The $\hat{\mathbf{r}}$ Unit Vector

1 The $\hat{\mathbf{r}}$ Unit Vector

One approach to finding the electric force between two charges and the electric field due to a point charge is to use $F = k|q_1 q_2|/r^2$ to find the magnitude and a diagram to write $\vec{\mathbf{F}}$ in the form $\vec{\mathbf{F}} = F_x \hat{\boldsymbol{\imath}} + F_y \hat{\boldsymbol{\jmath}}$.

An alternative approach is to use an equation for electric force and electric field using a unit vector $\hat{\mathbf{r}}$. This approach is sometimes used for finding the electric field due to a continuous charge distribution. In addition, the $\hat{\mathbf{r}}$ unit vector is often used when finding the magnetic field using the Biot–Savart law.

1.1 Electric Force

The equation for electric force (Coulomb's law) using a unit vector $\hat{\mathbf{r}}$ is

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1 q_2 \frac{\hat{\mathbf{r}}_{12}}{r^2}$$

where $\hat{\mathbf{r}}_{12}$ is the unit vector that points from the position of q_1 to the position of q_2, and r is the distance between q_1 and q_2.

To find $\hat{\mathbf{r}}_{12}$,

1. draw a vector, $\vec{\mathbf{r}}_{12}$, from q_1 to q_2;
2. Write $\vec{\mathbf{r}}_{12}$ in the form $\vec{\mathbf{r}}_{12} = r_x \hat{\boldsymbol{\imath}} + r_y \hat{\boldsymbol{\jmath}}$ using the diagram; then
3. $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

1.2 Electric Field

The equation for electric field using a unit vector $\hat{\mathbf{r}}$ is

$$\vec{\mathbf{E}}_{\text{due to } q} = kq \frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q to the point in space where we want to know $\vec{\mathbf{E}}$, and r is the distance between q and that point.

To find $\hat{\mathbf{r}}$,

1. draw a vector, $\vec{\mathbf{r}}$,from q to the point in space where you want to know $\vec{\mathbf{E}}$;
2. Write $\vec{\mathbf{r}}$ in the form $\vec{\mathbf{r}} = r_x \hat{\boldsymbol{\imath}} + r_y \hat{\boldsymbol{\jmath}}$ using the diagram; then
3. $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

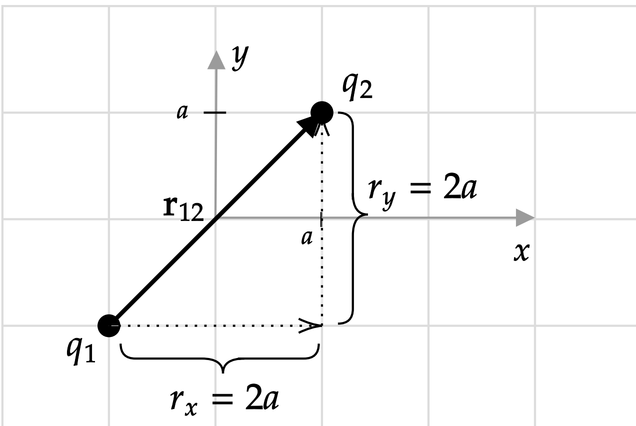
Note that in the equations for $\vec{\mathbf{F}}$ and $\vec{\mathbf{E}}$, we do not need to take the absolute value of the charges.

2 Example I

Charge q_1 is at $(x, y) = (-a, -a)$ and charge q_2 is at (a, a). Find

1. $\hat{\mathbf{r}}_{12}$
2. $\vec{\mathbf{F}}_{q_1 \text{ on } q_2}$
3. $F_{q_1 \text{ on } q_2}$

Solution



The sides of the right triangle have length $2a$, so the hypotenuse $r = \sqrt{(2a)^2 + (2a)^2} = \sqrt{8}a$.

From the diagram, $\vec{\mathbf{r}}_{12} = 2a\hat{\boldsymbol{\imath}} + 2a\hat{\boldsymbol{\jmath}}$, so $\hat{\mathbf{r}}_{12} = \dfrac{\vec{\mathbf{r}}_{12}}{r} = \dfrac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \dfrac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}$

Note that the magnitude of $\hat{\mathbf{r}}_{12} = 1$: $|\hat{\mathbf{r}}_{12}| = \sqrt{(1/2)^2 + (1/2)^2} = 1$

Substitution gives

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2\frac{\hat{\mathbf{r}}_{12}}{r^2} = \frac{kq_1q_2}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

Check: if q_1 and q_2 are both positive or both negative, the force on q_2 is upwards and to the right, as expected.

To calculate $|\vec{\mathbf{F}}|$, we can use $|\vec{\mathbf{F}}| = F = \sqrt{F_x^2 + F_y^2}$ and plug in $F_x = k\frac{q_1q_2}{8a^2}\frac{1}{\sqrt{2}}$ and $F_y = k\frac{q_1q_2}{8a^2}\frac{1}{\sqrt{2}}$ and use $\sqrt{c^2} = |c|$ (where c is a real number) to show that $F = k|q_1q_2|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{F}} = kq_2q_1\frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{F}}| = F = k|q_1q_2|\frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so $F = k|q_1q_2|\dfrac{1}{r^2} = k|q_1q_2|\dfrac{1}{8a^2}$.

3 Problem I

Charge q_1 is at $(x, y) = (-a, a)$ and charge q_2 is at $(a, 0)$. Find

1. $\hat{\mathbf{r}}_{12}$
2. $\vec{\mathbf{F}}_{q_1 \text{ on } q_2}$
3. $F_{q_1 \text{ on } q_2}$

Solution

$r = \sqrt{5}a$

$\vec{\mathbf{r}}_{12} = 2a\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}$

1. $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r = \frac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{5}}\hat{\boldsymbol{\jmath}}$
2. $\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2\dfrac{\hat{\mathbf{r}}_{12}}{r^2} = \dfrac{kq_1q_2}{5a^2}\left(\dfrac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \dfrac{1}{\sqrt{5}}\hat{\boldsymbol{\jmath}}\right)$
3. $F_{q_1 \text{ on } q_2} = k|q_1q_2|\dfrac{1}{r^2} = k|q_1q_2|\dfrac{1}{5a^2}$

4 Example II

If q_1 is at $(x, y) = (-a, -a)$, find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{E}}_{\text{at }(a,a)\text{ due to }q_1}$
3. $E_{\text{at }(a,a)\text{ due to }q_1}$

Solution

The calculation of $\hat{\mathbf{r}}$ is the same as that shown in the diagram Example I (except we do not need subscripts for the $\vec{\mathbf{E}}$ formula).

Substitution gives

$$\vec{\mathbf{E}}_{\text{at }(a,a)\text{ due to }q_1} = kq_1\frac{\hat{\mathbf{r}}}{r^2} = \frac{kq_1}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

Check: If a positive charge was placed at $(x, y) = (a, a)$, it would tend to move up and to the right, which is consitent with the signs on the components of the electric field found above.

To calculate $|\vec{\mathbf{E}}|$, we can use

$$|\vec{\mathbf{E}}| = E = \sqrt{E_x^2 + E_y^2}$$

and plug in $E_x = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and $E_y = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and use $\sqrt{c^2} = |c|$ (where c is a real number) to show that $E = k|q_1|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{E}} = kq_1\frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{E}}| = k|q_1|\frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so

$$|\vec{\mathbf{E}}| = k|q_1|\frac{1}{r^2} = \frac{k|q_1|}{8a^2}.$$

(Notice the relationship between the answers to this problem and the answers to Example I.)

5 Problem II

If q_1 is at $(x, y) = (-a, a)$, find

1. $\hat{\mathbf{r}}$
2. $\vec{\mathbf{E}}_{\text{at } (a,0) \text{ due to } q_1}$
3. $E_{\text{at } (a,0) \text{ due to } q_1}$

Solution:

(Notice the relationship between the answers to this problem and the answers to Problem I.)

$r = \sqrt{5}a$

$\vec{\mathbf{r}}_{12} = 2a\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}$

1. $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r = \frac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{5}}\hat{\boldsymbol{\jmath}}$
2. $\vec{\mathbf{E}} = kq_1\dfrac{\hat{\mathbf{r}}}{r^2} = \dfrac{kq_1}{5a^2}\left(\dfrac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \dfrac{1}{\sqrt{5}}\hat{\boldsymbol{\jmath}}\right)$
3. $E = k|q_1|\dfrac{1}{r^2} = k|q_1|\dfrac{1}{5a^2}$

6 Additional Problems

6.1 Computing $\hat{\mathbf{r}}$ for $\vec{\mathbf{F}}$ formula

If q_1 is at $(x, y) = (-a, 2a)$ and q_2 is at $(x, y) = (a, 0)$, find

1. $\hat{\mathbf{r}}_{12}$
2. $\hat{\mathbf{r}}_{21}$
3. r

6.2 Computing $\hat{\mathbf{r}}$ for $\vec{\mathbf{E}}$ formula

If q_1 is at $(x, y) = (a, 0)$ and the point where we want to compute $\vec{\mathbf{E}}$ is at $(x, y) = (-a, 2a)$, find

2. $\hat{\mathbf{r}}$
3. r

6.3 Finding $\hat{\mathbf{r}}$ given positions in polar form

Charge q_1 is a distance a from the origin and at an angle of 45° from the $+x$ axis (counterclockwise positive).

Charge q_2 is a distance $2a$ from the origin and at an angle of 135° from the $+x$ axis (counterclockwise positive).

Find

1. $\hat{\mathbf{r}}_{12}$
2. $\hat{\mathbf{r}}_{21}$

6.4 Problem I Follow-up

For the charge configuration given in Problem I, find

1. $\hat{\mathbf{r}}_{21}$
2. $\vec{\mathbf{F}}_{q_2 \text{ on } q_1}$
3. $F_{q_2 \text{ on } q_1}$