Capacitance

1 Overview

1.1 Technique

The general technique for computing capacitance when Gauss's law applies is:

1. Place an equal and opposite amount of charge, Q, on the conductors.
2. Use Gauss's law to compute the electric field between the conductors.
3. Use $V(b) - V(a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$ to find the potential difference between the conductors.
4. Use the equation for capacitance to find C: $C = Q/|\Delta V| = Q/|V(b) - V(a)|$.

1.2 Review of Related Topics

Conductors

1. When charges are placed on an isolated conductor (meaning it is far away from other charges), they will reconfigure themselves to make the electric field inside the conductor zero.
2. If a charged conductor is not isolated, the charges on the conductor will reconfigure themselves to make the electric field inside the conductor zero. The total field in the conductor is the field due to the charges on the conductor and charges elsewhere.

Electric Potential Energy and Electric Potential

The general formula for work is $W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$. If $\vec{\mathbf{F}}$ is a conservative force, such as the force due to a static electric field, we define a potential energy U according to

$\Delta U = U(b) - U(a) \equiv -W_{a\to b}$

The force on a charge q_o in an electric field $\vec{\mathbf{E}}$ is $\vec{\mathbf{F}} = q_o\vec{\mathbf{E}}$ and so we can write

$U(b) - U(a) = -\displaystyle\int_a^b q_o\vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$. Dividing both sides by q_o gives $\dfrac{U(b)}{q_o} - \dfrac{U(a)}{q_o} = -\displaystyle\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$

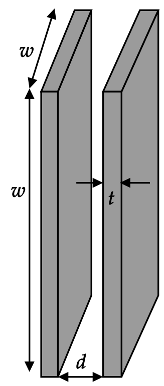
Defining electric potential as $V \equiv U/q_o$ gives $V(b) - V(a) = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$

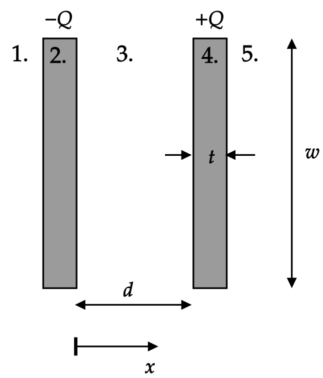
In summary, if you calculate the difference in potential between a and b (in Volts), you can also determine, for a charge q_o moved from a to b,

1. how much work (in Joules) the electric field did on the charge: $-q_o[V(b) - V(a)]$, and
2. the change in the charge's electric potential energy (in Joules): $+q_o[V(b) - V(a)]$.

2 Parallel Plates

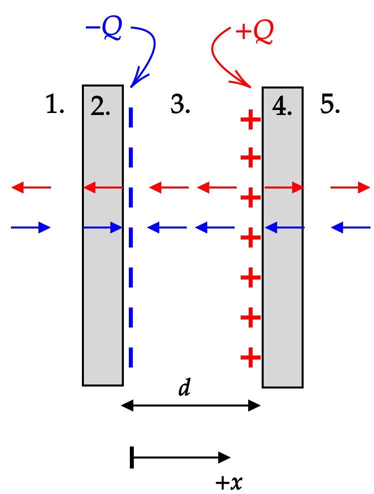
An equal and opposite amount of charge is placed on two conducting and parallel plates as shown on the left in the following figure. On the right, a side view of the plates is shown. The area of the plates, $A = w^2$, is much larger than shown such that the width, w, is much larger than the separation distance, d.





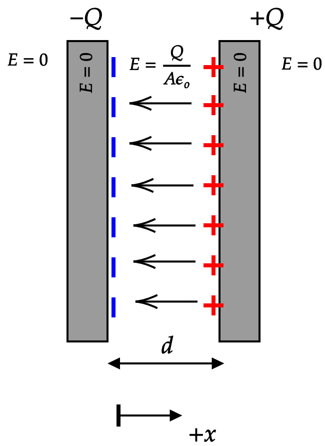
1. How will the charges distribute on each of the plates? That is, how much charge is on each of the four faces that have area A? Assume that no charge appears on the other (narrow, and having thickness t) faces of the plates, which have a much smaller area.

   Answer: The charges will move to the inner faces as shown in the following diagram. Electric field vectors for the positive an negative charges are shown. Because the charges are on a large plane, the field is indpendent of distance from the plane. With this charge configuration, inside the conductors, the field due to the positive charges cancels the field due to the negative charges giving a net field of zero.



2. What is the electric field in each of the five regions? (Hint: The magnitude of the field due to charges uniformly distributed on a plane is $|\sigma|/2\epsilon_o$. The field in each region will be the sum of the field due to charges on each plate. Your answer should be such that the electric field inside the conducting plates is zero!)

   Answer: The magnitude of the field due to each plate is $|\sigma|/2\epsilon_o$, where σ is the surface charge density. The surface charge densities are $\pm Q/A$.



From the diagram in the answer to part 1., the electric fields cancel except between the plates where it is

$$\frac{|+Q/A|}{2\epsilon_o} + \frac{|-Q/A|}{2\epsilon_o} = \frac{Q}{A\epsilon_o}$$

to the left, so

$$\vec{\mathbf{E}} = -\frac{Q}{A\epsilon_o}\hat{\boldsymbol{\imath}}$$

3. What is the electric potential difference, $V(d) - V(0)$, between the left and right plate? (Make sure the sign of your result matches your expectation based on the techniques covered in the last activity.)

   Answer: The general equation is

   $$V(b) - V(a) = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

   where b is the final position and a is the initial position. Using our variables,

   $$V(d) - V(0) = -\int_0^d \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

   The electric field is constant and in the same direction as x, so we know the result of the integration will be $\pm Ed = \pm Qd/A\epsilon_o$. Based on techniques covered in the last activity, we expect the potential to be higher at the right plate, so we choose the $+$ option. More formally,

   Using $d\mathbf{l} = dx\hat{\boldsymbol{\imath}}$ and $\vec{\mathbf{E}} = -\frac{Q}{A\epsilon_o}\hat{\boldsymbol{\imath}}$ gives

   $$V(d) - V(0) = -\int_0^d \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_0^d \left[-\frac{Q}{A\epsilon_o}\hat{\boldsymbol{\imath}}\right] \cdot dx\hat{\boldsymbol{\imath}} = \frac{Qd}{A\epsilon_o}$$

4. Use your answer to 3. to find the capacitance in terms of k, A, and d.

   Answer:

   $$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Qd}{A\epsilon_o}} = \frac{\epsilon_o A}{d}$$

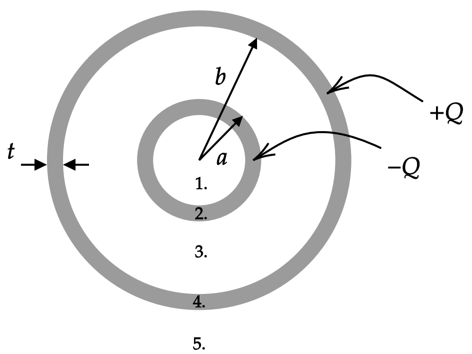
5. (Review question) How much work would the electric field do on a charge q_o that is moved from the left plate to the right plate? What would be the change in q_o's electric potential energy?

Answer: $W = -q_o \dfrac{\epsilon_o A}{d}$. Sign check: The force of the field is to the left and the displacement is to the right, so $\vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$ will be negative.

$\Delta U = -W = q_o \dfrac{\epsilon_o A}{d}$

3 Spherical

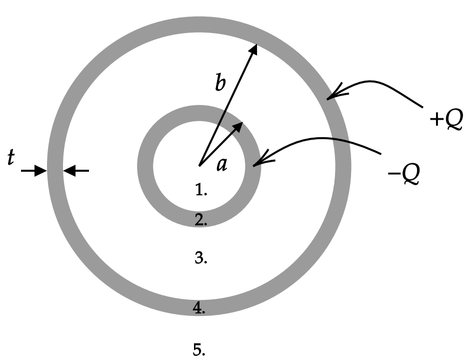
Charge placed on two spherical conducting shells, the cross–section of which is shown. Both shells have a thickness of t. The inner shell has an outer radius of a and a net charge of $-Q$. The outer shell has an inner radius of b and a net charge of $+Q$. Assume that Q is positive.



Using Gauss's law and the fact that the electric field inside a conductor must be zero, show that

1. there can be no charge on the inner surface of the inner conductor,
2. the charge on the inner surface of the outer conductor is $+Q$, and
3. there is no charge on the outer surface of the outer conductor.

Draw the Gaussian surfaces that you use to answer this question on the diagram above or on a new diagram in the space below.



4. What is the electric field in each of the 5 labeled regions? Region 1. is the empty volume inside of the inner conductor, region 2. is the inner conductor, region 3. is the empty volume between the conductors, region 4. is the outer conductor, and region 5. is the region outside of the outer

conductor. (Hint: Use Gauss's law several times; when not zero, the electric field should be proportional to $1/r^2$.)

Answer: 1. 0 2. 0 3. $E_r = -Q/4\pi\epsilon_o r^2$ 4. 0 5. 0

5. What is the potential difference, $V(b) - V(a)$? (Make sure the sign of your result matches your expectation based on the techniques covered in the last activity.)

   Answer: $V(b) - V(a) = \dfrac{Q}{4\pi\epsilon_o}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$

   Note that $V(b) - V(a)$ is positive, which is expected because moving from a to b we are moving against the direction of $\mathbf{E}$.

6. Find the capacitance in terms of k, a, and b.

   Answer: $C = \dfrac{Q}{V(b) - V(a)} = \dfrac{4\pi\epsilon_o}{\frac{1}{a} - \frac{1}{b}}$

7. (Review question) How much work would the electric field do on a charge q_o that is moved from $r = a$ to $r = b$? What would be the change in q_o's electric potential energy?

   Answer: $-q_o(V(b) - V(a))$ and $q_o(V(b) - V(a))$. Check: Work is negative b/c electric field direction is opposite the direction of movement. PE increases.