

# LRC AC Circuits

## 1 Introduction

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Previously, we have examined circuits with either capacitors and resistors or inductors and resistors that were powered by batteries that produce constant voltage. These are called a direct current (DC) circuits. When a switch was closed to complete a circuit, the current varied in time and the time dependence was exponential (the current had a term  $e^{-t/\tau}$ ).

If instead of a battery we use a voltage source that varies in time sinusoidally, the current will also vary in time sinusoidally, possibly with a different phase. (Technically, there will also be exponential terms, but after a short amount of time they approach zero.)

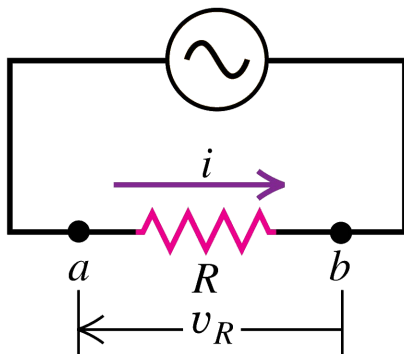
### 1.1 R Only

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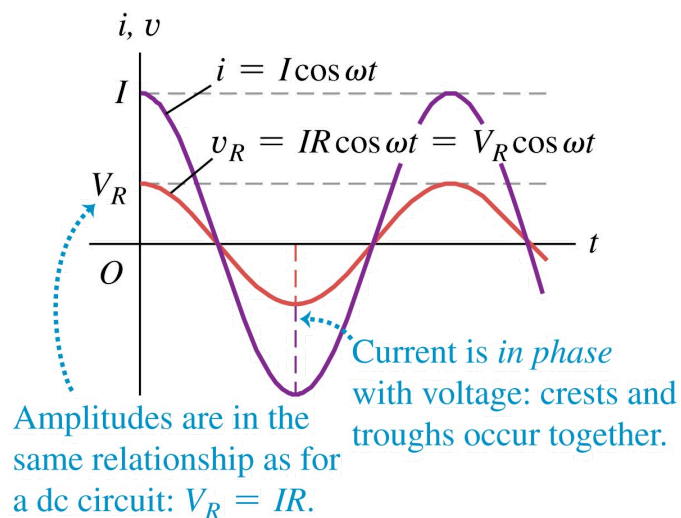
Figure (a) below shows a circuit in which an AC power source causes the current to vary sinusoidally in time according to  $i(t) = I \cos(\omega t)$ . By Ohm's law, the equation for the instantaneous voltage across the resistor is  $v_R(t) = IR \cos(\omega t) = V_R \cos(\omega t)$ .

Figure (b) shows the current and voltage. The voltage and current are “in phase” because the peaks, valleys, and zero crossings in the plots of  $i(t)$  and  $v_R(t)$  occur at the same time.

#### (a) Circuit with ac source and resistor



## (b) Graphs of current and voltage versus time



## 1.2 L Only

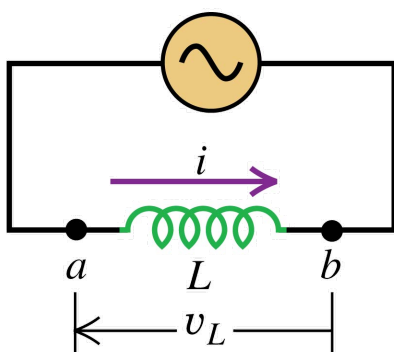
Figure (a) below shows a  $RL$  circuit in which an AC power source causes the current to vary sinusoidally in time according to  $i(t) = I \cos(\omega t)$ .

The voltage across the inductor varies according to  $v_L(t) = I\omega L \cos(\omega t + 90^\circ)$ .

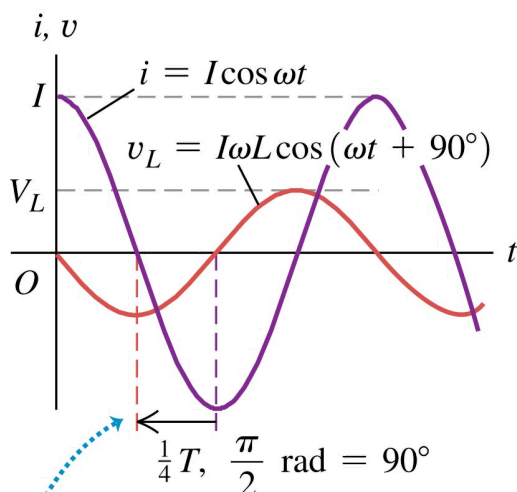
The term  $\omega L$  is called the inductive reactance:  $X_L \equiv \omega L$ . With this new variable,  $v_L(t) = IX_L \cos(\omega t + 90^\circ)$ .

Figure (b) shows the  $v(t)$  and  $i(t)$ . The voltage across the inductor “leads” the current by  $90^\circ$  (or, equivalently,  $T/4$ ) because the maxima (or minima) in  $v_L(t)$  occur before the maxima (or minima) in  $i(t)$ .

## (a) Circuit with ac source and inductor



(b) Graphs of current and voltage versus time



Voltage curve *leads* current curve by a quarter-cycle (corresponding to  $\phi = \pi/2 \text{ rad} = 90^\circ$ ).

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### 1.3 C Only

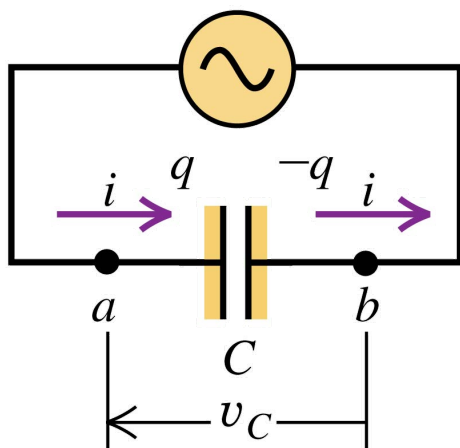
Figure (a) below shows a  $RC$  circuit in which an AC power source causes the current to vary sinusoidally in time according to  $i(t) = I \cos(\omega t)$ .

The voltage across the capacitor varies according to  $v_C(t) = \frac{I}{\omega C} \cos(\omega t - 90^\circ)$ .

The term  $1/(\omega C)$  is called the capacitive reactance:  $X_C \equiv 1/(\omega C)$ . With this new variable,  $v_C(t) = I X_C \cos(\omega t - 90^\circ)$ .

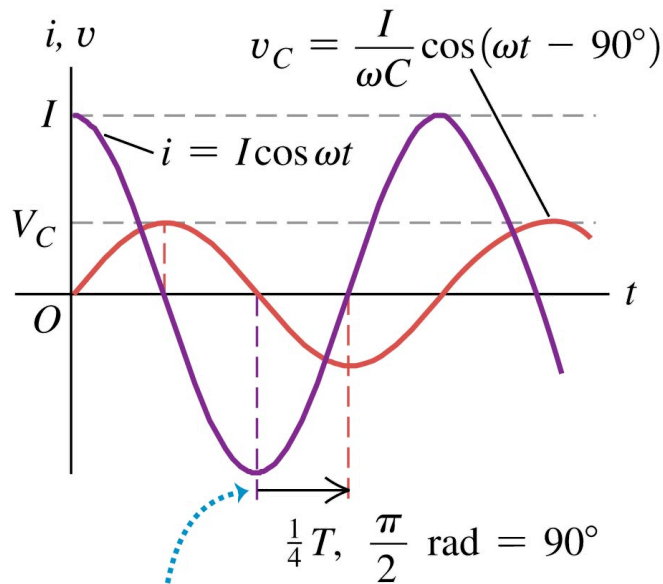
Figure (b) shows the  $v(t)$  and  $i(t)$ . The voltage across the inductor “lags” the current by  $90^\circ$  (or, equivalently,  $T/4$ ) because the maxima (or minima) in  $v_C(t)$  occur after the maxima (or minima) in  $i(t)$ .

### (a) Circuit with ac source and capacitor



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(b) Graphs of current and voltage versus time



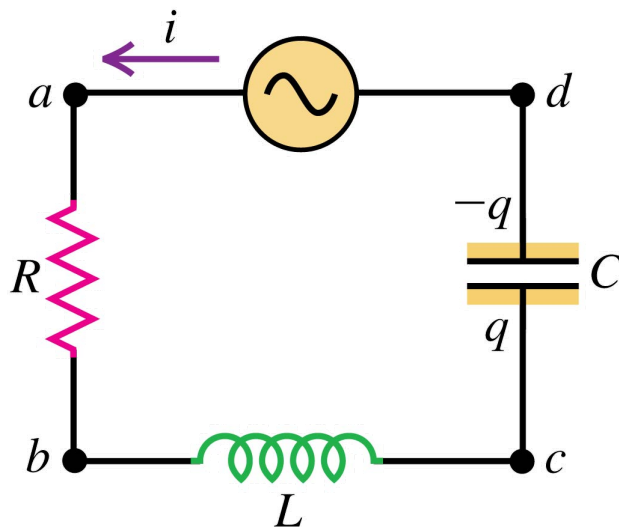
Voltage curve *lags* current curve by a quarter-cycle (corresponding to  $\phi = -\pi/2 \text{ rad} = -90^\circ$ ).

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## 1.4 Series LRC circuit

Suppose that we know the current  $i(t)$  in the following series LRC circuit is  $i(t) = I \cos(\omega t)$ .

(a)  $L$ - $R$ - $C$  series circuit



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We want to know the voltage across the AC power supply,  $v_d - v_a$ , which we call  $v$ . We know the voltage across each of the components from the discussion above:

$$v_R(t) = IR \cos(\omega t)$$

$$v_L(t) = I\omega L \cos(\omega t + 90^\circ) = IX_L \cos(\omega t + 90^\circ)$$

$$v_C(t) = \frac{I}{\omega C} \cos(\omega t - 90^\circ) = IX_C \cos(\omega t - 90^\circ)$$

From Kirchhoff's voltage law:

$$v(t) - v_R(t) - v_L(t) - v_C(t)$$

Substitution gives

$$v(t) = IR \cos(\omega t) + IX_L \cos(\omega t + 90^\circ) + IX_C \cos(\omega t - 90^\circ)$$

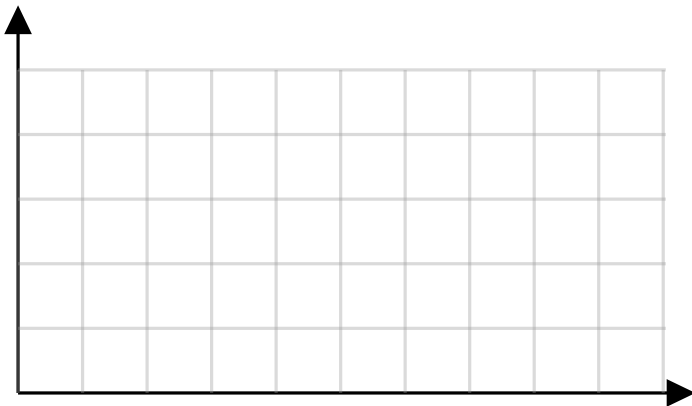
In this activity, you will compute  $v(t)$  in two ways. First, you will use the above formula and a trig identity to write  $v(t)$  in the form  $IZ \cos(\omega t + \phi)$ , where the constants  $Z$  and  $\phi$  depend on  $R$ ,  $L$ , and  $C$  (or equivalently,  $R$ ,  $X_L$ , and  $X_C$ ). Next, you will use a general formula to compute  $Z$  and  $\phi$ .

## 2 Problem I

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A series LRC circuit with known values of  $I$ ,  $R$ ,  $L$ , and  $C$  are such that  $IR = 1$  V,  $IX_L = 1$  V, and  $IX_C = 0$ . In addition, assume  $i(t) = (1 \text{ A}) \cos \omega t$  and  $\omega = 2\pi \text{ s}^{-1}$ .

Plot all quantities on the same graph.



1. Compute the period,  $T$ , of  $i(t)$
2. Plot  $i(t)$
3. Plot  $v_R(t)$
4. Plot  $v_L(t)$

5. Plot  $v_C(t)$

6. Starting with  $v(t) = v_R(t) + v_L(t) + v_C(t)$ , use the trig identity

$$A \cos(\theta) + B \cos(\theta + \pi/2) = \sqrt{A^2 + B^2} \cos(\theta + \tan^{-1}(B/A))$$

to write  $v(t)$  in the form  $v(t) = Z \cos(\omega t + \phi)$ .

That is, find the constants  $Z$  and  $\phi$ .

7. Plot  $v(t)$

### 3 Problem II

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In the previous problem, computing  $v(t)$  required the use of a trig identity to combine  $v_R$  and  $v_L$  and write  $v(t)$  in the form  $v(t) = Z \cos(\omega t + \phi)$ , where  $Z$  and  $\phi$  are constants that depend on  $L$  and  $R$ . When  $v_C$  is not zero, additional algebra is needed to compute  $v(t)$  (by using the trig identity again). However, there is formula that can be used to find  $v(t)$  in general so that trig identities are not needed to compute  $v(t)$ .

It can be shown that in general, the voltage across the AC power source is

$$v(t) = IZ \cos(\omega t + \phi)$$

Where the series LRC impedance  $Z$  is defined as

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

and

$$\phi = \tan^{-1} \left( \frac{X_L - X_C}{R} \right) = \tan^{-1} \left( \frac{\omega L - \frac{1}{\omega C}}{R} \right)$$

When  $\phi$  is positive,  $v(t)$  leads  $i(t)$ . When  $\phi$  is negative,  $v(t)$  lags  $i(t)$ . When  $\phi = 0$ ,  $v(t)$  is in phase with  $i(t)$ .

1. Using the parameters given in the previous problem, find  $v(t)$  using the above formula.

2. Does  $v(t)$  lead or lag  $i(t)$ ?