

# The $\hat{\mathbf{r}}$ Unit Vector

## 1 The $\hat{\mathbf{r}}$ Unit Vector

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One approach to finding the electric force between two charges and the electric field due to a point charge is to use  $F = k|q_1q_2|/r^2$  to find the magnitude and a diagram to write  $\vec{\mathbf{F}}$  in the form  $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}}$ .

An alternative approach is to use an equation for electric force and electric field using a unit vector  $\hat{\mathbf{r}}$ . This approach is sometimes used for finding the electric field due to a continuous charge distribution. In addition, the  $\hat{\mathbf{r}}$  unit vector is often used when finding the magnetic field using the Biot–Savart law.

### 1.1 Electric Force

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The equation for electric force (Coulomb’s law) using a unit vector  $\hat{\mathbf{r}}$  is

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2 \frac{\hat{\mathbf{r}}_{12}}{r^2}$$

where  $\hat{\mathbf{r}}_{12}$  is the unit vector that points from the position of  $q_1$  to the position of  $q_2$ , and  $r$  is the distance between  $q_1$  and  $q_2$ .

To find  $\hat{\mathbf{r}}_{12}$ ,

1. draw a vector,  $\vec{\mathbf{r}}_{12}$ , from  $q_1$  to  $q_2$ ;
2. Write  $\vec{\mathbf{r}}_{12}$  in the form  $\vec{\mathbf{r}}_{12} = r_x\hat{\mathbf{i}} + r_y\hat{\mathbf{j}}$  using the diagram; then
3.  $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r$ , where  $r = \sqrt{r_x^2 + r_y^2}$ .

### 1.2 Electric Field

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The equation for electric field using a unit vector  $\hat{\mathbf{r}}$  is

$$\vec{\mathbf{E}}_{\text{due to } q} = kq \frac{\hat{\mathbf{r}}}{r^2}$$

where  $\hat{\mathbf{r}}$  is the unit vector that points from the position of  $q$  to the point in space where we want to know  $\vec{\mathbf{E}}$ , and  $r$  is the distance between  $q$  and that point.

To find  $\hat{\mathbf{r}}$ ,

1. draw a vector,  $\vec{\mathbf{r}}$ , from  $q$  to the point in space where you want to know  $\vec{\mathbf{E}}$ ;
2. Write  $\vec{\mathbf{r}}$  in the form  $\vec{\mathbf{r}} = r_x\hat{\mathbf{i}} + r_y\hat{\mathbf{j}}$  using the diagram; then
3.  $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$ , where  $r = \sqrt{r_x^2 + r_y^2}$ .

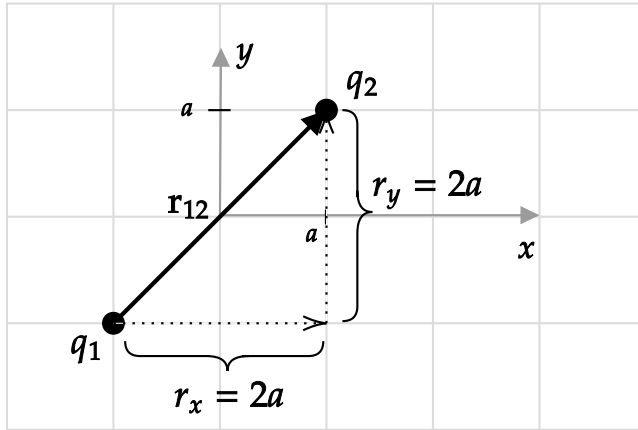
Note that in the equations for  $\vec{\mathbf{F}}$  and  $\vec{\mathbf{E}}$ , we do not need to take the absolute value of the charges.

## 2 Example I

Charge  $q_1$  is at  $(x, y) = (-a, -a)$  and charge  $q_2$  is at  $(a, a)$ . Find

1.  $\hat{\mathbf{r}}_{12}$
2.  $\vec{\mathbf{F}}_{q_1 \text{ on } q_2}$
3.  $F_{q_1 \text{ on } q_2}$

**Solution**



The sides of the right triangle have length  $2a$ , so the hypotenuse  $r = \sqrt{(2a)^2 + (2a)^2} = \sqrt{8}a$ .

From the diagram,  $\vec{\mathbf{r}}_{12} = 2a\hat{\mathbf{i}} + 2a\hat{\mathbf{j}}$ , so  $\hat{\mathbf{r}}_{12} = \frac{\vec{\mathbf{r}}_{12}}{r} = \frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}$

Note that the magnitude of  $\hat{\mathbf{r}}_{12} = 1$ :  $|\hat{\mathbf{r}}_{12}| = \sqrt{(1/2)^2 + (1/2)^2} = 1$

Substitution gives

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2 \frac{\hat{\mathbf{r}}_{12}}{r^2} = \frac{kq_1q_2}{8a^2} \left[ \frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}} \right]$$

Check: if  $q_1$  and  $q_2$  are both positive or both negative, the force on  $q_2$  is upwards and to the right, as expected.

To calculate  $|\vec{\mathbf{F}}|$ , we can use  $|\vec{\mathbf{F}}| = F = \sqrt{F_x^2 + F_y^2}$  and plug in  $F_x = k \frac{q_1q_2}{8a^2} \frac{1}{\sqrt{2}}$  and  $F_y = k \frac{q_1q_2}{8a^2} \frac{1}{\sqrt{2}}$  and use  $\sqrt{c^2} = |c|$  (where  $c$  is a real number) to show that  $F = k|q_1q_2|/8a^2$ . There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{F}} = kq_2q_1 \frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{F}}| = F = k|q_1q_2| \frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so  $F = k|q_1q_2| \frac{1}{r^2} = k|q_1q_2| \frac{1}{8a^2}$ .

### 3 Problem I

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Charge  $q_1$  is at  $(x, y) = (-a, a)$  and charge  $q_2$  is at  $(a, 0)$ . Find

1.  $\hat{\mathbf{r}}_{12}$

2.  $\vec{\mathbf{F}}_{q_1 \text{ on } q_2}$

3.  $F_{q_1 \text{ on } q_2}$

**Solution**

$$r = \sqrt{5}a$$

$$\vec{\mathbf{r}}_{12} = 2a\hat{\mathbf{i}} - a\hat{\mathbf{j}}$$

1.  $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r = \frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}}$

2.  $\vec{\mathbf{F}}_{q_1 \text{ on } q_2} = kq_1q_2 \frac{\hat{\mathbf{r}}_{12}}{r^2} = \frac{kq_1q_2}{5a^2} \left( \frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}} \right)$

3.  $F_{q_1 \text{ on } q_2} = k|q_1q_2| \frac{1}{r^2} = k|q_1q_2| \frac{1}{5a^2}$

## 4 Example II

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If  $q_1$  is at  $(x, y) = (-a, -a)$ , find

1.  $\hat{\mathbf{r}}$
2.  $\vec{\mathbf{E}}_{\text{at } (a,a) \text{ due to } q_1}$
3.  $E_{\text{at } (a,a) \text{ due to } q_1}$

### Solution

The calculation of  $\hat{\mathbf{r}}$  is the same as that shown in the diagram Example I (except we do not need subscripts for the  $\vec{\mathbf{E}}$  formula).

Substitution gives

$$\vec{\mathbf{E}}_{\text{at } (a,a) \text{ due to } q_1} = kq_1 \frac{\hat{\mathbf{r}}}{r^2} = \frac{kq_1}{8a^2} \left[ \frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}} \right]$$

Check: If a positive charge was placed at  $(x, y) = (a, a)$ , it would tend to move up and to the right, which is consistent with the signs on the components of the electric field found above.

To calculate  $|\vec{\mathbf{E}}|$ , we can use

$$|\vec{\mathbf{E}}| = E = \sqrt{E_x^2 + E_y^2}$$

and plug in  $E_x = k \frac{q_1}{8a^2} \frac{1}{\sqrt{2}}$  and  $E_y = k \frac{q_1}{8a^2} \frac{1}{\sqrt{2}}$  and use  $\sqrt{c^2} = |c|$  (where  $c$  is a real number) to show that  $E = k|q_1|/8a^2$ . There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{E}} = kq_1 \frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{E}}| = k|q_1| \frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so

$$|\vec{\mathbf{E}}| = k|q_1| \frac{1}{r^2} = \frac{k|q_1|}{8a^2}.$$

(Notice the relationship between the answers to this problem and the answers to Example I.)

## 5 Problem II

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If  $q_1$  is at  $(x, y) = (-a, a)$ , find

1.  $\hat{\mathbf{r}}$
2.  $\vec{\mathbf{E}}_{\text{at } (a,0)} \text{ due to } q_1$
3.  $E_{\text{at } (a,0)} \text{ due to } q_1$

**Solution:**

(Notice the relationship between the answers to this problem and the answers to Problem I.)

$$r = \sqrt{5}a$$

$$\vec{\mathbf{r}}_{12} = 2a\hat{\mathbf{i}} - a\hat{\mathbf{j}}$$

1.  $\hat{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{12}/r = \frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}}$
2.  $\vec{\mathbf{E}} = kq_1 \frac{\hat{\mathbf{r}}}{r^2} = \frac{kq_1}{5a^2} \left( \frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}} \right)$
3.  $E = k|q_1| \frac{1}{r^2} = k|q_1| \frac{1}{5a^2}$

## 6 Additional Problems

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### 6.1 Computing $\hat{\mathbf{r}}$ for $\vec{\mathbf{F}}$ formula

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If  $q_1$  is at  $(x, y) = (-a, 2a)$  and  $q_2$  is at  $(x, y) = (a, 0)$ , find

1.  $\hat{\mathbf{r}}_{12}$
2.  $\hat{\mathbf{r}}_{21}$
3.  $r$

### 6.2 Computing $\hat{\mathbf{r}}$ for $\vec{\mathbf{E}}$ formula

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If  $q_1$  is at  $(x, y) = (a, 0)$  and the point where we want to compute  $\vec{\mathbf{E}}$  is at  $(x, y) = (-a, 2a)$ , find

2.  $\hat{\mathbf{r}}$
3.  $r$

### 6.3 Finding $\hat{\mathbf{r}}$ given positions in polar form

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Charge  $q_1$  is a distance  $a$  from the origin and at an angle of  $45^\circ$  from the  $+x$  axis (counterclockwise positive).

Charge  $q_2$  is a distance  $2a$  from the origin and at an angle of  $135^\circ$  from the  $+x$  axis (counterclockwise positive).

Find

1.  $\hat{\mathbf{r}}_{12}$
2.  $\hat{\mathbf{r}}_{21}$

### 6.4 Problem I Follow-up

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For the charge configuration given in Problem I, find

1.  $\hat{\mathbf{r}}_{21}$
2.  $\vec{\mathbf{F}}_{q_2 \text{ on } q_1}$
3.  $F_{q_2 \text{ on } q_1}$