LRC AC Circuits

1 Introduction

Previously, we have examined circuits with either capacitors and resistors or inductors and resistors that were powered by batteries that produce constant voltage. These are called a direct current (DC) circuits. When a switch was closed to complete a circuit, the current varied in time and the time dependence was exponential (the current had a term $e^{-t/\tau}$).

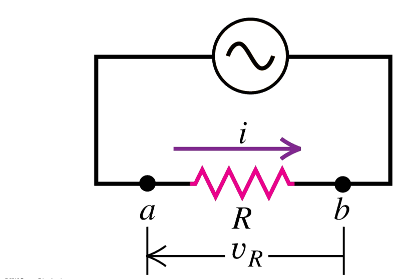
If instead of a battery, we use a voltage source that varies in time sinusouidally, the current will also vary in time sinusoidally, possibly with a differnt phase. There will also be a “transient” exponential variation for a short amount, but here we only discuss the variation after the transient variation is zero “steady–state”.

1.1 R Only

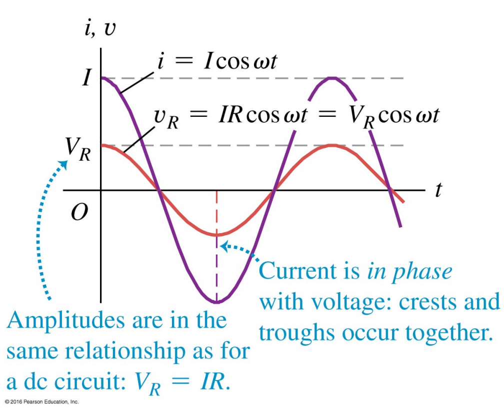
Figure (a) below shows a circuit in which an AC power source causes the current to vary sinusoidally in time according to $i(t) = I\cos(\omega t)$. By Ohm’s law, the equation for the instantaneous voltage across the resistor is $v_R(t) = IR\cos(\omega t) = V_R\cos(\omega t)$, where V_R is the amplitude of the voltage source.

Figure (b) shows the time variation of current and voltage for this circuit. The voltage and current are “in phase” because the peaks, valleys, and zero crossings of $i(t)$ and $v_R(t)$ occur at the same time.

(a) Circuit with ac source and resistor



(b) Graphs of current and voltage versus time



1.2 L Only

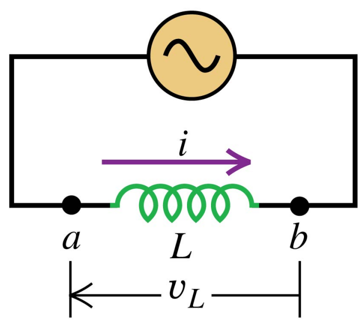
Figure (a) below shows a RL circuit in which an AC power source causes the current to vary sinusoidally in time according to $i(t) = I\cos(\omega t)$.

The voltage across the inductor varies in time according to $v_L(t) = I\omega L\cos(\omega t + 90^o)$.

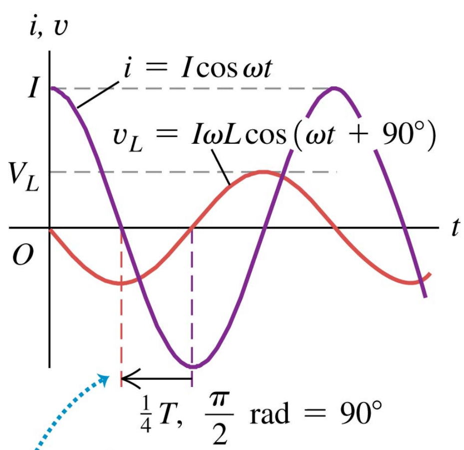
The term ωL is called the inductive reactance: $X_L \equiv \omega L$, which has units of Ohms when L has units of H and ω has units of $\mathrm{rad/s}$. With this new variable, $v_L(t) = IX_L\cos(\omega t + 90°)$.

Figure (b) shows the $v(t)$ and $i(t)$. The voltage across the inductor "leads" the current by 90° (or, equivalently, $T/4$, where $T = 2\pi/\omega$) because the maxima (or minima) in $v_L(t)$ occur before the maxima (or minima) in $i(t)$.

(a) Circuit with ac source and inductor



(b) Graphs of current and voltage versus time



Voltage curve *leads* current curve by a quarter-cycle (corresponding to $\phi = \pi/2$ rad = 90°).

1.3 C Only

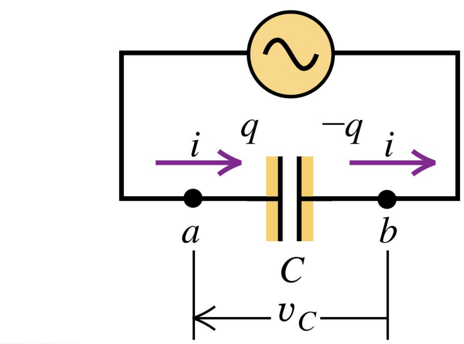
Figure (a) below shows a RC circuit in which an AC power source causes the current to vary sinusoidally in time according to $i(t) = I\cos(\omega t)$.

The voltage across the capacitor varies according to $v_C(t) = \frac{I}{\omega C}\cos(\omega t - 90^o)$.

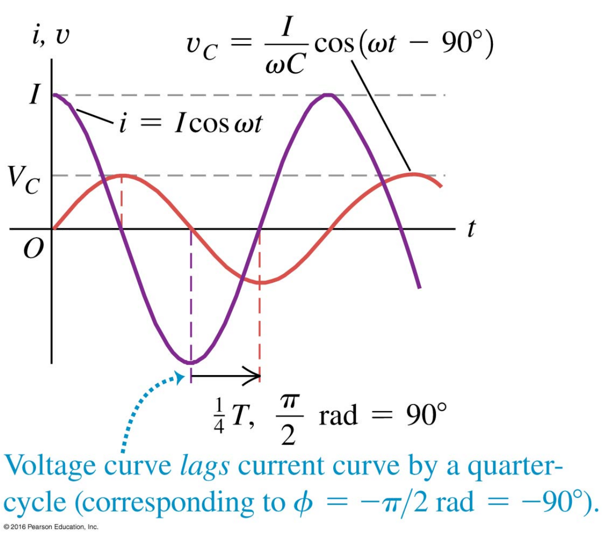
The term $1/(\omega C)$ is called the capacitive reactance: $X_C \equiv 1/(\omega C)$. With this new variable, $v_C(t) = IX_C\cos(\omega t - 90°)$.

Figure (b) shows the $v(t)$ and $i(t)$. The voltage across the inductor "lags" the current by 90° (or, equivalently, $T/4$) because the maxima (or minima) in $v_C(t)$ occur after the maxima (or minima) in $i(t)$.

(a) Circuit with ac source and capacitor



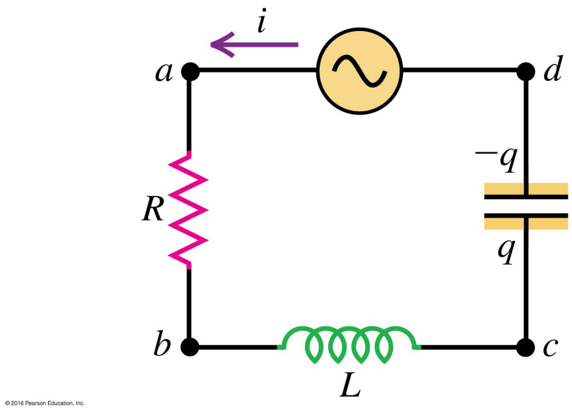
(b) Graphs of current and voltage versus time



1.4 Series LRC circuit

Suppose that we know the current $i(t)$ in the following series LRC circuit is $i(t) = I\cos(\omega t)$.

(a) L-R-C series circuit



We want to know the voltage across the AC power supply, $v_d - v_a$, which we call v. We know the voltage across each of the components from the discussion above:

$$v_R(t) = IR\cos(\omega t)$$

$$v_L(t) = I\omega L\cos(\omega t + 90^\circ) = IX_L\cos(\omega t + 90^\circ)$$

$$v_C(t) = \frac{I}{\omega C}\cos(\omega t - 90^\circ) = IX_C\cos(\omega t - 90^\circ)$$

From Kirchhoff's voltage law:

$$v(t) - v_R(t) - v_L(t) - v_C(t)$$

Substitution gives

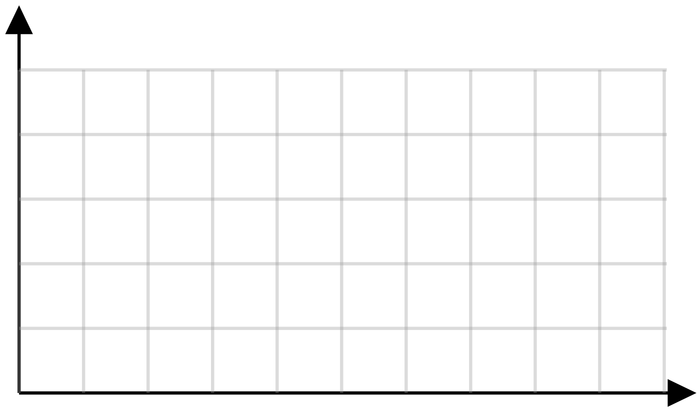
$$v(t) = IR\cos(\omega t) + IX_L\cos(\omega t + 90^\circ) + IX_C\cos(\omega t - 90^\circ)$$

In this activity, you will compute $v(t)$ in two ways. First, you will use the above formula and a trig identity to write $v(t)$ in the form $IZ\cos(\omega t + \phi)$, where the constants Z and ϕ depend on R, L, and C (or equivalently, R, X_L, and X_C). Next, you will use a general formula to compute Z and ϕ.

2 Problem I

A series LRC circuit with known values of I, R, L, and C are such that $IR = 1$ V, $IX_L = 1$ V, and $X_C = 0$. In addition, assume $i(t) = (1\text{ A})\cos\omega t$ and $\omega = 2\pi\ \text{s}^{-1}$.

Plot all quantites on the same graph.



1. Compute the period, T, of $i(t)$

   Answer: There are two ways to answer this.

   1. The current is $i(t) = \cos\omega t$.

      - At $t = 0$ s, $i(0\text{ s}) = 1$.
      - When $\omega t = 2\pi$ s, $i(2\pi\text{ s}) = 1$ again for the first time.

        So the time for the $i(t)$ to return to its starting value is t such that $\omega t = 2\pi \Rightarrow t = 2\pi/(2\pi\,\text{s}^{-1}) = 1$ s.

   2. Using the formula $T = 2\pi/\omega$.

2. Plot $i(t)$

   Answer: See Desmos plot

3. Plot $v_R(t)$

   Answer: See Desmos plot

4. Plot $v_L(t)$

   Answer: See Desmos plot

5. Plot $v_C(t)$

   Answer: See Desmos plot. Note that $v_C = 0$ when $X_C = 1/(\omega C) = 0$. To get $X_C \approx 0$ in the Desmos plot, we set $C = 1000$.

6. Starting with $v(t) = v_R(t) + v_L(t) + v_C(t)$, use the trig identity

   $A\cos(\theta) + B\cos(\theta + \pi/2) = \sqrt{A^2 + B^2}\cos\left(\theta + \tan^{-1}(B/A)\right)$

to write $v(t)$ in the form $v(t) = Z\cos(\omega t + \phi)$.

That is, find the constants Z and ϕ.

Answer: Here we have $v_R(t) = \cos\omega t$, $v_L(t) = \cos(\omega t + \pi/2)$, and $v_C(t) = 0$, so

$v(t) = \cos\omega t + \cos(\omega t + \pi/2)$.

Comparing this with the identity, $A = B = 1$ and we get

$v(t) = \sqrt{2}\cos(\omega t + \tan^{-1}(1/1)) = \sqrt{2}\cos(\omega t + \pi/4)$.

Because of the $+\pi/4$, we say that $v(t)$ leads $i(t)$ by $\pi/4$ (or 45° or $T/8$).

7. Plot $v(t)$

   Answer: See Desmos plot. Try to adjust the parameters R, L, and C to see how they change the curves (both amplitudes and phases).

3 Problem II

In the previous problem, computing $v(t)$ required the use of a trig identity to combine v_R and v_L and write $v(t)$ in the form $v(t) = Z\cos(\omega t + \phi)$, where Z and ϕ are constants that depend on L and R. When v_C is not zero, additional algebra is needed to compute $v(t)$ (by using the trig identity again). However, there is formula that can be used to find $v(t)$ in general so that trig identities are not needed to compute $v(t)$.

It can be shown that in general, the voltage across the AC power source is

$$v(t) = IZ\cos(\omega t + \phi)$$

Where the series LRC impedance Z is defined as

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

and

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

When ϕ is positive, $v(t)$ leads $i(t)$. When ϕ is negative, $v(t)$ lags $i(t)$. When $\phi = 0$, $v(t)$ is in phase with $i(t)$.

1. Using the parameters given in the previous problem, find $v(t)$ using the above formula.

   Answer: In the previous problem, we were given $I = 1$ and $\omega = 2\pi$, $IR = 1$, $IX_L = I\omega L = 1$, $X_C = 1/(\omega C)$. From this, we conclude $R = 1$, $\omega L = 1$, and $1/(\omega C) = 0$. Thus

   $$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{1^2 + (1-0)^2}$$

   and

   $$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{1-0}{1}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

   Thus,

$v(t) = IZ\cos(\omega t + \phi) = \sqrt{2}\cos(\omega t + \pi/4)$

which is the same as found in the previous problem.

2. Does $v(t)$ lead or lag $i(t)$?

   Answer: Lead. A plot of $i(t)$ and $v(t)$ shows that peaks in $v(t)$ occur **before** peaks in $i(t)$.