Prior to solving the problems in this tutorial, read Vector Fields.

You can experiment with the creation of vector fields using a GeoGebra app.

1 Vector Fields

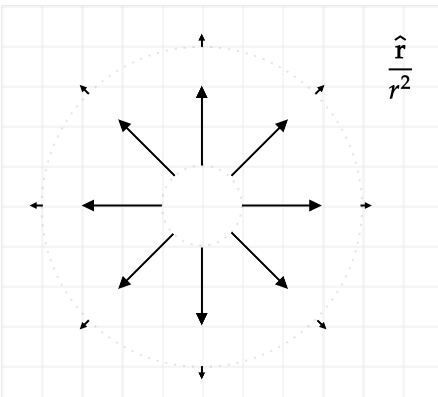
A vector field diagram is a plot showing the vectors associated with a vector function at select points in space.

Vector fields are fundamental to the theory of E&M. Many of the funamental theorems involve derivatives and integrals of vector functions that depend on position.

An example of a vector function is the electric field due to a positive point charge Q at the origin

$$\mathbf{E}(r) = \frac{kQ}{r^2}\hat{\boldsymbol{r}}$$

Th vector field diagram of $\mathbf{E}(r)/kQ = \hat{\boldsymbol{r}}/r^2$ is



Generally, only the relative lengths of the vectors is of interest and so a scale indicating a value for the length of a vector was omitted in this figure. Because the outer vectors are at a position that is 4x larger than the inner vectors, the outer vectors have a length that is 1/16 of the inner vectors.

When drawing a vector field diagram, points must be selected where a vector is drawn. The general rule is that just enough points should be selected so that the key patterns can be discerned – the reader should be able to determine the general length and direction of vectors at points without a vector on the diagram. In the previous figure, it is clear that the vectors are radial, the magnitude is is independent of radius, and the magnitude is proportional to $1/r^2$. These three facts are the key patterns associated with the vector function plotted.

1.1 Example – Vector Field with Cartesian Coordinates and Unit Vectors

Plot the vector field

$$\mathbf{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

at $x = 1$ and $y = -2, -1, 0, 1, 2$.

Answer: At the given points, $A = |\mathbf{A}| = \sqrt{5}, \sqrt{2}, 1, \sqrt{2}$, and $\sqrt{5}$. In the following diagram, the grid spacing has been assumed to be 1. Note that we could have also solved this probelm in cylindrical coordinates using $\mathbf{A} = s\hat{\mathbf{s}}$. In this case, the values of s would be given by the magnitudes of A already computed and we would need to determine the angles of the vector at each point. For example, the angle at $x = 1, y = -1$ would be $\tan^{-1}\left(\frac{+1}{-1}\right) = 135°$.

1.2 Example – Radial Field With Cylindrical Coordinate and Unit Vector

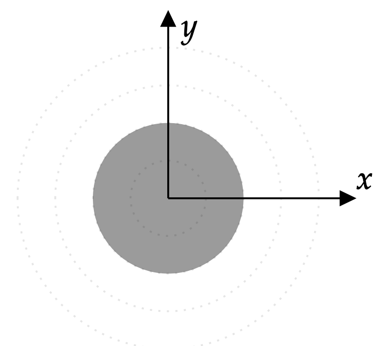
Outside of a solid and long cylinder of radius R with a uniform linear charge density of λ, the field is

$$\mathbf{E}(r) = 2k\lambda\frac{\hat{\boldsymbol{s}}}{s}$$

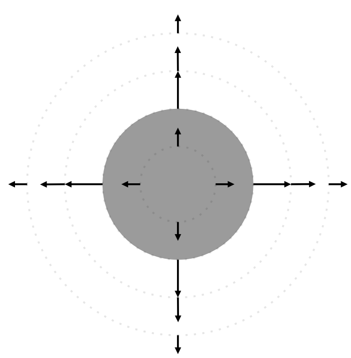
inside, it is

$$\mathbf{E}(r) = 2k\lambda\frac{s}{R^2}\hat{\boldsymbol{s}}$$

Sketch the vector field $\mathbf{A} \equiv \mathbf{E}/(2k\lambda)$ inside and outside of the cylinder if the cylinder is along the z–axis and centered on the origin as shown in the following diagram.



Answer: Assume that $R = 1$. $s = R/2$ is inside the cylinder and $\mathbf{A} = \hat{\boldsymbol{s}}/2$. At $s = R$, we can use either formula – they give the same result of $\mathbf{A} = \hat{\boldsymbol{s}}$. At $s = 1.5R$, $\mathbf{A} = 2\hat{\boldsymbol{s}}/3$. At $s = 2R$, $\mathbf{A} = \hat{\boldsymbol{s}}/2$. As one moves from the origin outwards, the vectors increase in length until $s = R$ at which point their length decreases as s increases. At a given ϕ, the vectors are all the same length.



1.3 Problem – Radial Field in Spherical

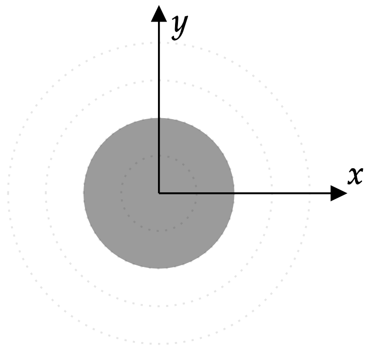
Outside of a solid sphere of radius R with uniformly distributed charge Q, the field is

$$\mathbf{E}(r) = kQ\frac{1}{r^2}\hat{\boldsymbol{r}}$$

inside, it is

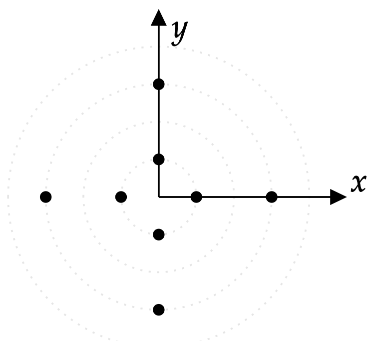
$$\mathbf{E}(r) = kQ\frac{r}{R^3}\hat{\boldsymbol{r}}$$

Sketch the vector field $\mathbf{A} \equiv \mathbf{E}/(kQ)$ inside and outside of the sphere.

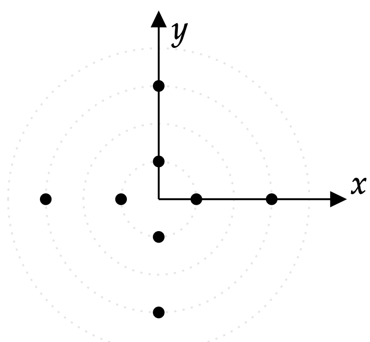


1.4 Problem – Fields in with Cartesian Unit Vectors and Cylindrical Coordinates

Plot the vector $\mathbf{A} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$ at the points shown in the following diagram.

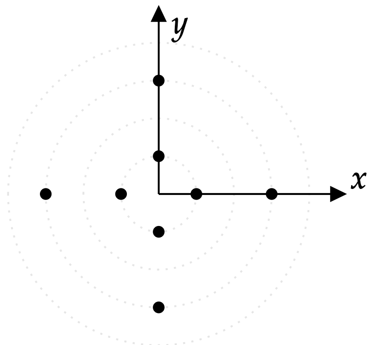


Plot the vector $\mathbf{A} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$ at the points shown in the following diagram.

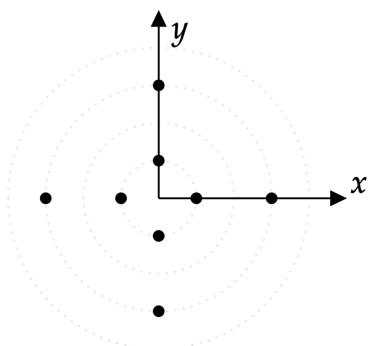


1.5 Fields with Cartesian Unit Vectors and Cartesian Coordinates

Plot the vector $\mathbf{A} = \frac{x}{s}\hat{\mathbf{x}} + \frac{y}{s}\hat{\mathbf{y}}$ at the points shown in the following diagram. Recall that the radial coordinate in cylindrical coordinates is s and $s = \sqrt{x^2 + y^2}$.

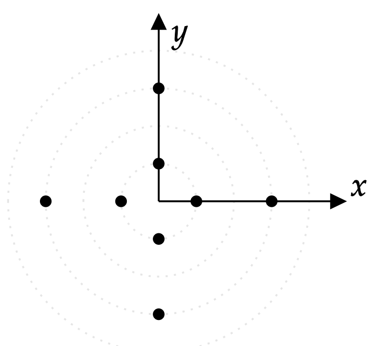


Plot the vector $\mathbf{A} = -\frac{y}{s}\hat{\mathbf{x}} + \frac{x}{s}\hat{\mathbf{y}}$ at the points shown in the following diagram.



1.6 Problem – Fields with Cylindrical Unit Vectors and Cylindrical Coordinates

Plot the vector $\mathbf{A} = s\hat{\boldsymbol{\phi}}$ at the points shown in the following diagram.



1.7 Problem – Field Due To Two Point Charges

Two point charges $\pm Q$ are at $\pm R$. The electric fields in the $x - y$ plane is

$$\mathbf{E}_{+}(r) = kQ\left[\frac{x+R}{[(x+R)^2+y^2]^{3/2}}\hat{\mathbf{x}} + \frac{y}{[(x+R)^2+y^2]^{3/2}}\hat{\mathbf{y}}\right]$$

$$\mathbf{E}_{-}(r) = -kQ\left[\frac{x-R}{[(x-R)^2+y^2]^{3/2}}\hat{\mathbf{x}} + \frac{y}{[(x-R)^2+y^2]^{3/2}}\hat{\mathbf{y}}\right]$$

Plot $\mathbf{A} \equiv (\mathbf{E}_{+} + \mathbf{E}_{-})/kQ$ at

- $x = 0, y = R, 2R, 3R$
- $x = 0, y = -R, -2R, -3R$