1 Introduction

The cross product of two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ is given by $\vec{\mathbf{v}} \times \vec{\mathbf{B}}$, which is a vector quantity with a magnitude of $vB\sin\phi$, where ϕ is the angle between the two vectors.

There two methods that are commonly used for computing a cross product.

1. If the angle ϕ is given or straightforward to compute and both vectors (a) lie in a coordinate plane or (b) both vectors point along a cartesian axis, the magnitude is given by $\vec{\mathbf{v}}\vec{\mathbf{B}}\sin\phi$ and the direction can be determiend using a right–hand rule or a diagram.

   Examples of case (a):

   - The cross product of $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}$ is zero because its magnitude is $|\hat{\boldsymbol{\imath}}||\hat{\boldsymbol{\imath}}|\sin 0 = 1 \cdot 1 \cdot 0$.
   - The cross product of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ has a magnitude of $|\hat{\boldsymbol{\imath}}||\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}|\sin 45^\circ = 1 \cdot \sqrt{2} \cdot 1/\sqrt{2} = 1$. (From a diagram, it is straightforward to see that the angle between $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ is 45°). The direction $\hat{\boldsymbol{k}}$, which can be determined using the right–hand rule.

   Example of case (b):

   - The cross product $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}$ has a magnitude $|\hat{\boldsymbol{\imath}}||\hat{\boldsymbol{\jmath}}|\sin 90^\circ$. The direction is $\hat{\boldsymbol{k}}$, which is determined using the right–hand rule.

2. Write the vectors in component form and then use either the (a) "multiply through" or (a) "determinant" method.

   "Multiply through" method: Express the product $(v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}) \times (B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}} + B_z\hat{\boldsymbol{k}})$ as 9 cross products (3 of which will be zero) and evaluate each of the six cross products.

   Example: $(v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}}) \times (B_x\hat{\boldsymbol{\imath}} + B_y)$

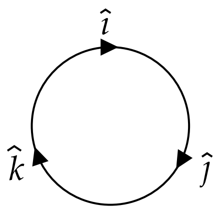
   "Determinant method":

2 Unit Vector Cross–Products

Computing the cross product of two vectors typically involves computation of cross products of unit vectors. The possible cross–products of unit vectors are

$$
\begin{array}{lll}
\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = 0 & \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = 0 & \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = 0 \\
\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} & \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \\
\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}} & \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}}
\end{array}
$$

There is a mnemonic (memory) device for remembering this table. Suppose you do the cross product of two consecutive unit vectors in the order indicated by the arrows in the circle shown in the following figure. In that case, the result is the remaining unit vector (the second row in the table). If you do the cross product of two unit vectors in reverse order, the result is the remaining unit vector with a negative sign (the third row in the table).



Example: The cross product of $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}$ requires going counterclockwise from $\hat{\boldsymbol{\jmath}}$ in the circle to $\hat{\boldsymbol{\imath}}$, so the result must be negative; the direction must be the unit vector that was not a part of the cross–product, which is $\hat{\boldsymbol{k}}$. Thus $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$.

3 Cross product review

Thus far, only the cross product of vectors with one component were considered, e.g.,

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = v_o \hat{\boldsymbol{\imath}} \times B_o \hat{\boldsymbol{\jmath}}$$

The most general case that you will encounter is

$$\mathbf{v} \times \mathbf{B} = (v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}} + v_z \hat{\boldsymbol{k}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}} + B_z \hat{\boldsymbol{k}})$$

which has a cross product of

$$\mathbf{v} \times \mathbf{B} = (v_y B_z - v_z B_y) \hat{\boldsymbol{\imath}} + (v_z B_x - v_x B_z) \hat{\boldsymbol{\jmath}} + (v_x B_y - v_y B_x) \hat{\boldsymbol{k}}$$

This formula is not easy to memorize. There are two ways to derive it

1. Using the “determinant method”: https://www.youtube.com/watch?v=2wTUqZa66ng
2. By splitting up $(v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}} + v_z \hat{\boldsymbol{k}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}} + B_z \hat{\boldsymbol{k}})$ into 9 cross products (3 of which will be zero).

When the number of components in $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ is two or less, I recommend using method 2.

In this section, you will compute the magnitude and direction of the force in two ways: 1. using $F = |q| v B \sin\phi$ and the cross–product right–hand rule and 2. using vector notation to find $\vec{\mathbf{F}} = q \vec{\mathbf{v}} \times \vec{\mathbf{B}}$ in component form and then computing the magnitude $|\vec{\mathbf{F}}|$.

The problems given are most easily solved using the first method. You are also asked to solve the problems using the second method as preparation for problems that are more easily solved this way.

3.1 Example

As an example of method 2., consider the case

$$(v_x \hat{\boldsymbol{\imath}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}} + B_z \hat{\boldsymbol{k}})$$

This can be re-written as (by doing the equivalent of “multiplying through”)

$$(v_x \hat{\boldsymbol{\imath}} \times B_x \hat{\boldsymbol{\imath}}) + (v_x \hat{\boldsymbol{\imath}} \times B_y \hat{\boldsymbol{\jmath}}) + (v_x \hat{\boldsymbol{\imath}} \times B_z \hat{\boldsymbol{k}})$$

After factoring out the constants, this is

$$v_x B_x(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}) + v_x B_y(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) + v_x B_z(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}})$$

The first term is zero because $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = 0$. The second term is $v_x B_y \hat{\boldsymbol{k}}$ because $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$. The third term is $-v_x B_z \hat{\boldsymbol{\jmath}}$ because $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}}$. Thus,

$$(v_x \hat{\boldsymbol{\imath}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}} + B_z \hat{\boldsymbol{k}}) = -v_x B_y \hat{\boldsymbol{\jmath}} + v_x B_y \hat{\boldsymbol{k}}$$

3.2 Problems

1. Use method 2. to find $(v_y \hat{\boldsymbol{\jmath}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}} + B_z \hat{\boldsymbol{k}})$
2. Use method 2. to find $(v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}})$

Answer

1. $(v_y \hat{\boldsymbol{\jmath}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}} + B_z \hat{\boldsymbol{k}}) = v_y B_x \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} + v_y B_y \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} + v_y B_z \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}$. The middle term is zero, leaving

   $v_y B_x \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} + v_y B_z \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = v_y B_z \hat{\boldsymbol{\imath}} - v_y B_x \hat{\boldsymbol{k}}$
2. $(v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}}) \times (B_x \hat{\boldsymbol{\imath}} + B_y \hat{\boldsymbol{\jmath}}) = v_x B_y \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} + v_y B_x \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}$ (The two terms that involve $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}$ have been omitted.). This leaves

   $v_x B_y \hat{\boldsymbol{k}} - v_y B_x \hat{\boldsymbol{k}} = (v_x B_y - v_y B_x) \hat{\boldsymbol{k}}$