Continuous Charge Distributions

1 Overview

Previously, you found the electric field at a location in space due to one or more point charges by finding the electric field due to each charge and then vectorially summing the fields to get the total field (this is "using superposition").

This process requires a significant amount of calculation if there are many charges. To reduce the number of calculations when charges are closely spaced, we sometimes assume they are continuously distributed; in this case, to compute the electric field, we need to evaluate an integral rather than a sum with many terms.

Section 21.5 of the textbook gives three examples for charges that are continuously distributed:

1. charges uniformly distributed along a straight line,
2. charges uniformly distributed on a circle, and
3. charges uniformly distributed on a disk.

If you read the textbook examples and the lecture notes, you should be able to identify the following steps (not necessarily in this order).

1. Identify answer features (e.g., what should the direction of $\vec{\mathbf{E}}$ be?).
2. Find $d\vec{\mathbf{E}}$ (or dE and its direction) for a dQ on the charged object. Use a diagram to check direction.
3. Find dQ in terms of coordinates (e.g., dx, dy, r, $d\theta$, etc.).
4. Simplify $d\vec{\mathbf{E}}$ (if possible) using symmetry arguments.
5. Integrate $d\vec{\mathbf{E}}$.
6. Check answer features.

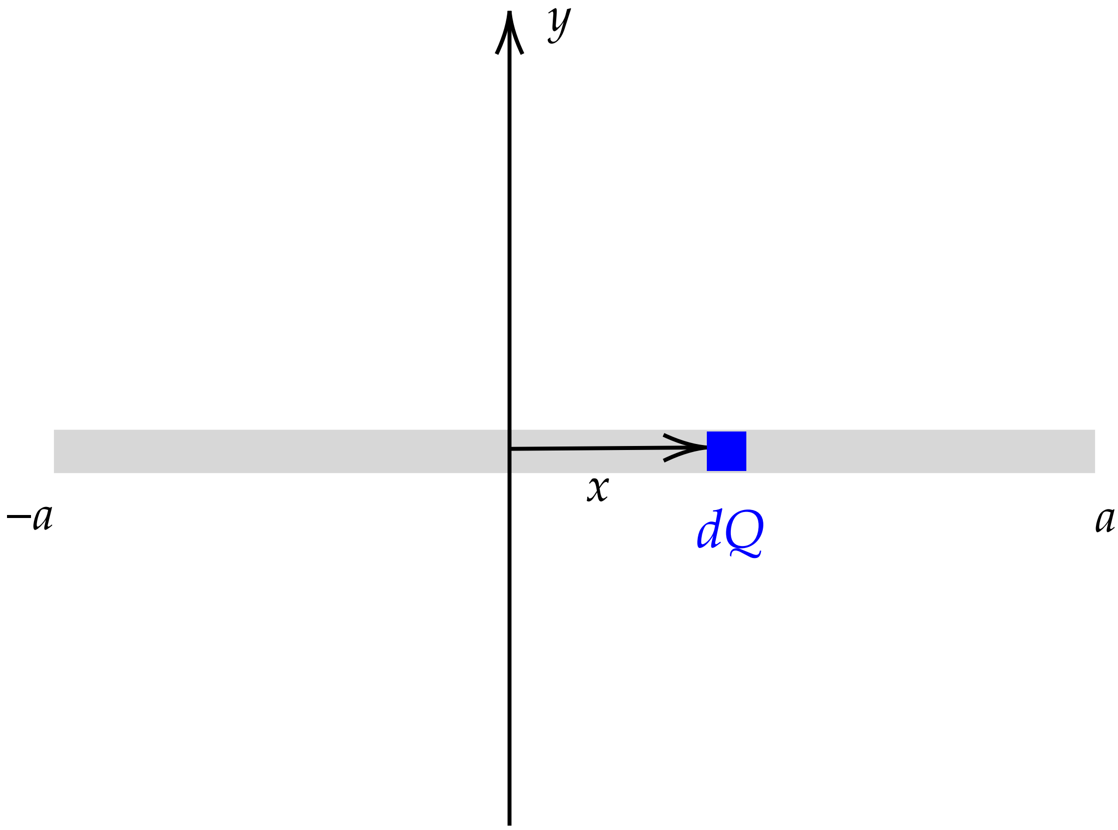
In this activity, you will explicitly address all of these steps for charges that are uniformly and continuously distributed along a straight line.

2 Finite Line of Charge

The following diagram shows a differential charge dQ at a location on the x–axis. Recall that the general equation for the electric field due to a point charge (with q replaced with dQ and $\vec{\mathbf{E}}$ replaced with $d\vec{\mathbf{E}}$) is

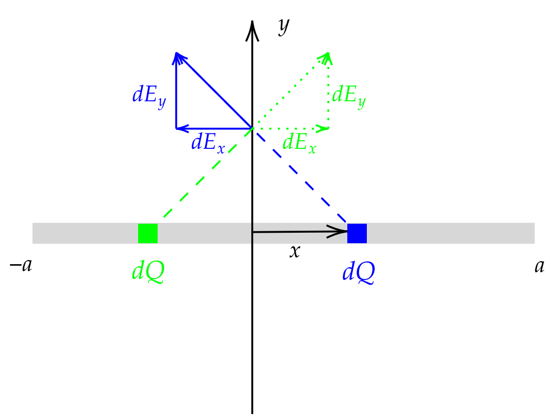
$d\mathbf{E} = k\frac{dQ}{r^2}\hat{\mathbf{r}}$, which has magnitude $dE = k\frac{dQ}{r^2}$

(In a previous activity, you found the components of $\vec{\mathbf{E}}$ using two methods that are used in the textbook. You may use either method for this problem.)



1. Draw the expected directions of dE_x and dE_y at a location on the $+y$–axis on the diagram given the location of dQ shown on the figure (assume dQ is positive).
2. In the diagram, the differential charge is at a positive x. At the same location on the $+y$–axis, draw the expected direction using dotted lines of dE_x and dE_y on the diagram with dotted lines if dQ is at $-x$.

   Solution

3. Based on your diagrams, do you expect any component of electric field due to dQ at $+x$ will cancel that due to dQ at $-x$?

   Answer: Yes, the x components will cancel leaving a total field in the $+y$ direction.

4. Find equations for the electric field components dE_x and dE_y at any location on the y–axis in terms of dQ, x, y, and k.

   Solution

   See the textbook for an alternative way of arriving at the same result.

   The vector from dQ to a point on the y–axis is $\mathbf{r} = -x\hat{\imath} + y\hat{\jmath}$. We need to find $d\mathbf{E} = kdQ/r^2\hat{\mathbf{r}}$, which can be written as $d\mathbf{E} = (kdQ/r^3)\mathbf{r}$ using $\hat{\mathbf{r}} = \mathbf{r}/r$. This equation allows us to bypass the need to compute $\hat{\mathbf{r}}$ explicitly. Using this and $\mathbf{r}$ from above gives

   $$d\mathbf{E} = k\frac{dQ}{r^3}\mathbf{r} = k\frac{dQ}{(\sqrt{x^2+y^2})^3}(-x\hat{\imath} + y\hat{\jmath})$$

   Thus,

   $$dE_x = -k\frac{xdQ}{(\sqrt{x^2+y^2})^3} \qquad dE_y = k\frac{ydQ}{(\sqrt{x^2+y^2})^3}$$

5. Suppose the charge dQ is at $x = a$ and you want to know the electric field at $y = a$. Plug these values into the equations for dE_x and dE_y found above. Are the signs of dE_x and dE_y consistent with what you drew for part 1.? Repeat supposing the charge dQ is at $x = -a$ and you want to know the electric field at $y = a$. Are the signs of dE_x and dE_y consistent with what you drew for part 2.?

6. If the differential charge dQ is at $x = 0$, do you expect dE_x or dE_y to be zero? Is your answer consistent with the result of plugging in $x = 0$ into your equations for dE_x and dE_y?

If the charge per unit length on the line is λ, then we can write $dQ = \lambda dx$. To find E_x and E_y, integrate dE_x and dE_y from $x = -a$ to a.

7. Write down the integrals that must be evaluated to find E_x and E_y. Do you expect either of the integrals to be zero?

   Solution

   $$E_x = -\int_{-a}^{a} k\frac{x\lambda dx}{(\sqrt{x^2+y^2})^3}$$

   $$E_y = \int_{-a}^{a} k\frac{y\lambda dx}{(\sqrt{x^2+y^2})^3} = ky\int_{-a}^{a}\frac{\lambda dx}{(\sqrt{x^2+y^2})^3}$$

   Based on 2., we expect E_x to be zero. Mathematically, the integrand for E_x is an odd function on the integration integral, which is another justification for E_x being zero. (The integrand for E_y is even, so we could change its limits to 0 to a and multiply by this integral by 2.)

   Note that with respect to integration, y can be treated as a constant because the integration is with respect to x, which is why y can be factored out as done in the equation for E_y.

8. From an integral table (or using trig substitutions), we know $\displaystyle\int \frac{du}{(u^2+a^2)^{3/2}} = \frac{u}{a^2\sqrt{u^2+a^2}} + C.$

   Use this to find E_y.

   Solution

In class, you were shown how to use trig substitution. Here we use an integral identity from a table. Replacing the integration variable u with x and the constant a with y, the identity can be written as

$$\int \frac{dx}{(x^2+y^2)^{3/2}} = \frac{x}{y^2\sqrt{x^2+y^2}} + C$$

We need to evaluate

$$E_y = ky\lambda \int_{-a}^{a} \frac{dx}{(\sqrt{x^2+y^2})^3}$$

Using the identity, this is

$$E_y = ky\lambda \left[\frac{x}{y^2\sqrt{x^2+y^2}}\right]_{x=-a}^{x=a} = ky\lambda\frac{2a}{y^2\sqrt{a^2+y^2}} = \frac{2k\lambda a}{y}\frac{1}{\sqrt{a^2+y^2}}$$

9. If y is negative (for example $y=-a$), is E_y positive or negative? Is this consistent with your answer to 3.?

10. The total charge on the line is $Q = 2a\lambda$ (length $\times$ charge/length). Suppose $y \gg a$ so that you can replace a^2+y^2 with y^2. Is your equation for E_y consistent with the electric field for a point charge $Q = 2a\lambda$ at the origin?

    Answer

    $$E_y = \frac{2k\lambda a}{y}\frac{1}{\sqrt{a^2+y^2}} = \frac{kQ}{y}\frac{1}{\sqrt{a^2+y^2}} \simeq \frac{kQ}{y}\frac{1}{\sqrt{y^2}} = \frac{kQ}{y}\frac{1}{|y|}$$

    If y is positive, $E_y = kQ/y^2$, which is positive. If y is negative, $E_y = -kQ/y^2$, which negative. (The textbook considers only the positive case, but given the above questions, it seems natural to ask this question.)

3 Long Line of Charge

Using the result of the previous problem, we can find E_y when $a \gg y$, which corresponds to a long line of charge. The result is

$$E_y = \frac{2\lambda k}{y}$$

This equation can be written more generally as

$$E = \frac{2\lambda k}{r}$$

where r is the perpendicular distance from the line and the direction of E is perpendicular the line with a direction that depends on the sign of the charge density λ.

A long line of charge lies along the line $y = b$.

1. Find the electric field magnitude and direction at the origin.

Answer

$E = \dfrac{2\lambda k}{b}$, direction downwards, or $\vec{\mathbf{E}} = -\dfrac{2\lambda k}{b}\hat{\boldsymbol{\jmath}}$

2. Find the electric field magnitude and direction at $(x, y) = (b, 0)$.

   Answer Same as 1. For a long line of charge, the electric field is always perpendicular to the line.