The $\hat{\mathbf{r}}$ Unit Vector

1 The $\hat{\mathbf{r}}$ Unit Vector

One approach to finding the electric force between two charges is to use $F = k|q_1q_2|/r^2$ to find the magnitude and a diagram to write $\vec{\mathbf{F}}$ in the form $\vec{\mathbf{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$.

An alternative, and more direct, approach is to use an equation for electric force using a unit vector $\hat{\mathbf{r}}$:

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2\frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q_1 to the position of q_2, and r is the distance between q_1 and q_2.

To find $\hat{\mathbf{r}}$,

1. draw a vector, $\vec{\mathbf{r}}$, from q_1 to q_2;
2. Write $\vec{\mathbf{r}}$ in the form $\vec{\mathbf{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$; then
3. $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

Similarly, the equation for electric field using a unit vector $\hat{\mathbf{r}}$ is

$$\vec{\mathbf{E}}_{\text{due to } q} = kq\frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q_1 to the point in space where we want to know $\vec{\mathbf{E}}$, and r is the distance between q_1 and that point.

To find $\hat{\mathbf{r}}$,

1. draw a vector, $\vec{\mathbf{r}}$ from q_1 to the point in space where you want to know $\vec{\mathbf{E}}$;
2. Write $\vec{\mathbf{r}}$ in the form $\vec{\mathbf{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$; then
3. $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

Note that in the equations for $\vec{\mathbf{F}}$ and $\vec{\mathbf{E}}$, we do not need to take the absolute value of the charges.

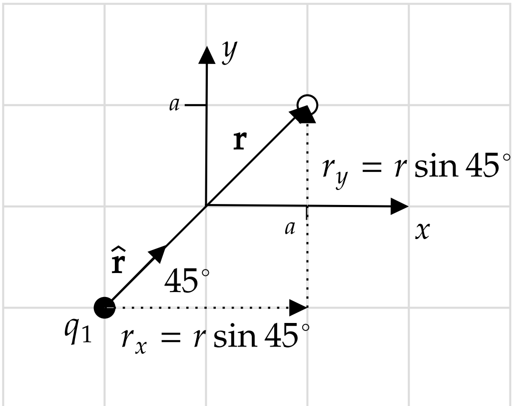
2 Example I

Charge q_1 is at $(x, y) = (-a, -a)$ and charge q_2 is at (a, a). Find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{F}}_{q_1 \text{ on } q_2}$
3. $F_{q_1 \text{ on } q_2}$

Solution

The calculation of $\hat{\mathbf{r}}$ is shown in the following diagram.



$$\mathbf{r} = r_x\mathbf{i} + r_y\mathrm{j} = r\cos 45^\circ\mathbf{i} + r\sin 45^\circ\ \mathrm{j}$$

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{\mathbf{r}}{r}\\ &= \frac{r\cos 45^\circ\mathbf{i} + r\sin 45^\circ\ \mathrm{j}}{r}\\ &= \cos 45^\circ\mathbf{i} + \sin 45^\circ\ \mathrm{j}\end{aligned}$$

Substitution gives

$$\vec{\mathbf{F}}_{q_1\text{ on }q_2} = kq_1q_2\frac{\hat{\mathbf{r}}}{r^2} = \frac{kq_1q_2}{8a^2}(\cos 45^\circ\,\hat{\imath} + \sin 45^\circ\,\hat{\jmath}) = \frac{kq_1q_2}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right]$$

Check: if q_1 and q_2 are both positive, the force on q_2 is upwards and to the right, as expected.

3 Problem I

Charge q_1 is at $(x, y) = (-a, a)$ and charge q_2 is at $(a, 0)$. Find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{F}}_{q_1\text{ on }q_2}$
3. $F_{q_1\text{ on }q_2}$

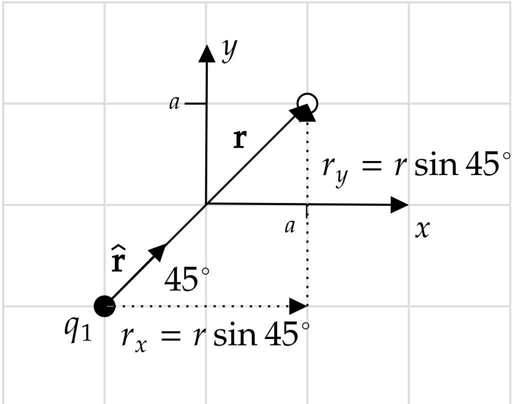
4 Example II

If q_1 is at $(x, y) = (-a, -a)$, find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{E}}_{\text{at }(a,a)\text{ due to }q_1}$
3. $E_{\text{at }(a,a)\text{ due to }q_1}$

Solution

The calculation of $\hat{\mathbf{r}}$ is shown in the following diagram.



$$\mathbf{r} = r_x\mathbf{i} + r_y\mathrm{j} = r\cos 45^\circ\mathbf{i} + r\sin 45^\circ\,\mathrm{j}$$

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{\mathbf{r}}{r}\\ &= \frac{r\cos 45^\circ\mathbf{i} + r\sin 45^\circ\,\mathrm{j}}{r}\\ &= \cos 45^\circ\mathbf{i} + \sin 45^\circ\,\mathrm{j}\end{aligned}$$

Substitution gives

$$\vec{\mathbf{E}}_{\text{at }(a,a)\text{ due to }q_1} = kq_1\frac{\hat{\mathbf{r}}}{r^2} = kq_1\frac{1}{8a^2}(\cos 45^\circ\,\hat{\boldsymbol{\imath}} + \sin 45^\circ\,\hat{\boldsymbol{\jmath}}) = \frac{kq_1}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

Check: If a positive charge was placed at $(x,y) = (a,a)$, it would tend to move up and to the right, which is consitent with the signs on the components of the electric field found above.

To calculate $|\vec{\mathbf{E}}|$, we can use

$$|\vec{\mathbf{E}}| = E = \sqrt{E_x^2 + E_y^2}$$

and plug in $E_x = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and $E_y = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and use $\sqrt{c^2} = |c|$ (where c is a real number) to show that $E = k|q_1|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{E}} = kq_1\frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{E}}| = k|q_1|\frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so

$$|\vec{\mathbf{E}}| = k|q_1|\frac{1}{r^2} = \frac{k|q_1|}{8a^2}.$$

5 Problem II

If q_1 is at $(x,y) = (-a,a)$, find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{E}}_{\text{at }(a,0)\text{ due to }q_1}$
3. $E_{\text{at }(a,0)\text{ due to }q_1}$