

The $\hat{\mathbf{r}}$ Unit Vector

1 The $\hat{\mathbf{r}}$ Unit Vector

One approach to finding the electric force between two charges is to use $F = k|q_1q_2|/r^2$ to find the magnitude and a diagram and trigonometry to write $\vec{\mathbf{F}}$ in the form $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}}$.

An alternative approach is to use an equation for electric force using a unit vector $\hat{\mathbf{r}}$:

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2 \frac{\hat{\mathbf{r}}_{12}}{r^2}$$

where $\hat{\mathbf{r}}_{12}$ is the unit vector that points from the position of q_1 to the position of q_2 , and r is the distance between q_1 and q_2 .

To find $\hat{\mathbf{r}}_{12}$,

1. draw a vector, $\vec{\mathbf{r}}_{12}$, from q_1 to q_2 ;
2. Write $\vec{\mathbf{r}}_{12}$ in the form $\vec{\mathbf{r}}_{12} = r_x\hat{\mathbf{i}} + r_y\hat{\mathbf{j}}$ using the diagram; then
3. $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

The equation for electric field using a unit vector $\hat{\mathbf{r}}$ is

$$\vec{\mathbf{E}}_{\text{due to } q} = kq \frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q to the point in space where we want to know $\vec{\mathbf{E}}$, and r is the distance between q and that point.

To find $\hat{\mathbf{r}}$,

1. draw a vector, $\vec{\mathbf{r}}$, from q to the point in space where you want to know $\vec{\mathbf{E}}$;
2. Write $\vec{\mathbf{r}}$ in the form $\vec{\mathbf{r}} = r_x\hat{\mathbf{i}} + r_y\hat{\mathbf{j}}$ using the diagram; then
3. $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

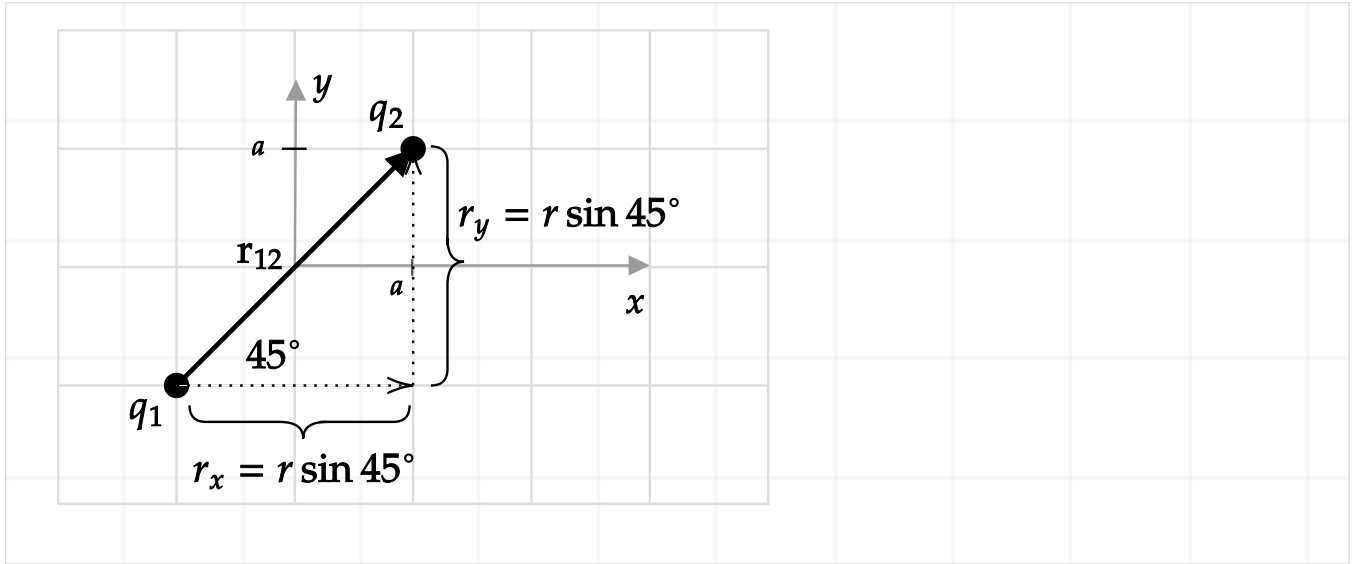
Note that in the equations for $\vec{\mathbf{F}}$ and $\vec{\mathbf{E}}$, we do not need to take the absolute value of the charges.

2 Example I

Charge q_1 is at $(x, y) = (-a, -a)$ and charge q_2 is at (a, a) . Find

1. $\hat{\mathbf{r}}_{12}$
2. $\vec{\mathbf{F}}_{q_1 \text{ on } q_2}$
3. $F_{q_1 \text{ on } q_2}$

Solution



$$\vec{\mathbf{r}}_{12} = r \cos(45^\circ) \hat{\mathbf{i}} + r \sin(45^\circ) \hat{\mathbf{j}}, \quad \hat{\mathbf{r}}_{12} = \frac{\vec{\mathbf{r}}}{r} = \cos(45^\circ) \hat{\mathbf{i}} + \sin(45^\circ) \hat{\mathbf{j}}$$

Note that the magnitude of $\hat{\mathbf{r}}_{12} = 1$: $|\hat{\mathbf{r}}_{12}| = \sqrt{\cos^2(45^\circ) + \sin^2(45^\circ)} = 1$

The sides of the right triangle have length $2a$, so the hypotenuse $r = \sqrt{8}a$. Substitution gives

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2 \frac{\hat{\mathbf{r}}_{12}}{r^2} = \frac{kq_1q_2}{8a^2} (\cos 45^\circ \hat{\mathbf{i}} + \sin 45^\circ \hat{\mathbf{j}}) = \frac{kq_1q_2}{8a^2} \left[\frac{1}{\sqrt{2}} \hat{\mathbf{i}} + \frac{1}{\sqrt{2}} \hat{\mathbf{j}} \right]$$

Check: if q_1 and q_2 are both positive or both negative, the force on q_2 is upwards and to the right, as expected.

To calculate $|\vec{\mathbf{F}}|$, we can use $|\vec{\mathbf{F}}| = F = \sqrt{F_x^2 + F_y^2}$ and plug in $F_x = k \frac{q_1q_2}{8a^2} \frac{1}{\sqrt{2}}$ and $F_y = k \frac{q_1q_2}{8a^2} \frac{1}{\sqrt{2}}$ and use $\sqrt{c^2} = |c|$ (where c is a real number) to show that $F = k|q_1q_2|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{F}} = kq_2q_1 \frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{F}}| = F = k|q_1q_2| \frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so $F = k|q_1q_2| \frac{1}{r^2} = k|q_1q_2| \frac{1}{8a^2}$.

3 Problem I

Charge q_1 is at $(x, y) = (-a, a)$ and charge q_2 is at $(a, 0)$. Find

1. $\hat{\mathbf{r}}_{12}$
2. $\vec{\mathbf{F}}_{q_1 \text{ on } q_2}$
3. $F_{q_1 \text{ on } q_2}$

4 Example II

If q_1 is at $(x, y) = (-a, -a)$, find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{E}}_{\text{at } (a,a) \text{ due to } q_1}$
3. $E_{\text{at } (a,a) \text{ due to } q_1}$

Solution

The calculation of $\hat{\mathbf{r}}$ is the same as that shown in the diagram Example I (except we do not need subscripts for the $\vec{\mathbf{E}}$ formula).

Substitution gives

$$\vec{\mathbf{E}}_{\text{at } (a,a) \text{ due to } q_1} = kq_1 \frac{\hat{\mathbf{r}}}{r^2} = kq_1 \frac{1}{8a^2} (\cos 45^\circ \hat{\mathbf{i}} + \sin 45^\circ \hat{\mathbf{j}}) = \frac{kq_1}{8a^2} \left[\frac{1}{\sqrt{2}} \hat{\mathbf{i}} + \frac{1}{\sqrt{2}} \hat{\mathbf{j}} \right]$$

(Notice the relationship between the answers to this problem and the answers to Example I.)

Check: If a positive charge was placed at $(x, y) = (a, a)$, it would tend to move up and to the right, which is consistent with the signs on the components of the electric field found above.

To calculate $|\vec{\mathbf{E}}|$, we can use

$$|\vec{\mathbf{E}}| = E = \sqrt{E_x^2 + E_y^2}$$

and plug in $E_x = k \frac{q_1}{8a^2} \frac{1}{\sqrt{2}}$ and $E_y = k \frac{q_1}{8a^2} \frac{1}{\sqrt{2}}$ and use $\sqrt{c^2} = |c|$ (where c is a real number) to show that $E = k|q_1|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{E}} = kq_1 \frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{E}}| = k|q_1| \frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so

$$|\vec{\mathbf{E}}| = k|q_1| \frac{1}{r^2} = \frac{k|q_1|}{8a^2}.$$

5 Problem II

If q_1 is at $(x, y) = (-a, a)$, find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{E}}_{\text{at}}(a,0)$ due to q_1
3. $E_{\text{at}}(a,0)$ due to q_1

6 Additional Problems

6.1 Computing $\hat{\mathbf{r}}$ for $\vec{\mathbf{F}}$ formula

If q_1 is at $(x, y) = (-a, 2a)$ and q_2 is at $(x, y) = (a, 0)$, find

1. $\hat{\mathbf{r}}_{12}$
2. $\hat{\mathbf{r}}_{21}$
3. r

6.2 Computing $\hat{\mathbf{r}}$ for $\vec{\mathbf{E}}$ formula

If q_1 is at $(x, y) = (a, 0)$ and the point where we want to compute $\vec{\mathbf{E}}$ is at $(x, y) = (-a, 2a)$, find

1. $\hat{\mathbf{r}}$
2. $\hat{\mathbf{r}}$
3. r

6.3 Problem I Follow-up

For the charge configuration given in Problem I, find

1. $\hat{\mathbf{r}}_{21}$
2. $\vec{\mathbf{F}}_{q_2 \text{ on } q_1}$
3. $F_{q_2 \text{ on } q_1}$