Electric Potential

1 Electric Potential Energy Differences, ΔU

Mathematically, the work done by a force $\vec{\mathbf{F}}$ in moving an object from position a to position b is

$$W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$$

Another way of writing $\vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$ is $|F|dl\cos\phi$, where ϕ is the angle between $\vec{\mathbf{F}}$ and $d\vec{\mathbf{l}}$.

There are three cases that you will encounter when evaluating this integral:

1. If a force is always perpendicular to the direction of movement, the work due to that force is zero. For example, a block sliding horizontally has a gravitational force exerted on it, but the gravitational force is downward, and so is perpendicular to the direction of motion. Thus, gravity does no work.
2. When the force on an object does not change when it is moved a distance L from a to b and the direction of force is always in the same direction as the direction of movement, then

   $$W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}} = (\pm)|\vec{\mathbf{F}}|L$$

   where L is positive; the $+$ sign is used for a force that is in the direction of movement, and the $-$ sign is used for a force that is in the opposite direction of movement. For example, if you lift a mass m upwards by a distance L, the force you exert is in the same direction of movement, so you do a work of mgL. The gravitational force on the mass is in the opposite direction of movement, so the work done by the gravitational force is $-mgL$. If, instead, you lower the mass, your force is upwards, and the direction of motion is downwards, so the work you do is now $-mgL$, and the work done by the gravitational force is $+mgL$.

3. When the direction of force relative to the direction of movement changes (so the dot product changes) and/or the magnitude of force changes. This is covered on page 755 of the textbook.

If $\vec{\mathbf{F}}$ is a special kind of force, called a *conservative* force, we do not need to perform integration to every time that we want to compute the work. For each conservative force, there is an equation for U (called potential energy, or PE) such that one needs to only know U at b and a. In this case,

$$W_{a\to b}^{\text{cons}} \equiv -\Delta U = -(U_b - U_a)$$

where the symbol $\equiv$ is used to indicate a definition.

In mechanics, you have encountered two conservative forces

1. A constant force (e.g., the force on a small mass near Earth's surface)
2. A force that varies according to $\hat{\mathbf{r}}/r^2$ (e.g., the gravitational force between two objects separated by a large distance)

In E&M, we encounter these same two types of conservative forces.

One of the most common difficulties in calculating work and change in potential energy is getting the correct sign for the answer. The following two problems have questions that help you determine the correct sign of work and changes in potential energy.

In general, determine the direction an object would move when released from rest (call this the “release direction”). If the object moves or is moved a small step in the release direction, its potential energy will decrease. If the object moves or is moved a small step in a direction opposite to the release direction, its potential energy will increase.

- The potential energy of an object increases when you do positive work on it. That is, when your force on the object is in the direction that you move the object. One way of determining if an object’s potential is higher is if it has more potential to do something. A mass lifted upwards has more potential to crush something below it. If you lower a mass, it will have less potential to crush something.
- Potential energy increases when a conservative force does negative work.

Example

Near Earth, the gravitation field is nearly constant. If point a is a point on the floor and point b is a point 1 m above a, using

$$W_{a\to b} = (\pm)|\vec{\mathbf{F}}|L$$

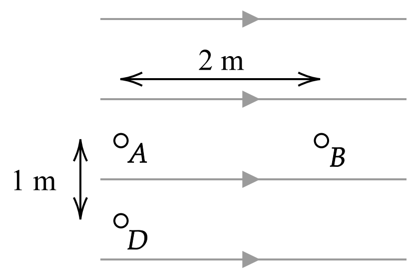
1. how much work is required by you to lift the object from a to b?;
2. how much work is done by the gravitational field?
3. The equation for U for a mass m in a constant gravitational field g is mgy. Use $W_{a\to b}^{\text{cons}} \equiv -\Delta U = -(U_b - U_a)$ to find the work done by the gravitational field when the object is moved from a to b.

Solution

1. Your force on the object must be upwards in order to lift it upwards, so your force is in the direction of movement. So $W_{a\to b} = (+)mgL$.
2. The gravitational force is downwards, so the force is opposite the direction of movement. So $W_{a\to b} = (-)mgL$
3. $U_b = mgy_b$ and $U_a = mgy_a$. Using
   $W_{a\to b}^{\text{cons}} \equiv -\Delta U = -(U_b - U_a)$
   gives
   $W_{a\to b}^{\text{cons}} = (mgy_b - mgy_a) = -mg(y_b - y_a) = -mgL$, which matches the answer to 2., as expected.

1.1 Problem – Uniform Field

The following diagram shows a region of space where the electric field is constant and has a value of 3 N/C.



1. A charge of $+3$ C is placed at point A. What happens to that charge when it is released?

2. A charge of $+3$ C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the charge changed?

3. A charge of -3 C is placed at point A. What happens to that charge when it is released?

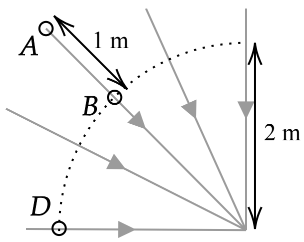
4. A charge of -3 C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the charge changed?

5. A charge of -3 C is moved straight downward from A to D. (a) How much work was done by the electric field? (b) By how much has the potential energy of the charge changed?

6. If a charge of -3 C is moved from A to D on a path that is not a straight line, will your answers to the previous problem change? If no, explain why. If yes, provide new answers.

1.2 Problem – Radial Field

In the previous problem, a charge was moved in a region of space where the electric field was constant and so the calculation of work did not require integration. In this problem, the electric field is not constant and so integration is required. The integration that must be performed to compute work in this case is given by Equation 23.8 in the textbook.



There is a charge of -6 C at the origin. Some electric field lines for this charge are shown. To simplify the calculations, use $k = 9 \cdot 10^9\ \mathrm{N \cdot m^2/C^2}$.

1. A charge of $+3$ C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

2. A charge of -3 C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

3.A charge of -3 C is moved from B to D along the dotted line. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

4. A charge of -3 C is moved from from D to B but deviate from the dotted line. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

2 Electric potential difference, ΔV

In the previous section, we considered moving an arbitrary amount of charge (either positive or negative) from point a to point b and computed its change in potential energy ΔU.

An electric potential difference ΔV is defined to be the change in electric potential energy of a (positive by convention) test charge, q_o when it is moved from point a to point b divided by q_o.

As a result, the only difference between the ΔU calculations performed previously and ΔV calculations is that we first compute ΔU for a $+1$ C charge. To get ΔV, we simply divide by ΔU by $+1$ C.

The definition of electric potential is similar to the definition of the electric field in that they both involve consideration of a test charge. That is, the electric field is the force on a test charge divided by the magnitude of the test charge:

$$\vec{\mathbf{E}} = \vec{\mathbf{F}}/q_o$$

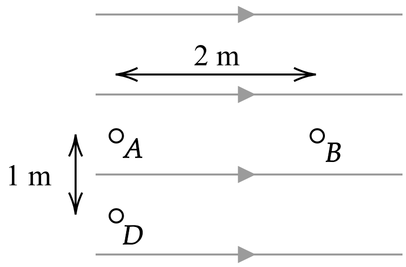
A change in electric potential is the change in electric potential energy of a positive test charge divided by the magnitude of the test charge's charge:

$$\Delta V = \Delta U/q_o$$

The advantage of using changes in electric potential (ΔV) as opposed to changes in electric potential energy (ΔU) of a specific amount of charge is that once the electric potential difference ΔV between two points is known for a test charge, the change in potential energy for an arbitrary amount of charge Q can be computed by simply multiplying ΔV and Q. This is similar to the advantage of the electric field. If we know the electric field at a given point, we can find the force on an arbitrary charge Q at that point by multiplying $\vec{\mathbf{E}}$ and Q.

2.1 Problem

The following diagram shows a region of space where the electric field is constant and has a value of 3~N/C.



1. What is the difference in electric potential $\Delta V = V_B - V_A$.

2. A charge of $+3$ C is moved from A to B. By how much has the electric potential energy of the moved charge changed?

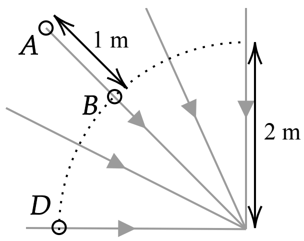
3. A charge of -3 C is moved from A to B. By how much has the electric potential energy of the moved charge changed?

4. A charge of -3 C is moved from B to D. By how much has the electric potential energy of the moved charge changed?

5. What is the difference in electric potential $\Delta V = V_B - V_D$.

2.2 Problem

There is a charge of -6 C at the origin. Some electric field lines for this charge are shown. To simplify the math, use $k = 9 \cdot 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$.



1. What is the difference in potential $\Delta V = V_B - V_A$?

2. A charge of -3 C is moved from A to B. By how much has the electric potential energy of the moved charge changed?

3. Aa charge of -3 C is moved from B to D. By how much has the electric potential energy of the moved charge changed?

4. What is difference in electric potential $\Delta V = V_D - V_A$.

3 U and V and Superposition

The electric potential energy of a charge q_0 that is a distance of r_1 from a charge q_1 is defined to be

$$U = k\frac{q_0 q_1}{r_1}$$

This corresponds to the work required to move q_0 from infinity to r_1. In this formula, if the charges have opposite signs then U is negative; if they have the same sign then U is positive. Note that there is a sign associated with the potential energy, but the direction of the vector that connects the charges does not matter; the equation for U only involves the values of the charges and the magnitude of the separation distance between them. As a result, we can also state that the formula above corresponds to the work required to move q_1 from infinity to a distance r_1 from q_0.

Consider next the potential energy of charge q_0 when it is a distance r_1 from charge q_1 and a distance r_2 from charge q_2. Because potential energy is a scalar and not a vector, the potential energy of q_0 is the **algebraic** sum, rather than the vector sum, of the potential energies due to q_1 and q_2

$$U = k\frac{q_0 q_1}{r_1} + k\frac{q_0 q_2}{r_2}$$

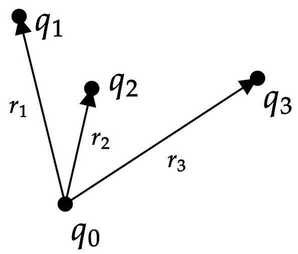
More generally, if there is a group of N charges, the potential energy of charge q_0 is

$$U = kq_0 \sum_{i=1}^{N} \frac{q_i}{r_i}$$

Similarly, the electric potential at a point in space due to a group of N charges is the **algebraic** sum of the potentials due to each of the charges at that point in space:

$$V = k \sum_{i=1}^{N} \frac{q_i}{r_i}$$

3.1 Problem



1. What is the potential energy of the charge q_0 in the diagram shown?

2. What is the potential at the position of q_0 if that charge was not there (i.e., the potential due to charges q_1, q_2, and q_3)?

3. Can you find the potential energy at the position of q_0 if that charge was not there? Why or why not?

4. Explain the difference between potential and potential energy.

3.2 Problem

Given a point charge q_1 at the origin:

1. Write the general equation for the electric potential at a distance r from q_1

2. Find the electric potential, V, at $(x, y) = (-d, 0)$ due to q_1.

3. If a charge q_2 is placed at $(x, y) = (d, 0)$, find the electric potential, V, at $(x, y) = (-d, 0)$ (hint – it is the sum of the electric potentials at $(x, y) = (-d, 0)$ due to q_1 and q_2).

4. How much work is required to place q_3 at $(x, y) = (-d, 0)$?

5. What is the potential energy, U, of q_3 when it is at $(x, y) = (-d, 0)$?

In summary, to find the work required to put a charge Q at point P (or, equivalently, the electric potential energy U of a single charge Q when it is at point P), find the potential V at point P due to all of the other charges and then $U = QV$.

4 Energy to Assemble a Collection of Charges

In the previous problem you computed the work required to move q_3 to $(x, y) = (-d, 0)$ after q_2 was in place. The total work required to assemble the system of three charges is larger than this work because it also took work to move q_2 into place. Given a point charge q_1 at origin, as in the previous question,

1. how much work is required to move q_2 to $(x, y) = (d, 0)$?;

2. how much work is required to move q_3 to $(x, y) = (-d, 0)$ if only q_1 is present?;

3. how much work is required to move q_3 to $(x, y) = (-d, 0)$ if only q_2 is present?

4. The total work required to assemble the system of three charges is the sum of the work from parts 1.-3.. Write the equation for this sum in terms of the given variables. (This sum is known as the total potential energy of the system of charges – see equation 23.11 of the textbook, which uses the same symbol U; ideally, they would have used U_c to indicate that it applies to a collection of charges and not a single charge).