t. \end{aligned} \tag{2.87}$$

*Therefore, the solution of the nonhomogeneous problem is the sum of the solution of the homogeneous problem and this particular solution:* $x(t) = 4\cos t + t\sin t$.

## 2.6 Cauchy-Euler Equations

ANOTHER CLASS OF SOLVABLE LINEAR DIFFERENTIAL EQUATIONS that is of interest are the Cauchy-Euler type of equations, also referred to in some books as Euler's equation. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{2.88}$$

Note that in such equations the power of $x$ in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (2.88), we have

$$[ar(r-1)+br+c]\,x^r = 0.$$

Since this has to be true for all $x$ in the problem domain, we obtain the characteristic equation

The solutions of Cauchy-Euler equations can be found using the characteristic equation $ar(r-1)+br+c=0$.

$$ar(r-1)+br+c = 0. \tag{2.89}$$

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.

For two real, distinct roots, the general solution takes the form

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}.$$

**Example 2.18.** *Find the general solution: $x^2y'' + 5xy' + 12y = 0$.*

*As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,*

$$\begin{aligned} 0 &= r(r-1)+5r+12 \\ &= r^2+4r+12 \\ &= (r+2)^2+8, \\ -8 &= (r+2)^2, \end{aligned} \tag{2.90}$$

*one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1\cos(2\sqrt{2}\ln|x|) + c_2\sin(2\sqrt{2}\ln|x|)\right]x^{-2}$*

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r-r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1)+br+c = 0.$$

Dividing this equation by $a$ and rewriting, we have

$$r^2 + (\frac{b}{a}-1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1-2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the Cauchy-Euler equation for this case can be written in the form

$$x^2y'' + (1-2r_1)xy' + r_1^2y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1v)x^{r_1-1}, \\ y_2''(x) &= (x^2v'' + 2r_1xv' + r_1(r_1-1)v)x^{r_1-2}. \end{aligned} \tag{2.91}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2y'' + (1-2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1+1}. \end{aligned} \quad (2.92)$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln|v'| = -\ln|x| + C,$$

where $A = \pm e^C$ absorbs $C$ and the signs from the absolute values. Exponentiating, we obtain one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A\ln|x| + k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1}\ln|x|.$$

Therefore, the general solution is found as $y(x) = (c_1 + c_2\ln|x|)x^r$.

For one root, $r_1 = r_2 = r$, the general solution is of the form

$$y(x) = (c_1 + c_2\ln|x|)x^r.$$

**Example 2.19.** *Solve the initial value problem: $t^2y'' + 3ty' + y = 0$, with the initial conditions $y(1) = 0$, $y'(1) = 1$.*

*For this example the characteristic equation takes the form*

$$r(r-1) + 3r + 1 = 0,$$

*or*

$$r^2 + 2r + 1 = 0.$$

*There is only one real root, $r = -1$. Therefore, the general solution is*

$$y(t) = (c_1 + c_2\ln|t|)t^{-1}.$$

*However, this problem is an initial value problem. At $t = 1$ we know the values of $y$ and $y'$. Using the general solution, we first have that*

$$0 = y(1) = c_1.$$

*Thus, we have so far that $y(t) = c_2\ln|t|t^{-1}$. Now, using the second condition and*

$$y'(t) = c_2(1 - \ln|t|)t^{-2},$$

*we have*

$$1 = y(1) = c_2.$$

*Therefore, the solution of the initial value problem is $y(t) = \ln|t|t^{-1}$.*

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha+i\beta}$. The key to obtaining real solutions is to first rewrite $x^y$ :

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^\alpha e^{i\beta \ln x}, \quad x > 0.$$

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form

$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|))$.

Recalling that

$$e^{i\beta \ln x} = \cos(\beta \ln |x|) + i \sin(\beta \ln |x|),$$

we can now find two real, linearly independent solutions, $x^\alpha \cos(\beta \ln |x|)$ and $x^\alpha \sin(\beta \ln |x|)$ following the same steps as earlier for the constant coefficient case. This gives the general solution as

$$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)).$$

**Example 2.20.** *Solve:* $x^2 y'' - xy' + 5y = 0$.

*The characteristic equation takes the form*

$$r(r-1) - r + 5 = 0,$$

*or*

$$r^2 - 2r + 5 = 0.$$

*The roots of this equation are complex,* $r_{1,2} = 1 \pm 2i$. *Therefore, the general solution is* $y(x) = x(c_1 \cos(2 \ln |x|) + c_2 \sin(2 \ln |x|))$.

The three cases are summarized in the table below.

**Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations**

1. Real, distinct roots $r_1, r_2$. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.
2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln |x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln |x|) x^r$.
3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln |x|)$ and $x^\alpha \sin(\beta \ln |x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|))$.

**Nonhomogeneous Cauchy-Euler Equations**

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients or the Method of Variation of Parameters. We will demonstrate this with a couple of examples.

**Example 2.21.** *Find the solution of* $x^2y'' - xy' - 3y = 2x^2$.

*First we find the solution of the homogeneous equation. The characteristic equation is* $r^2 - 2r - 3 = 0$. *So, the roots are* $r = -1, 3$ *and the solution is* $y_h(x) = c_1x^{-1} + c_2x^3$.

*We next need a particular solution. Let's guess* $y_p(x) = Ax^2$. *Inserting the guess into the nonhomogeneous differential equation, we have*

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{2.93}$$

*So,* $A = -2/3$. *Therefore, the general solution of the problem is*

$$y(x) = c_1x^{-1} + c_2x^3 - \frac{2}{3}x^2.$$

**Example 2.22.** *Find the solution of* $x^2y'' - xy' - 3y = 2x^3$.

*In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form*

$$ax^2y'' + bxy' + cy = dx^r,$$

*where* $r$ *is a solution of* $ar(r-1) + br + c = 0$. *Let's guess a solution of the form* $y = Ax^r \ln x$. *Then one finds that the differential equation reduces to* $Ax^r(2ar - a + b) = dx^r$. *[You should verify this for yourself.]*

*With this in mind, we can now solve the problem at hand. Let* $y_p = Ax^3 \ln x$. *Inserting into the equation, we obtain* $4Ax^3 = 2x^3$, *or* $A = 1/2$. *The general solution of the problem can now be written as*

$$y(x) = c_1x^{-1} + c_2x^3 + \frac{1}{2}x^3 \ln x.$$

**Example 2.23.** *Find the solution of* $x^2y'' - xy' - 3y = 2x^3$ *using Variation of Parameters.*

*As noted in the previous examples, the solution of the homogeneous problem has two linearly independent solutions,* $y_1(x) = x^{-1}$ *and* $y_2(x) = x^3$. *Assuming a particular solution of the form* $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, *we need to solve the system (2.60):*

$$\begin{aligned} c_1'(x)x^{-1} + c_2'(x)x^3 &= 0 \\ -c_1'(x)x^{-2} + 3c_2'(x)x^2 &= \frac{2x^3}{x^2} = 2x. \end{aligned} \tag{2.94}$$

*From the first equation of the system we have* $c_1'(x) = -x^4c_2'(x)$. *Substituting this into the second equation gives* $c_2'(x) = \frac{1}{2x}$. *So,* $c_2(x) = \frac{1}{2}\ln|x|$ *and, therefore,*

$c_1(x) = \frac{1}{8}x^4$. *The particular solution is*

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = \frac{1}{8}x^3 + \frac{1}{2}x^3 \ln|x|.$$

*Adding this to the homogeneous solution, we obtain the same solution as in the last example using the Method of Undetermined Coefficients. However, since* $\frac{1}{8}x^3$ *is a solution of the homogeneous problem, it can be absorbed into the first terms, leaving*

$$y(x) = c_1x^{-1} + c_2x^3 + \frac{1}{2}x^3 \ln x.$$

## *Problems*

**1.** Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0.$

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

d. $x'' - x' - 6x = 0$ for $x = x(t)$.

**2.** Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4.$

b. $xy'' - y' + 4x^3y = 0, \quad y_1(x) = \sin(x^2).$

c. $(1 - x^2)y'' - 2xy' + 2y = 0, \quad y_1(x) = x$. [Note: This is one solution of Legendre's differential equation in Example 4.4.]

d. $(x - 1)y'' - xy' + y = 0, \quad y_1(x) = e^x.$

**3.** Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of $y_1$ and $y_2$.

**4.** Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

**5.** Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$, Undetermined Coefficients.

b. $y'' + 2y' + y = 5 + 10\sin 2x$, Undetermined Coefficients.

c. $y'' - 5y' + 6y = 3e^x$, Reduction of Order.

d. $y'' + 5y' - 6y = 3e^x$, Reduction of Order.

e. $y'' + y = \sec^3 x$, Reduction of Order.

f. $y'' + y' = 3x^2$, Variation of Parameters.

g. $y'' - y = e^x + 1$, Variation of Parameters.

**6.** Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

b. $y'' - 4y' + 4y = 6xe^{2x}$.

c. $y'' - 2y' + y = \frac{e^{2x}}{(1+e^x)^2}$.

d. $y'' - 3y' + 2y = \cos(e^x)$.

**7.** Instead of assuming that $c_1'y_1 + c_2'y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1'y_1 + c_2'y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

**8.** Find all of the solutions of the second order differential equations for $x > 0$.. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2y'' + 3xy' + 2y = 0$.

b. $x^2y'' - 3xy' + 3y = 0$.

c. $x^2y'' + 5xy' + 4y = 0$.

d. $x^2y'' - 2xy' + 3y = 0$.

e. $x^2y'' + 3xy' - 3y = x^2$.

**9.** A spring fixed at its upper end is stretched six inches by a 10-pound weight attached at its lower end. The spring-mass system is suspended in a viscous medium so that the system is subjected to a damping force of $5\frac{dx}{dt}$ lbs. Describe the motion of the system if the weight is drawn down an additional 4 inches and released. What would happen if you changed the coefficient "5" to "4"? [You may need to consult your introductory physics text.]

**10.** Consider an LRC circuit with $L = 1.00$ H, $R = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ f, and $V = 1.00 \times 10^3$ V. Suppose that no charge is present and no current is flowing at time $t = 0$ when a battery of voltage $V$ is inserted. Find the current and the charge on the capacitor as functions of time. Describe how the system behaves over time.

**11.** Consider the problem of forced oscillations as described in section 2.5.2.

b. Plot the solutions in Equation (2.77) for the following cases: Let $c_1 = 0.5$, $c_2 = 0$, $F_0 = 1.0$ N, and $m = 1.0$ kg for $t \in [0, 100]$.

i. $\omega_0 = 2.0$ rad/s, $\omega = 0.1$ rad/s.

ii. $\omega_0 = 2.0$ rad/s, $\omega = 0.5$ rad/s.

iii. $\omega_0 = 2.0$ rad/s, $\omega = 1.5$ rad/s.

iv. $\omega_0 = 2.0$ rad/s, $\omega = 2.2$ rad/s.

v. $\omega_0 = 1.0$ rad/s, $\omega = 1.2$ rad/s.

vi. $\omega_0 = 1.5$ rad/s, $\omega = 1.5$ rad/s.

d. Confirm that the solution in Equation (2.78) is the same as the solution in Equation (2.77) for $F_0 = 2.0$ N, $m = 10.0$ kg, $\omega_0 = 1.5$ rad/s, and $\omega = 1.25$ rad/s, by plotting both solutions for $t \in [0, 100]$.

**12.** A certain model of the motion light plastic ball tossed into the air is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here $m$ is the mass of the ball, $g$=9.8 m/s$^2$ is the acceleration due to gravity and $c$ is a measure of the damping. Since there is no $x$ term, we can write this as a first order equation for the velocity $v(t) = x'(t)$ :

$$mv' + cv + mg = 0.$$

a. Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.

b. Use the solution of part a to find the general solution for the position $x(t)$.

c. Find an expression to determine how long it takes for the ball to reach it's maximum height?

d. Assume that $c/m = 5\ \text{s}^{-1}$. For $v_0 = 5, 10, 15, 20$ m/s, plot the solution, $x(t)$, versus the time.

e. From your plots and the expression in part c, determine the rise time. Do these answers agree?

f. What can you say about the time it takes for the ball to fall as compared to the rise time?

**13.** Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

b. $y'' + y = 2\sin 3x, \quad y(0) = 5, \quad y'(0) = 0.$

c. $y'' + y = 1 + 2\cos x, \quad y(0) = 2, \quad y'(0) = 0.$

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad y(1) = \pi, \quad y'(1) = 0.$

**14.** Use the initial value Green's function for $x'' + x = f(t)$, $x(0) = 4$, $x'(0) = 0$, to solve the following problems.

a. $x'' + x = 5t^2.$

b. $x'' + x = 2\tan t.$

**15.** For the problem $y'' - k^2y = f(x)$, $y(0) = 0$, $y'(0) = 1$,

a. Find the initial value Green's function.

b. Use the Green's function to solve $y'' - y = e^{-x}$.

c. Use the Green's function to solve $y'' - 4y = e^{2x}$.

**16.** Find and use the initial value Green's function to solve

$$x^2y'' + 3xy' - 15y = x^4e^x, \quad y(1) = 1, y'(1) = 0.$$

# 3
# Numerical Solutions

*"The laws of mathematics are not merely human inventions or creations. They simply 'are;' they exist quite independently of the human intellect." - M. C. Escher (1898-1972)*

So far we have seen some of the standard methods for solving first and second order differential equations. However, we have had to restrict ourselves to special cases in order to get nice analytical solutions to initial value problems. While these are not the only equations for which we can get exact results, there are many cases in which exact solutions are not possible. In such cases we have to rely on approximation techniques, including the numerical solution of the equation at hand.

The use of numerical methods to obtain approximate solutions of differential equations and systems of differential equations has been known for some time. However, with the advent of powerful computers and desktop computers, we can now solve many of these problems with relative ease. The simple ideas used to solve first order differential equations can be extended to the solutions of more complicated systems of partial differential equations, such as the large scale problems of modeling ocean dynamics, weather systems and even cosmological problems stemming from general relativity.

## 3.1 Euler's Method

In this section we will look at the simplest method for solving first order equations, Euler's Method. While it is not the most efficient method, it does provide us with a picture of how one proceeds and can be improved by introducing better techniques, which are typically covered in a numerical analysis text.

Let's consider the class of first order initial value problems of the form

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{3.1}$$

We are interested in finding the solution $y(x)$ of this equation which passes through the initial point $(x_0, y_0)$ in the $xy$-plane for values of $x$ in the interval

$[a,b]$, where $a = x_0$. We will seek approximations of the solution at $N$ points, labeled $x_n$ for $n = 1, \ldots, N$. For equally spaced points we have $\Delta x = x_1 - x_0 = x_2 - x_1$, etc. We can write these as

$$x_n = x_0 + n\Delta x.$$

In Figure 3.1 we show three such points on the $x$-axis.

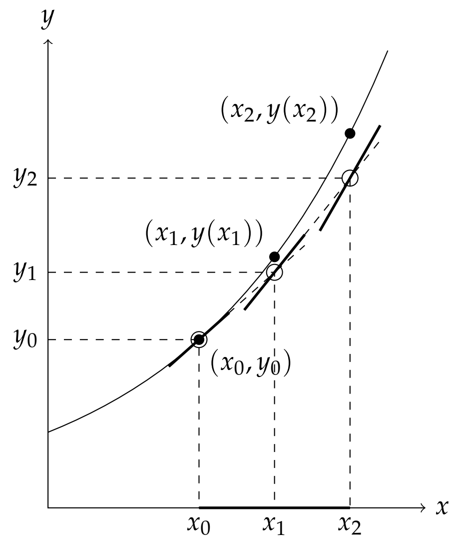



Figure 3.1: The basics of Euler's Method are shown. An interval of the $x$ axis is broken into $N$ subintervals. The approximations to the solutions are found using the slope of the tangent to the solution, given by $f(x,y)$. Knowing previous approximations at $(x_{n-1}, y_{n-1})$, one can determine the next approximation, $y_n$.

The first step of Euler's Method is to use the initial condition. We represent this as a point on the solution curve, $(x_0, y(x_0)) = (x_0, y_0)$, as shown in Figure 3.1. The next step is to develop a method for obtaining approximations to the solution for the other $x_n$'s.

We first note that the differential equation gives the slope of the tangent line at $(x, y(x))$ of the solution curve since the slope is the derivative, $y'(x)'$ From the differential equation the slope is $f(x, y(x))$. Referring to Figure 3.1, we see the tangent line drawn at $(x_0, y_0)$. We look now at $x = x_1$. The vertical line $x = x_1$ intersects both the solution curve and the tangent line passing through $(x_0, y_0)$. This is shown by a heavy dashed line.

While we do not know the solution at $x = x_1$, we can determine the tangent line and find the intersection point that it makes with the vertical. As seen in the figure, this intersection point is in theory close to the point on the solution curve. So, we will designate $y_1$ as the approximation of the solution $y(x_1)$. We just need to determine $y_1$.

The idea is simple. We approximate the derivative in the differential equation by its difference quotient:

$$\frac{dy}{dx} \approx \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{\Delta x}. \tag{3.2}$$

Since the slope of the tangent to the curve at $(x_0, y_0)$ is $y'(x_0) = f(x_0, y_0)$, we can write

$$\frac{y_1 - y_0}{\Delta x} \approx f(x_0, y_0). \tag{3.3}$$

Solving this equation for $y_1$, we obtain

$$y_1 = y_0 + \Delta x f(x_0, y_0). \tag{3.4}$$

This gives $y_1$ in terms of quantities that we know.

We now proceed to approximate $y(x_2)$. Referring to Figure 3.1, we see that this can be done by using the slope of the solution curve at $(x_1, y_1)$. The corresponding tangent line is shown passing though $(x_1, y_1)$ and we can then get the value of $y_2$ from the intersection of the tangent line with a vertical line, $x = x_2$. Following the previous arguments, we find that

$$y_2 = y_1 + \Delta x f(x_1, y_1). \tag{3.5}$$

Continuing this procedure for all $x_n$, $n = 1, \ldots N$, we arrive at the following scheme for determining a numerical solution to the initial value problem:

$$\begin{aligned} y_0 &= y(x_0), \\ y_n &= y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}), \quad n = 1, \ldots, N. \end{aligned} \tag{3.6}$$

This is referred to as Euler's Method.

**Example 3.1.** *Use Euler's Method to solve the initial value problem* $\frac{dy}{dx} = x + y$, $\quad y(0) = 1$ *and obtain an approximation for* $y(1)$.

*First, we will do this by hand. We break up the interval* $[0, 1]$, *since we want the solution at* $x = 1$ *and the initial value is at* $x = 0$. *Let* $\Delta x = 0.50$. *Then,* $x_0 = 0$, $x_1 = 0.5$ *and* $x_2 = 1.0$. *Note that there are* $N = \frac{b-a}{\Delta x} = 2$ *subintervals and thus three points.*

*We next carry out Euler's Method systematically by setting up a table for the needed values. Such a table is shown in Table 3.1. Note how the table is set up. There is a column for each* $x_n$ *and* $y_n$. *The first row is the initial condition. We also made use of the function* $f(x, y)$ *in computing the* $y_n$*'s from (3.6). This sometimes makes the computation easier. As a result, we find that the desired approximation is given as* $y_2 = 2.5$.

| $n$ | $x_n$ | $y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1} = 0.5x_{n-1} + 1.5y_{n-1}$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.5 | $0.5(0) + 1.5(1.0) = 1.5$ |
| 2 | 1.0 | $0.5(0.5) + 1.5(1.5) = 2.5$ |

Table 3.1: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.5$.

*Is this a good result? Well, we could make the spatial increments smaller. Let's repeat the procedure for* $\Delta x = 0.2$, *or* $N = 5$. *The results are in Table 3.2.*

*Now we see that the approximation is* $y_1 = 2.97664$. *So, it looks like the value is near 3, but we cannot say much more. Decreasing* $\Delta x$ *more shows that we are beginning to converge to a solution. We see this in Table 3.3.*

Of course, these values were not done by hand. The last computation would have taken 1000 lines in the table, or at least 40 pages! One could use a computer to do this. A simple code in Maple would look like the following:

Table 3.2: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.2$.

| $n$ | $x_n$ | $y_n = 0.2x_{n-1} + 1.2y_{n-1}$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.2 | $0.2(0) + 1.2(1.0) = 1.2$ |
| 2 | 0.4 | $0.2(0.2) + 1.2(1.2) = 1.48$ |
| 3 | 0.6 | $0.2(0.4) + 1.2(1.48) = 1.856$ |
| 4 | 0.8 | $0.2(0.6) + 1.2(1.856) = 2.3472$ |
| 5 | 1.0 | $0.2(0.8) + 1.2(2.3472) = 2.97664$ |

Table 3.3: Results of Euler's Method for $y' = x + y$, $y(0) = 1$ and varying $\Delta x$

| $\Delta x$ | $y_N \approx y(1)$ |
|---|---|
| 0.5 | 2.5 |
| 0.2 | 2.97664 |
| 0.1 | 3.187484920 |
| 0.01 | 3.409627659 |
| 0.001 | 3.433847864 |
| 0.0001 | 3.436291854 |

```
> restart:
> f:=(x,y)->y+x;
> a:=0:  b:=1:  N:=100:  h:=(b-a)/N;
> x[0]:=0: y[0]:=1:
  for i from 1 to N do
  y[i]:=y[i-1]+h*f(x[i-1],y[i-1]):
  x[i]:=x[0]+h*(i):
  od:
  evalf(y[N]);
```

In this case we could simply use the exact solution. The exact solution is easily found as

$$y(x) = 2e^x - x - 1.$$

(The reader can verify this.) So, the value we are seeking is

$$y(1) = 2e - 2 = 3.4365636\ldots.$$

Thus, even the last numerical solution was off by about 0.00027.

Adding a few extra lines for plotting, we can visually see how well the approximations compare to the exact solution. The Maple code for doing such a plot is given below.

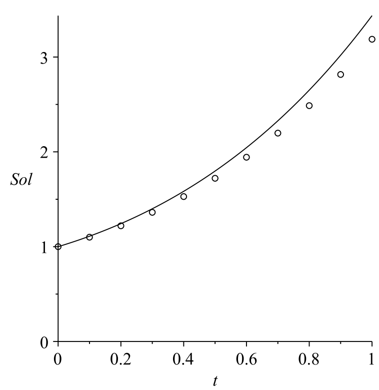



Figure 3.2: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 10$.

```
> with(plots):
> Data:=[seq([x[i],y[i]],i=0..N)]:
> P1:=pointplot(Data,symbol=DIAMOND):
> Sol:=t->-t-1+2*exp(t);
> P2:=plot(Sol(t),t=a..b,Sol=0..Sol(b)):
> display({P1,P2});
```

We show in Figures 3.2-3.3 the results for $N = 10$ and $N = 100$. In Figure 3.2 we can see how quickly the numerical solution diverges from the exact solution. In Figure 3.3 we can see that visually the solutions agree, but we note that from Table 3.3 that for $\Delta x = 0.01$, the solution is still off in the second decimal place with a relative error of about 0.8%.

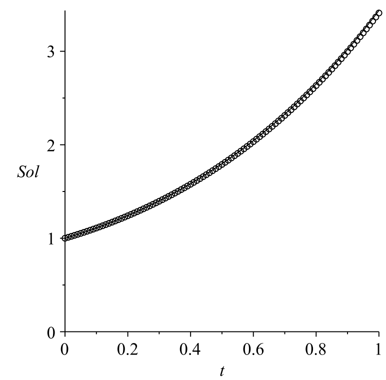



Figure 3.3: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 100$.

Why would we use a numerical method when we have the exact solution? Exact solutions can serve as test cases for our methods. We can make sure our code works before applying them to problems whose solution is not known.

There are many other methods for solving first order equations. One commonly used method is the fourth order Runge-Kutta method. This method has smaller errors at each step as compared to Euler's Method. It is well suited for programming and comes built-in in many packages like Maple and MATLAB. Typically, it is set up to handle systems of first order equations.

In fact, it is well known that $n$th order equations can be written as a system of $n$ first order equations. Consider the simple second order equation

$$y'' = f(x, y).$$

This is a larger class of equations than the second order constant coefficient equation. We can turn this into a system of two first order differential equations by letting $u = y$ and $v = y' = u'$. Then, $v' = y'' = f(x, u)$. So, we have the first order system

$$\begin{aligned} u' &= v, \\ v' &= f(x, u). \end{aligned} \tag{3.7}$$

We will not go further into the Runge-Kutta Method here. You can find more about it in a numerical analysis text. However, we will see that systems of differential equations do arise naturally in physics. Such systems are often coupled equations and lead to interesting behaviors.

## 3.2 *Higher Order Taylor Methods*

EULER'S METHOD FOR SOLVING DIFFERENTIAL EQUATIONS is easy to understand but is not efficient in the sense that it is what is called a first order method. The error at each step, the local truncation error, is of order $\Delta x$, for $x$ the independent variable. The accumulation of the local truncation errors results in what is called the global error. In order to generalize Euler's Method, we need to rederive it. Also, since these methods are typically used for initial value problems, we will cast the problem to be solved as

$$\frac{dy}{dt} = f(t, y), \quad y(a) = y_0, \quad t \in [a, b]. \tag{3.8}$$

The first step towards obtaining a numerical approximation to the solution of this problem is to divide the $t$-interval, $[a, b]$, into $N$ subintervals,

$$t_i = a + ih, \quad i = 0, 1, \ldots, N, \quad t_0 = a, \quad t_N = b,$$

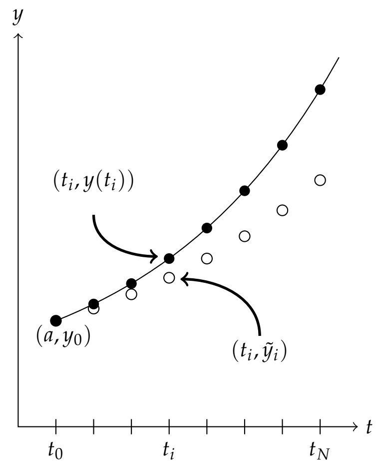



Figure 3.4: The interval $[a, b]$ is divided into $N$ equally spaced subintervals. The exact solution $y(t_i)$ is shown with the numerical solution, $\tilde{y}_i$ with $t_i = a + ih$, $i = 0, 1, \ldots, N$.

where

$$h = \frac{b-a}{N}.$$

We then seek the numerical solutions

$$\tilde{y}_i \approx y(t_i), \quad i = 1, 2, \ldots, N,$$

with $\tilde{y}_0 = y(t_0) = y_0$. Figure 3.4 graphically shows how these quantities are related.

Euler's Method can be derived using the Taylor series expansion of of the solution $y(t_i + h)$ about $t = t_i$ for $i = 1, 2, \ldots, N$. This is given by

$$\begin{aligned} y(t_{i+1}) &= y(t_i + h) \\ &= y(t_i) + y'(t_i)h + \frac{h^2}{2}y''(\xi_i), \quad \xi_i \in (t_i, t_{i+1}). \end{aligned} \tag{3.9}$$

Here the term $\frac{h^2}{2}y''(\xi_i)$ captures all of the higher order terms and represents the error made using a linear approximation to $y(t_i + h)$.

Dropping the remainder term, noting that $y'(t) = f(t, y)$, and defining the resulting numerical approximations by $\tilde{y}_i \approx y(t_i)$, we have

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hf(t_i, \tilde{y}_i), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y(a) = y_0. \end{aligned} \tag{3.10}$$

This is Euler's Method.

Euler's Method is not used in practice since the error is of order $h$. However, it is simple enough for understanding the idea of solving differential equations numerically. Also, it is easy to study the numerical error, which we will show next.

The error that results for a single step of the method is called the local truncation error, which is defined by

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - \tilde{y}_i}{h} - f(t_i, y_i).$$

A simple computation gives

$$\tau_{i+1}(h) = \frac{h}{2}y''(\xi_i), \quad \xi_i \in (t_i, t_{i+1}).$$

Since the local truncation error is of order $h$, this scheme is said to be of order one. More generally, for a numerical scheme of the form

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hF(t_i, \tilde{y}_i), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y(a) = y_0, \end{aligned} \tag{3.11}$$

The local truncation error.

the local truncation error is defined by

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - \tilde{y}_i}{h} - F(t_i, y_i).$$

The accumulation of these errors leads to the global error. In fact, one can show that if $f$ is continuous, satisfies the Lipschitz condition,

$$|f(t, y_2) - f(t, y_1)| \le L|y_2 - y_1|$$

for a particular domain $D \subset R^2$, and

$$|y''(t)| \leq M, \quad t \in [a,b],$$

then

$$|y(t_i) - \tilde{y}_|| \leq \frac{hM}{2L}\left(e^{L(t_i-a)} - 1\right), \quad i = 0,1,\ldots,N.$$

Furthermore, if one introduces round-off errors, bounded by $\delta$, in both the initial condition and at each step, the global error is modified as

$$|y(t_i) - \tilde{y}_|| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_i-a)} - 1\right) + |\delta_0|e^{L(t_i-a)}, \quad i = 0,1,\ldots,N.$$

Then for small enough steps $h$, there is a point when the round-off error will dominate the error. [See Burden and Faires, *Numerical Analysis* for the details.]

Can we improve upon Euler's Method? The natural next step towards finding a better scheme would be to keep more terms in the Taylor series expansion. This leads to Taylor series methods of order $n$.

Taylor series methods of order $n$ take the form

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hT^{(n)}(t_i,\tilde{y}_i), \quad i = 0,1,\ldots,N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{3.12}$$

where we have defined

$$T^{(n)}(t,y) = y'(t) + \frac{h}{2}y''(t) + \cdots + \frac{h^{(n-1)}}{n!}y^{(n)}(t).$$

However, since $y'(t) = f(t,y)$, we can write

$$T^{(n)}(t,y) = f(t,y) + \frac{h}{2}f'(t,y) + \cdots + \frac{h^{(n-1)}}{n!}f^{(n-1)}(t,y).$$

We note that for $n = 1$, we retrieve Euler's Method as a special case. We demonstrate a third order Taylor's Method in the next example.

**Example 3.2.** *Apply the third order Taylor's Method to*

$$\frac{dy}{dt} = t + y, \quad y(0) = 1$$

*and obtain an approximation for $y(1)$ for $h = 0.1$.*

*The third order Taylor's Method takes the form*

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hT^{(3)}(t_i,\tilde{y}_i), \quad i = 0,1,\ldots,N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{3.13}$$

*where*

$$T^{(3)}(t,y) = f(t,y) + \frac{h}{2}f'(t,y) + \frac{h^2}{3!}f''(t,y)$$

*and $f(t,y) = t + y(t)$.*

*In order to set up the scheme, we need the first and second derivative of $f(t,y)$ :*

$$\begin{aligned} f'(t,y) &= \frac{d}{dt}(t+y) \\ &= 1+y' \\ &= 1+t+y \end{aligned} \tag{3.14}$$

$$\begin{aligned} f''(t,y) &= \frac{d}{dt}(1+t+y) \\ &= 1+y' \\ &= 1+t+y \end{aligned} \tag{3.15}$$

*Inserting these expressions into the scheme, we have*

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + h\left[(t_i+y_i) + \frac{h}{2}(1+t_i+y_i) + \frac{h^2}{3!}(1+t_i+y_i)\right], \\ &= \tilde{y}_i + h(t_i+y_i) + h^2(\frac{1}{2}+\frac{h}{6})(1+t_i+y_i), \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{3.16}$$

*for $i = 0, 1, \ldots, N-1$.*

*In Figure 3.2 we show the results comparing Euler's Method, the 3rd Order Taylor's Method, and the exact solution for $N = 10$. In Table 3.4 we provide are the numerical values. The relative error in Euler's method is about 7% and that of the 3rd Order Taylor's Method is about 0.006%. Thus, the 3rd Order Taylor's Method is significantly better than Euler's Method.*

Table 3.4: Numerical values for Euler's Method, 3rd Order Taylor's Method, and exact solution for solving Example 3.2 with $N = 10$..

| Euler | Taylor | Exact |
|---|---|---|
| 1.0000 | 1.0000 | 1.0000 |
| 1.1000 | 1.1103 | 1.1103 |
| 1.2200 | 1.2428 | 1.2428 |
| 1.3620 | 1.3997 | 1.3997 |
| 1.5282 | 1.5836 | 1.5836 |
| 1.7210 | 1.7974 | 1.7974 |
| 1.9431 | 2.0442 | 2.0442 |
| 2.1974 | 2.3274 | 2.3275 |
| 2.4872 | 2.6509 | 2.6511 |
| 2.8159 | 3.0190 | 3.0192 |
| 3.1875 | 3.4364 | 3.4366 |

In the last section we provided some Maple code for performing Euler's method. A similar code in MATLAB looks like the following:

```
a=0;
b=1;
N=10;
h=(b-a)/N;

% Slope function
```

```
  f = inline('t+y','t','y');
  sol = inline('2*exp(t)-t-1','t');

% Initial Condition
  t(1)=0;
  y(1)=1;

% Euler's Method
  for i=2:N+1
      y(i)=y(i-1)+h*f(t(i-1),y(i-1));
      t(i)=t(i-1)+h;
  end
```

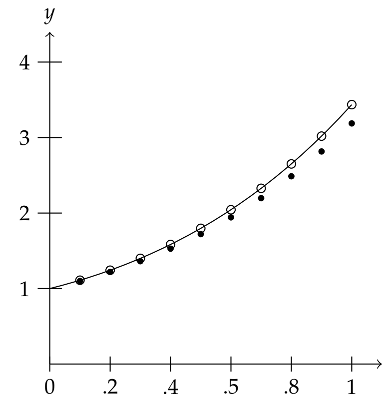


Figure 3.5: Numerical results for Euler's Method (filled circle) and 3rd Order Taylor's Method (open circle) for solving Example 3.2 as compared to exact solution (solid line).

A simple modification can be made for the 3rd Order Taylor's Method by replacing the Euler's method part of the preceding code by

```
% Taylor's Method, Order 3
  y(1)=1;
  h3 = h^2*(1/2+h/6);
  for i=2:N+1
      y(i)=y(i-1)+h*f(t(i-1),y(i-1))+h3*(1+t(i-1)+y(i-1));
      t(i)=t(i-1)+h;
  end
```

While the accuracy in the last example seemed sufficient, we have to remember that we only stopped at one unit of time. How can we be confident that the scheme would work as well if we carried out the computation for much longer times. For example, if the time unit were only a second, then one would need 86,400 times longer to predict a day forward. Of course, the scale matters. But, often we need to carry out numerical schemes for long times and we hope that the scheme not only converges to a solution, but that it coverges to the solution to the given problem. Also, the previous example was relatively easy to program because we could provide a relatively simple form for $T^{(3)}(t,y)$ with a quick computation of the derivatives of $f(t,y)$. This is not always the case and higher order Taylor methods in this form are not typically used. Instead, one can approximate $T^{(n)}(t,y)$ by evaluating the known function $f(t,y)$ at selected values of $t$ and $y$, leading to Runge-Kutta methods.

## 3.3 *Runge-Kutta Methods*

AS WE HAD SEEN IN THE LAST SECTION, we can use higher order Taylor methods to derive numerical schemes for solving

$$\frac{dy}{dt} = f(t,y), \quad y(a) = y_0, \quad t \in [a,b], \tag{3.17}$$

using a scheme of the form

$$\begin{aligned}\tilde{y}_{i+1} &= \tilde{y}_i + hT^{(n)}(t_i, \tilde{y}_i), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{3.18}$$

where we have defined

$$T^{(n)}(t, y) = y'(t) + \frac{h}{2} y''(t) + \cdots + \frac{h^{(n-1)}}{n!} y^{(n)}(t).$$

In this section we will find approximations of $T^{(n)}(t, y)$ which avoid the need for computing the derivatives.

For example, we could approximate

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2} fracdfdt(t, y)$$

by

$$T^{(2)}(t, y) \approx af(t + \alpha, y + \beta)$$

for selected values of $a$, $\alpha$, and $\beta$. This requires use of a generalization of Taylor's series to functions of two variables. In particular, for small $\alpha$ and $\beta$ we have

$$\begin{aligned} af(t + \alpha, y + \beta) &= a\left[ f(t, y) + \frac{\partial f}{\partial t}(t, y)\alpha + \frac{\partial f}{\partial y}(t, y)\beta \right. \\ &\quad \left. + \frac{1}{2}\left( \frac{\partial^2 f}{\partial t^2}(t, y)\alpha^2 + 2\frac{\partial^2 f}{\partial t \partial y}(t, y)\alpha\beta + \frac{\partial^2 f}{\partial y^2}(t, y)\beta^2 \right) \right] \\ &\quad + \text{ higher order terms.} \end{aligned} \tag{3.19}$$

Furthermore, we need $\frac{df}{dt}(t, y)$. Since $y = y(t)$, this can be found using a generalization of the Chain Rule from Calculus III:

$$\frac{df}{dt}(t, y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Thus,

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2}\left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right].$$

Comparing this expression to the linear (Taylor series) approximation of $af(t + \alpha, y + \beta)$, we have

$$\begin{aligned} T^{(2)} &\approx af(t + \alpha, y + \beta) \\ f + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2} f \frac{\partial f}{\partial y} &\approx af + a\alpha\frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial y}. \end{aligned} \tag{3.20}$$

We see that we can choose

$$a = 1, \quad \alpha = \frac{h}{2}, \quad \beta = \frac{h}{2} f.$$

The Midpoint or Second Order Runge-Kutta Method.

This leads to the numerical scheme

$$\begin{aligned} \tilde{y}_{i+1} &= \tilde{y}_i + hf\left( t_i + \frac{h}{2}, \tilde{y}_i + \frac{h}{2} f(t_i, \tilde{y}_i) \right), \quad i = 0, 1, \ldots, N-1, \\ \tilde{y}_0 &= y_0, \end{aligned} \tag{3.21}$$

This Runge-Kutta scheme is called the Midpoint Method, or Second Order Runge-Kutta Method, and it has order 2 if all second order derivatives of $f(t,y)$ are bounded.

Often, in implementing Runge-Kutta schemes, one computes the arguments separately as shown in the following MATLAB code snippet. (This code snippet could replace the Euler's Method section in the code in the last section.)

```
% Midpoint Method
  y(1)=1;
  for i=2:N+1
      k1=h/2*f(t(i-1),y(i-1));
      k2=h*f(t(i-1)+h/2,y(i-1)+k1);
      y(i)=y(i-1)+k2;
      t(i)=t(i-1)+h;
  end
```

**Example 3.3.** *Compare the Midpoint Method with the 2nd Order Taylor's Method for the problem*

$$y' = t^2 + y, \quad y(0) = 1, \quad t \in [0,1]. \tag{3.22}$$

*The solution to this problem is $y(t) = 3e^t - 2 - 2t - t^2$. In order to implement the 2nd Order Taylor's Method, we need*

$$\begin{aligned} T^{(2)} &= f(t,y) + \frac{h}{2} f'(t,y) \\ &= t^2 + y + \frac{h}{2}(2t + t^2 + y). \end{aligned} \tag{3.23}$$

*The results of the implementation are shown in Table 3.3.*

| Exact | Taylor | Error | Midpoint | Error |
|---|---|---|---|---|
| 1.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 |
| 1.1055 | 1.1050 | 0.0005 | 1.1053 | 0.0003 |
| 1.2242 | 1.2231 | 0.0011 | 1.2236 | 0.0006 |
| 1.3596 | 1.3577 | 0.0019 | 1.3585 | 0.0010 |
| 1.5155 | 1.5127 | 0.0028 | 1.5139 | 0.0016 |
| 1.6962 | 1.6923 | 0.0038 | 1.6939 | 0.0023 |
| 1.9064 | 1.9013 | 0.0051 | 1.9032 | 0.0031 |
| 2.1513 | 2.1447 | 0.0065 | 2.1471 | 0.0041 |
| 2.4366 | 2.4284 | 0.0083 | 2.4313 | 0.0053 |
| 2.7688 | 2.7585 | 0.0103 | 2.7620 | 0.0068 |
| 3.1548 | 3.1422 | 0.0126 | 3.1463 | 0.0085 |

Table 3.5: Numerical values for 2nd Order Taylor's Method, Midpoint Method, exact solution, and errors for solving Example 3.3 with $N = 10$..

There are other way to approximate higher order Taylor polynomials. For example, we can approximate $T^{(3)}(t,y)$ using four parameters by

$$T^{(3)}(t,y) \approx af(t,y) + bf(t+\alpha, y + \beta f(t,y).$$

Expanding this approximation and using

$$T^{(3)}(t,y) \approx f(t,y) + \frac{h}{2}\frac{df}{dt}(t,y) + \frac{h^2}{6}\frac{df}{dt}(t,y),$$

we find that we cannot get rid of $O(h^2)$ terms. Thus, the best we can do is derive second order schemes. In fact, following a procedure similar to the derivation of the Midpoint Method, we find that

$$a+b=1, \quad ,\alpha b = \frac{h}{2}, \beta = \alpha.$$

There are three equations and four unknowns. Therefore there are many second order methods. Two classic methods are given by the modified Euler method ($a = b = \frac{1}{2}$, $\alpha = \beta = h$) and Huen's method ($a = \frac{1}{4}$, $b = \frac{3}{4}$, $\alpha = \beta = \frac{2}{3}h$).

The Fourth Order Runge-Kutta.

The Fourth Order Runge-Kutta Method, which is most often used, is given by the scheme

$$\begin{aligned}
\tilde{y}_0 &= y_0, \\
k_1 &= hf(t_i, \tilde{y}_i), \\
k_2 &= hf(t_i + \frac{h}{2}, \tilde{y}_i + \frac{1}{2}k_1), \\
k_3 &= hf(t_i + \frac{h}{2}, \tilde{y}_i + \frac{1}{2}k_2), \\
k_4 &= hf(t_i + h, \tilde{y}_i + k_3), \\
\tilde{y}_{i+1} &= \tilde{y}_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad i = 0, 1, \ldots, N-1. \qquad (3.24)
\end{aligned}$$

Again, we can test this on Example 3.3 with $N = 10$. The MATLAB implementation is given by

```
% Runge-Kutta 4th Order to solve dy/dt = f(t,y), y(a)=y0, on [a,b]
  clear

  a=0;
  b=1;
  N=10;
  h=(b-a)/N;

% Slope function
  f = inline('t^2+y','t','y');
  sol = inline('-2-2*t-t^2+3*exp(t)','t');

% Initial Condition
  t(1)=0;
  y(1)=1;

% RK4 Method
  y1(1)=1;
```

```
for i=2:N+1
    k1=h*f(t(i-1),y1(i-1));
    k2=h*f(t(i-1)+h/2,y1(i-1)+k1/2);
    k3=h*f(t(i-1)+h/2,y1(i-1)+k2/2);
    k4=h*f(t(i-1)+h,y1(i-1)+k3);
    y1(i)=y1(i-1)+(k1+2*k2+2*k3+k4)/6;
    t(i)=t(i-1)+h;
end
```

MATLAB has built-in ODE solvers, such as **ode45** for a fourth order Runge-Kutta method. Its implementation is given by

MATLAB has built-in ODE solvers, as do other software packages, like Maple and Mathematica. You should also note that there are currently open source packages, such as Python based NumPy and Matplotlib, or Octave, of which some packages are contained within the Sage Project.

```
[t,y]=ode45(f,[0 1],1);
```

In this case $f$ is given by an inline function like in the above RK4 code. The time interval is enetered as $[0,1]$ and the 1 is the initial condition, $y(0) = 1$.

However, **ode45** is not a straight forward RK4 implementation. It is a hybrid method in which a combination of 4th and 5th order methods are combined allowing for adaptive methods to handled subintervals of the integration region which need more care. In this case, it implements a fourth order Runge-Kutta-Fehlberg method. Running this code for the above example actually results in values for $N = 41$ and not $N = 10$. If we wanted to have the routine output numerical solutions at specific times, then one could use the following form

```
tspan=0:h:1;
[t,y]=ode45(f,tspan,1);
```

In Table 3.6 we show the solutions which results for Example 3.3 comparing the RK4 snippet above with **ode45**. As you can see RK4 is much better than the previous implementation of the second order RK (Midpoint) Method. However, the MATLAB routine is two orders of magnitude better that RK4.

| Exact | Taylor | Error | Midpoint | Error |
|---|---|---|---|---|
| 1.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 |
| 1.1055 | 1.1055 | 4.5894e-08 | 1.1055 | -2.5083e-10 |
| 1.2242 | 1.2242 | 1.2335e-07 | 1.2242 | -6.0935e-10 |
| 1.3596 | 1.3596 | 2.3850e-07 | 1.3596 | -1.0954e-09 |
| 1.5155 | 1.5155 | 3.9843e-07 | 1.5155 | -1.7319e-09 |
| 1.6962 | 1.6962 | 6.1126e-07 | 1.6962 | -2.5451e-09 |
| 1.9064 | 1.9064 | 8.8636e-07 | 1.9064 | -3.5651e-09 |
| 2.1513 | 2.1513 | 1.2345e-06 | 2.1513 | -4.8265e-09 |
| 2.4366 | 2.4366 | 1.6679e-06 | 2.4366 | -6.3686e-09 |
| 2.7688 | 2.7688 | 2.2008e-06 | 2.7688 | -8.2366e-09 |
| 3.1548 | 3.1548 | 2.8492e-06 | 3.1548 | -1.0482e-08 |

Table 3.6: Numerical values for Fourth Order Runge-Kutta Method, rk45, exact solution, and errors for solving Example 3.3 with $N = 10$.

There are many ODE solvers in MATLAB. These are typically useful if RK4 is having difficulty solving particular problems. For the most part, one is fine using RK4, especially as a starting point. For example, there is **ode23**, which is similar to **ode45** but combining a second and third order scheme. Applying the results to Example 3.3 we obtain the results in Table 3.6. We compare these to the second order Runge-Kutta method. The code snippets are shown below.

```
% Second Order RK Method
  y1(1)=1;
  for i=2:N+1
      k1=h*f(t(i-1),y1(i-1));
      k2=h*f(t(i-1)+h/2,y1(i-1)+k1/2);
      y1(i)=y1(i-1)+k2;
      t(i)=t(i-1)+h;
  end


  tspan=0:h:1;
  [t,y]=ode23(f,tspan,1);
```

Table 3.7: Numerical values for Second Order Runge-Kutta Method, rk23, exact solution, and errors for solving Example 3.3 with $N = 10$.

| Exact | Taylor | Error | Midpoint | Error |
|---|---|---|---|---|
| 1.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 |
| 1.1055 | 1.1053 | 0.0003 | 1.1055 | 2.7409e-06 |
| 1.2242 | 1.2236 | 0.0006 | 1.2242 | 8.7114e-06 |
| 1.3596 | 1.3585 | 0.0010 | 1.3596 | 1.6792e-05 |
| 1.5155 | 1.5139 | 0.0016 | 1.5154 | 2.7361e-05 |
| 1.6962 | 1.6939 | 0.0023 | 1.6961 | 4.0853e-05 |
| 1.9064 | 1.9032 | 0.0031 | 1.9063 | 5.7764e-05 |
| 2.1513 | 2.1471 | 0.0041 | 2.1512 | 7.8665e-05 |
| 2.4366 | 2.4313 | 0.0053 | 2.4365 | 0.0001 |
| 2.7688 | 2.7620 | 0.0068 | 2.7687 | 0.0001 |
| 3.1548 | 3.1463 | 0.0085 | 3.1547 | 0.0002 |

We have seen several numerical schemes for solving initial value problems. There are other methods, or combinations of methods, which aim to refine the numerical approximations efficiently as if the step size in the current methods were taken to be much smaller. Some methods extrapolate solutions to obtain information outside of the solution interval. Others use one scheme to get a guess to the solution while refining, or correcting, this to obtain better solutions as the iteration through time proceeds. Such methods are described in courses in numerical analysis and in the literature. At this point we will apply these methods to several physics problems before continuing with analytical solutions.

## 3.4 Numerical Applications

In this section we apply various numerical methods to several physics problems after setting them up. We first describe how to work with second order equations, such as the nonlinear pendulum problem. We will see that there is a bit more to numerically solving differential equations than to just running standard routines. As we explore these problems, we will introduce other methods and provide some MATLAB code indicating how one might set up the system.

Other problems covered in these applications are various free fall problems beginning with a falling body from a large distance from the Earth, to flying soccer balls, and falling raindrops. We will also discuss the numerical solution of the two body problem and the Friedmann equation as nonterrestrial applications.

### 3.4.1 The Nonlinear Pendulum

Now we will investigate the use of numercial methods for solving the nonlinear pendulum problem.

**Example 3.4. Nonlinear pendulum** *Solve*

$$\ddot{\theta} = -\frac{g}{L}\sin\theta, \quad \theta(0) = \theta_0, \quad \omega(0) = 0, \quad t \in [0,8],$$

*using Euler's Method. Use the parameter values of* $m = 0.005$ *kg,* $L = 0.500$ *m, and* $g = 9.8$ *m/s*$^2$.

*This is a second order differential equation. As describe later, we can write this differential equation as a system of two first order differential equations,*

$$\begin{aligned} \dot{\theta} &= \omega, \\ \dot{\omega} &= -\frac{g}{L}\sin\theta. \end{aligned} \tag{3.25}$$

*Defining the vector*

$$\Theta(t) = \begin{pmatrix} \theta(t) \\ \omega(t) \end{pmatrix},$$

*we can write the first order system as*

$$\frac{d\Theta}{dt} = \mathbf{F}(t,\Theta), \quad \Theta(0) = \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix},$$

*where*

$$F(t,\Theta) = \begin{pmatrix} \omega(t) \\ -\frac{g}{L}\sin\theta(t) \end{pmatrix}.$$

*This allows us to use the the methods we have discussed on this first order equation for* $\Theta(t)$.

*For example, Euler's Method for this system becomes*

$$\Theta_{i+1} = \Theta_{i+1} + h\mathbf{F}(t_i, \Theta_i)$$

*with* $\Theta_0 = \Theta(0)$.

*We can write this scheme in component form as*

$$\begin{pmatrix} \theta_{i+1} \\ \omega_{i+1} \end{pmatrix} = \begin{pmatrix} \theta_i \\ \omega_i \end{pmatrix} + h \begin{pmatrix} \omega_i \\ -\frac{g}{L}\sin\theta_i \end{pmatrix},$$

*or*

$$\begin{aligned} \theta_{i+1} &= \theta_i + h\omega_i, \\ \omega_{i+1} &= \omega_i - h\frac{g}{L}\sin\theta_i, \end{aligned} \tag{3.26}$$

*starting with* $\theta_0 = \theta_0$ *and* $\omega_0 = 0$.

*The MATLAB code that can be used to implement this scheme takes the form*

```
g=9.8;
L=0.5;
m=0.005;

a=0;
b=8;
N=500;
h=(b-a)/N;

% Initial Condition
  t(1)=0;
  theta(1)=pi/6;
  omega(1)=0;

% Euler's Method
  for i=2:N+1
      omega(i)=omega(i-1)-g/L*h*sin(theta(i-1));
      theta(i)=theta(i-1)+h*omega(i-1);
      t(i)=t(i-1)+h;
  end
```

*In Figure 3.6 we plot the solution for a starting position of 30*$^o$ *with* $N = 500$. *Notice that the amplitude of oscillation is increasing, contrary to our experience. So, we increase N and see if that helps. In Figure 3.7 we show the results for* $N = 500$, $1000$, *and* $2000$ *points, or* $h = 0.016$, $0.008$, *and* $0.004$, *respectively. We note that the amplitude is not increasing as much.*

The problem with the solution is that Euler's Method is not an energy conserving method. As conservation of energy is important in physics, we would like to be able to seek problems which conserve energy. Such schemes

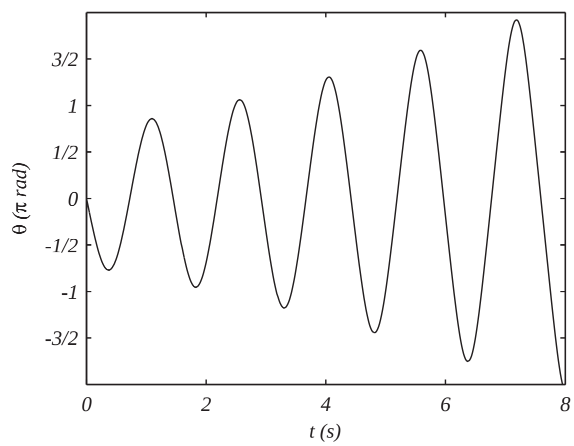



Figure 3.6: Solution for the nonlinear pendulum problem using Euler's Method on $t \in [0, 8]$ with $N = 500$.

used to solve oscillatory problems in classical mechanics are called symplectic integrators. A simple example is the Euler-Cromer, or semi-implicit Euler Method. We only need to make a small modification of Euler's Method. Namely, in the second equation of the method we use the updated value of the dependent variable as computed in the first line.

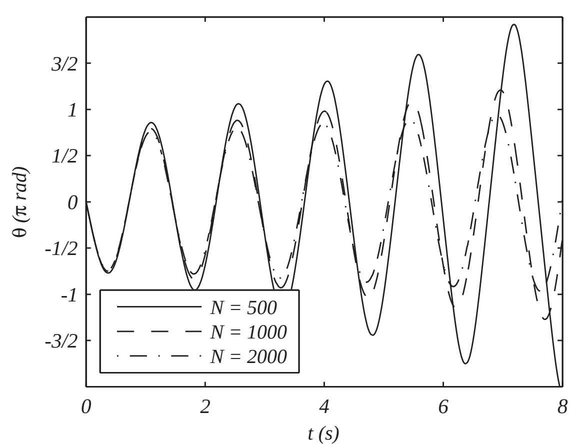



Figure 3.7: Solution for the nonlinear pendulum problem using Euler's Method on $t \in [0, 8]$ with $N = 500, 1000, 2000$.

Let's write the Euler scheme as

$$\begin{aligned} \omega_{i+1} &= \omega_i - h\frac{g}{L}\sin\theta_i, \\ \theta_{i+1} &= \theta_i + h\omega_i. \end{aligned} \tag{3.27}$$

Then, we replace $\omega_i$ in the second line by $\omega_{i+1}$ to obtain the new scheme

$$\begin{aligned} \omega_{i+1} &= \omega_i - h\frac{g}{L}\sin\theta_i, \\ \theta_{i+1} &= \theta_i + h\omega_{i+1}. \end{aligned} \tag{3.28}$$

The MATLAB code is easily changed as shown below.

```
g=9.8;
```

```
L=0.5;
m=0.005;

a=0;
b=8;
N=500;
h=(b-a)/N;

% Initial Condition
  t(1)=0;
  theta(1)=pi/6;
  omega(1)=0;

% Euler-Cromer Method
  for i=2:N+1
      omega(i)=omega(i-1)-g/L*h*sin(theta(i-1));
      theta(i)=theta(i-1)+h*omega(i);
      t(i)=t(i-1)+h;
  end
```

We then run the new scheme for $N = 500$ and compare this with what we obtained before. The results are shown in Figure 3.8. We see that the oscillation amplitude seems to be under control. However, the best test would be to investigate if the energy is conserved.

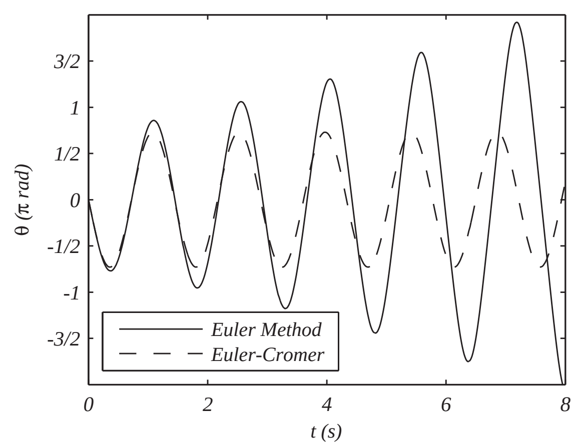



Figure 3.8: Solution for the nonlinear pendulum problem comparing Euler's Method and the Euler-Cromer Method on $t \in [0, 8]$ with $N = 500$.

Recall that the total mechanical energy for a pendulum consists of the kinetic and gravitational potential energies,

$$E = \frac{1}{2}mv^2 + mgh.$$

For the pendulum the tangential velocity is given by $v = L\omega$ and the height of the pendulum mass from the lowest point of the swing is $h = L(1 - \cos\theta)$.

Therefore, in terms of the dynamical variables, we have

$$E = \frac{1}{2}mL^2\omega^2 + mgL(1-\cos\theta).$$

We can compute the energy at each time step in the numerical simulation. In MATLAB it is easy to do using

```
E = 1/2*m*L^2*omega.^2+m*g*L*(1-cos(theta));
```

after implementing the scheme. In other programming environments one needs to loop through the times steps and compute the energy along the way. In Figure 3.9 we shown the results for Euler's Method for $N = 500, 1000, 2000$ and the Euler-Cromer Method for $N = 500$. It is clear that the Euler-Cromer Method does a much better job at maintaining energy conservation.

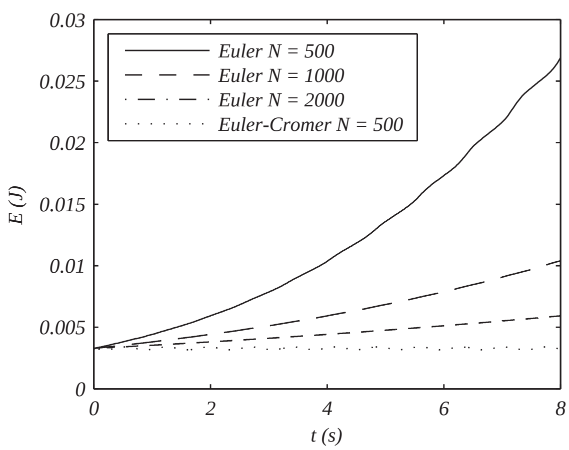



Figure 3.9: Total energy for the nonlinear pendulum problem.

### 3.4.2 *Extreme Sky Diving*

On October 14, 2012 Felix Baumgartner jumped from a helium balloon at an altitude of 39045 m (24.26 mi or 128100 ft). According preliminary data from the Red Bull Stratos Mission[1], as of November 6, 2012 Baumgartner experienced free fall until he opened his parachute at 1585 m after 4 minutes and 20 seconds. Within the first minute he had broken the record set by Joe Kittinger on August 16, 1960. Kittinger jumped from 102,800 feet (31 km) and fell freely for 4 minutes and 36 seconds to an altitude of 18,000 ft (5,500 m). Both set records for their times. Kittinger reached 614 mph (Mach 0.9) and Baumgartner reached 833.9 mph (Mach 1.24). Another record that was broken was that over 8 million watched the event on YouTube, breaking current live stream viewing events.

[1] The original estimated data was found at the Red Bull Stratos site, http://www.redbullstratos.com/. Some of the data has since been updated. The reader can redo the solution using the updated data.

This much attention also peaked interest in the physics of free fall. Free fall at constant $g$ through a height of $h$ should take a time of

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(36,529)}{9.8}} = 86 \text{ s}.$$

Of course, $g$ is not constant. In fact, at an altitude of 39 km, we have

$$g = \frac{GM}{R+h} = \frac{6.67 \times 10^{-11} \text{ N m}^2\text{kg}^2(5.97 \times 10^{24} \text{ kg})}{6375 + 39 \text{ km}} = 9.68 \text{ m/s}^2.$$

So, $g$ is roughly constant.

Next, we need to consider the drag force as one free falls through the atmosphere, $F_D = \frac{1}{2}CA\rho_a v^2$. One needs some values for the parameters in this problem. Let's take $m = 90$ kg, $A = 1.0$ m$^2$, and $\rho = 1.29$ kg/m$^3$, $C = 0.42$. Then, a simple model would give

$$m\dot{v} = -mg + \frac{1}{2}CA\rho v^2,$$

or

$$\dot{v} = -g + .0030v^2.$$

This gives a terminal velocity of 57.2 m/s, or 128 mph. However, we again have assumed that the drag coefficient and air density are constant. Since the Reynolds number is high, we expect $C$ is roughly constant. However, the density of the atmosphere is a function of altitude and we need to take this into account.

The Reynolds number is used several times in this chapter. It is defined as

$$Re = \frac{2rv}{\nu},$$

where $\nu$ is the kinematic viscosity. The kinematic viscosity of air at $60^o$ F is about $1.47 \times 10^{-5}$ m$^2$/s.

A simple model for $\rho = \rho(h)$ can be found at the NASA site.[2]. Using their data, we have

[2] http://www.grc.nasa.gov/WWW/k-12/rocket/atmos.html

$$\rho(h) = \begin{cases} \dfrac{101290(1.000 - 0.2253 \times 10^{-4}h)^{5.256}}{83007 - 1.8696h}, & h < 11000, \\ .3629e^{1.73-0.157\times10^{-3}h}, & h, < 25000 \\ \dfrac{2488}{(.6551 + 0.1380 \times 10^{-4}h)^{11.388}(40876 + .8614h)}, & h > 25000. \end{cases} \tag{3.29}$$

In Figure 3.10 the atmospheric density is shown as a function of altitude.

In order to use the methods for solving first order equations, we write the system of equations in the form

$$\begin{aligned} \frac{dh}{dt} &= v, \\ \frac{dv}{dt} &= -\frac{GM}{(R+h)^2} + \frac{1}{5}\rho(h)CAv^2. \end{aligned} \tag{3.30}$$

This is now in the form of a system of first order differential equations.

Then, we define a function to be called and store in as **gravf.m** as shown below.

```
function dy=gravf(t,y);

G=6.67E-11;
```

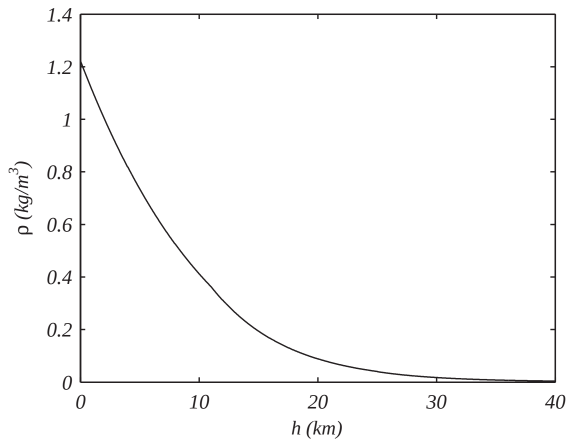



Figure 3.10: Atmospheric density as a function of altitude.

```
M=5.97E24;
R=6375000;
m=90;
C=.42;
A=1;

dy(1,1)=y(2);
dy(2,1)=-G*M/(R+y(1)).^2+.5*density2(y(1))*C*A*y(2).^2/m;
```

Now we are ready to call the function in our favorite routine.

```
h0=1000;
tmax=20;
tmin=0;
[t,y]=ode45('dgravf',[tmin tmax],[h0;0]);% Const rho
plot(t,y(:,1),'k--')
```

Here we are simulating free fall from an altitude of one kilometer. In Figure 3.11 we compare different models of free fall with $g$ taken as constant or derived from Newton's Law of Gravitation. We also consider constant density or the density dependence on the altitude as given earlier. We chose to keep the drag coefficient constant at $C = 0.42$.

We can see from these plots that the slight variation in the acceleration due to gravity does not have as much an effect as the variation of density with distance.

Now we can push the model to Baumgartner's jump from 39 km. In Figure 3.12 we compare the general model with that with no air resistance, though both taking into account the variation in $g$. As a body falls through the atmosphere we see the changing effects of the denser atmosphere on the free fall. For the parameters chosen, we find that it takes 238.8s, or a little less than four minutes to reach the point where Baumgartner opened his parachute. While not exactly the same as the real fall, it is amazingly close.

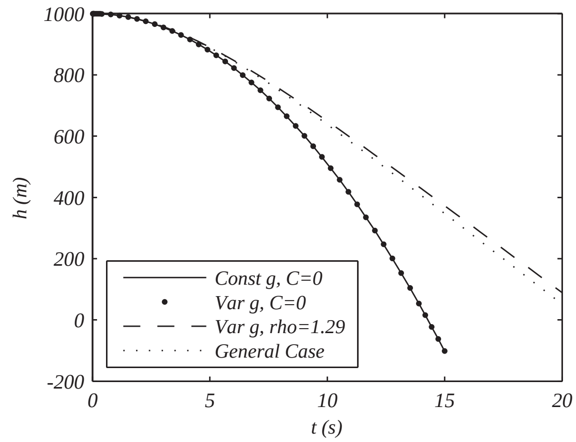



Figure 3.11: Comparison of different models of free fall from one kilometer above the Earth.

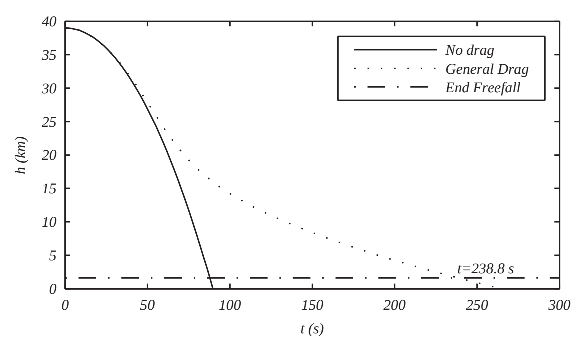



Figure 3.12: Free fall from 39 km at constant $g$ as compared to nonconstant $g$ and nonconstant atmospheric density with drag coefficient $C = .42$.

### 3.4.3 *The Flight of Sports Balls*

ANOTHER INTERESTING PROBLEM IS THE PROJECTILE MOTION of a sports ball. In an introductory physics course, one typically ignores air resistance and the path of the ball is a nice parabolic curve. However, adding air resistance complicates the problem significantly and cannot be solved analytically. Examples in sports are flying soccer balls, golf balls, ping pong balls, baseballs, and other spherical balls.

We will consider a ball moving in the $xz$-plane spinning about an axis perpendicular to the plane of motion. Such an analysis was reported in Goff and Carré, AJP 77(11) 1020. The typical trajectory of the ball is shown in Figure 3.13. The forces acting on the ball are the drag force, $\mathbf{F}_D$, the lift force, $\mathbf{F}_L$, and the gravitational force, $\mathbf{F}_W$. These are indicated in Figure 3.14. The equation of motion takes the form

$$m\mathbf{a} = \mathbf{F}_W + \mathbf{F}_D + \mathbf{F}_L.$$

Writing out the components, we have

$$ma_x = -F_D \cos\theta - F_L \sin\theta \quad (3.31)$$

$$ma_z = -mg - F_D \sin\theta + F_L \cos\theta. \quad (3.32)$$

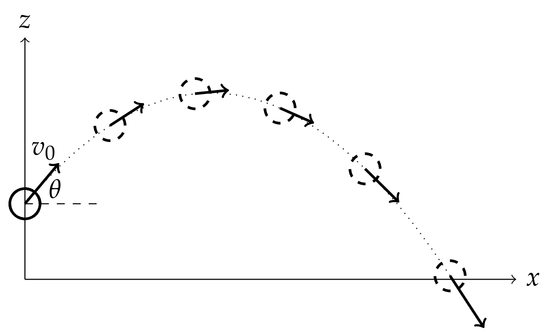



Figure 3.13: Projectile path.

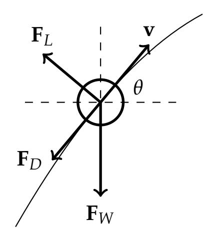



Figure 3.14: Forces acting on ball.

As we had seen before, the magnitude of the damping (drag) force is given by

$$F_D = \frac{1}{2} C_D \rho A v^2.$$

For the case of soccer ball dynamics, Goff and Carré noted that the Reynolds number, $Re = \frac{2rv}{\nu}$, is between 70000 and 490000 by using a kinematic viscosity of $\nu = 1.54 \times 10^{-5}$ m$^2$/s and typical speeds of $v = 4.5 - 31$ m/s. Their analysis gives $C_D \approx 0.2$. The parameters used for the ball were $m = 0.424$ kg and cross sectional area $A = 0.035$ m$^2$ and the density of air was taken as 1.2 kg/m$^3$.

The lift force takes a similar form,

$$F_L = \frac{1}{2} C_L \rho A v^2.$$

The sign of $C_L$ indicates if the ball has top spin ($C_L < 0$) or bottom spin *$C_L > 0$). The lift force is just one component of a more general Magnus force, which is the force on a spinning object in a fluid and is perpendicular to the motion. In this example we assume that the spin axis is perpendicular to the plane of motion. Allowing for spinning balls to veer from this plane would mean that we would also need a component of the Magnus force perpendicular to the plane of motion. This would lead to an additional sideways component (in the **k** direction) leading to a third acceleration equation. We will leave that case for the reader.

The lift coefficient can be related to the spin as

$$C_L = \frac{1}{2 + \frac{v}{v_{spin}}},$$

where $v_{spin} = r\omega$ is the peripheral speed of the ball. Here $R$ is the ball radius and $\omega$ is the angular speed in rad/s. If $v = 20$ m/s, $\omega = 200$ rad/s, and $r = 20$ mm, then $C_L = 0.45$.

So far, the problem has been reduced to

$$\frac{dv_x}{dt} = -\frac{\rho A}{2m}(C_D \cos\theta + C_L \sin\theta)v^2, \tag{3.33}$$

$$\frac{dv_z}{dt} = -g - \frac{\rho A}{2m}(C_D \sin\theta - C_L \cos\theta)v^2, \tag{3.34}$$

for $v_x$ and $v_z$ the components of the velocity. Also, $v^2 = v_x^2 + v_z^2$. Furthermore, from Figure 3.14, we can write

$$\cos\theta = \frac{v_x}{v}, \quad \sin\theta = \frac{v_z}{v}.$$

So, the equations can be written entirely as a system of differential equations for the velocity components,

$$\frac{dv_x}{dt} = -\alpha(C_D v_x + C_L v_z)(v_x^2 + v_z^2)^{1/2}, \tag{3.35}$$

$$\frac{dv_z}{dt} = -g - \alpha(C_D v_z - C_L v_x)(v_x^2 + v_z^2)^{1/2}, \tag{3.36}$$

where $\alpha = \rho A/2m = 0.0530$ m$^{-1}$.

Such systems of equations can be solved numerically by thinking of this as a vector differential equation,

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}(t, \mathbf{v}),$$

and applying one of the numerical methods for solving first order equations.

Since we are interested in the trajectory, $z = z(x)$, we would like to determine the parametric form of the path, $(x(t), z(t))$. So, instead of solving two first order equations for the velocity components, we can rewrite the two second order differential equations for $x(t)$ and $z(t)$ as four first order differential equations of the form

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(t, \mathbf{y}).$$

We first define

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ z(t) \\ v_x(t) \\ v_z(t) \end{bmatrix}$$

Then, the systems of first order differential equations becomes

$$\begin{aligned}
\frac{dy_1}{dt} &= y_3, \\
\frac{dy_2}{dt} &= y_4, \\
\frac{dy_3}{dt} &= -\alpha(C_D v_x + C_L v_z)(v_x^2 + v_z^2)^{1/2}, \\
\frac{dy_4}{dt} &= -g - \alpha(C_D v_z - C_L v_x)(v_x^2 + v_z^2)^{1/2}.
\end{aligned} \tag{3.37}$$

The system can be placed into a function file which can be called by an ODE solver, such as the MATLAB m-file below.

```
function dy = ballf(t,y)
global g CD CL alpha

dy = zeros(4,1);    % a column vector
v = sqrt(y(3).^2+y(4).^2);   % speed v

dy(1) = y(3);
dy(2) = y(4);
dy(3) = -alpha*v.*(CD*y(3)+CL*y(4));
dy(4) = alpha*v.*(-CD*y(4)+CL*y(3))-g;
```

Then, the solver can be called using

```
[T,Y] = ode45('ballf',[0 2.5],[x0,z0,v0x,v0z]);
```

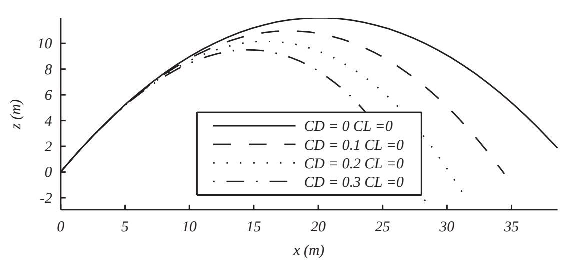



Figure 3.15: Example of soccer ball under the influence of drag.

In Figures 3.15 and 3.16 we indicate what typical solutions would look like for different values of drag and lift coefficients. In the case of nonzero lift coefficients, we indicate positive and negative values leading to flight with top spin, $C_L < 0$, or bottom spin, $C_L > 0$.

### 3.4.4 Falling Raindrops

A SIMPLE PROBLEM THAT APPEARS IN MECHANICS is that of a falling raindrop through a mist. The raindrop not only undergoes free fall, but the

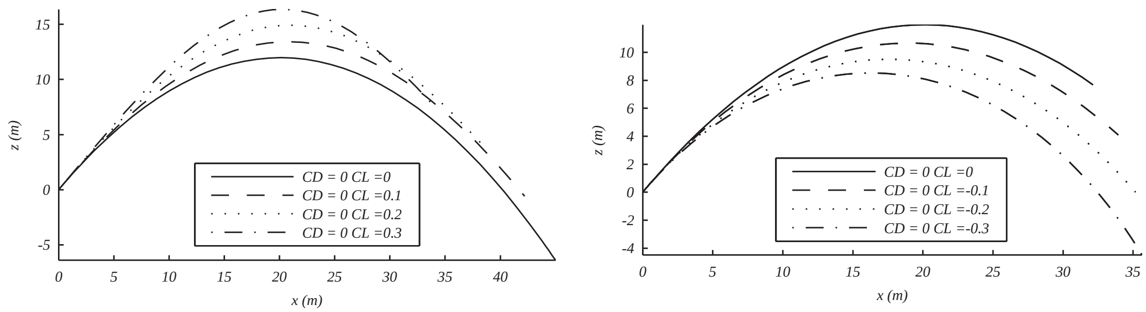



Figure 3.16: Example of soccer ball under the influence of lift with $CL > 0$ and $CL < 0$

mass of the drop grows as it interacts with the mist. There have been several papers written on this problem and it is a nice example to explore using numerical methods. In this section we look at models of a falling raindrop with and without air drag.

First we consider the case in which there is no air drag. A simple model of free fall from Newton's Second Law of Motion is

$$\frac{d(mv)}{dt} = mg.$$

In this discussion we will take downward as positive. Since the mass is not constant. we have

$$m\frac{dv}{dt} = mg - v\frac{dm}{dt}.$$

In order to proceed, we need to specify the rate at which the mass is changing. There are several models one can adapt.We will borrow some of the ideas and in some cases the numerical values from Sokal(2010)[3] and Edwards, Wilder, and Scime (2001).[4] These papers also quote other interesting work on the topic.

[3] A. D. Sokal, *The falling raindrop, revisited*, Am. J. Phys. 78, 643-645, (2010).

[4] B. F. Edwards, J. W. Wilder, and E. E. Scime, *Dynamics of Falling Raindrops*, Eur. J. Phys. 22, 113-118, (2001).

While $v$ and $m$ are functions of time, one can look for a way to eliminate time by assuming the rate of change of mass is an explicit function of $m$ and $v$ alone. For example, Sokal (2010) assumes the form

$$\frac{dm}{dt} = \lambda m^{\sigma} v^{\beta}, \quad \lambda > 0.$$

This contains two commonly assumed models of accretion:

1. $\sigma = 2/3$, $\beta = 0$. This corresponds to growth of the raindrop proportional to the surface area. Since $m \propto r^3$ and $A \propto r^2$, then $\dot{m} \propto A$ implies that $\dot{m} \propto m^{2/3}$.
2. $\sigma = 2/3$, $\beta = 1$. In this case the growth of the raindrop is proportional to the volume swept out along the path. Thus, $\Delta m \propto A(v\Delta t)$, where $A$ is the cross sectional area and $v\Delta t$ is the distance traveled in time $\Delta t$.

In both cases, the limiting value of the acceleration is a constant. It is $g/4$ in the first case and $g/7$ in the second case.

Another approach might be to use the effective radius of the drop, assuming that the raindrop remains close to spherical during the fall. According to Edwards, Wilder, and Scime (2001), raindrops with Reynolds number greater than 1000 and with radii larger than 1 mm will flatten. Even larger raindrops will break up when the drag force exceeds the surface tension. Therefore, they take 0.1 mm $< r <$ 1 mm and $10 < Re < 1000$. We will return to a discussion of the drag later.

It might seem more natural to make the radius the dynamic variable, than the mass. In this case, we can assume the accretion rate takes the form

$$\frac{dr}{dt} = \gamma r^{\alpha} v^{\beta}, \quad \gamma > 0.$$

Since, $m = \frac{4}{3}\pi\rho_d r^3$,

$$\frac{dm}{dt} \sim r^2 \frac{dr}{dt} \sim m^{2/3} \frac{dr}{dt}.$$

Therefore, the two special cases become

1. $\alpha = 0$, $\beta = 0$. This corresponds to a growth of the raindrop proportional to the surface area.
2. $\alpha = 0$, $\beta = 1$. In this case the growth of the raindrop is proportional to the volume swept out along the path.

Here $\rho_d$ is the density of the raindrop.

We will also need

$$\begin{aligned} \frac{v}{m}\frac{dm}{dt} &= \frac{4\pi\rho_d r^2}{\frac{4}{3}\pi\rho_d r^3} v \frac{dr}{dt} \\ &= 3\frac{v}{r}\frac{dr}{dt} \\ &= 3\gamma r^{\alpha-1} v^{\beta+1}. \end{aligned} \tag{3.38}$$

Putting this all together, we have a systems of two equations for $v(t)$ and $r(t)$ :

$$\begin{aligned} \frac{dv}{dt} &= g - 3\gamma r^{\alpha-1} v^{\beta+1}, \\ \frac{dr}{dt} &= \gamma r^{\alpha} v^{\beta}. \end{aligned} \tag{3.39}$$

**Example 3.5.** *Determine $v = v(r)$ for the case $\alpha = 0$, $\beta = 0$ and the initial conditions $r(0) = 0.1$ mm and $v(0) = 0$ m/s.*

*In this case Equations (3.39) become*

$$\begin{aligned} \frac{dv}{dt} &= g - 3\gamma r^{-1} v, \\ \frac{dr}{dt} &= \gamma. \end{aligned} \tag{3.40}$$

*Noting that*

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \gamma\frac{dv}{dr},$$

*we can convert the problem to one of finding the solution $v(r)$ subject to the equation*

$$\frac{dv}{dr} = \frac{g}{\gamma} - 3\frac{v}{r}$$

*with the initial condition $v(r_0) = 0$ m/s for $r_0 = 0.0001$ m.*

*Rearranging the differential equation, we find that it is a linear first order differential equation,*

$$\frac{dv}{dr} + \frac{3}{r}v = \frac{g}{\gamma}.$$

*This equation can be solved using an integrating factor, $\mu = r^3$, obtaining*

$$\frac{d}{dr}(r^3 v) = \frac{g}{\gamma}r^3.$$

*Integrating, we obtain the solution*

$$v(r) = \frac{g}{4\gamma}r\left(1 - \left(\frac{r_0}{r}\right)^4\right).$$

*Note that for large $r$, $v \sim \frac{g}{4\gamma}r$. Therefore, $\frac{dv}{dt} \sim \frac{g}{4}$.*

While this case was easily solved in terms of elementary operations, it is not always easy to generate solutions to Equations (3.39) analytically. Sokal (2010) derived a general solution in terms of incomplete Beta functions, though this does not help visualize the solution. Also, as we will see, adding air drag will lead to a nonintegrable system. So, we turn to numerical solutions.

In MATLAB, we can use the function in **raindropf.m** to capture the system (3.39). Here we put the velocity in y(1) and the radius in y(2).

```
function dy=raindropf(t,y);
global alpha beta gamma g

dy=[g-3*gamma*y(2)^(alpha-1)*y(1)^(beta+1); ...
    gamma*y(2)^alpha*y(1)^beta];
```

We then use the Runge-Kutta solver, **ode45**, to solve the system. An implementation is shown below which calls the function containing the system. The value $\gamma = 2.5 \times 10^{-7}$ is based on empirical results quoted by Edwards, Wilder, and Scime (2001).

```
clear
global alpha beta gamma g

alpha=0;
beta=0;
gamma=2.5e-07;
g=9.81;

r0=0.0001;
```

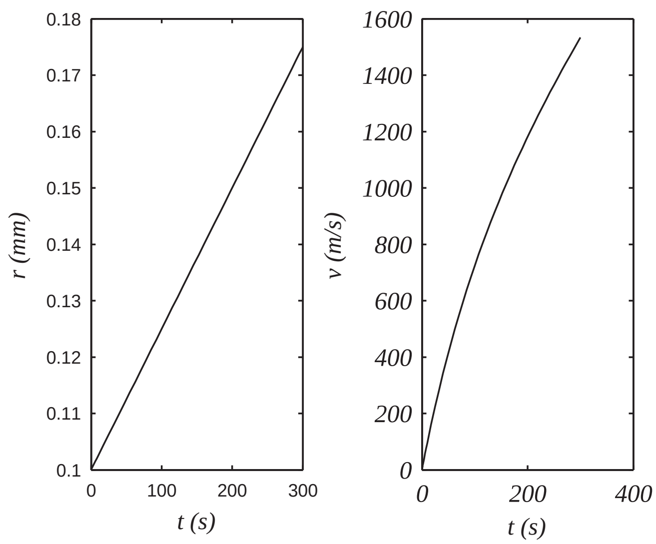



Figure 3.17: The plots of position and velocity as a function of time for $\alpha = \beta = 0$.

```
v0=0;
y0=[v0;r0];
tspan=[0 1000];

[t,y]=ode45(@raindropf,tspan,y0);
plot(1000*y(:,2),y(:,1),'k')
```

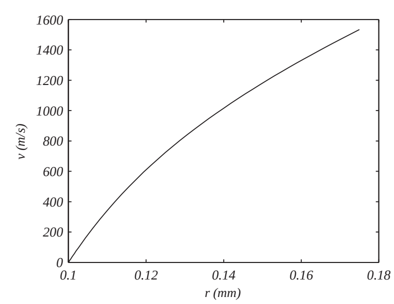



Figure 3.18: The plot the velocity as a function of position for $\alpha = \beta = 0$.

The resulting plots are shown in Figures 3.17-3.18. The plot of velocity as a function of position agrees with the exact solution, which we derived in the last example. We note that these drops do not grow much, but they seem to attain large speeds.

For the second case, $\alpha = 0, \beta = 1$, one can also obtain an exact solution. The result is

$$v(r) = \left[\frac{2g}{7\gamma} r \left(1 - \left(\frac{r_0}{r}\right)^7\right)\right]^{\frac{1}{2}}.$$

For large $r$ one can show that $\frac{dv}{dt} \sim \frac{g}{7}$. In Figures 3.20-3.19 we see again large velocities, though about a third as fast over the same time interval. However, we also see that the raindrop has significantly grown well past the point it would break up.

In this simple model of a falling raindrop we have not considered air drag. Earlier in the chapter we discussed the free fall of a body with air resistance and this lead to a terminal velocity. Recall that the drag force given by

$$f_D(v) = -\frac{1}{2} C_D A \rho_a v^2, \tag{3.41}$$

where $C_D$ is the drag coefficient, $A$ is the cross sectional area and $\rho_a$ is the air density. Also, we assume that the body is falling downward and downward is positive, so that $f_D(v) < 0$ so as to oppose the motion.

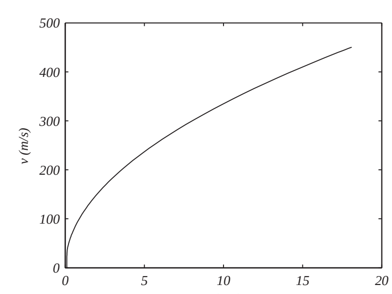



Figure 3.19: The plot the velocity as a function of position for $\alpha = 0, \beta = 1$.

We would like to incorporate this force into our model (3.39). The first equation came from the force law, which now becomes

$$m\frac{dv}{dt} = mg - v\frac{dm}{dt} - \frac{1}{2}C_D A\rho_a v^2,$$

or

$$\frac{dv}{dt} = g - \frac{v}{m}\frac{dm}{dt} - \frac{1}{2m}C_D A\rho_a v^2.$$

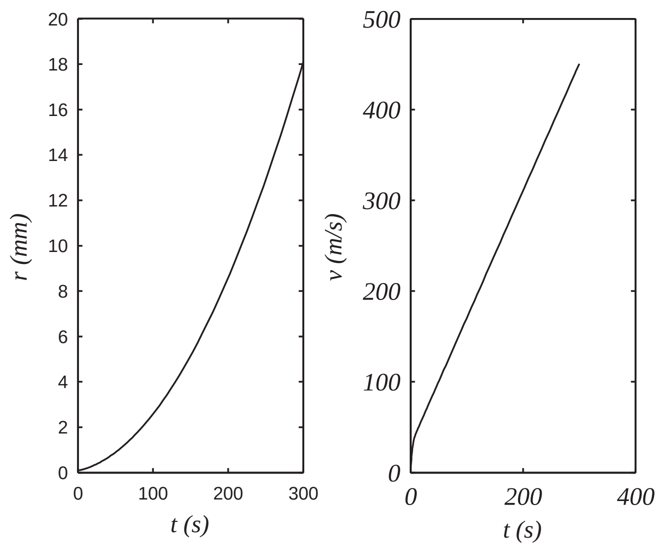



Figure 3.20: The plots of position and velocity as a function of time for $\alpha = 0, \beta = 1$.

The next step is to eliminate the dependence on the mass, $m$, in favor of the radius, $r$. The drag force term can be written as

$$\begin{aligned}\frac{f_D}{m} &= \frac{1}{2m}C_D A\rho_a v^2\\ &= \frac{1}{2}C_D\frac{\pi r^2}{\frac{4}{3}\pi\rho_d r^3}\rho_a v^2\\ &= \frac{3}{8}\frac{\rho_a}{\rho_d}C_D\frac{v^2}{r}.\end{aligned} \tag{3.42}$$

We had already done this for the second term; however, Edwards, Wilder, and Scime (2001) point to experimental data and propose that

$$\frac{dm}{dt} = \pi\rho_m r^2 v,$$

where $\rho_m$ is the mist density. So, the second terms leads to

$$\frac{v}{m}\frac{dm}{dt} = \frac{3}{4}\frac{\rho_m}{\rho_d}\frac{v^2}{r}.$$

But, since $m = \frac{4}{3}\pi\rho_d r^3$,

$$\frac{dm}{dt} = 4\pi\rho_d r^2\frac{dr}{dt}.$$

So,

$$\frac{dr}{dt} = \frac{\rho_m}{4\rho_d} v.$$

This suggests that their model corresponds to $\alpha = 0$, $\beta = 1$, and $\gamma = \frac{\rho_m}{4\rho_d}$.

Now we can write down the modified system

$$\begin{aligned} \frac{dv}{dt} &= g - 3\gamma r^{\alpha-1} v^{\beta+1} - \frac{3}{8}\frac{\rho_a}{\rho_d} C_D \frac{v^2}{r}, \\ \frac{dr}{dt} &= \gamma r^{\alpha} v^{\beta}. \end{aligned} \tag{3.43}$$

Edwards, Wilder, and Scime (2001) assume that the densities are constant with values $\rho_a = .856$ kg/m$^3$, $\rho_d = 1.000$ kg/m$^3$, and $\rho_m = 1.00 \times 10^{-3}$ kg/m$^3$. However, the drag coefficient is not constant. As described later in Section 3.4.7, there are various models indicating the dependence of $C_D$ on the Reynolds number,

$$Re = \frac{2rv}{\nu},$$

where $\nu$ is the kinematic viscosity, which Edwards, Wilder, and Scime (2001) set to $\nu = 2.06 \times 10^{-5}$ m$^2$/s. For raindrops of the range $r = 0.1$ mm to 1 mm, the Reynolds number is below 1000. Edwards, Wilder, and Scime (2001) modeled $C_D = 12Re^{-1/2}$. In the plots in Section 3.4.7 we include this model and see that this is a good approximation for these raindrops. In Chapter 10 we discuss least squares curve fitting and using these methods, one can use the models of Putnam (1961) and Schiller-Naumann (1933) to obtain a power law fit similar to that used here.

So, introducing

$$C_D = 12Re^{-1/2} = 12\left(\frac{2rv}{\nu}\right)^{-1/2}$$

and defining

$$\delta = \frac{9}{2^{3/2}}\frac{\rho_a}{\rho_d}\nu^{1/2},$$

we can write the system of equations (3.43) as

$$\begin{aligned} \frac{dv}{dt} &= g - 3\gamma \frac{v^2}{r} - \delta\left(\frac{v}{r}\right)^{\frac{3}{2}}, \\ \frac{dr}{dt} &= \gamma v. \end{aligned} \tag{3.44}$$

Now, we can modify the MATLAB code for the raindrop by adding the extra term to the first equation, setting $\alpha = 0$, $\beta = 1$, and using $\delta = 0.0124$ and $\gamma = 2.5 \times 10^{-7}$ from Edwards, Wilder, and Scime (2001).

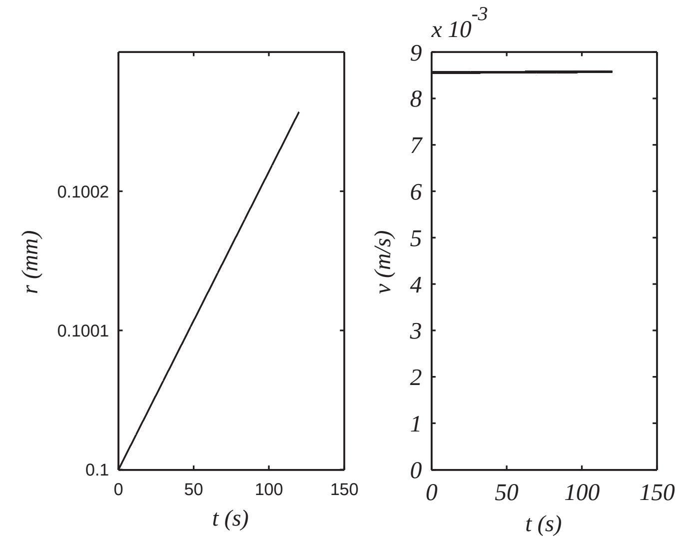



Figure 3.21: The plots of position and velocity as a function of time with air drag included.

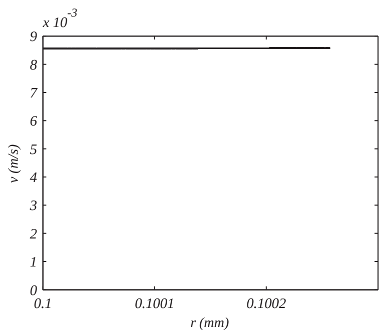



Figure 3.22: The plot the velocity as a function of position with air drag included.

In Figures 3.21-3.22 we see different behaviors as compared to the previous models. It appears that the velocity quickly reaches a terminal velocity and the radius continues to grow linearly in time, though at a slow rate.

We might be able to understand this behavior. Terminal, or constant $v$, would occur when

$$g - 3\gamma\frac{v^2}{r} - \delta\left(\frac{v}{r}\right)^{\frac{3}{2}} = 0.$$

Looking at these terms, one finds that the second term is significantly smaller than the other terms and thus

$$\delta\left(\frac{v}{r}\right)^{\frac{3}{2}} \approx g,$$

or

$$\frac{v}{r} \approx \left(\frac{g}{\delta}\right)^{2/3} \approx 85.54 \text{ s}^{-1}.$$

This agrees with the numerical data which gives the slope of the $v$ vs $r$ plot as $85.5236 \text{ s}^{-1}$.

### 3.4.5 *The Two-body Problem*

A STANDARD PROBLEM IN CLASSICAL DYNAMICS is the study of the motion of several bodies under the influence of Newton's Law of Gravitation. The so-called $n$-body problem is not solvable. However, the two body problem is. Such problems can model the motion of a planet around the sun, the moon around the Earth, or a satellite around the Earth. Further interesting, and more realistic problems, would involve perturbations of these orbits due to additional bodies. For example, one can study problems such as the influence of large planets on the asteroid belt. Since there are no analytic solutions to these problems, we have to resort to finding numerical solutions. We will look at the two body problem since we can compare the numerical methods to the exact solutions.

We consider two masses, $m_1$ and $m_2$, located at positions, $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, as shown in Figure 3.23. Newton's Law of Gravitation for the force between two masses separated by position vector $\mathbf{r}$ is given by

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\frac{\mathbf{r}}{r}.$$

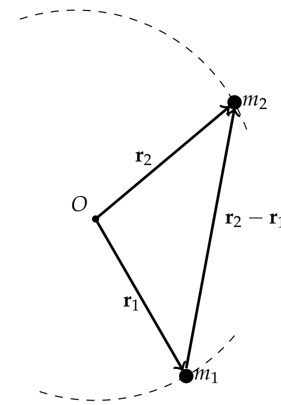



Figure 3.23: Two masses interact under Newton's Law of Gravitation.

Each mass experiences this force due to the other mass. This gives the system of equations

$$m_1\ddot{\mathbf{r}}_1 = -\frac{Gm_1m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.45}$$

$$m_2\ddot{\mathbf{r}}_2 = -\frac{Gm_1m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1). \tag{3.46}$$

Now we seek to set up this system so that we can find numerical solutions for the positions of the masses. From the conservation of angular momentum, we know that the motion takes place in a plane. [Note: The solution of the Kepler Problem is discussed in Chapter 9.] We will choose the orbital plane to be the $xy$-plane. We define $r_{12} = |\mathbf{r}_2 - \mathbf{r}_1|$, and $(x_i, y_i) = \mathbf{r}_i$, $i = 1, 2$. Furthermore, we write the two second order equations as four first order equations. So, defining the velocity components as $(u_i, v_i) = \mathbf{v}_i$, the system of equations can be written in the form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ -\alpha m_2(x_1 - x_2) \\ -\alpha m_2(y_1 - y_2) \\ -\alpha m_1(x_2 - x_1) \\ -\alpha m_1(y_2 - y_1). \end{pmatrix}, \tag{3.47}$$

where $\alpha = \frac{G}{r_{12}^3}$.

This system can be encoded in MATLAB as indicated in the function **twobody**:

```
function dz = twobody(t,z)
dz = zeros(8,1);
G = 1;
m1 = .1;
m2 = 2;
r=((z(1) - z(3)).^2 + (z(2) - z(4)).^2).^(3/2);
alpha=G/r;
dz(1) = z(5);
dz(2) = z(6);
dz(3) = z(7);
dz(4) = z(8);
```

```
dz(5) = alpha*m2*(z(3) - z(1));
dz(6) = alpha*m2*(z(4) - z(2));
dz(7) = alpha*m1*(z(1) - z(3));
dz(8) = alpha*m1*(z(2) - z(4));
```

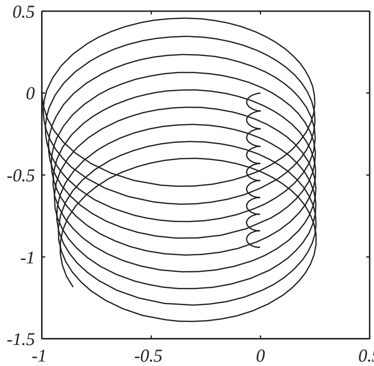


Figure 3.24: Simulation of two bodies under gravitational attraction.

In the above code we picked some seemingly nonphysical numbers for $G$ and the masses. Calling **ode45** with a set of initial conditions,

```
[t,z] = ode45('twobody',[0 20], [-1 0 0 0 0 -1 0 0]);
plot(z(:,1),z(:,2),'k',z(:,3),z(:,4),'k');
```

we obtain the plot shown in Figure 3.24. We see each mass moves along what looks like elliptical helices with the smaller body tracing out a larger orbit.

In the case of a very large body, most of the motion will be due to the smaller body. So, it might be better to plot the relative motion of the small body with respect to the larger body. Actually, an analysis of the two body problem shows that the center of mass

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

satisfies $\ddot{\mathbf{R}} = 0$. Therefore, the system moves with a constant velocity.

The relative position of the masses is defined through the variable $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Dividing the masses from the left hand side of Equations (3.46) and subtracting, we have the motion of $m_1$ about $m_2$

$$\ddot{\mathbf{r}} = -G(m_1 + m_2)\frac{\mathbf{r}}{r^3},$$

where $r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$. Note that $\mathbf{r} \times \ddot{\mathbf{r}} = 0$. Integrating, this gives $\mathbf{r} \times \dot{\mathbf{r}} =$ constant. This is just a statement of the conservation of angular momentum.

The orbiting body will remain in a plane and, therefore, we can take the $z$-axis perpendicular to $\mathbf{r} \times \dot{\mathbf{r}}$, the position as $\mathbf{r} = (x(t), y(t))$, and the velocity as $\dot{\mathbf{r}} = (u(t), v(t))$. Then, the equations of motion can be written as four first order equations:

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= -\mu\frac{x}{r^3} \\ \dot{v} &= -\mu\frac{y}{r^3}, \end{aligned} \tag{3.48}$$

where $\mu = G(m_1 + m_2)$ and $r = \sqrt{x^2 + y^2}$.

While we have established a system of equations which can be integrated, we should note a few results from the study of the Kepler problem in classical dynamics which we review in Chapter 9. Kepler's Laws of Planetary Motion state:

1. All planets travel in ellipses.

The polar equation for the path is given by

$$r = \frac{a(1-e^2)}{1+e\cos\phi},$$

where $e$ is the eccentricity and $a$ is the length of the semimajor axis. For $0 \le e < 1$, the orbit is an ellipse.

2. A planet sweeps out equal areas in equal times.
3. The square of the period of the orbit is proportional to the cube of the semimajor axis. In particular, one can show that

$$T^2 = \frac{4\pi^2}{\mu}a^3.$$

By an appropriate choice of units, we can make $\mu = G(m_1 + m_2)$ a reasonable number. For the Earth-Sun system,

$$\begin{aligned} \mu &= 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2} (1.99 \times 10^{30} + 5.97 \times 10^{24}) kg \\ &= 1.33 \times 10^{20} m^3 s^{-1}. \end{aligned}$$

That is a large number and can cause problems in the numerics. However, if one uses astronomical scales, such as putting lengths in astronomical units, 1 AU $= 1.50 \times 10^8$ km, and time in years, then

$$\mu = \frac{4\pi^2}{T^2}a^3 = 4\pi^2$$

in units of $AU^3/yr^2$.

Setting $\phi = 0$, the location of the perigee is given by

$$r = \frac{a(1-e^2)}{1+e} = a(1-e),$$

or

$$\mathbf{r} = (a(1-e), 0).$$

At this point the velocity is given by

$$\dot{\mathbf{r}} = \left(0, \sqrt{\frac{\mu}{a}\frac{1+e}{1-e}}\right).$$

Knowing the position and velocity at $\phi = 0$,, we can set the initial conditions for a bound orbit. The MATLAB code based on the above analysis is given below and the solution can be seen in Figure 3.25.

```
e=0.9;
tspan=[0 100];
z0=[1-e;0;0;sqrt((1+e)/(1-e))];
[t,z] = ode45('twobodyf',tspan, z0);
plot(z(:,1),z(:,2),'k');
axis equal
```

```
function dz = twobodyf(t,z)
dz = zeros(4,1);
GM = 1;

r=(z(1).^2 + z(2).^2).^(3/2);
alpha=GM/r;
dz(1) = z(3);
dz(2) = z(4);
dz(3) = -alpha*z(1);
dz(4) = -alpha*z(2);
```

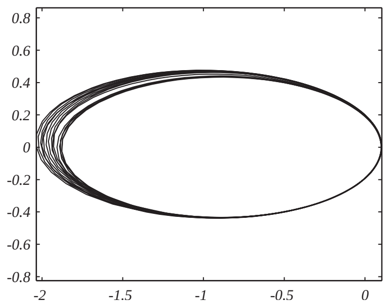


Figure 3.25: Simulation of one body orbiting a larger body under gravitational attraction.

While it is clear that the mass is following an elliptical orbit, we see that it will only do so for a finite period of time partly because the Runge-Kutta code does not conserve energy and it does not conserve the angular momentum. The conservation of energy is found (up to a factor of $m_1$) as

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\mu}{t} = -\frac{\mu}{2a}.$$

Similarly, the conservation of (specific) angular momentum is given by

$$\mathbf{r} \times \mathbf{v} = (x\dot{y} - y\dot{x})\mathbf{k} = \sqrt{\mu a(1-e^2)}\mathbf{k}.$$

As was the case with the nonlinear pendulum example, we saw that an implicit Euler method, or Cromer's method, was better at conserving energy. So, we compare the Euler's Method version with the Implicit-Euler Method. In general, we seek to solve the system

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{F}(\mathbf{r}, \mathbf{v}), \\ \dot{\mathbf{v}} &= \mathbf{G}(\mathbf{r}, \mathbf{v}). \end{aligned} \tag{3.49}$$

As we had seen earlier, Euler's Method is given by

$$\begin{aligned} \mathbf{v}_n &= \mathbf{v}_{n-1} + \Delta t * \mathbf{G}(t_{n-1}, \mathbf{x}_{n-1}), \\ \mathbf{r}_n &= \mathbf{r}_{n-1} + \Delta t * \mathbf{F}(t_{n-1}, \mathbf{v}_{n-1}). \end{aligned} \tag{3.50}$$

Euler's Method for the two body problem

For the two body problem, we can write out the Euler Method steps using $\mathbf{v} = (u, v)$, $\mathbf{r} = (x, y)$, $\mathbf{F} = (u, v)$, and $\mathbf{G} = -\frac{\mu}{r^3}(x, y)$. The MATLAB code would use the loop

```
for i=2:N+1
     alpha=mu/(x(i-1).^2 + y(i-1).^2).^(3/2);
     u(i)=u(i-1)-h*alpha*x(i-1);
     v(i)=v(i-1)-h*alpha*y(i-1);
     x(i)=x(i-1)+h*u(i-1);
     y(i)=y(i-1)+h*v(i-1);
     t(i)=t(i-1)+h;
 end
```

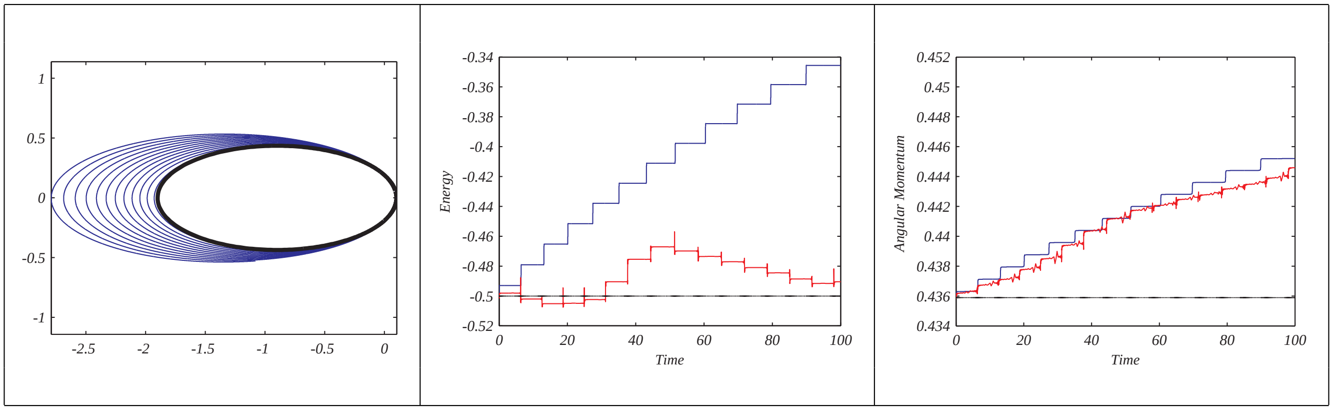



Table 3.8: Results for using Euler method for $N = 4000000$ and $t \in [0, 100]$. The parameters are $\mu = 1$, $e = 0, 9$, and $a = 1$.

Note that more compact forms can be used, but they are not readily adaptable to other packages or programming languages.

In Figure 3.8 we show the results along with the energy and angular momentum plots for $N = 4000000$ and $t \in [0, 100]$ for the case of $\mu = 1$, $e = 0, 9$, and $a = 1$. The orbit based on the exact solution is in the center of the figure on the left. The energy and angular momentum as a function of time are shown along with the similar plots obtained using **ode45**. In neither case are these two quantities conserved.

```
for i=2:N+1
     alpha=mu/(x(i-1).^2 + y(i-1).^2).^(3/2);
     u(i)=u(i-1)-h*alpha*x(i-1);
     v(i)=v(i-1)-h*alpha*y(i-1);
     x(i)=x(i-1)+h*u(i);
     y(i)=y(i-1)+h*v(i);
     t(i)=t(i-1)+h;
 end
```

Implicit-Euler Method for the two body problem

The Implicit-Euler Method is a slight modification to the Euler Method and has a better chance at handing the conserved quantities as the Implicit-Euler Method is one of many symplectic integrators. The modification uses the new value of the velocities in the updating of the position. Thus, we have

$$\begin{aligned} \mathbf{v}_n &= \mathbf{v}_{n-1} + \Delta t * \mathbf{G}(t_{n-1}, \mathbf{x}_{n-1}), \\ \mathbf{r}_n &= \mathbf{r}_{n-1} + \Delta t * \mathbf{F}(t_{n-1}, \mathbf{v}_n). \end{aligned} \tag{3.51}$$

It is a simple matter to update the MATLAB code. In Figure 3.9 we show the results along with the energy and angular momentum plots for $N = 200000$ and $t \in [0, 100]$ for the case of $\mu = 1$, $e = 0, 9$, and $a = 1$. The orbit based on the exact solution coincides with the orbit as seen in the left figure. The energy and angular momentum as functions of time are appear to be conserved. The energy fluctuates about $-0.5$ and the angular momentum remains constant. Again, the **ode45** results are shown in comparison. The number of time steps has been decreased from the Euler Method by a factor of 20.

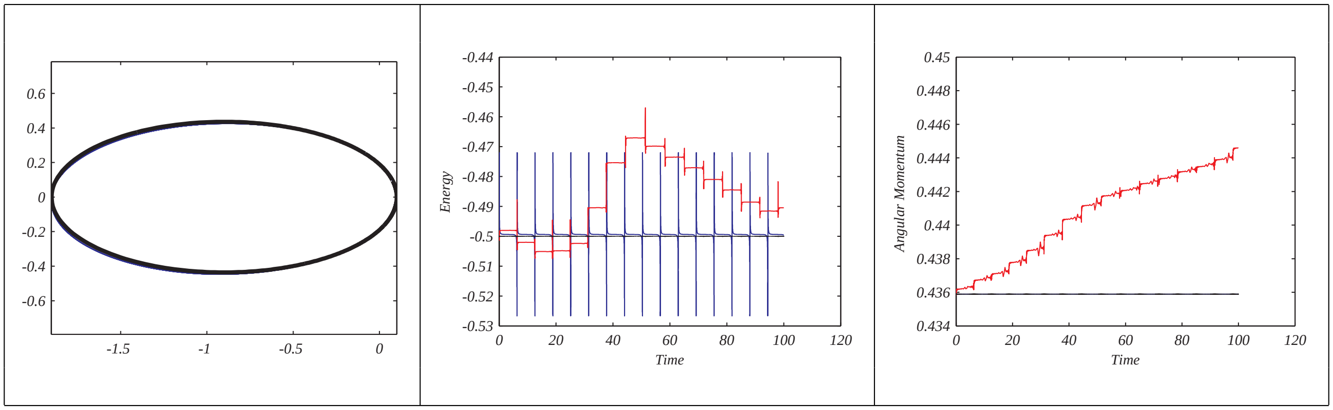



Table 3.9: Results for using the Implicit-Euler method for $N = 200000$ and $t \in [0, 100]$. The parameters are $\mu = 1$, $e = 0,9$, and $a = 1$.

The Euler and Implicit Euler are first order methods. We can attempt a faster and more accurate process which is also a symplectic method. As a final example, we introduce the velocity Verlet method for solving

$$\ddot{\mathbf{r}} = \mathbf{a}(\mathbf{r}(t)).$$

The derivation is based on a simple Taylor expansion:

$$\mathbf{r}(t + \Delta t) = \mathbf{r}(t) + \mathbf{v}(t)\Delta t + \frac{1}{2}\mathbf{a}(\mathbf{t})\Delta t^2 + \cdots.$$

Replace $\Delta t$ with $-\Delta t$ to obtain

$$\mathbf{r}(t - \Delta t) = \mathbf{r}(t) - \mathbf{v}(t)\Delta t + \frac{1}{2}\mathbf{a}(\mathbf{t})\Delta t^2 - \cdots.$$

Now, adding these expressions leads to some cancellations,

$$\mathbf{r}(t + \Delta t) = 2\mathbf{r}(t) - \mathbf{r}(t - \Delta t) + \mathbf{a}(\mathbf{t})\Delta t^2 + O(\Delta t^4).$$

Writing this in a more useful form, we have

$$\mathbf{r}_{n+1} = 2\mathbf{r}_n - \mathbf{r}_{n-1} + \mathbf{a}(\mathbf{r}_n)\Delta t^2.$$

Thus, we can find $\mathbf{r}_{n+1}$ from the previous two values without knowing the velocity. This method is called the Verlet, or Störmer-Verlet Method.

Loup Verlet (1931-) is a physicist who works on molecular dynamics and Fredrik Carl Mülertz Störmer (1874-1957) was a mathematician and physicist who modeled the motion of charged particles in his studies of the aurora borealis.

It is useful to know the velocity so that we can check energy conservation and angular momentum conservation. The Verlet Method can be rewritten in an equivalent form know as the velocity Verlet method. We use

$$\mathbf{r}(t) - \mathbf{r}(t - \Delta t) \approx \mathbf{v}(t)\Delta t - \frac{1}{2}\mathbf{a}\Delta t^2$$

in the Stömer-Verlet Method and write

$$\begin{aligned}
\mathbf{r}_n &= \mathbf{r}_{n-1} + \mathbf{v}_{n-1}u + \frac{h^2}{2}\mathbf{a}(\mathbf{r}_{n-1}), \\
\mathbf{v}_{n-1/2} &= \mathbf{v}_{n-1} + \frac{h}{2}\mathbf{a}(\mathbf{r}_{n-1}), \\
\mathbf{a}_n &= \mathbf{a}(\mathbf{r}_n), \\
\mathbf{v}_n &= \mathbf{v}_{n-1/2} + \frac{h}{2}\mathbf{a}_n,
\end{aligned} \tag{3.52}$$

where $h = \Delta t$. For the current problem, $\mathbf{a}(\mathbf{r}_n) = -\frac{\mu}{r_n^2}\mathbf{r}_n$.

The MATLAB snippet is given as

Störmer-Verlet Method for the two body problem.

```
for i=2:N+1
     alpha=mu/(x(i-1).^2 + y(i-1).^2).^(3/2);
     x(i)=x(i-1)+h*u(i-1)-h^2/2*alpha*x(i-1);
     y(i)=y(i-1)+h*v(i-1)-h^2/2*alpha*y(i-1);
     u(i)=u(i-1)-h/2*alpha*x(i-1);
     v(i)=v(i-1)-h/2*alpha*y(i-1);
     alpha=mu/(x(i).^2 + y(i).^2).^(3/2);
     u(i)=u(i)-h/2*alpha*x(i);
     v(i)=v(i)-h/2*alpha*y(i);
     t(i)=t(i-1)+h;
 end
```

The results using the velocity Verlet method are shown in Figure 3.10. For only $50,000$ steps we have much better results for the conservation laws and the orbit appears stable.

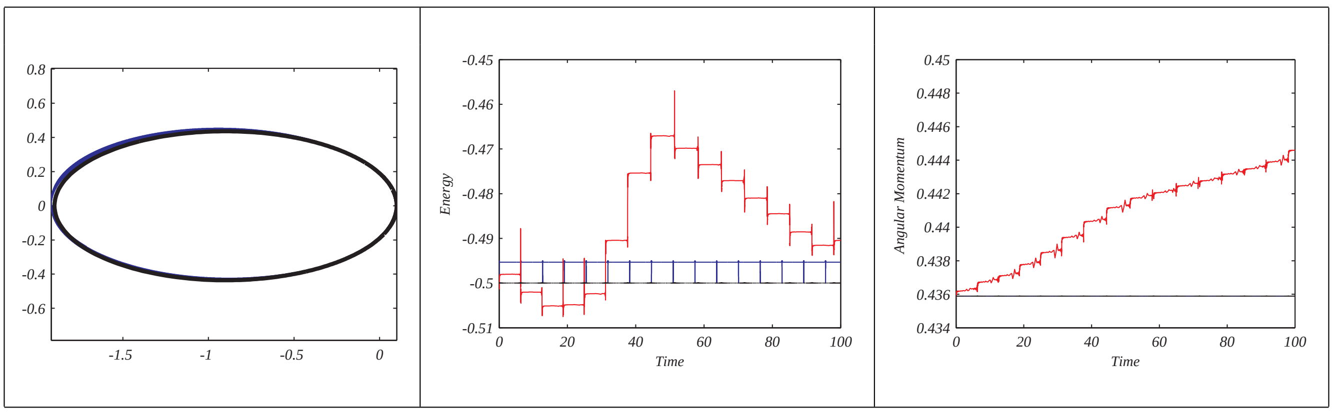



Table 3.10: Results for using velocity Verlet method for $N = 50000$ and $t \in [0, 100]$. The parameters are $\mu = 1$, $e = 0,9$, and $a = 1$.

### 3.4.6 *The Expanding Universe*

One of the remarkable stories of the twentieth century is the development of both the theory and the experimental data leading to our current understanding of the large scale structure of the universe. In 1916 Albert Einstein (1879-1955) published his general theory of relativity. It is a geometric theory of gravitation which relates the curvature of spacetime to its energy and momentum content. This relationship is embodied in the Einstein field equations, which are written compactly as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

The left side contains the curvature of spacetime as determined by the metric $g_{\mu\nu}$. The Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, is determined from the curvature tensor $R_{\mu\nu}$ and the scalar curvature, $R$. These in turn are obtained from the metric tensor. $\Lambda$ is the famous cosmological constant, which

Einstein originally introduced to maintain a static universe, which has since taken on a different role. The right hand side of Einstein's equation involves the familiar gravitational constant, the speed of light, and the stress-energy tensor, $T_{\mu\nu}$.

Georges Lemaître (1894-1966) had actually predicted the expansion of the universe in 1927 and proposed what later became known as the big bang theory.

In 1917 Einstein applied general relativity to cosmology. However, it was Alexander Alexandrovich Friedmann (1888-1925) who was the first to provide solutions to Einstein's equation based on the assumptions of homogeneity and isotropy and leading to the expansion of the universe. Unfortunately, Friedmann died in 1925 of typhoid.

In 1929 Edwin Hubble (1889-1953) showed that the radial velocities of galaxies are proportional to their distance, resulting in what is now called Hubble's Law. Hubble's Law takes the form

$$v = H_0 r,$$

where $H_0$ is the Hubble constant and indicates that the universe is expanding. The current values of the Hubble constant are $(70 \pm 7)$ km s$^{-1}$ Mpc $^{-1}$ and some recent WMAP results indicate it could be $(71.0 \pm 2.5)$ km s$^{-1}$ Mpc $^{-1}$.[5]

[5] These strange units are in common usage. Mpc stands for 1 megaparsec = $3.086 \times 10^{22}$ m and 1 km s$^{-1}$ Mpc $^{-1}$ = $3.24 \times 10^{-20}$ s$^{-1}$. The recent value was reported at the NASA website on March 25, 2013 http://map.gsfc.nasa.gov/universe/bb_tests_exp.html

In this section we are interested in Friedmann's Equation, which is the simple differential equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2} + \frac{\Lambda}{3}.$$

Here, $a(t)$ is the scale factor of the universe, which is taken to be one at present time; $\varepsilon(t)$ is the energy density; $R_0$ is the radius of curvature; and, $\kappa$ is the curvature constant, ($\kappa = +1$ for positively curved space, $\kappa = 0$ for flat space, $\kappa = -1$ for negatively curved space.) The cosmological constant,$\Lambda$, is now added to account for dark energy. The idea is that if we know the right side of Friedmann's equation, then we can say something about the future size of the unverse. This is a simple differential equation which comes from applying Einstein's equation to an isotropic, homogenous, and curved spacetime. Einstein's equation actually gives us a little more than this equation, but we will only focus on the (first) Friedmann equation. The reader can read more in books on cosmology, such as B. Ryden's *Introduction to Cosmology.*

Friedmann's equation can be written in a simpler form by taking into account the different contributions to the energy density. For $\Lambda = 0$ and zero curvature, one has

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t).$$

We define the Hubble parameter as $H(t) = \dot{a}/a$. At the current time, $t_0$, $H(t_0) = H_0$, Hubble's constant, and we take $a(t_0) = 1$. The energy density in this case is called the critical density,

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G}H(t)^2.$$

It is typical to introduce the density parameter, $\Omega =$ *varepsilon* $/\varepsilon_c$. Then, the Friedmann equation can be written as

$$1 - \Omega = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}.$$

Evaluating this expression at the current time, then

$$1 - \Omega_0 = -\frac{\kappa c^2}{R_0^2 H_0^2};$$

and, therefore,

$$1 - \Omega = -\frac{H_0^2(1 - \Omega_0)}{a^2 H^2}.$$

Solving for $H^2$, we have the differential equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega(t) + \frac{1 - \Omega_0}{a^2}\right],$$

where $\Omega$ takes into account the contributions to the energy density of the universe. These contributions are due to nonrelativistic matter density, contributions due to photons and neutrinos, and the cosmological constant, which might represent dark energy. This is discussed in Ryden (2003). In particular, $\Omega$ is a function of $a(t)$ for certain models. So, we write

$$\Omega = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0},$$

where current estimates (Ryden (2003)) are $\Omega_{r,0} = 8.4 \times 10^{-5}$, $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} \approx 0.7$. In general, We require

$$\Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} = \Omega_0.$$

So, in later examples, we will take this relationship into account.

The compact form of Friedmann's equation.

Therefore, the Friedmann equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2}\right]. \tag{3.53}$$

Taking the square root of this expression, we obtain a first order equation for the scale factor,

$$\dot{a} = \pm H_0 \sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + 1 - \Omega_0}.$$

The appropriate sign will be used when the scale factor is increasing or decreasing.

For special universes, by restricting the contributions to $\Omega$, one can get analytic solutions. But, in general one has to solve this equation numerically. We will leave most of these cases to the reader or for homework problems and will consider some simple examples.

**Example 3.6.** *Determine $a(t)$ for a flat universe with nonrelativistic matter only. (This is called an Einstein-de Sitter universe.)*

*In this case, we have $\Omega_{r,0} = 0$, $\Omega_{\Lambda,0} = 0$, and $\Omega_0 = 1$. Since $\Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} = \Omega_0$, $\Omega_{m,0} = 1$ and the Friedman equation takes the form*

$$\dot{a} = H_0\sqrt{\frac{1}{a}}.$$

*This is a simple separable first order equation. Thus,*

$$H_0\, dt = \sqrt{a} da.$$

*Integrating, we have*

$$H_0 t = \frac{2}{3}a^{3/2} + C.$$

*Taking $a(0) = 0$, we have*

$$a(t) = \left(\frac{t}{\frac{2}{3}H_0}\right)^{2/3}.$$

*Since $a(t_0) = 1$, we find*

$$t_0 = \frac{2}{3H_0}.$$

*This would give the age of the universe in this model as roughly $t_0 = 9.3$ Gyr.*

**Example 3.7.** *Determine $a(t)$ for a curved universe with nonrelativistic matter only.*

*We will consider $\Omega_0 > 1$. In this case, the Friedman equation takes the form*

$$\dot{a} = \pm H_0\sqrt{\frac{\Omega_0}{a} + (1 - \Omega_0)}.$$

*Note that there is an extremum $a_{max}$ which occurs for $\dot{a} = 0$. This occurs for*

$$a = a_{max} \equiv \frac{\Omega_0}{\Omega_0 - 1}.$$

*Analytic solutions are possible for this problem in parametric form. Note that we can write the differential equation in the form*

$$\begin{aligned}
\dot{a} &= \pm H_0\sqrt{\frac{\Omega_0}{a} + (1 - \Omega_0)} \\
&= \pm H_0\sqrt{\frac{\Omega_0}{a}}\sqrt{1 + \frac{a(1 - \Omega_0)}{\Omega_0}} \\
&= \pm H_0\sqrt{\frac{\Omega_0}{a}}\sqrt{1 - \frac{a}{a_{max}}}. \qquad (3.54)
\end{aligned}$$

*A separation of variables gives*

$$H_0\sqrt{\Omega_0}\, dt = \pm\frac{\sqrt{a}}{\sqrt{1 - \frac{a}{a_{max}}}}\, da.$$

*This form suggests a trigonometric substitution,*

$$\frac{a}{a_{max}} = \sin^2\theta$$

*with* $da = 2a_{max} \sin\theta \cos\theta\, d\theta$. *Thus, the integration becomes*

$$H_0\sqrt{\Omega_0}t = \pm \int \frac{\sqrt{a_{max}\sin^2\theta}}{\sqrt{\cos^2\theta}}\, 2a_{max}\sin\theta\cos\theta\, d\theta.$$

*In proceeding, we should be careful. Recall that for real numbers* $\sqrt{x^2} = |x|$. *In order to remove the roots of squares we need to consider the quadrant* $\theta$ *is in. Since* $a = 0$ *at* $t = 0$, *and it will vanish again for* $\theta = \pi$, *we will assume* $0 \le \theta \le \pi$. *For this range,* $\sin\theta \ge 0$. *However,* $\cos\theta$ *is not of one sign for this domain. In fact, a reaches it maximum at* $\theta = \pi/2$. *So,* $\dot{a} > 0$. *This corresponds to the upper sign in front of the integral. For* $\theta > \pi/2$, $\dot{a} < 0$ *and thus we need the lower sign and* $\sqrt{\cos^2\theta} = -\cos\theta$ *for that part of the domain. Thus, it is safe to simplify the square roots and we obtain*

$$\begin{aligned} H_0\sqrt{\Omega_0}t &= 2a_{max}^{3/2}\int \sin^2\theta, d\theta. \\ &= a_{max}^{3/2}\int (1-\cos 2\theta)\,, d\theta. \\ &= a_{max}^{3/2}\left(\theta - \frac{1}{2}\sin 2\theta\right) \end{aligned} \tag{3.55}$$

*for* $t = 0$ *at* $\theta = 0$.

*We have arrived at a parametric solution to the example,*

$$\begin{aligned} a &= = a_{max}\sin^2\theta, \\ t &= \frac{a_{max}^{3/2}}{H_0\sqrt{\Omega_0}}\left(\theta - \frac{1}{2}\sin 2\theta\right), \end{aligned} \tag{3.56}$$

*for* $0 \le \theta \le \pi$. *Letting,* $\phi = 2\theta$, *this solution can be written as*

$$\begin{aligned} a &= \frac{1}{2}a_{max}(1-\cos\phi), \\ t &= \frac{a_{max}^{3/2}}{2H_0\sqrt{\Omega_0}}\left(\phi - \sin\phi\right), \end{aligned} \tag{3.57}$$

*for* $0 \le \phi \le 2\pi$. *As we will see in Chapter 10, the curve described by these equations is a cycloid.*

*A similar computation can be performed for* $\Omega_0 < 1$. *This will be left as a homework exercise. The answer takes the form*

$$\begin{aligned} a &= \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\eta - 1), \\ t &= \frac{\Omega_0}{2H_0(1-\Omega)^{3/2}}\left(\sinh\eta - \eta\right), \end{aligned} \tag{3.58}$$

*for* $\eta \ge 0$.

**Example 3.8.** *Determine the numerical solution of Friedmann's equation for a curved universe with nonrelativistic matter only.*

*Since Friedmann's equation is a differential equation, we can use our favorite solver to obtain a solution. Not all universe types are amenable to obtaining an analytic solution as the last example. We can create a function in MATLAB for use in* **ode45***:*

```
function da=cosmosf(t,a)
global Omega
f=Omega./a+1-Omega;
da=sqrt(f);
end
```

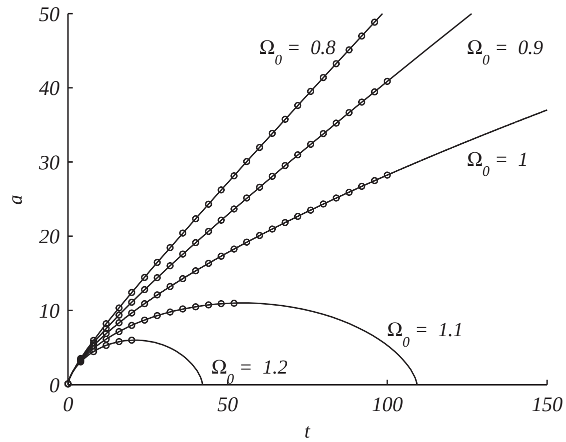



Figure 3.26: Numerical solution (circles) of the Friedmann equation superimposed on the analytic solutions for a matter plus curvature ($\Omega_0 \neq 1$) or no curvature ($\Omega_0 = 1$) universe.

*We can then solve the Friedmann equation and compare the solutions to the analytic forms in the last two examples. The code for doing this is given below:*

```
clear
global Omega

for Omega=0.8:.1:1.2;
    if Omega<1
        amax=50;
        tmax=100;
    elseif Omega==1
        amax=50;
        tmax=100;
    else
        amax=Omega/(Omega-1);
        tmax=Omega/(Omega-1)^1.5/2*pi;
    end

    tspan=0:4:tmax;
    a0=.1;
    [t,a]=ode45(@cosmosf,tspan,a0);
    plot(t,a,'ok')
    hold on

    if Omega<1
        eta=0:.1:4;
```

```
            a3 = Omega/(1-Omega)/2*(cosh(eta)-1);
            t3 = Omega/(1-Omega)^1.5/2*(sinh(eta)-eta);
            plot(t3,a3,'k')
            axis([0,max(t3),0,max(a3)])
            xlabel('t')
            ylabel('a')
        elseif Omega==1
            t3=0:.1:1.5*tmax;
            a3=(3*t3/2).^(2/3);
            plot(t3,a3,'k')
        else
            phi=0:.1:2*pi;
            a3 = Omega/(Omega-1)/2*(1-cos(phi));
            t3 = Omega/(Omega-1)^1.5/2*(phi-sin(phi));
            plot(t3,a3,'k')
        end
    end
    hold off
    axis([0,150,0,50])
    xlabel('t')
    ylabel('a')
```

*In Figure 3.26 we show the results. For $\Omega_0 > 1$ the solutions lie on the first half of the cycloid solution. The other solutions indicate that the universe continues to expand, leading to what is called the Big Chill. The analytic solutions to the $\Omega_0 > 1$ cases eventually collapse to $a = 0$ in finite time. These final states are what Stephen Hawking calls the Big Crunch.*

*The numerical solutions for $\Omega_0 > 1$ run into difficulty because the radicand in the square root is negative. But, this corresponds to when $\dot{a} < 0$. So, we have to modify the code by estimating the maximum on the curve and run the numerical algorithm with new initial conditions and using the fact that $\dot{a} < 0$ in the function* **cosmosf** *by setting da=-sqrt(f). The modified code is below and the resulting numerical solutions are shown in Figure 3.27.*

```
tspan=0:4:tmax;
    a0=.1;
    [t,a]=ode45(@cosmosf,tspan,a0);
    plot(t,a,'ok','MarkerSize',2)
    hold on
    if Omega>1
         tspan=tmax+.0001:4:2*tmax;
         a0=amax-.0001;
         [t2,a2]=ode45(@cosmosf2,tspan,a0);
         plot(t2,a2,'ok','MarkerSize',2)
    end
```

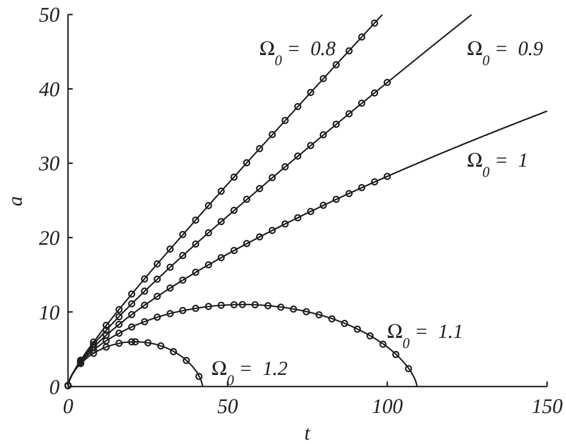



Figure 3.27: Modified numerical solution (circles) of the Friedmann equation superimposed on the analytic solutions for a matter plus curvature ($\Omega_0 \neq 1$) or no curvature ($\Omega_0 = 1$) universe with the extension past the maximum value of $a$ when $\Omega_0 > 1$..

### 3.4.7 *The Coefficient of Drag*

We have seen that air drag can play a role in interesting physics problems in differential equations. This also is an important concept in fluid flow and fluid mechanics when looking at flows around obstacles, or when the obstacle is moving with respect to the background fluid. The simplest such object is a sphere, such as a baseball, soccer ball, golf ball, or ideal spherical raindrop. The resistive force is characterized by the dimensionless drag coefficient

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L^2},$$

where $L$ and $U$ are the characteristic length and speed of the object moving through the fluid.

There has been much attention focussed on relating the drag coefficient to the Reynolds number. The Reynolds number is given by

The Reynolds number, *Re*, is named after Osborne Reynolds (1842-1912) who first determined it in 1883.

$$Re = \frac{\rho L U}{\eta} = \frac{LV}{\nu},$$

where $\eta$ is the viscosity and $\nu = \frac{\eta}{\rho}$ is the kinematic viscosity. It is a measure of the size of the kinematic to viscous forces in the problem at hand. There are different ranges of fluid behavior depending on the order of the Reynolds number. These range from laminar flow ($Re < 1000$) to turbulent flow ($Re > 2.5 \times 10^5$). There are a range of other types of flows such as creeping flow ($Re << 1$) and transitional flows, which are a mix of laminar and turbulent flow.

For low Reynolds number, the inertial forces are small compared to the viscous forces, leading to the Stokes drag force, $C_D = 24Re^{-1}$. This result can be determined analytically. Similarly, for large Reynolds number the drag coefficient is a constant. This is the Newtonian regime. Somewhere in between the form of the drag coefficient is found through empirical studies. There have been many empirical expressions developed and all are within a few percent of the data in the range of applicability. Some of the commonly used expressions are given below.

Models that are useful for $Re < 10^3$ :

$$C_2 = \frac{24}{Re} + \frac{4}{Re^{1/3}}, \qquad \text{Putnam (1961)}, \tag{3.59}$$

$$C_3 = \frac{24}{Re}\left(1 + 0.15Re^{0.687}\right), \quad \text{Schiller-Naumann (1933)}, \tag{3.60}$$

$$C_4 = 12Re^{-.5}; \qquad \text{Edwards et al. (2000)}, \tag{3.61}$$

(3.62)

Models that are useful for $Re < 2 \times 10^5$ are the White (1991) and Clift-Gavin (1970), respectively,

$$C_1 = \frac{24}{Re} + \frac{6.}{1+\sqrt{Re}} + 0.4, \tag{3.63}$$

$$C_5 = \frac{24}{Re}\left(1 + 0.15Re^{0.687}\right) + \frac{.42}{1 + 42500/Re^{1.16}}. \tag{3.64}$$

A more recent model was proposed by Morrison (2010) for $Re < 10^6$ :

$$C_6 = \frac{24}{Re} + \frac{2.6\left(\frac{Re}{5.0}\right)}{1+\left(\frac{Re}{5.0}\right)^{1.52}} + \frac{.411\left(\frac{Re}{263000}\right)^{-7.94}}{1+\left(\frac{Re}{263000}\right)^{-8.00}} + \frac{Re^{0.80}}{461000}. \tag{3.65}$$

Plots for these models are shown in Figures 3.28-3.29. In Figure 3.28 we see that the models differ significantly for large Reynolds numbers.

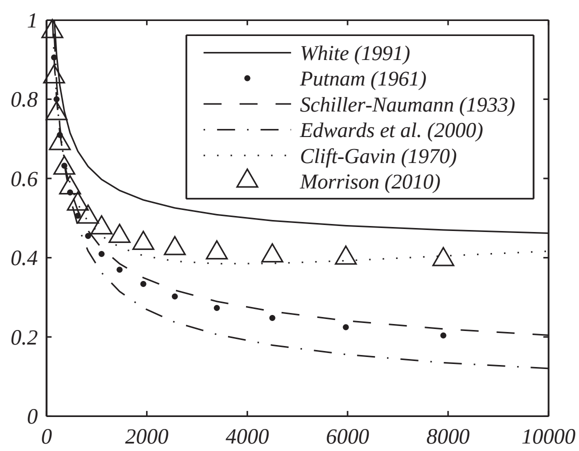



Figure 3.28: Drag coefficient as a function of Reynolds number for spheres.

Figure 3.29 shows a log-log plot of the drag coefficient as a function of Reynolds number. In Figure 3.30 we show a power law fit for Reynolds number less than 1000 confirming the model used by Edwards, Wilder, and Scime (2001) as described in the raindrop problem.

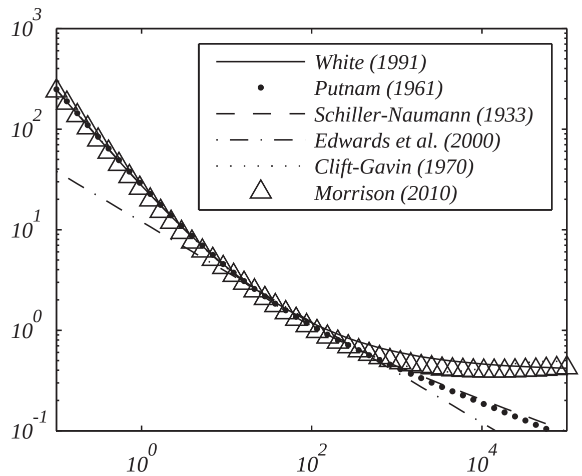



Figure 3.29: Log-log plot of the drag coefficient as a function of Reynolds number for spheres.

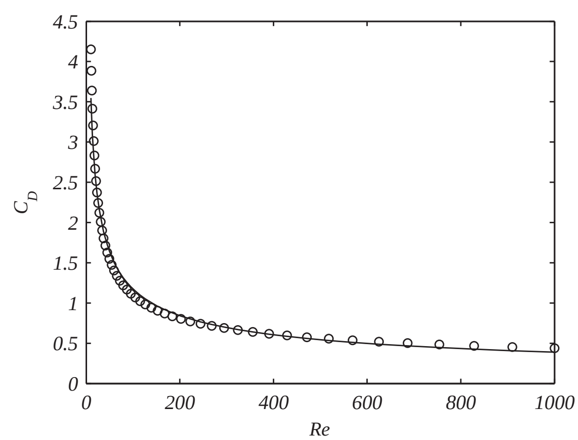



Figure 3.30: A power law fit for the drag coefficient as a function of Reynolds number using linearization and linear regression.

## *Problems*

**1.** Use i) Euler's Method and ii) the Midpoint Method to determine the given value of $y$ for the following problems

a. $\frac{dy}{dx} = 2y$, $y(0) = 2$. Find $y(1)$ with $h = 0.1$.

b. $\frac{dy}{dx} = x - y$, $y(0) = 1$. Find $y(2)$ with $h = 0.2$.

c. $\frac{dy}{dx} = x\sqrt{1-y^2}$, $y(1) = 0$. Find $y(2)$ with $h = 0.2$.

**2.** Numerically solve the nonlinear pendulum problem using the Euler-Cromer method for a pendulum with length $L = 0.5$ m using initial angles of $\theta_0 = 10^o$, and $\theta_0 = 70^o$. In each case run the routines long enough and with an appropriate $h$ such that you can determine the period in each case. Compare your results with the linear pendulum period.

**3.** For the Baumgartner sky dive we had obtained the results for his position as a function of time. There are other questions which could be asked.

a. Find the velocity as a function of time for the model developed in the text.

b. Find the velocity as a function of altitude for the model developed in the text.

c. What maximum velocity is obtained in the model? At what time and position?

d. Does the model indicate that terminal velocity was reached?

e. What speed is predicted for the point at which the parachute opened?

f. How do these numbers compare with reported data?

**4.** Consider the flight of a golf ball with mass 46 g and a diameter of 42.7 mm. Assume it is projected at $30^o$ with a speed of 36 m/s and no spin.

a. Ignoring air resistance, analytically find the path of the ball and determine the range, maximum height, and time of flight for it to land at the height that the ball had started.

b. Now consider a drag force $f_D = \frac{1}{2}C_D\rho\pi r^2 v^2$, with $C_D = 0.42$ and $\rho = 1.21$ kg/m$^3$. Determine the range, maximum height, and time of flight for the ball to land at the height that it had started.

c. Plot the Reynolds number as a function of time. [Take the kinematic viscosity of air, $\nu = 1.47 \times 10^{-5}$.

d. Based on the plot in part c, create a model to incorporate the change in Reynolds number and repeat part b. Compare the results from parts a, b and d.

**5.** Consider the flight of a tennis ball with mass 57 g and a diameter of 66.0 mm. Assume the ball is served 6.40 meters from the net at a speed of 50.0 m/s down the center line from a height of 2.8 m. It needs to just clear the net (0.914 m).

a. Ignoring air resistance and spin, analytically find the path of the ball assuming it just clears the net. Determine the angle to clear the net and the time of flight.

b. Find the angle to clear the net assuming the tennis ball is given a topspin with $\omega = 50$ rad/s.

c. Repeat part b assuming the tennis ball is given a bottom spin with $\omega = 50$ rad/s.

d. Repeat parts a, b, and c with a drag force, taking $C_D = 0.55$.

**6.** In Example 3.7 $a(t)$ was determined for a curved universe with nonrelativistic matter for $\Omega_0 > 1$. Derive the parametric equations for $\Omega_0 < 1$,

$$\begin{aligned} a &= \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\eta - 1), \\ t &= \frac{\Omega_0}{2H_0(1-\Omega)^{3/2}}(\sinh\eta - \eta), \end{aligned} \tag{3.66}$$

for $\eta \geq 0$.

**7.** Find numerical solutions for other models of the universe.

a. A flat universe with nonrelativistic matter only with $\Omega_{m,0} = 1$.

b. A curved universe with radiation only with curvature of different types.

c. A flat universe with nonrelativistic matter and radiation with several values of $\Omega_{m,0}$ and $\Omega_{r,0} + \Omega_{m,0} = 1$.

d. Look up the current values of $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, and $\kappa$. Use these values to predict future values of $a(t)$.

e. Investigate other types of universes of your choice, but different from the previous problems and examples.

# 4
# Series Solutions

*"In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure." - Hermann Hankel (1839-1873)*

## 4.1 Introduction to Power Series

AS NOTED A FEW TIMES, not all differential equations have exact solutions. So, we need to resort to seeking approximate solutions, or solutions i the neighborhood of the initial value. Before describing these methods, we need to recall power series. A power series expansion about $x = a$ with coefficient sequence $c_n$ is given by $\sum_{n=0}^{\infty} c_n(x-a)^n$. For now we will consider all constants to be real numbers with $x$ in some subset of the set of real numbers. We review power series in the appendix.

The two types of series encountered in calculus are Taylor and Maclaurin series. A Taylor series expansion of $f(x)$ about $x = a$ is the series

Taylor series expansion.

$$f(x) \sim \sum_{n=0}^{\infty} c_n(x-a)^n, \tag{4.1}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{4.2}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function.

A Maclaurin series expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

Maclaurin series expansion.

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{4.3}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{4.4}$$

We note that Maclaurin series are a special case of Taylor series for which the expansion is about $x = 0$. Typical Maclaurin series, which you should know, are given in Table 4.1.

Table 4.1: Common Mclaurin Series Expansions

| Series Expansions You Should Know | | | | |
|---|---|---|---|---|
| $e^x$ | $=$ | $1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$ | $=$ | $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ |
| $\cos x$ | $=$ | $1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots$ | $=$ | $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ |
| $\sin x$ | $=$ | $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$ | $=$ | $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ |
| $\cosh x$ | $=$ | $1+\frac{x^2}{2}+\frac{x^4}{4!}+\dots$ | $=$ | $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ |
| $\sinh x$ | $=$ | $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$ | $=$ | $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$ |
| $\frac{1}{1-x}$ | $=$ | $1+x+x^2+x^3+\dots$ | $=$ | $\sum_{n=0}^{\infty}x^n$ |
| $\frac{1}{1+x}$ | $=$ | $1-x+x^2-x^3+\dots$ | $=$ | $\sum_{n=0}^{\infty}(-x)^n$ |
| $\tan^{-1}x$ | $=$ | $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$ | $=$ | $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ |
| $\ln(1+x)$ | $=$ | $x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$ | $=$ | $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ |

A simple example of developing a series solution for a differential equation is given in the next example.

**Example 4.1.** $y'(x)=x+y(x)$, $y(0)=1$.

*We are interested in seeking solutions of this initial value problem. We note that this was already solved in Example 3.1.*

*Let's assume that we can write the solution as the Maclaurin series*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty}\frac{y^{(n)(x)}}{n!}x^n \\ &= y(0)+y'(0)x+\frac{1}{2}y''(0)x^2+\frac{1}{6}y'''(0)x^3+\dots. \end{aligned} \tag{4.5}$$

*We already know that $y(0)=1$. So, we know the first term in the series expansion. We can find the value of $y'(0)$ from the differential equation:*

$$y'(0)=0+y(0)=1.$$

*In order to obtain values of the higher order derivatives at $x=0$, we differentiate the differential equation several times:*

$$\begin{aligned} y''(x) &= 1+y'(x). \\ y''(0) &= 1+y'(0)=2. \\ y'''(x) &= y''(x)=2. \end{aligned} \tag{4.6}$$

*All other values of the derivatives are the same. Therefore, we have*

$$y(x)=1+x+2(\frac{1}{2}x^2+\frac{1}{3!}x^3+\dots).$$

*This solution can be summed as*

$$y(x) = 2(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots) - 1 - x = 2e^x - x - 1.$$

*This is the same result we had obtained before.*

## 4.2 Power Series Method

IN THE LAST EXAMPLE WE WERE ABLE to use the initial condition to produce a series solution to the given differential equation. Even if we specified more general initial conditions, are there other ways to obtain series solutions? Can we find a general solution in the form of power series? We will address these questions in the remaining sections. However, we will first begin with an example to demonstrate how we can find the general solution to a first order differential equation.

**Example 4.2.** *Find a general Maclaurin series solution to the ODE: $y' - 2xy = 0$.*

*Let's assume that the solution takes the form*

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

*The goal is to find the expansion coefficients, $c_n$, $n = 0, 1, \ldots$.*

*Differentiating, we have*

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

*Note that the index starts at $n = 1$, since there is no $n = 0$ term remaining.*

*Inserting the series for $y(x)$ and $y'(x)$ into the differential equation, we have*

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n \\ &= (c_1 + 2c_2 x + 3c_3 x^2 + 4x^3 + \ldots) \\ &\quad -2x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots) \\ &= c_1 + (2c_2 - c_0)x + (3c_3 - 2c_1)x^2 + (4c_4 - 2c_2)x^3 + \ldots. \end{aligned} \tag{4.7}$$

*Equating like powers of $x$ on both sides of this result, we have*

$$\begin{aligned} 0 &= c_1, \\ 0 &= 2c_2 - c_0, \\ 0 &= 3c_3 - c_1, \\ 0 &= 4c_4 - 2c_2, \ldots. \end{aligned} \tag{4.8}$$

*We can solve these sequentially for the coefficient of largest index:*

$$c_1 = 0, c_2 = c_0, c_3 = \frac{2}{3}c_1 = 0, c_3 = \frac{1}{2}c_2 = \frac{1}{2}c_0, \ldots.$$

*We note that the odd terms vanish and the even terms survive:*

$$\begin{aligned} y(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\ &= c_0 + c_0 x^2 + \frac{1}{2} c_0 x^4 + \ldots . \end{aligned} \tag{4.9}$$

*Thus, we have found a series solution, or at least the first several terms, up to a multiplicative constant.*

*Of course, it would be nice to obtain a few more terms and guess at the general form of the series solution. This could be done if we carried out the streps in a more general way. This is accomplished by keeping the summation notation and trying to combine all terms with like powers of $x$. We begin by inserting the series expansion into the differential equation and identifying the powers of $x$ :*

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 2 c_n x^{n+1}. \end{aligned} \tag{4.10}$$

Re-indexing a series.

*We note that the powers of $x$ in these two sums differ by 2. We can re-index the sums separately so that the powers are the same, say $k$. After all, when we had expanded these series earlier, the index, $n$, disappeared. Such an index is known as a dummy index since we could call the index anything, like $n-1$, $\ell - 1$, or even $k = n-1$ in the first series. So, we can let $k = n-1$, or $n = k+1$, to write*

$$\begin{aligned} \sum_{n=1}^{\infty} n c_n x^{n-1} &= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k \\ &= c_1 + 2c_2 x + 3c_3 x^2 + 4x^3 + \ldots . \end{aligned} \tag{4.11}$$

*Note, that re-indexing has not changed the terms in the series.*

*Similarly, we can let $k = n+1$, or $n = k-1$, in the second series to find*

$$\begin{aligned} \sum_{n=0}^{\infty} 2 c_n x^{n+1} &= \sum_{k=1}^{\infty} 2 c_{k-1} x^k \\ &= 2c_0 + 2c_1 x + 2c_2 x^2 + 2c_3 x^3 + \ldots . \end{aligned} \tag{4.12}$$

*Combining both series, we have*

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 2 c_n x^{n+1} \\ &= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=1}^{\infty} 2 c_{k-1} x^k \\ &= c_1 + \sum_{k=1}^{\infty} [(k+1) c_{k+1} - 2 c_{k-1}] x^k. \end{aligned} \tag{4.13}$$

*Here, we have combined the two series for $k = 1, 2, \ldots$. . The $k = 0$ term in the first series gives the constant term as shown.*

*We can now set the coefficients of powers of $x$ equal to zero since there are no terms on the left hand side of the equation. This gives $c_1 = 0$ and*

$$(k+1) c_{k+1} - 2 c_{k-1}, \quad k = 1, 2, \ldots .$$

*This last equation is called a recurrence relation. It can be used to find successive coefficients in terms of previous values. In particular, we have*

$$c_{k+1} = \frac{2}{k+1} c_{k-1}, \quad k = 1, 2, \ldots .$$

*Inserting different values of $k$, we have*

$$\begin{aligned}
k &= 1: \quad c_2 = \frac{2}{2} c_0 = c_0. \\
k &= 2: \quad c_3 = \frac{2}{3} c_1 = 0. \\
k &= 3: \quad c_4 = \frac{2}{4} c_2 = \frac{1}{2} c_0. \\
k &= 4: \quad c_5 = \frac{2}{5} c_3 = 0. \\
k &= 5: \quad c_6 = \frac{2}{6} c_4 = \frac{1}{3(2)} c_0.
\end{aligned} \tag{4.14}$$

*Continuing, we can see a pattern. Namely,*

$$c_k = \begin{cases} 0, & k = 2\ell + 1, \\ \frac{1}{\ell!}, & k = 2\ell. \end{cases}$$

*Thus,*

$$\begin{aligned}
y(x) &= \sum_{k=0}^{\infty} c_k x^k \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\
&= c_0 + c_0 x^2 + \frac{1}{2!} c_0 x^4 + \frac{1}{3!} c_0 x^6 + \ldots \\
&= c_0 (1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \ldots) \\
&= c_0 \sum_{\ell=0}^{\infty} \frac{1}{\ell!} x^{2\ell} \\
&= c_0 e^{x^2}.
\end{aligned} \tag{4.15}$$

This example demonstrated how we can solve a simple differential equation by first guessing that the solution was in the form of a power series. We would like to explore the use of power series for more general higher order equations. We will begin second order differential equations in the form

$$P(x) y''(x) + Q(x) y'(x) + R(x) y(x) = 0,$$

where $P(x)$, $Q(x)$, and $R(x)$ are polynomials in $x$. The point $x_0$ is called an ordinary point if $P(x_0) \neq 0$. Otherwise, $x_0$ is called a singular point.

Ordinary and singular points.

When $x_0$ is an ordinary point, then we can seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

For most of the examples, we will let $x_0 = 0$, in which case we seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

**Example 4.3.** *Find the general Maclaurin series solution to the ODE:*

$$y'' - xy' - y = 0.$$

*We will look for a solution of the form*

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

*The first and second derivatives of the series are given by*

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} c_n n x^{n-1} \\ y''(x) &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}. \end{aligned}$$

*Inserting these derivatives into the differential equation gives*

$$0 = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n.$$

*We want to combine the three sums into one sum and identify the coefficients of each power of $x$. The last two sums have similar powers of $x$. So, we need only re-index the first sum. We let $k = n - 2$, or $n = k + 2$. This gives*

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k.$$

*Inserting this sum, and setting $n = k$ in the other two sums, we have*

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_k k x^k - \sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) - c_k k - c_k\right] x^k + c_2(2)(1) - c_0 \\ &= \sum_{k=1}^{\infty} (k+1)\left[(k+2)c_{k+2} - c_k\right] x^k + 2c_2 - c_0. \end{aligned} \tag{4.16}$$

*Noting that the coefficients of powers $x^k$ have to vanish, we have $2c_2 - c_0 = 0$ and*

$$(k+1)\left[(k+2)c_{k+2} - c_k\right] = 0, \quad k = 1, 2, 3, \ldots,$$

*or*

$$\begin{aligned} c_2 &= \frac{1}{2}c_0, \\ c_{k+2} &= \frac{1}{k+2}c_k, \quad k = 1, 2, 3, \ldots. \end{aligned} \tag{4.17}$$

*Using this result, we can successively determine the coefficients to as many terms as we need.*

$$\begin{aligned} k &= 1: \quad c_3 = \frac{1}{3}c_1. \\ k &= 2: \quad c_4 = \frac{1}{4}c_2 = \frac{1}{8}c_0. \\ k &= 3: \quad c_5 = \frac{1}{5}c_3 = \frac{1}{15}c_1. \\ k &= 4: \quad c_6 = \frac{1}{6}c_4 = \frac{1}{48}c_0. \\ k &= 5: \quad c_7 = \frac{1}{7}c_5 = \frac{1}{105}c_1. \end{aligned} \tag{4.18}$$

*This gives the series solution as*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \\ &= c_0 + c_1 x + \frac{1}{2}c_0 x^2 + \frac{1}{3}c_1 x^3 + \frac{1}{8}c_0 x^4 + \frac{1}{15}c_1 x^5 + \frac{1}{48}c_0 x^6 + \dots \\ &= c_0\left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \dots\right) + c_1\left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \dots\right). \end{aligned} \tag{4.19}$$

*We note that the general solution to this second order differential equation has two arbitrary constants. The general solution is a linear combination of two linearly independent solutions obtained by setting one of the constants equal to one and the other equal to zero.*

**Example 4.4.** *Consider the Legendre equation* $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ *for* $p$ *an integer.*

*We first note that there are singular points for* $1 - x^2 = 0$*, or* $x = \pm 1$*. Therefore,* $x = 0$ *is an ordinary point and we can proceed to obtain solutions in the form of Maclaurin series expansions. Insert the series expansions*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n, \\ y'(x) &= \sum_{n=1}^{\infty} n c_n x^{n-1}, \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}, \end{aligned} \tag{4.20}$$

*into the differential equation to obtain*

$$\begin{aligned} 0 &= (1-x^2)y'' - 2xy' + p(p+1)y \\ &= (1-x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} n c_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} p(p+1)c_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} [p(p+1) - n(n+1)]c_n x^n. \end{aligned} \tag{4.21}$$

*Re-indexing the first sum with $k = n - 2$, we have*

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} [p(p+1) - n(n+1)]c_n x^n \\ &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty} [p(p+1) - k(k+1)]c_k x^k \\ &= 2c_2 + 6c_3 x + p(p+1)c_0 + p(p+1)c_1 x - 2c_1 x \\ &\quad + \sum_{k=2}^{\infty} \left((k+2)(k+1)c_{k+2} + [p(p+1) - k(k+1)]c_k\right) x^k. \end{aligned} \tag{4.22}$$

*Matching terms, we have*

$$\begin{aligned} k &= 0: \quad 2c_2 = -p(p+1)c_0. \\ k &= 1: \quad 6c_3 = [2 - p(p+1)]c_1. \\ k &\geq 2: \quad (k+2)(k+1)c_{k+2} = [k(k+1) - p(p+1)]c_k. \end{aligned} \tag{4.23}$$

*For $p = 0$, the first equation gives $c_2 = 0$ and the third equation gives $c_{2\ell} = 0$ for $\ell = 1, 2, 3, \ldots$. This leads to $y_1(x) = c_0$ is a solution for $p = 0$.*

*Similarly, for $p = 1$, the second equation gives $c_3 = 0$ and the third equation gives $c_{2\ell+1} = 0$ for $\ell = 1, 2, 3, \ldots$. Thus, $y_1(x) = c_1 x$ is a solution for $p = 1$.*

*In fact, for $p$ any nonnegative integer the series truncates. For example, if $p = 2$, then these equations reduce to*

$$\begin{aligned} k &= 0: \quad 2c_2 = -6c_0. \\ k &= 1: \quad 6c_3 = -4c_1. \\ k &\geq 2: \quad (k+2)(k+1)c_{k+2} = [k(k+1) - 2(3)]c_k. \end{aligned} \tag{4.24}$$

*For $k = 2$, we have $12c_4 = 0$. So, $c_6 = c_8 = \ldots = 0$. Also, we have $c_2 = -3c_0$. This gives*

$$y(x) = c_0(1 - 3x^2) + (c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + \ldots).$$

*Therefore, there is a polynomial solution of degree 2. The remaining coefficients are proportional to $c_1$, yielding the second linearly independent solution, which is not a polynomial.*

*For other nonnegative integer values of $p > 2$, we have*

$$c_{k+2} = \frac{k(k+1) - p(p+1)}{(k+2)(k+1)} c_k, \quad k \geq 2.$$

*When $k = p$, the right side of the equation vanishes, making the remaining coefficients vanish. Thus, we will be left with a polynomial of degree $p$. These are the Legendre polynomials, $P_p(x)$.*

## 4.3 Singular Points

The power series method does not alway give us the full general solution to a differential equation. Problems can arise when the differential equation has singular points. The simplest equations having singular points are Cauchy-Euler equations,

$$ax^2y'' + bxy' + cy = 0.$$

A few examples are sufficient to demonstrate the types of problems that can occur.

**Example 4.5.** *Find the series solutions for the Cauchy-Euler equation,*

$$ax^2y'' + bxy' + cy = 0,$$

*for the cases i. $a = 1$, $b = -4$, $c = 6$, ii. $a = 1$, $b = 2$, $c = -6$, and iii. $a = 1$, $b = 1$, $c = 6$.*

*As before, we insert*

$$y(x) = \sum_{n=0}^{\infty} d_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n d_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) d_n x^{n-2},$$

*into the differential equation to obtain*

$$\begin{aligned} 0 &= ax^2y'' + bxy' + cy \\ &= ax^2 \sum_{n=2}^{\infty} n(n-1)d_n x^{n-2} + bx \sum_{n=1}^{\infty} n d_n x^{n-1} + c \sum_{n=0}^{\infty} d_n x^n \\ &= a \sum_{n=0}^{\infty} n(n-1)d_n x^n + b \sum_{n=0}^{\infty} n d_n x^n + c \sum_{n=0}^{\infty} d_n x^n \\ &= \sum_{n=0}^{\infty} \left[an(n-1) + bn + c\right] d_n x^n. \qquad (4.25) \end{aligned}$$

*Here we changed the lower limits on the first sums as $n(n-1)$ vanishes for $n = 0, 1$ and the added terms all are zero.*

*Setting all coefficients to zero, we have*

$$\left[an^2 + (b-a)n + c\right] d_n, \quad n = 0, 1, \ldots.$$

*Therefore, all of the coefficients vanish, $d_n = 0$, except at the roots of $an^2 + (b - a)n + c = 0$.*

*In the first case, $a = 1$, $b = -4$, and $c = 6$, we have*

$$0 = n^2 + (-4-1)n + 6 = n^2 - 5n + 6 = (n-2)(n-3).$$

*Thus, $d_n = 0$, $n \neq 2, 3$. This leaves two terms in the series, reducing to the polynomial $y(x) = d_2x^2 + d_3x^3$.*

*In the second case, $a = 1$, $b = 2$, and $c = -6$, we have*

$$0 = n^2 + (2-1)n - 6 = n^2 + n - 6 = (n-2)(n+3).$$

*Thus, $d_n = 0$, $n \neq 2, -3$. Since the $n$'s are nonnegative, this leaves one term in the solution, $y(x) = d_2x^2$. So, we do not have the most general solution since we are missing a second linearly independent solution. We can use the Method* *of Reduction of Order from Section 2.5.3, or we could use what we know about Cauchy-Euler equations, to show that the general solution is*

$$y(x) = c_1 x^2 + c_2 x^{-3}.$$

*Finally, the third case has $a = 1$, $b = 1$, and $c = 6$, we have*

$$0 = n^2 + (1-1)n + 6 = n^2 + 6.$$

*Since there are no real solutions to this equation, $d_n = 0$ for all $n$. Again, we could use what we know about Cauchy-Euler equations, to show that the general solution is*

$$y(x) = c_1 \cos(\sqrt{6}\ln x) + c_2 \cos(\sqrt{6}\ln x).$$

In the last example, we have seen that the power series method does not always work. The key is to write the differential equation in the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

We already know that $x = 0$ is a singular point of the Cauchy-Euler equation. Putting the equation in the latter form, we have

$$y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0.$$

We see that $p(x) = a/x$ and $q(x) = b/x^2$ are not defined at $x = 0$. So, we do not expect a convergent power series solution in the neighborhood of $x = 0$.

**Theorem 4.1.** *The initial value problem*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad y(x_0) = \alpha, \quad y'(x_0) = \beta$$

*has a unique Taylor series solution converging in the interval $|x - x_0| < R$ if both $p(x)$ and $q(x)$ can be represented by convergent Taylor series converging for $|x - x_0| < R$. (Then, $p(x)$ and $q(x)$ are said to be analytic at $x = x_0$.) As noted earlier, $x_0$ is then called an ordinary point. Otherwise, if either, or both, $p(x)$ and $q(x)$ are not analytic at $x_0$, then $x_0$ is called a singular point.*

**Example 4.6.** *Determine if a power series solution exits for $xy'' + 2y' + xy = 0$ near $x = 0$.*

*Putting this equation in the form*

$$y'' + \frac{2}{x}y' + 2y = 0,$$

*we see that $a(x)$ is not defined at $x = 0$, so $x = 0$ is a singular point. Let's see how far we can get towards obtaining a series solution.*

*We let*

$$y(x) = \sum_{n=0}^{\infty} c_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2},$$

*into the differential equation to obtain*

$$\begin{aligned} 0 &= xy'' + 2y' + xy \\ &= x\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2\sum_{n=1}^{\infty} nc_n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= 2c_1 + \sum_{n=2}^{\infty} [n(n-1)+2n]c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1}. \end{aligned} \tag{4.26}$$

*Here we combined the first two series and pulled out the first term of the second series.*

*We can re-index the series. In the first series we let $k = n-1$ and in the second series we let $k = n+1$. This gives*

$$\begin{aligned} 0 &= 2c_1 + \sum_{n=2}^{\infty} n(n+1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= 2c_1 + \sum_{k=1}^{\infty} (k+1)(k+2)c_{k+1}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= 2c_1 + \sum_{k=1}^{\infty} [(k+1)(k+2)c_{k+1} + c_{k-1}]x^k. \end{aligned} \tag{4.27}$$

*Setting coefficients to zero, we have $c_1 = 0$ and*

$$c_{k+1} = \frac{1}{(k+1)(k+2)}c_{k-1}, \quad k = 1, 2, \ldots.$$

*Therefore, we have $c_n = 0$ for $n = 1, 3, 5, \ldots$. For the even indices, we have*

$$\begin{aligned} k &= 2: \quad c_2 = -\frac{1}{3(2)}c_0 = -\frac{c_0}{3!}. \\ k &= 4: \quad c_4 = -\frac{1}{5(4)}c_2 = \frac{c_0}{5!}. \\ k &= 6: \quad c_6 = -\frac{1}{7(6)}c_4 = -\frac{c_0}{7!}. \\ k &= 8: \quad c_8 = -\frac{1}{9(8)}c_6 = \frac{c_0}{9!}. \end{aligned} \tag{4.28}$$

*We can see the pattern and write the solution in closed form.*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\ &= c_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} \ldots\right) \\ &= c_0 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots\right) \\ &= c_0 \frac{\sin x}{x}. \end{aligned} \tag{4.29}$$

*We have another case where the power series method does not yield a general solution.*

In the last example we did not find the general solution. However, we did find one solution, $y_1(x) = \frac{\sin x}{x}$. So, we could use the Method of Reduction of Order to obtain the second linearly independent solution. This is carried out in the next example.

Use of the Method of Reduction of Order to obtain a second linearly independent solution. See Section 2.5.3

**Example 4.7.** *Let $y_1(x) = \frac{\sin x}{x}$ be one solution of $xy'' + 2y' + xy = 0$. Find a second linearly independent solution.*

*Let $y(x) = v(x)y_1(x)$. Inserting this into the differential equation, we have*

$$\begin{aligned}
0 &= xy'' + 2y' + xy \\
&= x(vy_1)'' + 2(vy_1)' + xvy_1 \\
&= x(v'y_1 + vy_1')' + 2(v'y_1 + vy_1') + xvy_1 \\
&= x(v''y_1 + 2v'y_1' + vy_1'') + 2(v'y_1 + vy_1') + xvy_1 \\
&= x(v''y_1 + 2v'y_1') + 2v'y_1 + v(xy_1'' + 2y_1' + xy_1) \\
&= x\left[\frac{\sin x}{x}v'' + 2\left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\right)v'\right] + 2\frac{\sin x}{x}v' \\
&= \sin x v'' + 2\cos x v'.
\end{aligned} \tag{4.30}$$

*This is a first order separable differential equation for $z = v'$. Thus,*

$$\sin x \frac{dz}{dx} = -2z\cos x,$$

*or*

$$\frac{dz}{z} = -2\cot x\, dx.$$

*Integrating, we have*

$$\ln|z| = 2\ln|\csc x| + C.$$

*Setting $C = 0$, we have $v' = z = \csc^2 x$, or $v = -\cot x$. This gives the second solution as*

$$y(x) = v(x)y_1(x) = -\cot x\frac{\sin x}{x} = -\frac{\cos x}{x}.$$

## 4.4 *The Frobenius Method*

### 4.4.1 *Introduction*

It might be possible to use power series to obtain solutions to differential equations in terms of series involving noninteger powers. For example, we found in Example 4.6 that $y_1(x) = \frac{\sin x}{x}$ and $y_2(x) = \frac{\cos x}{x}$ are solutions of the differential equation $xy'' + 2y' + xy = 0$. Series expansions about $x = 0$ are given by

$$\begin{aligned}
\frac{\sin x}{x} &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) \\
&= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots. \\
\frac{\cos x}{x} &= \frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)
\end{aligned} \tag{4.31}$$

$$= \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \ldots. \tag{4.32}$$

While the first series is a Taylor series, the second one is not due to the presence of the first term, $x^{-1}$. We would like to be able to capture such expansions. So, we seek solutions of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

for some real number $r$. This is the basis of the Frobenius Method.

Consider the differential equation,

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0.$$

If $xa(x)$ and $x^2b(x)$ are real analytic, i.e., have convergent Taylor series expansions about $x = 0$, then we can find a solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \tag{4.33}$$

for some constant $r$. Furthermore, $r$ is determined from the solution of an *indicial equation*.

If $x = 0$ is a regular singular point, then we can apply the Frobenius Method.

**Example 4.8.** *Show that $x = 0$ is a regular singular point of the equation*

$$x^2y'' + x(3+x)y' + (1+x)y = 0$$

*and then find a solution in the form $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$.*

*Rewriting the equation as*

$$y' + \frac{3+x}{x}y'' + \frac{(1+x)}{x^2}y = 0,$$

*we identify*

$$\begin{aligned} a(x) &= \frac{3+x}{x} \\ b(x) &= \frac{(1+x)}{x^2}. \end{aligned}$$

*So, $xa(x) = 3+x$ and $x^2b(x) = 1+x$ are polynomials is $x$ and are therefore real analytic. Thus, $x = 0$ is a regular singular point.*

*Now, we seek a solution to the differential equation using the Frobenius Method. We assume $y(x)$ and its derivatives are of the form*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^{n+r}, \\ y'(x) &= \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}. \end{aligned} \tag{4.34}$$

*Inserting these series into the differential equation, we have*

$$\begin{aligned}
0 &= x^2y'' + x(3+x)y' + (1+x)y = 0 \\
&= x^2 \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + x(3+x) \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} \\
&\quad +(1+x) \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r} + 3 \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} \\
&\quad + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r+1} + \sum_{n=0}^{\infty} c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1} \\
&= \sum_{n=0}^{\infty} c_n[(n+r)(n+r-1) + 3(n+r) + 1]x^{n+r} + \sum_{n=0}^{\infty} c_n[n+r+1]x^{n+r+1} \\
&= \sum_{n=0}^{\infty} c_n[(n+r)(n+r+2) + 1]x^{n+r} + \sum_{n=0}^{\infty} c_n(n+r+1)x^{n+r+1}.
\end{aligned}$$

*Next, we reindex the last sum using $k = n+1$ so that both sums involve the powers $x^{k+r}$. Therefore, we have*

$$\sum_{k=0}^{\infty} c_k[(k+r)(k+r+2) + 1]x^{k+r} + \sum_{k=1}^{\infty} c_{k-1}(k+r)x^{k+r} = 0. \tag{4.35}$$

*We can combine both sums for $k = 1, 2, \ldots$ if we set the coefficients in the $k = 0$ term to zero. Namely,*

$$c_0[r(r+2) + 1] = 0.$$

*If we assume that $c_0 \neq 0$, then*

$$r(r+2) + 1 = 0.$$

*This is the indicial equation Expanding, we have*

$$0 = r^2 + 2r + 1 = (r+1)^2.$$

*So, this gives $r = -1$.*

*Inserting $r = -1$ into Equation (4.35) and combining the remaining sums, we have*

$$\sum_{k=1}^{\infty} \left[k^2 c_k + (k-1)c_{k-1}\right] x^{k-1} = 0.$$

*Setting the coefficients equal to zero, we have found that*

$$c_k = \frac{1-k}{k^2} c_{k-1}, \quad k = 1, 2, 3, \ldots.$$

*So, each coefficient is a multiple of the previous one. In fact, for $k = 1$, we have that*

$$c_1 = (0), c_0 = 0.$$

*Therefore, all of the coefficients are zero except for $c_0$. This gives a solution as*

$$y_0(x) = \frac{c_0}{x}.$$

*We had assumed that $c_0 \neq 0$. What if $c_0 = 0$? Then, Equation (4.35) becomes*

$$\sum_{k=1}^{\infty}[((k+r)(k+r+2)+1)c_k + (k+r)c_{k-1}]x^{k+r} = 0.$$

*Setting the coefficients equal to zero, we have*

$$((k+r)(k+r+2)+1)c_k = -(k+r)c_{k-1}, \quad k = 1,2,3,\ldots.$$

*When $k = 1$,*

$$((1+r)(r+3)+1)c_1 = -(1+r)c_0 = 0.$$

*So $c_1 = 0$, or $0 = (1+r)(r+3)+1 = r^2+4r+4 = (r+2)^2$. If $c_1 \neq 0$, this gives $r = -2$ and*

$$c_k = -\frac{(k-2)}{(k+1)^2}c_{k-1}, \quad k = 2,3,4,\ldots.$$

*Then, we have $c_2 = 0$ and all other coefficient vanish, leaving the solution as*

$$y(x) = c_1 x^{1-2} = \frac{c_1}{x}.$$

*We only found one solution. We need a second linearly independent solution in order to find the general solution to the differential equation. This can be found using the Method of Reduction of Order from Section 2.5.3 For completeness we will seek a solution $y_2(x) = v(x)y_1(x)$, where $y_1(x) = x^{-1}$. Then,*

Method of Reduction of Order.

$$\begin{aligned}
0 &= x^2y_2'' + x(3+x)y_2' + (1+x)y_2 \\
&= x^2(vy_1)'' + x(3+x)(vy_1)' + (1+x)vy_1 \\
&= [x^2y_1'' + x(3+x)y_1' + (1+x)y_1]v \\
&\quad +[x^2v'' + x(3+x)v']y_1 + 2x^2v'y_1' \\
&= [x^2v'' + x(3+x)v']y_1 + 2x^2v'y_1' \\
&= [x^2v'' + x(3+x)v']x^{-1} - 2x^2v'x^{-2'} \\
&= xv'' + (3+x)v' - 2v' \\
&= xv'' + (1+x)v'. \qquad (4.36)
\end{aligned}$$

*Letting $z = v'$, the last equation can be written as*

$$x\frac{dz}{dx} + (1+x)z = 0.$$

*This is a separable first order equation. Separating variables and integrating, we have*

$$\int \frac{dz}{z} = -\int \frac{1+x}{x}\,dx,$$

*or*

$$\ln|z| = -\ln|x| - x + C.$$

*Exponentiating,*

$$z = \frac{dv}{dx} = A\frac{e^{-x}}{x}.$$

*Further integration yields*

$$v(x) = A \int \frac{e^{-x}}{x}\, dx + B.$$

*Thus,*

$$y_2(x) = \frac{1}{x} \int \frac{e^{-x}}{x}\, dx.$$

*Note that the integral does not have a simple antiderivative and defines the exponential function*

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^2 x^n}{n!n},$$

*where* $\gamma = 0.5772...$ *is the Euler-Mascheroni constant.*

*Thus, we have found the general solution*

$$y(x) = \frac{c_1}{x} + \frac{c_2}{x} E_1(x).$$

Another example is that of Bessel's equation. This is a famous equation which occurs in the solution of problems involving cylindrical symmetry. We discuss the solutions more generally in the last section of the chapter. Here we apply the Frobenius method to obtain the series solution.

**Example 4.9.** *Solve Bessel's equation using the Frobenius method,*

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0.$$

Bessel's equation.

*We first note that* $x = 0$ *is a regular singular point. We assume* $y(x)$ *and its derivatives are of the form*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^{n+r}, \\ y'(x) &= \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}. \end{aligned} \tag{4.37}$$

*Inserting these series into the differential equation, we have*

$$\begin{aligned} 0 &= x^2 y' + xy' + (x^2 - \nu^2) y \\ &= x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \\ &\quad + (x^2 - \nu^2) \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} \\ &\quad \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} \nu^2 c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} [(n+r)(n+r-1) + (n+r) - \nu^2] c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} \\ &= \sum_{n=0}^{\infty} [(n+r)^2 - \nu^2] c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2}. \end{aligned} \tag{4.38}$$

*We reindex the last sum with $k = n+2$, or $n = k-2$, to obtain*

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} \left( [(k+r)^2 - \nu^2] c_k + c_{k-2} \right) x^{k+r} \\ &\quad + (r^2 - \nu^2) c_0 x^r + [(1+r)^2 - \nu^2] c_1 x^{r+1}. \end{aligned} \tag{4.39}$$

*We again obtain the indicial equation from the $k = 0$ terms, $r^2 - \nu^2 = 0$. The solutions are $r = \pm\nu$.*

*We consider the case $r = \nu$. The $k = 1$ terms give*

$$\begin{aligned} 0 &= [(1+r)^2 - \nu^2] c_1 \\ &= [(1+\nu)^2 - \nu^2] c_1 \\ &= [1 + 2\nu] c_1 \end{aligned}$$

*For $1 + 2\nu \neq 0$, $c_1 = 0$. [In the next section we consider the case $\nu = -\frac{1}{2}$.]*

*For $k = 2, 3, \ldots$, we have*

$$[(k+\nu)^2 - \nu^2] c_k + c_{k-2} = 0,$$

*or*

$$c_k = -\frac{c_{k-2}}{k(k+2\nu)}.$$

*Noting that $c_k = 0$, $k = 1, 3, 5, \ldots$, we evaluate a few of the nonzero coefficients:*

$$\begin{aligned} k &= 2: \quad c_2 = -\frac{1}{2(2+2\nu)} c_0 = -\frac{1}{4(\nu+1)} c_0. \\ k &= 4: \quad c_4 = -\frac{1}{4(4+2\nu)} c_2 = -\frac{1}{8(\nu+2)} c_2 = \frac{1}{2^4(2)(\nu+2)(\nu+1)} c_0. \\ k &= 6: \quad c_6 = -\frac{1}{6(6+2\nu)} c_4 = -\frac{1}{12(\nu+3)} c_4 \\ &= -\frac{1}{2^6(6)(\nu+3))(\nu+2)(\nu+1)} c_0. \end{aligned}$$

*Continuing long enough, we see a pattern emerge,*

$$c_{2n} = \frac{(-1)^n}{2^{2n} n! (\nu+1)(\nu+2)\cdots(\nu+n)}, \quad n = 1, 2, \ldots.$$

*The solution is given by*

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! (\nu+1)(\nu+2)\cdots(\nu+n)} x^{2n+\nu}.$$

*As we will see later, picking the right value of $c_0$, this gives the Bessel function of the first kind of order $\nu$ provided $\nu$ is not a negative integer.*

*The case $r = -\nu$ is similar. The $k = 1$ terms give*

$$\begin{aligned} 0 &= [(1+r)^2 - \nu^2] c_1 \\ &= [(1-\nu)^2 - \nu^2] c_1 \\ &= [1 - 2\nu] c_1 \end{aligned}$$

*For $1 - 2\nu \neq 0$, $c_1 = 0$. [In the next section we consider the case $\nu = \frac{1}{2}$.]*

*For $k = 2, 3, \ldots$, we have*

$$[(k-\nu)^2 - \nu^2]c_k + c_{k-2} = 0,$$

*or*

$$c_k = \frac{c_{k-2}}{k(2\nu - k)}.$$

*Noting that $c_k = 0$, $k = 1, 3, 5, \ldots$, we evaluate a few of the nonzero coefficients:*

$$\begin{aligned}
k &= 2: \quad c_2 = \frac{1}{2(2\nu-2)}c_0 = \frac{1}{4(\nu-1)}c_0. \\
k &= 4: \quad c_4 = \frac{1}{4(2\nu-4)}c_2 = \frac{1}{8(\nu-2)}c_2 = \frac{1}{2^4(2)(\nu-2)(\nu-1)}c_0. \\
k &= 6: \quad c_6 = \frac{1}{6(2\nu-6)}c_4 = \frac{1}{12(\nu-3)}c_4 \\
&= \frac{1}{2^6(6)(\nu-3))(\nu-2)(\nu-1)}c_0.
\end{aligned}$$

*Continuing long enough, we see a pattern emerge,*

$$c_{2n} = \frac{1}{2^{2n} n!(\nu-1)(\nu-2)\cdots(\nu-n)}, \quad n = 1, 2, \ldots.$$

*The solution is given by*

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{1}{2^{2n} n!(\nu-1)(\nu-2)\cdots(\nu-n)} x^{2n+\nu}$$

*provided $\nu$ is not a positive integer. The example $\nu = 1$ is investigated in the next section.*

### 4.4.2 *Roots of the Indicial Equation*

In this section we will consider the types of solutions one can obtain of the differential equation,

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0,$$

when $x = 0$ is a regular singular point. In this case, we assume that $xa(x)$ and $x^2 b(x)$ have the convergent Maclaurin series expansions

$$\begin{aligned}
xa(x) &= a_0 + a_1 x + a_2 x^2 + \ldots \\
x^2 b(x) &= b_0 + b_1 x + b_2 x^2 + \ldots. 
\end{aligned} \tag{4.40}$$

Using the Frobenius Method, we assume $y(x)$ and its derivatives are of the form

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n x^{n+r}, \\
y'(x) &= \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}, \\
y''(x) &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}.
\end{aligned} \tag{4.41}$$

Inserting these series into the differential equation, we obtain

$$\sum_{n=0}^{\infty} c_n \left[ (n+r)(n+r-1) + (n+r)xa(x) + x^2 b(x) \right] x^{n+r-2} = 0.$$

Using the expansions for $xa(x)$ and $x^2 b(x)$, we note that the lowest power of $x$ is $n+r-2$ when $n=0$. The coefficient for this term must vanish,

$$c_0 \left[ r(r-1) + a + 0r + b_0 \right] = 0.$$

Assuming that $c_0 \neq 0$, we have the indicial equation

$$r(r-1) + a + 0r + b_0 = 0.$$

The roots of the indicial equation determine the type of behavior of the solution. This amounts to considering three different cases. Let the roots of the equation be $r_1$ and $r_2$. Then,

i. Distinct roots with $r_1 - r_2 \neq$ integer.

   In this case we have two linearly independent solutions,

$$\begin{aligned} y_1(x) &= |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1, \\ y_2(x) &= |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n, \quad d_0 = 1. \end{aligned}$$

ii. Equal roots: $r_1 = r_2 = r$.

   The form for $y_1(x)$ is the same, but one needs to use the Method of Reduction of Order to seek the second linearly independent solution.

$$\begin{aligned} y_1(x) &= |x|^{r} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1, \\ y_2(x) &= |x|^{r} \sum_{n=0}^{\infty} d_n x^n + y_1(x) \ln |x|, \quad d_0 = 1. \end{aligned}$$

iii. Distinct roots with $r_1 - r_2 =$ positive integer.

   Just as in the last case, one needs to find a second linearly independent solution.

$$\begin{aligned} y_1(x) &= |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1, \\ y_2(x) &= |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n + \alpha y_1(x) \ln |x|, \quad d_0 = 1. \end{aligned}$$

   The constant $\alpha$ can be subsequently determined and in some cases might vanish.

For solutions near regular singular points, $x = x_0$, one has a similar set of cases but for expansions of the form $y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n$.

**Example 4.10.** $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$

*In this example $x = 0$ is a singular point. We have $a(x) = 1/x$ and $b(x) = (x^2 - 1/4)/x^2$. Thus,*

$$\begin{aligned} xa(x) &= 1, \\ x^2b(x) &= x^2 - \frac{1}{4}. \end{aligned}$$

*So, $a_0 = 1$ and $b_0 = -\frac{1}{4}$. The indicial equation becomes*

$$r(r-1) + r - \frac{1}{4} = 0.$$

*Simplifying, we have $0 = r^2 - \frac{1}{4}$, or $r = \pm\frac{1}{2}$.*

*For $r = +\frac{1}{2}$, we insert the series $y(x) = \sqrt{x}\sum_{n=0}^{\infty} c_n x^n$ into the differential equation, collect like terms by reindexing, and find*

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} c_n \left[(n+\frac{1}{2})(n-\frac{1}{2}) + (n+\frac{1}{2}) + x^2 - \frac{1}{4}\right] x^{n-\frac{3}{2}} \\ &= \sum_{n=0}^{\infty} c_n \left[n^2 + n + x^2\right] x^{n-\frac{3}{2}} \\ &= \left[\sum_{n=0}^{\infty} n(n+1)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2}\right] x^{-\frac{3}{2}} \\ &= \left[\sum_{k=0}^{\infty} k(k+1)c_k x^k + \sum_{n=2}^{\infty} c_{k-2} x^k\right] x^{-\frac{3}{2}} \\ &= 2c_1 x + \sum_{n=2}^{\infty} \left[k(k+1)c_k + c_{k-2}\right] x^{k-\frac{3}{2}} \end{aligned} \tag{4.42}$$

*This gives $c_1 = 0$ and*

$$c_k = -\frac{1}{k(k+1)} c_{k-2}, \quad k \geq 2.$$

*Iterating, we have $c_k = 0$ for $k$ odd and*

$$\begin{aligned} k &= 2: \quad c_2 = -\frac{1}{3!}c_0. \\ k &= 4: \quad c_4 = -\frac{1}{20}c_2 = \frac{1}{5!}c_0. \\ k &= 6: \quad c_6 = -\frac{1}{42}c_4 = \frac{1}{7!}c_0. \end{aligned}$$

*This gives*

$$\begin{aligned} y_1(x) &= \sqrt{x}\sum_{n=0}^{\infty} c_n x^n \\ &= \sqrt{x}\left(c_0 - \frac{1}{3!}c_0 x^2 + \frac{1}{5!}c_0 x^4 - \ldots\right) \\ &= \frac{c_0}{\sqrt{x}}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots\right) = \frac{c_0}{\sqrt{x}}\sin x. \end{aligned}$$

*Similarly, for for* $r = -\frac{1}{2}$, *one obtains the second solution,* $y_2(x) = \frac{d_0}{\sqrt{x}} \cos x$. *Setting* $c + 0 = 1$ *and* $d_0 = 1$, *give the two linearly independent solutions. This differential equation is the Bessel equation of order one half and the solutions are Bessel functions of order one half:*

$$
\begin{aligned}
J_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \sin x,\ x > 0, \\
J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \cos x,\ x > 0.
\end{aligned}
$$

**Example 4.11.** $x^2 y'' + 3xy' + (1-2x)y = 0$, $x > 0$.

*For this problem* $xa(x) = 3$ *and* $x^2 b(x) = 1 - 2x$. *Thus, the indicial equation is*

$$0 = r(r-1) + 3r + 1 = (r+1)^2.$$

*This is a case with two equal roots,* $r = -1$. *A solution of the form*

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

*will only result in one solution. We will use this solution form to arrive at a second linearly independent solution.*

*We will not insert* $r = -1$ *into the solution form yet. Writing the differential equation in operator form,* $L[y] = 0$, *we have*

$$
\begin{aligned}
L[y] &= x^2 y'' + 3xy' + (1-2x)y \\
&= x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + 3x \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \\
&\quad + (1-2x) \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty} [(n+r)(n+r-1) + 2(n+r) + 1] c_n x^{n+r} - \sum_{n=0}^{\infty} 2c_n x^{n+r+1} \\
&= \sum_{n=0}^{\infty} (n+r+1)^2 c_n x^{n+r} - \sum_{n=1}^{\infty} 2c_{n-1} x^{n+r}.
\end{aligned}
$$

*Setting the coefficients of like terms equal to zero for* $n \geq 1$, *we have*

$$c_n = \frac{2}{(n+r+1)^2} a_{n-1}, \quad n \geq 1.$$

*Iterating, we find*

$$c_n = \frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2} c_0, \quad n \geq 1.$$

*Setting* $c_0 = 1$, *we have the expression*

$$y(x,r) = x^r + \sum_{n=1}^{\infty} \frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2} x^{n+r}.$$

*This is not a solution of the differential equation because we did not use the root* $r = -1$. *Instead, we have*

$$L[y(x,c)] = (r+1)^2 x^r \tag{4.43}$$

*from the $n = 0$ term. If $r = -1$, then $y(x, -1)$ is one solution of the differential equation. Namely,*

$$y_1(x) = x^{-1} + \sum_{n=1}^{\infty} c_n(-1)x^{n-1}.$$

*Now consider what happens if we differentiate Equation (4.43) with respect to r:*

$$\begin{aligned} \frac{\partial}{\partial r} L[y(x,r)] &= \left[\frac{\partial y(x,r)}{\partial r}\right] \\ &= 2(r+1)x^r + (r+1)^2 x^r \ln x, \quad x > 0. \end{aligned}$$

*Therefore, $\frac{\partial y(x,r)}{\partial r}$ is also a solution to the differential equation when $r = -1$.*

*Since*

$$y(x,r) = x^r + \sum_{n=1}^{\infty} c_n(r)x^{n+r},$$

*we have*

$$\begin{aligned} \frac{\partial y(x,r)}{\partial r} &= x^r \ln x + \sum_{n=1}^{\infty} c_n(r)x^{n+r} \ln x + \sum_{n=1}^{\infty} c_n'(r)x^{n+r} \\ &= y(x,r)\ln x + \sum_{n=1}^{\infty} c_n'(r)x^{n+r}. \end{aligned} \tag{4.44}$$

*Therefore, the second solution is given by*

$$y_2(x) = \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=-1} = y(x,-1)\ln x + \sum_{n=1}^{\infty} c_n'(-1)x^{n-1}.$$

*In order to determine the solutions, we need to evaluate $c_n(-1)$ and $c_n'(-1)$. Recall that (setting $c_0 = 1$)*

$$c_n(r) = \frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2}, \quad n \geq 1.$$

*Therefore,*

$$\begin{aligned} c_n(-1) &= \frac{2^n}{[(1)(2)\cdots(n)]^2}, \quad n \geq 1, \\ &= \frac{2^n}{(n!)^2}. \end{aligned} \tag{4.45}$$

*Next we compute $c_n'(-1)$. This can be done using logarithmic differentiation. We consider*

$$\begin{aligned} \ln c_n(r) &= \ln\left(\frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2}\right) \\ &= \ln 2^n - 2\ln(r+2) - 2\ln(r+3)\cdots - 2\ln(r+n+1). \end{aligned}$$

*Differentiating with respect to r and evaluating at $r = -1$, we have*

$$\begin{aligned} \frac{c_n'(r)}{c_n(r)} &= -2\left(\frac{1}{r+2} + \frac{1}{r+2} + \cdots + \frac{1}{r+3}\right) \\ c_n'(r) &= -2c_n(r)\left(\frac{1}{r+2} + \frac{1}{r+3} + \cdots + \frac{1}{r+n+1}\right). \\ c_n'(-1) &= -2c_n(-1)\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= -\frac{2^{n+1}}{(n!)^2} H_n, \end{aligned} \tag{4.46}$$

*where we have defined*

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

*This gives the second solution as*

$$y_2(x) = y_1(x) \ln x - \sum_{n=1}^{\infty} \frac{2^{n+1}}{(n!)^2} H_n x^{n-1}.$$

**Example 4.12.** $x^2y'' + xy' + (x^2 - 1)y = 0.$

*This equation is similar to the last example, but it is the Bessel equation of order one. The indicial equation is given by*

$$0 = r(r-1) + r - 1 = r^2 - 1.$$

*The roots are* $r_1 = 1$, $r_2 = -1$. *In this case the roots differ by an integer,* $r_1 - r_2 = 2$.

*The first solution can be obtained using*

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^{n+1}, \\ y'(x) &= \sum_{n=0}^{\infty} c_n (n+1) x^n, \\ y''(x) &= \sum_{n=0}^{\infty} c_n (n+1)(n) x^{n-1}. \end{aligned} \tag{4.47}$$

*Inserting these series into the differential equation, we have*

$$\begin{aligned} 0 &= x^2y'' + xy' + (x^2-1)y \\ &= x^2 \sum_{n=0}^{\infty} c_n n(n+1) x^{n-1} + x \sum_{n=0}^{\infty} c_n (n+1) x^n \\ &\quad +(x^2-1) \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= \sum_{n=0}^{\infty} c_n n(n+1) x^{n+1} + \sum_{n=0}^{\infty} c_n (n+1) x^{n+1} \\ &\quad \sum_{n=0}^{\infty} c_n x^{n+3} - \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= \sum_{n=0}^{\infty} [n(n+1) + (n+1) - 1] c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+3} \\ &= \sum_{n=0}^{\infty} [(n+1)^2 - 1] c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+3}. \end{aligned} \tag{4.48}$$

*We reindex the last sum with* $k = n+2$, *or* $n = k-2$, *to obtain*

$$\sum_{n=2}^{\infty} \left( [(k+1)^2 - 1] c_k + c_{k-2} \right) x^{k+1} + 3c_1 x^2 = 0.$$

*Thus,* $c_1 = 0$ *and*

$$c_k = -\frac{1}{(k+1)^2 - 1} c_{k-2} = -\frac{1}{k(k+2)} c_{k-2}, \quad k \geq 2.$$

*Since $c_1 = 0$, all $c_k$'s vanish for odd $k$. For $k = 2n$, we have*

$$c_{2n} = -\frac{1}{2n(2n+2)}c_{2n-2} = -\frac{1}{4n(n+1)}c_{2(n-1)}, \quad n \geq 1.$$

$$\begin{aligned}
n &= 1: \quad c_2 = -\frac{1}{4(1)(2)}c_0. \\
n &= 2: \quad c_4 = -\frac{1}{4(2)(3)}c_2 = \frac{1}{4^2 2!3!}c_0. \\
n &= 3: \quad c_6 = -\frac{1}{4(3)(4)}c_4 = \frac{1}{4^3 3!4!}c_0.
\end{aligned}$$

*Continuing, this gives*

$$c_{2n} = \frac{(-1)^n}{4^n n!(n+1)!}c_0$$

*and the first solution is*

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+1)!} x^{2n+1}.$$

*Now we look for a second linearly independent solution of the form*

$$y_2(x) = \sum_{n=0}^{\infty} d_n x^{n-1} + \alpha y_1(x) \ln x, \quad x > 0.$$

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} d_n x^{n-1} + \alpha y_1(x) \ln x, \\
y_2'(x) &= \sum_{n=0}^{\infty} (n-1) d_n x^{n-2} + \alpha [y_1'(x) \ln x + y_1(x) x^{-1}], \\
y_2''(x) &= \sum_{n=0}^{\infty} (n-1) d_n (n-2) x^{n-3} + \alpha [y_1''(x) \ln x + 2y_1'(x) x^{-1} - y_1(x) x^{-2}].
\end{aligned}$$

*Inserting these series into the differential equation, we have*

$$\begin{aligned}
0 &= x^2 y_2'' + x y_2' + (x^2 - 1) y_2 \\
&= x^2 \sum_{n=0}^{\infty} (n-1)(n-2) d_n x^{n-3} + \alpha x^2 [y_1''(x) \ln x + 2y_1'(x) x^{-1} - y_1(x) x^{-2}] \\
&\quad + x \sum_{n=0}^{\infty} (n-1) d_n x^{n-2} + \alpha x [y_1'(x) \ln x + y_1(x) x^{-1}] \\
&\quad + (x^2 - 1) \left[ \sum_{n=0}^{\infty} d_n x^{n-1} + \alpha y_1(x) \ln x \right] \\
&= \sum_{n=0}^{\infty} [(n-1)(n-2) + (n-1) - 1] d_n x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} \\
&\quad + \alpha [x^2 y_1''(x) + x y_1'(x) + (x^2 - 1) y_1(x)] \ln x \\
&\quad + \alpha [2x y_1'(x) - y_1(x)] + \alpha y_1(x) \\
&= \sum_{n=0}^{\infty} n(n-2) d_n x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} + 2\alpha x y_1'(x). \\
&= \sum_{n=0}^{\infty} n(n-2) d_n x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} + 2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{4^n (n+1)! n!} x^{2n+1}. \quad (4.49)
\end{aligned}$$

*We now try to combine like powers of* $x$*. First, we combine the terms involving* $d_n$*'s,*

$$-d_1 + d_0 x + \sum_{k=3}^{\infty} [k(k-2)d_k + d_{k-2}]x^{k-1} = -2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{4^n n!(n+1)!} x^{2n+1}.$$

*Since there are no even powers on the right hand side of the equation, we find* $d_1 = 0$*, and* $k(k-2)d_k + d_{k-2} = 0$*,* $k \geq 3$ *and* $k$ *odd. Therefore, all odd* $d_k$*'s vanish.*

*Next, we set* $k - 1 = 2n + 1$*, or* $k = 2n + 2$*, in the remaining terms, obtaining*

$$d_0 x + \sum_{n=1}^{\infty} [(2n+2)(2n)d_{2n+2} + d_{2n}]x^{2n+1} = -2\alpha x - 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{4^n (n+1)!n!} x^{2n+1}.$$

*Thus,* $d_0 = -2\alpha$*. We choose* $\alpha = -\frac{1}{2}$*, making* $d_0 = 1$*. The remaining terms satisfy the relation*

$$(2n+2)(2n)d_{2n+2} + d_{2n} = \frac{(-1)^n (2n+1)}{4^n n!(n+1)!}, \quad n \geq 1$$

*or*

$$d_{2n+2} = \frac{d_{2n}}{4(n+1)(n)} + \frac{(-1)^n (2n+1)}{(n+1)(n)4^{n+1}(n+1)!n!}, \quad n \geq 1.$$

$$\begin{aligned}
d_4 &= -\frac{1}{4(2)(1)} d_2 - \frac{3}{(2)(1)4^2 2!1!} \\
&= -\frac{1}{4^2 2!1!}\left(4d_2 + \frac{3}{2}\right). \\
d_6 &= -\frac{1}{4(3)(2)} d_4 + \frac{5}{(3)(2)4^3 3!2!} \\
&= -\frac{1}{4(3)(2)}\left(-\frac{1}{4^2 2!1!}\left(4d_2 + \frac{3}{2}\right)\right) + \frac{5}{(3)(2)4^3 3!2!} \\
&= \frac{1}{4^3 3!2!}\left(4d_2 + \frac{3}{2} + \frac{5}{6}\right). \\
d_8 &= -\frac{1}{4(4)(3)} d_6 - \frac{7}{(4)(3)4^4 4!3!} \\
&= -\frac{1}{4^4 4!3!}\left(4d_2 + \frac{3}{2} + \frac{5}{6}\right) - \frac{7}{(4)(3)4^4 4!3!} \\
&= -\frac{1}{4^4 4!3!}\left(4d_2 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12}\right).
\end{aligned} \tag{4.50}$$

*Choosing* $4d_2 = 1$*, the coefficients take an interesting form. Namely,*

$$\begin{aligned}
1 + \frac{3}{2} &= 1 + \frac{1}{2} + 1 \\
1 + \frac{3}{2} + \frac{5}{6} &= 1 + \frac{1}{2}\frac{1}{3} + 1 + \frac{1}{2} \\
1 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12} &= 1 + \frac{1}{2}\frac{1}{3} + \frac{1}{4} + 1 + \frac{1}{2} + \frac{1}{3}.
\end{aligned} \tag{4.51}$$

*Defining the partial sums of the harmonic series,*

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad H_0 = 0,$$

*these coefficients become $H_n + H_{n-1}$ and the coefficients in the expansion are*

$$d_{2n} = \frac{(-1)^{n-1}(H_n + H_{n-1})}{4^n n!(n-1)!}, n = 1, 2, \ldots.$$

*We can verify this by computing $d_{10}$:*

$$\begin{aligned}
d_{10} &= -\frac{1}{4(4)(3)} d_8 + \frac{9}{(5)(4)4^5 5!4!} \\
&= \frac{1}{4^5 5!4!}\left(4d_2 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12}\right) + \frac{9}{(5)(4)4^5 5!4!} \\
&= \frac{1}{4^5 5!4!}\left(4d_2 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12} + \frac{9}{20}\right) \\
&= \frac{1}{4^5 5!4!}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \\
&= \frac{1}{4^5 5!4!}\left(H_5 + H_4\right).
\end{aligned}$$

*This gives the second solution as*

$$y_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(H_n + H_{n-1})}{4^n n!(n-1)!} x^{2n-1} - \frac{1}{2} y_1(x) \ln x + x^{-1}.$$

## 4.5 Legendre Polynomials

[1] Adrien-Marie Legendre (1752-1833) was a French mathematician who made many contributions to analysis and algebra.

THE LEGENDRE[1] POLYNOMIALS are one of a set of classical orthogonal polynomials . These polynomials satisfy a second-order linear differential equation. This differential equation occurs naturally in the solution of initial-boundary value problems in three dimensions which possess some spherical symmetry. Legendre polynomials, or Legendre functions of the first kind, are solutions of the differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

A generalization of the Legendre equation is given by $(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0$. Solutions to this equation, $P_n^m(x)$ and $Q_n^m(x)$, are called the associated Legendre functions of the first and second kind.

In Example **??** we found that for $n$ an integer, there are polynomial solutions. The first of these are given by $P_0(x) = c_0$, $P_1(x) = c_1 x$, and $P_2(x) = c_2(1 - 3x^2)$. As the Legendre equation is a linear second-order differential equation, we expect two linearly independent solutions. The second solution, called the Legendre function of the second kind, is given by $Q_n(x)$ and is not well behaved at $x = \pm 1$. For example,

$$Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

We will mostly focus on the Legendre polynomials and some of their properties in this section.

### 4.5.1 Properties of Legendre Polynomials

LEGENDRE POLYNOMIALS BELONG TO THE CLASS of classical orthogonal polynomials. Members of this class satisfy similar properties. First, we have the Rodrigues Formula for Legendre polynomials:

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N_0. \qquad (4.52)}$$

From the Rodrigues formula, one can show that $P_n(x)$ is an $n$th degree polynomial. Also, for $n$ odd, the polynomial is an odd function and for $n$ even, the polynomial is an even function.

**Example 4.13.** *Determine $P_2(x)$ from the Rodrigues Formula:*

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 \\ &= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) \\ &= \frac{1}{8}\frac{d}{dx}(4x^3-4x) \\ &= \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned} \tag{4.53}$$

*Note that we get the same result as we found in the last section using orthogonalization.*

| $n$ | $(x^2-1)^n$ | $\frac{d^n}{dx^n}(x^2-1)^n$ | $\frac{1}{2^n n!}$ | $P_n(x)$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | $x^2-1$ | $2x$ | $\frac{1}{2}$ | $x$ |
| 2 | $x^4-2x^2+1$ | $12x^2-4$ | $\frac{1}{8}$ | $\frac{1}{2}(3x^2-1)$ |
| 3 | $x^6-3x^4+3x^2-1$ | $120x^3-72x$ | $\frac{1}{48}$ | $\frac{1}{2}(5x^3-3x)$ |

Table 4.2: Tabular computation of the Legendre polynomials using the Rodrigues Formula.

The first several Legendre polynomials are given in Table 4.2. In Figure 4.1 we show plots of these Legendre polynomials.

The Three-Term Recursion Formula.

The classical orthogonal polynomials also satisfy a three-term recursion formula (or, recurrence relation or formula). In the case of the Legendre polynomials, we have

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots. \tag{4.54}$$

This can also be rewritten by replacing $n$ with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{4.55}$$

**Example 4.14.** *Use the recursion formula to find $P_2(x)$ and $P_3(x)$, given that $P_0(x) = 1$ and $P_1(x) = x$.*

*We first begin by inserting $n = 1$ into Equation (4.54):*

$$2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1.$$

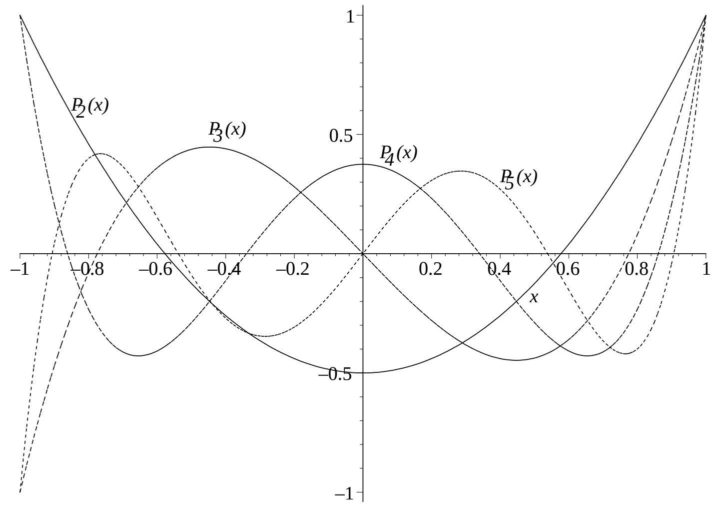



Figure 4.1: Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

*So,* $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

*For* $n = 2$*, we have*

$$\begin{aligned} 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ &= \frac{5}{2}x(3x^2 - 1) - 2x \\ &= \frac{1}{2}(15x^3 - 9x). \end{aligned} \tag{4.56}$$

*This gives* $P_3(x) = \frac{1}{2}(5x^3 - 3x)$*. These expressions agree with the earlier results.*

### 4.5.2 *The Generating Function for Legendre Polynomials*

A proof of the three-term recursion formula can be obtained from the generating function of the Legendre polynomials. Many special functions have such generating functions. In this case, it is given by

$$g(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \le 1, |t| < 1. \tag{4.57}$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions.

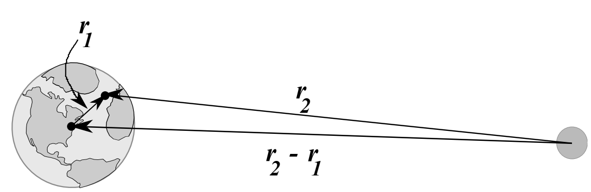



Figure 4.2: The position vectors used to describe the tidal force on the Earth due to the moon.

For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 4.2. The tidal potential $\Phi$ is proportional to

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1 r_2 \cos\theta + r_2^2}},$$

where $\theta$ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1 r_2 \cos\theta + r_2^2}} = \frac{1}{r_2} \frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have that the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion, $\frac{1}{r_2}$, is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the gravitational potential for mass $m$ at distance $r$ from $M$ is given by $\Phi = -\frac{GMm}{r}$ and that the force is the gradient of the potential, $\mathbf{F} = -\nabla\Phi \propto \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

**Example 4.15.** *Evaluate $P_n(0)$ using the generating function. $P_n(0)$ is found by considering $g(0,t)$. Setting $x = 0$ in Equation (4.57), we have*

$$\begin{aligned} g(0,t) &= \frac{1}{\sqrt{1+t^2}} \\ &= \sum_{n=0}^{\infty} P_n(0) t^n \\ &= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \ldots. \end{aligned} \tag{4.58}$$

*We can use the binomial expansion to find the final answer. Namely, we have*

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \ldots.$$

*Comparing these expansions, we have the $P_n(0) = 0$ for $n$ odd and for even integers one can show that*[2]

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{4.59}$$

[2] This example can be finished by first proving that

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

*where $n!!$ is the double factorial,*

$$n!! = \begin{cases} n(n-2)\ldots(3)1, & n > 0, odd, \\ n(n-2)\ldots(4)2, & n > 0, even, \\ 1, & n = 0, -1. \end{cases}.$$

**Example 4.16.** *Evaluate $P_n(-1)$. This is a simpler problem. In this case we have*

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots.$$

*Therefore, $P_n(-1) = (-1)^n$.*

Proof of the three-term recursion formula using the generating function.

**Example 4.17.** *Prove the three-term recursion formula,*

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots,$$

*using the generating function.*

*We can also use the generating function to find recurrence relations. To prove the three term recursion (4.54) that we introduced above, then we need only differentiate the generating function with respect to $t$ in Equation (4.57) and rearrange the result. First note that*

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2} g(x,t).$$

*Combining this with*

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1},$$

*we have*

$$(x-t)g(x,t) = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

*Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain*

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

*Multiplying out the $x - t$ factor and rearranging, leads to three separate sums:*

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0. \quad (4.60)$$

*Each term contains powers of $t$ that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k = n - 1$. Then, the first sum can be written*

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k.$$

*Using different indices is just another way of writing out the terms. Note that*

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

*and*

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

*actually give the same sum. The indices are sometimes referred to as dummy indices because they do not show up in the expanded expression and can be replaced with another letter.*

*If we want to do so, we could now replace all the $k$'s with $n$'s. However, we will leave the $k$'s in the first term and now reindex the next sums in Equation (4.60). The second sum just needs the replacement $n = k$ and the last sum we re-index using $k = n+1$. Therefore, Equation (4.60) becomes*

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty}kP_{k-1}(x)t^k = 0. \quad (4.61)$$

*We can now combine all the terms, noting the $k = -1$ term is automatically zero and the $k = 0$ terms give*

$$P_1(x) - xP_0(x) = 0. \quad (4.62)$$

*Of course, we know this already. So, that leaves the $k > 0$ terms:*

$$\sum_{k=1}^{\infty}\left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right]t^k = 0. \quad (4.63)$$

*Since this is true for all $t$, the coefficients of the $t^k$'s are zero, or*

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

*While this is the standard form for the three-term recurrence relation, the earlier form is obtained by setting $k = n-1$.*

There are other recursion relations that we list in the box below. Equation (4.64) was derived using the generating function. Differentiating it with respect to $x$, we find Equation (4.65). Equation (4.66) can be proven using the generating function by differentiating $g(x,t)$ with respect to $x$ and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 9. Combining this result with Equation (4.64), we can derive Equations (4.67) and (4.68). Adding and subtracting these equations yields Equations (4.69) and (4.70).

| **Recursion Formulae for Legendre Polynomials for $n = 1, 2, \ldots$.** | |
|---|---|
| $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ | (4.64) |
| $(n+1)P'_{n+1}(x) = (2n+1)[P_n(x) + xP'_n(x)] - nP'_{n-1}(x)$ | (4.65) |
| $P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$ | (4.66) |
| $P'_{n-1}(x) = xP'_n(x) - nP_n(x)$ | (4.67) |
| $P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$ | (4.68) |
| $P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x).$ | (4.69) |
| $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$ | (4.70) |
| $(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$ | (4.71) |

Finally, Equation (4.71) can be obtained using Equations (4.67) and (4.68). Just multiply Equation (4.67) by $x$,

$$x^2P'_n(x) - nxP_n(x) = xP'_{n-1}(x).$$

Now use Equation (4.68), but first replace $n$ with $n-1$ to eliminate the $xP'_{n-1}(x)$ term:

$$x^2P'_n(x) - nxP_n(x) = P'_n(x) - nP_{n-1}(x).$$

Rearranging gives the Equation (4.71).

**Example 4.18.** *Use the generating function to prove*

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

*Another use of the generating function is to obtain the normalization constant. This can be done by first squaring the generating function in order to get the products $P_n(x)P_m(x)$, and then integrating over $x$.*

The normalization constant.

*Squaring the generating function must be done with care, as we need to make proper use of the dummy summation index. So, we first write*

$$\begin{aligned} \frac{1}{1-2xt+t^2} &= \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \end{aligned} \tag{4.72}$$

*Integrating from $x = -1$ to $x = 1$ and using the orthogonality of the Legendre polynomials, we have*

$$\begin{aligned} \int_{-1}^{1} \frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m} \int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n^2(x)\,dx. \end{aligned} \tag{4.73}$$

*However, one can show that*[3]

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

*Expanding this expression about $t=0$, we obtain*[4]

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}.$$

*Comparing this result with Equation (4.73), we find that*

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}. \tag{4.74}$$

Finally, we can use the properties of the Legendre polynomials to obtain the Legendre differential equation. We begin by differentiating Equation (4.71) and using Equation (4.67) to simplify:

$$\begin{aligned} \frac{d}{dx}\left((x^2-1)P_n'(x)\right) &= nP_n(x) + nxP_n'(x) - nP_{n-1}'(x) \\ &= nP_n(x) + n^2P_n(x) \\ &= n(n+1)P_n(x). \end{aligned} \tag{4.75}$$

## 4.6 Bessel Functions

BESSEL FUNCTIONS ARISE IN MANY PROBLEMS in physics possessing cylindrical symmetry, such as the vibrations of circular drumheads and the radial modes in optical fibers. They also provide us with another orthogonal set of basis functions.

The first occurrence of Bessel functions (zeroth order) was in the work of Daniel Bernoulli on heavy chains (1738). More general Bessel functions were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764. Joseph Fourier found them in the study of heat conduction in solid cylinders and Siméon Poisson (1781-1840) in heat conduction of spheres (1823).

The history of Bessel functions, did not just originate in the study of the wave and heat equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G. N. Watson in his *Treatise on Bessel Functions*, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, $E$, as functions of time. Lagrange found expressions for the coefficients in the expansions of $r$ and $E$ in trigonometric functions of time. However, he only computed the first few coefficients. In 1816, Friedrich Wilhelm Bessel (1784-1846) had shown that the coefficients in the expansion for $r$ could be given an integral representation. In 1824, he presented a thorough study of these functions, which are now called Bessel functions.

[4] You will need the series expansion

$$\begin{aligned} \ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots. \end{aligned}$$

Bessel functions have a long history and were named after Friedrich Wilhelm Bessel (1784-1846).

You might have seen Bessel functions in a course on differential equations as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{4.76}$$

Solutions to this equation are obtained in the form of series expansions. Namely, one seeks solutions of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$$

by determining the form the coefficients must take. We will leave this for a homework exercise and simply report the results.

One solution of the differential equation is the *Bessel function of the first kind of order* $p$, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{4.77}$$

Here $\Gamma(x)$ s the Gamma function, satisfying $\Gamma(x+1) = x\Gamma(x)$. It is a generalization of the factorial and is discussed in the next section.

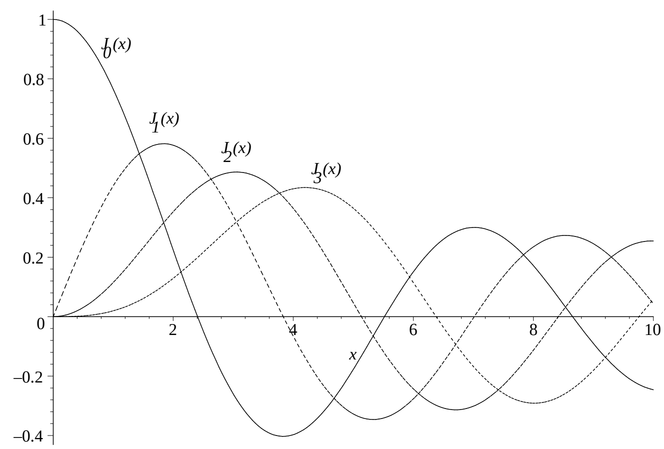



Figure 4.3: Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

In Figure 4.3, we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

A second linearly independent solution is obtained for $p$ not an integer as $J_{-p}(x)$. However, for $p$ an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when $p$ is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{4.78}$$

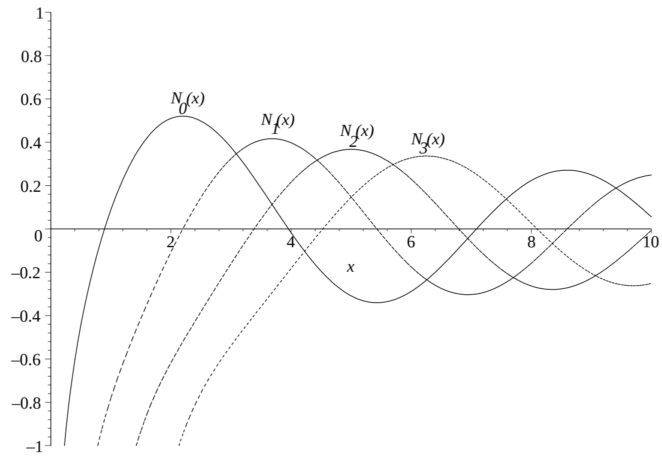



Figure 4.4: Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.

These functions are called the Neumann functions, or Bessel functions of the second kind of order $p$.

In Figure 4.4, we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

In many applications, one desires bounded solutions at $x = 0$. These functions do not satisfy this boundary condition. For example, one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates. The radial equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

**Derivative Identities** These identities follow directly from the manipulation of the series solution.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{4.79}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x). \tag{4.80}$$

**Recursion Formulae** The next identities follow from adding, or subtracting, the derivative identities.

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x). \tag{4.81}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{4.82}$$

**Orthogonality** One can recast the Bessel equation into an eigenvalue problem whose solutions form an orthogonal basis of functions on $L^2_x(0,a)$. Using Sturm-Liouville Theory, one can show that

$$\int_0^a xJ_p(j_{pn}\frac{x}{a})J_p(j_{pm}\frac{x}{a})\,dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn})\right]^2\delta_{n,m}, \tag{4.83}$$

where $j_{pn}$ is the $n$th root of $J_p(x)$, $J_p(j_{pn}) = 0$, $n = 1, 2, \ldots$. A list of some of these roots is provided in Table 4.3.

Table 4.3: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

| $n$ | $m=0$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ |
|---|---|---|---|---|---|---|
| 1 | 2.405 | 3.832 | 5.136 | 6.380 | 7.588 | 8.771 |
| 2 | 5.520 | 7.016 | 8.417 | 9.761 | 11.065 | 12.339 |
| 3 | 8.654 | 10.173 | 11.620 | 13.015 | 14.373 | 15.700 |
| 4 | 11.792 | 13.324 | 14.796 | 16.223 | 17.616 | 18.980 |
| 5 | 14.931 | 16.471 | 17.960 | 19.409 | 20.827 | 22.218 |
| 6 | 18.071 | 19.616 | 21.117 | 22.583 | 24.019 | 25.430 |
| 7 | 21.212 | 22.760 | 24.270 | 25.748 | 27.199 | 28.627 |
| 8 | 24.352 | 25.904 | 27.421 | 28.908 | 30.371 | 31.812 |
| 9 | 27.493 | 29.047 | 30.569 | 32.065 | 33.537 | 34.989 |

**Generating Function**

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad x > 0, t \neq 0. \tag{4.84}$$

**Integral Representation**

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{4.85}$$

## 4.7 *Gamma Function*

The name and symbol for the Gamma function were first given by Legendre in 1811. However, the search for a generalization of the factorial extends back to the 1720's when Euler provided the first representation of the factorial as an infinite product, later to be modified by others like Gauß, Weierstraß, and Legendre.

A function that often occurs in the study of special functions is the Gamma function. We will need the Gamma function in the next section on Fourier-Bessel series.

For $x > 0$ we define the Gamma function as

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt, \quad x > 0. \tag{4.86}$$

The Gamma function is a generalization of the factorial function and a plot is shown in Figure 4.5. In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove this identity by simply performing an integration by parts. (See Problem 11.) In particular, for integers $n \in Z^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

We can also define the Gamma function for negative, non-integer values of $x$. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad -n < x < 0.$$

Note that the Gamma function is undefined at zero and the negative integers.

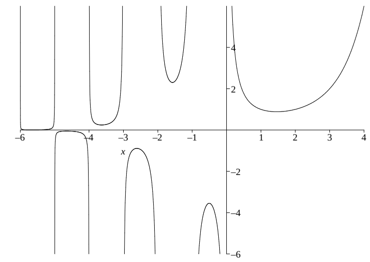


Figure 4.5: Plot of the Gamma function.

**Example 4.19.** *We now prove that*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

*This is done by direct computation of the integral:*

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt.$$

*Letting $t = z^2$, we have*

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2}\, dz.$$

*Due to the symmetry of the integrand, we obtain the classic integral*

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-z^2}\, dz,$$

*which can be performed using a standard trick.*[5] *Consider the integral*

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx.$$

*Then,*

$$I^2 = \int_{-\infty}^\infty e^{-x^2}\, dx \int_{-\infty}^\infty e^{-y^2}\, dy.$$

*Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:*

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\, dxdy.$$

*This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes*

$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2}\, rdrd\theta.$$

*This is simple to integrate and we have $I^2 = \pi$. So, the final result is found by taking the square root of both sides:*

$$\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}.$$

[5] Using a substitution $x^2 = \beta y^2$, we can show the more general result:

$$\int_{-\infty}^\infty e^{-\beta y^2}\, dy = \sqrt{\frac{\pi}{\beta}}.$$

In Problem 13, the reader will prove the more general identity

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

The are many other important relations, including infinite products, which we will not need at this point. The reader is encouraged to read about these elsewhere. In the meantime, we move on to the discussion of another important special function in physics and mathematics.

## 4.8 Hypergeometric Functions

Hypergeometric functions are probably the most useful, but least understood, class of functions. They typically do not make it into the undergraduate curriculum and seldom in graduate curriculum. Most functions that you know can be expressed using hypergeometric functions. There are many approaches to these functions and the literature can fill books. [6]

[6] See for example *Special Functions* by G. E. Andrews, R. Askey, and R. Roy, 1999, Cambridge University Press.

In 1812 Gauss published a study of the *hypergeometric series*

$$\begin{aligned} y(x) = 1 \quad &+ \quad \frac{\alpha\beta}{\gamma}x + \frac{\alpha(1+\alpha)(1+\beta)}{2!\gamma(1+\gamma)}x^2 \\ &+ \quad \frac{\alpha(1+\alpha)(2+\alpha)\beta(1+\beta)(2+\beta)}{3!\gamma(1+\gamma)(2+\gamma)}x^3 + \ldots . \end{aligned} \tag{4.87}$$

Here $\alpha, \beta, \gamma$, and $x$ are real numbers. If one sets $\alpha = 1$ and $\beta = \gamma$, this series reduces to the familiar geometric series

$$y(x) = 1 + x + x^2 + x^3 + \ldots .$$

The hypergeometric series is actually a solution of the differential equation

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]\, y' - \alpha\beta y = 0. \tag{4.88}$$

This equation was first introduced by Euler and latter studied extensively by Gauss, Kummer and Riemann. It is sometimes called Gauss' equation. Note that there is a symmetry in that $\alpha$ and $\beta$ may be interchanged without changing the equation. The points $x = 0$ and $x = 1$ are regular singular points. Series solutions may be sought using the Frobenius method. It can be confirmed that the above hypergeometric series results.

A more compact form for the hypergeometric series may be obtained by introducing new notation. One typically introduces the *Pochhammer symbol,* $(\alpha)_n$, satisfying (i) $(\alpha)_0 = 1$ if $\alpha \neq 0$. and (ii) $(\alpha)_k = \alpha(1+\alpha)\ldots(k-1+\alpha)$, for $k = 1, 2, \ldots$.

Consider $(1)_n$. For $n = 0$, $(1)_0 = 1$. For $n > 0$,

$$(1)_n = 1(1+1)(2+1)\ldots[(n-1)+1].$$

This reduces to $(1)_n = n!$. In fact, one can show that

$$(k)_n = \frac{(n+k-1)!}{(k-1)!}$$

for $k$ and $n$ positive integers. In fact, one can extend this result to noninteger values for $k$ by introducing the gamma function:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

We can now write the hypergeometric series in standard notation as

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n.$$

Using this one can show that the general solution of Gauss' equation is

$$y(x) = A_2F_1(\alpha,\beta;\gamma;x) + B_2x_2^{1-\gamma}F_1(1-\gamma+\alpha,1-\gamma+\beta;2-\gamma;x).$$

By carefully letting $\beta$ approach $\infty$, one obtains what is called the *confluent hypergeometric function*. This in effect changes the nature of the differential equation. Gauss' equation has three regular singular points at $x = 0, 1, \infty$. One can transform Gauss' equation by letting $x = u/\beta$. This changes the regular singular points to $u = 0, \beta, \infty$. Letting $\beta \to \infty$, two of the singular points merge.

The new confluent hypergeometric function is then given as

$${}_1F_1(\alpha;\gamma;u) = \lim_{\beta\to\infty} {}_2F_1\left(\alpha,\beta;\gamma;\frac{u}{\beta}\right).$$

This function satisfies the differential equation

$$xy'' + (\gamma - x)y' - \alpha y = 0.$$

The purpose of this section is only to introduce the hypergeometric function. Many other special functions are related to the hypergeometric function after making some variable transformations. For example, the Legendre polynomials are given by

$$P_n(x) =_2 F_1(-n, n+1; 1; \frac{1-x}{2}).$$

In fact, one can also show that

$$\sin^{-1} x = x_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right).$$

The Bessel function $J_p(x)$ can be written in terms of confluent geometric functions as

$$J_p(x) = \frac{1}{\Gamma(p+1)}\left(\frac{z}{2}\right)^p e^{-iz}\,{}_1F_1\left(\frac{1}{2}+p, 1+2p; 2iz\right).$$

These are just a few connections of the powerful hypergeometric functions to some of the elementary functions that you know.

## *Problems*

**1.** Find the first four terms in the Taylor series expansion of the solution to $y'(x) = \sqrt{x^2 + y^2(x)}$, $y(0) = 1$.

**2.** Use the power series method to obtain power series solutions about the given point.

a. $y' = y - x$, $y(0) = 2$, $x_0 = 0$.

b. $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 0$, $x_0 = 0$.

c. $y'' + 2x^2y' + xy = 0$, $x_0 = 0$.

d. $y'' - xy' + 3y = 0$, $y(0) = 2$, $x_0 = 0$.

e. $xy'' - xy' + y = e^x$, $y(0) = 1$, $y'(0) = 2$, $x_0 = 0$.

f. $x^2y'' - xy' + y = 0$, $x_0 = 1$.

**3.** In Example 4.3 we found the general Maclaurin series solution to $y'' - xy' - y = 0$.

a. Show that one solution of this problem is $y_1(x) = e^{x^2/2}$.

b. Find the first five nonzero terms of the Maclaurin series expansion for $y_1(x)$ and

c. According to Maple, a second solution is $\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) e^{x^2/2}$. Use the Method of Reduction of Order to find this second linearly independent solution. Note: The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

d. Verify that this second solution is consistent with the solution found in Example 4.3.

**4.** Find at least one solution about the singular point $x = 0$ using the power series method. Determine the second solution using the method of reduction of order.

a. $x^2y'' + 2xy' - 2y = 0$.

b. $xy'' + (1 - x)y' - y = 0$.

c. $x^2y'' - x(1 - x)y' + y = 0$.

**5.** List the singular points in the finite plane of the following:

a. $(1 - x^2)y'' + \frac{3}{x+2}y' + \frac{(1-x)^2}{x+3}y = 0$.

b. $\frac{1}{x}y'' + \frac{3(x-4)}{x+6}y' + \frac{x^2(x-2)}{x-1}y = 0$.

c. $y'' + xy = 0$.

d. $x^2(x - 2)y'' + 4(x - 2)y' + 3y = 0$.

**6.** Sometimes one is interested in solutions for large $x$. This leads to the concept of the *point at infinity*.

a. Let $z = \frac{1}{x}$ and $y(x) = v(z)$. Using the Chain Rule, show that

$$\begin{aligned}\frac{dy}{dx} &= -z^2 \frac{dv}{dz}, \\ \frac{d^2y}{dx^2} &= z^4 \frac{d^2v}{dz^2} + 2z^2 \frac{dv}{dz}.\end{aligned}$$

b. Use the transformation in part (a) to transform the differential equation $x^2 y'' + y = 0$ into an equation for $w(z)$ and classify the point at infinity by determining if $w = 0$ is an ordinary point, a regular singular point, or an irregular singular point.

c. Classify the point at infinity for the following equations:

i. $y'' + xy = 0$.

ii. $x^2(x-2)y'' + 4(x-2)y' + 3y = 0$.

**7.** Find the general solution of the following equations using the Method of Frobenius at $x = 0$.

a. $4xy'' + 2y' + y = 0$.

b. $y'' + \frac{1}{4x^2} y = 0$.

c. $xy'' + 2y' + xy = 0$.

d. $y'' + \frac{1}{2x} y' - \frac{x+1}{2x^2} y = 0$.

d. $4x^2 y'' + 4xy' + (4x^2 - 1)y = 0$.

e. $2x(x+1)y'' + 3(x+1)y' - y = 0$.

f. $x^2 y'' - x(1+x)y' + y = 0$.

g. $xy'' - (4+x)y' + 2y = 0$.

**8.** Find $P_4(x)$ using

a. The Rodrigues Formula in Equation (4.52).

b. The three-term recursion formula in Equation (4.54).

**9.** In Equations (4.64) through (4.71) we provide several identities for Legendre polynomials. Derive the results in Equations (4.65) through (4.71) as described in the text. Namely,

a. Differentiating Equation (4.64) with respect to $x$, derive Equation (4.65).

b. Derive Equation (4.66) by differentiating $g(x, t)$ with respect to $x$ and rearranging the resulting infinite series.

c. Combining the previous result with Equation (4.64), derive Equations (4.67) and (4.68).

d. Adding and subtracting Equations (4.67) and (4.68), obtain Equations (4.69) and (4.70).

e. Derive Equation (4.71) using some of the other identities.

**10.** Use the recursion relation (4.54) to evaluate $\int_{-1}^{1} x P_n(x) P_m(x)\, dx$, $n \le m$.

**11.** Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

**12.** Prove the double factorial identities:

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

**13.** Using the property $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, prove that

$$\Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

**14.** Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\int_0^\infty x^{2/3} e^{-x}\, dx$.

b. $\int_0^\infty x^5 e^{-x^2}\, dx$.

c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n dx$.

**15.** A solution of Bessel's equation, $x^2 y'' + xy' + (x^2 - n^2)y = 0,$ , can be found using the guess $y(x) = \sum_{j=0}^\infty a_j x^{j+n}$. One obtains the recurrence relation $a_j = \frac{-1}{j(2n+j)} a_{j-2}$. Show that for $a_0 = (n!2^n)^{-1}$, we get the Bessel function of the first kind of order $n$ from the even values $j = 2k$:

$$J_n(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}.$$

**16.** Use the infinite series in Problem 15 to derive the derivative identities (4.79) and (4.80):

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

b. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

**17.** Prove the following identities based on those in Problem 16.

a. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$.

b. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$.

**18.** Use the derivative identities of Bessel functions, (4.79) and (4.80), and integration by parts to show that

$$\int x^3 J_0(x)\, dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$$

**19.** We can rewrite Bessel functions, $J_\nu(x)$, in a form which will allow the order to be non-integer by using the gamma function. You will need

$$\Gamma\left(k + \frac{1}{2}\right) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}$$

.

a. Extend the series definition of the Bessel function of the first kind of order $\nu$, $J_\nu(x)$, for $\nu \geq 0$ by writing the series solution for $y(x)$ in Problem 15 using the gamma function.

b. Extend the series to $J_{-\nu}(x)$, for $\nu \geq 0$. Discuss the resulting series and what happens when $\nu$ is a positive integer.

c. Use these results to obtain the closed form expressions

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

d. Use the results in part c with the recursion formula for Bessel functions to obtain a closed form for $J_{3/2}(x)$.

**20.** Show that setting $\alpha = 1$ and $\beta = \gamma$ in ${}_2F_1(\alpha, \beta; \gamma; x)$ leads to the geometric series.

**21.** Prove the following:

a. $(a)_n = (a)_{n-1}(a+n-1)$, $n = 1, 2, \ldots, a \neq 0$.

b. $(a)_n = a(a+1)_{n-1}$, $n = 1, 2, \ldots, a \neq 0$.

c. $J_p(x) = \frac{1}{\Gamma(p+1)} \left(\frac{z}{2}\right)^p e^{-iz} {}_1F_1\left(\frac{1}{2} + p, 1 + 2p; 2iz\right)$.

**22.** Verify the following relations by transforming the hypergeometric equation into the equation satisfied by each function.

a. $P_n(x) =_2 F_1(-n, n+1; 1; \frac{1-x}{2})$.

b. $\sin^{-1} x = x_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$.

c. $J_p(x) = \frac{1}{\Gamma(p+1)} \left(\frac{z}{2}\right)^p e^{-iz} {}_1F_1\left(\frac{1}{2} + p, 1 + 2p; 2iz\right)$.

# 5
# Laplace Transforms

*"We could, of course, use any notation we want; do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations." - Richard P. Feynman (1918-1988)*

## 5.1 The Laplace Transform

The Laplace transform is named after Pierre-Simon de Laplace (1749 - 1827). Laplace made major contributions, especially to celestial mechanics, tidal analysis, and probability.

UP TO THIS POINT WE HAVE ONLY EXPLORED Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the Laplace transform, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving initial value problems for ordinary differential equations.

Integral transform on $[a, b]$ with respect to the integral kernel, $K(x, k)$.

The Fourier and Laplace transforms are examples of a broader class of transforms known as integral transforms. For a function $f(x)$ defined on an interval $(a, b)$, we define the integral transform

$$F(k) = \int_a^b K(x,k) f(x)\, dx,$$

where $K(x, k)$ is a specified kernel of the transform. Looking at the Fourier transform, we see that the interval is stretched over the entire real axis and the kernel is of the form, $K(x, k) = e^{ikx}$. In Table 5.1 we show several types of integral transforms.

| | |
|---|---|
| Laplace Transform | $F(s) = \int_0^\infty e^{-sx} f(x)\, dx$ |
| Fourier Transform | $F(k) = \int_{-\infty}^\infty e^{ikx} f(x)\, dx$ |
| Fourier Cosine Transform | $F(k) = \int_0^\infty \cos(kx) f(x)\, dx$ |
| Fourier Sine Transform | $F(k) = \int_0^\infty \sin(kx) f(x)\, dx$ |
| Mellin Transform | $F(k) = \int_0^\infty x^{k-1} f(x)\, dx$ |
| Hankel Transform | $F(k) = \int_0^\infty x J_n(kx) f(x)\, dx$ |

Table 5.1: A Table of Common Integral Transforms.

It should be noted that these integral transforms inherit the linearity of integration. Namely, let $h(x) = \alpha f(x) + \beta g(x)$, where $\alpha$ and $\beta$ are constants.

Then,

$$\begin{aligned} H(k) &= \int_a^b K(x,k)h(x)\,dx, \\ &= \int_a^b K(x,k)(\alpha f(x)+\beta g(x))\,dx, \\ &= \alpha\int_a^b K(x,k)f(x)\,dx+\beta\int_a^b K(x,k)g(x)\,dx, \\ &= \alpha F(x)+\beta G(x). \end{aligned} \tag{5.1}$$

Therefore, we have shown linearity of the integral transforms. We have seen the linearity property used for Fourier transforms and we will use linearity in the study of Laplace transforms.

The Laplace transform of $f$, $F = \mathcal{L}[f]$.

We now turn to Laplace transforms. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0. \tag{5.2}$$

This is an improper integral and one needs

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

to guarantee convergence.

Laplace transforms also have proven useful in engineering for solving circuit problems and doing systems analysis. In Figure 5.1 it is shown that a signal $x(t)$ is provided as input to a linear system, indicated by $h(t)$. One is interested in the system output, $y(t)$, which is given by a convolution of the input and system functions. By considering the transforms of $x(t)$ and $h(t)$, the transform of the output is given as a product of the Laplace transforms in the $s$-domain. In order to obtain the output, one needs to compute a convolution product for Laplace transforms similar to the convolution operation we had seen for Fourier transforms earlier in the chapter. Of course, for us to do this in practice, we have to know how to compute Laplace transforms.

$$\begin{tikzcd} x(t) \arrow[r, dashed] \arrow[d, "\text{Laplace Transform}"'] & \boxed{h(t)} \arrow[r, dashed] \arrow[d] & y(t) = h(t) * x(t) \\ X(s) \arrow[r] & \boxed{H(s)} \arrow[r] & Y(s) = H(s)X(s) \arrow[u, "\text{Inverse Laplace Transform}"'] \end{tikzcd}$$

Figure 5.1: A schematic depicting the use of Laplace transforms in systems theory.

## 5.2 Properties and Examples of Laplace Transforms

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table 5.2.

Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 5.3, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these transforms can be used to sum infinite series and to solve initial value problems for ordinary differential equations.

| $f(t)$ | $F(s)$ | $f(t)$ | $F(s)$ |
|---|---|---|---|
| $c$ | $\frac{c}{s}$ | $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| $t^n$ | $\frac{n!}{s^{n+1}}, \quad s > 0$ | $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ | $e^{at} \sin \omega t$ | $\frac{\omega}{(s-a)^2+\omega^2}$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ | $e^{at} \cos \omega t$ | $\frac{s-a}{(s-a)^2+\omega^2}$ |
| $t \sin \omega t$ | $\frac{2\omega s}{(s^2+\omega^2)^2}$ | $t \cos \omega t$ | $\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ |
| $\sinh at$ | $\frac{a}{s^2-a^2}$ | $\cosh at$ | $\frac{s}{s^2-a^2}$ |
| $H(t-a)$ | $\frac{e^{-as}}{s}, \quad s > 0$ | $\delta(t-a)$ | $e^{-as}, \quad a \geq 0, s > 0$ |

Table 5.2: Table of Selected Laplace Transform Pairs.

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

**Example 5.1.** *Show that* $\mathcal{L}[1] = \frac{1}{s}$.

*For this example, we insert* $f(t) = 1$ *into the definition of the Laplace transform:*

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt.$$

*This is an improper integral and the computation is understood by introducing an upper limit of* $a$ *and then letting* $a \to \infty$. *We will not always write this limit, but it will be understood that this is how one computes such improper integrals. Proceeding with the computation, we have*

$$\begin{aligned} \mathcal{L}[1] &= \int_0^\infty e^{-st}\, dt \\ &= \lim_{a\to\infty} \int_0^a e^{-st}\, dt \\ &= \lim_{a\to\infty} \left( -\frac{1}{s} e^{-st} \right)_0^a \\ &= \lim_{a\to\infty} \left( -\frac{1}{s} e^{-sa} + \frac{1}{s} \right) = \frac{1}{s}. \end{aligned} \tag{5.3}$$

Thus, we have found that the Laplace transform of 1 is $\frac{1}{s}$. This result can be extended to any constant $c$, using the linearity of the transform, $\mathcal{L}[c] = c\mathcal{L}[1]$. Therefore,

$$\mathcal{L}[c] = \frac{c}{s}.$$

**Example 5.2.** *Show that* $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, *for* $s > a$.

*For this example, we can easily compute the transform. Again, we only need to compute the integral of an exponential function.*

$$\begin{aligned} \mathcal{L}[e^{at}] &= \int_0^\infty e^{at}e^{-st}\,dt \\ &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \left(\frac{1}{a-s}e^{(a-s)t}\right)_0^\infty \\ &= \lim_{t\to\infty}\frac{1}{a-s}e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a}. \end{aligned} \tag{5.4}$$

*Note that the last limit was computed as* $\lim_{t\to\infty} e^{(a-s)t} = 0$. *This is only true if* $a - s < 0$, *or* $s > a$. *[Actually,* $a$ *could be complex. In this case we would only need* $s$ *to be greater than the real part of* $a$, $s > Re(a)$.*]*

**Example 5.3.** *Show that* $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$ *and* $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$.

*For these examples, we could again insert the trigonometric functions directly into the transform and integrate. For example,*

$$\mathcal{L}[\cos at] = \int_0^\infty e^{-st}\cos at\,dt.$$

*Recall how one evaluates integrals involving the product of a trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the original unknown integral. Rearranging the resulting integral expressions, one arrives at the desired result. However, there is a much simpler way to compute these transforms.*

*Recall that* $e^{iat} = \cos at + i\sin at$. *Making use of the linearity of the Laplace transform, we have*

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].$$

*Thus, transforming this complex exponential will simultaneously provide the Laplace transforms for the sine and cosine functions!*

*The transform is simply computed as*

$$\mathcal{L}[e^{iat}] = \int_0^\infty e^{iat}e^{-st}\,dt = \int_0^\infty e^{-(s-ia)t}\,dt = \frac{1}{s-ia}.$$

*Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case,* $s > Re(ia) = 0$.

*We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:*

$$\frac{1}{s-ia} = \frac{1}{s-ia}\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

*Reading off the real and imaginary parts, we find the sought-after transforms,*

$$\begin{aligned} \mathcal{L}[\cos at] &= \frac{s}{s^2+a^2}, \\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}. \end{aligned} \tag{5.5}$$

**Example 5.4.** *Show that* $\mathcal{L}[t] = \frac{1}{s^2}$.

*For this example we evaluate*

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

*This integral can be evaluated using the method of integration by parts:*

$$\begin{aligned}\int_0^\infty te^{-st}\,dt &= -t\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2}. \end{aligned} \tag{5.6}$$

**Example 5.5.** *Show that* $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ *for nonnegative integer* $n$.

*We have seen the* $n = 0$ *and* $n = 1$ *cases:* $\mathcal{L}[1] = \frac{1}{s}$ *and* $\mathcal{L}[t] = \frac{1}{s^2}$. *We now generalize these results to nonnegative integer powers,* $n > 1$, *of* $t$. *We consider the integral*

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt.$$

*Following the previous example, we again integrate by parts:*[1]

$$\begin{aligned}\int_0^\infty t^n e^{-st}\,dt &= -t^n\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt \\ &= \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt. \end{aligned} \tag{5.7}$$

*We could continue to integrate by parts until the final integral is computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of* $t^{n-1}$. *So, we can write the result as*

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

*This is an example of a recursive definition of a sequence. In this case, we have a sequence of integrals. Denoting*

$$I_n = \mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

*and noting that* $I_0 = \mathcal{L}[1] = \frac{1}{s}$, *we have the following:*

$$I_n = \frac{n}{s}I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{5.8}$$

*This is also what is called a difference equation. It is a first-order difference equation with an "initial condition,"* $I_0$. *The next step is to solve this difference equation.*

*Finding the solution of this first-order difference equation is easy to do using simple iteration. Note that replacing* $n$ *with* $n-1$, *we have*

$$I_{n-1} = \frac{n-1}{s}I_{n-2}.$$

*Repeating the process, we find*

$$\begin{aligned} I_n &= \frac{n}{s}I_{n-1} \\ &= \frac{n}{s}\left(\frac{n-1}{s}I_{n-2}\right) \\ &= \frac{n(n-1)}{s^2}I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3}I_{n-3}. \end{aligned} \tag{5.9}$$

[1] This integral can just as easily be done using differentiation. We note that

$$\left(-\frac{d}{ds}\right)^n \int_0^\infty e^{-st}\,dt = \int_0^\infty t^n e^{-st}\,dt.$$

Since

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s},$$

$$\int_0^\infty t^n e^{-st}\,dt = \left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

We compute $\int_0^\infty t^n e^{-st}\,dt$ by turning it into an initial value problem for a first-order difference equation and finding the solution using an iterative method.

*We can repeat this process until we get to $I_0$, which we know. We have to carefully count the number of iterations. We do this by iterating k times and then figure out how many steps will get us to the known initial value. A list of iterates is easily written out:*

$$\begin{aligned}
I_n &= \frac{n}{s} I_{n-1} \\
&= \frac{n(n-1)}{s^2} I_{n-2} \\
&= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\
&= \ldots \\
&= \frac{n(n-1)(n-2)\ldots(n-k+1)}{s^k} I_{n-k}.
\end{aligned} \tag{5.10}$$

*Since we know $I_0 = \frac{1}{s}$, we choose to stop at $k = n$ obtaining*

$$I_n = \frac{n(n-1)(n-2)\ldots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

*Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$.*

*Such iterative techniques are useful in obtaining a variety of integrals, such as $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$.*

As a final note, one can extend this result to cases when $n$ is not an integer. To do this, we use the Gamma function, which was discussed in Section 4.7. Recall that the Gamma function is the generalization of the factorial function and is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{5.11}$$

Note the similarity to the Laplace transform of $t^{x-1}$ :

$$\mathcal{L}[t^{x-1}] = \int_0^\infty t^{x-1} e^{-st}\, dt.$$

For $x-1$ an integer and $s = 1$, we have that

$$\Gamma(x) = (x-1)!.$$

Thus, the Gamma function can be viewed as a generalization of the factorial and we have shown that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for $p > -1$.

Now we are ready to introduce additional properties of the Laplace transform in Table 5.3. We have already discussed the first property, which is a consequence of the linearity of integral transforms. We will prove the other properties in this and the following sections.

**Example 5.6.** *Show that $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$.*

| Laplace Transform Properties |
|---|
| $\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$ |
| $\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$ |
| $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0)$ |
| $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0)-f'(0)$ |
| $\mathcal{L}[e^{at}f(t)] = F(s-a)$ |
| $\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$ |
| $\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\,du] = F(s)G(s)$ |

Table 5.3: Table of selected Laplace transform properties.

*We have to compute*

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}\,dt.$$

*We can move the derivative off $f$ by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Letting $u = e^{-st}$ and $v = f(t)$, we have*

$$\begin{aligned}\mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt}e^{-st}\,dt \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0)+sF(s). \end{aligned} \tag{5.12}$$

*Here we have assumed that $f(t)e^{-st}$ vanishes for large $t$.*

*The final result is that*

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0).$$

**Example 6:** *Show that $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0)-f'(0)$.*

*We can compute this Laplace transform using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, we have*

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s)-g(0) = sG(s)-f'(0).$$

*But,*

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0).$$

*So,*

$$\begin{aligned}\mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= sG(s)-f'(0) \\ &= s\left[sF(s)-f(0)\right]-f'(0) \\ &= s^2F(s)-sf(0)-f'(0). \end{aligned} \tag{5.13}$$

We will return to the other properties in Table 5.3 after looking at a few applications.

## 5.3 Solution of ODEs Using Laplace Transforms

One of the typical applications of Laplace transforms is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of $s$. Then, one transforms back into $t$-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 5.2.

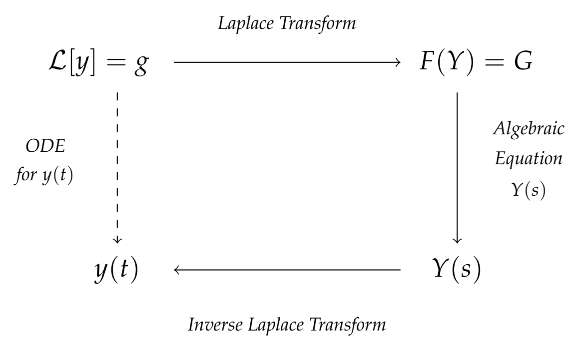



Figure 5.2: The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform gives the solution to the initial value problem.

**Example 5.7.** *Solve the initial value problem $y' + 3y = e^{2t}$, $y(0) = 1$.*

*The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is*

$$\mathcal{L}[y' + 3y] = sY - y(0) + 3Y = (s+3)Y - 1.$$

*Transforming the righthand side, we have*

$$\mathcal{L}[e^{2t}] = \frac{1}{s-2}.$$

*Combining these two results, we obtain*

$$(s+3)Y - 1 = \frac{1}{s-2}.$$

*The next step is to solve for $Y(s)$ :*

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

*Now we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. We will use the tables of Laplace transform pairs. Later we will show that there are other methods for carrying out the Laplace transform inversion.*

*The inverse transform of the first term is $e^{-3t}$. However, we have not seen anything that looks like the second form in the table of transforms that we have compiled, but we can rewrite the second term using a partial fraction decomposition. Let's recall how to do this.*

*The goal is to find constants A and B such that*

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}. \tag{5.14}$$

*We picked this form because we know that recombining the two terms into one term will have the same denominator. We just need to make sure the numerators agree afterward. So, adding the two terms, we have*

This is an example of carrying out a partial fraction decomposition.

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}.$$

*Equating numerators,*

$$1 = A(s+3) + B(s-2).$$

*There are several ways to proceed at this point.*

a. *Method 1.*

   *We can rewrite the equation by gathering terms with common powers of s, we have*

$$(A+B)s + 3A - 2B = 1.$$

   *The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B:*

$$\begin{aligned} A + B &= 0, \\ 3A - 2B &= 1. \end{aligned} \tag{5.15}$$

   *The first equation gives* $A = -B$, *so the second equation becomes* $-5B = 1$. *The solution is then* $A = -B = \frac{1}{5}$.

b. *Method 2.*

   *Since the equation* $\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$ *is true for all s, we can pick specific values. For* $s = 2$, *we find* $1 = 5A$, *or* $A = \frac{1}{5}$. *For* $s = -3$, *we find* $1 = -5B$, *or* $B = -\frac{1}{5}$. *Thus, we obtain the same result as Method 1, but much quicker.*

c. *Method 3.*

   *We could just inspect the original partial fraction problem. Since the numerator has no s terms, we might guess the form*

$$\frac{1}{(s-2)(s+3)} = \frac{1}{s-2} - \frac{1}{s+3}.$$

   *But, recombining the terms on the right hand side, we see that*

$$\frac{1}{s-2} - \frac{1}{s+3} = \frac{5}{(s-2)(s+3)}.$$

   *Since we were off by 5, we divide the partial fractions by 5 to obtain*

$$\frac{1}{(s-2)(s+3)} = \frac{1}{5}\left[\frac{1}{s-2} - \frac{1}{s+3}\right],$$

   *which once again gives the desired form.*

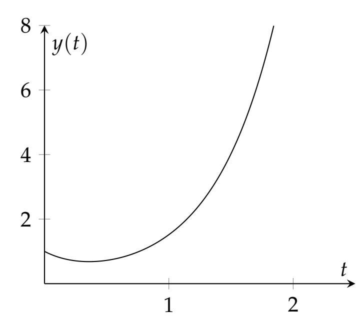


Figure 5.3: A plot of the solution to Example 5.7.

*Returning to the problem, we have found that*

$$Y(s) = \frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right).$$

*We can now see that the function with this Laplace transform is given by*

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right)\right] = e^{-3t} + \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

*works. Simplifying, we have the solution of the initial value problem*

$$y(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}.$$

*We can verify that we have solved the initial value problem.*

$$y' + 3y = \frac{2}{5}e^{2t} - \frac{12}{5}e^{-3t} + 3(\frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}) = e^{2t}$$

*and* $y(0) = \frac{1}{5} + \frac{4}{5} = 1$.

**Example 5.8.** *Solve the initial value problem* $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

*We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have*

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2+4)Y - s - 3. \end{aligned} \tag{5.16}$$

*Solving for* $Y$*, we have*

$$Y(s) = \frac{s+3}{s^2+4}.$$

*We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine. We just need to play with the numerator. Splitting the expression into two terms, we have*

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}.$$

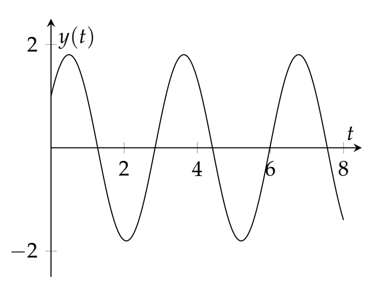


Figure 5.4: A plot of the solution to Example 5.8.

*The first term is now recognizable as the transform of* $\cos 2t$*. The second term is not the transform of* $\sin 2t$*. It would be if the numerator were a 2. This can be corrected by multiplying and dividing by 2:*

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right).$$

*The solution is then found as*

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right)\right] = \cos 2t + \frac{3}{2}\sin 2t.$$

*The reader can verify that this is the solution of the initial value problem.*

## 5.4 Step and Impulse Functions

OFTEN, THE INITIAL VALUE PROBLEMS THAT ONE FACES in differential equations courses can be solved using either the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using the latter can be messy and involves some skill with integration. Many circuit designs can be modeled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems, and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing impulse forces.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{5.17}$$

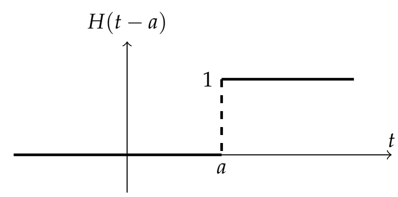



Figure 5.5: A shifted Heaviside function, $H(t-a)$.

A more general version of the step function is the horizontally shifted step function, $H(t-a)$. This function is shown in Figure 5.5. The Laplace transform of this function is found for $a > 0$ as

$$\begin{aligned} \mathcal{L}[H(t-a)] &= \int_0^\infty H(t-a)e^{-st}\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \left.\frac{e^{-st}}{s}\right|_a^\infty = \frac{e^{-as}}{s}. \end{aligned} \tag{5.18}$$

Just like the Fourier transform, the Laplace transform has two Shift Theorems involving the multiplication of the function, $f(t)$, or its transform, $F(s)$, by exponentials. The First and Second Shift Properties/Theorems are given by

The Shift Theorems.

$$\boxed{\begin{aligned} \mathcal{L}[e^{at}f(t)] &= F(s-a), & (5.19) \\ \mathcal{L}[f(t-a)H(t-a)] &= e^{-as}F(s). & (5.20) \end{aligned}}$$

We prove the First Shift Theorem and leave the other proof as an exercise for the reader. Namely,

$$\begin{aligned} \mathcal{L}[e^{at}f(t)] &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a). \end{aligned} \tag{5.21}$$

**Example 5.9.** *Compute the Laplace transform of* $e^{-at}\sin\omega t$.

*This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The First Shift Theorem* *tells us that we first need the transform of the sine function. So, for $f(t) = \sin \omega t$, we have*

$$F(s) = \frac{\omega}{s^2 + \omega^2}.$$

*Using this transform, we can obtain the solution to this problem as*

$$\mathcal{L}[e^{-at} \sin \omega t] = F(s+a) = \frac{\omega}{(s+a)^2 + \omega^2}.$$

More interesting examples can be found using piecewise defined functions. First we consider the function $H(t) - H(t-a)$. For $t < 0$, both terms are zero. In the interval $[0, a]$, the function $H(t) = 1$ and $H(t-a) = 0$. Therefore, $H(t) - H(t-a) = 1$ for $t \in [0, a]$. Finally, for $t > a$, both functions are one and therefore the difference is zero. The graph of $H(t) - H(t-a)$ is shown in Figure 5.6.

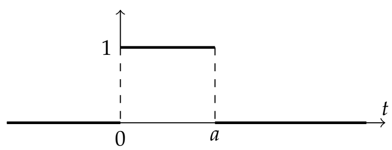


Figure 5.6: The box function, $H(t) - H(t-a)$.

We now consider the piecewise defined function:

$$g(t) = \begin{cases} f(t), & 0 \leq t \leq a, \\ 0, & t < 0, t > a. \end{cases}$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function,

$$g(t) = f(t)[H(t) - H(t-a)].$$

We depict this in Figure 5.7.

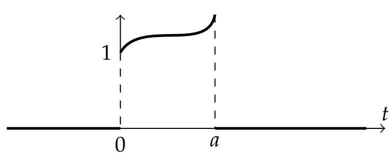


Figure 5.7: Formation of a piecewise function, $f(t)[H(t) - H(t-a)]$.

Even more complicated functions can be written in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a) - H(t-b)]$ for $b > a$. This is similar to a box function. It is nonzero between $a$ and $b$ and has height $f(t)$.

We show as an example the square wave function in Figure 5.8. It can be represented as a sum of an infinite number of boxes,

$$f(t) = \sum_{n=-\infty}^{\infty} [H(t-2na) - H(t-(2n+1)a)],$$

for $a > 0$.

**Example 5.10.** *Find the Laplace Transform of a square wave "turned on" at $t = 0$.*

.

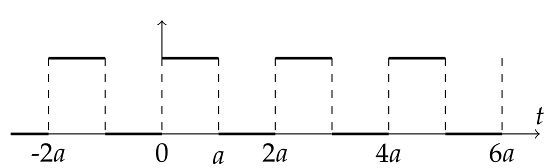


Figure 5.8: A square wave, $f(t) = \sum_{n=-\infty}^{\infty} [H(t-2na) - H(t-(2n+1)a)]$.

*We let*

$$f(t) = \sum_{n=0}^{\infty} [H(t-2na) - H(t-(2n+1)a)], \quad a > 0.$$

*Using the properties of the Heaviside function, we have*

$$
\begin{aligned}
\mathcal{L}[f(t)] &= \sum_{n=0}^{\infty}\left[\mathcal{L}[H(t-2na)]-\mathcal{L}[H(t-(2n+1)a)]\right] \\
&= \sum_{n=0}^{\infty}\left[\frac{e^{-2nas}}{s}-\frac{e^{-(2n+1)as}}{s}\right] \\
&= \frac{1-e^{-as}}{s}\sum_{n=0}^{\infty}\left(e^{-2as}\right)^n \\
&= \frac{1-e^{-as}}{s}\left(\frac{1}{1-e^{-2as}}\right) \\
&= \frac{1-e^{-as}}{s(1-e^{-2as})}. \qquad (5.22)
\end{aligned}
$$

*Note that the third line in the derivation is a geometric series. We summed this series to get the answer in a compact form since* $e^{-2as} < 1$.

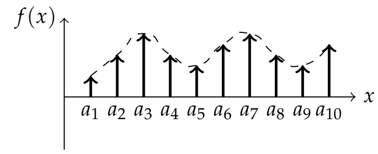



Figure 5.9: Plot representing impulse forces of height $f(a_i)$. The sum $\sum_{i=1}^{n} f(a_i)\delta(x-a_i)$ describes a general impulse function.

Other interesting examples are provided by the delta function. The Dirac delta function can be used to represent a unit impulse. Summing over a number of impulses, or point sources, we can describe a general function as shown in Figure 5.9. The sum of impulses located at points $a_i$, $i = 1, \ldots, n$, with strengths $f(a_i)$ would be given by

$$f(x) = \sum_{i=1}^{n} f(a_i)\delta(x-a_i).$$

A continuous sum could be written as

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\, d\xi.$$

This is simply an application of the sifting property of the delta function. We will investigate a case when one would use a single impulse. While a mass on a spring is undergoing simple harmonic motion, we hit it for an instant at time $t = a$. In such a case, we could represent the force as a multiple of $\delta(t-a)$.

$\mathcal{L}[\delta(t-a)] = e^{-as}$.

One would then need the Laplace transform of the delta function to solve the associated initial value problem. Inserting the delta function into the Laplace transform, we find that for $a > 0$,

$$
\begin{aligned}
\mathcal{L}[\delta(t-a)] &= \int_0^{\infty} \delta(t-a)e^{-st}\, dt \\
&= \int_{-\infty}^{\infty} \delta(t-a)e^{-st}\, dt \\
&= e^{-as}. \qquad (5.23)
\end{aligned}
$$

**Example 5.11.** *Solve the initial value problem* $y'' + 4\pi^2 y = \delta(t-2)$, $y(0) = y'(0) = 0$.

*This initial value problem models a spring oscillation with an impulse force. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at* $t = 2$. *Of course,* *we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.*

*First, we transform the differential equation:*

$$s^2Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

*Inserting the initial conditions, we have*

$$(s^2 + 4\pi^2)Y = e^{-2s}.$$

*Solving for $Y(s)$, we obtain*

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

*We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some Laplace transform, $F(s)$. Thus, we need the Second Shift Theorem since the solution is of the form $Y(s) = e^{-2s}F(s)$ for*

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

*We need to find the corresponding $f(t)$ of the Laplace transform pair. The denominator in $F(s)$ suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have*

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2 + 4\pi^2}.$$

*So, we write*

$$F(s) = \frac{1}{2\pi} \frac{2\pi}{s^2 + 4\pi^2}.$$

*This gives $f(t) = (2\pi)^{-1} \sin 2\pi t$.*

*We now apply the Second Shift Theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$, or*

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-2s}F(s)\right] \\ &= H(t-2)f(t-2) \\ &= \frac{1}{2\pi} H(t-2) \sin 2\pi(t-2). \end{aligned} \tag{5.24}$$

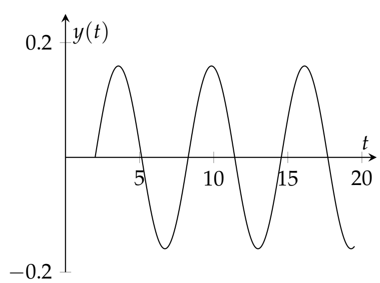


Figure 5.10: A plot of the solution to Example 5.11 in which a spring at rest experiences an impulse force at $t = 2$.

*This solution tells us that the mass is at rest until $t = 2$ and then begins to oscillate at its natural frequency. A plot of this solution is shown in Figure 5.10*

**Example 5.12.** *Solve the initial value problem*

$$y'' + y = f(t), \quad y(0) = 0, y'(0) = 0,$$

*where*

$$f(t) = \begin{cases} cos \pi t, & 0 \le t \le 2, \\ 0, & otherwise. \end{cases}$$

*We need the Laplace transform of $f(t)$. This function can be written in terms of a Heaviside function, $f(t) = \cos \pi t H(t-2)$. In order to apply the Second*

*Shift Theorem, we need a shifted version of the cosine function. We find the shifted version by noting that* $\cos \pi(t-2) = \cos \pi t$. *Thus, we have*

$$\begin{aligned} f(t) &= \cos \pi t \, [H(t) - H(t-2)] \\ &= \cos \pi t - \cos \pi(t-2) H(t-2), \quad t \geq 0. \end{aligned} \tag{5.25}$$

*The Laplace transform of this driving term is*

$$F(s) = (1 - e^{-2s}) \mathcal{L}[\cos \pi t] = (1 - e^{-2s}) \frac{s}{s^2 + \pi^2}.$$

*Now we can proceed to solve the initial value problem. The Laplace transform of the initial value problem yields*

$$(s^2 + 1) Y(s) = (1 - e^{-2s}) \frac{s}{s^2 + \pi^2}.$$

*Therefore,*

$$Y(s) = (1 - e^{-2s}) \frac{s}{(s^2 + \pi^2)(s^2 + 1)}.$$

*We can retrieve the solution to the initial value problem using the Second Shift Theorem. The solution is of the form* $Y(s) = (1 - e^{-2s}) G(s)$ *for*

$$G(s) = \frac{s}{(s^2 + \pi^2)(s^2 + 1)}.$$

*Then, the final solution takes the form*

$$y(t) = g(t) - g(t-2) H(t-2).$$

*We only need to find* $g(t)$ *in order to finish the problem. This is easily done using the partial fraction decomposition*

$$G(s) = \frac{s}{(s^2 + \pi^2)(s^2 + 1)} = \frac{1}{\pi^2 - 1} \left[ \frac{s}{s^2 + 1} - \frac{s}{s^2 + \pi^2} \right].$$

*Then,*

$$g(t) = \mathcal{L}^{-1} \left[ \frac{s}{(s^2 + \pi^2)(s^2 + 1)} \right] = \frac{1}{\pi^2 - 1} \left( \cos t - \cos \pi t \right).$$

*The final solution is then given by*

$$y(t) = \frac{1}{\pi^2 - 1} \left[ \cos t - \cos \pi t - H(t-2)(\cos(t-2) - \cos \pi t) \right].$$

*A plot of this solution is shown in Figure 5.11*

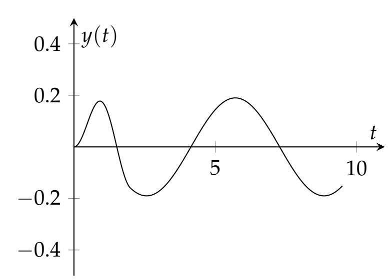



Figure 5.11: A plot of the solution to Example 5.12 in which a spring at rest experiences an piecewise defined force.

## 5.5 *The Convolution Theorem*

Finally, we consider the convolution of two functions. Often, we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say we have obtained $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve an initial value problem. In this case, we could find a partial fraction decomposition. But, there are other ways to find the inverse transform, especially if we cannot perform a partial fraction decomposition. We could use the Convolution Theorem for Laplace transforms or we could compute the inverse transform directly. We will look into these methods in the next two sections.We begin with defining the convolution.

We define the convolution of two functions defined on $[0, \infty)$ much the same way as we had done for the Fourier transform. The convolution $f * g$ is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)\, du. \tag{5.26}$$

Note that the convolution integral has finite limits as opposed to the Fourier transform case.

The convolution operation has two important properties:

The convolution is commutative.

1. The convolution is commutative: $f * g = g * f$

   *Proof.* The key is to make a substitution $y = t - u$ in the integral. This makes $f$ a simple function of the integration variable.

$$\begin{aligned} (g * f)(t) &= \int_0^t g(u)f(t-u)\, du \\ &= -\int_t^0 g(t-y)f(y)\, dy \\ &= \int_0^t f(y)g(t-y)\, dy \\ &= (f * g)(t). \end{aligned} \tag{5.27}$$

   □

The Convolution Theorem for Laplace transforms.

2. The Convolution Theorem: The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f * g] = F(s)G(s).$$

   *Proof.* Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that the functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Second, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.

   Carrying out the computation, we have

$$\begin{aligned} \mathcal{L}[f * g] &= \int_0^\infty \left( \int_0^t f(u)g(t-u)\, du \right) e^{-st}\, dt \\ &= \int_0^\infty \left( \int_0^\infty f(u)g(t-u)\, du \right) e^{-st}\, dt \\ &= \int_0^\infty f(u) \left( \int_0^\infty g(t-u)e^{-st}\, dt \right) du \end{aligned} \tag{5.28}$$

Now, make the substitution $\tau = t - u$. We note that

$$int_0^\infty f(u)\left(\int_0^\infty g(t-u)e^{-st}\,dt\right)\,du = \int_0^\infty f(u)\left(\int_{-u}^\infty g(\tau)e^{-s(\tau+u)}\,d\tau\right)\,du$$

However, since $g(\tau)$ is a causal function, we have that it vanishes for $\tau < 0$ and we can change the integration interval to $[0,\infty)$. So, after a little rearranging, we can proceed to the result.

$$\begin{aligned}
\mathcal{L}[f*g] &= \int_0^\infty f(u)\left(\int_0^\infty g(\tau)e^{-s(\tau+u)}\,d\tau\right)\,du \\
&= \int_0^\infty f(u)e^{-su}\left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right)\,du \\
&= \left(\int_0^\infty f(u)e^{-su}\,du\right)\left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right) \\
&= F(s)G(s). \qquad (5.29)
\end{aligned}$$

□

We make use of the Convolution Theorem to do the following examples.

**Example 5.13.** *Find* $y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)}\right]$.

*We note that this is a product of two functions:*

$$Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-1}\frac{1}{s-2} = F(s)G(s).$$

*We know the inverse transforms of the factors:* $f(t) = e^t$ *and* $g(t) = e^{2t}$.

*Using the Convolution Theorem, we find* $y(t) = (f*g)(t)$. *We compute the convolution:*

$$\begin{aligned}
y(t) &= \int_0^t f(u)g(t-u)\,du \\
&= \int_0^t e^u e^{2(t-u)}\,du \\
&= e^{2t}\int_0^t e^{-u}\,du \\
&= e^{2t}[-e^t+1] = e^{2t}-e^t. \qquad (5.30)
\end{aligned}$$

*One can also confirm this by carrying out a partial fraction decomposition.*

**Example 5.14.** *Consider the initial value problem,* $y'' + 9y = 2\sin 3t$, $y(0) = 1$, $y'(0) = 0$.

*The Laplace transform of this problem is given by*

$$(s^2+9)Y - s = \frac{6}{s^2+9}.$$

*Solving for* $Y(s)$, *we obtain*

$$Y(s) = \frac{6}{(s^2+9)^2} + \frac{s}{s^2+9}.$$

*The inverse Laplace transform of the second term is easily found as* $\cos(3t)$; *however, the first term is more complicated.*

*We can use the Convolution Theorem to find the Laplace transform of the first term. We note that*

$$\frac{6}{(s^2+9)^2} = \frac{2}{3}\frac{3}{(s^2+9)}\frac{3}{(s^2+9)}$$

*is a product of two Laplace transforms (up to the constant factor). Thus,*

$$\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] = \frac{2}{3}(f*g)(t),$$

*where $f(t) = g(t) = sin3t$. Evaluating this convolution product, we have*

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] &= \frac{2}{3}(f*g)(t) \\
&= \frac{2}{3}\int_0^t \sin 3u \sin 3(t-u)\,du \\
&= \frac{1}{3}\int_0^t [\cos 3(2u-t) - \cos 3t]\,du \\
&= \frac{1}{3}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t \\
&= \frac{1}{9}\sin 3t - \frac{1}{3}t\cos 3t. \qquad (5.31)
\end{aligned}$$

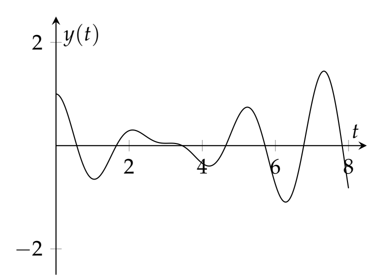


Figure 5.12: Plot of the solution to Example 5.14 showing a resonance.

*Combining this with the inverse transform of the second term of $Y(s)$, the solution to the initial value problem is*

$$y(t) = -\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t + \cos 3t.$$

*Note that the amplitude of the solution will grow in time from the first term. You can see this in Figure 5.12. This is known as a resonance.*

**Example 5.15.** *Find $\mathcal{L}^{-1}[\frac{6}{(s^2+9)^2}]$ using partial fraction decomposition.*

*If we look at Table 5.2, we see that the Laplace transform pairs with the denominator $(s^2+\omega^2)^2$ are*

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2},$$

*and*

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

*So, we might consider rewriting a partial fraction decomposition as*

$$\frac{6}{(s^2+9)^2} = \frac{A6s}{(s^2+9)^2} + \frac{B(s^2-9)}{(s^2+9)^2} + \frac{Cs+D}{s^2+9}.$$

*Combining the terms on the right over a common denominator, we find*

$$6 = 6As + B(s^2-9) + (Cs+D)(s^2+9).$$

*Collecting like powers of s, we have*

$$Cs^3 + (D+B)s^2 + 6As + (D-B) = 6.$$

*Therefore, $C = 0$, $A = 0$, $D+B = 0$, and $D-B = \frac{2}{3}$. Solving the last two equations, we find $D = -B = \frac{1}{3}$.*

*Using these results, we find*

$$\frac{6}{(s^2+9)^2} = -\frac{1}{3}\frac{(s^2-9)}{(s^2+9)^2} + \frac{1}{3}\frac{1}{s^2+9}.$$

*This is the result we had obtained in the last example using the Convolution Theorem.*

## 5.6 Systems of ODEs

Lapace transforms are also useful for solving systems of differential equations. We will study linear systems of differential equation in Chapter 6. For now, we will just look at simple examples of the application of Laplace transforms.

An example of a system of two differential equations for two unknown functions, $x(t)$ and $y(t)$, is given by the pair of coupled differential equations

$$\begin{aligned} x' &= 3x + 4y, \\ y' &= 2x + y. \end{aligned} \tag{5.32}$$

Neither equation can be solved on its own without knowledge of the other unknown function. This is why they are called couple. We will also need initial values for the system. We will choose $x(0) = 1$ and $y(0) = 0$.

Now, what would happen if we were to take the Laplace transform of each equation? We can apply the rules as before. Letting the Laplace transforms of $x(t)$ and $y(t)$ be $X(t)$ and $Y(t)$, respectively, we have

$$\begin{aligned} sX - 1 &= 3X + 4Y, \\ sY &= 2X + Y. \end{aligned} \tag{5.33}$$

We have obtained a system of algebraic equations for $X$ and $Y$. Using standard methods, like Cramer's Method, we can solve this system of two equations and two unknowns. First, we rewrite the equations as

$$\begin{aligned} (s-3)X - 4Y &= 1, \\ -2X + (s-1)Y &= 0. \end{aligned} \tag{5.34}$$

Using Cramer's (determinant) Rule for solving such systems, we have

$$X = \frac{\begin{vmatrix} 1 & -4 \\ 0 & s-1 \end{vmatrix}}{\begin{vmatrix} s-3 & -4 \\ -2 & s-1 \end{vmatrix}}, \qquad Y = \frac{\begin{vmatrix} s-3 & 1 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} s-3 & -4 \\ -2 & s-1 \end{vmatrix}}. \tag{5.35}$$

Note that the denominator in each solution is a $2 \times 2$ determinant consisting of the coefficients of $X$ and $Y$ in the appropriate order. The numerators are the same determinant but with the right hand side of the equation replacing the respective columns.

Computing the determinants, using

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

we have

$$X = \frac{1}{(s-3)(s-1)-8}, \quad Y = \frac{2}{(s-3)(s-1)-8},$$

or

$$X = \frac{s-1}{s^2-4s-5}, \quad Y = \frac{2}{s^2-4s-5}.$$

We now know the Laplace transforms of the solutions, so a simple inverse Laplace transform is in order. The denominators are the same,

$$s^2 - 4s - 5 = (s-5)(s+1).$$

We can apply a partial fraction decomposition to each function to obtain

$$\begin{aligned}
X &= \frac{s-1}{(s-5)(s+1)} \\
&= \frac{s-5+4}{(s-5)(s+1)} \\
&= \frac{1}{s+1} + \frac{4}{(s-5)(s+1)} \\
&= \frac{1}{s+1} + \frac{2}{3}\left[\frac{1}{s-5} - \frac{1}{s+1}\right] \\
&= \frac{2}{3}\frac{1}{s-5} + \frac{1}{3}\frac{1}{s+1}. \\
Y &= \frac{2}{(s-5)(s+1)} \\
&= \frac{1}{3}\left[\frac{1}{s-5} - \frac{1}{s+1}\right].
\end{aligned}$$

So, the solutions to the system of differential equations is given by

$$\begin{aligned}
x(t) &= \frac{2}{3}e^5t + \frac{1}{3}e^{-t}. \\
y(t) &= \frac{1}{3}(e^{5t} - e^{-t}).
\end{aligned}$$

We can verify that $x(0) = 1$ and $y(0) = 0$.

$$\begin{aligned}
x' &= \frac{10}{3}e^{5t} - \frac{1}{3}e^{-t} \\
3x + 4y &= (2e^{5t} + e^{-t}) + \frac{4}{3}(e^{5t} - e^{-t}) \\
&= \frac{10}{3}e^{5t} - \frac{1}{3}e^{-t}. \\
y' &= \frac{5}{3}e^{5t} + \frac{1}{3}e^{-t} \\
2x + y &= (\frac{4}{3}e^{5t} + \frac{2}{3}e^{-t}) + \frac{1}{3}(e^{5t} - e^{-t}) \\
&= \frac{5}{3}e^{5t} + \frac{1}{3}e^{-t}.
\end{aligned} \tag{5.36}$$

**Example 5.16.** *Determine the current in Figure 5.13 for the following values:* $i_1(0) = i_2(0) = i_3(0) = 0$ *and*

$$v(t) = \begin{cases} v_0, & 0 \le t \le 3.0 \\ 0, & \text{otherwise.} \end{cases}$$

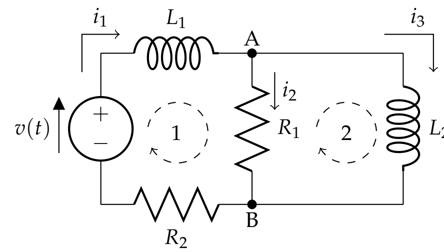



Figure 5.13: A circuit with two loops containing two resistors and two inductors in parallel.

*The problem can be modeled by a system of differential equations. In Figure 5.13 there are three currents indicated. Kirchoff's Point (Junction) Rule indicates that* $i_1 = i_2 + i_3$.

*In order to apply Kirchoff's Loop Rule , we need to tally the potential drops and rises. For resistors, these come from Ohm's Law,* $v = iR$*, and for inductors, this comes from Faraday's Law,* $v = L\frac{di}{dt}$*. For the left loop (2), we have*

$$L_2 i_3' = R_1 i_2,$$

*where the prime denotes the time derivative. For the right loop (1), we have*

$$L_1 i_1' + R_1 i_2 + R_2 i_1 = v(t).$$

*We can use the Point Rule to eliminate one of the currents,* $i_2 = i_1 - i_3$*, leaving the model as two first order differential equations,*

$$\begin{aligned} L_2 i_3' - R_1(i_1 - i_3) &= 0 \\ L_1 i_1' + R_1(i_1 - i_3) + R_2 i_1 &= v(t), \end{aligned}$$

*or*

$$\begin{aligned} L_2 i_3' - R_1 i_1 + R_1 i_3 &= 0 \\ L_1 i_1' + (R_1 + R_2) i_1 - R_1 i_3 &= v_0(1 - H(t-3)), \end{aligned}$$

*where* $H(t)$ *is the Heaviside function.*

*Taking the Laplace transform, assuming that* $i_1(0) = i_2(0) = 0$*, we obtain the algebraic system of equations*

$$\begin{aligned} -R_1 I_1 + (sL_2 + R_1) I_3 &= 0 \\ (sL_1 + R_1 + R_2) I_1 - R_1 I_3 &= \frac{v_0}{s}\left(1 - e^{-3s}\right). \end{aligned}$$

*Here* $I_1(s)$ *and* $I_3(s)$ *are the Laplace transforms of* $i_1(t)$ *and* $i_3(t)$*, respectively.*

*As before, we use Cramer's Rule to find the solutions.*

$$\begin{aligned} I_1 &= \frac{\begin{vmatrix} 0 & sL_2 + R_1 \\ \frac{v_0}{s}\left(1 - e^{-3s}\right) & -R_1 \end{vmatrix}}{\begin{vmatrix} -R_1 & sL_2 + R_1 \\ sL_1 + R_1 + R_2 & -R_1 \end{vmatrix}}, \\ &= \frac{-v_0(sL_2 + R_1)\left(1 - e^{-3s}\right)}{s[R_1^2 - (sL_2 + R_1)(sL_1 + R_1 + R_2)]} \\ &= \frac{v_0(sL_2 + R_1)\left(1 - e^{-3s}\right)}{s[(L_1 L_2)s^2 + (R_1 L_1 + L_2(R_1 + R_2))s + R_1 R_2]}. \end{aligned}$$

$$
\begin{aligned}
I_3 &= \frac{\begin{vmatrix} -R_1 & 0 \\ sL_1 + R_1 + R_2 & \frac{v_0}{s}\left(1 - e^{-3s}\right) \end{vmatrix}}{\begin{vmatrix} -R_1 & sL_2 + R_1 \\ sL_1 + R_1 + R_2 & -R_1 \end{vmatrix}} \\
&= \frac{v_0 R_1 \left(1 - e^{-3s}\right)}{s[R_1^2 - (sL_2 + R_1)(sL_1 + R_1 + R_2)]} \\
&= \frac{-v_0 R_1 \left(1 - e^{-3s}\right)}{s[(L_1 L_2)s^2 + (R_1 L_1 + L_2(R_1 + R_2))s + R_1 R_2]}. \qquad (5.37)
\end{aligned}
$$

*The denominator in these expressions cannot be factored. So, to make any further progress, one needs specific values for the constants. Let* $R_1 = 2.00\Omega$, $R_2 = 18.0\Omega$, $L_1 = 48.0$ *H,* $L_2 = 6.00$ *H. and* $v_0 = 18$ *V. Then,*

$$
\begin{aligned}
I_1 &= \frac{3s+1}{s(2s+1)(4s+1)}\left(1 - e^{-3s}\right) \\
I_3 &= -\frac{1}{s(2s+1)(4s+1)}\left(1 - e^{-3s}\right)
\end{aligned}
$$

*Using partial fractions on the coefficient of* $\left(1 - e^{-3s}\right)$*, we find that*

$$
\begin{aligned}
\frac{3s+1}{s(2s+1)(4s+1)} &= \frac{1}{s} - \frac{2}{4s+1} - \frac{1}{2s+1}, \\
\frac{1}{s(2s+1)(4s+1)} &= \frac{1}{s} - \frac{8}{4s+1} + \frac{2}{2s+1}.
\end{aligned}
$$

*This gives*

$$
\begin{aligned}
I_1 &= \left(\frac{1}{s} - \frac{1}{2}\frac{1}{s+\frac{1}{4}} - \frac{1}{2}\frac{1}{s+\frac{1}{2}}\right)(1 - e^{-3s}) \\
I_3 &= \left(-\frac{1}{s} + \frac{2}{s+\frac{1}{4}} - \frac{1}{s+\frac{1}{2}}\right)(1 - e^{-3s})
\end{aligned}
$$

*Taking the inverse Laplace transform, we find the solutions*

$$
\begin{aligned}
i_1 &= 1 - \frac{1}{2}e^{-\frac{t}{4}} - \frac{1}{2}e^{-\frac{t}{2}} + \left(-1 + \frac{1}{2}e^{-\frac{t-3}{4}} + \frac{1}{2}e^{-\frac{t-3}{2}}\right) H(t-3) \\
&= \begin{cases} 1 - \frac{1}{2}e^{-\frac{t}{4}} - \frac{1}{2}e^{-\frac{t}{2}}, & t \le 3, \\ -\frac{1}{2}(1 - e^{\frac{3}{4}})e^{-\frac{t}{4}} - \frac{1}{2}(1 - e^{\frac{3}{2}})e^{-\frac{t}{2}}, & t \ge 3. \end{cases} \\
i_3 &= -1 + 2e^{-\frac{t}{4}} - e^{-\frac{t}{2}} + \left(1 - 2e^{-\frac{t-3}{4}} + e^{-\frac{t-3}{2}}\right) H(t-3). \\
&= \begin{cases} -1 + 2e^{-\frac{t}{4}} - e^{-\frac{t}{2}}, & t \le 3, \\ 2(1 - e^{\frac{3}{4}})e^{-\frac{t}{4}} - (1 - e^{\frac{3}{2}})e^{-\frac{t}{2}}, & t \ge 3. \end{cases}
\end{aligned}
$$

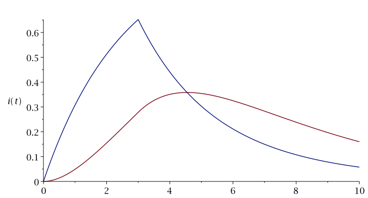



Figure 5.14: A plot of the currents vs time for Example 5.16 with the voltage $v(t) = v_0(1 - H(t-3))$. The taller curve represents $i_1$ and the other curve is $-i_3$.

*In Figure 5.14 we plot the currents vs time. The taller curve represents* $i_1$ *and the other curve is* $-i_3$*. We note that the derived current,* $i_3$*, is negative, indicating a flow in reverse of the direction shown in Figure 5.13. Not the sudden change in* $i[1]$ *at* $t = 3$*, the time that the voltage is turned on.*

*One can easily change the time that the voltage is applied. Namely, if*

$$
v(t) = \begin{cases} v_0, & 0 \le t \le t_0 \\ 0, & \textit{otherwise}, \end{cases}
$$

*then the solutions are given by*

$$\begin{aligned} i_1 &= 1 - \frac{1}{2}e^{-\frac{t}{4}} - \frac{1}{2}e^{-\frac{t}{2}} + \left(-1 + \frac{1}{2}e^{-\frac{t-t_0}{4}} + \frac{1}{2}e^{-\frac{t-t_0}{2}}\right) H(t-t_0) \\ i_3 &= -1 + 2e^{-\frac{t}{4}} - e^{-\frac{t}{2}} + \left(1 - 2e^{-\frac{t-t_0}{4}} + e^{-\frac{t-t_0}{2}}\right) H(t-t_0). \end{aligned}$$

*A plot of the currents for $t_0 = 10$ are shown in Figure 5.15.*

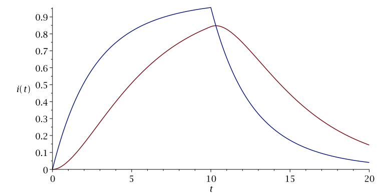



Figure 5.15: A plot of the currents vs time for Example 5.16 for the voltage given by $v(t) = v_0(1 - H(t-10))$. The taller curve represents $i_1$ and the other curve is $-i_3$.

## Problems

**1.** Find the Laplace transform of the following functions:

a. $f(t) = 9t^2 - 7$.

b. $f(t) = e^{5t-3}$.

c. $f(t) = \cos 7t$.

d. $f(t) = e^{4t}\sin 2t$.

e. $f(t) = e^{2t}(t + \cosh t)$.

f. $f(t) = t^2 H(t-1)$.

g. $f(t) = \begin{cases} \sin t, & t < 4\pi, \\ \sin t + \cos t, & t > 4\pi. \end{cases}$

h. $f(t) = \int_0^t (t-u)^2 \sin u \, du$.

i. $f(t) = (t+5)^2 + te^{2t}\cos 3t$ and write the answer in the simplest form.

**2.** Find the inverse Laplace transform of the following functions using the properties of Laplace transforms and the table of Laplace transform pairs.

a. $F(s) = \dfrac{18}{s^3} + \dfrac{7}{s}$.

b. $F(s) = \dfrac{1}{s-5} - \dfrac{2}{s^2+4}$.

c. $F(s) = \dfrac{s+1}{s^2+1}$.

d. $F(s) = \dfrac{3}{s^2+2s+2}$.

e. $F(s) = \dfrac{1}{(s-1)^2}$.

f. $F(s) = \dfrac{e^{-3s}}{s^2-1}$.

g. $F(s) = \dfrac{1}{s^2+4s-5}$.

h. $F(s) = \dfrac{s+3}{s^2+8s+17}$.

**3.** Compute the convolution $(f * g)(t)$ (in the Laplace transform sense) and its corresponding Laplace transform $\mathcal{L}[f * g]$ for the following functions:

a. $f(t) = t^2$, $g(t) = t^3$.
b. $f(t) = t^2$, $g(t) = \cos 2t$.
c. $f(t) = 3t^2 - 2t + 1$, $g(t) = e^{-3t}$.
d. $f(t) = \delta\left(t - \frac{\pi}{4}\right)$, $g(t) = \sin 5t$.

**4.** For the following problems, draw the given function and find the Laplace transform in closed form.

a. $f(t) = 1 + \sum_{n=1}^{\infty}(-1)^n H(t-n)$.
b. $f(t) = \sum_{n=0}^{\infty}[H(t-2n+1) - H(t-2n)]$.
c. $f(t) = \sum_{n=0}^{\infty}(t-2n)[H(t-2n) - H(t-2n-1)] + (2n+2-t)[H(t-2n-1) - H(t-2n-2)]$.

**5.** Use the Convolution Theorem to compute the inverse transform of the following:

a. $F(s) = \dfrac{2}{s^2(s^2+1)}$.
b. $F(s) = \dfrac{e^{-3s}}{s^2}$.
c. $F(s) = \dfrac{1}{s(s^2+2s+5)}$.

**6.** Find the inverse Laplace transform in two different ways: (i) Use tables. (ii) Use the Convolution Theorem.

a. $F(s) = \dfrac{1}{s^3(s+4)^2}$.
b. $F(s) = \dfrac{1}{s^2-4s-5}$.
c. $F(s) = \dfrac{s+3}{s^2+8s+17}$.
d. $F(s) = \dfrac{s+1}{(s-2)^2(s+4)}$.
e. $F(s) = \dfrac{s^2+8s-3}{(s^2+2s+1)(s^2+1)}$.

**7.** Use Laplace transforms to solve the following initial value problems. Where possible, describe the solution behavior in terms of oscillation and decay.

a. $y'' - 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 0$.
b. $y'' - y = te^{2t}$, $y(0) = 0$, $y'(0) = 1$.
c. $y'' + 4y = \delta(t-1)$, $y(0) = 3$, $y'(0) = 0$.
d. $y'' + 6y' + 18y = 2H(\pi - t)$, $y(0) = 0$, $y'(0) = 0$.

**8.** Use Laplace transforms to convert the following system of differential equations into an algebraic system and find the solution of the differential equations.

$$\begin{aligned} x'' &= 3x - 6y, \quad x(0) = 1, \quad x'(0) = 0, \\ y'' &= x + y, \quad\;\;\; y(0) = 0, \quad y'(0) = 0. \end{aligned}$$

**9.** Use Laplace transforms to convert the following nonhomogeneous systems of differential equations into an algebraic system and find the solutions of the differential equations.

a.

$$\begin{aligned} x' &= 2x + 3y + 2\sin 2t, \quad x(0) = 1, \\ y' &= -3x + 2y, \quad y(0) = 0. \end{aligned}$$

b.

$$\begin{aligned} x' &= -4x - y + e^{-t}, \quad x(0) = 2, \\ y' &= x - 2y + 2e^{-3t}, \quad y(0) = -1. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - y + 2\cos t, \quad x(0) = 3, \\ y' &= x + y - 3\sin t, \quad y(0) = 2. \end{aligned}$$

**10.** Redo Example 5.16 using the values $R_1 = 1.00\Omega$, $R_2 = 1.40\Omega$, $L_1 = 0.80$ H, $L_2 = 1.00$ H. and $v_0 = 100$ V in $v(t) = v_0(1 - H(t - t_0))$. Plot the currents as a function of time for several values of $t_0$.

# 6

# Linear Systems of Differential Equations

*"Do not worry too much about your difficulties in mathematics, I can assure you that mine are still greater." - Albert Einstein* (1879-1955)

## 6.1 Linear Systems

### 6.1.1 Coupled Oscillators

In the last section we saw that the numerical solution of second order equations, or higher, can be cast into systems of first order equations. Such systems are typically coupled in the sense that the solution of at least one of the equations in the system depends on knowing one of the other solutions in the system. In many physical systems this coupling takes place naturally. We will introduce a simple model in this section to illustrate the coupling of simple oscillators.

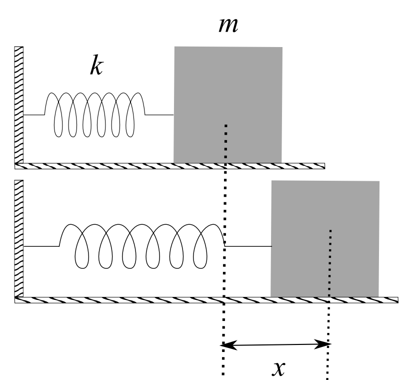


Figure 6.1: Spring-Mass system.

There are many problems in physics that result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. Thus,

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$ we have a system of second order differential equations in general for three dimensional problems, or one second order differential equation for one dimensional problems for a single mass.

We have already seen the simple problem of a mass on a spring as shown in Figure 2.1. Recall that the net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant and $x$ is the elongation of the spring. When the spring constant is positive, the spring force is negative and when the spring constant is negative the spring force is positive. The equation for simple harmonic motion for the mass-spring system was found to be given by

$$m\ddot{x} + kx = 0.$$

This second order equation can be written as a system of two first order equations in terms of the unknown position and velocity. We first set $y = \dot{x}$. Noting that $\ddot{x} = \dot{y}$, we rewrite the second order equation in terms of $x$ and $\dot{y}$. Thus, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{6.1}$$

One can look at more complicated spring-mass systems. Consider two blocks attached with two springs as in Figure 6.2. In this case we apply Newton's second law for each block. We will designate the elongations of each spring from equilibrium as $x_1$ and $x_2$. These are shown in Figure 6.2.

For mass $m_1$, the forces acting on it are due to each spring. The first spring with spring constant $k_1$ provides a force on $m_1$ of $-k_1x_1$. The second spring is stretched, or compressed, based upon the relative locations of the two masses. So, the second spring will exert a force on $m_1$ of $k_2(x_2 - x_1)$.

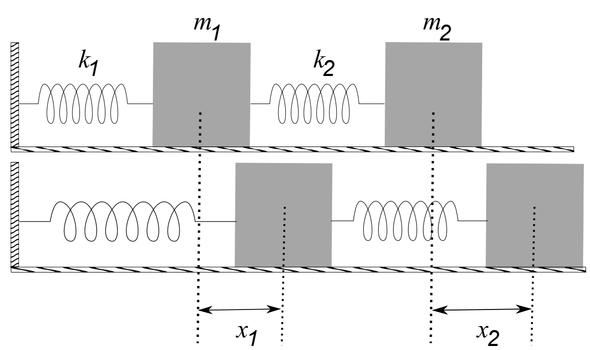



Figure 6.2: System of two masses and two springs.

Similarly, the only force acting directly on mass $m_2$ is provided by the restoring force from spring 2. So, that force is given by $-k_2(x_2 - x_1)$. The reader should think about the signs in each case.

Putting this all together, we apply Newton's Second Law to both masses. We obtain the two equations

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{6.2}$$

Thus, we see that we have a coupled system of two second order differential equations. Each equation depends on the unknowns $x_1$ and $x_2$.

One can rewrite this system of two second order equations as a system of four first order equations by letting $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. This leads to the system

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \end{aligned}$$

$$\dot{x}_4 = -\frac{k_2}{m_2}(x_2 - x_1). \tag{6.3}$$

As we will see in the next chapter, this system can be written more compactly in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{6.4}$$

We can solve this system of first order equations using matrix methods. However, we will first need to recall a few things from linear algebra. This will be done in the next chapter. For now, we will return to simpler systems and explore the behavior of typical solutions in planar systems.

### 6.1.2 *Planar Systems*

We now consider examples of solving a coupled system of first order differential equations in the plane. We will focus on the theory of linear systems with constant coefficients. Understanding these simple systems will help in the study of nonlinear systems, which contain much more interesting behaviors, such as the onset of chaos. In the next chapter we will return to these systems and describe a matrix approach to obtaining the solutions.

A general form for first order systems in the plane is given by a system of two equations for unknowns $x(t)$ and $y(t)$ :

$$\begin{aligned} x'(t) &= P(x,y,t) \\ y'(t) &= Q(x,y,t). \end{aligned} \tag{6.5}$$

An autonomous system is one in which there is no explicit time dependence:

Autonomous systems.

$$\begin{aligned} x'(t) &= P(x,y) \\ y'(t) &= Q(x,y). \end{aligned} \tag{6.6}$$

Otherwise the system is called nonautonomous.

A *linear system* takes the form

$$\begin{aligned} x' &= a(t)x + b(t)y + e(t) \\ y' &= c(t)x + d(t)y + f(t). \end{aligned} \tag{6.7}$$

A homogeneous linear system results when $e(t) = 0$ and $f(t) = 0$.

A linear, constant coefficient system of first order differential equations is given by

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f. \end{aligned} \tag{6.8}$$

We will focus on linear, homogeneous systems of constant coefficient first order differential equations:

A linear, homogeneous system of constant coefficient first order differential equations in the plane.

$$\boxed{\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned}} \tag{6.9}$$

As we will see later, such systems can result by a simple translation of the unknown functions. These equations are said to be coupled if either $b \neq 0$ or $c \neq 0$.

We begin by noting that the system (6.9) can be rewritten as a second order constant coefficient linear differential equation, which we already know how to solve. We differentiate the first equation in system (6.9) and systematically replace occurrences of $y$ and $y'$, since we also know from the first equation that $y = \frac{1}{b}(x' - ax)$. Thus, we have

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + b(cx + dy) \\ &= ax' + bcx + d(x' - ax). \end{aligned} \tag{6.10}$$

Rewriting the last line, we have

$$\boxed{x'' - (a+d)x' + (ad - bc)x = 0.} \tag{6.11}$$

This is a linear, homogeneous, constant coefficient ordinary differential equation. We know that we can solve this by first looking at the roots of the characteristic equation

$$r^2 - (a+d)r + ad - bc = 0 \tag{6.12}$$

and writing down the appropriate general solution for $x(t)$. Then we can find $y(t)$ using Equation (6.9):

$$y = \frac{1}{b}(x' - ax).$$

We now demonstrate this for a specific example.

**Example 6.1.** *Consider the system of differential equations*

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{6.13}$$

*Carrying out the above outlined steps, we have that* $x'' + 3x' - 4x = 0$. *This can be shown as follows:*

$$\begin{aligned} x'' &= -x' + 6y' \\ &= -x' + 6(x - 2y) \\ &= -x' + 6x - 12\left(\frac{x' + x}{6}\right) \\ &= -3x' + 4x \end{aligned} \tag{6.14}$$

*The resulting differential equation has a characteristic equation of* $r^2 + 3r - 4 = 0$. *The roots of this equation are* $r = 1, -4$. *Therefore,* $x(t) = c_1 e^t + c_2 e^{-4t}$. *But, we still need* $y(t)$. *From the first equation of the system we have*

$$y(t) = \frac{1}{6}(x' + x) = \frac{1}{6}(2c_1 e^t - 3c_2 e^{-4t}).$$

*Thus, the solution to the system is*

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \tfrac{1}{3} c_1 e^t - \tfrac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{6.15}$$

Sometimes one needs initial conditions. For these systems we would specify conditions like $x(0) = x_0$ and $y(0) = y_0$. These would allow the determination of the arbitrary constants as before.

Solving systems with initial conditions.

**Example 6.2.** *Solve*

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{6.16}$$

*given* $x(0) = 2$, $y(0) = 0$.

*We already have the general solution of this system in (6.15). Inserting the initial conditions, we have*

$$\begin{aligned} 2 &= c_1 + c_2, \\ 0 &= \tfrac{1}{3} c_1 - \tfrac{1}{2} c_2. \end{aligned} \tag{6.17}$$

*Solving for* $c_1$ *and* $c_2$ *gives* $c_1 = 6/5$ *and* $c_2 = 4/5$. *Therefore, the solution of the initial value problem is*

$$\begin{aligned} x(t) &= \tfrac{2}{5}\left(3e^t + 2e^{-4t}\right), \\ y(t) &= \tfrac{2}{5}\left(e^t - e^{-4t}\right). \end{aligned} \tag{6.18}$$

### 6.1.3 Equilibrium Solutions and Nearby Behaviors

In studying systems of differential equations, it is often useful to study the behavior of solutions without obtaining an algebraic form for the solution. This is done by exploring equilibrium solutions and solutions nearby equilibrium solutions. Such techniques will be seen to be useful later in studying nonlinear systems.

We begin this section by studying equilibrium solutions of system (6.8). For equilibrium solutions the system does not change in time. Therefore, equilibrium solutions satisfy the equations $x' = 0$ and $y' = 0$. Of course, this can only happen for constant solutions. Let $x_0$ and $y_0$ be the (constant) equilibrium solutions. Then, $x_0$ and $y_0$ must satisfy the system

Equilibrium solutions.

$$\begin{aligned} 0 &= ax_0 + by_0 + e, \\ 0 &= cx_0 + dy_0 + f. \end{aligned} \tag{6.19}$$

This is a linear system of nonhomogeneous algebraic equations. One only has a unique solution when the determinant of the system is not zero, i.e., $ad - bc \neq 0$. Using Cramer's (determinant) Rule for solving such systems, we have

$$\boxed{x_0 = -\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y_0 = -\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.} \tag{6.20}$$

If the system is homogeneous, $e = f = 0$, then we have that the origin is the equilibrium solution; i.e., $(x_0, y_0) = (0, 0)$. Often we will have this case since one can always make a change of coordinates from $(x, y)$ to $(u, v)$ by $u = x - x_0$ and $v = y - y_0$. Then, $u_0 = v_0 = 0$.

Next we are interested in the behavior of solutions near the equilibrium solutions. Later this behavior will be useful in analyzing more complicated nonlinear systems. We will look at some simple systems that are readily solved.

**Example 6.3. Stable Node (sink)**

*Consider the system*

$$\begin{aligned} x' &= -2x \\ y' &= -y. \end{aligned} \tag{6.21}$$

*This is a simple uncoupled system. Each equation is simply solved to give*

$$x(t) = c_1 e^{-2t} \text{ and } y(t) = c_2 e^{-t}.$$

*In this case we see that all solutions tend towards the equilibrium point,* $(0, 0)$. *This will be called a stable node, or a sink.*

Before looking at other types of solutions, we will explore the stable node in the above example. There are several methods of looking at the behavior of solutions. We can look at solution plots of the dependent versus the independent variables, or we can look in the $xy$-plane at the parametric curves $(x(t), y(t))$.

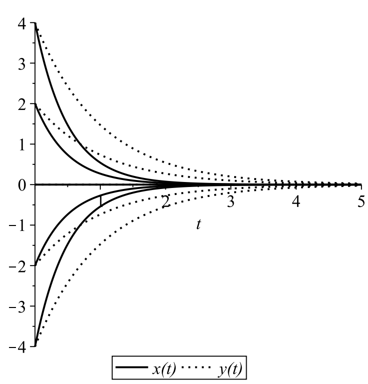



Figure 6.3: Plots of solutions of Example 6.3 for several initial conditions.

**Solution Plots:** One can plot each solution as a function of $t$ given a set of initial conditions. Examples are shown in Figure 6.3 for several initial conditions. Note that the solutions decay for large $t$. Special cases result for various initial conditions. Note that for $t = 0$, $x(0) = c_1$ and $y(0) = c_2$. (Of course, one can provide initial conditions at any $t = t_0$. It is generally easier to pick $t = 0$ in our general explanations.) If we pick an initial condition with $c_1$=0, then $x(t) = 0$ for all $t$. One obtains similar results when setting $y(0) = 0$.

**Phase Portrait:** There are other types of plots which can provide additional information about the solutions even if we cannot find the exact solutions as we can for these simple examples. In particular, one can consider the solutions $x(t)$ and $y(t)$ as the coordinates along a parameterized path, or curve, in the plane: $\mathbf{r} = (x(t), y(t))$ Such curves are called trajectories or orbits. The $xy$-plane is called the phase plane and a collection of such orbits gives a phase portrait for the family of solutions of the given system.

One method for determining the equations of the orbits in the phase plane is to eliminate the parameter $t$ between the known solutions to get a relationship between $x$ and $y$. Since the solutions are known for the last example, we can do this, since the solutions are known. In particular, we have

$$x = c_1 e^{-2t} = c_1 \left(\frac{y}{c_2}\right)^2 \equiv Ay^2.$$

Another way to obtain information about the orbits comes from noting that the slopes of the orbits in the $xy$-plane are given by $dy/dx$. For autonomous systems, we can write this slope just in terms of $x$ and $y$. This leads to a first order differential equation, which possibly could be solved analytically or numerically.

First we will obtain the orbits for Example 6.3 by solving the corresponding slope equation. Recall that for trajectories defined parametrically by $x = x(t)$ and $y = y(t)$, we have from the Chain Rule for $y = y(x(t))$ that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Therefore,

The Slope of a parametric curve.

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.} \tag{6.22}$$

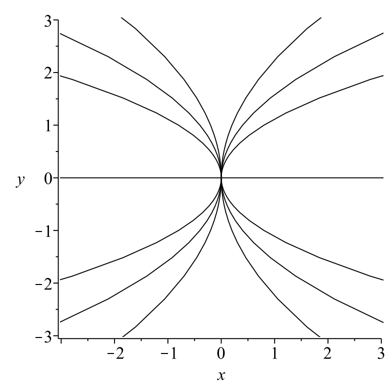


Figure 6.4: Orbits for Example 6.3.

For the system in (6.21) we use Equation (6.22) to obtain the equation for the slope at a point on the orbit:

$$\frac{dy}{dx} = \frac{y}{2x}.$$

The general solution of this first order differential equation is found using separation of variables as $x = Ay^2$ for $A$ an arbitrary constant. Plots of these solutions in the phase plane are given in Figure 6.4. [Note that this is the same form for the orbits that we had obtained above by eliminating $t$ from the solution of the system.]

Once one has solutions to differential equations, we often are interested in the long time behavior of the solutions. Given a particular initial condition $(x_0, y_0)$, how does the solution behave as time increases? For orbits near an equilibrium solution, do the solutions tend towards, or away from, the equilibrium point? The answer is obvious when one has the exact solutions $x(t)$ and $y(t)$. However, this is not always the case.

Let's consider the above example for initial conditions in the first quadrant of the phase plane. For a point in the first quadrant we have that

$$dx/dt = -2x < 0,$$

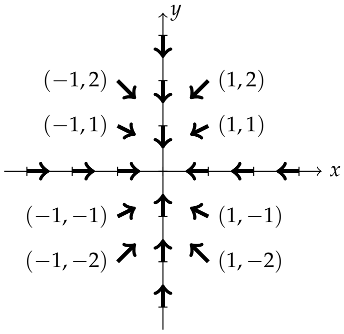



Figure 6.5: Sketch of tangent vectors using Example 6.3.

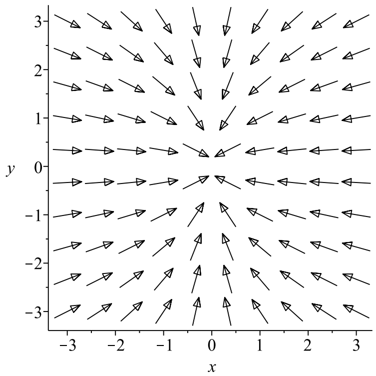



Figure 6.6: Direction field for Example 6.3.

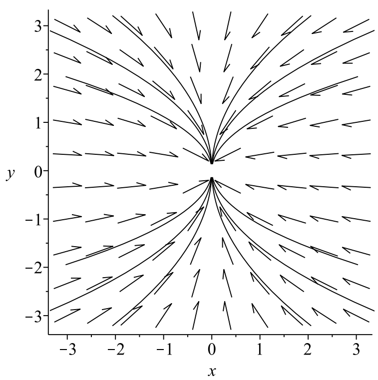



Figure 6.7: Phase portrait for Example 6.3. This is a stable node, or sink

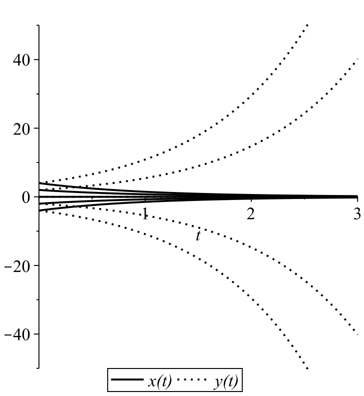



Figure 6.8: Plots of solutions of Example 6.5 for several initial conditions.

meaning that as $t \to \infty$, $x(t)$ get more negative. Similarly,

$$dy/dt = -y < 0,$$

indicating that $y(t)$ is also getting smaller for this problem. Thus, these orbits tend towards the origin as $t \to \infty$. This qualitative information was obtained without relying on the known solutions to the problem.

**Direction Fields:** Another way to determine the behavior of the solutions of the system of differential equations is to draw the direction field. A direction field is a vector field in which one plots arrows in the direction of tangents to the orbits at selected points in the plane. This is done because the slopes of the tangent lines are given by $dy/dx$. For the general system (6.9), the slope is

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

This is a first order differential equation which can be solved as we show in the following examples.

**Example 6.4.** *Draw the direction field for Example 6.3.*

*We can use software to draw direction fields. However, one can sketch these fields by hand. We have that the slope of the tangent at this point is given by*

$$\frac{dy}{dx} = \frac{-y}{-2x} = \frac{y}{2x}.$$

*For each point in the plane one draws a piece of tangent line with this slope. In Figure 6.5 we show a few of these. For $(x, y) = (1, 1)$ the slope is $dy/dx = 1/2$. So, we draw an arrow with slope $1/2$ at this point. From system (6.21), we have that $x'$ and $y'$ are both negative at this point. Therefore, the vector points down and to the left.*

*We can do this for several points, as shown in Figure 6.5. Sometimes one can quickly sketch vectors with the same slope. For this example, when $y = 0$, the slope is zero and when $x = 0$ the slope is infinite. So, several vectors can be provided. Such vectors are tangent to curves known as isoclines in which $\frac{dy}{dx}$ =constant.*

It is often difficult to provide an accurate sketch of a direction field. Computer software can be used to provide a better rendition. For Example 6.3 the direction field is shown in Figure 6.6. Looking at this direction field, one can begin to "see" the orbits by following the tangent vectors.

Of course, one can superimpose the orbits on the direction field. This is shown in Figure 6.7. Are these the patterns you saw in Figure 6.6?

In this example we see all orbits "flow" towards the origin, or equilibrium point. Again, this is an example of what is called a *stable node* or a *sink*. (Imagine what happens to the water in a sink when the drain is unplugged.)

**Example 6.5. Saddle** *Consider the system*

$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned} \tag{6.23}$$

*This is another uncoupled system. The solutions are again simply gotten by integration. We have that* $x(t) = c_1e^{-t}$ *and* $y(t) = c_2e^t$. *Here we have that* $x$ *decays as* $t$ *gets large and* $y$ *increases as* $t$ *gets large. In particular, if one picks initial conditions with* $c_2 = 0$, *then orbits follow the* $x$*-axis towards the origin. For initial points with* $c_1 = 0$, *orbits originating on the* $y$*-axis will flow away from the origin. Of course, in these cases the origin is an equilibrium point and once at equilibrium, one remains there.*

*In fact, there is only one line on which to pick initial conditions such that the orbit leads towards the equilibrium point. No matter how small* $c_2$ *is, sooner, or later, the exponential growth term will dominate the solution. One can see this behavior in Figure 6.8.*

Similar to the first example, we can look at plots of solutions orbits in the phase plane. These are given by Figures 6.8-6.9. The orbits can be obtained from the system as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x}.$$

The solution is $y = \frac{A}{x}$. For different values of $A \neq 0$ we obtain a family of hyperbolae. These are the same curves one might obtain for the level curves of a surface known as a saddle surface, $z = xy$. Thus, this type of equilibrium point is classified as a saddle point. From the phase portrait we can verify that there are many orbits that lead away from the origin (equilibrium point), but there is one line of initial conditions that leads to the origin and that is the $x$-axis. In this case, the line of initial conditions is given by the $x$-axis.

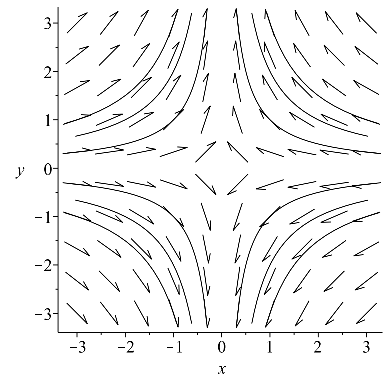



Figure 6.9: Phase portrait for Example 6.5. This is a saddle.

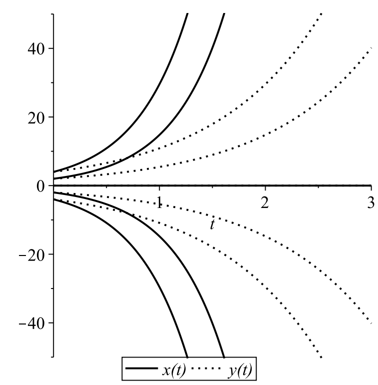



Figure 6.10: Plots of solutions of Example 6.6 for several initial conditions.

**Example 6.6. Unstable Node (source)**

$$\begin{aligned} x' &= 2x \\ y' &= y. \end{aligned} \tag{6.24}$$

*This example is similar to Example 6.3. The solutions are obtained by replacing* $t$ *with* $-t$. *The solutions, orbits, and direction fields can be seen in Figures 6.10-6.11. This is once again a node, but all orbits lead away from the equilibrium point. It is called an unstable node or a source.*

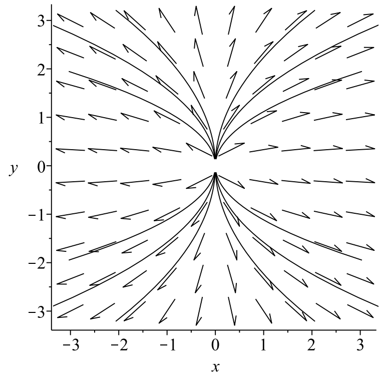



Figure 6.11: Phase portrait for Example 6.6, an unstable node or source.

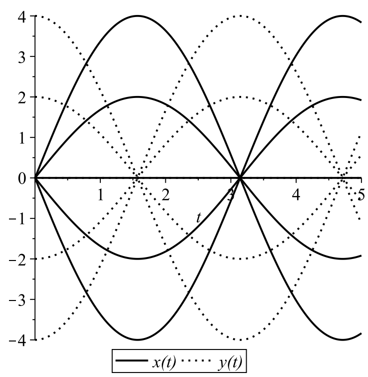



Figure 6.12: Plots of solutions of Example 6.7 for several initial conditions.

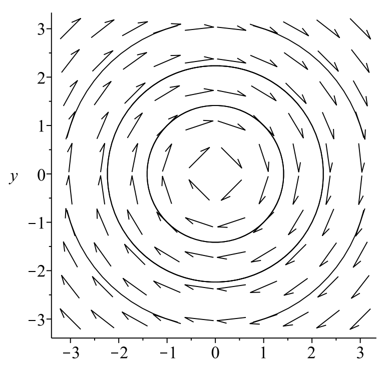



Figure 6.13: Phase portrait for Example 6.7, a center.

**Example 6.7. Center**

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{6.25}$$

*This system is a simple, coupled system. Neither equation can be solved without some information about the other unknown function. However, we can differentiate the first equation and use the second equation to obtain*

$$x'' + x = 0.$$

*We recognize this equation as one that appears in the study of simple harmonic motion. The solutions are pure sinusoidal oscillations:*

$$x(t) = c_1 \cos t + c_2 \sin t, \quad y(t) = -c_1 \sin t + c_2 \cos t.$$

*In the phase plane the trajectories can be determined either by looking at the direction field, or solving the first order equation*

$$\frac{dy}{dx} = -\frac{x}{y}.$$

*Performing a separation of variables and integrating, we find that*

$$x^2 + y^2 = C.$$

*Thus, we have a family of circles for $C > 0$. (Can you prove this using the general solution?) Looking at the results graphically in Figures 6.12-6.13 confirms this result. This type of point is called a center.*

**Example 6.8. Focus (spiral)**

$$\begin{aligned} x' &= \alpha x + y \\ y' &= -x. \end{aligned} \tag{6.26}$$

*In this example, we will see an additional set of behaviors of equilibrium points in planar systems. We have added one term, $\alpha x$, to the system in Example 6.7. We will consider the effects for two specific values of the parameter: $\alpha = 0.1, -0.2$. The resulting behaviors are shown in the Figures 6.15-6.18. We see orbits that look like spirals. These orbits are stable and unstable spirals (or foci, the plural of focus.)*

*We can understand these behaviors by once again relating the system of first order differential equations to a second order differential equation. Using the usual method for obtaining a second order equation form a system, we find that $x(t)$ satisfies the differential equation*

$$x'' - \alpha x' + x = 0.$$

*We recall from our first course that this is a form of damped simple harmonic motion. The characteristic equation is $r^2 - \alpha r + 1 = 0$. The solution of this quadratic equation is*

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

*There are five special cases to consider as shown in the below classification.*

**Classification of Solutions of $x'' - \alpha x' + x = 0$**

1. $\alpha = -2$. *There is one real solution. This case is called critical damping since the solution $r = -1$ leads to exponential decay. The solution is $x(t) = (c_1 + c_2 t)e^{-t}$.*
2. $\alpha < -2$. *There are two real, negative solutions, $r = -\mu, -\nu$, $\mu, \nu > 0$. The solution is $x(t) = c_1 e^{-\mu t} + c_2 e^{-\nu t}$. In this case we have what is called overdamped motion. There are no oscillations*
3. $-2 < \alpha < 0$. *There are two complex conjugate solutions $r = \alpha/2 \pm i\beta$ with real part less than zero and $\beta = \frac{\sqrt{4-\alpha^2}}{2}$. The solution is $x(t) = (c_1 \cos\beta t + c_2 \sin\beta t)e^{\alpha t/2}$. Since $\alpha < 0$, this consists of a decaying exponential times oscillations. This is often called an underdamped oscillation.*
4. $\alpha = 0$. *This leads to simple harmonic motion.*
5. $0 < \alpha < 2$. *This is similar to the underdamped case, except $\alpha > 0$. The solutions are growing oscillations.*
6. $\alpha = 2$. *There is one real solution. The solution is $x(t) = (c_1 + c_2 t)e^{t}$. It leads to unbounded growth in time.*
7. *For $\alpha > 2$. There are two real, positive solutions $r = \mu, \nu > 0$. The solution is $x(t) = c_1 e^{\mu t} + c_2 e^{\nu t}$, which grows in time.*

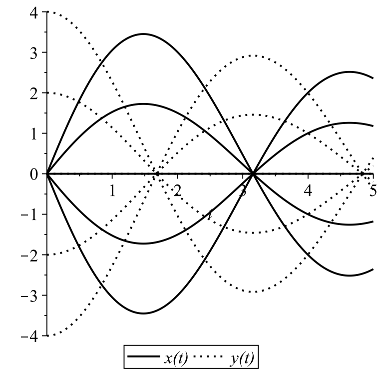



Figure 6.14: Plots of solutions of Example 6.8 for several initial conditions, $\alpha = -0.2$.

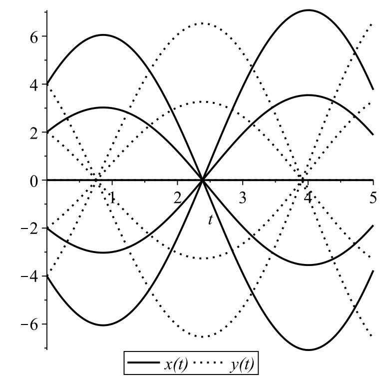



Figure 6.15: Plots of solutions of Example 6.8 for several initial conditions, $\alpha = 0.1$.

*For $\alpha < 0$ the solutions are losing energy, so the solutions can oscillate with a diminishing amplitude. (See Figure 6.14.) For $\alpha > 0$, there is a growth in the amplitude, which is not typical. (See Figure 6.15.) Of course, there can be overdamped motion if the magnitude of $\alpha$ is too large.*

**Example 6.9. Degenerate Node** *For this example, we will write out the solutions. It is a coupled system for which only the second equation is coupled.*

$$\begin{aligned} x' &= -x \\ y' &= -2x - y. \end{aligned} \tag{6.27}$$

*There are two possible approaches:*

*a. We could solve the first equation to find $x(t) = c_1 e^{-t}$. Inserting this solution into the second equation, we have*

$$y' + y = -2c_1 e^{-t}.$$

*This is a relatively simple linear first order equation for $y = y(t)$. The integrating factor is $\mu = e^t$. The solution is found as $y(t) = (c_2 - 2c_1 t)e^{-t}$.*

*b. Another method would be to proceed to rewrite this as a second order equation. Computing $x''$ does not get us very far. So, we look at*

$$\begin{aligned} y'' &= -2x' - y' \\ &= 2x - y' \\ &= -2y' - y. \end{aligned} \tag{6.28}$$

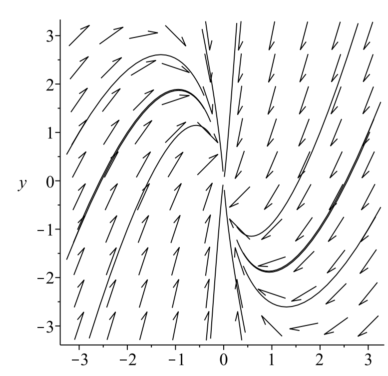



Figure 6.16: Phase portrait for 6.9. This is a degenerate node.

*Therefore, y satisfies*

$$y'' + 2y' + y = 0.$$

*The characteristic equation has one real root, $r = -1$. So, we write*

$$y(t) = (k_1 + k_2 t)e^{-t}.$$

*This is a stable degenerate node. Combining this with the solution $x(t) = c_1 e^{-t}$, we can show that $y(t) = (c_2 - 2c_1 t)e^{-t}$ as before.*

*In Figure 6.16 we see several orbits in this system. It differs from the stable node show in Figure 6.4 in that there is only one direction along which the orbits approach the origin instead of two. If one picks $c_1 = 0$, then $x(t) = 0$ and $y(t) = c_2 e^{-t}$. This leads to orbits running along the y-axis as seen in the figure.*

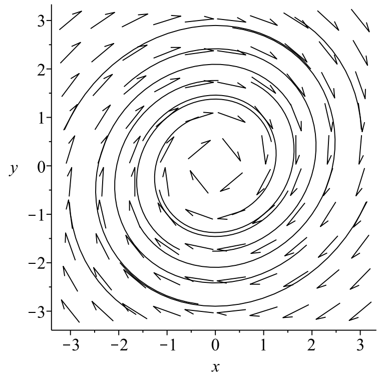



Figure 6.17: Phase portrait for Example 6.8 with $\alpha = -0.2$. This is a stable focus, or spiral.

**Example 6.10. A Line of Equilibria, Zero Root**

$$\begin{aligned} x' &= 2x - y \\ y' &= -2x + y. \end{aligned} \tag{6.29}$$

*In this last example, we have a coupled set of equations. We rewrite it as a second order differential equation:*

$$\begin{aligned} x'' &= 2x' - y' \\ &= 2x' - (-2x + y) \\ &= 2x' + 2x + (x' - 2x) = 3x'. \end{aligned} \tag{6.30}$$

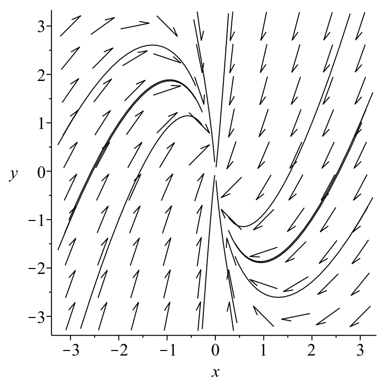



Figure 6.18: Phase portrait for Example 6.9. This is a degenerate node.

*So, the second order equation is*

$$x'' - 3x' = 0$$

*and the characteristic equation is $0 = r(r-3)$. This gives the general solution as*

$$x(t) = c_1 + c_2 e^{3t}$$

*and thus*

$$y = 2x - x' = 2(c_1 + c_2 e^3 t) - (3c_2 e^{3t}) = 2c_1 - c_2 e^{3t}.$$

*In Figure 6.19 we show the direction field. The constant slope field seen in this example is confirmed by a simple computation:*

$$\frac{dy}{dx} = \frac{-2x + y}{2x - y} = -1.$$

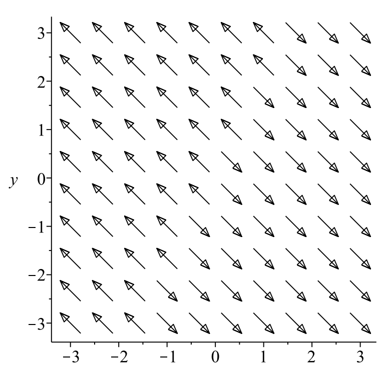



Figure 6.19: Plots of direction field of Example 6.10.

*Furthermore, looking at initial conditions with $y = 2x$, we have at $t = 0$,*

$$2c_1 - c_2 = 2(c_1 + c_2) \quad \Rightarrow \quad c_2 = 0.$$

*Therefore, points on this line remain on this line forever, $(x, y) = (c_1, 2c_1)$. This line of fixed points is called a line of equilibria.*

### 6.1.4 Polar Representation of Spirals

IN THE EXAMPLES WITH A CENTER OR A SPIRAL, one might be able to write the solutions in polar coordinates. Recall that a point in the plane can be described by either Cartesian $(x, y)$ or polar $(r, \theta)$ coordinates. Given the polar form, one can find the Cartesian components using

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Given the Cartesian coordinates, one can find the polar coordinates using

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}. \tag{6.31}$$

Since $x$ and $y$ are functions of $t$, then naturally we can think of $r$ and $\theta$ as functions of $t$. Converting a system of equations in the plane for $x'$ and $y'$ to polar form requires knowing $r'$ and $\theta'$. So, we first find expressions for $r'$ and $\theta'$ in terms of $x'$ and $y'$.

Differentiating the first equation in (6.31) gives

$$rr' = xx' + yy'.$$

Inserting the expressions for $x'$ and $y'$ from system 6.9, we have

$$rr' = x(ax + by) + y(cx + dy).$$

In some cases this may be written entirely in terms of $r$'s. Similarly, we have that

$$\theta' = \frac{xy' - yx'}{r^2},$$

which the reader can prove for homework.

In summary, when converting first order equations from rectangular to polar form, one needs the relations below.

**Derivatives of Polar Variables**

$$\begin{aligned} r' &= \frac{xx' + yy'}{r}, \\ \theta' &= \frac{xy' - yx'}{r^2}. \end{aligned} \tag{6.32}$$

**Example 6.11.** *Rewrite the following system in polar form and solve the resulting system.*

$$\begin{aligned} x' &= ax + by \\ y' &= -bx + ay. \end{aligned} \tag{6.33}$$

*We first compute $r'$ and $\theta'$:*

$$rr' = xx' + yy' = x(ax + by) + y(-bx + ay) = ar^2.$$

$$r^2\theta' = xy' - yx' = x(-bx + ay) - y(ax + by) = -br^2.$$

*This leads to simpler system*

$$\begin{aligned} r' &= ar \\ \theta' &= -b. \end{aligned} \tag{6.34}$$

*This system is uncoupled. The second equation in this system indicates that we traverse the orbit at a constant rate in the clockwise direction. Solving these equations, we have that* $r(t) = r_0 e^{at}, \quad \theta(t) = \theta_0 - bt$. *Eliminating* $t$ *between these solutions, we finally find the polar equation of the orbits:*

$$r = r_0 e^{-a(\theta-\theta_0)t/b}.$$

*If you graph this for* $a \neq 0$*, you will get stable or unstable spirals.*

**Example 6.12.** *Consider the specific system*

$$\begin{aligned} x' &= -y + x \\ y' &= x + y. \end{aligned} \tag{6.35}$$

*In order to convert this system into polar form, we compute*

$$rr' = xx' + yy' = x(-y + x) + y(x + y) = r^2.$$

$$r^2\theta' = -xy' - yx' = x(x + y) - y(-y + x) = r^2.$$

*This leads to simpler system*

$$\begin{aligned} r' &= r \\ \theta' &= 1. \end{aligned} \tag{6.36}$$

*Solving these equations yields*

$$r(t) = r_0 e^t, \quad \theta(t) = t + \theta_0.$$

*Eliminating* $t$ *from this solution gives the orbits in the phase plane,* $r(\theta) = r_0 e^{\theta-\theta_0}$.

A more complicated example arises for a nonlinear system of differential equations. Consider the following example.

**Example 6.13.**

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2). \end{aligned} \tag{6.37}$$

Transforming to polar coordinates, one can show that in order to convert this system into polar form, we compute

$$r' = r(1 - r^2), \quad \theta' = 1.$$

This uncoupled system can be solved and this is left to the reader.

## 6.2 *Applications*

In this section we will describe some simple applications leading to systems of differential equations which can be solved using the methods in this chapter. These systems are left for homework problems and the as the start of further explorations for student projects.

### 6.2.1 *Mass-Spring Systems*

The first examples that we had seen involved masses on springs. Recall that for a simple mass on a spring we studied simple harmonic motion, which is governed by the equation

$$m\ddot{x} + kx = 0.$$

This second order equation can be written as two first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x, \end{aligned} \tag{6.38}$$

or

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x, \end{aligned} \tag{6.39}$$

where $\omega^2 = \frac{k}{m}$. The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

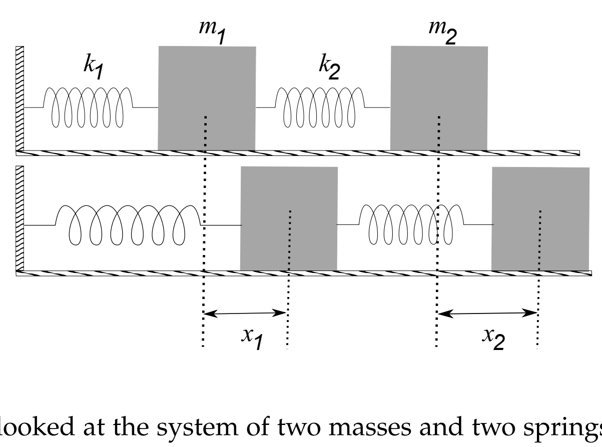



Figure 6.20: System of two masses and two springs.

We also looked at the system of two masses and two springs as shown in Figure 6.20. The equations governing the motion of the masses is

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{6.40}$$

We can rewrite this system as four first order equations

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\
\dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1).
\end{aligned} \tag{6.41}$$

The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}.$$

We can study this system for specific values of the constants using the methods covered in the last sections.

Writing the spring-block system as a second order vector system.

Actually, one can also put the system (6.40) in the matrix form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6.42}$$

This system can then be written compactly as

$$M\ddot{\mathbf{x}} = -K\mathbf{x}, \tag{6.43}$$

where

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}.$$

This system can be solved by guessing a form for the solution. We could guess

$$\mathbf{x} = \mathbf{a}e^{i\omega t}$$

or

$$\mathbf{x} = \begin{pmatrix} a_1\cos(\omega t - \delta_1) \\ a_2\cos(\omega t - \delta_2) \end{pmatrix},$$

where $\delta_i$ are phase shifts determined from initial conditions.

Inserting $\mathbf{x} = \mathbf{a}e^{i\omega t}$ into the system gives

$$(K - \omega^2 M)\mathbf{a} = \mathbf{0}.$$

This is a homogeneous system. It is a generalized eigenvalue problem for eigenvalues $\omega^2$ and eigenvectors $\mathbf{a}$. We solve this in a similar way to the standard matrix eigenvalue problems. The eigenvalue equation is found as

$$\det\,(K - \omega^2 M) = 0.$$

Once the eigenvalues are found, then one determines the eigenvectors and constructs the solution.

**Example 6.14.** *Let $m_1 = m_2 = m$ and $k_1 = k_2 = k$. Then, we have to solve the system*

$$\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

*The eigenvalue equation is given by*

$$\begin{aligned} 0 &= \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} \\ &= (2k - m\omega^2)(k - m\omega^2) - k^2 \\ &= m^2\omega^4 - 3km\omega^2 + k^2. \end{aligned} \tag{6.44}$$

*Solving this quadratic equation for $\omega^2$, we have*

$$\omega^2 = \frac{3 \pm 1}{2} \frac{k}{m}.$$

*For positive values of $\omega$, one can show that*

$$\omega = \frac{1}{2}\left(\pm 1 + \sqrt{5}\right)\sqrt{\frac{k}{m}}.$$

*The eigenvectors can be found for each eigenvalue by solving the homogeneous system*

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.$$

*The eigenvectors are given by*

$$\mathbf{a}_1 = \begin{pmatrix} -\frac{\sqrt{5}+1}{2} \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}.$$

*We are now ready to construct the real solutions to the problem. Similar to solving two first order systems with complex roots, we take the real and imaginary parts and take a linear combination of the solutions. In this problem there are four terms, giving the solution in the form*

$$\mathbf{x}(t) = c_1\mathbf{a_1} cos\omega_1 t + c_2\mathbf{a_1} sin\omega_1 t + c_3\mathbf{a_2} cos\omega_2 t + c_4\mathbf{a_2} sin\omega_2 t,$$

*where the $\omega$'s are the eigenvalues and the* **a***'s are the corresponding eigenvectors. The constants are determined from the initial conditions, $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$.*

### *6.2.2 Circuits*

In the last chapter we investigated simple series LRC circuits. More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This results in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 6.21. This is not a problem that can be covered in the first year physics course.

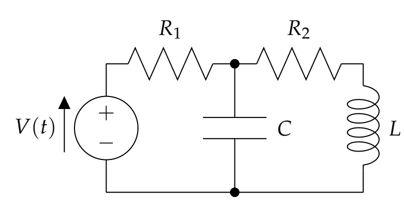



Figure 6.21: A circuit with two loops containing several different circuit elements.

There are two loops, indicated in Figure 6.22 as traversed clockwise. For each loop we need to apply Kirchoff's Loop Rule. There are three oriented currents, labeled $I_i$, $i = 1, 2, 3$. Corresponding to each current is a changing charge, $q_i$ such that

$$I_i = \frac{dq_i}{dt}, \quad i = 1, 2, 3.$$

We have for loop one

$$I_1 R_1 + \frac{q_2}{C} = V(t) \tag{6.45}$$

and for loop two

$$I_3 R_2 + L\frac{dI_3}{dt} = \frac{q_2}{C}. \tag{6.46}$$

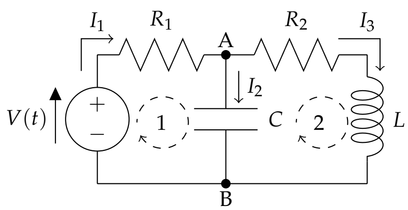


Figure 6.22: The previous parallel circuit with the directions indicated for traversing the loops in Kirchoff's Laws.

There are three unknown functions for the charge. Once we know the charge functions, differentiation will yield the three currents. However, we only have two equations. We need a third equation. This equation is found from Kirchoff's Point (Junction) Rule.

Consider the points A and B in Figure 6.22. Any charge (current) entering these junctions must be the same as the total charge (current) leaving the junctions. For point A we have

$$I_1 = I_2 + I_3, \tag{6.47}$$

or

$$\dot{q}_1 = \dot{q}_2 + \dot{q}_3. \tag{6.48}$$

Equations (6.45), (6.46), and (6.48) form a coupled system of differential equations for this problem. There are both first and second order derivatives involved. We can write the whole system in terms of charges as

$$\begin{aligned} R_1\dot{q}_1 + \frac{q_2}{C} &= V(t) \\ R_2\dot{q}_3 + L\ddot{q}_3 &= \frac{q_2}{C} \\ \dot{q}_1 &= \dot{q}_2 + \dot{q}_3. \end{aligned} \tag{6.49}$$

The question is whether, or not, we can write this as a system of first order differential equations. Since there is only one second order derivative, we can introduce the new variable $q_4 = \dot{q}_3$. The first equation can be solved for $\dot{q}_1$. The third equation can be solved for $\dot{q}_2$ with appropriate substitutions for the other terms. $\dot{q}_3$ is gotten from the definition of $q_4$ and the second equation can be solved for $\ddot{q}_3$ and substitutions made to obtain the system

$$\begin{aligned} \dot{q}_1 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} \\ \dot{q}_2 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} - q_4 \\ \dot{q}_3 &= q_4 \\ \dot{q}_4 &= \frac{q_2}{LC} - \frac{R_2}{L} q_4. \end{aligned}$$

So, we have a nonhomogeneous first order system of differential equations.

### 6.2.3 Mixture Problems

There are many types of mixture problems. Such problems are standard in a first course on differential equations as examples of first order differential equations. Typically these examples consist of a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

$$\boxed{\text{Rate of change of substance} = \text{Rate In} - \text{Rate Out.}}$$

This can be generalized to the case of two interconnected tanks. We provide some examples.

**Example 6.15. Single Tank Problem**

*A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. [See Figure 6.23.] At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time t. In all such problems one assumes that the solution is well mixed at each instant of time.*

*Let $x(t)$ be the amount of salt at time $t$. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that $dx/dt$ has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:*

$$\left(2\frac{pounds}{gal}\right)\left(5\frac{gal}{min}\right) = 10\frac{pounds}{min}.$$

*Similarly, one can determine the rate out as*

$$\left(\frac{x\ pounds}{50\ gal}\right)\left(5\frac{gal}{min}\right) = \frac{x}{10}\frac{pounds}{min}.$$

*Thus, we have*

$$\frac{dx}{dt} = 10 - \frac{x}{10}, \quad x(0) = 0.$$

*This equation is easily solved using the methods for first order equations. The general solution is given by*

$$x(t) = 100 + Ae^{-t/10}.$$

Using the initial condition, one finds the particular solution

$$x(t) = 100(1 - e^{-t/10}).$$

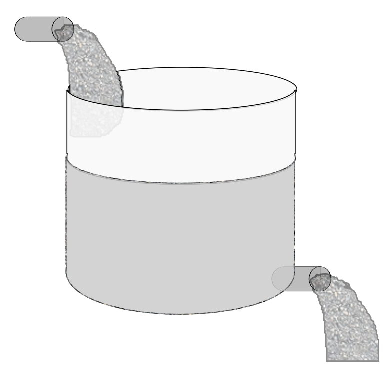


Figure 6.23: A typical mixing problem.

Often one is interested in the long time behavior of a system. In this case we have that $\lim_{t\to\infty} x(t) = 100$ lb. This makes sense because 2 pounds per gallon enter during this time to eventually leave the entire 50 gallons with this concentration. Thus,

$$50gal \times 2\frac{\text{lb}}{\text{gal}} = 100\text{lb}.$$

**Example 6.16. Double Tank Problem**

*One has two tanks connected together, labeled tank X and tank Y, as shown in Figure 6.24.*

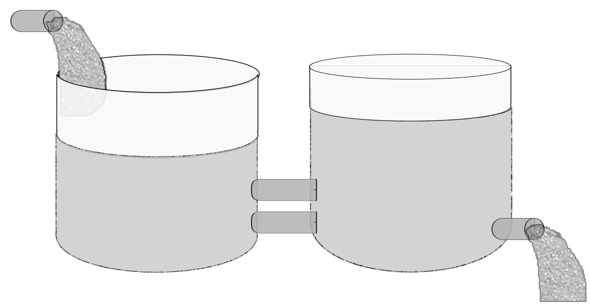


Figure 6.24: The two tank problem.

*Let tank X initially have 100 gallons of brine made with 100 pounds of salt. Tank Y initially has 100 gallons of pure water. Pure water is pumped into tank X at a rate of 2.0 gallons per minute. Some of the mixture of brine and pure water flows into tank Y at 3 gallons per minute. To keep the tank levels the same, one gallon of the Y mixture flows back into tank X at a rate of one gallon per minute and 2.0 gallons per minute drains out. Find the amount of salt at any given time in the tanks. What happens over a long period of time?*

*In this problem we set up two equations. Let $x(t)$ be the amount of salt in tank X and $y(t)$ the amount of salt in tank Y. Again, we carefully look at the rates into and out of each tank in order to set up the system of differential equations. We obtain the system*

$$\begin{aligned} \frac{dx}{dt} &= \frac{y}{100} - \frac{3x}{100} \\ \frac{dy}{dt} &= \frac{3x}{100} - \frac{3y}{100}. \end{aligned} \tag{6.50}$$

*This is a linear, homogenous constant coefficient system of two first order equations, which we know how to solve. The matrix form of the system is given by*

$$\dot{\mathbf{x}} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

*The eigenvalues for the problem are given by $\lambda = -3 \pm \sqrt{3}$ and the eigenvectors are*

$$\begin{pmatrix} 1 \\ \pm\sqrt{3} \end{pmatrix}.$$

*Since the eigenvalues are real and distinct, the general solution is easily written down:*

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{(-3+\sqrt{3})t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{(-3-\sqrt{3})t}.$$

*Finally, we need to satisfy the initial conditions. So,*

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix},$$

*or*

$$c_1 + c_2 = 100, \quad (c_1 - c_2)\sqrt{3} = 0.$$

*So,* $c_2 = c_1 = 50$. *The final solution is*

$$\mathbf{x}(t) = 50 \left( \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{(-3+\sqrt{3})t} + \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{(-3-\sqrt{3})t} \right),$$

*or*

$$\begin{aligned} x(t) &= 50\left(e^{(-3+\sqrt{3})t} + e^{(-3-\sqrt{3})t}\right) \\ y(t) &= 50\sqrt{3}\left(e^{(-3+\sqrt{3})t} - e^{(-3-\sqrt{3})t}\right). \end{aligned} \tag{6.51}$$

### 6.2.4 Chemical Kinetics

THERE ARE MANY PROBLEMS IN THE CHEMISTRY of chemical reactions which lead to systems of differential equations. The simplest reaction is when a chemical $A$ turns into chemical $B$. This happens at a certain rate, $k > 0$. This reaction can be represented by the chemical formula

$$\boxed{A \xrightarrow[k]{} B.}$$

In this case we have that the rates of change of the concentrations of $A$, $[A]$, and $B$, $[B]$, are given by

The chemical reactions used in these examples are first order reactions. Second order reactions have rates proportional to the square of the concentration.

$$\begin{aligned} \frac{d[A]}{dt} &= -k[A] \\ \frac{d[B]}{dt} &= k[A] \end{aligned} \tag{6.52}$$

Think about this as it is a key to understanding the next reactions.

A more complicated reaction is given by

$$\boxed{A \xrightarrow[k_1]{} B \xrightarrow[k_2]{} C.}$$

Here there are three concentrations and two rates of change. The system of equations governing the reaction is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{6.53}$$

The more complication rate of change is when [B] increases from [A] changing to [B] and decrease when [B] changes to [C]. Thus, there are two terms in the rate of change equation for concentration [B].

One can further consider reactions in which a reverse reaction is possible. Thus, a further generalization occurs for the reaction

$$\boxed{A \underset{k_1}{\overset{k_3}{\rightleftarrows}} B \xrightarrow[k_2]{} C.}$$

The reverse reaction rates contribute to the reaction equations for [A] and [B]. The resulting system of equations is

$$\begin{aligned}\frac{d[A]}{dt} &= -k_1[A] + k_3[B],\\ \frac{d[B]}{dt} &= k_1[A] - k_2[B] - k_3[B],\\ \frac{d[C]}{dt} &= k_2[B].\end{aligned} \tag{6.54}$$

Nonlinear chemical reactions will be discussed in the next chapter.

### 6.2.5 *Predator Prey Models*

Another common population model is that describing the coexistence of species. For example, we could consider a population of rabbits and foxes. Left to themselves, rabbits would tend to multiply, thus

$$\frac{dR}{dt} = aR,$$

with $a > 0$. In such a model the rabbit population would grow exponentially. Similarly, a population of foxes would decay without the rabbits to feed on. So, we have that

$$\frac{dF}{dt} = -bF$$

for $b > 0$.

Now, if we put these populations together on a deserted island, they would interact. The more foxes, the rabbit population would decrease. However, the more rabbits, the foxes would have plenty to eat and the population would thrive. Thus, we could model the competing populations as

$$\begin{aligned}\frac{dR}{dt} &= aR - cF,\\ \frac{dF}{dt} &= -bF + dR,\end{aligned} \tag{6.55}$$

where all of the constants are positive numbers. Studying this coupled system would lead to a study of the dynamics of these populations. The nonlinear version of this system, the Lotka-Volterra model, will be discussed in the next chapter.

### 6.2.6 Love Affairs

The next application is one that was introduced in 1988 by Strogatz as a cute system involving relationships.[1] One considers what happens to the affections that two people have for each other over time. Let $R$ denote the affection that Romeo has for Juliet and $J$ be the affection that Juliet has for Romeo. Positive values indicate love and negative values indicate dislike.

[1] Steven H. Strogatz introduced this problem as an interesting example of systems of differential equations in *Mathematics Magazine*, Vol. 61, No. 1 (Feb. 1988) p 35. He also describes it in his book *Nonlinear Dynamics and Chaos (1994)*.

One possible model is given by

$$\begin{aligned}\frac{dR}{dt} &= bJ\\ \frac{dJ}{dt} &= cR\end{aligned} \tag{6.56}$$

with $b > 0$ and $c < 0$. In this case Romeo loves Juliet the more she likes him. But Juliet backs away when she finds his love for her increasing.

A typical system relating the combined changes in affection can be modeled as

$$\begin{aligned}\frac{dR}{dt} &= aR + bJ\\ \frac{dJ}{dt} &= cR + dJ.\end{aligned} \tag{6.57}$$

Several scenarios are possible for various choices of the constants. For example, if $a > 0$ and $b > 0$, Romeo gets more and more excited by Juliet's love for him. If $c > 0$ and $d < 0$, Juliet is being cautious about her relationship with Romeo. For specific values of the parameters and initial conditions, one can explore this match of an overly zealous lover with a cautious lover.

### 6.2.7 Epidemics

Another interesting area of application of differential equation is in predicting the spread of disease. Typically, one has a population of susceptible people or animals. Several infected individuals are introduced into the population and one is interested in how the infection spreads and if the number of infected people drastically increases or dies off. Such models are typically nonlinear and we will look at what is called the SIR model in the next chapter. In this section we will model a simple linear model.

Let us break the population into three classes. First, we let $S(t)$ represent the healthy people, who are susceptible to infection. Let $I(t)$ be the number of infected people. Of these infected people, some will die from the infection and others could recover. We will consider the case that initially there is one infected person and the rest, say $N$, are healthy. Can we predict how many deaths have occurred by time t?

We model this problem using the compartmental analysis we had seen for mixing problems. The total rate of change of any population would be due to those entering the group less those leaving the group. For example, the number of healthy people decreases due infection and can increase when some of the infected group recovers. Let's assume that a) the rate of infection is proportional to the number of healthy people, $aS$, and b) the number who recover is proportional to the number of infected people, $rI$. Thus, the rate of change of healthy people is found as

$$\frac{dS}{dt} = -aS + rI.$$

Let the number of deaths be $D(t)$. Then, the death rate could be taken to be proportional to the number of infected people. So,

$$\frac{dD}{dt} = dI$$

Finally, the rate of change of infected people is due to healthy people getting infected and the infected people who either recover or die. Using the corresponding terms in the other equations, we can write the rate of change of infected people as

$$\frac{dI}{dt} = aS - rI - dI.$$

This linear system of differential equations can be written in matrix form.

$$\frac{d}{dt}\begin{pmatrix} S \\ I \\ D \end{pmatrix} = \begin{pmatrix} -a & r & 0 \\ a & -d-r & 0 \\ 0 & d & 0 \end{pmatrix}\begin{pmatrix} S \\ I \\ D \end{pmatrix}. \tag{6.58}$$

The reader can find the solutions of this system and determine if this is a realistic model.

## 6.3 Matrix Formulation

We have investigated several linear systems in the plane and in the next chapter we will use some of these ideas to investigate nonlinear systems. We need a deeper insight into the solutions of planar systems. So, in this section we will recast the first order linear systems into matrix form. This will lead to a better understanding of first order systems and allow for extensions to higher dimensions and the solution of nonhomogeneous equations later in this chapter.

We start with the usual homogeneous system in Equation (6.9). Let the unknowns be represented by the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then we have that

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$$

Here we have introduced the *coefficient matrix A*. This is a first order vector differential equation,

$$\mathbf{x}' = A\mathbf{x}.$$

Formerly, we can write the solution as

$$\mathbf{x} = \mathbf{x}_0 e^{At}.$$

2

We would like to investigate the solution of our system. Our investigations will lead to new techniques for solving linear systems using matrix methods.

We begin by recalling the solution to the specific problem (6.16). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{6.60}$$

This can be rewritten using matrix operations. Namely, we first write the solution in vector form.

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t \\ \frac{1}{3} c_1 e^t \end{pmatrix} + \begin{pmatrix} c_2 e^{-4t} \\ -\frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{-4t}. \end{aligned} \tag{6.61}$$

We see that our solution is in the form of a linear combination of vectors of the form

$$\mathbf{x} = \mathbf{v} e^{\lambda t}$$

with $\mathbf{v}$ a constant vector and $\lambda$ a constant number. This is similar to how we began to find solutions to second order constant coefficient equations. So, for the general problem (6.3) we insert this guess. Thus,

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \Rightarrow \\ \lambda \mathbf{v} e^{\lambda t} &= A \mathbf{v} e^{\lambda t}. \end{aligned} \tag{6.62}$$

For this to be true for all $t$, we have that

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{6.63}$$

This is an eigenvalue problem. $A$ is a $2 \times 2$ matrix for our problem, but could easily be generalized to a system of $n$ first order differential equations. We will confine our remarks for now to planar systems. However, we need to recall how to solve eigenvalue problems and then see how solutions of eigenvalue problems can be used to obtain solutions to our systems of differential equations..

[2] The *exponential of a matrix* is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

So, we define

$$e^A = \sum_{k=0}^{\infty} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{6.59}$$

In general, it is difficult computing $e^A$ unless $A$ is diagonal.

## 6.4 Eigenvalue Problems

We seek *nontrivial solutions* to the eigenvalue problem

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{6.64}$$

We note that $\mathbf{v} = \mathbf{0}$ is an obvious solution. Furthermore, it does not lead to anything useful. So, it is called a *trivial solution*. Typically, we are given the matrix $A$ and have to determine the *eigenvalues*, $\lambda$, and the associated *eigenvectors*, $\mathbf{v}$, satisfying the above eigenvalue problem. Later in the course we will explore other types of eigenvalue problems.

For now we begin to solve the eigenvalue problem for $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Inserting this into Equation (6.64), we obtain the homogeneous algebraic system

$$\begin{aligned} (a-\lambda)v_1 + bv_2 &= 0, \\ cv_1 + (d-\lambda)v_2 &= 0. \end{aligned} \tag{6.65}$$

The solution of such a system would be unique if the determinant of the system is not zero. However, this would give the trivial solution $v_1 = 0$, $v_2 = 0$. To get a nontrivial solution, we need to force the determinant to be zero. This yields the eigenvalue equation

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc.$$

This is a quadratic equation for the eigenvalues that would lead to nontrivial solutions. If we expand the right side of the equation, we find that

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

This is the same equation as the characteristic equation (6.12) for the general constant coefficient differential equation considered in the first chapter. Thus, the eigenvalues correspond to the solutions of the characteristic polynomial for the system.

Once we find the eigenvalues, then there are possibly an infinite number solutions to the algebraic system. We will see this in the examples.

So, the process is to

a) Write the coefficient matrix;

b) Find the eigenvalues from the equation $\det(A - \lambda I) = 0$; and,

c) Find the eigenvectors by solving the linear system $(A - \lambda I)\mathbf{v} = 0$ for each $\lambda$.

## 6.5 Solving Constant Coefficient Systems in 2D

Before proceeding to examples, we first indicate the types of solutions that could result from the solution of a homogeneous, constant coefficient system of first order differential equations.

We begin with the linear system of differential equations in matrix form.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} = A\mathbf{x}. \tag{6.66}$$

The type of behavior depends upon the eigenvalues of matrix $A$. The procedure is to determine the eigenvalues and eigenvectors and use them to construct the general solution.

If we have an initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$, we can determine the two arbitrary constants in the general solution in order to obtain the particular solution. Thus, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutions[3], then the general solution is given as

[3] Recall that linear independence means $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$ if and only if $c_1, c_2 = 0$. The reader should derive the condition on the $\mathbf{x}_i$ for linear independence.

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Then, setting $t = 0$, we get two linear equations for $c_1$ and $c_2$:

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0.$$

The major work is in finding the linearly independent solutions. This depends upon the different types of eigenvalues that one obtains from solving the eigenvalue equation, $\det(A - \lambda I) = 0$. The nature of these roots indicate the form of the general solution. On the next page we summarize the classification of solutions in terms of the eigenvalues of the coefficient matrix. We first make some general remarks about the plausibility of these solutions and then provide examples in the following section to clarify the matrix methods for our two dimensional systems.

The construction of the general solution in Case I is straight forward. However, the other two cases need a little explanation.

**Classification of the Solutions for Two Linear First Order Differential Equations**

1. **Case I: Two real, distinct roots.**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue obtaining two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then write the general solution as a linear combination $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$

2. **Case II: One Repeated Root**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for one eigenvalue $\lambda$, obtaining the first eigenvector $\mathbf{v}_1$. One then needs a second linearly independent solution. This is obtained by solving the nonhomogeneous problem $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.

   The general solution is then given by $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$.

3. **Case III: Two complex conjugate roots.**

   Solve the eigenvalue problem $\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$ for one eigenvalue, $\lambda = \alpha + i\beta$, obtaining one eigenvector $\mathbf{v}$. Note that this eigenvector may have complex entries. Thus, one can write the vector $\mathbf{y}(t) = e^{\lambda t}\mathbf{v} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}$. Now, construct two linearly independent solutions to the problem using the real and imaginary parts of $\mathbf{y}(t)$ : $\mathbf{y}_1(t) = Re(\mathbf{y}(t))$ and $\mathbf{y}_2(t) = Im(\mathbf{y}(t))$. Then the general solution can be written as $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

Let's consider Case III. Note that since the original system of equations does not have any $i$'s, then we would expect real solutions. So, we look at the real and imaginary parts of the complex solution. We have that the complex solution satisfies the equation

$$\frac{d}{dt}\left[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))\right] = A[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))].$$

Differentiating the sum and splitting the real and imaginary parts of the equation, gives

$$\frac{d}{dt}Re(\mathbf{y}(t)) + i\frac{d}{dt}Im(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))] + iA[Im(\mathbf{y}(t))].$$

Setting the real and imaginary parts equal, we have

$$\frac{d}{dt}Re(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))],$$

and

$$\frac{d}{dt}Im(\mathbf{y}(t)) = A[Im(\mathbf{y}(t))].$$

Therefore, the real and imaginary parts each are linearly independent solutions of the system and the general solution can be written as a linear combination of these expressions.

We now turn to Case II. Writing the system of first order equations as a second order equation for $x(t)$ with the sole solution of the characteristic equation, $\lambda = \frac{1}{2}(a+d)$, we have that the general solution takes the form

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

This suggests that the second linearly independent solution involves a term of the form $\mathbf{v}te^{\lambda t}$. It turns out that the guess that works is

$$\mathbf{x} = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2.$$

Inserting this guess into the system $\mathbf{x}' = A\mathbf{x}$ yields

$$\begin{aligned}
(te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2)' &= A\left[te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2\right].\\
e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 &= \lambda te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}A\mathbf{v}_2.\\
e^{\lambda t}(\mathbf{v}_1 + \lambda\mathbf{v}_2) &= e^{\lambda t}A\mathbf{v}_2.
\end{aligned} \tag{6.67}$$

Noting this is true for all $t$, we find that

$$\mathbf{v}_1 + \lambda\mathbf{v}_2 = A\mathbf{v}_2. \tag{6.68}$$

Therefore,

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

We know everything except for $\mathbf{v}_2$. So, we just solve for it and obtain the second linearly independent solution.

## 6.6 *Examples of the Matrix Method*

Here we will give some examples for typical systems for the three cases mentioned in the last section.

**Example 6.17.** $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

***Eigenvalues:*** *We first determine the eigenvalues.*

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} \tag{6.69}$$

*Therefore,*

$$\begin{aligned}
0 &= (4-\lambda)(3-\lambda) - 6\\
0 &= \lambda^2 - 7\lambda + 6\\
0 &= (\lambda - 1)(\lambda - 6)
\end{aligned} \tag{6.70}$$

*The eigenvalues are then $\lambda = 1, 6$. This is an example of Case I.*

***Eigenvectors:*** *Next we determine the eigenvectors associated with each of these eigenvalues. We have to solve the system $A\mathbf{v} = \lambda\mathbf{v}$ in each case.*

*Case* $\lambda = 1$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{6.71}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.72}$$

*This gives* $3v_1 + 2v_2 = 0$. *One possible solution yields an eigenvector of*

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

*Case* $\lambda = 6$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 6\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{6.73}$$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.74}$$

*For this case we need to solve* $-2v_1 + 2v_2 = 0$. *This yields*

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

***General Solution:*** *We can now construct the general solution.*

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= c_1 e^t \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 e^t + c_2 e^{6t} \\ -3c_1 e^t + c_2 e^{6t} \end{pmatrix}. \end{aligned} \tag{6.75}$$

**Example 6.18.** $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

***Eigenvalues:*** *Again, one solves the eigenvalue equation.*

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} \tag{6.76}$$

*Therefore,*

$$\begin{aligned} 0 &= (3-\lambda)(-1-\lambda) + 5 \\ 0 &= \lambda^2 - 2\lambda + 2 \\ \lambda &= \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2} = 1 \pm i. \end{aligned} \tag{6.77}$$

*The eigenvalues are then* $\lambda = 1 + i, 1 - i$. *This is an example of Case III.*

***Eigenvectors:*** *In order to find the general solution, we need only find the eigenvector associated with $1+i$.*

$$\begin{aligned}\begin{pmatrix}3 & -5\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} &= (1+i)\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\ \begin{pmatrix}2-i & -5\\ 1 & -2-i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}.\end{aligned} \tag{6.78}$$

*We need to solve $(2-i)v_1 - 5v_2 = 0$. Thus,*

$$\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}2+i\\ 1\end{pmatrix}. \tag{6.79}$$

***Complex Solution:*** *In order to get the two real linearly independent solutions, we need to compute the real and imaginary parts of $\mathbf{v}e^{\lambda t}$.*

$$\begin{aligned}e^{\lambda t}\begin{pmatrix}2+i\\ 1\end{pmatrix} &= e^{(1+i)t}\begin{pmatrix}2+i\\ 1\end{pmatrix}\\ &= e^t(\cos t + i\sin t)\begin{pmatrix}2+i\\ 1\end{pmatrix}\\ &= e^t\begin{pmatrix}(2+i)(\cos t + i\sin t)\\ \cos t + i\sin t\end{pmatrix}\\ &= e^t\begin{pmatrix}(2\cos t - \sin t) + i(\cos t + 2\sin t)\\ \cos t + i\sin t\end{pmatrix}\\ &= e^t\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix} + ie^t\begin{pmatrix}\cos t + 2\sin t\\ \sin t\end{pmatrix}.\end{aligned}$$

***General Solution:*** *Now we can construct the general solution.*

$$\begin{aligned}\mathbf{x}(t) &= c_1e^t\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix} + c_2e^t\begin{pmatrix}\cos t + 2\sin t\\ \sin t\end{pmatrix}\\ &= e^t\begin{pmatrix}c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t)\\ c_1\cos t + c_2\sin t\end{pmatrix}.\end{aligned} \tag{6.80}$$

*Note: This can be rewritten as*

$$\mathbf{x}(t) = e^t\cos t\begin{pmatrix}2c_1 + c_2\\ c_1\end{pmatrix} + e^t\sin t\begin{pmatrix}2c_2 - c_1\\ c_2\end{pmatrix}.$$

**Example 6.19.** $A = \begin{pmatrix}7 & -1\\ 9 & 1\end{pmatrix}$.

***Eigenvalues:***

$$0 = \begin{vmatrix}7-\lambda & -1\\ 9 & 1-\lambda\end{vmatrix} \tag{6.81}$$

*Therefore,*

$$\begin{aligned} 0 &= (7-\lambda)(1-\lambda)+9 \\ 0 &= \lambda^2-8\lambda+16 \\ 0 &= (\lambda-4)^2. \end{aligned} \tag{6.82}$$

*There is only one real eigenvalue, $\lambda = 4$. This is an example of Case II.*

***Eigenvectors:*** *In this case we first solve for $\mathbf{v}_1$ and then get the second linearly independent vector.*

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{6.83}$$

*Therefore, we have*

$$3v_1 - v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

***Second Linearly Independent Solution:***

*Now we need to solve $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$.*

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - 4\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned} \tag{6.84}$$

*Expanding the matrix product, we obtain the system of equations*

$$\begin{aligned} 3u_1 - u_2 &= 1 \\ 9u_1 - 3u_2 &= 3. \end{aligned} \tag{6.85}$$

*The solution of this system is* $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

***General Solution:*** *We construct the general solution as*

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1). \\ &= c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix} c_1 + c_2(1+t) \\ 3c_1 + c_2(2+3t) \end{pmatrix}. \end{aligned} \tag{6.86}$$

### 6.6.1 Planar Systems - Summary

The reader should have noted by now that there is a connection between the behavior of the solutions obtained in Section 2.2 and the eigenvalues found

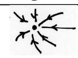
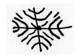
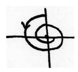
| Type | Figure | Eigenvalues | Stability |
|---|---|---|---|
| Node |  | Real $\lambda$, same signs | $\lambda < 0$, stable |
| Saddle |  | Real $\lambda$ opposite signs | Mostly Unstable |
| Center |  | $\lambda$ pure imaginary | — |
| Focus/Spiral |  | Complex $\lambda$, $\mathrm{Re}(\lambda) \neq 0$ | $\mathrm{Re}(\lambda < 0)$, stable |
| Degenerate Node | | Repeated roots, $\lambda > 0$, stable | |
| Line of Equilibria | | One zero eigenvalue | $\lambda < 0$, stable |

Table 6.1: List of typical behaviors in planar systems.

from the coefficient matrices in the previous examples. Here we summarize some of these cases.

The connection, as we have seen, is that the characteristic equation for the associated second order differential equation is the same as the eigenvalue equation of the coefficient matrix for the linear system. However, one should be a little careful in cases in which the coefficient matrix in not diagonalizable. In Table 6.2 are three examples of systems with repeated roots. The reader should look at these systems and look at the commonalities and differences in these systems and their solutions. In these cases one has unstable nodes, though they are degenerate in that there is only one accessible eigenvector.

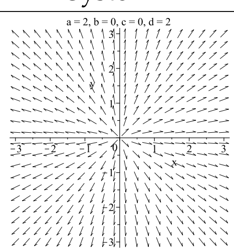
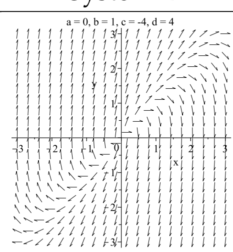
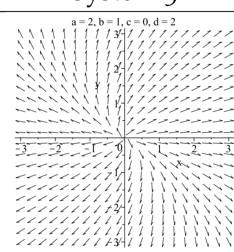
| System 1 | System 2 | System 3 |
|---|---|---|
|  |  |  |
| $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$ | $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \mathbf{x}$ | $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$ |

Table 6.2: Three examples of systems with a repeated root of $\lambda = 2$.

## 6.7 *Theory of Homogeneous Constant Coefficient Systems*

There is a general theory for solving homogeneous, constant coefficient systems of first order differential equations. We begin by once again recalling the specific problem (6.16). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{6.87}$$

This time we rewrite the solution as

$$\begin{aligned}
\mathbf{x} &= \begin{pmatrix} c_1e^t + c_2e^{-4t} \\ \frac{1}{3}c_1e^t - \frac{1}{2}c_2e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} e^t & e^{-4t} \\ \frac{1}{3}e^t & -\frac{1}{2}e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&\equiv \Phi(t)\mathbf{C}. \qquad (6.88)
\end{aligned}$$

Thus, we can write the general solution as a $2 \times 2$ matrix $\Phi$ times an arbitrary constant vector. The matrix $\Phi$ consists of two columns that are linearly independent solutions of the original system. This matrix is an example of what we will define as the *Fundamental Matrix* of solutions of the system. So, determining the Fundamental Matrix will allow us to find the general solution of the system upon multiplication by a constant matrix. In fact, we will see that it will also lead to a simple representation of the solution of the initial value problem for our system. We will outline the general theory.

Consider the homogeneous, constant coefficient system of first order differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \qquad (6.89)
\end{aligned}$$

As we have seen, this can be written in the matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now, consider $m$ vector solutions of this system: $\phi_1(t)$, $\phi_2(t), \ldots \phi_m(t)$. These solutions are said to be *linearly independent* on some domain if

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_m\phi_m(t) = 0$$

for all $t$ in the domain implies that $c_1 = c_2 = \ldots = c_m = 0$.

Let $\phi_1(t)$, $\phi_2(t), \ldots \phi_n(t)$ be a set of $n$ linearly independent set of solutions of our system, called a *fundamental set of solutions*. We construct a matrix from these solutions using these solutions as the column of that matrix. We define this matrix to be the *fundamental matrix solution*. This matrix takes the form

$$\Phi = \begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}.$$

What do we mean by a "matrix" solution? We have assumed that each $\phi_k$ is a solution of our system. Therefore, we have that $\phi_k' = A\phi_k$, for $k = 1, \dots, n$. We say that $\Phi$ is a matrix solution because we can show that $\Phi$ also satisfies the matrix formulation of the system of differential equations. We can show this using the properties of matrices.

$$\begin{aligned} \frac{d}{dt}\Phi &= \begin{pmatrix} \phi_1' & \dots & \phi_n' \end{pmatrix} \\ &= \begin{pmatrix} A\phi_1 & \dots & A\phi_n \end{pmatrix} \\ &= A\begin{pmatrix} \phi_1 & \dots & \phi_n \end{pmatrix} \\ &= A\Phi. \end{aligned} \tag{6.90}$$

Given a set of vector solutions of the system, when are they linearly independent? We consider a matrix solution $\Omega(t)$ of the system in which we have $n$ vector solutions. Then, we define the *Wronskian* of $\Omega(t)$ to be

$$W = \det \Omega(t).$$

If $W(t) \neq 0$, then $\Omega(t)$ is a fundamental matrix solution.

Before continuing, we list the fundamental matrix solutions for the set of examples in the last section. (Refer to the solutions from those examples.) Furthermore, note that the fundamental matrix solutions are not unique as one can multiply any column by a nonzero constant and still have a fundamental matrix solution.

**Example 6.17** $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{pmatrix}.$$

We should note in this case that the Wronskian is found as

$$\begin{aligned} W &= \det \Phi(t) \\ &= \begin{vmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{vmatrix} \\ &= 5e^{7t} \neq 0. \end{aligned} \tag{6.91}$$

**Example 6.18** $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^t(2\cos t - \sin t) & e^t(\cos t + 2\sin t) \\ e^t \cos t & e^t \sin t \end{pmatrix}.$$

**Example 6.19** $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^{4t} & e^{4t}(1+t) \\ 3e^{4t} & e^{4t}(2+3t) \end{pmatrix}.$$

So far we have only determined the general solution. This is done by the following steps:

**Procedure for Determining the General Solution**

1. Solve the eigenvalue problem $(A - \lambda I)\mathbf{v} = 0$.
2. Construct vector solutions from $\mathbf{v}e^{\lambda t}$. The method depends if one has real or complex conjugate eigenvalues.
3. Form the fundamental solution matrix $\Phi(t)$ from the vector solution.
4. The general solution is given by $\mathbf{x}(t) = \Phi(t)\mathbf{C}$ for $\mathbf{C}$ an arbitrary constant vector.

We are now ready to solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Starting with the general solution, we have that

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

As usual, we need to solve for the $c_k$'s. Using matrix methods, this is now easy. Since the Wronskian is not zero, then we can invert $\Phi$ at any value of $t$. So, we have

$$\mathbf{C} = \Phi^{-1}(t_0)\mathbf{x}_0.$$

Putting $\mathbf{C}$ back into the general solution, we obtain the solution to the initial value problem:

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0.$$

You can easily verify that this is a solution of the system and satisfies the initial condition at $t = t_0$.

The matrix combination $\Phi(t)\Phi^{-1}(t_0)$ is useful. So, we will define the resulting product to be the *principal matrix solution*, denoting it by

$$\Psi(t) = \Phi(t)\Phi^{-1}(t_0).$$

Thus, the solution of the initial value problem is $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$. Furthermore, we note that $\Psi(t)$ is a solution to the matrix initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = I,$$

where $I$ is the $n \times n$ identity matrix.

**Matrix Solution of the Homogeneous Problem**

In summary, the matrix solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0,$$

where $\Phi(t)$ is the fundamental matrix solution and $\Psi(t)$ is the principal matrix solution.

**Example 6.20.** *Let's consider the matrix initial value problem*

$$\begin{aligned} x' &= 5x + 3y \\ y' &= -6x - 4y, \end{aligned} \tag{6.92}$$

*satisfying* $x(0) = 1$, $y(0) = 2$. *Find the solution of this problem.*

*We first note that the coefficient matrix is*

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

*The eigenvalue equation is easily found from*

$$\begin{aligned} 0 &= -(5-\lambda)(4+\lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1). \end{aligned} \tag{6.93}$$

*So, the eigenvalues are* $\lambda = -1, 2$. *The corresponding eigenvectors are found to be*

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

*Now we construct the fundamental matrix solution. The columns are obtained using the eigenvectors and the exponentials,* $e^{\lambda t}$ :

$$\phi_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \qquad \phi_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

*So, the fundamental matrix solution is*

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}.$$

*The general solution to our problem is then*

$$\mathbf{x}(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \mathbf{C}$$

*for* $\mathbf{C}$ *is an arbitrary constant vector.*

*In order to find the particular solution of the initial value problem, we need the principal matrix solution. We first evaluate $\Phi(0)$, then we invert it:*

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

*The particular solution is then*

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-t} + 4e^{2t} \\ 6e^{-t} - 4e^{2t} \end{pmatrix} \end{aligned} \tag{6.94}$$

*Thus, $x(t) = -3e^{-t} + 4e^{2t}$ and $y(t) = 6e^{-t} - 4e^{2t}$.*

## 6.8 Nonhomogeneous Systems

Before leaving the theory of systems of linear, constant coefficient systems, we will discuss nonhomogeneous systems. We would like to solve systems of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t). \tag{6.95}$$

We will assume that we have found the fundamental matrix solution of the homogeneous equation. Furthermore, we will assume that $A(t)$ and $\mathbf{f}(t)$ are continuous on some common domain.

As with second order equations, we can look for solutions that are a sum of the general solution to the homogeneous problem plus a particular solution of the nonhomogeneous problem. Namely, we can write the general solution as

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \mathbf{x}_p(t),$$

where $\mathbf{C}$ is an arbitrary constant vector, $\Phi(t)$ is the fundamental matrix solution of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$\mathbf{x}_p' = A(t)\mathbf{x}_p + \mathbf{f}(t).$$

Such a representation is easily verified.

We need to find the particular solution, $\mathbf{x}_p(t)$. We can do this by applying *The Method of Variation of Parameters for Systems*. We consider a solution in the form of the solution of the homogeneous problem, but replace the constant vector by unknown parameter functions. Namely, we assume that

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{c}(t).$$

Differentiating, we have that

$$\mathbf{x}_p' = \Phi'\mathbf{c} + \Phi\mathbf{c}' = A\Phi\mathbf{c} + \Phi\mathbf{c}',$$

or

$$\mathbf{x}_p' - A\mathbf{x}_p = \Phi\mathbf{c}'.$$

But the left side is $\mathbf{f}$. So, we have that,

$$\Phi\mathbf{c}' = \mathbf{f},$$

or, since $\Phi$ is invertible (why?),

$$\mathbf{c}' = \Phi^{-1}\mathbf{f}.$$

In principle, this can be integrated to give $\mathbf{c}$. Therefore, the particular solution can be written as

$$\mathbf{x}_p(t) = \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds. \tag{6.96}$$

This is the *variation of parameters formula.*

The general solution of Equation (6.95) has been found as

$$\boxed{\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds.} \tag{6.97}$$

We can use the general solution to find the particular solution of an initial value problem consisting of Equation (6.95) and the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. This condition is satisfied for a solution of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{6.98}$$

provided

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

This can be solved for $\mathbf{C}$ as in the last section. Inserting the solution back into the general solution (6.98), we have

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{6.99}$$

This solution can be written a little neater in terms of the principal matrix solution, $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$ :

$$\boxed{\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^t \Psi^{-1}(s)\mathbf{f}(s)\,ds} \tag{6.100}$$

Finally, one further simplification occurs when $A$ is a constant matrix, which are the only types of problems we have solved in this chapter. In this case, we have that $\Psi^{-1}(t) = \Psi(-t)$. So, computing $\Psi^{-1}(t)$ is relatively easy.

**Example 6.21.** $x'' + x = 2\cos t$, $x(0) = 4$, $x'(0) = 0$. *This example can be solved using the Method of Undetermined Coefficients. However, we will use the matrix method described in this section.*

*First, we write the problem in matrix form. The system can be written as*

$$\begin{aligned} x' &= y \\ y' &= -x + 2\cos t. \end{aligned} \tag{6.101}$$

*Thus, we have a nonhomogeneous system of the form*

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos t \end{pmatrix}.$$

*Next we need the fundamental matrix of solutions of the homogeneous problem. We have that*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

*The eigenvalues of this matrix are $\lambda = \pm i$. An eigenvector associated with $\lambda = i$ is easily found as $\begin{pmatrix} 1 \\ i \end{pmatrix}$. This leads to a complex solution*

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{pmatrix}.$$

*From this solution we can construct the fundamental solution matrix*

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

*So, the general solution to the homogeneous problem is*

$$\mathbf{x}_h = \Phi(t)\mathbf{C} = \begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix}.$$

*Next we seek a particular solution to the nonhomogeneous problem. From Equation (6.98) we see that we need $\Phi^{-1}(s)\mathbf{f}(s)$. Thus, we have*

$$\begin{aligned} \Phi^{-1}(s)\mathbf{f}(s) &= \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}\begin{pmatrix} 0 \\ 2\cos s \end{pmatrix} \\ &= \begin{pmatrix} -2\sin s\cos s \\ 2\cos^2 s \end{pmatrix}. \end{aligned} \tag{6.102}$$

*We now compute*

$$\begin{aligned} \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\int_{t_0}^{t} \begin{pmatrix} -2\sin s\cos s \\ 2\cos^2 s \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} -\sin^2 t \\ t + \frac{1}{2}\sin(2t) \end{pmatrix} \\ &= \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}. \end{aligned} \tag{6.103}$$

*therefore, the general solution is*

$$\mathbf{x} = \begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix} + \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}.$$

*The solution to the initial value problem is*

$$\mathbf{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} t \sin t \\ \sin t + t \cos t \end{pmatrix},$$

*or*

$$\mathbf{x} = \begin{pmatrix} 4 \cos t + t \sin t \\ -3 \sin t + t \cos t \end{pmatrix}.$$

## *Problems*

**1.** Consider the system

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

a. Determine the second order differential equation satisfied by $x(t)$.

b. Solve the differential equation for $x(t)$.

c. Using this solution, find $y(t)$.

d. Verify your solutions for $x(t)$ and $y(t)$.

e. Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

**2.** Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

a.

$$\begin{aligned} x' &= 3x \\ y' &= -2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= 2y \\ y' &= -3x. \end{aligned}$$

d.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

e.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

**3.** Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right].$$

**4.** Consider the system of equations in Example 6.13.

a. Derive the polar form of the system.

b. Solve the radial equation, $r' = r(1 - r^2)$, for the initial values $r(0) = 0, 0.5, 1.0, 2.0$.

c. Based upon these solutions, plot and describe the behavior of all solutions to the original system in Cartesian coordinates.

**5.** Consider the following systems. For each system determine the coefficient matrix. When possible, solve the eigenvalue problem for each matrix and use the eigenvalues and eigenfunctions to provide solutions to the given systems. Finally, in the common cases which you investigated in Problem 2, make comparisons with your previous answers, such as what type of eigenvalues correspond to stable nodes.

a.

$$\begin{aligned} x' &= 3x - y \\ y' &= 2x - 2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

d.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

e.

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

f.

$$\begin{aligned} x' &= x - y \\ y' &= x + y. \end{aligned}$$

**6.** For the given matrix, evaluate $e^{tA}$, using the definition

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \ldots,$$

and simplifying.

a. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

b. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$.

c. $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

d. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

e. $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. item[f.] $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

**7.** Find the principal matrix solution for the system $\mathbf{x}' = A\mathbf{x}$ where matrix $A$ is given. If an initial condition is provided, find the solution of the initial value problem using the principal matrix.

a. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$.

b. $A = \begin{pmatrix} 12 & -15 \\ 4 & -4 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$.

d. $A = \begin{pmatrix} 4 & -13 \\ 2 & -6 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

e. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

f. $A = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$.

g. $A = \begin{pmatrix} 8 & -5 \\ 16 & 8 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

h. $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$.

i. $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

**8.** Solve the following initial value problems using Equation (6.100), the solution of a nonhomogeneous system using the principal matrix solution.

a. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b. $\mathbf{x}' = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

**9.** Add a third spring connected to mass two in the coupled system shown in Figure 6.2 to a wall on the far right. Assume that the masses are the same and the springs are the same.

a. Model this system with a set of first order differential equations.

b. If the masses are all 2.0 kg and the spring constants are all 10.0 N/m, then find the general solution for the system.

c. Move mass one to the left (of equilibrium) 10.0 cm and mass two to the right 5.0 cm. Let them go. find the solution and plot it as a function of time. Where is each mass at 5.0 seconds?

d. Model this initial value problem with a set of two second order differential equations. Set up the system in the form $M\ddot{\mathbf{x}} = -K\mathbf{x}$ and solve using the values in part b.

**10.** In Example 6.14 we investigated a couple mass-spring system as a pair of second order differential equations.

a. In that problem we used $\sqrt{\frac{3\pm\sqrt{5}}{2}} = \frac{\sqrt{5}\pm 1}{2}$. Prove this result.

b. Rewrite the system as a system of four first order equations.

c. Find the eigenvalues and eigenfunctions for the system of equations in part b to arrive at the solution found in Example 6.14.

d. Let $k = 5.00$ N/m and $m = 0.250$ kg. Assume that the masses are initially at rest and plot the positions as a function of time if initially i) $x_1(0) = x_2(0) = 10.0$ cm and i) $x_1(0) = -x_2(0) = 10.0$ cm. Describe the resulting motion.

**11.** Consider the series circuit in Figure 2.4 with $L = 1.00$ H, $R = 1.00 \times 10^2$ $\Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of two first order differential equations for the charge and the current.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when $V_0$ is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

**12.** Consider the series circuit in Figure 6.21 with $L = 1.00$ H, $R_1 = R_2 = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of first order differential equations for the charges and the currents in each loop.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when $V_0$ is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

**13.** Initially a 200 gallon tank is filled with pure water. At time $t = 0$ a salt concentration with 3 pounds of salt per gallon is added to the container at the rate of 4 gallons per minute, and the well-stirred mixture is drained from the container at the same rate.

a. Find the number of pounds of salt in the container as a function of time.

b. How many minutes does it take for the concentration to reach 2 pounds per gallon?

c. What does the concentration in the container approach for large values of time? Does this agree with your intuition?

d. Assuming that the tank holds much more than 200 gallons, and everything is the same except that the mixture is drained at 3 gallons per minute, what would the answers to parts a and b become?

**14.** You make two quarts of salsa for a party. The recipe calls for five teaspoons of lime juice per quart, but you had accidentally put in five tablespoons per quart. You decide to feed your guests the salsa anyway. Assume that the guests take a quarter cup of salsa per minute and that you replace what was taken with chopped tomatoes and onions without any lime juice. [1 quart = 4 cups and 1 Tb = 3 tsp.]

a. Write down the differential equation and initial condition for the amount of lime juice as a function of time in this mixture-type problem.

b. Solve this initial value problem.

c. How long will it take to get the salsa back to the recipe's suggested concentration?

**15.** Consider the chemical reaction leading to the system in (6.54). Let the rate constants be $k_1 = 0.20\ \text{ms}^{-1}$, $k_2 = 0.05\ \text{ms}^{-1}$, and $k_3 = 0.10\ \text{ms}^{-1}$. What do the eigenvalues of the coefficient matrix say about the behavior of the system? Find the solution of the system assuming $[A](0) = A_0 = 1.0$ $\mu$mol, $[B](0) = 0$, and $[C](0) = 0$. Plot the solutions for $t = 0.0$ to 50.0 ms and describe what is happening over this time.

**16.** Consider the epidemic model leading to the system in (6.58). Choose the constants as $a = 2.0$ days$^{-1}$, $d = 3.0$ days$^{-1}$, and $r = 1.0$ days$^{-1}$. What are the eigenvalues of the coefficient matrix? Find the solution of the system assuming an initial population of $1,000$ and one infected individual. Plot the solutions for $t = 0.0$ to 5.0 days and describe what is happening over this time. Is this model realistic?

**17.** Find and classify any equilibrium points in the Romeo and Juliet problem for the following cases. Solve the systems and describe their affections as a function of time.

a. $a = 0$, $b = 2$, $c = -1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

b. $a = 0$, $b = 2$, $c = 1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

c. $a = -1$, $b = 2$, $c = -1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

# 7
# Nonlinear Systems

*"The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful." - Jules Henri Poincaré (1854-1912)*

## 7.1 Introduction

Some of the most interesting phenomena in the world are modeled by nonlinear systems. These systems can be modeled by differential equations when time is considered as a continuous variable or difference equations when time is treated in discrete steps. Applications involving differential equations can be found in many physical systems such as planetary systems, weather prediction, electrical circuits, and kinetics. Even in some simple dynamical systems a combination of damping and a driving force can lead to chaotic behavior. Namely, small changes in initial conditions could lead to very different outcomes. In this chapter we will explore a few different nonlinear systems and introduce some of the tools needed to investigate them. These tools are based on some of the material in Chapters 2 and 3 for linear systems of differential equations.

Nonlinear differential equations are either integrable, but difficult to solve, or they are not integrable and can only be solved numerically. We will see that we can sometimes approximate the solutions of nonlinear systems with linear systems in small regions of phase space and determine the qualitative behavior of the system without knowledge of the exact solution.

Nonlinear problems occur naturally. We will see problems from many of the same fields we explored in Section 6.2. We will concentrate mainly on continuous dynamical systems. We will begin with a simple population model and look at the behavior of equilibrium solutions of first order autonomous differential equations. We will then look at nonlinear systems in the plane, such as the nonlinear pendulum and other nonlinear oscillations. We will conclude by discussing a few other interesting physical examples stressing some of the key ideas of nonlinear dynamics.

## 7.2 *The Logistic Equation*

In this section we will explore a simple nonlinear population model. Typically, we want to model the growth of a given population, $y(t)$, and the differential equation governing the growth behavior of this population is developed in a manner similar to that used previously for mixing problems. Namely, we note that the rate of change of the population is given by an equation of the form

$$\frac{dy}{dt} = \text{Rate In} - \text{Rate Out}.$$

The *Rate In* could be due to the number of births per unit time and the *Rate Out* by the number of deaths per unit time. While there are other potential contributions to these rates we will consider the birth and death rates in the simplest examples.

A simple population model can be obtained if one assumes that these rates are linear in the population. Thus, we assume that the

$$\text{Rate In } = by \text{ and the } \text{ Rate Out } = my.$$

Here we have denoted the birth rate as $b$ and the mortality rate as $m$. This gives the rate of change of population as

$$\frac{dy}{dt} = by - my. \tag{7.1}$$

Generally, these rates could depend on the time. In the case that they are both constant rates, we can define $k = b - m$ and obtain the familiar exponential model of population growth:

Malthusian population growth.

$$\frac{dy}{dt} = ky.$$

This is easily solved and one obtains exponential growth $(k > 0)$ or decay $(k < 0)$. This Malthusian growth model has been named after Thomas Robert Malthus (1766-1834), a clergyman who used this model to warn of the impending doom of the human race if its reproductive practices continued.

When populations get large enough, there is competition for resources, such as space and food, which can lead to a higher mortality rate. Thus, the mortality rate may be a function of the population size, $m = m(y)$. The simplest model would be a linear dependence, $m = \tilde{m} + cy$. Then, the previous exponential model takes the form

$$\frac{dy}{dt} = ky - cy^2, \tag{7.2}$$

where $k = b - \tilde{m}$. This is known as the *logistic model* of population growth. Typically, $c$ is small and the added nonlinear term does not really kick in until the population gets large enough.

The logistic model was first published in 1838 by Pierre François Verhulst (1804-1849) in the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where $N$ is the population at time $t$, $r$ is the growth rate, and $K$ is what is called the carrying capacity. Note that in our model $r = k = Kc$.

**Example 7.1.** *Show that Equation (7.2) can be written in the form*

$$z' = kz(1-z)$$

*which has only one parameter.*

*We carry this out be rescaling the population, $y(t) = \alpha z(t)$, where $\alpha$ is to be determined. Inserting this transformation, we have*

$$\begin{aligned} y' &= ky - cy^2 \\ \alpha z' &= \alpha kz - c\alpha^2 z^2, \end{aligned}$$

*or*

$$z' = kz\left(1 - \alpha\frac{c}{k}z\right).$$

*Thus, we obtain the result, $z' = kz(1-z)$, if we pick $\alpha = \frac{k}{c}$.*

Before we obtain the exact solution, it is instructive to study the qualitative behavior of the solutions without actually writing down any explicit solutions. Such methods are useful for more difficult nonlinear equations as we will see later in this chapter.

We will demonstrate this analysis with a simple logistic equation example. We will first look for constant solutions, called equilibrium solutions, satisfying $y'(t) = 0$. Then, we will look at the behavior of solutions near the equilibrium solutions, or fixed points, and determine the stability of the equilibrium solutions. In the next section we will extend these ideas to other first order differential equations.

**Example 7.2.** *Find and classify the equilibrium solutions of the logistic equation,*

$$\frac{dy}{dt} = y - y^2. \tag{7.3}$$

*First, we determine the equilibrium, or constant, solutions given by $y' = 0$. For this case, we have $y - y^2 = 0$. So, the equilibrium solutions are $y = 0$ and $y = 1$.*

*These solutions divide the ty-plane into three regions, $y < 0$, $0 < y < 1$, and $y > 1$. Solutions that originate in one of these regions at $t = t_0$ will remain in that region for all $t > t_0$ since solutions of this differential equation cannot intersect.*

Note: If two solutions of the differential equation intersect then they have common values $y_1$ at time $t_1$. Using this information, we could set up an initial value problem for which the initial condition is $y(t_1) = y_1$. Since the two different solutions intersect at this point in the phase plane, we would have an initial value problem with two different solutions. This would violate the uniqueness theorem for initial value problems.

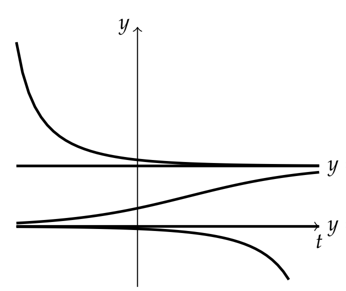



Figure 7.1: Representative solution behavior for $y' = y - y^2$.

*Next, we determine the behavior of solutions in the three regions. Noting that $y'(t)$ gives the slope of any solution in the plane, then we find that the solutions are monotonic in each region. Namely, in regions where $y'(t) > 0$, we have monotonically increasing functions and in regions where $y'(t) < 0$, we have monotonically decreasing functions. We determine the sign of $y'(t)$ from the right hand side of the differential equation.*

*For example, in this problem $y - y^2 > 0$ only for the middle region and $y - y^2 < 0$ for the other two regions. Thus, the slope is positive in the middle region, giving a rising solution as shown in Figure 7.1. Note that this solution does not cross the equilibrium solutions. Similar statements can be made about the solutions in the other regions.*

Stable and unstable equilibria.

*We further note that the solutions on either side of the equilibrium solution $y = 1$ tend to approach this equilibrium solution for large values of $t$. In fact, no matter* *how far these solutions are from $y = 1$, as long as $y(t) > 0$, the solutions will eventually approach this equilibrium solution as $t \to \infty$. We then say that the equilibrium solution, $y = 1$, is a stable equilibrium.*

*Similarly, we note that the solutions on either side of the equilibrium solution $y = 0$ tend away from $y = 0$ for large values of $t$. No matter how close a solution is to $y = 0$ at some given time, eventually these solutions will diverge as $t \to \infty$. We say that such equilibrium solutions are unstable equilibria.*

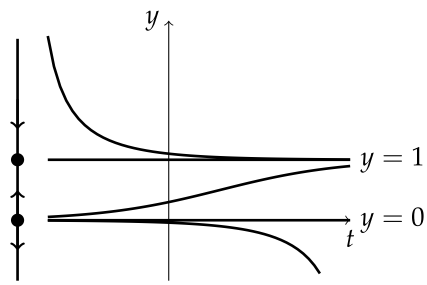



Figure 7.2: Representative solution behavior and the phase line for $y' = y - y^2$.

Phase lines.

*If we are only interested in the behavior of the equilibrium solutions, we could just display a phase line. In Figure 7.2 we place a vertical line to the right of the ty-plane plot. On this line we first place dots at the corresponding equilibrium solutions and label the solutions. These points divide the phase line into three intervals.*

*In each interval we then place arrows pointing upward or downward indicating solutions with positive or negative slopes, respectively. For example, for the interval $y > 1$ there is a downward pointing arrow indicating that the slope is negative in that region.*

*Looking at the resulting phase line we can determine if a given equilibrium is stable (arrows pointing towards the point) or unstable (arrows pointing away from the point). In Figure 7.3 we draw the final phase line by itself. We see that $y = 1$ is a stable equilibrium point and $y = 0$ is an unstable equilibrium point.*

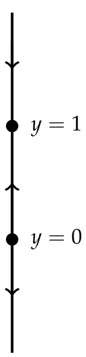



Figure 7.3: Phase line for $y' = y - y^2$.

### 7.2.1 The Riccati Equation

We have seen that one does not need an explicit solution of the logistic equation (7.2) in order to study the behavior of its solutions. However, the logistic equation is an example of a nonlinear first order equation that is solvable. It is also an example of a general Riccati equation, a first order differential equation quadratic in the unknown function.

The Riccati equation is named after the Italian mathematician Jacopo Francesco Riccati (1676-1754). When $a(t) = 0$, the equation becomes a Bernoulli equation.

The general form of the *Riccati equation* is

$$\frac{dy}{dt} = a(t) + b(t)y + c(t)y^2. \tag{7.4}$$

As long as $c(t) \neq 0$, this equation can be reduced to a second order linear differential equation through the transformation

$$y(t) = -\frac{1}{c(t)} \frac{x'(t)}{x(t)}.$$

We will demonstrate the use of this transformation in obtaining the solution of the logistic equation.

**Example 7.3.** *Solve the logistic equation*

$$\frac{dy}{dt} = ky - cy^2 \tag{7.5}$$

*using the transformation*

$$y = \frac{1}{c}\frac{x'}{x}.$$

*differentiating this transformation with respect to $t$, we obtain*

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{c}\left[\frac{x''}{x} - \left(\frac{x'}{x}\right)^2\right] \\ &= \frac{1}{c}\left[\frac{x''}{x} - (cy)^2\right] \\ &= \frac{1}{c}\frac{x''}{x} - cy^2. \end{aligned} \tag{7.6}$$

*Inserting this result into the logistic equation (7.5), we have*

$$\frac{1}{c}\frac{x''}{x} - cy^2 = k\frac{1}{c}\left(\frac{x'}{x}\right) - cy^2.$$

*Simplifying, we see that the logistic equation has been reduced to a second order linear, differential equation,*

$$x'' = kx'.$$

*This equation is readily solved. One integration gives*

$$x'(t) = Be^{kt}.$$

*A second integration gives*

$$x(t) = A + Be^{kt},$$

*where $A$ and $B$ are two arbitrary constants.*

*Inserting this result into the Riccati transformation, we obtain*

$$y(t) = \frac{1}{c}\frac{x'}{x} = \frac{kBe^{kt}}{c(A + Be^{kt})}.$$

*It appears that we have two arbitrary constants. However, we started out with a first order differential equation and so we expect only one arbitrary constant. We can resolve this dilemma by dividing[1] the numerator and denominator by $Be^{kt}$ and defining $C = \frac{A}{B}$. Then, we have the solution*

$$y(t) = \frac{k/c}{1 + Ce^{-kt}}, \tag{7.7}$$

*showing that there really is only one arbitrary constant in the solution.*

[1] This general solution holds for $B \neq 0$. If $B = 0$, then we have $x(t) = A$ and, thus, $y(t)$ is the constant equilibrium solution.

Plots of the solution (7.7) of the logistic equation for different initial conditions gives the solutions seen in the last section. In particular, setting all of the constants to unity, we have the sigmoid function,

$$y(t) = \frac{1}{1+e^{-t}}.$$

This is the signature $S$-shaped curve of the logistic model as shown in Figure 7.4. We should note that this is not the only way to obtain the solution to the logistic equation, though this approach has provided us with an introduction to Riccati equations. A more direct approach would be to use separation of variables on the logistic equation, which is Problem 1.

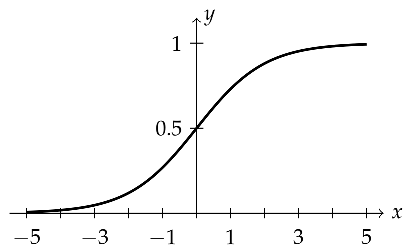


Figure 7.4: Plot of the sigmoid function.

## 7.3 *Autonomous First Order Equations*

In this section we will study the stability of nonlinear first order autonomous equations. We will then extend this study in the next section to looking at families of first order equations which are connected through a parameter.

Recall that a first order autonomous equation is given in the form

$$\frac{dy}{dt} = f(y). \tag{7.8}$$

We will assume that $f$ and $\frac{\partial f}{\partial y}$ are continuous functions of $y$, so that we know that solutions of initial value problems exist and are unique.

A solution $y(t)$ of Equation (7.8) is called an *equilibrium solution*, or a *fixed point* solution, if it is a constant solution satisfying $y'(t) = 0$. Such solutions are the roots of the right hand side of the differential equation, $f(y) = 0$.

**Example 7.4.** *Find the equilibrium solutions of $y' = 1 - y^2$.*

*The equilibrium solutions are the roots of $f(y) = 1 - y^2 = 0$. The equilibria are found to be $y = \pm 1$.*

Once we have determined the equilibrium solutions, we would like to classify them. Are they stable or unstable? As we had seen previously, we are interested in the behavior of solutions near the equilibria. This classification can be determined using a linearization of the given equation. This will provide an analytic criteria to establish the stability of equilibrium solutions without geometrically drawing the phase lines as we had done previously.

Linearization of first order equations.

Let $y^*$ be an equilibrium solution of Equation (7.8). Then, any solution can be written in the form

$$y(t) = y^* + \xi(t),$$

where $\xi(t)$ measures how far the solution is from the equilibrium at any given time.

Inserting this form into Equation (7.8), we have

$$\frac{d\xi}{dt} = f(y^* + \xi).$$

We now consider small $\xi(t)$ in order to study solutions near the equilibrium solution. For such solutions, we can expand $f(y)$ about the equilibrium solution,

$$f(y^* + \xi) = f(y^*) + f'(y^*)\xi + \frac{1}{2!}f''(y^*)\xi^2 + \cdots.$$

Since $y^*$ is an equilibrium solution, $f(y^*) = 0$, the first term in the Taylor series vanishes. If the first derivative does not vanish, then for solutions close to equilibrium, we can neglect higher order terms in the expansion. Then, $\xi(t)$ approximately satisfies the differential equation

$$\frac{d\xi}{dt} = f'(y^*)\xi. \tag{7.9}$$

This is called a linearization of the original nonlinear equation about the equilibrium point. This equation has exponential solutions for $f'(y^*) \neq 0$,

$$\xi(t) = \xi_0 e^{f'(y^*)t}.$$

Now we see how the stability criteria arise. If $f'(y^*) > 0$, $\xi(t)$ grows in time. Therefore, nearby solutions stray from the equilibrium solution for large times. On the other hand, if $f'(y^*) < 0$, $\xi(t)$ decays in time and nearby solutions approach the equilibrium solution for large $t$. Thus, we have the results:

The stability criteria for equilibrium solutions of a first order differential equation.

$$\boxed{\begin{array}{ll} f'(y^*) < 0, & y^* \text{ is stable.} \\ f'(y^*) > 0, & y^* \text{ is unstable.} \end{array}} \tag{7.10}$$

**Example 7.5.** *Determine the stability of the equilibrium solutions of $y' = 1 - y^2$.*

*In the last example we found the equilibrium solutions, $y^* = \pm 1$. The stability criteria require computing*

$$f'(y^*) = -2y^*.$$

*For this problem we have $f'(\pm 1) = \mp 2$. Therefore, $y^* = 1$ is a stable equilibrium and $y^* = -1$ is an unstable equilibrium.*

**Example 7.6.** *Find and classify the equilibria for the logistic equation $y' = y - y^2$.*

*We had already investigated this problem using phase lines. There are two equilibria, $y = 0$ and $y = 1$.*

*We next apply the stability criteria. Noting that $f'(y) = 1 - 2y$, the first equilibrium solution gives $f'(0) = 1$. So, $y = 0$ is an unstable equilibrium. Since $f'(1) = -1 < 0$, we see that $y = 1$ is a stable equilibrium. These results are the same as we hade determined earlier using phase lines.*

## *7.4 Bifurcations for First Order Equations*

We now consider families of first order autonomous differential equations of the form

$$\frac{dy}{dt} = f(y;\mu).$$

Here $\mu$ is a parameter that we can change and then observe the resulting behaviors of the solutions of the differential equation. When a small change in the parameter leads to changes in the behavior of the solution, then the system is said to undergo a *bifurcation*. The value of the parameter, $\mu$, at which the bifurcation occurs is called a *bifurcation point*.

Bifurcations and bifurcation points.

We will consider several generic examples, leading to special classes of bifurcations of first order autonomous differential equations. We will study the stability of equilibrium solutions using both phase lines and the stability criteria developed in the last section

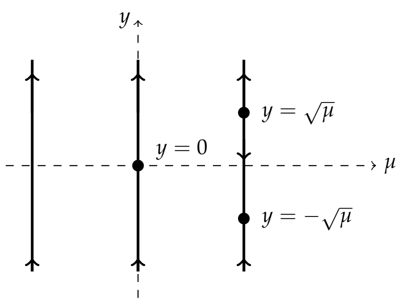



Figure 7.5: Phase lines for $y' = y^2 - \mu$. On the right $\mu > 0$ and on the left $\mu < 0$.

**Example 7.7.** $y' = y^2 - \mu$.

*First note that equilibrium solutions occur for $y^2 = \mu$. In this problem, there are three cases to consider.*

1. $\mu > 0$.
   *In this case there are two real solutions of $y^2 = \mu$, $y = \pm\sqrt{\mu}$. Note that $y^2 - \mu < 0$ for $|y| < \sqrt{\mu}$. So, we have the right phase line in Figure 7.5.*
2. $\mu = 0$.
   *There is only one equilibrium point at $y = 0$. The equation becomes $y' = y^2$. It is obvious that the right side of this equation is never negative. So, the phase line, which is shown as the middle line in Figure 7.5, has upward pointing arrows.*
3. $\mu < 0$.
   *In this case there are no equilibrium solutions. Since $y^2 - \mu > 0$, the slopes for all solutions are positive as indicated by the last phase line in Figure 7.5.*

*We can also confirm the behaviors of the equilibrium points by noting that $f'(y) = 2y$. Then, $f'(\pm\sqrt{\mu}) = \pm 2\sqrt{\mu}$ for $\mu \geq 0$. Therefore, the equilibria $y = +\sqrt{\mu}$ are unstable equilibria for $\mu > 0$. Similarly, the equilibria $y = -\sqrt{\mu}$ are stable equilibria for $\mu > 0$.*

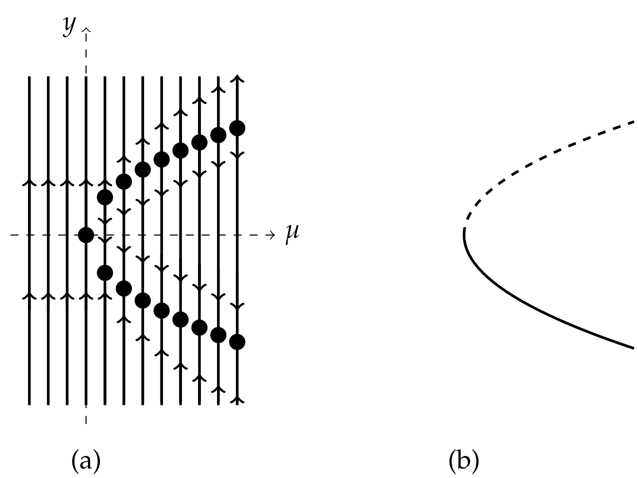



Figure 7.6: (a) The typical phase lines for $y' = y^2 - \mu$. (b) Bifurcation diagram for $y' = y^2 - \mu$. This is an example of a saddle-node bifurcation.

*We can combine these results for the phase lines into one diagram known as a bifurcation diagram. We will plot the equilibrium solutions and their phase lines*

*$y = \pm\sqrt{\mu}$ in the $\mu y$-plane. We begin by lining up the phase lines for various $\mu$'s. These are shown on the left side of Figure 7.6. Note the pattern of equilibrium points lies on the parabolic curve $y^2 = \mu$. The upper branch of this curve is a collection of unstable equilibria and the bottom is a stable branch. So, we can dispose of the phase lines and just keep the equilibria. However, we will draw the unstable branch as a dashed line and the stable branch as a solid line.*

*The bifurcation diagram is displayed on the right side of Figure 7.6. This type of bifurcation is called a saddle-node bifurcation. The point $\mu = 0$ at which the behavior changes is the bifurcation point. As $\mu$ changes from negative to positive values, the system goes from having no equilibria to having one stable and one unstable equilibrium point.*

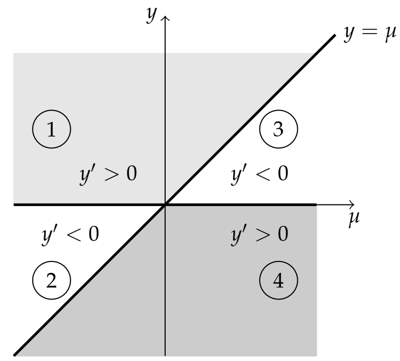



Figure 7.7: The regions indicating the different signs of the derivative for $y' = y^2 - \mu y$.

**Example 7.8.** $y' = y^2 - \mu y$.

*Writing this equation in factored form, $y' = y(y - \mu)$, we see that there are two equilibrium points, $y = 0$ and $y = \mu$. The behavior of the solutions depends upon the sign of $y' = y(y - \mu)$. This leads to four cases with the indicated signs of the derivative. The regions indicating the signs of $y'$ are shown in Figure 7.7.*

1. $y > 0, y - \mu > 0 \Rightarrow y' > 0$.
2. $y < 0, y - \mu > 0 \Rightarrow y' < 0$.
3. $y > 0, y - \mu < 0 \Rightarrow y' < 0$.
4. $y < 0, y - \mu < 0 \Rightarrow y' > 0$.

*The corresponding phase lines and superimposed bifurcation diagram are shown in figure 7.8. The bifurcation diagram is on the right side of Figure 7.8 and this type of bifurcation is called a transcritical bifurcation.*

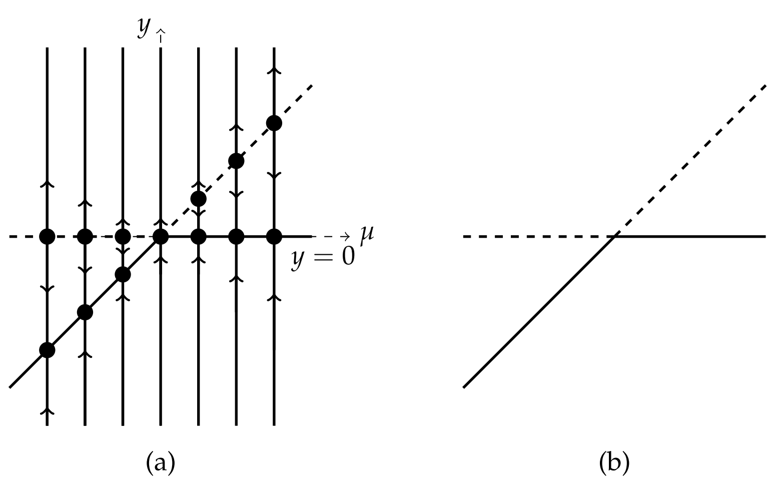


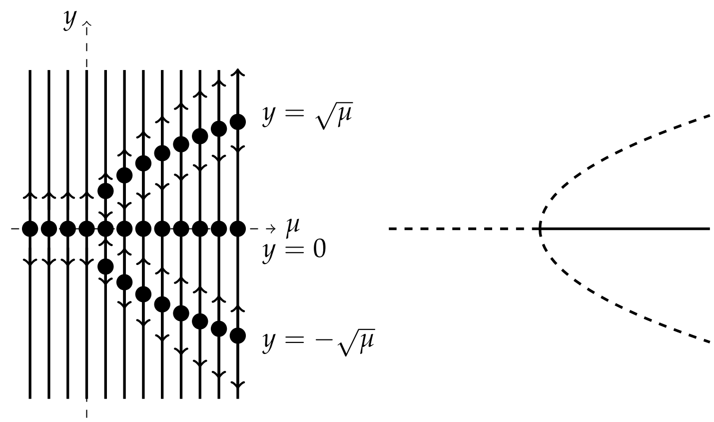

Figure 7.8: (a) Collection of phase lines for $y' = y^2 - \mu y$. (b) Bifurcation diagram for $y' = y^2 - \mu y$. This is an example of a transcritical bifurcation.

*Again, the stability can be determined from the derivative $f'(y) = 2y - \mu$ evaluated at $y = 0, \mu$. From $f'(0) = -\mu$, we see that $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. Similarly, $f'(\mu) = \mu$ implies that $y = \mu$ is unstable for $\mu > 0$ and stable for $\mu < 0$. These results are consistent with the phase line plots.*

**Example 7.9.** $y' = y^3 - \mu y$.

*For this last example, we find from $y^3 - \mu y = y(y^2 - \mu) = 0$ that there are two cases.*

1. *$\mu < 0$. In this case there is only one equilibrium point at $y = 0$. For positive values of $y$ we have that $y' > 0$ and for negative values of $y$ we have that $y' < 0$. Therefore, this is an unstable equilibrium point.*
2. *$\mu > 0$. Here we have three equilibria, $y = 0, \pm\sqrt{\mu}$. A careful investigation shows that $y = 0$ is a stable equilibrium point and that the other two equilibria are unstable.*




(a) (b)

Figure 7.9: (a) The phase lines for $y' = y^3 - \mu y$. The left one corresponds to $\mu < 0$ and the right phase line is for $\mu > 0$. (b)Bifurcation diagram for $y' = y^3 - \mu y$. This is an example of a pitchfork bifurcation.

*In Figure 7.9 we show the phase lines for these two cases. The corresponding bifurcation diagram is then sketched on the right side of Figure 7.9. For obvious reasons this has been labeled a pitchfork bifurcation.*

When two of the prongs of the pitchfork are unstable branches, the bifurcation is called a subcritical pitchfork bifurcation. When two prongs are stable branches, the bifurcation is a supercritical pitchfork bifurcation.

*Since $f'(y) = 3y^2 - \mu$, the stability analysis gives that $f'(0) = -\mu$. So, $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. For $\mu > 0$, we have that $f'(\pm\sqrt{\mu}) = 2\mu$. Therefore, $y = \pm\sqrt{\mu}$, $\mu > 0$, is unstable. Thus, we have a subcritical pitchfork bifurcation.*

## 7.5 Nonlinear Pendulum

In this section we return to the nonlinear pendulum as an example of periodic motion in a nonlinear system. Oscillations are important in many areas of physics. We have already seen the motion of a mass on a spring, leading to simple, damped, and forced harmonic motions. Later we will explore these effects on a simple nonlinear system. In this section we will investigate the nonlinear pendulum equation (2.4) and determine its period of oscillation.

Recall that the derivation of the pendulum equation was based upon a simple point mass $m$ hanging on a string of length $L$ from some support as shown in Figure 7.10. One pulls the mass back to some starting angle, $\theta_0$, and releases it. The goal is to find the angular position as a function of time, $\theta(t)$.

In Chapter 2 we derived the nonlinear pendulum equation,

$$L\ddot{\theta} + g\sin\theta = 0. \tag{7.11}$$

There are several variations of Equation (7.11) which we have used in this text. The first one is the linear pendulum, which was obtained using a small angle approximation,

$$L\ddot{\theta} + g\theta = 0. \tag{7.12}$$

We also made the system more realistic by adding damping and forcing. A variety of these oscillation problems are summarized in the table below.

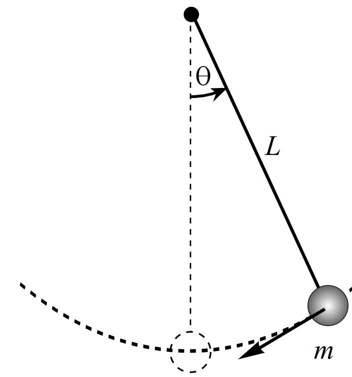



Figure 7.10: A simple pendulum consists of a point mass $m$ attached to a string of length $L$. It is released from an angle $\theta_0$.

**Equations for Pendulum Motion**

1. Nonlinear Pendulum: $L\ddot{\theta} + g\sin\theta = 0$.
2. Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0$.
3. Linear Pendulum: $L\ddot{\theta} + g\theta = 0$.
4. Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = 0$.
5. Forced Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t$.
6. Forced Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = F\cos\omega t$.

### 7.5.1 *The Period of the Nonlinear Pendulum*

RECALL THAT THE PERIOD OF THE SIMPLE PENDULUM is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{7.13}$$

for

$$\omega \equiv \sqrt{\frac{g}{L}}. \tag{7.14}$$

This was based upon the solving the linear pendulum equation (7.12). This equation was derived assuming a small angle approximation. How good is this approximation? What is meant by a *small* angle?

We recall that the Taylor series approximation of $\sin\theta$ about $\theta = 0$ :

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots. \tag{7.15}$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error } = \left|\frac{\sin\theta - \theta}{\sin\theta}\right|.$$

A plot of the relative error is given in Figure 7.11. Thus for $\theta \approx 0.4$ radians (or, $23^o$) we have that the relative error is about 2.6%.

Relative error in $\sin\theta$ approximation.

We would like to do better than this. So, we now turn to the nonlinear pendulum equation (7.11) in the simpler form

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{7.16}$$

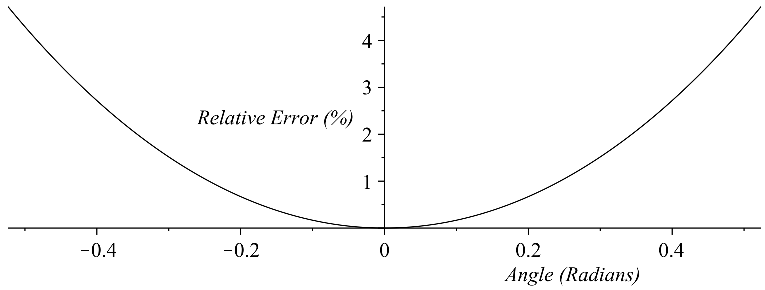



Figure 7.11: The relative error in percent when approximating $\sin\theta$ by $\theta$.

Solution of nonlinear pendulum equation.

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\,d\phi\right] = (\ddot{\theta} + F(\theta))\dot{\theta}.$$

For the nonlinear pendulum problem, we multiply Equation (7.16) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin\theta\dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{7.17}$$

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2\cos\theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2\cos\theta)}}.$$

Of course, we need to be able to do the integral. When one finds a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral.

In fact, the above integral can be transformed into what is known as an elliptic integral of the first kind. We will rewrite this result and then use it to obtain an approximation to the period of oscillation of the nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (7.17). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. The kinetic energy of the mass on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where $h$ is the height that the mass is from the bottom of the swing. A little trigonometry gives that $h = L(1 - \cos\theta)$. So,

$$U = mgL(1 - \cos\theta).$$

Total mechanical energy for the nonlinear pendulum.

So, the total mechanical energy is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta). \tag{7.18}$$

We note that a little rearranging shows that we can relate this result to Equation (7.17). Dividing by $m$ and $L^2$ and using the definition of $\omega^2 = g/L$, we have

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \frac{1}{mL^2}E - \omega^2.$$

Therefore, we have determined the integration constant in terms of the total mechanical energy,

$$c = \frac{1}{mL^2}E - \omega^2.$$

We can use Equation (7.18) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total mechanical energy is pure potential energy. Letting $\theta_0$ denote the angle at the highest angular position, we have that

$$E = mgL(1 - \cos\theta_0) = mL^2\omega^2(1 - \cos\theta_0).$$

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0. \tag{7.19}$$

We can solve for $\dot{\theta}$ and integrate the differential equation to obtain

$$t = \int dt = \int \frac{d\theta}{\omega\sqrt{2(\cos\theta - \cos\theta_0)}}.$$

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta),$$

we can rewrite the argument in the radical as

$$\cos\theta - \cos\theta_0 = 2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right].$$

Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \tag{7.20}$$

This result can now be transformed into an elliptic integral.[2] We define

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

and

$$k = \sin\frac{\theta_0}{2}.$$

Then, Equation (7.20) becomes

$$T = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{7.21}$$

This is done by noting that $dz = \frac{1}{2k}\cos\frac{\theta}{2}\,d\theta = \frac{1}{2k}(1-k^2z^2)^{1/2}\,d\theta$ and that $\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is called the complete elliptic integral of the first kind.

The complete elliptic integral of the first kind.

We note that the incomplete elliptic integral of the first kind is defined as

The incomplete elliptic integral of the first kind.

$$F(\phi, k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Then, the complete elliptic integral of the first kind is given by $K(k) = F(\frac{\pi}{2}, k)$, or

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Therefore, the period of the nonlinear pendulum is given by

$$T = \frac{4}{\omega}K\left(\sin\frac{\theta_0}{2}\right). \tag{7.22}$$

There are table of values for elliptic integrals. However, one can use a computer algebra system to compute values of such integrals. We will look for small angle approximations.

For small angles ($\theta_0 \ll \frac{\pi}{2}$), we have that $k$ is small. So, we can develop a series expansion for the period, $T$, for small $k$. This is simply done by using the binomial expansion,

$$(1-k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6)$$

Inserting this expansion into the integrand for the complete elliptic integral and integrating term by term, we find that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]. \tag{7.23}$$

[2] Elliptic integrals were first studied by Leonhard Euler and Giulio Carlo de' Toschi di Fagnano (1682-1766) , who studied the lengths of curves such as the ellipse and the lemniscate,

$$(x^2 + y^2)^2 = x^2 - y^2.$$

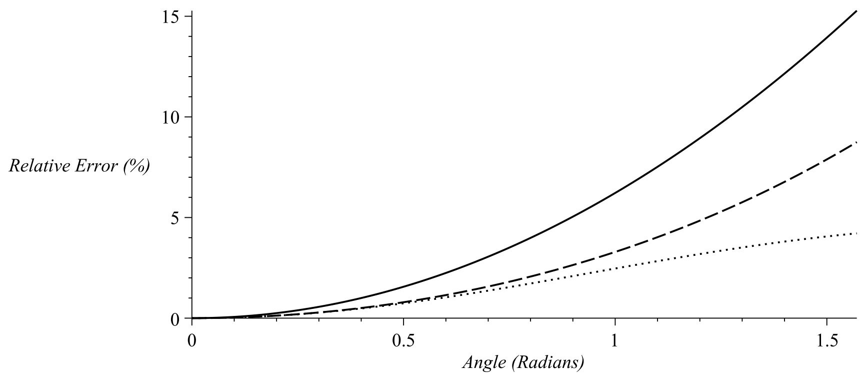



Figure 7.12: The relative error in percent when approximating the exact period of a nonlinear pendulum with one (solid), two (dashed), or three (dotted) terms in Equation (7.23).

The first term of the expansion gives the well known period of the simple pendulum for small angles. The next terms in the expression give further corrections to the linear result which are useful for larger amplitudes of oscillation. In Figure 7.12 we show the relative errors incurred when keeping the $k^2$ (quadratic) and $k^4$ (quartic) terms as compared to the exact value of the period.

## 7.6 The Stability of Fixed Points in Nonlinear Systems

We next investigate the stability of the equilibrium solutions of the nonlinear pendulum. Along the way we will develop some basic methods for studying the stability of equilibria in nonlinear systems in general.

There are two simple systems that we will consider, the damped linear pendulum,

$$x'' + bx' + \omega^2 x = 0$$

and the the damped nonlinear pendulum,

$$x'' + bx' + \omega^2 \sin x = 0.$$

These are second order differential equations and can be cast as a system of two first order differential equations using the methods of Chapter 2.

The linear equation can be written as

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 x. \end{aligned} \tag{7.24}$$

This system has only one equilibrium solution, $x = 0$, $y = 0$.

The damped nonlinear pendulum takes the form

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{7.25}$$

This system also has the equilibrium solution $x = 0$, $y = 0$. However, there are actually an infinite number of solutions. The equilibria are determined from

$$y = 0 \text{ and } -by - \omega^2 \sin x = 0. \tag{7.26}$$

These equations imply that $y = 0$ and $\sin x = 0$. There are an infinite number of solutions to the latter equation: $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. So, this system has an infinite number of equilibria, $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

The next step is to determine the stability of the equilibrium solutions these systems. This can be accomplished just as we had done for first order equations. To do this we need a more general theory for nonlinear systems. So, we will develop the needed machinery.

We begin with the $n-$dimensional system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n. \tag{7.27}$$

Linear stability analysis of systems.

Here $\mathbf{f}: R^n \to R^n$ is a mapping from $R^n$ to $R^n$. We define the equilibrium solutions, or fixed points, of this system as the points $\mathbf{x}^*$ satisfying $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

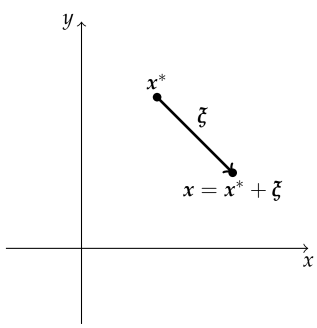



Figure 7.13: A general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \xi$,

The stability in the neighborhood of equilibria will now be determined. We are interested in what happens to solutions of the system with initial conditions starting near a fixed point. We will represent a general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$. We note that the length of $\boldsymbol{\xi}$ gives an indication of how close we are to the fixed point. So, we consider that initially, $|\boldsymbol{\xi}| \ll 1$.

As the system evolves, $\boldsymbol{\xi}$ will change. The change of $\boldsymbol{\xi}$ in time is in turn governed by a system of equations. We can approximate this evolution as follows. First, we note that

$$\mathbf{x}' = \boldsymbol{\xi}'.$$

Next, we have that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}).$$

We can expand the right side about the fixed point using a multidimensional version of Taylor's Theorem. Thus, we have that

$$\mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}) = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2).$$

The Jacobian matrix.

Here $D\mathbf{f}(\mathbf{x})$ is the *Jacobian matrix*, defined as

$$D\mathbf{f}(\mathbf{x}^*) \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Linearization of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.

Noting that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, we then have that system (7.27) becomes

$$\xi' \approx D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi}. \tag{7.28}$$

It is this equation which describes the behavior of the system near the fixed point. As with first order equations, we say that system (7.27) has been linearized or that Equation (7.28) is the linearization of system (7.27).

The stability of the equilibrium point of the nonlinear system is now reduced to analyzing the behavior of the linearized system given by Equation (7.28). We can use the methods from the last two chapters to investigate the eigenvalues of the Jacobian matrix evaluated at each equilibrium point. We will demonstrate this procedure with several examples.

**Example 7.10.** *Determine the equilibrium points and their stability for the system*

$$\begin{aligned} x' &= -2x - 3xy, \\ y' &= 3y - y^2. \end{aligned} \tag{7.29}$$

*We first determine the fixed points. Setting the right hand side equal to zero and factoring, we have*

$$\begin{aligned} -x(2+3y) &= 0, \\ y(3-y) &= 0. \end{aligned} \tag{7.30}$$

*From the second equation, we see that either $y = 0$ or $y = 3$. The first equation then gives $x = 0$ in either case. So, there are two fixed points: $(0,0)$ and $(0,3)$.*

*Next, we linearize the system of differential equations about each fixed point. First, we note that the Jacobian matrix is given by*

$$D\mathbf{f}(x,y) = \begin{pmatrix} -2-3y & -3x \\ 0 & 3-2y \end{pmatrix}. \tag{7.31}$$

1. *Case I Equilibrium point $(0,0)$.*

   *In this case we find that*

   $$D\mathbf{f}(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{7.32}$$

   *Therefore, the linearized equation becomes*

   $$\xi' = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \boldsymbol{\xi}. \tag{7.33}$$

   *This is equivalently written out as the system*

   $$\begin{aligned} \xi_1' &= -2\xi_1, \\ \xi_2' &= 3\xi_2. \end{aligned} \tag{7.34}$$

   *This is the linearized system about the origin. Note the similarity with the original system.*

   *We should emphasize that the linearized equations are constant coefficient equations and we can use matrix methods to determine the nature of the equilibrium point. The eigenvalues of this system are obviously $\lambda = -2, 3$. Therefore, we have that the origin is a saddle point.*

2. *Case II Equilibrium point $(0,3)$.*

*Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes*

$$D\mathbf{f}(0,3) = \begin{pmatrix} -11 & 0 \\ 0 & -3 \end{pmatrix}. \tag{7.35}$$

*The eigenvalues are $\lambda = -11, -3$. So, this fixed point is a stable node.*

*This analysis has given us a saddle and a stable node. We know what the behavior is like near each fixed point, but we have to resort to other means to say anything about the behavior far from these points. The phase portrait for this system is given in Figure 7.14. You should be able to locate the saddle point and the node in the figure. Notice how solutions behave in regions far from these points.*

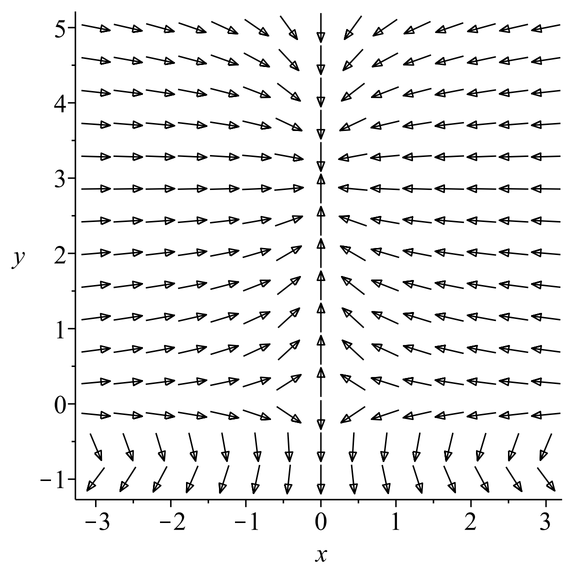



Figure 7.14: Phase plane for the system $x' = -2x - 3xy$, $y' = 3y - y^2$.

We can expect to be able to perform a linearization under general conditions. These are given in the *Hartman-Großman Theorem*:

**Theorem 7.1.** *A continuous map exists between the linear and nonlinear systems when $D\mathbf{f}(\mathbf{x}^*)$ does not have any eigenvalues with zero real part.*

Generally, there are several types of behavior that one can see in nonlinear systems. One can see sinks or sources, hyperbolic (saddle) points, elliptic points (centers) or foci. We have defined some of these for planar systems. In general, if at least two eigenvalues have real parts with opposite signs, then the fixed point is a *hyperbolic point*. If the real part of a nonzero eigenvalue is zero, then we have a center, or *elliptic point.*

For linear systems in the plane, this classification was done in Chapter 3. The Jacobian matrix evaluated at the equilibrium points is simply a $2 \times 2$ matrix.

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7.36}$$

Here we are using $J = D\mathbf{f}(\mathbf{x}^*)$.

The eigenvalue equation is given by

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

However, $a+d$ is the trace, $\text{tr}(J)$ and $\det(J) = ad - bc$. Therefore, we can write the eigenvalue equation as

$$\lambda^2 - \text{tr}(J)\lambda + \det(J) = 0.$$

The solution of this equation is found using the quadratic formula,

$$\lambda = \frac{1}{2}\left[-\text{tr}(J) \pm \sqrt{\text{tr}^2(J) - 4\det(J)}\right].$$

We had seen in Chapters 2 and 3 that equilibrium points in planar systems can be classified as nodes, saddles, centers, or spirals (foci). The type of behavior can be determined from solutions of the eigenvalue equation. Since the nature of the eigenvalues depends on the trace and determinant of the Jacobian matrix at the equilibrium point, we can relate the types of equilibria to points in the det-tr plane. This is shown in Figure 7.15.

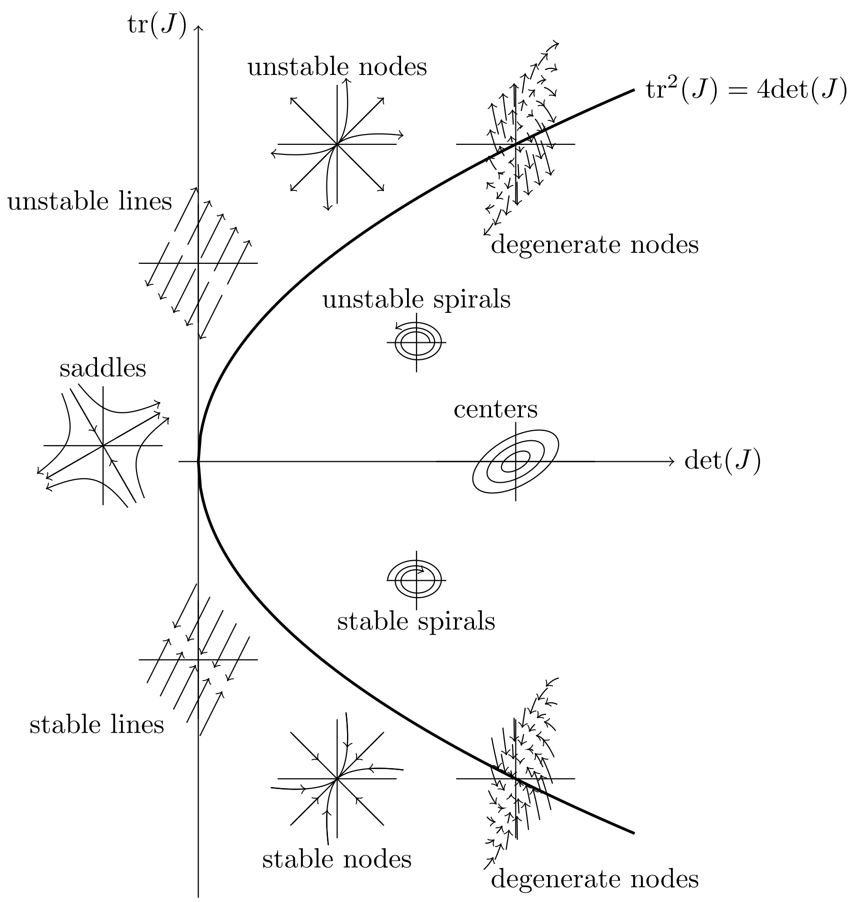



Figure 7.15: Diagram indicating the behavior of equilibrium points in the det − tr plane. The parabolic curve

$$\text{tr}^2(J) = 4\det(J)$$

indicates where the discriminant vanishes.

In Figure 7.15 the parabola $\text{tr}^2(J) = 4\det(J)$ divides the det-tr plane. Points on this curve give a vanishing discriminant in the computation of the eigenvalues. In these cases one finds repeated roots, or eigenvalues. Along this curve one can find stable and unstable degenerate nodes. Also along this line are stable and unstable proper nodes, called star nodes. These arise from systems of the form $x' = ax$, $y' = ay$.

In the case that $\det(J) < 0$, we have that the discriminant

$$\Delta \equiv \operatorname{tr}^2(J) - 4\det(J)$$

is positive. Not only that, $\Delta > \operatorname{tr}^2(J)$. Thus, we obtain two real and distinct eigenvalues with opposite signs. These lead to saddle points.

In the case that $\det(J) > 0$, we can have either $\Delta > 0$ or $\Delta < 0$. The discriminant is negative for points inside the parabolic curve. It is in this region that one finds centers and spirals, corresponding to complex eigenvalues. When $\operatorname{tr}(J) > 0$, there are unstable spirals. There are stable spirals when $\operatorname{tr}(J) < 0$. For the case that $\operatorname{tr}(J) = 0$, the eigenvalues are pure imaginary, giving centers.

There are several other types of behavior depicted in the figure, but we will now turn to studying a few of examples.

**Example 7.11.** *Find and classify all of the equilibrium solutions of the nonlinear system*

$$\begin{aligned} x' &= 2x - y + 2xy + 3(x^2 - y^2), \\ y' &= x - 3y + xy - 3(x^2 - y^2). \end{aligned} \tag{7.37}$$

*In Figure 7.16 we show the direction field for this system. Try to locate and classify the equilibrium points visually. After the stability analysis, you should return to this figure and determine if you identified the equilibrium points correctly.*

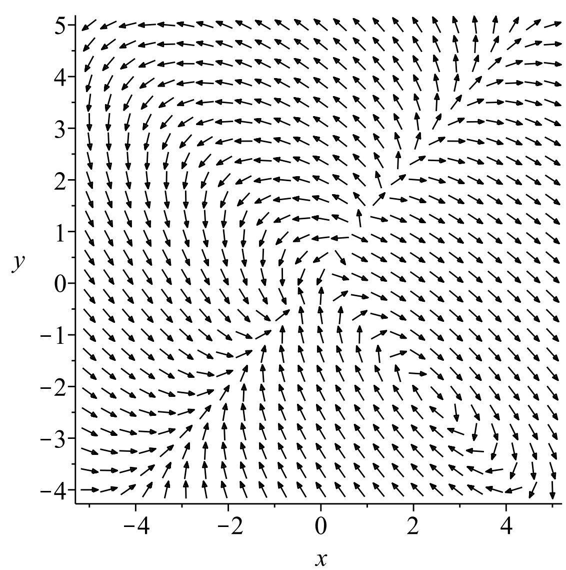



Figure 7.16: Phase plane for the system $x' = 2x - y + 2xy + 3(x^2 - y^2)$, $y' = x - 3y + xy - 3(x^2 - y^2)$.

*We will first determine the equilibrium points. Setting the right hand side of each differential equation to zero, we have*

$$2x - y + 2xy + 3(x^2 - y^2) = 0,$$

$$x - 3y + xy - 3(x^2 - y^2) \quad = \quad 0. \tag{7.38}$$

*This system of algebraic equations can be solved exactly. Adding the equations, we have*

$$3x - 4y + 3xy = 0.$$

*Solving for* $x$*,*

$$x = \frac{4y}{3(1+y)},$$

*and substituting the result for* $x$ *into the first algebraic equation, we find an equation for* $y$ *:*

$$\frac{y(1-y)(9y^2 + 22y + 5)}{3(1+y)^2} = 0.$$

*The solutions to this equation are*

$$y = 0, 1, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}.$$

*The corresponding values for* $x$ *are*

$$x = 0, \frac{2}{3}, 1 \mp \frac{\sqrt{19}}{3}.$$

*Now that we have located the equilibria, we can classify them. The Jacobian matrix is given by*

$$D\mathbf{f}(x, y) = \begin{pmatrix} 6x + 2y + 2 & 2x - 6y - 1 \\ -6x + y + 1 & x + 6y - 3 \end{pmatrix}. \tag{7.39}$$

*Now, we evaluate the Jacobian at each equilibrium point and find the eigenvalues.*

1. *Case I. Equilibrium point* $(0, 0)$*.*

   *In this case we find that*

   $$D\mathbf{f}(0, 0) = \begin{pmatrix} -2 & -1 \\ 1 & -3 \end{pmatrix}. \tag{7.40}$$

   *The eigenvalues of this matrix are* $\lambda = -\frac{1}{2} \pm \frac{\sqrt{21}}{2}$*. Therefore, the origin is a saddle point.*

2. *Case II. Equilibrium point* $(\frac{2}{3}, 1)$*.*

   *Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes*

   $$D\mathbf{f}\left(\frac{2}{3}, 1\right) = \begin{pmatrix} 8 & -\frac{17}{3} \\ -2 & \frac{11}{3} \end{pmatrix}. \tag{7.41}$$

   *The eigenvalues are* $\lambda = \frac{35}{6} \pm \frac{\sqrt{577}}{6} \approx 9.84, 1.83$*. This fixed point is an unstable node.*

3. *Case III. Equilibrium point* $(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19})$.
   *The Jacobian matrix for this case becomes*

$$D\mathbf{f}\left(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}\right) = \begin{pmatrix} \frac{50}{9} \mp \frac{14}{9}\sqrt{19} & \frac{25}{3} \mp 2\sqrt{19} \\ -\frac{56}{9} \pm \frac{20}{9}\sqrt{19} & -\frac{28}{3} \pm \sqrt{19} \end{pmatrix}. \tag{7.42}$$

   *There are two equilibrium points under this case. The first is given by*

$$(1 - \frac{\sqrt{19}}{3}, -\frac{11}{9} + \frac{2}{9}\sqrt{19}) \approx (0.453, -0.254).$$

   *The eigenvalues for this point are*

$$\lambda = -\frac{17}{9} - \frac{5}{18}\sqrt{19} \pm \frac{1}{18}\sqrt{3868\sqrt{19} - 16153}.$$

   *These are approximately* $-4.58$ *and* $-1.62$ *So, this equilibrium point is a stable node.*
   *The other equilibrium is* $(1 + \frac{\sqrt{19}}{3}, -\frac{11}{9} - \frac{2}{9}\sqrt{19}) \approx (2.45, -2.19)$. *The corresponding eigenvalues are complex with negative real parts,*

$$\lambda = -\frac{17}{9} + \frac{5}{18}\sqrt{19} \pm \frac{i}{18}\sqrt{16153 + 3868\sqrt{19}},$$

   *or* $\lambda \approx -0.678 \pm 10.1i$. *This point is a stable spiral.*

*Plots of the phase plane are given in Figures 7.14 and 7.16. The reader can look at the direction field and verify these results for the behavior of equilibrium solutions. A zoomed in view is shown in Figure 7.17 with several orbits indicated.*

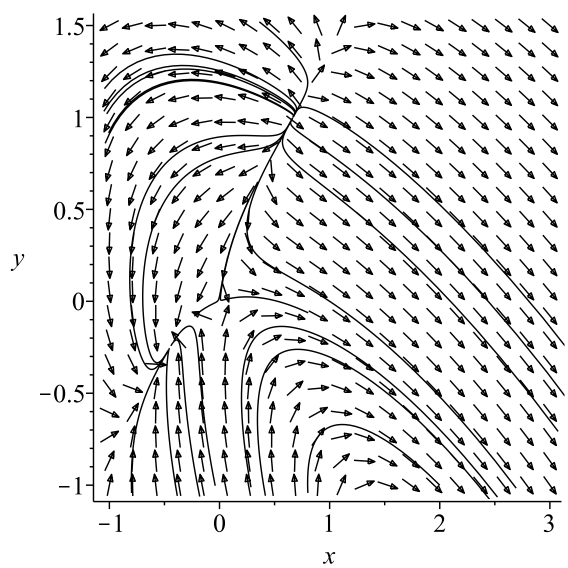



Figure 7.17: A closer look at the phase plane for the system

$$x' = 2x - y + 2xy + 3(x^2 - y^2),$$
$$y' = x - 3y + xy - 3(x^2 - y^2)$$

with a few trajectories shown.

**Example 7.12. Damped Nonlinear Pendulum Equilibria**

*We are now ready to establish the behavior of the fixed points of the damped nonlinear pendulum system in Equation (7.25). Recall that the system for the* *damped nonlinear pendulum was given by*

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{7.43}$$

*For a damped system, we will need $b > 0$. We had found that there are an infinite number of equilibrium points at $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.*

*The Jacobian matrix for this systems is*

$$D\mathbf{f}(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{pmatrix}. \tag{7.44}$$

*Evaluating this matrix at the fixed points, we find that*

$$D\mathbf{f}(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\omega^2 & -b \end{pmatrix}. \tag{7.45}$$

*The eigenvalue equation is given by*

$$\lambda^2 + b\lambda + (-1)^n \omega^2 = 0.$$

*There are two cases to consider: $n$ even and $n$ odd. For the first case, we find the eigenvalues*

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2}.$$

*For $b^2 < 4\omega^2$, we have two complex conjugate roots with a negative real part. Thus, we have stable foci for even $n$ values. If there is no damping, then we obtain centers ($\lambda = \pm i\omega$).*

*In the second case, $n$ odd, we find*

$$\lambda = \frac{-b \pm \sqrt{b^2 + 4\omega^2}}{2}.$$

*Since $b^2 + 4\omega^2 > b^2$, these roots will be real with opposite signs. Thus, we have hyperbolic points, or saddles. If there is no damping, the eigenvalues reduce to $\lambda = \pm\omega$.*

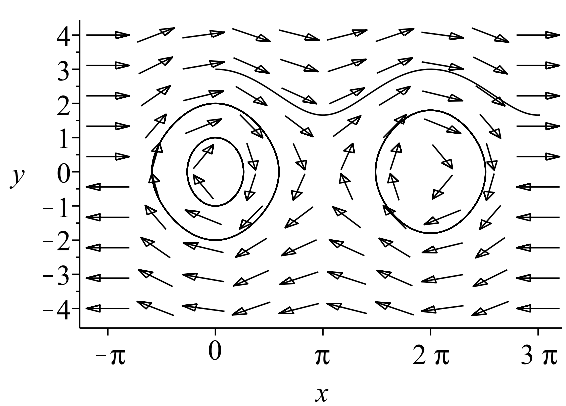



Figure 7.18: Phase plane for the undamped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

*In Figure (7.18) we show the phase plane for the undamped nonlinear pendulum with $\omega = 1.25$. We see that we have a mixture of centers and saddles. There are orbits for which there is periodic motion. In the case that $\theta = \pi$ we have an inverted pendulum. This is an unstable position and this is reflected in the presence of saddle points, especially if the pendulum is constructed using a massless rod.*

*There are also unbounded orbits, going through all possible angles. These correspond to the mass spinning around the pivot in one direction forever due to initially having large enough energies.*

*We have indicated in the figure solution curves with the initial conditions* $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$. *These show the various types of motions that we have described.*

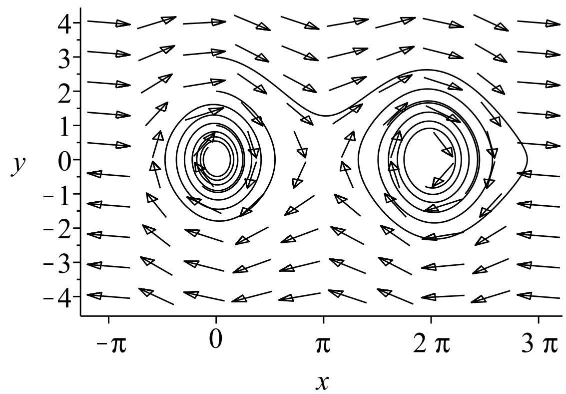



Figure 7.19: Phase plane for the damped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

When there is damping, we see that we can have a variety of other behaviors as seen in Figure (7.19). In this example we have set $b = 0.08$ and $\omega = 1.25$. We see that energy loss results in the mass settling around one of the stable fixed points. This leads to an understanding as to why there are an infinite number of equilibria, even though physically the mass traces out a bound set of Cartesian points. We have indicated in the Figure (7.19) solution curves with the initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

In Figure 7.20 we show a region of the phase plane which corresponds to oscillations about $x = 0$. For small angles the pendulum oscillates following somewhat elliptical orbits. As the angles get larger, due to greater initial energies, these orbits begin to change from ellipses to other periodic orbits. There is a limiting orbit, beyond which one has unbounded motion. The limiting orbit connects the saddle points on either side of the center. The curve is called a separatrix and being that these trajectories connect two saddles, they are often referred to as heteroclinic orbits.

Heteroclinc orbits and separatrices.

In Figures 7.21-7.21 we show more orbits, including both bound and unbound motion beyond the interval $x \in [-\pi, \pi]$. For both plots we have chosen $\omega = 5$ and the same set of initial conditions, $x(0) = \pi k/10$, $k = -20, \ldots, 20$. for $y(0) = 0, \pm 10$. The time interval is taken for $t \in [-3, 3]$. The only difference is that in the damped case we have $b = 0.5$. In these plots one can see what happens to the heteroclinic orbits and nearby unbounded orbits under damping.

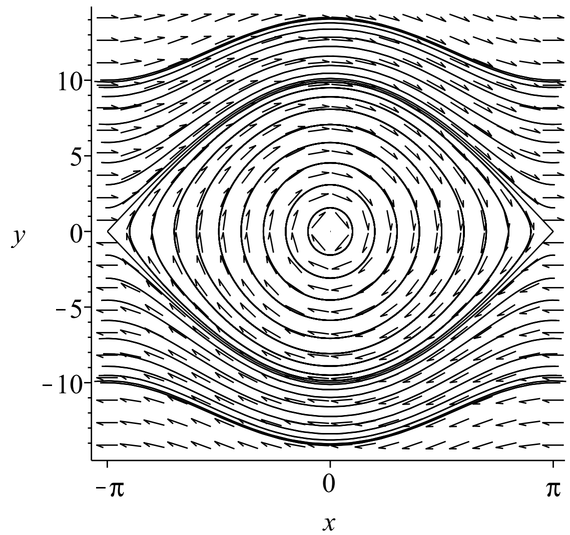



Figure 7.20: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$. The orbits surround a center at $(0,0)$. At the edges there are saddle points, $(\pm\pi, 0)$.

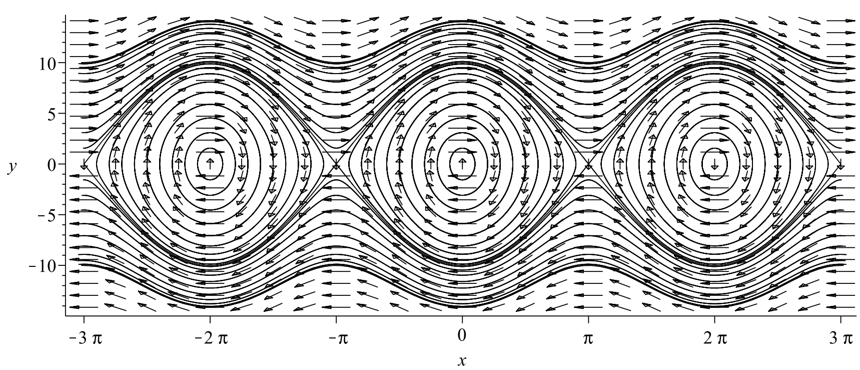



Figure 7.21: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$.

Before leaving this problem, we should note that the orbits in the phase plane for the undamped nonlinear pendulum can be obtained graphically. Recall from Equation (7.18), the total mechanical energy for the nonlinear pendulum is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1-\cos\theta).$$

From this equation we obtained Equation (7.19),

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

Letting $y = \dot{\theta}$, $x = \theta$, and defining $z = -\omega^2\cos\theta_0$, this equation can be written as

$$\frac{1}{2}y^2 - \omega^2\cos x = z. \tag{7.46}$$

For each energy ($z$), this gives a constant energy curve. Plotting the family of energy curves we obtain the phase portrait shown in Figure 7.23.

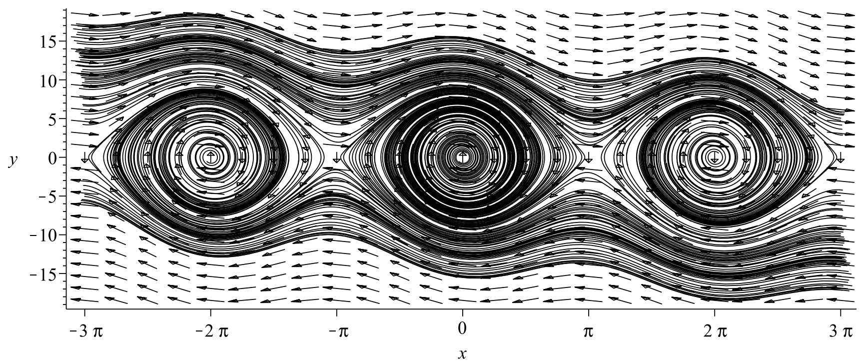



Figure 7.22: Several orbits in the phase plane for the damped nonlinear pendulum with $\omega = 5.0$ and $b = 0.5$.

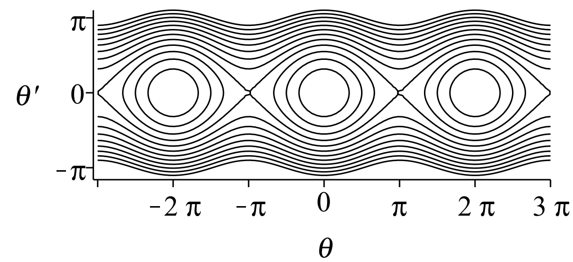



Figure 7.23: A family of energy curves in the phase plane for $\frac{1}{2}\dot{\theta}^2 - \omega^2 \cos\theta = z$. Here we took $\omega = 1.0$ and $z \in [-5, 15]$.

## 7.7 Nonlinear Population Models

We have already encountered several models of population dynamics in this and previous chapters. Of course, one could dream up several other examples. While such models might seem far from applications in physics, it turns out that these models lead to systems od differential equations which also appear in physical systems such as the coupling of waves in lasers, in plasma physics, and in chemical reactions.

Two well-known nonlinear population models are the predator-prey and competing species models. In the predator-prey model, one typically has one species, the predator, feeding on the other, the prey. We will look at the standard Lotka-Volterra model in this section. The competing species model looks similar, except there are a few sign changes, since one species is not feeding on the other. Also, we can build in logistic terms into our model. We will save this latter type of model for the homework.

The Lotka-Volterra model is named after Alfred James Lotka (1880-1949) and Vito Volterra (1860-1940).

The Lotka-Volterra model of population dynamics.

The Lotka-Volterra model takes the form

$$\begin{aligned} \dot{x} &= ax - bxy, \\ \dot{y} &= -dy + cxy, \end{aligned} \tag{7.47}$$

where $a$, $b$, $c$, and $d$ are positive constants. In this model, we can think of $x$ as the population of rabbits (prey) and $y$ is the population of foxes (predators). Choosing all constants to be positive, we can describe the terms.

- $ax$: When left alone, the rabbit population will grow. Thus $a$ is the natural growth rate without predators.
- $-dy$: When there are no rabbits, the fox population should decay. Thus, the coefficient needs to be negative.
- $-bxy$: We add a nonlinear term corresponding to the depletion of the rabbits when the foxes are around.
- $cxy$: The more rabbits there are, the more food for the foxes. So, we add a nonlinear term giving rise to an increase in fox population.

**Example 7.13.** *Determine the equilibrium points and their stability for the Lotka-Volterra system.*

*The analysis of the Lotka-Volterra model begins with determining the fixed points. So, we have from Equation (7.47)*

$$\begin{aligned} x(a-by) &= 0, \\ y(-d+cx) &= 0. \end{aligned} \tag{7.48}$$

*Therefore, the origin,* $(0,0)$, *and* $(\frac{d}{c}, \frac{a}{b})$ *are the fixed points.*

*Next, we determine their stability, by linearization about the fixed points. We can use the Jacobian matrix, or we could just expand the right hand side of each equation in (7.47) about the equilibrium points as shown in he next example. The Jacobian matrix for this system is*

$$Df(x,y) = \begin{pmatrix} a-by & -bx \\ cy & -d+cx \end{pmatrix}.$$

*Evaluating at each fixed point, we have*

$$Df(0,0) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \tag{7.49}$$

$$Df\left(\frac{d}{c}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{pmatrix}. \tag{7.50}$$

*The eigenvalues of (7.49) are* $\lambda = a, -d$. *So, the origin is a saddle point.*

*The eigenvalues of (7.50) satisfy* $\lambda^2 + ad = 0$. *So, the other point is a center. In Figure 7.24 we show a sample direction field for the Lotka-Volterra system.*

Another way to carry out the linearization of the system of differential equations is to expand the equations about the fixed points. For fixed points $(x^*, y^*)$, we let

$$(x,y) = (x^* + u, y^* + v).$$

Inserting this translation of the origin into the equations of the system, and dropping nonlinear terms in $u$ and $v$, results in the linearized system. This method is equivalent to analyzing the Jacobian matrix for each fixed point.

Direct linearization of a system is carried out by introducing $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$, or $(x,y) = (x^* + u, y^* + v)$ into the system and dropping nonlinear terms in $u$ and $v$.

**Example 7.14.** *Expand the Lotka-Volterra system about the equilibrium points.*

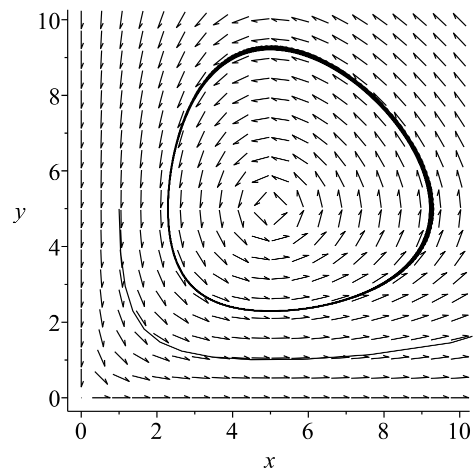



Figure 7.24: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$. Solution curves are shown for initial conditions $(x_0, y_0) = (8,3), (1,5)$.

*For the origin $(0,0)$ the linearization about the origin amounts to simply dropping the nonlinear terms. In this case we have*

$$\begin{aligned} \dot{u} &= au, \\ \dot{v} &= -dv. \end{aligned} \tag{7.51}$$

*The coefficient matrix for this system is the same as $Df(0,0)$.*

*For the second fixed point, we let*

$$(x, y) = \left(\frac{d}{c} + u, \frac{a}{b} + v\right).$$

*Inserting this transformation into the system gives*

$$\begin{aligned} \dot{u} &= a\left(\frac{d}{c} + u\right) - b\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right), \\ \dot{v} &= -d\left(\frac{a}{b} + v\right) + c\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right). \end{aligned} \tag{7.52}$$

*Expanding, we obtain*

$$\begin{aligned} \dot{u} &= \frac{ad}{c} + au - b\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right), \\ \dot{v} &= -\frac{ad}{b} - dv + c\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right). \end{aligned} \tag{7.53}$$

*In both equations the constant terms cancel and linearization is simply getting rid of the $uv$ terms. This leaves the linearized system*

$$\begin{aligned} \dot{u} &= au - b\left(\frac{d}{c}v + \frac{a}{b}u\right), \\ \dot{v} &= -dv + c\left(+\frac{d}{c}v + \frac{a}{b}u\right), \end{aligned} \tag{7.54}$$

*or*

$$\begin{aligned} \dot{u} &= -\frac{bd}{c}v, \\ \dot{v} &= \frac{ac}{b}u. \end{aligned} \tag{7.55}$$

*The coefficient matrix for this linearized system is the same as* $Df\left(\frac{d}{c}, \frac{a}{b}\right)$. *In fact, for nearby orbits, they are almost circular orbits. From this linearized system, we have* $\ddot{u} + adu = 0$.

*We can take* $u = A\cos(\sqrt{ad}t + \phi)$, *where* $A$ *and* $\phi$ *can be determined from the initial conditions. Then,*

$$\begin{aligned} v &= -\frac{c}{bd}\dot{u} \\ &= \frac{c}{bd}A\sqrt{ad}\sin(\sqrt{ad}t + \phi) \\ &= \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi). \end{aligned} \tag{7.56}$$

*Therefore, the solutions near the center are given by*

$$(x, y) = \left(\frac{d}{c} + A\cos(\sqrt{ad}t + \phi), \frac{a}{b} + \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi)\right).$$

For $a = d = 1$, $b = c = 0.2$, and initial values of $(x_0, y_0) = (5.5, 5)$, these solutions become

$$x(t) = 5.0 + 0.5\cos t, \quad y(t) = 5.0 + 0.5\sin t.$$

Plots of these solutions are shown in Figure (7.25).

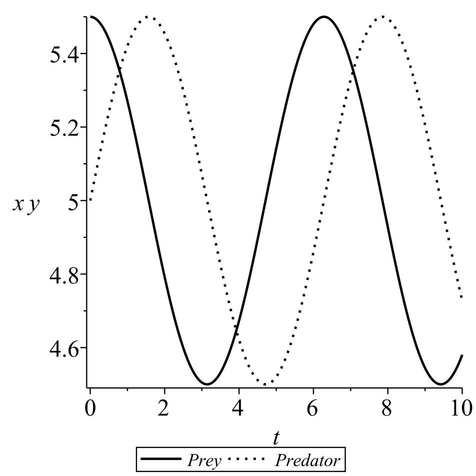



Figure 7.25: The linearized solutions of Lotka-Volterra system $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ for the initial conditions $(x_0, y_0) = (5.5, 5)$.

It is also possible to find a first integral of the Lotka-Volterra system whose level curves give the phase portrait of the system. As we had done in Chapter 2, we can write

$$\begin{aligned} \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} \\ &= \frac{-dy + cxy}{ax - bxy} \\ &= \frac{y(-d + cx)}{x(a - by)}. \end{aligned} \tag{7.57}$$

This is an equation of the form seen in Problem 2.13. This equation is now a separable differential equation. The solution this differential equation is given in implicit form as

$$a \ln y + d \ln x - cx - by = C,$$

where $C$ is an arbitrary constant. This expression is known as the first integral of the Lotka-Volterra system. These level curves are shown in Figure 7.26.

The first integral of the Lotka-Volterra system.

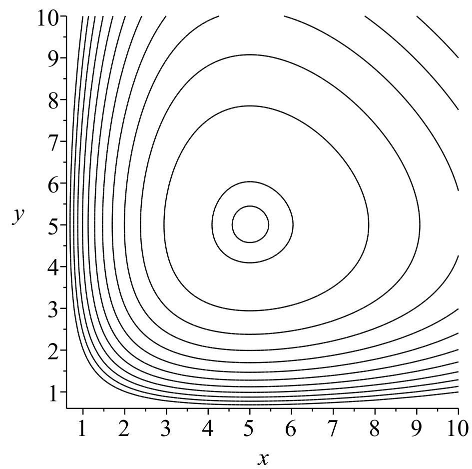



Figure 7.26: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ based upon the first integral of the system.

## 7.8 Limit Cycles

So far we have just been concerned with equilibrium solutions and their behavior. However, asymptotically stable fixed points are not the only attractors. There are other types of solutions, known as limit cycles, towards which a solution may tend. In this section we will look at some examples of these periodic solutions.

Such solutions are common in nature. Rayleigh investigated the problem

$$x'' + c\left(\frac{1}{3}(x')^2 - 1\right)x' + x = 0 \tag{7.58}$$

in the study of the vibrations of a violin string. Balthasar van der Pol (1889-1959) studied an electrical circuit, modeling this behavior. Others have looked into biological systems, such as neural systems, chemical reactions, such as Michaelis-Menten kinetics, and other chemical systems leading to chemical oscillations. One of the most important models in the historical study of dynamical systems is that of planetary motion and investigating the stability of planetary orbits. As is well known, these orbits are periodic.

Limit cycles are isolated periodic solutions towards which neighboring states might tend when stable. A key example exhibiting a limit cycle is given in the next example.

**Example 7.15.** *Find the limit cycle in the system*

$$\begin{aligned} x' &= \mu x - y - x(x^2+y^2) \\ y' &= x + \mu y - y(x^2+y^2). \end{aligned} \tag{7.59}$$

*It is clear that the origin is a fixed point. The Jacobian matrix is given as*

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}. \tag{7.60}$$

*The eigenvalues are found to be $\lambda = \mu \pm i$. For $\mu = 0$ we have a center. For $\mu < 0$ we have a stable spiral and for $\mu > 0$ we have an unstable spiral. However, this spiral does not wander off to infinity. We see in Figure 7.27 that the equilibrium point is a spiral. However, in Figure 7.28 it is clear that the solution does not spiral out to infinity. It is bounded by a circle.*

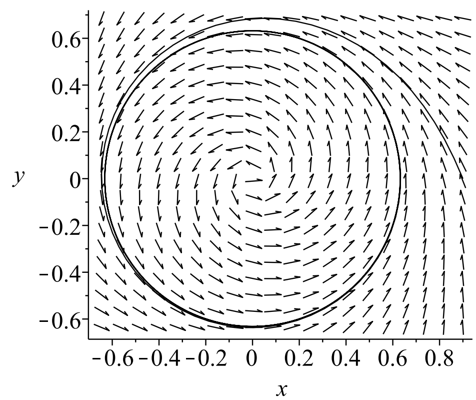



Figure 7.27: Phase plane for system (7.59) with $\mu = 0.4$.

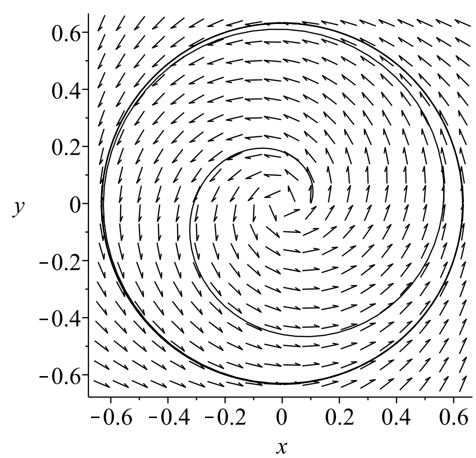



Figure 7.28: Phase plane for system (7.59) with $\mu = 0.4$ showing that the inner spiral is bounded by a limit cycle.

*One can actually find the radius of this circle. This requires rewriting the system in polar form. Recall from Chapter 2 that we can change derivatives of Cartesian coordinates to derivatives of polar coordinates by using the relations*

$$rr' = xx' + yy', \tag{7.61}$$

$$r^2\theta' = xy' - yx'. \tag{7.62}$$

*Inserting the system (7.59) into these expressions, we have*

$$rr' = \mu r^2 - r^4, \qquad r^2\theta' = r^2.$$

*This leads to the system*

$$\begin{aligned} r' &= \mu r - r^3, \\ \theta' &= 1. \end{aligned} \tag{7.63}$$

*Of course, for a circle the radius is constant, $r = const$. Therefore, in order to find the limit cycle, we need to look at the equilibrium solutions of Equation (7.63). This amounts to finding the constant solutions of $\mu r - r^3 = 0$. The equilibrium solutions are $r = 0, \pm\sqrt{\mu}$. The limit cycle corresponds to the positive radius solution, $r = \sqrt{\mu}$.*

*In Figures 7.27-7.28 we take $\mu = 0.4$. In this case we expect a circle with $r = \sqrt{0.4} \approx 0.63$. From the $\theta$ equation, we have that $\theta' > 0$. This means that we follow the limit cycle in a counterclockwise direction as time increases.*

Limit cycles are not always circles. In Figures 7.29-7.30 we show the behavior of the Rayleigh system (7.58) for $c = 0.4$ and $c = 2.0$. In this case we see that solutions tend towards a noncircular limit cycle in a clockwise direction.

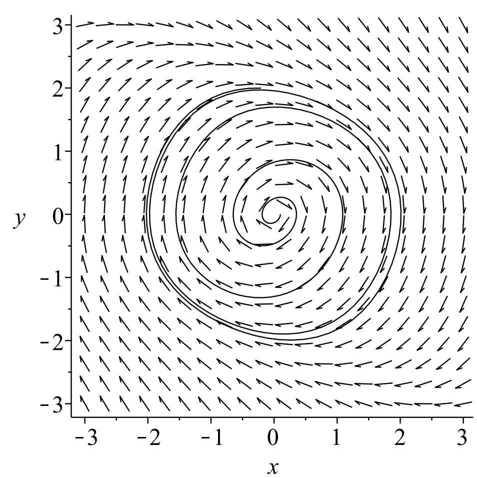



Figure 7.29: Phase plane for the Rayleigh system (7.58) with $c = 0.4$.

A slight change of the Rayleigh system leads to the van der Pol equation:

$$x'' + c(x^2 - 1)x' + x = 0 \tag{7.64}$$

The van der Pol system.

The limit cycle for $c = 2.0$ is shown in Figure 7.31.

Can one determine ahead of time if a given nonlinear system will have a limit cycle? In order to answer this question, we will introduce some definitions.

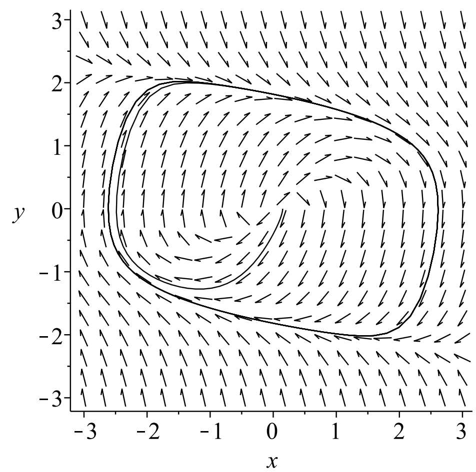



Figure 7.30: Phase plane for the van der Pol system (7.64) with $c = 2.0$.

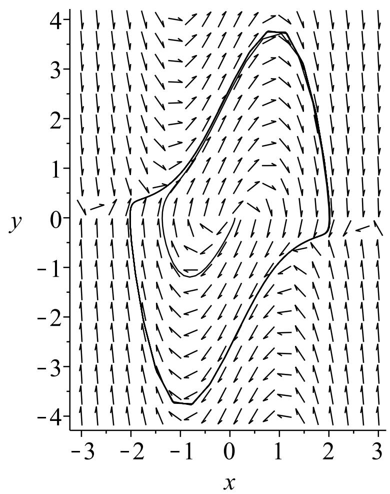



Figure 7.31: Phase plane for the van der Pol system (7.64) with $c = 0.4$.

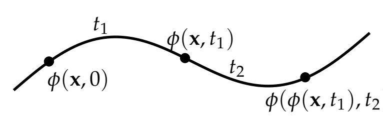



Figure 7.32: A sketch depicting the idea of trajectory, or orbit, passing through $\mathbf{x}$.

Flows.

We first describe different trajectories and families of trajectories. A *flow* on $R^2$ is a function $\phi$ that satisfies the following

1. $\phi(\mathbf{x}, t)$ is continuous in both arguments.
2. $\phi(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in R^2$.
3. $\phi(\phi(\mathbf{x}, t_1), t_2) = \phi(\mathbf{x}, t_1 + t_2)$.

Orbits and trajectories.

The *orbit*, or *trajectory*, through $\mathbf{x}$ is defined as $\gamma = \{\phi(\mathbf{x}, t) | t \in I\}$. In Figure 7.32 we demonstrate these properties. For $t = 0$, $\phi(\mathbf{x}, 0) = \mathbf{x}$. Increasing $t$, one follows the trajectory until one reaches the point $\phi(\mathbf{x}, t_1)$. Continuing $t_2$ further, one is then at $\phi(\phi(\mathbf{x}, t_1), t_2)$. By the third property, this is the same as going from $\mathbf{x}$ to $\phi(\mathbf{x}, t_1 + t_2)$ for $t = t_1 + t_2$.

Having defined the orbits, we need to define the asymptotic behavior of the orbit for both positive and negative large times. We define the *positive semiorbit* through $\mathbf{x}$ as $\gamma^+ = \{\phi(\mathbf{x}, t) | t > 0\}$. The *negative semiorbit* through $\mathbf{x}$ is defined as $\gamma^- = \{\phi(\mathbf{x}, t) | t < 0\}$. Thus, we have $\gamma = \gamma^+ \cup \gamma^-$.

Limit sets and limit points.

The *positive limit set*, or *ω-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^+ = \{\mathbf{y}| \text{ there exists a sequence of } t_n \to \infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}.$$

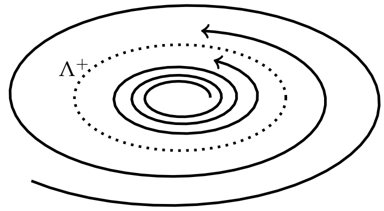



Figure 7.33: A sketch depicting an $\omega$-limit set. Note that the orbits tend towards the set as $t$ increases.

The $\mathbf{y}$'s are referred to as *ω-limit points*. This is shown in Figure 7.33.

Similarly, we define the *negative limit set*, or the *alpha-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^- = \{\mathbf{y}| \text{ there exists a sequences of } t_n \to -\infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}$$

and the corresponding $\mathbf{y}$'s are *α-limit points*. This is shown in Figure 7.34.

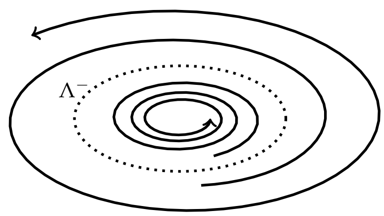



Figure 7.34: A sketch depicting an $\alpha$-limit set. Note that the orbits tend away from the set as $t$ increases.

Cycles and periodic orbits.

There are several types of orbits that a system might possess. A *cycle* or *periodic orbit* is any closed orbit which is not an equilibrium point. A periodic orbit is stable if for every neighborhood of the orbit such that all nearby orbits stay inside the neighborhood. Otherwise, it is unstable. The orbit is asymptotically stable if all nearby orbits converge to the periodic orbit.

A limit cycle is a cycle which is the $\alpha$ or $\omega$-limit set of some trajectory other than the limit cycle. A limit cycle $\Gamma$ is stable if $\Lambda^+ = \Gamma$ for all $\mathbf{x}$ in some neighborhood of $\Gamma$. A limit cycle $\Gamma$ is unstable if $\Lambda^- = \Gamma$ for all $\mathbf{x}$ in some neighborhood of $\Gamma$. Finally, a limit cycle is semistable if it is attracting on one side and repelling on the other side. In the previous examples, we saw limit cycles that were stable. Figures 7.33 and 7.34 depict stable and unstable limit cycles, respectively.

We now state a theorem which describes the type of orbits we might find in our system.

**Theorem 7.2. Poincaré-Bendixon Theorem** *Let $\gamma^+$ be contained in a bounded region in which there are finitely many critical points. Then $\Lambda^+$ is either*

1. *a single critical point;*
2. *a single closed orbit;*
3. *a set of critical points joined by heteroclinic orbits.*
   *[Compare Figures 7.35 and 7.36.]*

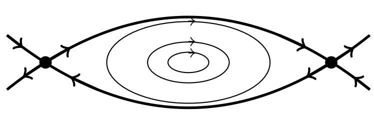


Figure 7.35: A heteroclinic orbit connecting two critical points.

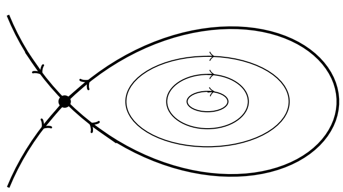


Figure 7.36: A homoclinic orbit returning to the point it left.

We are interested in determining when limit cycles may, or may not, exist. A consequence of the Poincaré-Bendixon Theorem is given by the following corollary.

**Corollary** Let $D$ be a bounded closed set containing no critical points and suppose that $\gamma^+ \subset D$. Then there exists a limit cycle contained in $D$.

More specific criteria allow us to determine if there is a limit cycle in a given region. These are given by Dulac's Criteria and Bendixon's Criteria.

**Dulac's Criteria** Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

and a continuously differentiable function $\psi$ defined on an annular region $D$ contained in some open set. If

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g)$$

does not change sign in $D$, then there is at most one limit cycle contained entirely in $D$.

**Bendixon's Criteria** Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

defined on a simply connected domain $D$ such that

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g) \neq 0$$

in $D$. Then, there are no limit cycles entirely in $D$.

*Proof.* These are easily proved using Green's Theorem in the Plane. (See your calculus text.) We prove Bendixon's Criteria. Let $\mathbf{f} = (f,g)$. Assume that $\Gamma$ is a closed orbit lying in $D$. Let $S$ be the interior of $\Gamma$. Then

$$\begin{aligned}
\int_S \nabla \cdot \mathbf{f}\, dxdy &= \oint_\Gamma (f\,dy - g\,dx) \\
&= \int_0^T (f\dot{y} - g\dot{x})dt \\
&= \int_0^T (fg - gf)dt = 0. \qquad (7.65)
\end{aligned}$$

So, if $\nabla \cdot \mathbf{f}$ is not identically zero and does not change sign in $S$, then from the continuity of $\nabla \cdot \mathbf{f}$ in $S$ we have that the right side above is either positive or negative. Thus, we have a contradiction and there is no closed orbit lying in $D$ □

**Example 7.16.** *Consider the earlier example in (7.59) with $\mu = 1$.*

$$\begin{aligned}
x' &= x - y - x(x^2 + y^2) \\
y' &= x + y - y(x^2 + y^2). \qquad (7.66)
\end{aligned}$$

*We already know that a limit cycle exists at $x^2 + y^2 = 1$. A simple computation gives that*

$$\nabla \cdot \mathbf{f} = 2 - 4x^2 - 4y^2.$$

*For an arbitrary annulus $a < x^2 + y^2 < b$, we have*

$$2 - 4b < \nabla \cdot \mathbf{f} < 2 - 4a.$$

*For $a = 3/4$ and $b = 5/4$, $-3 < \nabla \cdot \mathbf{f} < -1$. Thus, $\nabla \cdot \mathbf{f} < 0$ in the annulus $3/4 < x^2 + y^2 < 5/4$. Therefore, by Dulac's Criteria there is at most one limit cycle in this annulus.*

**Example 7.17.** *Consider the system*

$$\begin{aligned} x' &= y \\ y' &= -ax - by + cx^2 + dy^2. \end{aligned} \tag{7.67}$$

*Let $\psi(x, y) = e^{-2dx}$. Then,*

$$\frac{\partial}{\partial x}(\psi y) + \frac{\partial}{\partial y}(\psi(-ax - by + cx^2 + dy^2)) = -be^{-2dx} \neq 0.$$

*We conclude by Bendixon's Criteria that there are no limit cycles for this system.*

## 7.9 Nonautonomous Nonlinear Systems

In this section we discuss nonautonomous systems. Recall that an autonomous system is one in which there is no explicit time dependence. A simple example is the forced nonlinear pendulum given by the nonhomogeneous equation

$$\ddot{x} + \omega^2 \sin x = f(t). \tag{7.68}$$

We can set this up as a system of two first order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(t). \end{aligned} \tag{7.69}$$

This system is not in a form for which we could use the earlier methods. Namely, it is a nonautonomous system. However, we introduce a new variable $z(t) = t$ and turn it into an autonomous system in one more dimension. The new system takes the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(z). \\ \dot{z} &= 1. \end{aligned} \tag{7.70}$$

The system is now a three dimensional autonomous, possibly nonlinear, system and can be explored using methods from Chapters 2 and 3.

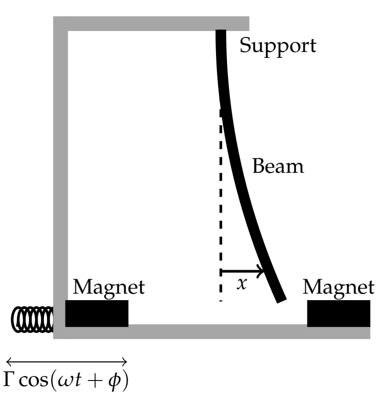



Figure 7.37: One model of the Duffing equation describes a periodically forced beam which interacts with two magnets.

A more interesting model is provided by the Duffing Equation. This equation, named after Georg Wilhelm Christian Caspar Duffing (1861-1944), models hard spring and soft spring oscillations. It also models a periodically forced beam as shown in Figure 7.37. It is of interest because it is a simple system which exhibits chaotic dynamics and will motivate us towards using new visualization methods for nonautonomous systems.

The most general form of Duffing's equation is given by the damped, forced system

$$\ddot{x} + k\dot{x} + (\beta x^3 \pm \omega_0^2 x) = \Gamma \cos(\omega t + \phi). \tag{7.71}$$

This equation models hard spring, ($\beta > 0$), and soft spring, ($\beta < 0$), oscillations. However, we will use the simpler version of the Duffing equation:

$$\ddot{x} + k\dot{x} + x^3 - x = \Gamma \cos \omega t. \tag{7.72}$$

An equation of this form can be obtained by setting $\phi = 0$ and rescaling $x$ and $t$ in the original equation. We will explore the behavior of the system as we vary the remaining parameters. In Figures 7.38-7.41 we show some typical solution plots superimposed on the direction field.

The undamped, unforced Duffing equation.

We start with the undamped ($k = 0$) and unforced ($\Gamma = 0$) Duffing equation,

$$\ddot{x} + x^3 - x = 0.$$

We can write this second order equation as the autonomous system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1 - x^2). \end{aligned} \tag{7.73}$$

We see that there are three equilibrium points at $(0, 0)$ and $(\pm 1, 0)$. In Figure 7.38 we plot several orbits for $k = 0$, and $\Gamma = 0$. We see that the three equilibrium points consist of two centers and a saddle.

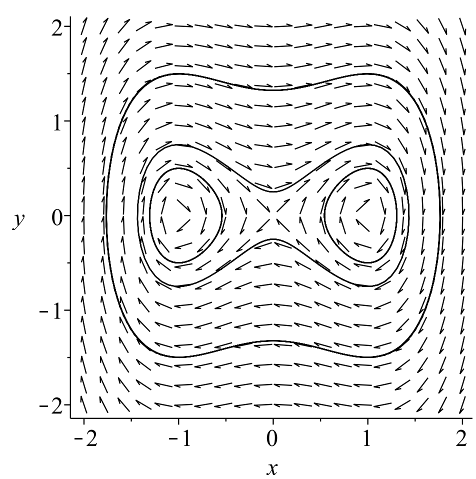



Figure 7.38: Phase plane for the undamped, unforced Duffing equation ($k = 0, \Gamma = 0$).

The unforced Duffing equation.

We now turn on the damping. The system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ky + x(1 - x^2). \end{aligned} \tag{7.74}$$

In Figures 7.39 and 7.40 we show what happens when $k = 0.1$. These plots are reminiscent of the plots for the nonlinear pendulum; however, there are fewer equilibria. Note that the centers become stable spirals for $k > 0$.

The damped, forced Duffing equation.

Next we turn on the forcing to obtain a damped, forced Duffing equation. The system is now nonautonomous.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1 - x^2) + \Gamma \cos \omega t. \end{aligned} \tag{7.75}$$

In Figure 7.41 we only show one orbit with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. The solution intersects itself and look a bit messy. We can imagine what we would get if we added any more orbits. For completeness, we show in Figure 7.42 an example with four different orbits.

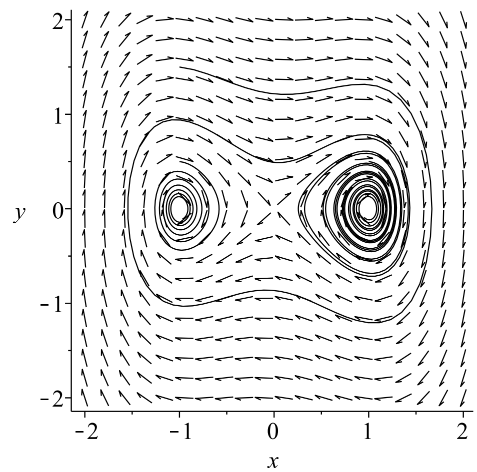



Figure 7.39: Phase plane for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

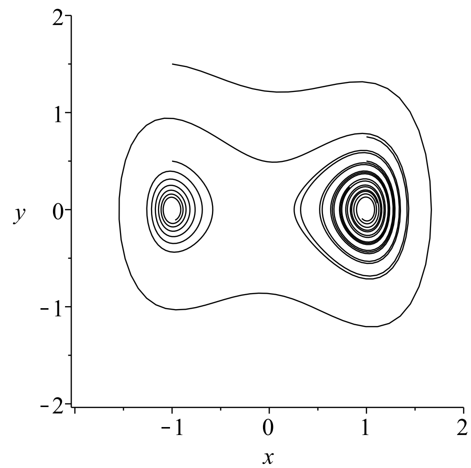



Figure 7.40: Display of two orbits for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

In cases for which one has periodic orbits such as the Duffing equation, Poincaré introduced the notion of *surfaces of section*. One embeds the orbit in a higher dimensional space so that there are no self intersections, like we saw in Figures 7.41 and 7.42. In Figure 7.44 we show an example where a simple orbit is shown as it periodically pierces a given surface.

In order to simplify the resulting pictures, one only plots the points at which the orbit pierces the surface as sketched in Figure 7.43. In practice, there is a natural frequency, such as $\omega$ in the forced Duffing equation. Then, one plots points at times that are multiples of the period, $T = \frac{2\pi}{\omega}$. In Figure 7.45 we show what the plot for one orbit would look like for the damped, unforced Duffing equation.

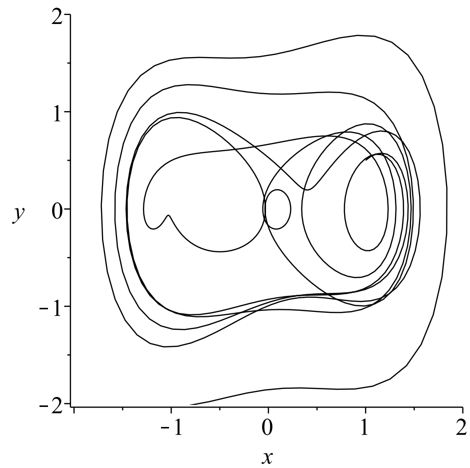



Figure 7.41: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case we show only one orbit which was generated from the initial condition $(x_0 = 1.0, \quad y_0 = 0.5)$.

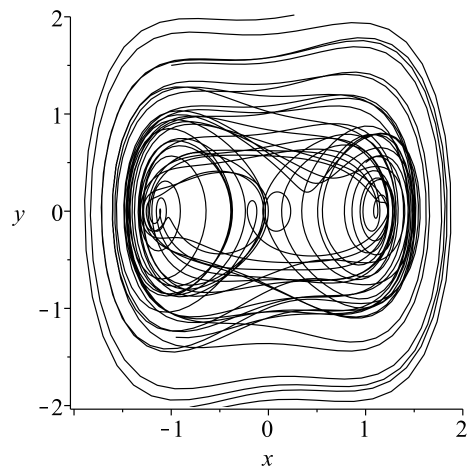



Figure 7.42: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case four initial conditions were used to generate four orbits.

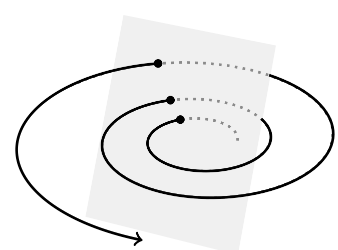


Figure 7.43: As an orbit pierces the surface of section, one plots the point of intersection in that plane to produce the surface of section plot.

The more interesting case, is when there is forcing and damping. In this case the surface of section plot is given in Figure 7.46. While this is not as busy as the solution plot in Figure 7.41, it still provides some interesting behavior. What one finds is what is called a strange attractor. Plotting many orbits, we find that after a long time, all of the orbits are attracted to a small region in the plane, much like a stable node attracts nearby orbits. However, this set consists of more than one point. Also, the flow on the attractor is chaotic in nature. Thus, points wander in an irregular way throughout the attractor. This is one of the interesting topics in chaos theory and this whole theory of dynamical systems has only been touched in this text leaving the reader to wander of into further depth into this fascinating field.

The surface of section plots at the end of the last section were obtained using code from S. Lynch's book *Dynamical Systems with Applications Using Maple*. For reference, the plots in Figures 7.38 and 7.39 were generated in Maple using the following commands:

```
> with(DEtools):
```

Figure 7.44: Poincaré's surface of section. One notes each time the orbit pierces the surface.

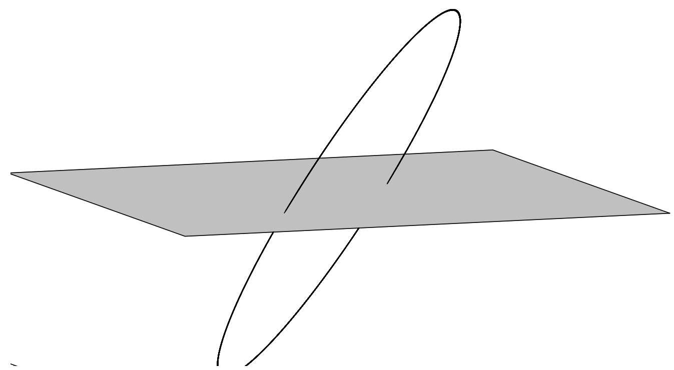


Figure 7.45: Poincaré's surface of section plot for the damped, unforced Duffing equation.

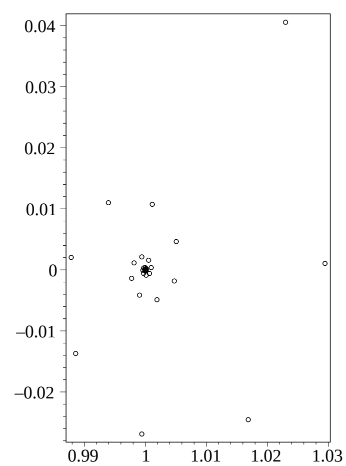
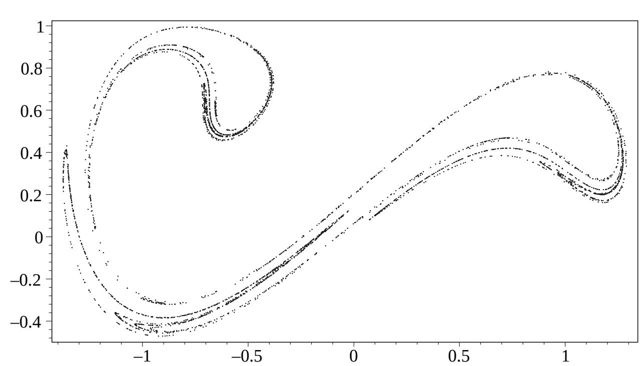



Figure 7.46: Poincaré's surface of section plot for the damped, forced Duffing equation. This leads to what is known as a strange attractor.



```
> Gamma:=0.5:omega:=1.25:k:=0.1:
> DEplot([diff(x(t),t)=y(t), diff(y(t),t)=x(t)-k*y(t)-(x(t))^3
  + Gamma*cos(omega*t)], [x(t),y(t)],t=0..500,[[x(0)=1,y(0)=0.5],
  [x(0)=-1,y(0)=0.5], [x(0)=1,y(0)=0.75], [x(0)=-1,y(0)=1.5]],
  x=-2..2,y=-2..2, stepsize=0.1, linecolor=blue, thickness=1,
  color=black);
```

## 7.10 *Exact Solutions Using Elliptic Functions*

THE SOLUTION OF THE NONLINEAR PENDULUM EQUATION led to the introduction of elliptic integrals. The incomplete elliptic integral of the first kind is defined as

$$F(\phi,k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{7.76}$$

The complete integral of the first kind is given by $K(k) = F(\frac{\pi}{2},k)$, or

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{1} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Recall, a first integration of the nonlinear pendulum equation from Equation (7.18),

$$\left(\frac{d\theta}{dt}\right)^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

or

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega^2\left[\sin^2\frac{\theta}{2} - \sin^2\frac{\theta_0}{2}\right].$$

Letting

$$kz = \sin\frac{\theta}{2} \text{ and } k = \sin\frac{\theta_0}{2},$$

the differential equation becomes

$$\frac{dz}{d\tau} = \pm\omega\sqrt{1-z^2}\sqrt{1-k^2z^2}.$$

Applying separation of variables, we find

$$\begin{aligned} \pm\omega(t-t_0) &= \frac{1}{\omega}\int_1^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} && (7.77)\\ &= \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} - \int_0^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} && (7.78)\\ &= K(k) - F(\sin^{-1}(k^{-1}\sin\theta),k). && (7.79) \end{aligned}$$

The solution, $\theta(t)$, is then found by solving for $z$ and using $kz = \sin\frac{\theta}{2}$ to solve for $\theta$. This requires that we know how to invert the elliptic integral, $F(z,k)$.

The inverse of Equation (7.76) is defined as $\phi = F^{-1}(u,k) = \mathrm{am}(u,k)$, where $u = \sin\phi$. The function $\mathrm{am}(u,k)$ is called the Jacobi amplitude function and $k$ is the elliptic modulus. [In some references and software like

MATLAB packages, $m = k^2$ is used as the parameter.] Three of the Jacobi elliptic functions, shown in Figure 7.47, can be defined in terms of the amplitude function by

$$\operatorname{sn}(u,k) = \sin \operatorname{am}(u,k) = \sin\phi,$$

$$\operatorname{cn}(u,k) = \cos \operatorname{am}(u,k) = \cos\phi,$$

Jacobi elliptic functions.

and the delta amplitude

$$\operatorname{dn}(u,k) = \sqrt{1-k^2\sin^2\phi}.$$

They are related through the identities

$$\operatorname{cn}^2(u,k) + \operatorname{sn}^2(u,k) = 1, \tag{7.80}$$

$$\operatorname{dn}^2(u,k) + k^2\operatorname{sn}^2(u,k) = 1. \tag{7.81}$$

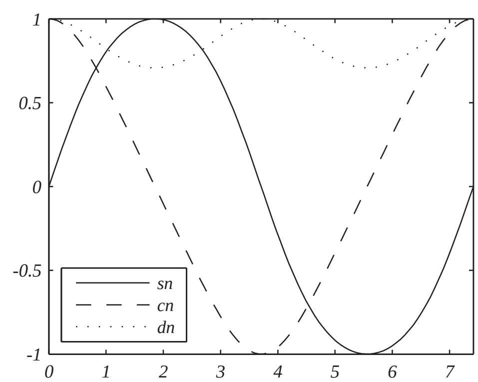



Figure 7.47: Plots of the Jacobi elliptic functions $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, and $\operatorname{dn}(u,k)$ for $m = k^2 = 0.5$. Here $K(k) = 1.8541$.

Also, we see that these functions are periodic. The period is given in terms of the complete elliptic integral of the first kind, $K(k)$. Namely,

The elliptic functions can be extended to the complex plane. In this case the functions are doubly periodic. However, we will not need to consider this in the current text.

$$\operatorname{sn}(u+K,k) = \frac{\operatorname{cn} u}{\operatorname{dn} u}, \quad \operatorname{sn}(u+2K,k) = -\operatorname{sn} u,$$

$$\operatorname{cn}(u+K,k) = -\sqrt{1-k^2}\frac{\operatorname{sn} u}{\operatorname{dn} u}, \quad \operatorname{dn}(u+2K,k) = -\operatorname{cn} u,$$

$$\operatorname{dn}(u+K,k) = \frac{\sqrt{1-k^2}}{\operatorname{dn} u}, \quad \operatorname{dn}(u+2K,k) = \operatorname{dn} u.$$

Therefore, dn and cn have a period of $4K$ and dn has a period of $2K$.

Special values found in Figure 7.47 are seen as

$$\operatorname{sn}(K,k) = 1,$$

$$\operatorname{cn}(K,k) = 0,$$

$$\operatorname{dn}(K,k) = \sqrt{1-k^2} = k',$$

where $k'$ is called the complementary modulus.

Important to this section are the derivatives of these elliptic functions,

$$\frac{\partial}{\partial u}\operatorname{sn}(u,k) = \operatorname{cn}(u,k)\operatorname{dn}(u,k),$$

$$\frac{\partial}{\partial u}\operatorname{cn}(u,k) = -\operatorname{sn}(u,k)\operatorname{dn}(u,k),$$

$$\frac{\partial}{\partial u}\operatorname{dn}(u,k) = -k^2\operatorname{sn}(u,k)\operatorname{cn}(u,k),$$

and the amplitude function

$$\frac{\partial}{\partial u}\operatorname{am}(u,k) = \operatorname{dn}(u,k).$$

Sometimes the Jacobi elliptic functions are displayed without reference to the elliptic modulus, such as $\operatorname{sn}(u) = \operatorname{sn}(u,k)$. When $k$ is understood, we can do the same.

**Example 7.18.** *Show that* $\operatorname{sn}(u)$ *satisfies the differential equation*

$$y'' + (1+k^2)y = 2k^2y^3.$$

*From the above derivatives, we have that*

$$\begin{aligned}\frac{d^2}{du^2}\operatorname{sn}(u) &= \frac{d}{du}(\operatorname{cn}(u)\operatorname{dn}(u)) \\ &= -\operatorname{sn}(u)\operatorname{dn}^2(u) - k^2\operatorname{sn}(u)\operatorname{cn}^2(u). \qquad (7.82)\end{aligned}$$

*Letting* $y(u) = \operatorname{sn}(u)$ *and using the identities (7.80)-(7.81), we have that*

$$y'' = -y(1-k^2y^2) - k^2y(1-y^2) = -(1+k^2)y + 2k^2y^3.$$

*This is the desired result.*

**Example 7.19.** *Show that* $\theta(t) = 2\sin^{-1}(k\operatorname{sn} t)$ *is a solution of the equation* $\ddot{\theta} + \sin\theta = 0$.

*Differentiating* $\theta(t) = 2\sin^{-1}(k\operatorname{sn} t)$, *we have*

$$\begin{aligned}\frac{d^2}{dt^2}\left(2\sin^{-1}(k\operatorname{sn} t)\right) &= \frac{d}{dt}\left(2\frac{k\operatorname{cn} t\operatorname{dn} t}{\sqrt{1-k^2\operatorname{sn}^2 t}}\right) \\ &= \frac{d}{dt}(2k\operatorname{cn} t) \\ &= -2k\operatorname{sn} t\operatorname{dn} t. \qquad (7.83)\end{aligned}$$

*However, we can evaluate* $\sin\theta$ *for a range of* $\theta$. *Thus, we have*

$$\begin{aligned}\sin\theta &= \sin(2\sin^{-1}(k\operatorname{sn} t)) \\ &= 2\sin(\sin^{-1}(k\operatorname{sn} t))\cos(\sin^{-1}(k\operatorname{sn} t)) \\ &= 2k\operatorname{sn} t\sqrt{1-k^2\operatorname{sn}^2 t} \\ &= 2k\operatorname{sn} t\operatorname{dn} t. \qquad (7.84)\end{aligned}$$

*Comparing these results, we have shown that* $\ddot{\theta} + \sin\theta = 0$.

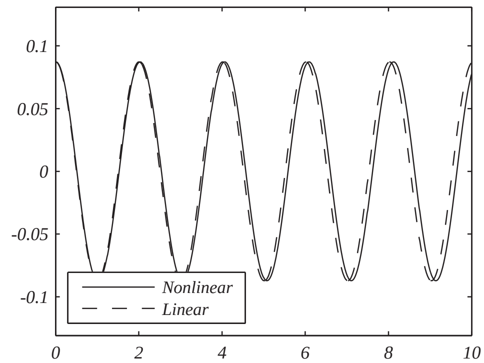



Figure 7.48: Comparison of exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 10^o$.

The solution to the last example can be used to obtain the exact solution to the nonlinear pendulum problem, $\ddot{\theta} + \omega^2 \sin\theta = 0$, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$. The general solution is given by $\theta(t) = 2\sin^{-1}(k \operatorname{sn}(\omega t + \phi))$ where $\phi$ has to be determined from the initial conditions. We note that

$$\begin{aligned} \frac{d \operatorname{sn}(u+K)}{du} &= \operatorname{cn}(u+K)\operatorname{dn}(u+K) \\ &= \left(-\sqrt{1-k^2}\frac{\operatorname{sn} u}{\operatorname{dn} u}\right)\left(\frac{\sqrt{1-k^2}}{\operatorname{dn} u}\right) \\ &= -(1-k^2)\frac{\operatorname{sn} u}{\operatorname{dn}^2 u}. \end{aligned} \tag{7.85}$$

Evaluating at $u = 0$, we have $\operatorname{sn}'(K) = 0$.

Therefore, if we pick $\phi = K$, then $\dot{\theta}(0) = 0$ and the solution is

$$\theta(t) = 2\sin^{-1}(k \operatorname{sn}(\omega t + K)).$$

Furthermore, the other initial value is found to be

$$\theta(0) = 2\sin^{-1}(k \operatorname{sn} K) = 2\sin^{-1} k.$$

Thus, $k = \sin\frac{\theta_0}{2}$, as we had seen in the earlier derivation of the elliptic integral solution. The solution is given as

$$\theta(t) = 2\sin^{-1}(\sin\frac{\theta_0}{2}\operatorname{sn}(\omega t + K)).$$

In Figures 7.48-7.49 we show comparisons of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and initial angles $\theta_0 = 10^o$ and $\theta_0 = 50^o$.

## *Problems*

1. Solve the general logistic problem,

$$\frac{dy}{dt} = ky - cy^2, \quad y(0) = y_0 \tag{7.86}$$

using separation of variables.

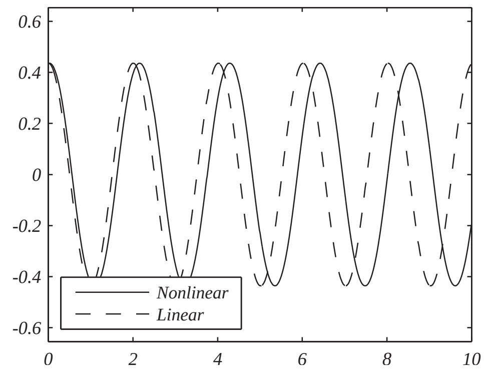



Figure 7.49: Comparison of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 50^o$.

**2.** Find the equilibrium solutions and determine their stability for the following systems. For each case draw representative solutions and phase lines.

a. $y' = y^2 - 6y - 16$.

b. $y' = \cos y$.

c. $y' = y(y-2)(y+3)$.

d. $y' = y^2(y+1)(y-4)$.

**3.** For $y' = y - y^2$, find the general solution corresponding to $y(0) = y_0$. Provide specific solutions for the following initial conditions and sketch them: a. $y(0) = 0.25$, b. $y(0) = 1.5$, and c. $y(0) = -0.5$.

**4.** For each problem determine equilibrium points, bifurcation points and construct a bifurcation diagram. Discuss the different behaviors in each system.

a. $y' = y - \mu y^2$

b. $y' = y(\mu - y)(\mu - 2y)$

c. $x' = \mu - x^3$

d. $x' = x - \frac{\mu x}{1+x^2}$

**5.** Consider the family of differential equations $x' = x^3 + \delta x^2 - \mu x$.

a. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta = 0$.

b. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta > 0$.

Hint: Pick a few values of $\delta$ and $\mu$ in order to get a feel for how this system behaves.

**6.** System 7.63 can be solved exactly. Integrate the $r$-equation using separation of variables. For initial conditions a) $r(0) = 0.25$, $\theta(0) = 0$, and b) $r(0) = 1.5$, $\theta(0) = 0$, and $\mu = 1.0$, find and plot the solutions in the $xy$-plane showing the approach to a limit cycle.

**7.** Consider the system

$$\begin{aligned} x' &= -y + x\left[\mu - x^2 - y^2\right], \\ y' &= x + y\left[\mu - x^2 - y^2\right]. \end{aligned}$$

Rewrite this system in polar form. Look at the behavior of the $r$ equation and construct a bifurcation diagram in $\mu r$ space. What might this diagram look like in the three dimensional $\mu x y$ space? (Think about the symmetry in this problem.) This leads to what is called a *Hopf bifurcation.*

**8.** Find the fixed points of the following systems. Linearize the system about each fixed point and determine the nature and stability in the neighborhood of each fixed point, when possible. Verify your findings by plotting phase portraits using a computer.

a.

$$\begin{aligned} x' &= x(100 - x - 2y), \\ y' &= y(150 - x - 6y). \end{aligned}$$

b.

$$\begin{aligned} x' &= x + x^3, \\ y' &= y + y^3. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - x^2 + xy, \\ y' &= 2y - xy - 6y^2. \end{aligned}$$

d.

$$\begin{aligned} x' &= -2xy, \\ y' &= -x + y + xy - y^3. \end{aligned}$$

**9.** Plot phase portraits for the Lienard system

$$\begin{aligned} x' &= y - \mu(x^3 - x) \\ y' &= -x. \end{aligned}$$

for a small and a not so small value of $\mu$. Describe what happens as one varies $\mu$.

**10.** Consider the period of a nonlinear pendulum. Let the length be $L = 1.0$ m and $g = 9.8\ \mathrm{m/s^2}$. Sketch $T$ vs the initial angle $\theta_0$ and compare the linear and nonlinear values for the period. For what angles can you use the linear approximation confidently?

**11.** Another population model is one in which species compete for resources, such as a limited food supply. Such a model is given by

$$x' = ax - bx^2 - cxy,$$
$$y' = dy - ey^2 - fxy.$$

In this case, assume that all constants are positive.

a Describe the effects/purpose of each terms.

b Find the fixed points of the model.

c Linearize the system about each fixed point and determine the stability.

d From the above, describe the types of solution behavior you might expect, in terms of the model.

**12.** Consider a model of a food chain of three species. Assume that each population on its own can be modeled by logistic growth. Let the species be labeled by $x(t)$, $y(t)$, and $z(t)$. Assume that population $x$ is at the bottom of the chain. That population will be depleted by population $y$. Population $y$ is sustained by $x$'s, but eaten by $z$'s. A simple, but scaled, model for this system can be given by the system

$$\begin{aligned} x' &= x(1-x) - xy \\ y' &= y(1-y) + xy - yz \\ z' &= z(1-z) + yz. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Find the eigenvalues and eigenvectors.

d. Describe the solution behavior near each equilibrium point.

e. Which of these equilibria are important in the study of the population model and describe the interactions of the species in the neighborhood of these point(s).

**13.** Derive the first integral of the Lotka-Volterra system, $a \ln y + d \ln x - cx - by = C$.

**14.** Show that the system $x' = x - y - x^3$, $y' = x + y - y^3$, has a unique limit cycle by picking an appropriate $\psi(x, y)$ in Dulac's Criteria.

**15.** The Lorenz model is a simple model for atmospheric convection developed by Edward Lorenz in 1963. The system is given by the three equations

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Determine any bifurcation points and describe what happens near the bifurcation point(s). Consider $\sigma = 10$, $\beta = 8/3$, and vary $\rho$.

d. This system is know to exhibit chaotic behavior. Lorenz found a so-called strange attractor for parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. Using a computer, locate this strange attractor.

**16.** The Michaelis-Menten kinetics reaction is given by

$$\boxed{E + S \underset{k_1}{\overset{k_3}{\longleftrightarrow}} ES \xrightarrow[k_2]{} E + P.}$$

The resulting system of equations for the chemical concentrations is

$$\begin{aligned} \frac{d[S]}{dt} &= -k_1[E][S] + k_3[ES], \\ \frac{d[E]}{dt} &= -k_1[E][S] + (k_2 + k_2)[ES], \\ \frac{d[ES]}{dt} &= k_1[E][S] - (k_2 + k_2)[ES], \\ \frac{d[P]}{dt} &= k_3[ES]. \end{aligned} \tag{7.87}$$

In chemical kinetics one seeks to determine the rate of product formation ($v = d[P]/dt = k_3[ES]$). Assuming that $[ES]$ is a constant, find $v$ as a function of $[S]$ and the total enzyme concentration $[E_T] = [E] + [ES]$. As a nonlinear dynamical system, what are the equilibrium points?

**17.** In Equation (6.58) we saw a linear version of an epidemic model. The commonly used nonlinear SIR model is given by

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I, \end{aligned} \tag{7.88}$$

where $S$ is the number of susceptible individuals, $I$ is the number of infected individuals, and $R$ are the number who have been removed from the the other groups, either by recovering or dying.

a. Let $N = S + I + R$ be the total population. Prove that $N =$ constant. Thus, one need only solve the first two equations and find $R = N - S - I$ afterwards.

b. Find and classify the equilibria. Describe the equilibria in terms of the population behavior.

c. Let $\beta = 0.05$ and $\gamma = 0.2$. Assume that in a population of 100 there is one infected person. Numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

d. The equations can be modified by adding constant birth and death rates. Assuming these are te same, one would have a new system.

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI + \mu(N - S) \\ \frac{dI}{dt} &= \beta SI - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned} \tag{7.89}$$

How does this affect any equilibrium solutions?

e. Again, let $\beta = 0.05$ and $\gamma = 0.2$. Let $\mu = 0.1$ For a population of 100 with one infected person numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

**18.** An undamped, unforced Duffing equation, $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$, can be solved exactly in terms of elliptic functions. Using the results of Exercise 7.18, determine the solution of this equation and determine if there are any restrictions on the parameters.

**19.** Determine the circumference of an ellipse in terms of an elliptic integral.

**20.** Evaluate the following in terms of elliptic integrals and compute the values to four decimal places.

a. $\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$.

b. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}}$.

c. $\int_0^{2} \frac{dx}{\sqrt{(9-x^2)(4-x^2)}}$.

d. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$.

e. $\int_1^{\infty} \frac{dx}{\sqrt{x^4-1}}$.

# A
# Calculus Review

*"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." Bertrand Russell (1872-1970)*

Before you begin our study of differential equations perhaps you should review some things from calculus. You definitely need to know something before taking this class. It is assumed that you have taken Calculus and are comfortable with differentiation and integration. Of course, you are not expected to know every detail from these courses. However, there are some topics and methods that will come up and it would be useful to have a handy reference to what it is you should know.

Most importantly, you should still have your calculus text to which you can refer throughout the course. Looking back on that old material, you will find that it appears easier than when you first encountered the material. That is the nature of learning mathematics and other subjects. Your understanding is continually evolving as you explore topics more in depth. It does not always sink in the first time you see it. In this chapter we will give a quick review of these topics. We will also mention a few new methods that might be interesting.

## A.1 What Do I Need To Know From Calculus?

### A.1.1 Introduction

There are two main topics in calculus: derivatives and integrals. You learned that derivatives are useful in providing rates of change in either time or space. Integrals provide areas under curves, but also are useful in providing other types of sums over continuous bodies, such as lengths, areas, volumes, moments of inertia, or flux integrals. In physics, one can look at graphs of position versus time and the slope (derivative) of such a function gives the velocity. (See Figure A.1.) By plotting velocity versus time you can either look at the derivative to obtain acceleration, or you could look

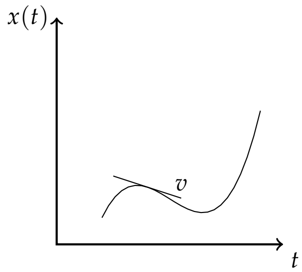



Figure A.1: Plot of position vs time.

at the area under the curve and get the displacement:

$$x = \int_{t_0}^{t} v\, dt. \tag{A.1}$$

This is shown in Figure A.2.

Of course, you need to know how to differentiate and integrate given functions. Even before getting into differentiation and integration, you need to have a bag of functions useful in physics. Common functions are the polynomial and rational functions. You should be fairly familiar with these. Polynomial functions take the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{A.2}$$

where $a_n \neq 0$. This is the form of a polynomial of degree $n$. Rational functions, $f(x) = \frac{g(x)}{h(x)}$, consist of ratios of polynomials. Their graphs can exhibit vertical and horizontal asymptotes.

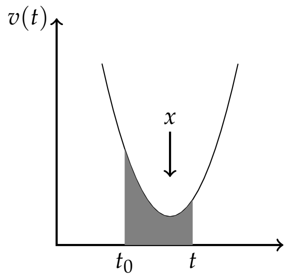



Figure A.2: Plot of velocity vs time.

Next are the exponential and logarithmic functions. The most common are the natural exponential and the natural logarithm. The natural exponential is given by $f(x) = e^x$, where $e \approx 2.718281828\ldots$. The natural logarithm is the inverse to the exponential, denoted by $\ln x$. (One needs to be careful, because some mathematics and physics books use log to mean natural exponential, whereas many of us were first trained to use this notation to mean the common logarithm, which is the 'log base 10'. Here we will use $\ln x$ for the natural logarithm.)

Exponential properties.

The properties of the exponential function follow from the basic properties for exponents. Namely, we have:

$$\boxed{\begin{aligned} e^0 &= 1, && \text{(A.3)}\\ e^{-a} &= \frac{1}{e^a} && \text{(A.4)}\\ e^a e^b &= e^{a+b}, && \text{(A.5)}\\ (e^a)^b &= e^{ab}. && \text{(A.6)} \end{aligned}}$$

The relation between the natural logarithm and natural exponential is given by

$$y = e^x \Leftrightarrow x = \ln y. \tag{A.7}$$

Logarithmic properties.

Some common logarithmic properties are

$$\boxed{\begin{aligned} \ln 1 &= 0, && \text{(A.8)}\\ \ln \frac{1}{a} &= -\ln a, && \text{(A.9)}\\ \ln(ab) &= \ln a + \ln b, && \text{(A.10)}\\ \ln \frac{a}{b} &= \ln a - \ln b, && \text{(A.11)}\\ \ln \frac{1}{b} &= -\ln b. && \text{(A.12)} \end{aligned}}$$

We will see applications of these relations as we progress through the course.

### A.1.2 Trigonometric Functions

ANOTHER SET OF USEFUL FUNCTIONS are the trigonometric functions. These functions have probably plagued you since high school. They have their origins as far back as the building of the pyramids. Typical applications in your introductory math classes probably have included finding the heights of trees, flag poles, or buildings. It was recognized a long time ago that similar right triangles have fixed ratios of any pair of sides of the two similar triangles. These ratios only change when the non-right angles change.

Thus, the ratio of two sides of a right triangle only depends upon the angle. Since there are six possible ratios (think about it!), then there are six possible functions. These are designated as sine, cosine, tangent and their reciprocals (cosecant, secant and cotangent). In your introductory physics class, you really only needed the first three. You also learned that they are represented as the ratios of the opposite to hypotenuse, adjacent to hypotenuse, etc. Hopefully, you have this down by now.

You should also know the exact values of these basic trigonometric functions for the special angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, and their corresponding angles in the second, third and fourth quadrants. This becomes internalized after much use, but we provide these values in Table A.1 just in case you need a reminder.

Table A.1: Table of Trigonometric Values

| $\theta$ | $\cos\theta$ | $\sin\theta$ | $\tan\theta$ |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| $\frac{\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $\frac{\pi}{3}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| $\frac{\pi}{2}$ | 0 | 1 | undefined |

The problems students often have using trigonometric functions in later courses stem from using, or recalling, identities. We will have many an occasion to do so in this class as well. What is an identity? It is a relation that holds true all of the time. For example, the most common identity for trigonometric functions is the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1. \tag{A.13}$$

This holds true for every angle $\theta$! An even simpler identity is

$$\tan\theta = \frac{\sin\theta}{\cos\theta}. \tag{A.14}$$

Other simple identities can be derived from the Pythagorean identity. Dividing the identity by $\cos^2\theta$, or $\sin^2\theta$, yields

$$\boxed{\begin{aligned} \tan^2\theta + 1 &= \sec^2\theta, && \text{(A.15)} \\ 1 + \cot^2\theta &= \csc^2\theta. && \text{(A.16)} \end{aligned}}$$

Sum and difference identities.

Several other useful identities stem from the use of the sine and cosine of the sum and difference of two angles. Namely, we have that

$$\boxed{\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A, & \text{(A.17)}\\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B. & \text{(A.18)} \end{aligned}}$$

Note that the upper (lower) signs are taken together.

**Example A.1.** *Evaluate* $\sin \frac{\pi}{12}$.

$$\begin{aligned} \sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \frac{1}{2} \\ &= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right). \end{aligned} \tag{A.19}$$

Double angle formulae.

The double angle formulae are found by setting $A = B$ :

$$\boxed{\begin{aligned} \sin(2A) &= 2 \sin A \cos B, & \text{(A.20)}\\ \cos(2A) &= \cos^2 A - \sin^2 A. & \text{(A.21)} \end{aligned}}$$

Using Equation (A.13), we can rewrite (A.21) as

$$\begin{aligned} \cos(2A) &= 2\cos^2 A - 1, & \text{(A.22)}\\ &= 1 - 2\sin^2 A. & \text{(A.23)} \end{aligned}$$

Half angle formulae.

These, in turn, lead to the half angle formulae. Solving for $\cos^2 A$ and $\sin^2 A$, we find that

$$\boxed{\begin{aligned} \sin^2 A &= \frac{1 - \cos 2A}{2}, & \text{(A.24)}\\ \cos^2 A &= \frac{1 + \cos 2A}{2}. & \text{(A.25)} \end{aligned}}$$

**Example A.2.** *Evaluate* $\cos \frac{\pi}{12}$. *In the last example, we used the sum/difference identities to evaluate a similar expression. We could have also used a half angle identity. In this example, we have*

$$\begin{aligned} \cos^2 \frac{\pi}{12} &= \frac{1}{2}\left(1 + \cos \frac{\pi}{6}\right) \\ &= \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{4}\left(2 + \sqrt{3}\right) \end{aligned} \tag{A.26}$$

*So,* $\cos\frac{\pi}{12} = \frac{1}{2}\sqrt{2+\sqrt{3}}$*. This is not the simplest form and is called a nested radical. In fact, if we proceeded using the difference identity for cosines, then we would obtain*

$$\cos\frac{\pi}{12} = \frac{\sqrt{2}}{4}(1+\sqrt{3}).$$

*So, how does one show that these answers are the same?*

*Let's focus on the factor* $\sqrt{2+\sqrt{3}}$*. We seek to write this in the form* $c + d\sqrt{3}$*. Equating the two expressions and squaring, we have*

$$\begin{aligned} 2+\sqrt{3} &= (c+d\sqrt{3})^2 \\ &= c^2 + 3d^2 + 2cd\sqrt{3}. \end{aligned} \tag{A.27}$$

It is useful at times to know when one can reduce square roots of such radicals, called denesting. More generally, one seeks to write $\sqrt{a+b\sqrt{q}} = c + d\sqrt{q}$. Following the procedure in this example, one has $d = \frac{b}{2c}$ and

$$c^2 = \frac{1}{2}\left(a \pm \sqrt{a^2 - qb^2}\right).$$

As long as $a^2 - qb^2$ is a perfect square, there is a chance to reduce the expression to a simpler form.

*In order to solve for* $c$ *and* $d$*, it would seem natural to equate the coefficients of* $\sqrt{3}$ *and the remaining terms. We obtain a system of two nonlinear algebraic equations,*

$$c^2 + 3d^2 = 2 \tag{A.28}$$

$$2cd = 1. \tag{A.29}$$

*Solving the second equation for* $d = 1/2c$*, and substituting the result into the first equation, we find*

$$4c^4 - 8c^2 + 3 = 0.$$

*This fourth order equation has four solutions,*

$$c = \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{6}}{2}$$

*and*

$$b = \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{6}}{6}.$$

*Thus,*

$$\begin{aligned} \cos\frac{\pi}{12} &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\ &= \pm\frac{1}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\sqrt{3}\right) \\ &= \pm\frac{\sqrt{2}}{4}(1+\sqrt{3}) \end{aligned} \tag{A.30}$$

*and*

$$\begin{aligned} \cos\frac{\pi}{12} &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\ &= \pm\frac{1}{2}\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{6}\sqrt{3}\right) \\ &= \pm\frac{\sqrt{6}}{12}(3+\sqrt{3}). \end{aligned} \tag{A.31}$$

*Of the four solutions, two are negative and we know the value of the cosine for this angle has to be positive. The remaining two solutions are actually equal! A quick* *computation will verify this:*

$$\begin{aligned}\frac{\sqrt{6}}{12}(3+\sqrt{3}) &= \frac{\sqrt{3}\sqrt{2}}{12}(3+\sqrt{3}) \\ &= \frac{\sqrt{2}}{12}(3\sqrt{3}+3) \\ &= \frac{\sqrt{2}}{4}(\sqrt{3}+1). \end{aligned} \tag{A.32}$$

*We could have bypassed this situation be requiring that the solutions for b and c were not simply proportional to $\sqrt{3}$ like they are in the second case.*

Product Identities

Finally, another useful set of identities are the product identities. For example, if we add the identities for $\sin(A+B)$ and $\sin(A-B)$, the second terms cancel and we have

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B.$$

Thus, we have that

$$\boxed{\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B)).} \tag{A.33}$$

Similarly, we have

$$\boxed{\cos A\cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B)).} \tag{A.34}$$

and

$$\boxed{\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B)).} \tag{A.35}$$

Know the above boxed identities!

These boxed equations are the most common trigonometric identities. They appear often and should just roll off of your tongue.

We will also need to understand the behaviors of trigonometric functions. In particular, we know that the sine and cosine functions are periodic. They are not the only periodic functions, as we shall see. [Just visualize the teeth on a carpenter's saw.] However, they are the most common periodic functions.

Periodic functions.

A periodic function $f(x)$ satisfies the relation

$$f(x+p) = f(x), \quad \text{for all } x$$

for some constant $p$. If $p$ is the smallest such number, then $p$ is called the period. Both the sine and cosine functions have period $2\pi$. This means that the graph repeats its form every $2\pi$ units. Similarly, $\sin bx$ and $\cos bx$ have the common period $p = \frac{2\pi}{b}$. We will make use of this fact in later chapters.

In Feynman's *Surely You're Joking Mr. Feynman!*, Richard Feynman (1918-1988) talks about his invention of his own notation for both trigonometric and inverse trigonometric functions as the standard notation did not make sense to him.

Related to these are the inverse trigonometric functions. For example, $f(x) = \sin^{-1} x$, or $f(x) = \arcsin x$. Inverse functions give back angles, so you should think

$$\theta = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin\theta. \tag{A.36}$$

Also, you should recall that $y = \sin^{-1} x = \arcsin x$ is only a function if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Similar relations exist for $y = \cos^{-1} x = \arccos x$ and $\tan^{-1} x = \arctan x$.

Once you think about these functions as providing angles, then you can make sense out of more complicated looking expressions, like $\tan(\sin^{-1} x)$. Such expressions often pop up in evaluations of integrals. We can untangle this in order to produce a simpler form by referring to expression (A.36). $\theta = \sin^{-1} x$ is simple an angle whose sine is $x$. Knowing the sine is the opposite side of a right triangle divided by its hypotenuse, then one just draws a triangle in this proportion as shown in Figure A.3. Namely, the side opposite the angle has length $x$ and the hypotenuse has length 1. Using the Pythagorean Theorem, the missing side (adjacent to the angle) is simply $\sqrt{1-x^2}$. Having obtained the lengths for all three sides, we can now produce the tangent of the angle as

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}.$$

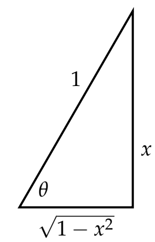


Figure A.3: $\theta = \sin^{-1} x \Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}}$

### A.1.3 Hyperbolic Functions

So, ARE THERE ANY OTHER FUNCTIONS that are useful in physics? Actually, there are many more. However, you have probably not see many of them to date. We will see by the end of the semester that there are many important functions that arise as solutions of some fairly generic, but important, physics problems. In your calculus classes you have also seen that some relations are represented in parametric form. However, there is at least one other set of elementary functions, which you should already know about. These are the hyperbolic functions. Such functions are useful in representing hanging cables, unbounded orbits, and special traveling waves called solitons. They also play a role in special and general relativity.

Solitons are special solutions to some generic nonlinear wave equations. They typically experience elastic collisions and play special roles in a variety of fields in physics, such as hydrodynamics and optics. A simple soliton solution is of the form

$$u(x,t) = 2\eta^2 \operatorname{sech}^2 \eta(x - 4\eta^2 t).$$

Hyperbolic functions.

We recall a few definitions and identities of hyperbolic functions: the hyperbolic sine and hyperbolic cosine (shown in Figure A.4):

$$\boxed{\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, && \text{(A.37)} \\ \cosh x &= \frac{e^x + e^{-x}}{2}. && \text{(A.38)} \end{aligned}}$$

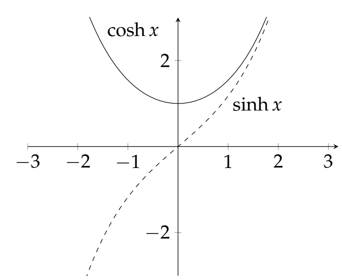


Figure A.4: Plots of $\cosh x$ and $\sinh x$. Note that $\sinh 0 = 0$, $\cosh 0 = 1$, and $\cosh x \geq 1$.

Hyperbolic functions are related to the trigonometric functions. We can see this from the relations

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \tag{A.39}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{A.40}$$

. Letting $\theta = ix$ we have $\sin ix = i \sinh x$ and $\cos ix = \cosh x$..

There are four other hyperbolic functions. These are defined in terms of the above functions similar to the relations between the trigonometric functions. Namely, just as all of the trigonometric functions can be built from the sine and the cosine, the hyperbolic functions can be defined in terms of the hyperbolic sine and hyperbolic cosine. We have

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{A.41}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{A.42}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \tag{A.43}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \tag{A.44}$$

There are also a whole set of identities, similar to those for the trigonometric functions. For example, the Pythagorean identity for trigonometric functions, $\sin^2\theta + \cos^2\theta = 1$, is replaced by the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

This is easily shown by simply using the definitions of these functions. This identity is also useful for providing a parametric set of equations describing hyperbolae. Letting $x = a\cosh t$ and $y = b\sinh t$, one has

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2 t - \sinh^2 t = 1.$$

Hyperbolic identities.

A list of commonly needed hyperbolic function identities are given by the following:

$$\cosh^2 x - \sinh^2 x = 1, \tag{A.45}$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1, \tag{A.46}$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B, \tag{A.47}$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A, \tag{A.48}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x, \tag{A.49}$$

$$\sinh 2x = 2\sinh x \cosh x, \tag{A.50}$$

$$\cosh^2 x = \frac{1}{2}\left(1 + \cosh 2x\right), \tag{A.51}$$

$$\sinh^2 x = \frac{1}{2}\left(\cosh 2x - 1\right). \tag{A.52}$$

Note the similarity with the trigonometric identities. Other identities can be derived from these.

There also exist inverse hyperbolic functions and these can be written in terms of logarithms. As with the inverse trigonometric functions, we begin with the definition

$$y = \sinh^{-1} x \quad \Leftrightarrow \quad x = \sinh y. \tag{A.53}$$

The aim is to write $y$ in terms of $x$ without using the inverse function. First, we note that

$$x = \frac{1}{2}\left(e^y - e^{-y}\right). \tag{A.54}$$

Next we solve for $e^y$. This is done by noting that $e^{-y} = \frac{1}{e^y}$ and rewriting the previous equation as

$$0 = (e^y)^2 - 2xe^y - 1. \tag{A.55}$$

This equation is in quadratic form which we can solve using the quadratic formula as

$$e^y = x + \sqrt{1+x^2}.$$

(There is only one root as we expect the exponential to be positive.)

The final step is to solve for $y$,

$$y = \ln\left(x + \sqrt{1+x^2}\right). \tag{A.56}$$

The inverse hyperbolic functions care given by

$$\begin{aligned} \sinh^{-1} x &= \ln\left(x + \sqrt{1+x^2}\right), \\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right), \\ \tanh^{-1} x &= \frac{1}{2}\ln\frac{1+x}{1-x}. \end{aligned}$$

### *A.1.4 Derivatives*

Now that we know some elementary functions, we seek their derivatives. We will not spend time exploring the appropriate limits in any rigorous way. We are only interested in the results. We provide these in Table A.2. We expect that you know the meaning of the derivative and all of the usual rules, such as the product and quotient rules.

| Function | Derivative |
|---|---|
| $a$ | $0$ |
| $x^n$ | $nx^{n-1}$ |
| $e^{ax}$ | $ae^{ax}$ |
| $\ln ax$ | $\frac{1}{x}$ |
| $\sin ax$ | $a\cos ax$ |
| $\cos ax$ | $-a\sin ax$ |
| $\tan ax$ | $a\sec^2 ax$ |
| $\csc ax$ | $-a\csc ax\cot ax$ |
| $\sec ax$ | $a\sec ax\tan ax$ |
| $\cot ax$ | $-a\csc^2 ax$ |
| $\sinh ax$ | $a\cosh ax$ |
| $\cosh ax$ | $a\sinh ax$ |
| $\tanh ax$ | $a\,\mathrm{sech}^2\, ax$ |
| $\mathrm{csch}\, ax$ | $-a\,\mathrm{csch}\, ax\coth ax$ |
| $\mathrm{sech}\, ax$ | $-a\,\mathrm{sech}\, ax\tanh ax$ |
| $\coth ax$ | $-a\,\mathrm{csch}^2\, ax$ |

Table A.2: Table of Common Derivatives ($a$ is a constant).

Also, you should be familiar with the Chain Rule. Recall that this rule tells us that if we have a composition of functions, such as the elementary functions above, then we can compute the derivative of the composite function. Namely, if $h(x) = f(g(x))$, then

$$\frac{dh}{dx} = \frac{d}{dx}\left(f(g(x))\right) = \left.\frac{df}{dg}\right|_{g(x)}\frac{dg}{dx} = f'(g(x))g'(x). \tag{A.57}$$

**Example A.3.** *Differentiate* $H(x) = 5\cos\left(\pi \tanh 2x^2\right)$.

*This is a composition of three functions,* $H(x) = f(g(h(x)))$*, where* $f(x) = 5\cos x$, $g(x) = \pi \tanh x$*, and* $h(x) = 2x^2$*. Then the derivative becomes*

$$\begin{aligned} H'(x) &= 5\left(-\sin\left(\pi\tanh 2x^2\right)\right)\frac{d}{dx}\left(\left(\pi\tanh 2x^2\right)\right) \\ &= -5\pi\sin\left(\pi\tanh 2x^2\right)\operatorname{sech}^2 2x^2\frac{d}{dx}\left(2x^2\right) \\ &= -20\pi x\sin\left(\pi\tanh 2x^2\right)\operatorname{sech}^2 2x^2. \end{aligned} \tag{A.58}$$

### A.1.5 Integrals

Integration is typically a bit harder. Imagine being given the last result in (A.58) and having to figure out what was differentiated in order to get the given function. As you may recall from the Fundamental Theorem of Calculus, the integral is the inverse operation to differentiation:

$$\int \frac{df}{dx}\,dx = f(x) + C. \tag{A.59}$$

It is not always easy to evaluate a given integral. In fact some integrals are not even doable! However, you learned in calculus that there are some methods that could yield an answer. While you might be happier using a computer algebra system, such as Maple or WolframAlpha.com, or a fancy calculator, you should know a few basic integrals and know how to use tables for some of the more complicated ones. In fact, it can be exhilarating when you can do a given integral without reference to a computer or a Table of Integrals. However, you should be prepared to do some integrals using what you have been taught in calculus. We will review a few of these methods and some of the standard integrals in this section.

First of all, there are some integrals you are expected to know without doing any work. These integrals appear often and are just an application of the Fundamental Theorem of Calculus to the previous Table A.2. The basic integrals that students should know off the top of their heads are given in Table A.3.

These are not the only integrals you should be able to do. We can expand the list by recalling a few of the techniques that you learned in calculus, the Method of Substitution, Integration by Parts, integration using partial fraction decomposition, and trigonometric integrals, and trigonometric substitution. There are also a few other techniques that you had not seen before. We will look at several examples.

**Example A.4.** *Evaluate* $\int \frac{x}{\sqrt{x^2+1}}\,dx$.

*When confronted with an integral, you should first ask if a simple substitution would reduce the integral to one you know how to do.*

*The ugly part of this integral is the $x^2+1$ under the square root. So, we let $u = x^2 + 1$.*

*Noting that when $u = f(x)$, we have $du = f'(x)\,dx$. For our example, $du = 2x\,dx$.*

*Looking at the integral, part of the integrand can be written as $x\,dx = \frac{1}{2}u\,du$. Then, the integral becomes*

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}.$$

*The substitution has converted our integral into an integral over $u$. Also, this integral is doable! It is one of the integrals we should know. Namely, we can write it as*

$$\frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du.$$

*This is now easily finished after integrating and using the substitution variable to give*

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + C = \sqrt{x^2+1} + C.$$

*Note that we have added the required integration constant and that the derivative of the result easily gives the original integrand (after employing the Chain Rule).*

| Function | Indefinite Integral |
|---|---|
| $a$ | $ax$ |
| $x^n$ | $\frac{x^{n+1}}{n+1}$ |
| $e^{ax}$ | $\frac{1}{a}e^{ax}$ |
| $\frac{1}{x}$ | $\ln x$ |
| $\sin ax$ | $-\frac{1}{a}\cos ax$ |
| $\cos ax$ | $\frac{1}{a}\sin ax$ |
| $\sec^2 ax$ | $\frac{1}{a}\tan ax$ |
| $\sinh ax$ | $\frac{1}{a}\cosh ax$ |
| $\cosh ax$ | $\frac{1}{a}\sinh ax$ |
| $\operatorname{sech}^2 ax$ | $\frac{1}{a}\tanh ax$ |
| $\sec x$ | $\ln\lvert\sec x + \tan x\rvert$ |
| $\frac{1}{a+bx}$ | $\frac{1}{b}\ln(a+bx)$ |
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a}\tan^{-1}\frac{x}{a}$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\frac{x}{a}$ |
| $\frac{1}{x\sqrt{x^2-a^2}}$ | $\frac{1}{a}\sec^{-1}\frac{x}{a}$ |
| $\frac{1}{\sqrt{x^2-a^2}}$ | $\cosh^{-1}\frac{x}{a} = \ln\lvert\sqrt{x^2-a^2}+x\rvert$ |

Table A.3: Table of Common Integrals.

Often we are faced with definite integrals, in which we integrate between two limits. There are several ways to use these limits. However, students often forget that a change of variables generally means that the limits have to change.

**Example A.5.** *Evaluate* $\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx$.

*This is the last example but with integration limits added. We proceed as before. We let* $u = x^2+1$. *As* $x$ *goes from 0 to 2,* $u$ *takes values from 1 to 5. So, this substitution gives*

$$\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int_1^5 \frac{du}{\sqrt{u}} = \sqrt{u}\big|_1^5 = \sqrt{5}-1.$$

When you becomes proficient at integration, you can bypass some of these steps. In the next example we try to demonstrate the thought process involved in using substitution without explicitly using the substitution variable.

**Example A.6.** *Evaluate* $\int_0^2 \frac{x}{\sqrt{9+4x^2}}\,dx$

*As with the previous example, one sees that the derivative of* $9+4x^2$ *is proportional to* $x$*, which is in the numerator of the integrand. Thus a substitution would give an integrand of the form* $u^{-1/2}$*. So, we expect the answer to be proportional to* $\sqrt{u} = \sqrt{9+4x^2}$*. The starting point is therefore,*

$$\int \frac{x}{\sqrt{9+4x^2}}\,dx = A\sqrt{9+4x^2},$$

*where* $A$ *is a constant to be determined.*

*We can determine* $A$ *through differentiation since the derivative of the answer should be the integrand. Thus,*

$$\begin{aligned}\frac{d}{dx}A(9+4x^2)^{\frac{1}{2}} &= A(9+4x^2)^{-\frac{1}{2}}\left(\frac{1}{2}\right)(8x)\\ &= 4xA(9+4x^2)^{-\frac{1}{2}}.\end{aligned} \tag{A.60}$$

*Comparing this result with the integrand, we see that the integrand is obtained when* $A = \frac{1}{4}$*. Therefore,*

$$\int \frac{x}{\sqrt{9+4x^2}}\,dx = \frac{1}{4}\sqrt{9+4x^2}.$$

*We now complete the integral,*

$$\int_0^2 \frac{x}{\sqrt{9+4x^2}}\,dx = \frac{1}{4}[5-3] = \frac{1}{2}.$$

The function

$$\mathrm{gd}(x) = \int_0^x \frac{dx}{\cosh x} = 2\tan^{-1} e^x - \frac{\pi}{2}$$

is called the Gudermannian and connects trigonometric and hyperbolic functions. This function was named after Christoph Gudermann (1798-1852), but introduced by Johann Heinrich Lambert (1728-1777), who was one of the first to introduce hyperbolic functions.

**Example A.7.** *Evaluate* $\int \frac{dx}{\cosh x}$.

*This integral can be performed by first using the definition of* $\cosh x$ *followed by a simple substitution.*

$$\begin{aligned}\int \frac{dx}{\cosh x} &= \int \frac{2}{e^x+e^{-x}}\,dx\\ &= \int \frac{2e^x}{e^{2x}+1}\,dx.\end{aligned} \tag{A.61}$$

*Now, we let* $u = e^x$ *and* $du = e^x dx$*. Then,*

$$\begin{aligned}\int \frac{dx}{\cosh x} &= \int \frac{2}{1+u^2}\,du\\ &= 2\tan^{-1} u + C\\ &= 2\tan^{-1} e^x + C.\end{aligned} \tag{A.62}$$

**Integration by Parts**

When the Method of Substitution fails, there are other methods you can try. One of the most used is the Method of Integration by Parts. Recall the Integration by Parts Formula:

Integration by Parts Formula.

$$\boxed{\int u\,dv = uv - \int v\,du. \qquad \text{(A.63)}}$$

The idea is that you are given the integral on the left and you can relate it to an integral on the right. Hopefully, the new integral is one you can do, or at least it is an easier integral than the one you are trying to evaluate.

However, you are not usually given the functions $u$ and $v$. You have to determine them. The integral form that you really have is a function of another variable, say $x$. Another form of the Integration by Parts Formula can be written as

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx. \qquad \text{(A.64)}}$$

This form is a bit more complicated in appearance, though it is clearer than the $u$-$v$ form as to what is happening. The derivative has been moved from one function to the other. Recall that this formula was derived by integrating the product rule for differentiation. (See your calculus text.)

**Note:** Often in physics one needs to move a derivative between functions inside an integrand. The key - use integration by parts to move the derivative from one function to the other under an integral.

These two formulae can be related by using the differential relations

$$\begin{aligned} u = f(x) \quad &\to \quad du = f'(x)\,dx, \\ v = g(x) \quad &\to \quad dv = g'(x)\,dx. \end{aligned} \qquad \text{(A.65)}$$

This also gives a method for applying the Integration by Parts Formula.

**Example A.8.** *Consider the integral* $\int x \sin 2x\,dx$. *We choose* $u = x$ *and* $dv = \sin 2x\,dx$. *This gives the correct left side of the Integration by Parts Formula. We next determine* $v$ *and* $du$*:*

$$du = \frac{du}{dx}dx = dx,$$

$$v = \int dv = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x.$$

*We note that one usually does not need the integration constant. Inserting these expressions into the Integration by Parts Formula, we have*

$$\int x \sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx.$$

*We see that the new integral is easier to do than the original integral. Had we picked* $u = \sin 2x$ *and* $dv = x\,dx$, *then the formula still works, but the resulting integral is not easier.*

*For completeness, we finish the integration. The result is*

$$\int x \sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

*As always, you can check your answer by differentiating the result, a step students often forget to do. Namely,*

$$\begin{aligned} \frac{d}{dx}\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C\right) &= -\frac{1}{2}\cos 2x + x\sin 2x + \frac{1}{4}(2\cos 2x) \\ &= x\sin 2x. \end{aligned} \tag{A.66}$$

*So, we do get back the integrand in the original integral.*

We can also perform integration by parts on definite integrals. The general formula is written as

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx. \qquad \text{(A.67)}}$$

Integration by Parts for Definite Integrals.

**Example A.9.** *Consider the integral*

$$\int_0^\pi x^2\cos x\,dx.$$

*This will require two integrations by parts. First, we let $u = x^2$ and $dv = \cos x$. Then,*

$$du = 2x\,dx. \quad v = \sin x.$$

*Inserting into the Integration by Parts Formula, we have*

$$\begin{aligned} \int_0^\pi x^2\cos x\,dx &= x^2\sin x\Big|_0^\pi - 2\int_0^\pi x\sin x\,dx \\ &= -2\int_0^\pi x\sin x\,dx. \end{aligned} \tag{A.68}$$

*We note that the resulting integral is easier that the given integral, but we still cannot do the integral off the top of our head (unless we look at Example 3!). So, we need to integrate by parts again. (Note: In your calculus class you may recall that there is a tabular method for carrying out multiple applications of the formula. We will show this method in the next example.)*

*We apply integration by parts by letting $U = x$ and $dV = \sin x\,dx$. This gives $dU = dx$ and $V = -\cos x$. Therefore, we have*

$$\begin{aligned} \int_0^\pi x\sin x\,dx &= -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx \\ &= \pi + \sin x\Big|_0^\pi \\ &= \pi. \end{aligned} \tag{A.69}$$

*The final result is*

$$\int_0^\pi x^2\cos x\,dx = -2\pi.$$

There are other ways to compute integrals of this type. First of all, there is the Tabular Method to perform integration by parts. A second method is to use differentiation of parameters under the integral. We will demonstrate this using examples.

**Example A.10.** *Compute the integral* $\int_0^\pi x^2 \cos x \, dx$ *using the Tabular Method.*

*First we identify the two functions under the integral,* $x^2$ *and* $\cos x$*. We then write the two functions and list the derivatives and integrals of each, respectively. This is shown in Table A.4. Note that we stopped when we reached zero in the left column.*

*Next, one draws diagonal arrows, as indicated, with alternating signs attached, starting with* $+$*. The indefinite integral is then obtained by summing the products of the functions at the ends of the arrows along with the signs on each arrow:*

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

*To find the definite integral, one evaluates the antiderivative at the given limits.*

$$\begin{aligned} \int_0^\pi x^2 \cos x \, dx &= \left[ x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^\pi \\ &= (\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi) - 0 \\ &= -2\pi. \end{aligned} \tag{A.70}$$

Actually, the Tabular Method works even if a zero does not appear in the left column. One can go as far as possible, and if a zero does not appear, then one needs only integrate, if possible, the product of the functions in the last row, adding the next sign in the alternating sign progression. The next example shows how this works.

| $D$ | | $I$ |
|---|---|---|
| $x^2$ | $+$ | $\cos x$ |
| $2x$ | $-$ | $\sin x$ |
| $2$ | $+$ | $-\cos x$ |
| $0$ | | $-\sin x$ |

Table A.4: Tabular Method

**Example A.11.** *Use the Tabular Method to compute* $\int e^{2x} \sin 3x \, dx$*.*

*As before, we first set up the table as shown in Table A.5.*

*Putting together the pieces, noting that the derivatives in the left column will never vanish, we have*

$$\int e^{2x} \sin 3x \, dx = (\frac{1}{2} \sin 3x - \frac{3}{4} \cos 3x) e^{2x} + \int (-9 \sin 3x) \left( \frac{1}{4} e^{2x} \right) \, dx.$$

*The integral on the right is a multiple of the one on the left, so we can combine them,*

$$\frac{13}{4} \int e^{2x} \sin 3x \, dx = (\frac{1}{2} \sin 3x - \frac{3}{4} \cos 3x) e^{2x},$$

*or*

$$\int e^{2x} \sin 3x \, dx = (\frac{2}{13} \sin 3x - \frac{3}{13} \cos 3x) e^{2x}.$$

Table A.5: Tabular Method, showing a nonterminating example.

| $D$ | | $I$ |
|---|---|---|
| $\sin 3x$ | | $e^{2x}$ |
| | $+$ | |
| $3\cos 3x$ | | $\frac{1}{2}e^{2x}$ |
| | $-$ | |
| $-9\sin 3x$ | | $\frac{1}{4}e^{2x}$ |

**Differentiation Under the Integral**

Differentiation Under the Integral Sign and Feynman's trick.

Another method that one can use to evaluate this integral is to differentiate under the integral sign. This is mentioned in the Richard Feynman's memoir *Surely You're Joking, Mr. Feynman!*. In the book Feynman recounts using this "trick" to be able to do integrals that his MIT classmates could not do. This is based on a theorem found in Advanced Calculus texts. Reader's unfamiliar with partial derivatives should be able to grasp their use in the following example.

**Theorem A.1.** *Let the functions $f(x,t)$ and $\frac{\partial f(x,t)}{\partial x}$ be continuous in both $t$, and $x$, in the region of the $(t,x)$ plane which includes $a(x) \le t \le b(x)$, $x_0 \le x \le x_1$, where the functions $a(x)$ and $b(x)$ are continuous and have continuous derivatives for $x_0 \le x \le x_1$. Defining*

$$F(x) \equiv \int_{a(x)}^{b(x)} f(x,t)\,dt,$$

*then*

$$\begin{aligned} \frac{dF(x)}{dx} &= \left(\frac{\partial F}{\partial b}\right)\frac{db}{dx} + \left(\frac{\partial F}{\partial a}\right)\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt \\ &= f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt. \end{aligned} \tag{A.71}$$

*for $x_0 \le x \le x_1$. This is a generalized version of the Fundamental Theorem of Calculus.*

In the next examples we show how we can use this theorem to bypass integration by parts.

**Example A.12.** *Use differentiation under the integral sign to evaluate $\int xe^x\,dx$. First, consider the integral*

$$I(x,a) = \int e^{ax}\,dx = \frac{e^{ax}}{a}.$$

*Then,*

$$\frac{\partial I(x,a)}{\partial a} = \int xe^{ax}\,dx.$$

*So,*

$$\begin{aligned}
\int x e^{ax}\,dx &= \frac{\partial I(x,a)}{\partial a} \\
&= \frac{\partial}{\partial a}\left(\int e^{ax}\,dx\right) \\
&= \frac{\partial}{\partial a}\left(\frac{e^{ax}}{a}\right) \\
&= \left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}. \qquad \text{(A.72)}
\end{aligned}$$

*Evaluating this result at $a = 1$, we have*

$$\int x e^x\,dx = (x-1)e^x.$$

*The reader can verify this result by employing the previous methods or by just differentiating the result.*

**Example A.13.** *We will do the integral $\int_0^\pi x^2\cos x\,dx$ once more. First, consider the integral*

$$\begin{aligned}
I(a) &\equiv \int_0^\pi \cos ax\,dx \\
&= \left.\frac{\sin ax}{a}\right|_0^\pi \\
&= \frac{\sin a\pi}{a}. \qquad \text{(A.73)}
\end{aligned}$$

*Differentiating the integral $I(a)$ with respect to $a$ twice gives*

$$\frac{d^2I(a)}{da^2} = -\int_0^\pi x^2\cos ax\,dx. \qquad \text{(A.74)}$$

*Evaluation of this result at $a = 1$ leads to the desired result. Namely,*

$$\begin{aligned}
\int_0^\pi x^2\cos x\,dx &= -\left.\frac{d^2I(a)}{da^2}\right|_{a=1} \\
&= -\frac{d^2}{da^2}\left(\frac{\sin a\pi}{a}\right)\Big|_{a=1} \\
&= -\frac{d}{da}\left(\frac{a\pi\cos a\pi-\sin a\pi}{a^2}\right)\Big|_{a=1} \\
&= -\left(\frac{a^2\pi^2\sin a\pi+2a\pi\cos a\pi-2\sin a\pi}{a^3}\right)\Big|_{a=1} \\
&= -2\pi. \qquad \text{(A.75)}
\end{aligned}$$

### Trigonometric Integrals

Other types of integrals that you will see often are trigonometric integrals. In particular, integrals involving powers of sines and cosines. For odd powers, a simple substitution will turn the integrals into simple powers.

**Example A.14.** *For example, consider*

$$\int \cos^3 x \, dx.$$

*This can be rewritten as*

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx.$$

Integration of odd powers of sine and cosine.

*Let* $u = \sin x$. *Then,* $du = \cos x \, dx$. *Since* $\cos^2 x = 1 - \sin^2 x$, *we have*

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\ &= \int (1 - u^2) \, du \\ &= u - \frac{1}{3} u^3 + C \\ &= \sin x - \frac{1}{3} \sin^3 x + C. \end{aligned} \tag{A.76}$$

*A quick check confirms the answer:*

$$\begin{aligned} \frac{d}{dx} \left( \sin x - \frac{1}{3} \sin^3 x + C \right) &= \cos x - \sin^2 x \cos x \\ &= \cos x (1 - \sin^2 x) \\ &= \cos^3 x. \end{aligned} \tag{A.77}$$

Even powers of sines and cosines are a little more complicated, but doable. In these cases we need the half angle formulae (A.24)-(A.25).

Integration of even powers of sine and cosine.

**Example A.15.** *As an example, we will compute*

$$\int_0^{2\pi} \cos^2 x \, dx.$$

*Substituting the half angle formula for* $\cos^2 x$, *we have*

$$\begin{aligned} \int_0^{2\pi} \cos^2 x \, dx &= \frac{1}{2} \int_0^{2\pi} (1 + \cos 2x) \, dx \\ &= \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right)_0^{2\pi} \\ &= \pi. \end{aligned} \tag{A.78}$$

Recall that RMS averages refer to the root mean square average. This is computed by first computing the average, or mean, of the square of some quantity. Then one takes the square root. Typical examples are RMS voltage, RMS current, and the average energy in an electromagnetic wave. AC currents oscillate so fast that the measured value is the RMS voltage.

We note that this result appears often in physics. When looking at root mean square averages of sinusoidal waves, one needs the average of the square of sines and cosines. Recall that the average of a function on interval $[a, b]$ is given as

$$f\text{ave} = \frac{1}{b - a} \int_a^b f(x) \, dx. \tag{A.79}$$

So, the average of $\cos^2 x$ over one period is

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 x \, dx = \frac{1}{2}. \tag{A.80}$$

The root mean square is then found by taking the square root, $\frac{1}{\sqrt{2}}$.

**Trigonometric Function Substitution**

Another class of integrals typically studied in calculus are those involving the forms $\sqrt{1-x^2}$, $\sqrt{1+x^2}$, or $\sqrt{x^2-1}$. These can be simplified through the use of trigonometric substitutions. The idea is to combine the two terms under the radical into one term using trigonometric identities. We will consider some typical examples.

**Example A.16.** *Evaluate* $\int \sqrt{1-x^2}\,dx$.

*Since* $1-\sin^2\theta = \cos^2\theta$, *we perform the sine substitution*

$$x = \sin\theta, \quad dx = \cos\theta\, d\theta.$$

*Then,*

$$\begin{aligned}\int \sqrt{1-x^2}\,dx &= \int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta \\ &= \int \cos^2\theta\, d\theta. \end{aligned} \tag{A.81}$$

In any of these computations careful attention has to be paid to simplifying the radical. This is because

$$\sqrt{x^2} = |x|.$$

For example, $\sqrt{(-5)^2} = \sqrt{25} = 5$. For $x = \sin\theta$, one typically specifies the domain $-\pi/2 \le \theta \le \pi/2$. In this domain we have $|\cos\theta| = \cos\theta$.

*Using the last example, we have*

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C.$$

*However, we need to write the answer in terms of $x$. We do this by first using the double angle formula for* $\sin 2\theta$ *and* $\cos\theta = \sqrt{1-x^2}$ *to obtain*

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1}x - x\sqrt{1-x^2}\right) + C.$$

Similar trigonometric substitutions result for integrands involving $\sqrt{1+x^2}$ and $\sqrt{x^2-1}$. The substitutions are summarized in Table A.6. The simplification of the given form is then obtained using trigonometric identities. This can also be accomplished by referring to the right triangles shown in Figure A.5.

| Form | Substitution | Differential |
|---|---|---|
| $\sqrt{a^2-x^2}$ | $x = a\sin\theta$ | $dx = a\cos\theta\, d\theta$ |
| $\sqrt{a^2+x^2}$ | $x = a\tan\theta$ | $dx = a\sec^2\theta\, d\theta$ |
| $\sqrt{x^2-a^2}$ | $x = a\sec\theta$ | $dx = a\sec\theta\tan\theta\, d\theta$ |

Table A.6: Standard trigonometric substitutions.

**Example A.17.** *Evaluate* $\int_0^2 \sqrt{x^2+4}\,dx$.

*Let* $x = 2\tan\theta$. *Then,* $dx = 2\sec^2\theta\, d\theta$ *and*

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sec\theta.$$

*So, the integral becomes*

$$\int_0^2 \sqrt{x^2+4}\,dx = 4\int_0^{\pi/4} \sec^3\theta\, d\theta.$$

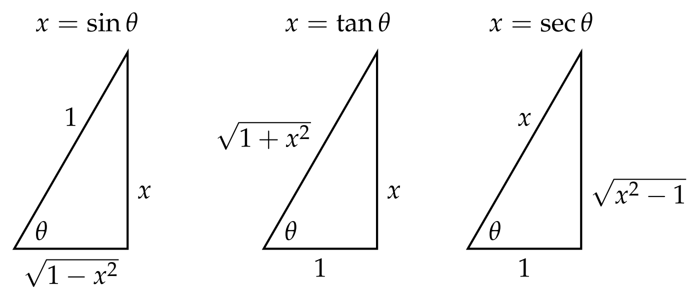



Figure A.5: Geometric relations used in trigonometric substitution.

*One has to recall, or look up,*

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\left(\tan\theta\sec\theta + \ln|\sec\theta + \tan\theta|\right) + C.$$

*This gives*

$$\begin{aligned}\int_0^2 \sqrt{x^2+4}\, dx &= 2\left[\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} \\ &= 2\left(\sqrt{2} + \ln|\sqrt{2}+1| - (0+\ln(1))\right) \\ &= 2(\sqrt{2}+\ln(\sqrt{2}+1)). \qquad \text{(A.82)}\end{aligned}$$

**Example A.18.** *Evaluate* $\int \frac{dx}{\sqrt{x^2-1}}$, $x \geq 1$.

*In this case one needs the secant substitution. This yields*

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\sec^2\theta - 1}} \\ &= \int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta} \\ &= \int \sec\theta d\theta \\ &= \ln(\sec\theta + \tan\theta) + C \\ &= \ln(x + \sqrt{x^2-1}) + C. \qquad \text{(A.83)}\end{aligned}$$

**Example A.19.** *Evaluate* $\int \frac{dx}{x\sqrt{x^2-1}}$, $x \geq 1$.

*Again we can use a secant substitution. This yields*

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} \\ &= \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\, d\theta \\ &= \int d\theta = \theta + C = \sec^{-1}x + C. \qquad \text{(A.84)}\end{aligned}$$

### Hyperbolic Function Substitution

Even though trigonometric substitution plays a role in the calculus program, students often see hyperbolic function substitution used in physics courses. The reason might be because hyperbolic function substitution is sometimes simpler. The idea is the same as for trigonometric substitution. We use an identity to simplify the radical.

**Example A.20.** *Evaluate* $\int_0^2 \sqrt{x^2+4}\,dx$ *using the substitution* $x = 2\sinh u$.

*Since* $x = 2\sinh u$*, we have* $dx = 2\cosh u\,du$*. Also, we can use the identity* $\cosh^2 u - \sinh^2 u = 1$ *to rewrite*

$$\sqrt{x^2+4} = \sqrt{4\sinh^2 u + 4} = 2\cosh u.$$

*The integral can be now be evaluated using these substitutions and some hyperbolic function identities,*

$$\begin{aligned}
\int_0^2 \sqrt{x^2+4}\,dx &= 4\int_0^{\sinh^{-1} 1} \cosh^2 u\,du \\
&= 2\int_0^{\sinh^{-1} 1} (1+\cosh 2u)\,du \\
&= 2\left[u + \frac{1}{2}\sinh 2u\right]_0^{\sinh^{-1} 1} \\
&= 2\left[u + \sinh u\cosh u\right]_0^{\sinh^{-1} 1} \\
&= 2\left(\sinh^{-1} 1 + \sqrt{2}\right). \qquad \text{(A.85)}
\end{aligned}$$

*In Example A.17 we used a trigonometric substitution and found*

$$\int_0^2 \sqrt{x^2+4} = 2(\sqrt{2} + \ln(\sqrt{2}+1)).$$

*This is the same result since* $\sinh^{-1} 1 = \ln(1+\sqrt{2})$.

**Example A.21.** *Evaluate* $\int \frac{dx}{\sqrt{x^2-1}}$ *for* $x \geq 1$ *using hyperbolic function substitution.*

*This integral was evaluated in Example A.19 using the trigonometric substitution* $x = \sec\theta$ *and the resulting integral of* $\sec\theta$ *had to be recalled. Here we will use the substitution*

$$x = \cosh u, \quad dx = \sinh u\,du, \quad \sqrt{x^2-1} = \sqrt{\cosh^2 u - 1} = \sinh u.$$

*Then,*

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sinh u\,du}{\sinh u} \\
&= \int du = u + C \\
&= \cosh^{-1} x + C \\
&= \frac{1}{2}\ln(x + \sqrt{x^2-1}) + C, \quad x \geq 1. \qquad \text{(A.86)}
\end{aligned}$$

*This is the same result as we had obtained previously, but this derivation was a little cleaner.*

*Also, we can extend this result to values* $x \leq -1$ *by letting* $x = -\cosh u$*. This gives*

$$\int \frac{dx}{\sqrt{x^2-1}} = \frac{1}{2}\ln(x + \sqrt{x^2-1}) + C, \quad x \leq -1.$$

*Combining these results, we have shown*

$$\int \frac{dx}{\sqrt{x^2-1}} = \frac{1}{2}\ln(|x| + \sqrt{x^2-1}) + C, \quad x^2 \geq 1.$$

We have seen in the last example that the use of hyperbolic function substitution allows us to bypass integrating the secant function in Example A.19 when using trigonometric substitutions. In fact, we can use hyperbolic substitutions to evaluate integrals of powers of secants. Comparing Examples A.19 and A.21, we consider the transformation $\sec\theta = \cosh u$. The relation between differentials is found by differentiation, giving

$$\sec\theta\tan\theta\, d\theta = \sinh u\, du.$$

Since

$$\tanh^2\theta = \sec^2\theta - 1,$$

we have $\tan\theta = \sinh u$, therefore

$$d\theta = \frac{du}{\cosh u}.$$

Evaluation of $\int \sec\theta\, d\theta$.

In the next example we show how useful this transformation is.

**Example A.22.** *Evaluate* $\int \sec\theta\, d\theta$ *using hyperbolic function substitution.*

*From the discussion in the last paragraph, we have*

$$\begin{aligned} \int \sec\theta\, d\theta &= \int du \\ &= u + C \\ &= \cosh^{-1}(\sec\theta) + C \end{aligned} \tag{A.87}$$

*We can express this result in the usual form by using the logarithmic form of the inverse hyperbolic cosine,*

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}).$$

*The result is*

$$\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta).$$

This example was fairly simple using the transformation $\sec\theta = \cosh u$. Another common integral that arises often is integrations of $\sec^3\theta$. In a typical calculus class this integral is evaluated using integration by parts. However. that leads to a tricky manipulation that is a bit scary the first time it is encountered (and probably upon several more encounters.) In the next example, we will show how hyperbolic function substitution is simpler.

Evaluation of $\int \sec^3\theta\, d\theta$.

**Example A.23.** *Evaluate* $\int \sec^3\theta\, d\theta$ *using hyperbolic function substitution.*

*First, we consider the transformation* $\sec\theta = \cosh u$ *with* $d\theta = \frac{du}{\cosh u}$. *Then,*

$$\int \sec^3\theta\, d\theta = \int \frac{du}{\cosh u}.$$

*This integral was done in Example A.7, leading to*

$$\int \sec^3\theta\, d\theta = 2\tan^{-1} e^u + C.$$

*While correct, this is not the form usually encountered. Instead, we make the slightly different transformation* $\tan\theta = \sinh u$. *Since* $\sec^2\theta = 1+\tan^2\theta$, *we find* $\sec\theta = \cosh u$. *As before, we find*

$$d\theta = \frac{du}{\cosh u}.$$

*Using this transformation and several identities, the integral becomes*

$$\begin{aligned}
\int \sec^3\theta\, d\theta &= \int \cosh^2 u\, du \\
&= \frac{1}{2}\int (1+\cosh 2u)\, du \\
&= \frac{1}{2}\left(u + \frac{1}{2}\sinh 2u\right) \\
&= \frac{1}{2}\left(u + \sinh u \cosh u\right) \\
&= \frac{1}{2}\left(\cosh^{-1}(\sec\theta) + \tan\theta\sec\theta\right) \\
&= \frac{1}{2}\left(\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)\right). \qquad \text{(A.88)}
\end{aligned}$$

There are many other integration methods, some of which we will visit in other parts of the book, such as partial fraction decomposition and numerical integration. Another topic which we will revisit is power series.

### *A.1.6 Geometric Series*

INFINITE SERIES OCCUR OFTEN in mathematics and physics. Two series which occur often are the geometric series and the binomial series. we will discuss these next.

Geometric series are fairly common and will be used throughout the book. You should learn to recognize them and work with them.

A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots. \qquad \text{(A.89)}$$

Here $a$ is the first term and $r$ is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is $r$.

**Example A.24.** *For example,*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

*is an example of a geometric series. We can write this using summation notation,*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} 1\left(\frac{1}{2}\right)^n.$$

*Thus,* $a = 1$ *is the first term and* $r = \frac{1}{2}$ *is the common ratio of successive terms. Next, we seek the sum of this infinite series, if it exists.*

The sum of a geometric series, when it exists, can easily be determined. We consider the $n$th partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{A.90}$$

Now, multiply this equation by $r$.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{A.91}$$

Subtracting these two equations, while noting the many cancelations, we have

$$\begin{aligned}(1-r)s_n &= (a + ar + \ldots + ar^{n-2} + ar^{n-1}) \\ &\quad -(ar + ar^2 + \ldots + ar^{n-1} + ar^n) \\ &= a - ar^n \\ &= a(1-r^n).\end{aligned} \tag{A.92}$$

Thus, the $n$th partial sums can be written in the compact form

$$s_n = \frac{a(1-r^n)}{1-r}. \tag{A.93}$$

The sum, if it exists, is given by $S = \lim_{n\to\infty} s_n$. Letting $n$ get large in the partial sum (A.93), we need only evaluate $\lim_{n\to\infty} r^n$. From the special limits in the Appendix we know that this limit is zero for $|r| < 1$. Thus, we have

**Geometric Series**

The sum of the geometric series exists for $|r| < 1$ and is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad |r| < 1. \tag{A.94}$$

The reader should verify that the geometric series diverges for all other values of $r$. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

**Example A.25.** $\sum_{n=0}^{\infty} \frac{1}{2^n}$

*In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is*

$$S = \frac{1}{1-\frac{1}{2}} = 2.$$

**Example A.26.** $\sum_{k=2}^{\infty} \frac{4}{3^k}$

*In this example we first note that the first term occurs for $k = 2$. It sometimes helps to write out the terms of the series,*

$$\sum_{k=2}^{\infty} \frac{4}{3^k} = \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \frac{4}{3^5} + \ldots.$$

*Looking at the series, we see that* $a = \frac{4}{9}$ *and* $r = \frac{1}{3}$. *Since* $|r|<1$, *the geometric series converges. So, the sum of the series is given by*

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

**Example A.27.** $\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right)$

*Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed* [1]. *Thus, we have*

[1] A rearrangement of terms in an infinite series is allowed when the series is absolutely convergent. (See the Appendix.)

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

*Now we can add both geometric series to obtain*

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \frac{\frac{3}{2}}{1 - \frac{1}{2}} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

Geometric series are important because they are easily recognized and summed. Other series which can be summed include special cases of Taylor series and *telescoping series.* Next, we show an example of a telescoping series.

**Example A.28.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *The first few terms of this series are*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots.$$

*It does not appear that we can sum this infinite series. However, if we used the partial fraction expansion*

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

*then we find the kth partial sum can be written as*

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right). \end{aligned} \tag{A.95}$$

*We see that there are many cancelations of neighboring terms, leading to the series collapsing (like a retractable telescope) to something manageable:*

$$s_k = 1 - \frac{1}{k+1}.$$

*Taking the limit as* $k \to \infty$, *we find* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

### A.1.7 Power Series

Actually, what are now known as Taylor and Maclaurin series were known long before they were named. James Gregory (1638-1675) has been recognized for discovering Taylor series, which were later named after Brook Taylor (1685-1731) . Similarly, Colin Maclaurin (1698-1746) did not actually discover Maclaurin series, but the name was adopted because of his particular use of series.

ANOTHER EXAMPLE OF AN INFINITE SERIES that the student has encountered in previous courses is the power series. Examples of such series are provided by Taylor and Maclaurin series.

A power series expansion about $x = a$ with coefficient sequence $c_n$ is given by $\sum_{n=0}^{\infty} c_n(x-a)^n$. For now we will consider all constants to be real numbers with $x$ in some subset of the set of real numbers.

Consider the following expansion about $x = 0$ :

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots. \tag{A.96}$$

We would like to make sense of such expansions. For what values of $x$ will this infinite series converge? Until now we did not pay much attention to which infinite series might converge. However, this particular series is already familiar to us. It is a geometric series. Note that each term is gotten from the previous one through multiplication by $r = x$. The first term is $a = 1$. So, from Equation (A.94), we have that the sum of the series is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

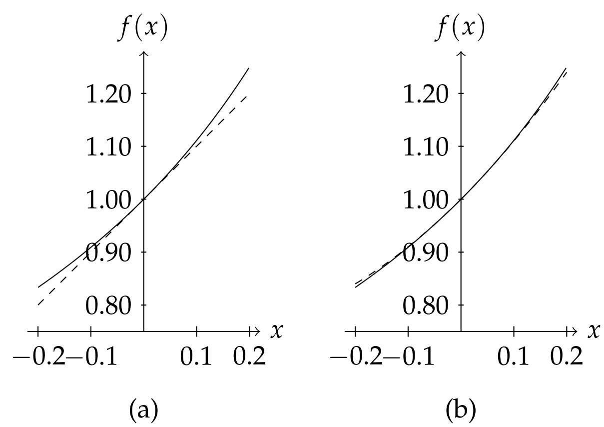



Figure A.6: (a) Comparison of $\frac{1}{1-x}$ (solid) to $1 + x$ (dashed) for $x \in [-0.2, 0.2]$. (b) Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) for $x \in [-0.2, 0.2]$.

In this case we see that the sum, when it exists, is a simple function. In fact, when $x$ is small, we can use this infinite series to provide approximations to the function $(1-x)^{-1}$. If $x$ is small enough, we can write

$$(1-x)^{-1} \approx 1 + x.$$

In Figure A.6a we see that for small values of $x$ these functions do agree.

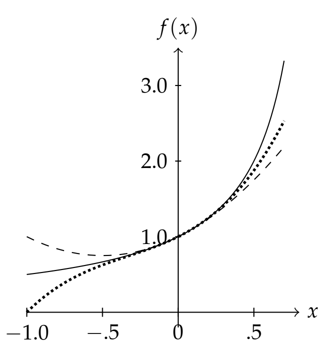



Figure A.7: Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) and $1 + x + x^2 + x^3$ (dotted) for $x \in [-1.0, 0.7]$.

Of course, if we want better agreement, we select more terms. In Figure A.6b we see what happens when we do so. The agreement is much better. But extending the interval, we see in Figure A.7 that keeping only quadratic terms may not be good enough. Keeping the cubic terms gives better agreement over the interval.

Finally, in Figure A.8 we show the sum of the first 21 terms over the entire interval $[-1,1]$. Note that there are problems with approximations near the endpoints of the interval, $x = \pm 1$.

Such polynomial approximations are called Taylor polynomials. Thus, $T_3(x) = 1 + x + x^2 + x^3$ is the third order Taylor polynomial approximation of $f(x) = \frac{1}{1-x}$.

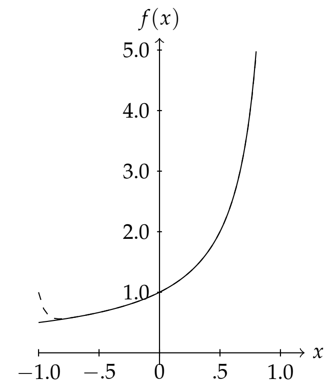



Figure A.8: Comparison of $\frac{1}{1-x}$ (solid) to $\sum_{n=0}^{20} x^n$ for $x \in [-1,1]$.

With this example we have seen how useful a series representation might be for a given function. However, the series representation was a simple geometric series, which we already knew how to sum. Is there a way to begin with a function and then find its series representation? Once we have such a representation, will the series converge to the function with which we started? For what values of $x$ will it converge? These questions can be answered by recalling the definitions of Taylor and Maclaurin series.

Taylor series expansion.

A Taylor series expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{A.97}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{A.98}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function. A special class of series are those Taylor series for which the expansion is about $x = 0$. These are called Maclaurin series.

Maclaurin series expansion.

A Maclaurin series expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{A.99}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{A.100}$$

**Example A.29.** *Expand $f(x) = e^x$ about $x = 0$.*

*We begin by creating a table. In order to compute the expansion coefficients, $c_n$, we will need to perform repeated differentiations of $f(x)$. So, we provide a table for these derivatives. Then, we only need to evaluate the second column at $x = 0$ and divide by $n!$.*

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $c_n$ |
|---|---|---|---|
| 0 | $e^x$ | $e^0 = 1$ | $\frac{1}{0!} = 1$ |
| 1 | $e^x$ | $e^0 = 1$ | $\frac{1}{1!} = 1$ |
| 2 | $e^x$ | $e^0 = 1$ | $\frac{1}{2!}$ |
| 3 | $e^x$ | $e^0 = 1$ | $\frac{1}{3!}$ |

*Next, we look at the last column and try to determine a pattern so that we can write down the general term of the series. If there is only a need to get a polynomial approximation, then the first few terms may be sufficient. In this case, the pattern is obvious:* $c_n = \frac{1}{n!}$. *So,*

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

**Example A.30.** *Expand* $f(x) = e^x$ *about* $x = 1$.

*Here we seek an expansion of the form* $e^x \sim \sum_{n=0}^{\infty} c_n (x-1)^n$. *We could create a table like the last example. In fact, the last column would have values of the form* $\frac{e}{n!}$. *(You should confirm this.) However, we will make use of the Maclaurin series expansion for* $e^x$ *and get the result quicker. Note that* $e^x = e^{x-1+1} = ee^{x-1}$. *Now, apply the known expansion for* $e^x$ *:*

$$e^x \sim e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \ldots\right) = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}.$$

**Example A.31.** *Expand* $f(x) = \frac{1}{1-x}$ *about* $x = 0$.

*This is the example with which we started our discussion. We can set up a table in order to find the Maclaurin series coefficients. We see from the last column of the table that we get back the geometric series (A.96).*

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $c_n$ |
|---|---|---|---|
| 0 | $\frac{1}{1-x}$ | 1 | $\frac{1}{0!} = 1$ |
| 1 | $\frac{1}{(1-x)^2}$ | 1 | $\frac{1}{1!} = 1$ |
| 2 | $\frac{2(1)}{(1-x)^3}$ | $2(1)$ | $\frac{2!}{2!} = 1$ |
| 3 | $\frac{3(2)(1)}{(1-x)^4}$ | $3(2)(1)$ | $\frac{3!}{3!} = 1$ |

*So, we have found*

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n.$$

We can replace $\sim$ by equality if we can determine the range of $x$-values for which the resulting infinite series converges. We will investigate such convergence shortly.

Series expansions for many elementary functions arise in a variety of applications. Some common expansions are provided in Table A.7.

We still need to determine the values of $x$ for which a given power series converges. The first five of the above expansions converge for all reals, but the others only converge for $|x| < 1$.

We consider the convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$. For $x = a$ the series obviously converges. Will it converge for other points? One can prove

**Theorem A.2.** *If* $\sum_{n=0}^{\infty} c_n (b-a)^n$ *converges for* $b \neq a$, *then* $\sum_{n=0}^{\infty} c_n (x-a)^n$ *converges absolutely for all* $x$ *satisfying* $|x-a| < |b-a|$.

Table A.7: Common Mclaurin Series Expansions

| **Series Expansions You Should Know** | | | | |
|---|---|---|---|---|
| $e^x$ | = | $1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$ | = | $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ |
| $\cos x$ | = | $1-\frac{x^2}{2}+\frac{x^4}{4!}-\ldots$ | = | $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ |
| $\sin x$ | = | $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$ | = | $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ |
| $\cosh x$ | = | $1+\frac{x^2}{2}+\frac{x^4}{4!}+\ldots$ | = | $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ |
| $\sinh x$ | = | $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$ | = | $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$ |
| $\frac{1}{1-x}$ | = | $1+x+x^2+x^3+\ldots$ | = | $\sum_{n=0}^{\infty}x^n$ |
| $\frac{1}{1+x}$ | = | $1-x+x^2-x^3+\ldots$ | = | $\sum_{n=0}^{\infty}(-x)^n$ |
| $\tan^{-1}x$ | = | $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$ | = | $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ |
| $\ln(1+x)$ | = | $x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$ | = | $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ |

This leads to three possibilities

1. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may only converge at $x=a$.
2. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge for all real numbers.
3. $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a|<R$ and diverges for $|x-a|>R$.

The number $R$ is called the radius of convergence of the power series and $(a-R, a+R)$ is called the interval of convergence. Convergence at the endpoints of this interval has to be tested for each power series.

Interval and radius of convergence.

In order to determine the interval of convergence, one needs only note that when a power series converges, it does so absolutely. So, we need only test the convergence of $\sum_{n=0}^{\infty}|c_n(x-a)^n| = \sum_{n=0}^{\infty}|c_n||x-a|^n$. This is easily done using either the ratio test or the $n$th root test. We first identify the nonnegative terms $a_n = |c_n||x-a|^n$, and then we apply one of the convergence tests from the calculus curriculum.

For example, the $n$th Root Test gives the convergence condition for $a_n = |c_n||x-a|^n$,

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{|c_n|}|x-a| < 1.$$

Since $|x-a|$ is independent of $n$,, we can factor it out of the limit and divide the value of the limit to obtain

$$|x-a| < \left(\lim_{n\to\infty}\sqrt[n]{|c_n|}\right)^{-1} \equiv R.$$

Thus, we have found the radius of convergence, $R$.

Similarly, we can apply the Ratio Test.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}|x-a| < 1.$$

Again, we rewrite this result to determine the radius of convergence:

$$|x-a| < \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)^{-1} \equiv R.$$

**Example A.32.** *Find the radius of convergence of the series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.*

*Since there is a factorial, we will use the Ratio Test.*

$$\rho = \lim_{n\to\infty} \frac{|n!|}{|(n+1)!|}|x| = \lim_{n\to\infty} \frac{1}{n+1}|x| = 0.$$

*Since $\rho = 0$, it is independent of $|x|$ and thus the series converges for all $x$. We also can say that the radius of convergence is infinite.*

**Example A.33.** *Find the radius of convergence of the series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.*

*In this example we will use the nth Root Test.*

$$\rho = \lim_{n\to\infty} \sqrt[n]{1}|x| = |x| < 1.$$

*Thus, we find that we have absolute convergence for $|x| < 1$. Setting $x = 1$ or $x = -1$, we find that the resulting series do not converge. So, the endpoints are not included in the complete interval of convergence.*

*In this example we could have also used the Ratio Test. Thus,*

$$\rho = \lim_{n\to\infty} \frac{1}{1}|x| = |x| < 1.$$

*We have obtained the same result as when we used the nth Root Test.*

**Example A.34.** *Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$.*

*In this example, we have an expansion about $x = 2$. Using the nth Root Test we find that*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n}}|x-2| = 3|x-2| < 1.$$

*Solving for $|x-2|$ in this inequality, we find $|x-2| < \frac{1}{3}$. Thus, the radius of convergence is $R = \frac{1}{3}$ and the interval of convergence is $\left(2 - \frac{1}{3}, 2 + \frac{1}{3}\right) = \left(\frac{5}{3}, \frac{7}{3}\right)$.*

*As for the endpoints, we first test the point $x = \frac{7}{3}$. The resulting series is $\sum_{n=1}^{\infty} \frac{3^n(\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, and thus it does not converge. Inserting $x = \frac{5}{3}$, we get the alternating harmonic series. This series does converge. So, we have convergence on $[\frac{5}{3}, \frac{7}{3})$. However, it is only conditionally convergent at the left endpoint, $x = \frac{5}{3}$.*

**Example A.35.** *Find an expansion of $f(x) = \frac{1}{x+2}$ about $x = 1$.*

*Instead of explicitly computing the Taylor series expansion for this function, we can make use of an already known function. We first write $f(x)$ as a function of $x - 1$, since we are expanding about $x = 1$; i.e., we are seeking a series whose terms are powers of $x - 1$.*

*This expansion is easily done by noting that* $\frac{1}{x+2} = \frac{1}{(x-1)+3}$*. Factoring out a* 3*, we can rewrite this expression as a sum of a geometric series. Namely, we use the expansion for*

$$\begin{aligned} g(z) &= \frac{1}{1+z} \\ &= 1 - z + z^2 - z^3 + \dots. \end{aligned} \tag{A.101}$$

*and then we rewrite* $f(x)$ *as*

$$\begin{aligned} f(x) &= \frac{1}{x+2} \\ &= \frac{1}{(x-1)+3} \\ &= \frac{1}{3[1+\frac{1}{3}(x-1)]} \\ &= \frac{1}{3}\frac{1}{1+\frac{1}{3}(x-1)}. \end{aligned} \tag{A.102}$$

*Note that* $f(x) = \frac{1}{3}g(\frac{1}{3}(x-1))$ *for* $g(z) = \frac{1}{1+z}$*. So, the expansion becomes*

$$f(x) = \frac{1}{3}\left[1 - \frac{1}{3}(x-1) + \left(\frac{1}{3}(x-1)\right)^2 - \left(\frac{1}{3}(x-1)\right)^3 + \dots\right].$$

*This can further be simplified as*

$$f(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \dots.$$

*Convergence is easily established. The expansion for* $g(z)$ *converges for* $|z| < 1$*. So, the expansion for* $f(x)$ *converges for* $|-\frac{1}{3}(x-1)| < 1$*. This implies that* $|x-1| < 3$*. Putting this inequality in interval notation, we have that the power series converges absolutely for* $x \in (-2, 4)$*. Inserting the endpoints, one can show that the series diverges for both* $x = -2$ *and* $x = 4$*. You should verify this!*

**Example A.36.** *Prove Euler's Formula:* $e^{i\theta} = \cos\theta + i\sin\theta$.

*As a final application, we can derive Euler's Formula ,*

Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, is an important formula and is used throughout the text.

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

*where* $i = \sqrt{-1}$*. We naively use the expansion for* $e^x$ *with* $x = i\theta$*. This leads us to*

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots.$$

*Next we note that each term has a power of* $i$*. The sequence of powers of* $i$ *is given as* $\{1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \dots\}$*. See the pattern? We conclude that*

$$i^n = i^r, \text{ where } r = \text{ remainder after dividing } n \text{ by } 4.$$

*This gives*

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right).$$

*We recognize the expansions in the parentheses as those for the cosine and sine functions. Thus, we end with Euler's Formula.*

We further derive relations from this result, which will be important for our next studies. From Euler's formula we have that for integer $n$:

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

We also have

$$e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n.$$

Equating these two expressions, we are led to de Moivre's Formula, named after Abraham de Moivre (1667-1754),

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \quad (A.103)$$

de Moivre's Formula.

This formula is useful for deriving identities relating powers of sines or cosines to simple functions. For example, if we take $n = 2$ in Equation (A.103), we find

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

Looking at the real and imaginary parts of this result leads to the well known double angle identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Here we see elegant proofs of well known trigonometric identities.

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta, \quad (A.104) \\ \sin 2\theta &= 2\sin\theta\cos\theta, \\ \cos^2\theta &= \frac{1}{2}(1+\cos 2\theta), \\ \sin^2\theta &= \frac{1}{2}(1-\cos 2\theta). \end{aligned}$$

Replacing $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$ leads to the half angle formulae:

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta).$$

We can also use Euler's Formula to write sines and cosines in terms of complex exponentials. We first note that due to the fact that the cosine is an even function and the sine is an odd function, we have

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Trigonometric functions can be written in terms of complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Combining this with Euler's Formula, we have that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Hyperbolic functions and trigonometric functions are intimately related.

$$\cos(ix) = \cosh x,$$
$$\sin(ix) = -i\sinh x.$$

We finally note that there is a simple relationship between hyperbolic functions and trigonometric functions. Recall that

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

If we let $x = i\theta$, then we have that $\cosh(i\theta) = \cos\theta$ and $\cos(ix) = \cosh x$. Similarly, we can show that $\sinh(i\theta) = i\sin\theta$ and $\sin(ix) = -i\sinh x$.

### A.1.8 The Binomial Expansion

ANOTHER SERIES EXPANSION WHICH OCCURS often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$ in powers of $a$ and $b$. We will investigate this expansion first for nonnegative integer powers $p$ and then derive the expansion for other values of $p$. While the binomial expansion can be obtained using Taylor series, we will provide a more intuitive derivation to show that

The binomial expansion is a special series expansion used to approximate expressions of the form $(a+b)^p$ for $b \ll a$, or $(1+x)^p$ for $|x| \ll 1$.

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r, \qquad \text{(A.105)}$$

where the $C_r^n$ are called the *binomial coefficients*.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
&\cdots
\end{aligned} \qquad \text{(A.106)}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of $a$ and a power of $b$. The powers of $a$ are decreasing from $n$ to 0 in the expansion of $(a+b)^n$. Similarly, the powers of $b$ increase from 0 to $n$. The sums of the exponents in each term is $n$. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not yet know the numerical coefficients in the expansion.

Let's list the coefficients for the above expansions.

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \qquad \text{(A.107)}$$

This pattern is the famous Pascal's triangle.[2] There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. The second term and next to last term have a coefficient of $n$. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows $n=2$ and $n=3$ that $1+2=3$ and $2+1=3$:

$$\begin{array}{lccccccc}
n=2: & & 1 & & 2 & & 1 & \\
 & & & \searrow \ \swarrow & & \searrow \ \swarrow & & \\
n=3: & 1 & & 3 & & 3 & & 1
\end{array} \qquad \text{(A.108)}$$

[2] Pascal's triangle is named after Blaise Pascal (1623-1662). While such configurations of numbers were known earlier in history, Pascal published them and applied them to probability theory.

Pascal's triangle has many unusual properties and a variety of uses:

- Horizontal rows add to powers of 2.
- The horizontal rows are powers of 11 (1, 11, 121, 1331, etc.).
- Adding any two successive numbers in the diagonal 1-3-6-10-15-21-28... results in a perfect square.
- When the first number to the right of the 1 in any row is a prime number, all numbers in that row are divisible by that prime number. The reader can readily check this for the $n=5$ and $n=7$ rows.
- Sums along certain diagonals leads to the Fibonacci sequence. These diagonals are parallel to the line connecting the first 1 for $n=3$ row and the 2 in the $n=2$ row.

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{llllllllllllll} n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\ n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{A.109}$$

So, we use the numbers in row $n=4$ to generate entries in row $n=5$ : $1+4=5$, $4+6=10$. We then use row $n=5$ to get row $n=6$, etc.

Of course, it would take a while to compute each row up to the desired $n$. Fortunately, there is a simple expression for computing a specific coefficient. Consider the $k$th term in the expansion of $(a+b)^n$. Let $r=k-1$, for $k=1,\ldots,n+1$. Then this term is of the form $C_r^n a^{n-r}b^r$. We have seen that the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the binomial coefficients, $C_r^n$, have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \left(\begin{array}{c} n \\ r \end{array}\right).$$

This is nothing other than the combinatoric symbol for determining how to choose $n$ objects $r$ at a time. In the binomial expansions this makes sense. We have to count the number of ways that we can arrange $r$ products of $b$ with $n-r$ products of $a$. There are $n$ slots to place the $b$'s. For example, the $r=2$ case for $n=4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it is natural to use this notation.

Andreas Freiherr von Ettingshausen (1796-1878) was a German mathematician and physicist who in 1826 introduced the notation $\left(\begin{array}{c} n \\ r \end{array}\right)$. However, the binomial coefficients were known by the Hindus centuries beforehand.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) a^{n-r}b^r. \tag{A.110}$$

Now consider the geometric series $1+x+x^2+\ldots$. We have seen that such this geometric series converges for $|x|<1$, giving

$$1+x+x^2+\ldots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$. This is a binomial to a power, but the power is not an integer.

It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation(A.110). This example suggests that our sum may no longer be finite. So, for $p$ a real number, $a=1$ and $b=x$, we generalize Equation(A.110) as

$$(1+x)^p = \sum_{r=0}^{\infty} \left(\begin{array}{c} p \\ r \end{array}\right) x^r \tag{A.111}$$

and see if the resulting series makes sense. However, we quickly run into problems with the coefficients in the series.

Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned} \begin{pmatrix} p \\ r \end{pmatrix} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{A.112}$$

With this in mind we now state the theorem:

**General Binomial Expansion**

The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (A.110). For $p$ real, we have the following binomial series:

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r, \quad |x| < 1. \tag{A.113}$$

Often in physics we only need the first few terms for the case that $x \ll 1$ :

$$\boxed{(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3).} \tag{A.114}$$

**Example A.37.** *Approximate* $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ *for* $v \ll c$.

The factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is important in special relativity. Namely, this is the factor relating differences in time and length measurements by observers moving relative inertial frames. For terrestrial speeds, this gives an appropriate approximation.

*For* $v \ll c$ *the first approximation is found inserting* $v/c = 0$. *Thus, one obtains* $\gamma = 1$. *This is the Newtonian approximation and does not provide enough of an approximation for terrestrial speeds. Thus, we need to expand* $\gamma$ *in powers of* $v/c$.

*First, we rewrite* $\gamma$ *as*

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}.$$

*Using the binomial expansion for* $p = -1/2$, *we have*

$$\gamma \approx 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2}.$$

**Example A.38.** *Time Dilation Example*

*The average speed of a large commercial jet airliner is about 500 mph. If you flew for an hour (measured from the ground), then how much younger would you be than if you had not taken the flight, assuming these reference frames obeyed the postulates of special relativity?*

*This is the problem of time dilation. Let $\Delta t$ be the elapsed time in a stationary reference frame on the ground and $\Delta\tau$ be that in the frame of the moving plane. Then from the Theory of Special Relativity these are related by*

$$\Delta t = \gamma \Delta\tau.$$

*The time differences would then be*

$$\begin{aligned}\Delta t - \Delta\tau &= (1-\gamma^{-1})\Delta t \\ &= \left(1-\sqrt{1-\frac{v^2}{c^2}}\right)\Delta t. \qquad \text{(A.115)}\end{aligned}$$

*The plane speed, 500 mph, is roughly 225 m/s and $c = 3.00\times 10^8$ m/s. Since $V \ll c$, we would need to use the binomial approximation to get a nonzero result.*

$$\begin{aligned}\Delta t - \Delta\tau &= \left(1-\sqrt{1-\frac{v^2}{c^2}}\right)\Delta t \\ &= \left(1-\left(1-\frac{v^2}{2c^2}+\dots\right)\right)\Delta t \\ &\approx \frac{v^2}{2c^2}\Delta t \\ &= \frac{(225)^2}{2(3.00\times 10^8)^2}(1\ h) = 1.01\ ns. \qquad \text{(A.116)}\end{aligned}$$

*Thus, you have aged one nanosecond less than if you did not take the flight.*

**Example A.39. Small differences in large numbers:** *Compute $f(R,h) = \sqrt{R^2+h^2}-R$ for $R = 6378.164$ km and $h = 1.0$ m.*

*Inserting these values into a scientific calculator, one finds that*

$$f(6378164, 1) = \sqrt{6378164^2+1} - 6378164 = 1\times 10^{-7}\ m.$$

*In some calculators one might obtain 0, in other calculators, or computer algebra systems like Maple, one might obtain other answers. What answer do you get and how accurate is your answer?*

*The problem with this computation is that $R \gg h$. Therefore, the computation of $f(R,h)$ depends on how many digits the computing device can handle. The best way to get an answer is to use the binomial approximation. Writing $h = Rx$, or $x = \frac{h}{R}$, we have*

$$\begin{aligned}f(R,h) &= \sqrt{R^2+h^2}-R \\ &= R\sqrt{1+x^2}-R \\ &\simeq R\left[1+\frac{1}{2}x^2\right]-R\end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}Rx^2 \\ &= \frac{1}{2}\frac{h}{R^2} = 7.83926 \times 10^{-8}\ m. \end{aligned} \tag{A.117}$$

*Of course, you should verify how many digits should be kept in reporting the result.*

In the next examples, we generalize this example. Such general computations appear in proofs involving general expansions without specific numerical values given.

**Example A.40.** *Obtain an approximation to $(a+b)^p$ when $a$ is much larger than $b$, denoted by $a \gg b$.*

*If we neglect $b$ then $(a+b)^p \simeq a^p$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion. Namely, what is the power of $b/a$ of the first neglected term in this expansion?*

*In order to do this we first divide out $a$ as*

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p.$$

*Now we have a small parameter, $\frac{b}{a}$. According to what we have seen earlier, we can use the binomial expansion to write*

$$\left(1+\frac{b}{a}\right)^n = \sum_{r=0}^{\infty}\begin{pmatrix} p \\ r \end{pmatrix}\left(\frac{b}{a}\right)^r. \tag{A.118}$$

*Thus, we have a sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write*

$$\left(1+\frac{b}{a}\right)^p = 1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

*Here we used $O()$, big-Oh notation, to indicate the size of the first neglected term.*

*Summarizing, we have*

$$\begin{aligned} (a+b)^p &= a^p\left(1+\frac{b}{a}\right)^p \\ &= a^p\left(1+p\frac{b}{a}+O\left(\left(\frac{b}{a}\right)^2\right)\right) \\ &= a^p + pa^p\frac{b}{a} + a^pO\left(\left(\frac{b}{a}\right)^2\right). \end{aligned} \tag{A.119}$$

*Therefore, we can approximate $(a+b)^p \simeq a^p + pba^{p-1}$, with an error on the order of $b^2a^{p-2}$. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a+b)^p \simeq a^p$, but it is not typically good enough in applications because the error in this case is of the order $ba^{p-1}$.*

**Example A.41.** *Approximate $f(x) = (a+x)^p - a^p$ for $x \ll a$.*

*In an earlier example we computed $f(R,h) = \sqrt{R^2+h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m. We can make use of the binomial expansion to determine the behavior of similar functions in the form $f(x) = (a+x)^p - a^p$. Inserting the binomial expression into $f(x)$, we have as $\frac{x}{a} \to 0$ that*

$$\begin{aligned} f(x) &= (a+x)^p - a^p \\ &= a^p\left[\left(1+\frac{x}{a}\right)^p - 1\right] \\ &= a^p\left[\frac{px}{a} + O\left(\left(\frac{x}{a}\right)^2\right)\right] \\ &= O\left(\frac{x}{a}\right) \quad \text{as } \frac{x}{a} \to 0. \end{aligned} \tag{A.120}$$

*This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as*

$$(a+x)^p - a^p = a^{p-1}px + O\left(\left(\frac{x}{a}\right)^2\right) \quad \text{as } \frac{x}{a} \to 0.$$

*We now use this approximation to compute $f(R,h) = \sqrt{R^2+h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m in the earlier example. We let $a = R^2$, $x = 1$ and $p = \frac{1}{2}$. Then, the leading order approximation would be of order*

$$O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4 \times 10^{-14}.$$

*Thus, we have*

$$\sqrt{6378164^2+1} - 6378164 \approx a^{p-1}px$$

*where*

$$a^{p-1}px = (6378164^2)^{-1/2}(0.5)1 = 7.83926 \times 10^{-8}.$$

*This is the same result we had obtained before. However, we have also an estimate of the size of the error and this might be useful in indicating how many digits we should trust in the answer.*

## Problems

**1.** Prove the following identities using only the definitions of the trigonometric functions, the Pythagorean identity, or the identities for sines and cosines of sums of angles.

a. $\cos 2x = 2\cos^2 x - 1$.

b. $\sin 3x = A\sin^3 x + B\sin x$, for what values of $A$ and $B$?

c. $\sec\theta + \tan\theta = \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$.

**2.** Determine the exact values of

a. $\sin\frac{\pi}{8}$.

b. $\tan 15^o$.

c. $\cos 105^o$.

**3.** Denest the following if possible.

a. $\sqrt{3-2\sqrt{2}}$.

b. $\sqrt{1+\sqrt{2}}$.

c. $\sqrt{5+2\sqrt{6}}$.

d. $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$.

e. Find the roots of $x^2+6x-4\sqrt{5}=0$ in simplified form.

**4.** Determine the exact values of

a. $\sin\left(\cos^{-1}\dfrac{3}{5}\right)$.

b. $\tan\left(\sin^{-1}\dfrac{x}{7}\right)$.

c. $\sin^{-1}\left(\sin\dfrac{3\pi}{2}\right)$.

**5.** Do the following.

a. Write $(\cosh x - \sinh x)^6$ in terms of exponentials.

b. Prove $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$ using the exponential forms of the hyperbolic functions.

c. Prove $\cosh 2x = \cosh^2 x + \sinh^2 x$.

d. If $\cosh x = \dfrac{13}{12}$ and $x<0$, find $\sinh x$ and $\tanh x$.

e. Find the exact value of $\sinh(\text{arccosh } 3)$.

**6.** Prove that the inverse hyperbolic functions are the following logarithms:

a. $\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$.

b. $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$.

**7.** Write the following in terms of logarithms:

a. $\cosh^{-1}\frac{4}{3}$.

b. $\tanh^{-1}\frac{1}{2}$.

c. $\sinh^{-1} 2$.

**8.** Solve the following equations for $x$.

a. $\cosh(x+\ln 3) = 3$.

b. $2\tanh^{-1}\frac{x-2}{x-1} = \ln 2$.

c. $\sinh^2 x - 7\cosh x + 13 = 0$.

**9.** Compute the following integrals.

a. $\int xe^{2x^2}\,dx$.

b. $\int_0^3 \dfrac{5x}{\sqrt{x^2+16}} dx$.

c. $\int x^3 \sin 3x \, dx$. (Do this using integration by parts, the Tabular Method, and differentiation under the integral sign.)

d. $\int \cos^4 3x \, dx$.

e. $\int_0^{\pi/4} \sec^3 x \, dx$.

f. $\int e^x \sinh x \, dx$.

g. $\int \sqrt{9-x^2} \, dx$

h. $\int \dfrac{dx}{(4-x^2)^2}$, using the substitution $x = 2\tanh u$.

i. $\int_0^4 \dfrac{dx}{\sqrt{9+x^2}}$, using a hyperbolic function substitution.

j. $\int \dfrac{dx}{1-x^2}$, using the substitution $x = \tanh u$.

k. $\int \dfrac{dx}{(x^2+4)^{3/2}}$, using the substitutions $x = 2\tan\theta$ and $x = 2\sinh u$.

l. $\int \dfrac{dx}{\sqrt{3x^2-6x+4}}$.

**10.** Find the sum for each of the series:

a. $5 + \frac{25}{7} + \frac{125}{49} + \frac{625}{343} + \cdots$.

b. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 3}{4^n}$.

c. $\sum_{n=2}^{\infty} \dfrac{2}{5^n}$.

d. $\sum_{n=-1}^{\infty} (-1)^{n+1} \left(\dfrac{e}{\pi}\right)^n$.

e. $\sum_{n=0}^{\infty} \left(\dfrac{5}{2^n} + \dfrac{1}{3^n}\right)$.

f. $\sum_{n=1}^{\infty} \dfrac{3}{n(n+3)}$.

g. What is $0.56\bar{9}$?

**11.** A superball is dropped from a 2.00 m height. After it rebounds, it reaches a new height of 1.65 m. Assuming a constant coefficient of restitution, find the (ideal) total distance the ball will travel as it keeps bouncing.

**12.** Here are some telescoping series problems.

a. Verify that

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1}\right).$$

b. Find the $n$th partial sum of the series $\sum_{n=1}^{\infty} \left(\dfrac{n+1}{n+2} - \dfrac{n}{n+1}\right)$ and use it to determine the sum of the resulting *telescoping* series.

c. Sum the series $\sum_{n=1}^{\infty}\left[\tan^{-1} n - \tan^{-1}(n+1)\right]$ by first writing the $N$th partial sum and then computing $\lim_{N\to\infty} s_N$.

**13.** Determine the radius and interval of convergence of the following infinite series:

a. $\sum_{n=1}^{\infty}(-1)^n \dfrac{(x-1)^n}{n}$.

b. $\sum_{n=1}^{\infty} \dfrac{x^n}{2^n n!}$.

c. $\sum_{n=1}^{\infty} \dfrac{1}{n}\left(\dfrac{x}{5}\right)^n$.

d. $\sum_{n=1}^{\infty}(-1)^n \dfrac{x^n}{\sqrt{n}}$.

**14.** Find the Taylor series centered at $x = a$ and its corresponding radius of convergence for the given function. In most cases, you need not employ the direct method of computation of the Taylor coefficients.

a. $f(x) = \sinh x$, $a = 0$.

b. $f(x) = \sqrt{1+x}$, $a = 0$.

c. $f(x) = \ln \dfrac{1+x}{1-x}$, $a = 0$.

d. $f(x) = xe^x$, $a = 1$.

e. $f(x) = \dfrac{1}{\sqrt{x}}$, $a = 1$.

f. $f(x) = x^4 + x - 2$, $a = 2$.

g. $f(x) = \dfrac{x-1}{2+x}$, $a = 1$.

**15.** Consider Gregory's expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

a. Derive Gregory's expansion by using the definition

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2},$$

expanding the integrand in a Maclaurin series, and integrating the resulting series term by term.

b. From this result, derive Gregory's series for $\pi$ by inserting an appropriate value for $x$ in the series expansion for $\tan^{-1} x$.

**16.** In the event that a series converges uniformly, one can consider the derivative of the series to arrive at the summation of other infinite series.

a. Differentiate the series representation for $f(x) = \frac{1}{1-x}$ to sum the series $\sum_{n=1}^{\infty} nx^n$, $|x| < 1$.

b. Use the result from part a to sum the series $\sum_{n=1}^{\infty} \dfrac{n}{5^n}$.

c. Sum the series $\sum_{n=2}^{\infty} n(n-1)x^n$, $|x| < 1$.

d. Use the result from part c to sum the series $\sum_{n=2}^{\infty} \dfrac{n^2 - n}{5^n}$.

e. Use the results from this problem to sum the series $\sum_{n=4}^{\infty} \dfrac{n^2}{5^n}$.

**17.** Evaluate the integral $\int_0^{\pi/6} \sin^2 x\, dx$ by doing the following:

a. Compute the integral exactly.

b. Integrate the first three terms of the Maclaurin series expansion of the integrand and compare with the exact result.

**18.** Determine the next term in the time dilation example, A.38. That is, find the $\frac{v^4}{c^2}$ term and determine a better approximation to the time difference of 1 ns.

**19.** Evaluate the following expressions at the given point. Use your calculator or your computer (such as Maple). Then use series expansions to find an approximation to the value of the expression to as many places as you trust.

a. $\dfrac{1}{\sqrt{1+x^3}} - \cos x^2$ at $x = 0.015$.

b. $\ln \sqrt{\dfrac{1+x}{1-x}} - \tan x$ at $x = 0.0015$.

c. $f(x) = \dfrac{1}{\sqrt{1+2x^2}} - 1 + x^2$ at $x = 5.00 \times 10^{-3}$.

d. $f(R, h) = R - \sqrt{R^2 + h^2}$ for $R = 1.374 \times 10^3$ km and $h = 1.00$ m.

e. $f(x) = 1 - \dfrac{1}{\sqrt{1-x}}$ for $x = 2.5 \times 10^{-13}$.

# Index