Electric Potential

1 Electric Potential Energy Differences, ΔU

Mathematically, the work done by a force $\vec{\mathbf{F}}$ in moving an object from position a to position b is

$$W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$$

Another way of writing $\vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$ is $|F|dl\cos\phi$, where ϕ is the angle between $\vec{\mathbf{F}}$ and $d\vec{\mathbf{l}}$. There are three cases that you will encounter when evaluating this integral:

1. If a force is always perpendicular to the direction of movement, the work due to that force is zero. For example, a block sliding horizontally has a gravitational force exerted on it, but the gravitational force is downward, and so is perpendicular to the direction of motion. Thus, gravity does no work.
2. When the force on an object does not change when it is moved a distance L from a to b and the direction of force is always in the same direction as the direction of movement, then

   $$W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}} = (\pm)|\vec{\mathbf{F}}|L$$

   where L is positive; the $+$ sign is used for a force that is in the direction of movement, and the $-$ sign is used for a force that is in the opposite direction of movement. For example, if you lift a mass m upwards by a distance L, the force you exert is in the same direction of movement, so you do a work of mgL on the mass. The gravitational force on the mass is in the opposite direction of movement, so the work done by the gravitational force is $-mgL$. If, instead, you lower the mass, your force is upwards, and the direction of motion is downwards, so the work you do is now $-mgL$, and the work done by the gravitational force is $+mgL$.
3. When the direction of force relative to the direction of movement changes (so the dot product changes) and/or the magnitude of force changes. This is covered on page 755 of the textbook.

If $\vec{\mathbf{F}}$ is a special kind of force, called a *conservative* force, we do not need to perform integration to every time that we want to compute the work. For each conservative force, there is an equation for U (called potential energy, or PE) such that one needs to only know U at b and a. In this case,

$$W^{\text{cons}}_{a\to b} \equiv -\Delta U = -(U_b - U_a)$$

where the symbol $\equiv$ is used to indicate a definition and the superscript cons indicates that the equation applies only to a conservative force.

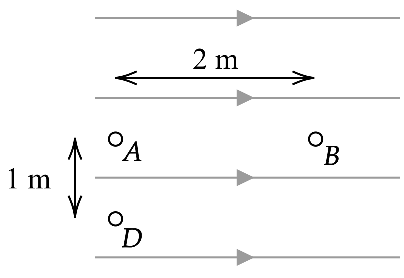
In mechanics, you encountered conservative forces of

1. A force that is constant in magnitude and direction (e.g., the force on a small mass near Earth's surface)
2. A force that varies according to $\hat{\mathbf{r}}/r^2$ (e.g., the gravitational force between two objects separated by a large distance)

In E&M, we encounter these same two types of conservative forces.

1.1 Problem – Uniform Field

The following diagram shows a region of space where the electric field is constant and has a value of 3 N/C and points to the right. Field lines are shown as lines with arrows.



1. A charge of $+3$ C is placed at point A. What happens to that charge when it is released from rest?

   Answer: Moves to right. By convention, electric field lines point in the direction of the force on a positive charge.

2. A charge of $+3$ C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the charge changed?

   Answer: The charge naturally wants to move from A to B because it is a positive charge, and the field line direction indicates the direction of the force on a positive charge. The force the field exerts on the charge is $\vec{\mathbf{F}} = Q\vec{\mathbf{E}} = (3\text{ C})(3\text{ N/C})\hat{\boldsymbol{\imath}} = (9\text{ N})\hat{\boldsymbol{\imath}}$, where $\hat{\boldsymbol{\imath}}$ points to the right.

   (a) $W = +|\vec{\mathbf{F}}|L = +(9\text{ N})(2\text{ m}) = +18$ J, and (b) $\Delta U = -18$ J (the change in PE, ΔU, is equal and opposite to the work done by the field).

3. A charge of -3 C is placed at point A. What happens to that charge when it is released from rest?

   Answer: Moves to left.

4. A charge of -3 C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the charge changed?

   Answer: (a) -18 J, and (b) $\Delta U = +18$ J (the change in PE, ΔU, is equal and opposite to the work done by the field).

5. A charge of -3 C is moved straight downward from A to D. (a) How much work was done by the electric field? (b) By how much has the potential energy of the charge changed?

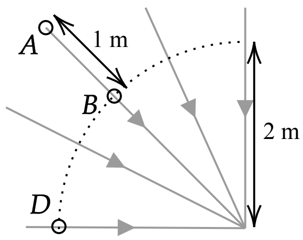
   Answer: The force due to the field is always perpendicular to the direction of movement. So the work done is zero: (a) 0 J; (b) 0 J.

6. If a charge of -3 C is moved from A to D on a path that is not a straight line, will your answers to the previous problem change? If no, explain why. If yes, provide new answers.

   Answer: No, they won't change. Think of movement along a smooth and curved line as being made of a series of tiny and equal-sized steps in vertical and horizontal directions. There is no work associated with the vertical steps. There is positive work associated with steps to the right and negative work associated with steps to the left. To get from A to D along an arbitrary path, you must take an equal number of steps to the left and to the right. See also Figure 23.1 and 23.2 in the textbook, which describes how the work done by a conservative force does not depend on the path.

1.2 Problem – Radial Field

In the previous problem, a charge was moved in a region of space where the electric field was constant and so the calculation of work did not require integration. In this problem, the electric field is not constant and so integration is required. The integration that must be performed to compute work in this case is given by Equation 23.8 in the textbook.



There is a charge of -6 C at the origin. Some electric field lines for this charge are shown. To simplify the calculations, use $k = 9 \cdot 10^9\ \mathrm{N \cdot m^2/C^2}$.

1. A charge of $+3$ C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

   Answer: According to equation 23.8, the work done by the field, labeled W^E here, when a charge q_0 is moved from a distance r_a to a distance r_b from a charge q is

   $$W^E_{a\rightarrow b} = kqq_o\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

   For this problem, $q = -6$ C, $q_0 = 3$ C, $r_a = 3$ m, and $r_b = 2$ m.

   $$W^E_{a\rightarrow b} = (9 \cdot 10^9)(-6)(+3)\left(\frac{1}{3} - \frac{1}{2}\right)\ \mathrm{J} = +27 \cdot 10^9\ \mathrm{J}$$

   This is the answer for (a). Note that the work is positive as expected – the force of the electric field on the charge is in the same direction as its movement. The change in electric potential energy is equal to and opposite of the work done by the field, so (b) $-27 \cdot 10^9$ J. Note that the answer of $27 \cdot 10^9$ J is unphysically large; it is the amount of energy that you would need to lift $27 \cdot 10^9$ kg (about 5 million elephants) by 1 m.

2. A charge of -3 C is moved from A to B. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

   Answer: (a) $-27 \cdot 10^9$ J, (b) $+27 \cdot 10^9$ J.

3. A charge of -3 C is moved from B to D along the dotted curve. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

   Answer: (a) 0 J, (b) 0 J

4. A charge of -3 C is moved from from D to B but along a path that deviates from the dotted curve. (a) How much work was done by the electric field? (b) By how much has the potential energy of the moved charge changed?

   Answer: (a) 0 J, (b) 0 J

2 Electric potential difference, ΔV

In the previous section, we considered moving an arbitrary amount of charge (either positive or negative) from point a to point b and computed its change in potential energy ΔU.

An electric potential difference ΔV is defined to be the change in electric potential energy of a test charge, q_o when it is moved from point a to point b divided by q_o.

As a result, the only difference between the ΔU calculations performed previously and ΔV calculations is that we first compute ΔU for a $+1$ C charge. To get ΔV, we simply divide by ΔU by $+1$ C.

The definition of electric potential is similar to the definition of the electric field in that they both involve consideration of a test charge. That is, the electric field is the force on a test charge divided by the magnitude of the test charge:

$\vec{\mathbf{E}} = \vec{\mathbf{F}}/q_o$

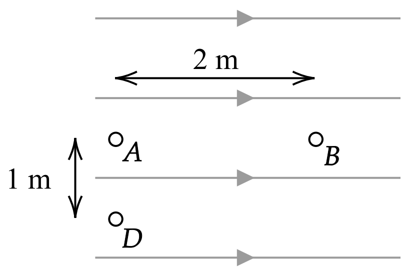
A change in electric potential is the change in electric potential energy of a positive test charge divided by the magnitude of the test charge's charge:

$\Delta V = \Delta U/q_o$

The advantage of using changes in electric potential (ΔV) as opposed to changes in electric potential energy (ΔU) of a specific amount of charge is that once the electric potential difference ΔV between two points is known for a test charge, the change in potential energy for an arbitrary amount of charge Q can be computed by simply multiplying ΔV by Q. This is similar to the advantage of the electric field. If we know the electric field at a given point, we can find the force on an arbitrary charge Q at that point by multiplying $\vec{\mathbf{E}}$ by Q.

2.1 Problem

The following diagram shows a region of space where the electric field is constant and has a value of 3~N/C.



1. What is the difference in electric potential $\Delta V = V_B - V_A$.

   Answer: As discussed in the introduction to this section, the difference in electric potential can be determined by first computing ΔU for a $+1$ C charge. Then $\Delta V = \Delta U/(1\text{ C})$. The steps for computing ΔV for a $+1$ C charge will be identical to those in problem 1.1.2 of this tutorial. Using these steps, the result for this problem is $\Delta U = -6$ J. Thus,

   $$\Delta V = \frac{-6\text{ J}}{+1\text{ C}} = -6\text{ volts}$$

   A faster way of solving this is to simply compute ΔV by using ΔU for the 3 C charge considered in 1.1.2 (-18 J) and dividing by 3 C. This is also valid and makes sense – a difference in potential from point A to point B corresponds to the change in potential energy associated with moving a positive unit of charge from point A to point B.

2. A charge of $+3$ C is moved from A to B. By how much has the electric potential energy of the moved charge changed?

   Answer: This question was already answered previously in 1.1.2. But given ΔV, we can compute ΔU:

   $$\Delta U = Q\Delta V = (3\text{ C})(-6\text{ volt}) = (3\text{ C})\left(-6\frac{\text{J}}{\text{C}}\right) = -18\text{ J}$$

3. A charge of -3 C is moved from A to B. By how much has the electric potential energy of the moved charge changed?

   Answer: This question was already answered previously in 1.1.4. But given ΔV, we can compute ΔU:

   $$\Delta U = Q\Delta V = (-3\text{ C})(-6\text{ volt}) = (-3\text{ C})\left(-6\frac{\text{J}}{\text{C}}\right) = +18\text{ J}$$

4. A charge of -3 C is moved from B to D. By how much has the electric potential energy of the moved charge changed?

   Answer: The move from B to D can be made by moving from B to A and then moving from A to D. The change in potential when moving from B to A is opposite to the change in potential when moving from A to B, which was found to be -6 Volts. The change in potential in going from A to D is zero. Thus,

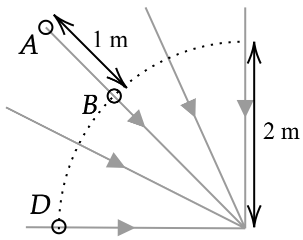
   $$\Delta U = Q\Delta V = (-3\text{ C})(+6\text{ volt}) = -18\text{ J}$$

5. What is the difference in electric potential $\Delta V = V_B - V_D$.

Answer: -6 volt. This is the same as $(V_B - V_A) + (V_A - V_D) = -6 \text{ volt} + 0 = -6 \text{ volt}$

2.2 Problem

There is a charge of -6 C at the origin. Some electric field lines for this charge are shown. To simplify the math, use $k = 9 \cdot 10^9\ \mathrm{N \cdot m^2/C^2}$.



1. What is the difference in potential $\Delta V = V_B - V_A$?

   Answer:

   The potential at a point in space due to a point charge q a distance r from that point is $V = kq/r$.

   $V_A = \dfrac{k(-6\ \mathrm{C})}{3\ \mathrm{m}} = -18 \cdot 10^9$ volt

   $V_B = \dfrac{k(-6\ \mathrm{C})}{2\ \mathrm{m}} = -27 \cdot 10^9$ volt

   $V_B - V_A = -9 \cdot 10^9$ volt

   A negative value is expected because a positive charge placed at A will tend to move towards B, which has a lower potential.

2. A charge of -3 C is moved from A to B. By how much has the electric potential energy of the moved charge changed?

   Answer:

   This is most easily found using

   $\Delta U = Q\Delta V$

   From part 1., $V_B - V_A = -9 \cdot 10^9$ volt. As a result,

   $U_B - U_A = (-3\ \mathrm{C})(V_B - V_A) = +27 \cdot 10^9$

   This answer matches the answer to the identical question of problem 1.1.2.

3. Aa charge of -3 C is moved from B to D. By how much has the electric potential energy of the moved charge changed?

   Answer: 0 J.

4. What is difference in electric potential $\Delta V = V_D - V_A$.

   Answer: The potential at D is the same as the potential at B. This can be seen mathematically from the equation $V = \frac{kq}{r}$ – at D and B, r is the same. Physically, in moving from B to D along the dotted line, no work is required because the electric field is perpendicular to this path. The problem statement did not require moving from B to D along the dotted line, but was established earlier, the changes in potential energy are independent of the path for a conservative force, so we are free to choose a convenient path. Based on this, $V_D - V_A = V_B - V_A$ and so the answer is the same as that for part 1. of this problem: $-9 \cdot 10^9$ volt.

3 U and V and Superposition

The electric potential energy of a charge q_0 that is a distance of r_1 from a charge q_1 is defined to be

$$U = k\frac{q_0 q_1}{r_1}$$

This corresponds to the work required to move q_0 from infinity to r_1. In this formula, if the charges have opposite signs then U is negative; if they have the same sign then U is positive. Note that there is a sign associated with the potential energy, but the direction of the vector that connects the charges does not matter; the equation for U only involves the values of the charges and the magnitude of the separation distance between them. As a result, we can also state that the formula above corresponds to the work required to move q_1 from infinity to a distance r_1 from q_0.

Consider next the potential energy of charge q_0 when it is a distance r_1 from charge q_1 and a distance r_2 from charge q_2. The potential energy of q_0 is the sum

$$U = k\frac{q_0 q_1}{r_1} + k\frac{q_0 q_2}{r_2}$$

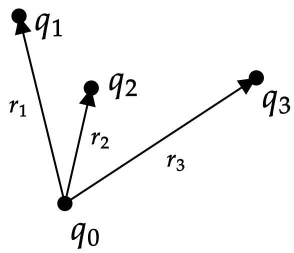
More generally, if q_o is among a group of N other charges, the potential energy of q_0 is

$$U = kq_0 \sum_{i=1}^{N} \frac{q_i}{r_i}$$

Dividing by q_o gives the electric potential at a point in space due to a group of N charges:

$$V = k \sum_{i=1}^{N} \frac{q_i}{r_i}$$

3.1 Problem



1. What is the electric potential energy of the charge q_0 in the diagram shown?

   Answer

   $U = \frac{kq_oq_1}{r_1} + \frac{kq_oq_2}{r_2} + \frac{kq_oq_3}{r_3}$

   This represents the amount of energy it would take to move charge q_o from infinity to its position on the diagram.

2. What is the electric potential at the position of q_0 if q_0 was not there?

   Answer

   $V = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3}$

3. Can you find the potential energy at the position of q_0 if that charge was not there? Why or why not?

   Answer: No. It does not make sense to ask what the potential energy is at a point in space. Only physical objects (e.g., masses, charges) have potential energy.

4. Explain the difference between potential and potential energy.

   Answer: Potential energy is the energy associated with an object at a given location in space. The electric potential energy of a charge is the energy required to move it from a large distance away from all other charges to a given location in space. The electric potential energy of a charge Q at a point in space is related to the electric potential at that point in space by $U = QV$.

3.2 Problem

Given a point charge q_1 at the origin:

1. Write the general equation for the electric potential at a distance r from q_1

   Answer: $V = kq_1/r$

2. Find the electric potential, V_1, at $(x, y) = (-d, 0)$ due to q_1.

   Answer: $V_1(d, 0) = kq_1/d$

3. If a charge q_2 is placed at $(x, y) = (d, 0)$, find the electric potential, V, at $(x, y) = (-d, 0)$ (hint – it is the sum of the electric potentials at due to q_1 and q_2).

   Answer: $V(-d, 0) = V_1(-d, 0) + V_2(-d, 0) = kq_1/d + kq_2/2d$

4. How much work is required to place charge q_3 at $(x, y) = (-d, 0)$?

   Answer: $U = W = q_3V(-d, 0) = q_3(kq_1/d + kq_2/2d)$

5. What is the potential energy, U, of q_3 when it is at $(x, y) = (-d, 0)$?

   Answer: See 4.

4 Energy to Assemble a Collection of Charges

In the previous problem you computed the work required to move q_3 to $(x, y) = (-d, 0)$ after q_2 was in place. The total work required to assemble the system of three charges is larger than this work because it also took work to move q_2 into place. Given a point charge q_1 at origin, as in the previous question,

1. how much work is required to move q_2 to $(x, y) = (d, 0)$?;

   Answer: $W = q_2 V_1(0, d) = q_2(kq_1/d)$

2. how much work is required to move q_3 to $(x, y) = (-d, 0)$ if only q_1 is present?;

   Answer: $W = q_3 V_1(-d, 0) = q_3(kq_1/d)$

3. how much work is required to move q_3 to $(x, y) = (-d, 0)$ if only q_2 is present?

   Answer: $W = q_3 V_2(-d, 0) = q_3 k q_2/2d$

4. The total work required to assemble the system of three charges is the sum of the work from parts 1.-3.. Write the equation for this sum in terms of the given variables. (This sum is known as the total potential energy of the system of charges – see equation 23.11 of the textbook.)

   Answer: $W = q_2 k q_1/d + q_3(kq_1/d) + q_3 k q_2/2d$