Magnetic Field of a Moving Charge

1 Introduction

In previous activities, you computed the force on moving charges in a region of space where there is a magnetic field. No mention was made of how the magnetic field was created.

In this activity, you compute the magnetic field created by moving charges.

The magnetic field due to a point charge q moving with velocity $\vec{\mathbf{v}}$ (when $|\vec{\mathbf{v}}|$ is small compared to the speed of light) is

$$\vec{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector that points from the position of q to the point in space where we want to know $\vec{\mathbf{B}}$, and r is the distance between q and that point.

To find $\hat{\mathbf{r}}$ (see also the $\hat{\mathbf{r}}$ Unit Vector activity),

1. draw a vector, $\vec{\mathbf{r}}$, from q to the point in space where you want to know $\vec{\mathbf{B}}$;
2. Write $\vec{\mathbf{r}}$ in the form $\vec{\mathbf{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$; then
3. $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$, where $r = \sqrt{r_x^2 + r_y^2}$.

In this activity, the examples and solutions are given using the above approach for computing $\vec{\mathbf{B}}$. An alternative is to use the fact that $\vec{\mathbf{v}}\times\hat{\mathbf{r}} = |\vec{\mathbf{v}}|\sin\phi = v\sin\phi$, where ϕ is the angle between $\vec{\mathbf{v}}$ and $\hat{\mathbf{r}}$ and $0 \leq \phi \leq 180^\circ$. With this, the magnitude of the magnetic field is

$$B = \frac{\mu_o}{4\pi}\frac{|q|v\sin\phi}{r^2}$$

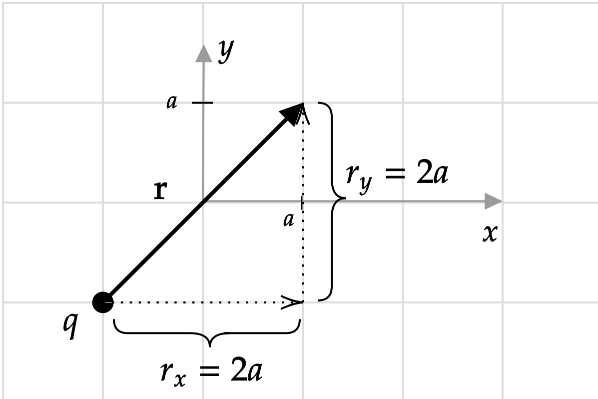
and the right–hand rule can be used to determine the direction of $\vec{\mathbf{B}}$. See the Cross Products activity for a discussion of when and how to compute the cross-product using this method.

2 Example

If q is at $(x, y) = (-a, -a)$ and has a velocity of $\vec{\mathbf{v}} = v_o\hat{\boldsymbol{\imath}}$, find $\vec{\mathbf{B}}$ at $(x, y) = (a, a)$.

Solution

To find $\hat{\mathbf{r}}$, we draw a vector from q to the point where we want to compute $\vec{\mathbf{B}}$.



Based on the diagram, $\vec{\mathbf{r}} = 2a\hat{\boldsymbol{\imath}} + 2a\hat{\boldsymbol{\jmath}}$ and $r = \sqrt{(2a)^2 + (2a)^2} = 2\sqrt{2}a$, so

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

The cross-product is

$$\vec{\mathbf{v}} \times \hat{\mathbf{r}} = v_o\hat{\boldsymbol{\imath}} \times \left[\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right] = \frac{v_o}{\sqrt{2}}(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) = \frac{v_o}{\sqrt{2}}\hat{\boldsymbol{k}}$$

Substitution into

$$\vec{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

gives

$$\vec{\mathbf{B}}(a,a) = \frac{\mu_o}{4\pi}\frac{q\frac{v_o}{\sqrt{2}}\hat{\boldsymbol{k}}}{(2\sqrt{2}a)^2} = \frac{\mu_o}{4\pi}\frac{qv_o}{(8\sqrt{2})a^2}\hat{\boldsymbol{k}}$$

Check: Use the right–hand rule for cross products on $\vec{\mathbf{v}} \times \hat{\mathbf{r}}$ to verify that the result is out of the page. (Why do we know that the $\hat{\boldsymbol{k}}$ direction is out of the page?)

3 Problem I

If q is at $(x,y) = (a,a)$ and has a velocity of $\vec{\mathbf{v}} = v_o\hat{\boldsymbol{\imath}}$, find $\vec{\mathbf{B}}$ at $(x,y) = (-a,-a)$.

Solution

$\vec{\mathbf{r}} = -2a\hat{\boldsymbol{\imath}} - 2a\hat{\boldsymbol{\jmath}}$ and $r = 2\sqrt{2}a$, so

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \left[-\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right]$$

The cross–product is

$$\vec{\mathbf{v}} \times \hat{\mathbf{r}} = v_o\hat{\boldsymbol{\imath}} \times \left[-\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}\right] = -\frac{v_o}{\sqrt{2}}(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) = -\frac{v_o}{\sqrt{2}}\hat{\boldsymbol{k}}$$

Substitution into

$$\vec{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

gives

$$\vec{\mathbf{B}}(-a,-a) = -\frac{\mu_o}{4\pi}\frac{qv_o}{(8\sqrt{2})a^2}\hat{\boldsymbol{k}}$$

Check: Using the right–hand rule for cross products on $\vec{\mathbf{v}} \times \hat{\mathbf{r}}$ confirms that the result is into the page.

4 Problem II

If q is at $(x,y) = (a,0)$ and has a velocity of $\vec{\mathbf{v}} = v_o\hat{\boldsymbol{\jmath}}$, find $\vec{\mathbf{B}}$ vector at $(x,y) = (a,a)$.

Answer: $\vec{\mathbf{B}}(a,a) = 0$ (From a diagram, $\vec{\mathbf{v}}$ and $\hat{\mathbf{r}}$ are parallel, so their cross product is zero.)

5 Problem III

If q is at $(x,y) = (a,2a)$ and has a velocity of $\vec{\mathbf{v}} = v_o\hat{\boldsymbol{\jmath}}$, find $\vec{\mathbf{B}}$ at $(x,y) = (-a,-a)$.

Answer:

$$\vec{\mathbf{B}}(-a,-a) = \frac{\mu_o}{4\pi}\frac{2qv_o\hat{\boldsymbol{k}}}{13\sqrt{13}a^2}$$

6 Problem IV

If q is at the position (x_o,y_o) and has a velocity of $\vec{\mathbf{v}} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}}$,

$$\vec{\mathbf{B}}(x,y) = \frac{\mu_o}{4\pi}\frac{q}{r^3}\left[v_x(y-y_o) - v_y(x-x_o)\right]\hat{\boldsymbol{k}}$$

where

$$r = \sqrt{(x-x_o)^2 + (y-y_o)^2}$$

1. Explain why $\vec{\mathbf{B}}$ only has a $\hat{\boldsymbol{k}}$ component.

   Answer: $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are in the x–y plane, and the result of a cross–product is a vector that is perpendicular to the plane to the two crossed vectors.

2. Use this formula to find $\vec{\mathbf{B}}$ for the example problem in section 2.

   Answer: In the example problem, q is at $(-a,-a)$ and has a velocity of $\vec{\mathbf{v}} = v_o\hat{\boldsymbol{\imath}}$, and we want to know $\vec{\mathbf{B}}$ at (a,a). In terms of the variables for the given equation, the position of the charge is $(x_o,y_o) = (-a,-a)$, the location where where we want to know $\vec{\mathbf{B}}$ is $(x,y) = (a,a)$, $v_x = v_o$, and $v_y = 0$. Substituting these values into

$r = \sqrt{(x - x_o)^2 + (y - y_o)^2}$ and $\vec{\mathbf{B}}(x, y) = \dfrac{\mu_o}{4\pi}\dfrac{q}{r^3}\left[v_x(y - y_o) - v_y(x - x_o)\right]\hat{\boldsymbol{k}}$

gives $r = \sqrt{(a - -a)^2 + (a - -a)^2} = \sqrt{8}a$ and $\vec{\mathbf{B}}(a, a) = \dfrac{\mu_o}{4\pi}\dfrac{q}{(\sqrt{8}a)^3} v_o(a - -a)\hat{\boldsymbol{k}}$

Simplification gives the same result found in the example:

$$\vec{\mathbf{B}}(a, a) = \frac{\mu_o}{4\pi}\frac{q v_o}{(8\sqrt{2})a^2}\hat{\boldsymbol{k}}$$

3. Derive this formula.

Answer: The vector from the position of q, (x_o, y_o), to the point where we want to know the field, (x, y), is

$\vec{\mathbf{r}} = (x - x_o)\hat{\boldsymbol{\imath}} + (y - y_o)\hat{\boldsymbol{\jmath}}$, so $r = \sqrt{(x - x_o)^2 + (y - y_o)^2}$.

Using this with $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$ and $\vec{\mathbf{v}} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}}$ in $\vec{\mathbf{B}} = \dfrac{\mu_o}{4\pi}\dfrac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$

gives

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi}\frac{q(v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}}) \times \dfrac{(x - x_o)\hat{\boldsymbol{\imath}} + (y - y_o)\hat{\boldsymbol{\jmath}}}{r}}{r^2}$$

or

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi}\frac{q}{r^3}(v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}}) \times \left[(x - x_o)\hat{\boldsymbol{\imath}} + (y - y_o)\hat{\boldsymbol{\jmath}}\right]$$

Using the Multiply Through method for cross–products (and dropping the terms involving $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}$) gives

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi}\frac{q}{r^3}\left[v_x\hat{\boldsymbol{\imath}} \times (y - y_o)\hat{\boldsymbol{\jmath}} + v_y\hat{\boldsymbol{\jmath}} \times (x - x_o)\hat{\boldsymbol{\imath}}\right]$$

Evaluation of the cross–products gives

$$\vec{\mathbf{B}}(x, y) = \frac{\mu_o}{4\pi}\frac{1}{r^3}\left[v_x(y - y_o) - v_y(x - x_o)\right]\hat{\boldsymbol{k}}$$