Electric Field and $\hat{\mathbf{r}}$

1 Overview

This activity covers topics in Section 21.4 of Young and Freedman 2015, 14th Edition.

The electric field vector, $\vec{\mathbf{E}}$, is a quantity assigned to a point in space. Given this quantity, we can compute the force on a charge Q will experience if it is placed at that point in space using the equation $\vec{\mathbf{F}} = Q\vec{\mathbf{E}}$. The direction of $\mathbf{E}$ is also the direction a charge will begin to move if released from rest.

To find $\vec{\mathbf{E}}$ at any point in space, compute the force $\vec{\mathbf{F}}$ due to all other charges on a hypothetical (or "test") charge q_o at a point where you want to know $\vec{\mathbf{E}}$. To find $\vec{\mathbf{E}}$ at that point, divide $\vec{\mathbf{F}}$ by q_o.

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_o}$$

2 Example I

Charge q_1 is at $(x, y) = (-a, -a)$. Find the electric field at $(x, y) = (a, a)$ in the form $\vec{\mathbf{E}} = E_x\hat{\boldsymbol{\imath}} + E_y\hat{\boldsymbol{\jmath}}$. Also, find E. (Note that E and $|\mathbf{E}|$ are used interchangebly.)

Solution

To find the electric field at a point in space, we put a hypothetical "test" charge q_o at that point, compute the force on it due to all other charges, and then use

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_o}$$

The force a charge q_1 at $(x, y) = (-a, -a)$ exerts on a charge q_2 at $(x, y) = (a, a)$ was computed in a previous activity. We can use that answer after replacing q_2 with q_o. The result is

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_o} = k\frac{|q_1 q_o|}{8a^2}(\cos 45^\circ\hat{\boldsymbol{\imath}} + \sin 45^\circ\hat{\boldsymbol{\jmath}})$$

We also found that this equation applies when q_1 and q_o are both positive or both are negative. If q_1 was positive and q_o was negative, or vice-versa, we found the sign changed:

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_o} = -k\frac{|q_1 q_o|}{8a^2}(\cos 45^\circ\hat{\boldsymbol{\imath}} + \sin 45^\circ\hat{\boldsymbol{\jmath}})$$

Based on this, we can write a single equation for all possibilities:

$$\vec{\mathbf{F}}_{q_1 \text{ on } q_o} = k\frac{q_1 q_o}{8a^2}(\cos 45^\circ\hat{\boldsymbol{\imath}} + \sin 45^\circ\hat{\boldsymbol{\jmath}})$$

The electric field at the location of q_o is then

$$\vec{\mathbf{E}}_{\text{at }(a,a)\text{ due to }q_1} = \frac{\vec{\mathbf{F}}}{q_o} = \frac{kq_1}{8a^2}(\cos 45^\circ\,\hat{\boldsymbol{\imath}} + \sin 45^\circ\,\hat{\boldsymbol{\jmath}}) = \frac{kq_1}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

where the fact that $\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2}$ was used.

Sign check: When computing electric fields and forces, it is easy to make a sign error. The electric field vector points in the direction a positive charge will move if released there from rest. Suppose q_1 is positive. Our equation predicts that a charge released from rest at (a,a) will move up and to the right. Suppose q_1 is negative. Our equation predicts that the charge will move down and to the left. This is consistent with the fact that like charges repel and unlike charges attract.

3 Problem I

Charge q_1 is at $(x,y) = (-a,a)$. At $(x,y) = (a,0)$, find $\mathbf{E}$ in the form $\vec{\mathbf{E}} = E_x\hat{\boldsymbol{\imath}} + E_y\hat{\boldsymbol{\jmath}}$. Check signs of the components of $\mathbf{E}$ using the technique used in Example I. Also, find E.

Answer:

From Problem II on the Electric Force Activity, if q_2 is at $(x,y) = (a,0)$ (and using the arguments in the previous example to drop the absolute value sign),

$$\vec{\mathbf{F}}_{q_1\text{ on }q_2} = k\frac{q_1 q_2}{5a^2}\left(\frac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \frac{2}{\sqrt{5}}\hat{\boldsymbol{\jmath}}\right)$$

Replacing q_2 with a test charge q_o,

$$\vec{\mathbf{F}}_{q_1\text{ on }q_o} = k\frac{q_1 q_o}{5a^2}\left(\frac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \frac{2}{\sqrt{5}}\hat{\boldsymbol{\jmath}}\right)$$

$$\vec{\mathbf{E}}_{\text{at }(a,0)\text{ due to }q_1} = \frac{\vec{\mathbf{F}}}{q_o} = \frac{kq_1}{5a^2}\left(\frac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \frac{2}{\sqrt{5}}\hat{\boldsymbol{\jmath}}\right)$$

4 The $\hat{\mathbf{r}}$ Unit Vector

Previously, when computing the electric force between two charges, you used the formula $F = k|q_1 q_2|/r^2$ to find the magnitude of the force and then used a diagram to write $\mathbf{F}$ in the form $\vec{\mathbf{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$. A similar process was used for computing $\vec{\mathbf{E}}$ above (because we calculated $\mathbf{F}$ as part of the process). The textbook provides an equation for the electric field that requires a slightly different calculation method.

The equation for the electric field using a unit vector is

$$\vec{\mathbf{E}}_{\text{due to } q_1} = kq_1\frac{\hat{\mathbf{r}}}{r^2}\ ,$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q_1 to the point in space where we want to know $\mathbf{E}$, and r is the distance between q_1 and that point.

To find $\hat{\mathbf{r}}$,

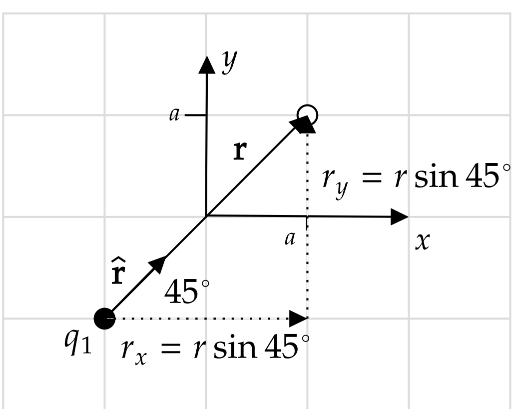
1. draw a vector, $\mathbf{r}$ from q_1 to the point in space where you want to know $\mathbf{E}$;
2. Write $\mathbf{r}$ in the form $\mathbf{r} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$; then
3. $\hat{\mathbf{r}} = \mathbf{r}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

5 Example II

If q_1 is at $(x, y) = (-a, -a)$, find the electric field at $(x, y) = (a, a)$ using $\vec{\mathbf{E}}_{\text{due to } q_1} = kq_1\hat{\mathbf{r}}/r^2$. Also, find E.

Solution

The calculation of $\hat{\mathbf{r}}$ is shown in the following diagram.



$$\mathbf{r} = r_x\mathbf{i} + r_y\mathrm{j} = r\cos 45°\mathbf{i} + r\sin 45°\ \mathrm{j}$$

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{\mathbf{r}}{r}\\ &= \frac{r\cos 45°\mathbf{i} + r\sin 45°\ \mathrm{j}}{r}\\ &= \cos 45°\mathbf{i} + \sin 45°\ \mathrm{j}\end{aligned}$$

Substitution gives

$$\vec{\mathbf{E}}_{\text{at }(a,a)\text{ due to } q_1} = kq_1\frac{1}{r^2}\hat{\mathbf{r}} = kq_1\frac{1}{8a^2}(\cos 45^\circ\hat{\boldsymbol{\imath}} + \sin 45^\circ\hat{\boldsymbol{\jmath}}) = \frac{kq_1}{8a^2}\left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]\ ,$$

which is the same result obtained in the previous example, as expected.

To calculate $\mathbf{E}$, we can use

$$|\mathbf{E}| = E = \sqrt{E_x^2 + E_y^2}$$

and plug in $E_x = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and $E_y = k\frac{q_1}{8a^2}\frac{1}{\sqrt{2}}$ and use $\sqrt{c^2} = |c|$ (where c is a real number) to show that $E = k|q_1|/8a^2$. There is an easier way. Taking the magnitude of both sides of

$$\vec{\mathbf{E}} = kq_1\frac{\hat{\mathbf{r}}}{r^2} \quad \text{gives} \quad |\vec{\mathbf{E}}| = k|q_1|\frac{|\hat{\mathbf{r}}|}{r^2}.$$

The magnitude of a unit vector is 1, so

$$|\vec{\mathbf{E}}| = k|q_1|\frac{1}{r^2} = \frac{k|q_1|}{8a^2}, \text{ as before.}$$

6 Problem II

Charge q_1 is at $(x, y) = (-a, a)$. Find the electric field at $(x, y) = (a, 0)$ using $\vec{\mathbf{E}}_{\text{at }(a,0)\text{ due to }q_1} = kq_1\hat{\mathbf{r}}/r^2$. Check signs of the components of $\mathbf{E}$ using the technique used in Example I. Also, find E.

Answer:

$$\mathbf{r} = 2a\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}, \qquad \hat{\mathbf{r}} = \frac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{5}}\hat{\boldsymbol{\jmath}}, \qquad r^2 = 5a^2$$

$$\vec{\mathbf{E}}_{\text{at }(a,0)\text{ due to }q_1} = \frac{kq_1}{5a^2}\left(\frac{2}{\sqrt{5}}\hat{\boldsymbol{\imath}} - \frac{2}{\sqrt{5}}\hat{\boldsymbol{\jmath}}\right), \text{ which matches the solution to Problem I, as expected.}$$

7 Problem III - Superposition

In the previous examples, only one charge was responsible for creating the electric field $\vec{\mathbf{E}}$. When there are more charges, superposition can be used to find the total electric field by summing $\mathbf{E}$ due to each charge.

Charge $q_1 = +q$ is at $(x, y) = (a, 0)$, charge $q_2 = +q$ is at $(x, y) = (-a, 0)$, and charge $q_3 = -q$ is at $(x, y) = (0, a)$. Assume that q is a positive number.

1. Draw this charge configuration below.

2. Why does it not make sense to ask what the electric *force* is at the origin?

There is no charge at the origin. (The electric field can be used to find the foce on a charge *if* it was placed at the origin.)

In the following,

3. Find the electric field at the origin due to q_1. Write your answer in the form $\vec{\mathbf{E}}_1 = E_{x1}\hat{\boldsymbol{\imath}} + E_{y1}\hat{\boldsymbol{\jmath}}$.

   Answer: $\vec{\mathbf{E}}_1 = -\frac{kq}{a^2}\hat{\boldsymbol{\imath}}$

4. Find the electric field at the origin due to q_2. Write your answer in the form $\vec{\mathbf{E}}_2 = E_{x2}\hat{\boldsymbol{\imath}} + E_{y2}\hat{\boldsymbol{\jmath}}$.

   Answer: $\vec{\mathbf{E}}_2 = +\frac{kq}{a^2}\hat{\boldsymbol{\imath}}$

5. Find the electric field at the origin due to q_3. Write your answer in the form $\vec{\mathbf{E}}_3 = E_{x3}\hat{\boldsymbol{\imath}} + E_{y3}\hat{\boldsymbol{\jmath}}$.

   Answer: $\vec{\mathbf{E}}_3 = -\frac{kq}{a^2}\hat{\boldsymbol{\jmath}}$

6. Find the total electric field at the origin by adding $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$. Write your answer in the form $\vec{\mathbf{E}} = E_x\hat{\boldsymbol{\imath}} + E_y\hat{\boldsymbol{\jmath}}$.

   Answer: $\vec{\mathbf{E}} = -\frac{kq}{a^2}\hat{\boldsymbol{\jmath}}$

7. Will your answers to 3.–6. change if the problem had asked for the electric field at a different

position? If so, which answers?

Yes, all answers. The electric field at a given location due to each charge depends on the distance to the location. If the location changes, the distance changes.

8. Find the electric field at the origin if charge $q_1 = 2q$ (instead of q).

Answer: $\vec{\mathbf{E}} = -\frac{kq}{a^2}\hat{\boldsymbol{\imath}} - \frac{kq}{a^2}\hat{\boldsymbol{\jmath}}$

9. Find the electric field at the origin if charge $q_1 = -2q$ (instead of q).

Answer: $\vec{\mathbf{E}} = +\frac{3kq}{a^2}\hat{\boldsymbol{\imath}} - \frac{kq}{a^2}\hat{\boldsymbol{\jmath}}$