LRC AC Circuits

1 Introduction

Previously, we have examined circuits with either capacitors and resistors or inductors and resistors that were powered by batteries that produce constant voltage. These are called a direct current (DC) circuits. When a switch was closed to complete a circuit, the current varied in time and the time dependence was exponential (the current had a term $e^{-t/\tau}$.

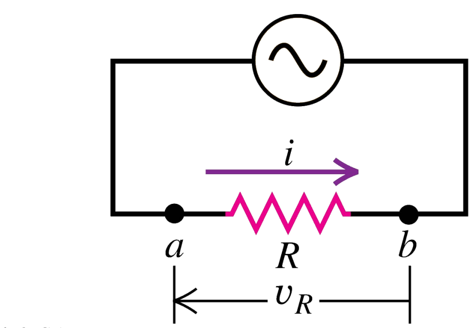
If instead of a battery we use a voltage source that varies in time sinusouidally, the current will also vary in time sinusoidally, possibly with a diffrent phase. (Technically, there will also be exponential terms, but after a short amount of time they approach zero.)

1.1 R Only

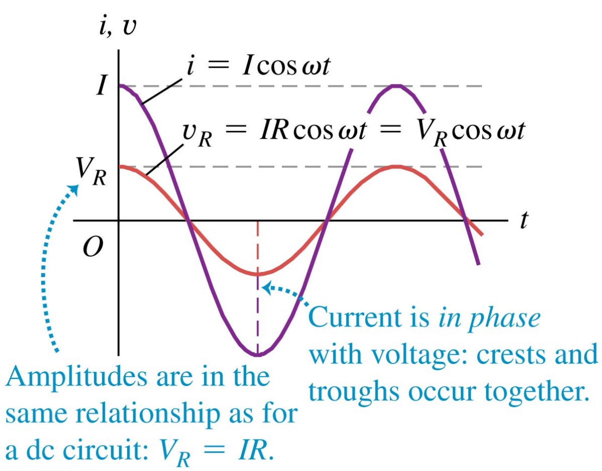
Figure (a) below shows a circuit in which an AC power source causes the current to vary sinusoidally in time according to $i(t) = I\cos(\omega t)$. By Ohm's law, the equation for the instantaneous voltage across the resistor is $v_R(t) = IR\cos(\omega t) = V_R\cos(\omega t)$.

Figure (b) shows the current and voltage. The voltage and current are "in phase" because the peaks, valleys, and zero crossings in the plots of $i(t)$ and $v_R(t)$ occur at the same time.

(a) Circuit with ac source and resistor



(b) Graphs of current and voltage versus time



1.2 L Only

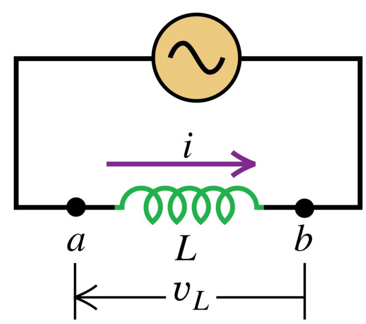
Figure (a) below shows a RL circuit in which an AC power source causes the current to vary sinusoidally in time according to $i(t) = I\cos(\omega t)$.

The voltage across the inductor varies according to $v_L(t) = I\omega L\cos(\omega t + 90^o)$.

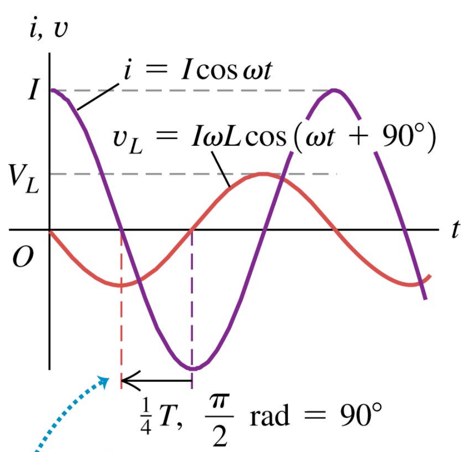
The term ωL is called the inductive reactance: $X_L \equiv \omega L$. With this new variable, $v_L(t) = IX_L\cos(\omega t + 90°)$.

Figure (b) shows the $v(t)$ and $i(t)$. The voltage across the inductor "leads" the current by 90° (or, equivalently, $T/4$) because the maxima (or minima) in $v_L(t)$ occur before the maxima (or minima) in $i(t)$.

(a) Circuit with ac source and inductor



(b) Graphs of current and voltage versus time



Voltage curve *leads* current curve by a quarter-cycle (corresponding to $\phi = \pi/2$ rad $= 90°$).

1.3 C Only

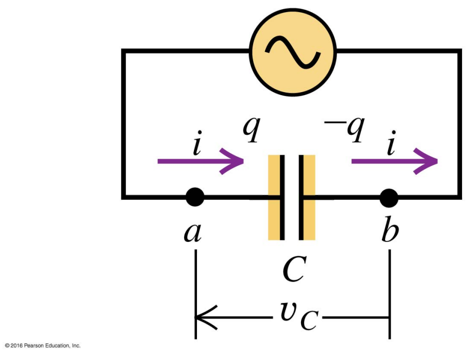
Figure (a) below shows a RC circuit in which an AC power source causes the current to vary sinusoidally in time according to $i(t) = I\cos(\omega t)$.

The voltage across the capacitor varies according to $v_C(t) = \frac{I}{\omega C}\cos(\omega t - 90^o)$.

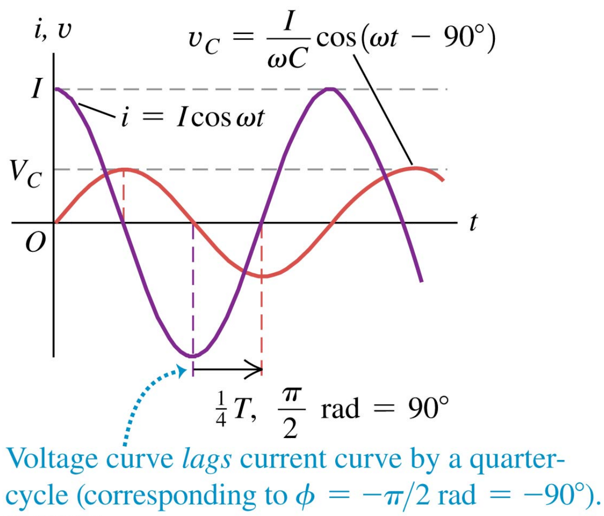
The term $1/(\omega C)$ is called the capacitive reactance: $X_C \equiv 1/(\omega C)$. With this new variable, $v_C(t) = IX_C\cos(\omega t - 90^\circ)$.

Figure (b) shows the $v(t)$ and $i(t)$. The voltage across the inductor "lags" the current by 90° (or, equivalently, $T/4$) because the maxima (or minima) in $v_C(t)$ occur after the maxima (or minima) in $i(t)$.

(a) Circuit with ac source and capacitor



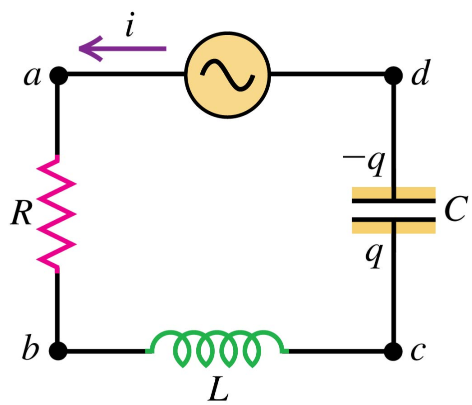
(b) Graphs of current and voltage versus time



1.4 Series LRC circuit

Suppose that we know the current $i(t)$ in the following series LRC circuit is $i(t) = I\cos(\omega t)$.

(a) L-R-C series circuit



We want to know the voltage across the AC power supply, $v_d - v_a$, which we call v. We know the voltage across each of the components from the discussion above:

$$v_R(t) = IR\cos(\omega t)$$

$$v_L(t) = I\omega L\cos(\omega t + 90^\circ) = IX_L\cos(\omega t + 90^\circ)$$

$$v_C(t) = \frac{I}{\omega C}\cos(\omega t - 90^\circ) = IX_C\cos(\omega t - 90^\circ)$$

From Kirchhoff's voltage law:

$$v(t) - v_R(t) - v_L(t) - v_C(t)$$

Substitution gives

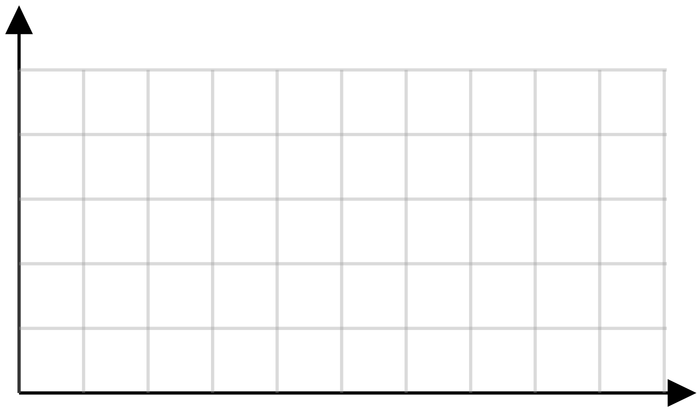
$$v(t) = IR\cos(\omega t) + IX_L\cos(\omega t + 90^\circ) + IX_C\cos(\omega t - 90^\circ)$$

In this activity, you will compute $v(t)$ in two ways. First, you will use the above formula and a trig identity to write $v(t)$ in the form $IZ\cos(\omega t + \phi)$, where the constants Z and ϕ depend on R, L, and C (or equivalently, R, X_L, and X_C). Next, you will use a general formula to compute Z and ϕ.

2 Problem I

A series LRC circuit with known values of I, R, L, and C are such that $IR = 1$ V, $IX_L = 1$ V, and $X_C = 0$. In addition, assume $i(t) = (1\text{ A})\cos\omega t$ and $\omega = 2\pi\ \text{s}^{-1}$.

Plot all quantites on the same graph.



1. Compute the period, T, of $i(t)$

   Answer: There are two ways to answer this.

   1. The current is $i(t) = \cos \omega t$.

      - At $t = 0$ s, $i(0\text{ s}) = 1$.
      - When $\omega t = 2\pi$ s, $i(2\pi\text{ s}) = 1$ again for the first time.

        So the time for the $i(t)$ to return to its starting value is t such that $\omega t = 2\pi \Rightarrow t = 2\pi/(2\pi\text{ s}^{-1}) = 1$ s.

   2. Using the formula $T = 2\pi/\omega$.

2. Plot $i(t)$

   Answer: See Desmos plot

3. Plot $v_R(t)$

   Answer: See Desmos plot

4. Plot $v_L(t)$

   Answer: See Desmos plot

5. Plot $v_C(t)$

   Answer: See Desmos plot. Note that $v_C = 0$ when $X_C = 1/(\omega C) = 0$. To get $X_C \approx 0$ in the Desmos plot, we set $C = 1000$.

6. Starting with $v(t) = v_R(t) + v_L(t) + v_C(t)$, use the trig identity

   $A\cos(\theta) + B\cos(\theta + \pi/2) = \sqrt{A^2 + B^2}\cos\left(\theta + \tan^{-1}(B/A)\right)$

   to write $v(t)$ in the form $v(t) = Z\cos(\omega t + \phi)$.

   That is, find the constants Z and ϕ.

   Answer: Here we have $v_R(t) = \cos \omega t$, $v_L(t) = \cos(\omega t + \pi/2)$, and $v_C(t) = 0$, so

   $v(t) = \cos \omega t + \cos(\omega t + \pi/2)$.

   Comparing this with the identity, $A = B = 1$ and we get

   $v(t) = \sqrt{2}\cos(\omega t + \tan^{-1}(1/1)) = \sqrt{2}\cos(\omega t + \pi/4)$.

Because of the $+\pi/4$, we say that $v(t)$ leads $i(t)$ by $\pi/4$ (or $45°$ or $T/8$).

7. Plot $v(t)$

   Answer: See Desmos plot. Try to adjust the parameters R, L, and C to see how they change the curves (both amplitudes and phases).

3 Problem II

In the previous problem, computing $v(t)$ required the use of a trig identity to combine v_R and v_L and write $v(t)$ in the form $v(t) = Z\cos(\omega t + \phi)$, where Z and ϕ are constants that depend on L and R. When v_C is not zero, additional algebra is needed to compute $v(t)$ (by using the trig identity again). However, there is formula that can be used to find $v(t)$ in general so that trig identities are not needed to compute $v(t)$.

It can be shown that in general, the voltage across the AC power source is

$$v(t) = IZ\cos(\omega t + \phi)$$

Where the series LRC impedance Z is defined as

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

and

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

When ϕ is positive, $v(t)$ leads $i(t)$. When ϕ is negative, $v(t)$ lags $i(t)$. When $\phi = 0$, $v(t)$ is in phase with $i(t)$.

1. Using the parameters given in the previous problem, find $v(t)$ using the above formula.

   Answer: In the previous problem, we were given $I = 1$ and $\omega = 2\pi$, $IR = 1$, $IX_L = I\omega L = 1$, $X_C = 1/(\omega C)$. From this, we conclude $R = 1$, $\omega L = 1$, and $1/(\omega C) = 0$. Thus

   $$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{1^2 + (1-0)^2}$$

   and

   $$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{1-0}{1}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

   Thus,

   $$v(t) = IZ\cos(\omega t + \phi) = \sqrt{2}\cos(\omega t + \pi/4)$$

   which is the same as found in the previous problem.

2. Does $v(t)$ lead or lag $i(t)$?

   Answer: Lead. A plot of $i(t)$ and $v(t)$ shows that peaks in $v(t)$ occur **before** peaks in $i(t)$.