Electric Field and $\hat{\mathbf{r}}$

1 Overview

This activity covers topics in Section 21.4 of Young and Freedman 2015, 14th Edition.

The electric field vector, $\vec{\mathbf{E}}$, is a quantity that we assign to a point in space. Given this quantity, we can compute the force on a charge Q will experience if it is placed at that point in space using the equation $\vec{\mathbf{F}} = Q\vec{\mathbf{E}}$. The direction of $\mathbf{E}$ is also the direction a charge will begin to move if released from rest.

To find $\vec{\mathbf{E}}$ at any point in space, compute the force $\vec{\mathbf{F}}$ due to all other charges on a hypothetical (or "test") charge q_o at a point where you want to know $\vec{\mathbf{E}}$. To find $\vec{\mathbf{E}}$ at that point, divide $\vec{\mathbf{F}}$ by q_o.

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_o}$$

2 Example

Charge q_1 is at $(x, y) = (-a, -a)$. Find the electric field at $(x, y) = (a, a)$ in the form $\vec{\mathbf{E}} = E_x\hat{\boldsymbol{\imath}} + E_y\hat{\boldsymbol{\jmath}}$. Also find E. (Note that E and $|\mathbf{E}|$ are used interchangebly.)

Solution

To find the electric field at a point in space, we put a hypothetical "test" charge q_o at that point, compute the force on it due to all other charges, and then use

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_o}$$

The force a charge q_1 at $(x, y) = (-a, -a)$ exerts on a charge q_2 at $(x, y) = (a, a)$ was computed in a previous activity. We can use that answer after the replacement of q_2 with q_o. The result is

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_o} = k\frac{|q_1 q_o|}{8a^2}(\cos 45^\circ\,\hat{\boldsymbol{\imath}} + \sin 45^\circ\,\hat{\boldsymbol{\jmath}})$$

We also found that we got the same result if q_1 and q_o are are positive or both are negative. If q_1 was positive and q_o is negative, or vice-versa, we found the sign changed:

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_o} = -k\frac{|q_1 q_o|}{8a^2}(\cos 45^\circ\,\hat{\boldsymbol{\imath}} + \sin 45^\circ\,\hat{\boldsymbol{\jmath}})$$

Based on this, we can write a single equation for all possibilities:

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_o} = k\frac{q_1 q_o}{8a^2}(\cos 45^\circ\,\hat{\boldsymbol{\imath}} + \sin 45^\circ\,\hat{\boldsymbol{\jmath}})$$

The electric field at the location of q_o is then

$$\vec{\mathbf{E}}_{\text{at }(a,a)\text{ due to }q_1} = \frac{\vec{\mathbf{F}}}{q_o} = k\frac{q_1}{8a^2}(\cos 45^\circ\,\hat{\boldsymbol{\imath}} + \sin 45^\circ\,\hat{\boldsymbol{\jmath}}) = k\frac{q_1}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

where the fact that $\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2}$ was used.

Sign check: It is easy to make a sign error when computing electric fields and forces. The electric field at points in the direction a positve charge will move if released from rest. Suppose q_1 is positive. Our equation predicts that a charge will move up and to the right. Suppose q_1 is negative. Our equation predicts that a charge will move down and to the left.

3 Problem I

Charge q_1 is at $(x, y) = (-a, a)$. At $(x, y) = (a, 0)$, find $\mathbf{E}$ in the form $\vec{\mathbf{E}} = E_x\hat{\boldsymbol{\imath}} + E_y\hat{\boldsymbol{\jmath}}$ and E.

4 Unit Vector

Previously, when computing the electric force between two charges, you used the formula $F = k|q_1 q_2|/r^2$ to find the magnitude of the force and then used a diagram to write $\mathbf{F}$ in the form $\vec{\mathbf{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$. A similar process was used for computing $\vec{\mathbf{E}}$ above (because we computed $\mathbf{F}$ as part of the process). The textbook provides an equation for the electric field that requires a slightly different method of calculation.

The equation for the electric field using a unit vector is

$$\vec{\mathbf{E}}_{\text{due to } q_1} = kq_1\frac{\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q_1 to the point in space where we want to know $\mathbf{E}$ and r is the distance between q_1 and that point.

To find $\hat{\mathbf{r}}$,

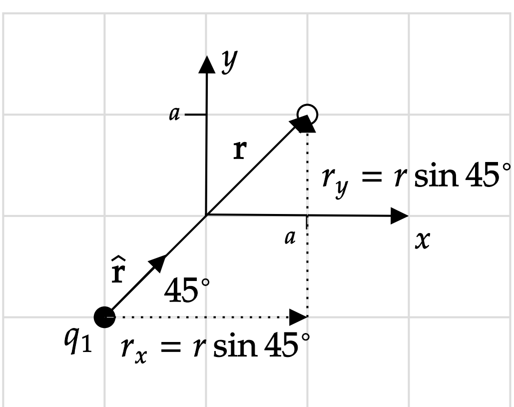
1. draw a vector, $\mathbf{r}$ from q_1 to the point in space where you want to know $\mathbf{E}$;
2. Write $\mathbf{r}$ in the form $\mathbf{r} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\mathbf{y}}$; then
3. $\hat{\mathbf{r}} = \mathbf{r}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

4.1 Example

If q_1 is at $(x, y) = (-a, -a)$, find the electric field using $\vec{\mathbf{E}}_{\text{due to } q_1} = kq_1\hat{\mathbf{r}}/r^2$ at $(x, y) = (a, a)$.

Solution

The calculation of $\hat{\mathbf{r}}$ is shown in the following diagram.



$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} = r\cos 45°\mathbf{i} + r\sin 45°\,\mathbf{j}$$

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{\mathbf{r}}{r} \\ &= \frac{r\cos 45°\mathbf{i} + r\sin 45°\,\mathbf{j}}{r} \\ &= \cos 45°\mathbf{i} + \sin 45°\,\mathbf{j} \\ &= \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})\end{aligned}$$

Substitution gives

$$\vec{\mathbf{E}}_{\text{at } (a,a) \text{ due to } q_1} = kq_1\frac{\hat{\mathbf{r}}}{r^2} = k\frac{q_1}{8a^2}(\cos 45°\,\hat{\boldsymbol{\imath}} + \sin 45°\,\hat{\boldsymbol{\jmath}}) = k\frac{q_1}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

which is the same result obtained in the previous example, as expected.

Note that we did not need to compute the magnitude of the electric field, E, to compute $\mathbf{E}$. It can be computed from the above equation using

$$|\mathbf{E}| = E = \sqrt{E_x^2 + E_y^2}$$

One can plug in $E_x = k\frac{q_1}{8a^2}\cos 45^\circ$ and $E_y = k\frac{q_1}{8a^2}\sin 45^\circ$ and use the identities $\sqrt{c^2} = |c|$ (where c is a real number) and $\sin^2\theta + \cos^2\theta = 1$ to show that $E = k|q_1|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{E}} = kq_1\frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{E}}| = k|q_1|\frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1 (hence the name), so

$$|\vec{\mathbf{E}}| = k|q_1|\frac{1}{r^2} = \frac{k|q_1|}{8a^2}, \text{ as before.}$$

5 Problem II

Charge q_1 is at $(x, y) = (-a, a)$. Find the electric field at $(x, y) = (a, 0)$ using $\vec{\mathbf{E}}_{\text{at }(a,0)\text{ due to }q_1} = kq_1\hat{\mathbf{r}}/r^2$ at $(x, y) = (a, a)$.

6 Problem III - Superposition

In the previous examples, only one charge was responsible for creating the electric field $\vec{\mathbf{E}}$. When there are more charges, superposition can be used.

Charge $q_1 = +q$ is at $(x, y) = (a, 0)$, charge $q_2 = +q$ is at $(x, y) = (-a, 0)$, and charge $q_3 = -q$ is at $(x, y) = (0, a)$. Assume that q is a positive number.

1. Draw this charge configuration below.

2. Why does it not make sense to ask what the electric force is at the origin?

In the following,

3. Find the electric field at the origin due to q_1. Write your answer in the form $\vec{\mathbf{E}}_1 = E_{x1}\hat{\boldsymbol{\imath}} + E_{y1}\hat{\boldsymbol{\jmath}}$.

4. Find the electric field at the origin due to q_2. Write your answer in the form $\vec{\mathbf{E}}_2 = E_{x2}\hat{\boldsymbol{\imath}} + E_{y2}\hat{\boldsymbol{\jmath}}$.

5. Find the electric field at the origin due to q_3. Write your answer in the form $\vec{\mathbf{E}}_3 = E_{x3}\hat{\boldsymbol{\imath}} + E_{y3}\hat{\boldsymbol{\jmath}}$.

6. Find the electric field at the origin. Write your answer in the form $\vec{\mathbf{E}} = E_x \hat{\boldsymbol{\imath}} + E_y \hat{\boldsymbol{\jmath}}$.

7. Will your answers to 3.-6. change if the problem had asked for the electric field at a different position? If so, which answers?

8. Find the electric field at the origin if charge $q_1 = 2q$ (instead of q).

9. Find the electric field at the origin if charge $q_1 = -2q$ (instead of q).