Logarithms

1 Introduction – Base 10 Logarithms

The motivation for the base 10 logarithm is that it reduces numbers raised by a power of 10 to the power the number was raised to. So 10^2 becomes 2, 10^3 becomes 3, etc. The base 10 logarithm is sometimes called the "common logarithm".

In mathematical notation,

$\log_{10}(10^x) = x$

For example,

$\log_{10}(10^{-5}) = -5$ and $\log_{10}(10^7) = 7$

(To take the base 10 logarithm of a number that is not exactly a power of 10, use a calculator.)

Several identities follow as a result:

1. If you raise a base 10 logged number by 10, you get back the number that was logged.

   $10^{\log_{10}(x)} = x$

   For example,

   $10^{\log_{10}(7)} = 7$ and $10^{\log_{10}(8.8)} = 8.8$
2. The sum of two logged numbers is the log of the product of the numbers:

   $\log_{10} y + \log_{10} x = \log_{10}(yx)$;

   For example,

   $\log_{10} 10 + \log_{10} 100 = \log_{10} 10 \cdot 100 = \log_{10} 10^3 = 3$
3. The difference between two logged number is the log of the ratio of the numbers:

   $\log_{10} y - \log_{10} x = \log_{10}(y/x)$

   For example,

   $\log_{10} 10 - \log_{10} 100 = \log_{10}(10/100) = \log_{10} 10^{-1} = -1$

1.1 Problems

1. What is $\log_{10}(0.000000001)$?

   Answer: $\log_{10}(0.000000001) = \log_{10}(10^{-9}) = -9$
2. What is $\log_{10}(10,000)$?

   Answer: $\log_{10}(10,000) = \log_{10}(10^4) = 4$
3. $\log_{10}(10,000) + \log_{10}(0.000000001) = \log_{10}(x)$. Find x.

   Answer: $\log_{10}(10^4) + \log_{10}(10^{-9}) = \log_{10}(10^4 \cdot 10^{-9}) = \log_{10}(10^{-5}) \Rightarrow x = 10^{-9}$
4. $\log_{10}(10,000) - \log_{10}(0.000000001) = \log_{10}(x)$. Find x.

Answer: $\log_{10}(10^4) - \log_{10}(10^{-9}) = \log_{10}(10^4/10^{-9}) = \log_{10}(10^{13}) \Rightarrow x = 10^{13}$

5. If $x = x_o \log_{10}(y/y_o)$, solve for y.

   Answer: $x/x_o = \log_{10}(y/y_o)$; $10^{x/x_o} = 10^{\log_{10}(y/y_o)} = y/y_o \Rightarrow y = y_o 10^{x/x_o}$

2 Introduction – Base e Logarithm

The base 10 logarithm reduces numbers raised by a power of 10 to the power the number was raised to.

The base e logarithm reduces numbers raised by a power of e to the power the number was raised to. It is represented by $\log_e x$, or more commonly, $\ln(x)$.

“ln” represents the “natural logarithm”. The term “natural” is used because the exponential e appears in many natural problems, for example, some populations grow in proportion to $e^{t/\tau}$, where the constant τ is a growth rate.

In mathematical notation,

$\ln(e^x) = x$; for example $\ln(e^{-5}) = -5$ and $\ln(e^7) = 7$

Several identities follow as a result:

1. If you raise a base e logged number by e, you get back the number that was logged.

   $e^{\ln(x)} = x$
2. The sum of two logged numbers is the log of the product of the numbers:

   $\ln(y) + \ln(x) = \ln(yx)$;
3. The difference between two logged number is the log of the ratio of the numbers:

   $\ln(y) - \ln(x) = \ln(y/x)$

2.1 Problems

1. What is $\ln(e^3)$?

   Answer: 3
2. What is $\ln(1/e^3)$?

   Answer: -3
3. $\ln(e^{-4}) + \ln(e^3) = \ln(x)$. Find x

   Answer: $\ln(e^{-4}) + \ln(e^3) = \ln(e^{-4} \cdot e^3) = \ln(e^{-1}) = -1$
4. $\ln(e^{-4}) - \ln(e^3) = \ln(x)$. Find x.

   Answer: $\ln(e^{-4}) - \ln(e^3) = \ln(e^{-4}/e^3) = \ln(e^{-7}) = -7$
5. If $x = x_o \ln(y/y_o)$, solve for y in terms of x.

   Answer: $x/x_o = \ln(y/y_o)$; $e^{x/x_o} = e^{\ln(y/y_o)} = y/y_o \Rightarrow y = y_o e^{x/x_o}$