The $\hat{\mathbf{r}}$ Unit Vector

1 The $\hat{\mathbf{r}}$ Unit Vector

Previously, when computing the electric force between two charges, you used the formula $E = k|q|/r^2$ to find the electric field and then used a diagram to write $\vec{\mathbf{E}}$ in the form $\vec{\mathbf{E}} = E_x\hat{\boldsymbol{\imath}} + E_y\hat{\boldsymbol{\jmath}}$.

An alternative, and more direct, approach is to use an equation for electric field using a unit vector $\hat{\mathbf{r}}$:

$$\vec{\mathbf{E}}_{\text{due to } q} = kq\frac{\hat{\mathbf{r}}}{r^2},$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q_1 to the point in space where we want to know $\vec{\mathbf{E}}$, and r is the distance between q_1 and that point. Note that in this equation, we use q and not $|q|$.

To find $\hat{\mathbf{r}}$,

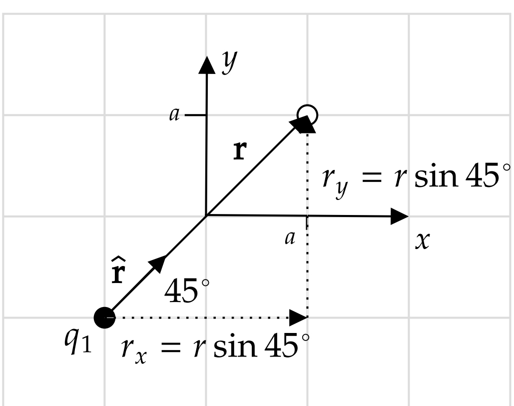
1. draw a vector, $\vec{\mathbf{r}}$ from q_1 to the point in space where you want to know $\vec{\mathbf{E}}$;
2. Write $\vec{\mathbf{r}}$ in the form $\vec{\mathbf{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$; then
3. $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

2 Example

If q_1 is at $(x, y) = (-a, -a)$, find the electric field at $(x, y) = (a, a)$ using $\vec{\mathbf{E}}_{\text{due to } q_1} = kq_1\hat{\mathbf{r}}/r^2$. Also, find E.

Solution

The calculation of $\hat{\mathbf{r}}$ is shown in the following diagram.



$$\mathbf{r} = r_x\mathbf{i} + r_y\mathrm{j} = r\cos 45°\mathbf{i} + r\sin 45°\ \mathrm{j}$$

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{\mathbf{r}}{r}\\ &= \frac{r\cos 45°\mathbf{i} + r\sin 45°\ \mathrm{j}}{r}\\ &= \cos 45°\mathbf{i} + \sin 45°\ \mathrm{j}\end{aligned}$$

Substitution gives

$$\vec{\mathbf{E}}_{\text{at } (a,a) \text{ due to } q_1} = kq_1\frac{1}{r^2}\hat{\mathbf{r}} = kq_1\frac{1}{8a^2}(\cos 45°\hat{\boldsymbol{\imath}} + \sin 45°\hat{\boldsymbol{\jmath}}) = \frac{kq_1}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right],$$

which is the same result obtained in the previous example, as expected.

To calculate $|\vec{\mathbf{E}}|$, we can use

$$|\vec{\mathbf{E}}| = E = \sqrt{E_x^2 + E_y^2}$$

and plug in $E_x = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and $E_y = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and use $\sqrt{c^2} = |c|$ (where c is a real number) to show that $E = k|q_1|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{E}} = kq_1\frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{E}}| = k|q_1|\frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so

$$|\vec{\mathbf{E}}| = k|q_1|\frac{1}{r^2} = \frac{k|q_1|}{8a^2}, \text{ as before.}$$

3 Problem

Charge q_1 is at $(x, y) = (-a, a)$. Find the electric field at $(x, y) = (a, 0)$ using $\vec{\mathbf{E}}_{\text{at }(a,0)\text{ due to }q_1} = kq_1\hat{\mathbf{r}}/r^2$. Check signs of the components of $\vec{\mathbf{E}}$ using the technique used in the Example. Also, find $|\vec{\mathbf{E}}|$.

Answer:

$$\vec{\mathbf{r}} = 2a\hat{\imath} - a\hat{\jmath}, \qquad \hat{\mathbf{r}} = \frac{2}{\sqrt{5}}\hat{\imath} - \frac{1}{\sqrt{5}}\hat{\jmath}, \qquad r^2 = 5a^2$$

$$\vec{\mathbf{E}}_{\text{at }(a,0)\text{ due to }q_1} = \frac{kq_1}{5a^2}\left(\frac{2}{\sqrt{5}}\hat{\imath} - \frac{2}{\sqrt{5}}\hat{\jmath}\right), \text{ which matches the solution to Problem I, as expected.}$$

4 Question

Can you solve an Electric Force (Coulomb's Law) problem using $\hat{\mathbf{r}}$ notation? If yes, what is the equation for the electric force using $\hat{\mathbf{r}}$?