

Magnetic Field of a Moving Charge

1 Introduction

In previous activities, you computed the force on moving charges in a region of space where there is a magnetic field. No mention was made of how the magnetic field was created.

In this activity, you compute the magnetic field created by moving charges.

The magnetic field due to a point charge q moving with velocity \vec{v} (when $|\vec{v}|$ is small compared to the speed of light) is

$$\vec{B} = \frac{\mu_o}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}$$

where \hat{r} is the unit vector that points from the position of q to the point in space where we want to know \vec{B} , and r is the distance between q and that point.

To find \hat{r} (see also the \hat{r} Unit Vector activity),

1. draw a vector, \vec{r} , from q to the point in space where you want to know \vec{B} ;
2. Write \vec{r} in the form $\vec{r} = r_x\hat{i} + r_y\hat{j}$; then
3. $\hat{r} = \vec{r}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

In this activity, the examples and solutions are given using the above approach for computing \vec{B} . An alternative is to use the fact that $\vec{v} \times \hat{r} = |\vec{v}| \sin \phi = v \sin \phi$, where ϕ is the angle between \vec{v} and \hat{r} and $0 \leq \phi \leq 180^\circ$. With this, the magnitude of the magnetic field is

$$B = \frac{\mu_o}{4\pi} \frac{|q|v \sin \phi}{r^2}$$

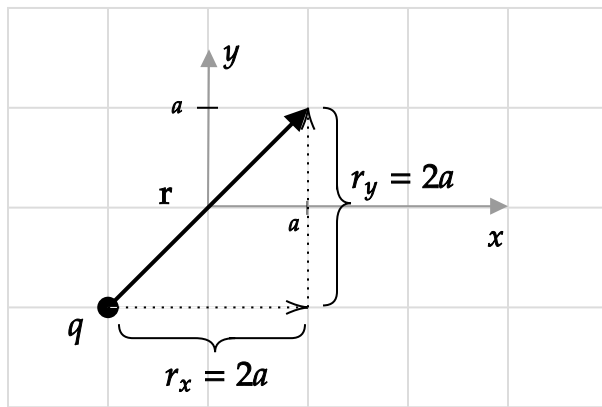
and the right-hand rule can be used to determine the direction of \vec{B} . See the Cross Products activity for a discussion of when and how to compute the cross-product using this method.

2 Example

If q is at $(x, y) = (-a, -a)$ and has a velocity of $\vec{v} = v_o\hat{i}$, find \vec{B} at $(x, y) = (a, a)$.

Solution

To find \hat{r} , we draw a vector from q to the point where we want to compute \vec{B} .



Based on the diagram, $\vec{r} = 2a\hat{i} + 2a\hat{j}$ and $r = \sqrt{(2a)^2 + (2a)^2} = 2\sqrt{2}a$, so

$$\hat{\mathbf{r}} = \frac{\vec{r}}{r} = \left[\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} \right]$$

The cross-product is

$$\vec{\mathbf{v}} \times \hat{\mathbf{r}} = v_o \hat{\mathbf{i}} \times \left[\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} \right] = \frac{v_o}{\sqrt{2}} (\hat{i} \times \hat{j}) = \frac{v_o}{\sqrt{2}} \hat{\mathbf{k}}$$

Substitution into

$$\vec{\mathbf{B}} = \frac{\mu_o}{4\pi} \frac{q \vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

gives

$$\vec{\mathbf{B}}(a, a) = \frac{\mu_o}{4\pi} \frac{q \frac{v_o}{\sqrt{2}} \hat{\mathbf{k}}}{(2\sqrt{2}a)^2} = \frac{\mu_o}{4\pi} \frac{qv_o}{(8\sqrt{2})a^2} \hat{\mathbf{k}}$$

Check: Use the right-hand rule for cross products on $\vec{\mathbf{v}} \times \hat{\mathbf{r}}$ to verify that the result is out of the page. (Why do we know that the $\hat{\mathbf{k}}$ direction is out of the page?)

3 Problem I

If q is at $(x, y) = (a, a)$ and has a velocity of $\vec{\mathbf{v}} = v_o \hat{\mathbf{i}}$, find $\vec{\mathbf{B}}$ at $(x, y) = (-a, -a)$.

Solution

$\vec{r} = -2a\hat{i} - 2a\hat{j}$ and $r = 2\sqrt{2}a$, so

$$\hat{\mathbf{r}} = \frac{\vec{r}}{r} = \left[-\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} \right]$$

The cross-product is

$$\vec{v} \times \hat{r} = v_o \hat{i} \times \left[-\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right] = -\frac{v_o}{\sqrt{2}} (\hat{i} \times \hat{j}) = -\frac{v_o}{\sqrt{2}} \hat{k}$$

Substitution into

$$\vec{B} = \frac{\mu_o}{4\pi} \frac{q \vec{v} \times \hat{r}}{r^2}$$

gives

$$\vec{B}(-a, -a) = -\frac{\mu_o}{4\pi} \frac{qv_o}{(8\sqrt{2})a^2} \hat{k}$$

Check: Using the right-hand rule for cross products on $\vec{v} \times \hat{r}$ confirms that the result is into the page.

4 Problem II

If q is at $(x, y) = (a, 0)$ and has a velocity of $\vec{v} = v_o \hat{j}$, find \vec{B} at $(x, y) = (a, a)$.

Answer: $\vec{B}(a, a) = 0$ (From a diagram, \vec{v} and \hat{r} are parallel, so their cross product is zero.)

5 Problem III

If q is at $(x, y) = (a, 2a)$ and has a velocity of $\vec{v} = v_o \hat{j}$, find \vec{B} at $(x, y) = (-a, -a)$.

Answer:

$$\vec{B}(-a, -a) = \frac{\mu_o}{4\pi} \frac{2qv_o \hat{k}}{13\sqrt{13}a^2}$$

6 Problem IV

If q is at the position (x_o, y_o) and has a velocity of $\vec{v} = v_x \hat{i} + v_y \hat{j}$,

$$\vec{B}(x, y) = \frac{\mu_o}{4\pi} \frac{q}{r^3} [v_x(y - y_o) - v_y(x - x_o)] \hat{k}$$

where

$$r = \sqrt{(x - x_o)^2 + (y - y_o)^2}$$

1. Explain why \vec{B} only has a \hat{k} component.

Answer: \vec{r} and \vec{v} are in the x - y plane, and the result of a cross-product is a vector that is perpendicular to the plane to the two crossed vectors.

2. Use this formula to find \vec{B} for the example problem in section 2.

Answer: In the example problem, q is at $(-a, -a)$ and has a velocity of $\vec{v} = v_o \hat{i}$, and we want to know \vec{B} at (a, a) . In terms of the variables for the given equation, the position of the charge is $(x_o, y_o) = (-a, -a)$, the location where we want to know \vec{B} is $(x, y) = (a, a)$, $v_x = v_o$, and $v_y = 0$. Substituting these values into

$$r = \sqrt{(x - x_o)^2 + (y - y_o)^2} \text{ and } \vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi} \frac{q}{r^3} [v_x(y - y_o) - v_y(x - x_o)] \hat{\mathbf{k}}$$

$$\text{gives } r = \sqrt{(a - -a)^2 + (a - -a)^2} = \sqrt{8}a \text{ and } \vec{\mathbf{B}}(a, a) = \frac{\mu_o}{4\pi} \frac{q}{(\sqrt{8}a)^3} v_o(a - -a) \hat{\mathbf{k}}$$

Simplification gives the same result found in the example:

$$\vec{\mathbf{B}}(a, a) = \frac{\mu_o}{4\pi} \frac{qv_o}{(8\sqrt{2})a^2} \hat{\mathbf{k}}$$

3. Derive this formula.

Answer: The vector from the position of q , (x_o, y_o) , to the point where we want to know the field, (x, y) , is

$$\vec{\mathbf{r}} = (x - x_o)\hat{\mathbf{i}} + (y - y_o)\hat{\mathbf{j}}, \text{ so } r = \sqrt{(x - x_o)^2 + (y - y_o)^2}.$$

$$\text{Using this with } \hat{\mathbf{r}} = \vec{\mathbf{r}}/r \text{ and } \vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \text{ in } \vec{\mathbf{B}} = \frac{\mu_o}{4\pi} \frac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

gives

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi} \frac{q(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}) \times \frac{(x - x_o)\hat{\mathbf{i}} + (y - y_o)\hat{\mathbf{j}}}{r}}{r^2}$$

or

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi} \frac{q}{r^3} (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}) \times [(x - x_o)\hat{\mathbf{i}} + (y - y_o)\hat{\mathbf{j}}]$$

Using the Multiply Through method for cross-products (and dropping the terms involving $\hat{\mathbf{i}} \times \hat{\mathbf{i}}$ and $\hat{\mathbf{j}} \times \hat{\mathbf{j}}$) gives

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi} \frac{q}{r^3} [v_x\hat{\mathbf{i}} \times (y - y_o)\hat{\mathbf{j}} + v_y\hat{\mathbf{j}} \times (x - x_o)\hat{\mathbf{i}}]$$

Evaluation of the cross-products gives

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi} \frac{1}{r^3} [v_x(y - y_o) - v_y(x - x_o)] \hat{\mathbf{k}}$$